2

Controlled Systems

2.1 The Time-Continuous Case

2.1.1 The Problem of Controllability

We start with a system of differential equations of the form

$$\dot{x}_i = f_i(x,u), \qquad i = 1,\ldots,n \tag{2.1}$$

where $x \in \mathrm{I\!R}^n,\ u \in \mathrm{I\!R}^m$,

$$f_i : \mathrm{I\!R}^n \times \mathrm{I\!R}^m \to \mathrm{I\!R}$$

with $f_i \in \mathcal{C}\left(\mathrm{I\!R}^{n+m}, \mathrm{I\!R}\right)$ and $f_i(\cdot,u) \in \mathcal{C}^1\left(\mathrm{I\!R}^n, \mathrm{I\!R}\right)$ for every $u \in \mathrm{I\!R}^m$ and for $i = 1,\ldots,n$.

Assumption: For every function $u \in \mathcal{C}^1\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ and every point $x_0 \in \mathrm{I\!R}^n$ there is exactly one function $x \in \mathcal{C}^1\left(\mathrm{I\!R}, \mathrm{I\!R}^n\right)$ with

$$\dot{x}_i(t) = f_i(x(t), u(t)), \quad t \in \mathrm{I\!R} \tag{2.2}$$

$$\text{for} \quad i = 1,\ldots,n \quad \text{and}$$

$$x(0) = x_0. \tag{2.3}$$

We consider every function $u \in \mathcal{C}\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ as a control of the system (2.1) that is used in order to transfer every initial state $x_0 \in \mathrm{I\!R}^n$ within a given time interval $[0,T]$ into a final state $x_T \in \mathrm{I\!R}^n$, i.e., we look for a control function $u \in \mathcal{C}\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ such that the unique solution $x \in \mathcal{C}^1\left(\mathrm{I\!R}, \mathrm{I\!R}^n\right)$ of (2.2), (2.3) satisfies the condition

$$x(T) = x_T. \tag{2.4}$$

In the following we are mainly interested in final states that are rest points of the system (2.1) for $u = \Theta_m$ =null vector of $\mathrm{I\!R}^m$ (which we also call the uncontrolled system).

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We therefore assume that the system (2.1) for $u = \Theta_m$ possesses a rest point $\widehat{x} \in \mathbb{R}^n$ which is a solution of the equations

$$f_i\left(\widehat{x}, \Theta_m\right) = 0 \qquad \text{for} \quad i = 1, \ldots, n. \tag{2.5}$$

Let Ω be a non-empty subset of $\mathbb{R}^m$ with $\Theta_m \in \Omega$.

Definition.

(1) The system (2.1) is called $\Omega-$controllable on the interval $[0, T]$ with $T > 0$, if for every pair $x_0, x_T \in \mathbb{R}^n$ there is a control $u \in \mathcal{C}\left(\mathbb{R}, \mathbb{R}^m\right)$ with

$$u(t) \in \Omega \qquad \textit{for all} \quad t \in [0, T]$$

such that the unique solution $x \in \mathcal{C}^1\left(\mathbb{R}, \mathbb{R}^n\right)$ of (2.2), (2.3) satisfies condition (2.4).

(2) The system (2.1) is called $\Omega-$controllable, if there exists some $T > 0$ such that it is $\Omega-$controllable on the interval $[0, T]$.

(3) The system (2.1) is called completely $\Omega-$controllable, if it is $\Omega-$controllable on every interval $[0, T]$, $T > 0$.

The complete $\Omega-$controllability is obviously the strongest property. In the next section we will present sufficient conditions for linear controlled systems to be completely $\Omega-$controllable.

In the case of non-linear systems we will concentrate on the following problem: For a given $x_0 \in \mathbb{R}^n$ and $x_T = \widehat{x}$ =solution of (2.5) and a time $T > 0$ find a control function $u \in \mathcal{C}\left(\mathbb{R}, \mathbb{R}^m\right)$ such that the unique solution $x \in \mathcal{C}^1\left(\mathbb{R}, \mathbb{R}^n\right)$ of (2.2), (2.3) satisfies condition (2.4).

2.1.2 Controllability of Linear Systems

Instead of (2.1) we consider a system of the form

$$\dot{x} = A\,x + B\,u \tag{2.6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A =$ real $n \times n-$matrix and $B =$ real $n \times m-$matrix. If we put

$$f(x, u) = A\,x + B\,u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

then (2.6) is of the form (2.1) and $f \in \mathcal{C}^1\left(\mathbb{R}^{n+m}, \mathbb{R}^n\right)$.

The assumption in Section 2.1.1 is satisfied.

For every $x_0 \in \mathbb{R}^n$ and every function $u \in \mathcal{C}\left(\mathbb{R}, \mathbb{R}^m\right)$ the unique solution $x \in \mathcal{C}^1\left(\mathbb{R}, \mathbb{R}^n\right)$ of (2.6) with $x(0) = x_0$ is given by

$$x(t) = \mathrm{e}^{tA}\left(x_0 + \int_0^t \mathrm{e}^{-sA}\,B\,u(s)\,\mathrm{d}s\right), \qquad t \in \mathbb{R}. \tag{2.7}$$

We put

$$Y(t) = \mathrm{e}^{-tA}\,B \qquad \text{for all} \quad t \in \mathbb{R} \tag{2.8}$$

and consider control functions of the form

$$u(t) = Y^T\,x, \qquad t \in \mathbb{R}, \tag{2.9}$$

where $x \in \mathbb{R}^n$ is chosen arbitrarily. Insertion into (2.7) gives

$$x(t) = \mathrm{e}^{tA}\left(x_0 + \int_0^t Y(s)\,Y(s)^T\,x\;\mathrm{d}s\right), \qquad t \in \mathbb{R}.$$

Let, for some given $T > 0$, the $n \times n-$matrix

$$M(T) = \int_0^T Y(t)\,Y(t)^T\;\mathrm{d}t \tag{2.10}$$

(which is symmetric and positive semi-definite) be non-singular.

Then, for every vector $x_T \in \mathbb{R}^n$, there is exactly one $z_T \in \mathbb{R}^n$ such that

$$x_T = \mathrm{e}^{TA}\,(x_0 + M(T)\,z_T) \quad\Longleftrightarrow\quad M(T)\,z_T = \mathrm{e}^{-TA}\,x_T - x_0$$

which implies that the system (2.6) is $\mathbb{R}^m-$controllable on $[0, T]$.

Lemma 2.1. *For some $T > 0$ the matrix $M(T)$ (2.10) is non-singular, if and only if the following implication holds true*

$$Y(t)^T\,x = \Theta_m \qquad \textit{for all} \quad t \in [0, T] \quad\Longrightarrow\quad x = \Theta_n. \tag{2.11}$$

Proof.
(a) Let $M(T)$ be non-singular. If there were some $z \in \mathbb{R}^n$ with $z \neq \Theta_n$ such that $Y(t)^T\,z = \Theta_n$ for all $t \in [0, T]$, then it would follow that $M(T)\,z = \Theta_n$ which is impossible.

(b) Let the implication (2.11) hold true. If $M(T)$ were singular, then there would exist some $z \in \mathbb{R}^n,\ z \neq \Theta_n$, with $M(T)\,z = \Theta_n$.

This implies

$$z^T\,M(T)\,z = \int_0^T z^T\,Y(t)\,Y(t)^T\,z\;\mathrm{d}t = 0$$

from which it follows that

$$Y(t)^T\,z = \Theta_m \qquad \text{for all} \quad t \in [0, T].$$

However, this implies $z = \Theta_n$ contradicting $z \neq \Theta_n$. Therefore, $M(T)$ must be non-singular. □

Theorem 2.2. *The implication (2.11) holds true for every $T > 0$, if and only if the so called Kalman condition*

$$\operatorname{rank}\left(B \mid AB \mid \ldots \mid A^{n-1}B\right) = n \tag{2.12}$$

is satisfied.

Proof.
(a) Let the implication (2.11) be violated for some $T > 0$. Then there is some $z \in \mathbb{R}^n, \; z \neq \Theta_n$ with

$$Y(t)^T z = \Theta_m \qquad \text{for all} \quad t \in [0, T].$$

If one differentiates $z^T Y(t)$ successively by t and puts $t = 0$ then it follows that

$$z^T B = \Theta_m^T, \; z^T AB = \Theta_m^T, \ldots, \; z^T A^{n-1} B = \Theta_m^T \tag{2.13}$$

which contradicts condition (2.12).
(b) Let condition (2.12) be violated. Then there is a vector $z \in \mathbb{R}^n$ with $z \neq \Theta_n$ such that (2.13) holds true. Now let

$$\Phi(-\lambda) = a_0 + a_1 (-\lambda) + \ldots + a_{n-1} (-\lambda)^n$$

be the characteristic polynomial of A. Then it follows, by the theorem of Cayley-Hamilton, that $\Phi(-A) = 0$ and, therefore,

$$A^n = b_0 I + b_1 A + \ldots + b_{n-1} A^{n-1}$$

with suitable coefficients $b_0, \ldots, b_{n-1} \in \mathbb{R}$. This implies

$$A^k = b_0 A^{k-n} + b_1 A^{k+1-n} + \ldots + b_{n-1} A^{k-1}$$

for all $k \geq n$ and on using (2.13) one concludes that

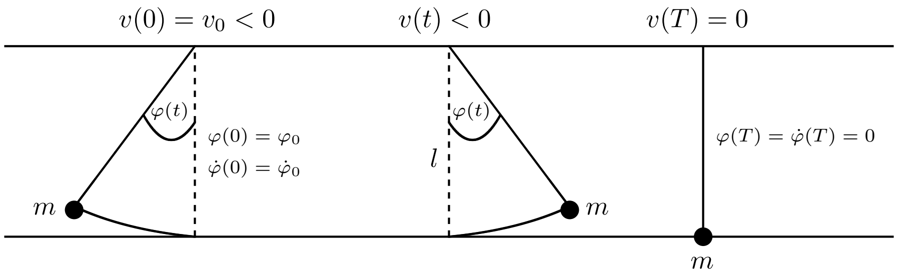
$$z^T A^k B = \Theta_m^T \quad \text{for all} \quad k \geq 0 \qquad \Longrightarrow \qquad z^T Y(t) = 0 \quad \text{for all} \quad t \in \mathbb{R},$$

i.e., the condition (2.11) is violated for all $T > 0$. □

As a consequence of the above considerations we obtain

Theorem 2.3. *If the Kalman condition (2.12) is satisfied, then the system (2.6) is completely* $\mathbb{R}^m$*−controllable.*

Let us demonstrate this result by an example. We consider a moving linear pendulum as being depicted in the following picture:



The movement of the pendulum is described by the differential equation

$$\ddot{\varphi}(t) = -\frac{g}{l}\,\varphi(t) - \frac{\ddot{v}(t)}{l}, \qquad t \in \mathbb{R}, \tag{2.14}$$

with the initial conditions

$$\varphi(0) = \varphi_0, \qquad \dot{\varphi}(0) = \dot{\varphi}_0$$

where $\varphi = \varphi(t)$ denotes the deviation angle at the time t and $v = v(t)$ is the location of the suspension point at the time t. This is moving with speed $\dot{v}(t)$ and acceleration $\ddot{v}(t)$ at the time t. The problem to be solved consists of finding a function $v = v(t)$ by which an initial state $(\varphi(0), \dot{\varphi}(0), v(0), \dot{v}(0)) = (\varphi_0, \dot{\varphi}_0, v_0, \dot{v}_0)$ of the pendulum is transferred in the final (rest) state $(\varphi(T), \dot{\varphi}(T), v(T), \dot{v}(T)) = (0, 0, 0, 0)$ at a given time $T > 0$.

If we define a vector function $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ by putting $x_1(t) = \varphi(t),\ x_2(t) = \dot{\varphi}(t),\ x_3(t) = v(t),\ x_4(t) = \dot{v}(t)$, then the differential equation (2.14) can be rewritten in the form

$$\dot{x}(t) = A\,x(t) + B\,u(t), \qquad t \in \mathbb{R}, \tag{2.15}$$

where $u(t) = \ddot{v}(t),\ t \in \mathbb{R}$, is the control and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -g/l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1/l \\ 0 \\ 1 \end{pmatrix}.$$

In this case we obtain

$$\big(B \mid AB \mid A^2B \mid A^3B\big) = \begin{pmatrix} 0 & -1/l & 0 & g/l^2 \\ -1/l & 0 & g/l^2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

hence

$$\operatorname{rank}\big(B \mid AB \mid A^2B \mid A^3B\big) = 4,$$

i.e., the Kalman condition (2.12) is satisfied.

By Theorem 2.3 the system (2.15) is completely $\mathbb{R}^4-$controllable. This means that every initial state $(\varphi(0), \dot{\varphi}(0), v(0), \dot{v}(0))^T \in \mathbb{R}^4$ of the pendulum can be controlled within every time interval $[0, T],\ T > 0$, by a suitable control function $u(t) = \ddot{v}(t),\ t \in [0, T]$ with $u \in \mathcal{C}^1(\mathbb{R})$ into every final state $(\varphi(T), \dot{\varphi}(T), v(T), \dot{v}(T))^T \in \mathbb{R}^4$.

2.1.3 Restricted Null-Controllability of Linear Systems

We start with a system

$$\dot{x}(t) = A\,x(t), \qquad t \in \mathrm{I\!R}, \tag{2.16}$$

where $x \in \mathrm{I\!R}^n$ and $A =$real $n \times n-$matrix.

This system has $\widehat{x} = \Theta_n$ as rest point. In the following we make the special choice $\Omega = [-1, +1]^m$ and ask for the existence of some time $T > 0$ and a control function $u \in \mathcal{C}\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ with

$$u(t) \in \Omega \qquad \text{for all} \quad t \in [0, T] \tag{2.17}$$

such that for every initial state $x_0 \in \mathrm{I\!R}^n$ the unique solution $x \in \mathcal{C}^1\left(\mathrm{I\!R}, \mathrm{I\!R}^n\right)$ of (2.6) with

$$x(0) = x_0 \tag{2.3}$$

satisfies the end condition

$$x(T) = \widehat{x} = \Theta_n. \tag{2.18}$$

In order to give an affirmative answer to this question we replace the space $\mathcal{C}\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ of control functions by the space $L^\infty\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ of measurable and essentially bounded $m-$vector functions u on $\mathrm{I\!R}$. Condition (2.17) is replaced by

$$||u(t)||_\infty \leq 1 \qquad \text{for almost all} \quad t \in [0, T] \tag{2.19}$$

where $||\cdot||_\infty$ denotes the maximum norm in $\mathrm{I\!R}^m$.

For every $T > 0$ we define

$$\mathcal{U}_T = \left\{ u \in L^\infty\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right) \mid u \text{ satisfies } (2.19) \right\}. \tag{2.20}$$

From the solution formula (2.7) which also holds true for the unique absolutely continuous solution $x : \mathrm{I\!R} \to \mathrm{I\!R}^n$ of (2.6) with (2.3) in the case that $u \in L^\infty\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ it follows that the end condition (2.18) is equivalent to

$$\int_0^T Y(t)\,u(t)\;\mathrm{d}t = -x_0 \tag{2.21}$$

where $Y(t)$ is given by (2.8).

This leads to the
Problem of Restricted Null-Controllability: Let $x_0 \in \mathrm{I\!R}^n$ be given arbitrarily. Find some $T > 0$ and $u \in \mathcal{U}_T$ (2.20) such that condition (2.21) is satisfied.

If this problem has a solution, we call the system (2.6) restricted null-controllable.

If we define, for every $T > 0$, a so called reachable set

$$E(T) = \left\{ x = \int_0^T Y(t)\, u(t)\ \mathrm{d}t \mid u \in \mathcal{U}_T \right\} \subseteq \mathrm{I\!R}^n, \tag{2.22}$$

then the problem of restricted null-controllability is solvable, if and only if

$$E = \bigcup_{T>0} E(T) = \mathrm{I\!R}^n. \tag{2.23}$$

Obviously every reachable set $E(T)$ is convex and we have the following implication

$$0 \leq T_1 < T_2 \quad \Longrightarrow \quad E\,(T_1) \subseteq E\,(T_2)\,.$$

This implies that E is also convex.

Assumption: $E \neq \mathrm{I\!R}^n$.

Then there exists some $\widehat{x} \in \mathrm{I\!R}^n$ with $\widehat{x} \notin E$. By a well known separation theorem for convex sets there exists a number $\alpha \in \mathrm{I\!R}$ and a vector $y \in \mathrm{I\!R}^n,\ y \neq \Theta_n$ such that

$$y^T x \leq \alpha < y^T \widehat{x} \qquad \text{for all} \quad x \in E. \tag{2.24}$$

Because of $\Theta_n \in E$ it follows that $\alpha \geq 0$. Further the left side of (2.24) is equivalent to

$$\int_0^T y^T\, Y(t)\, u(t)\ \mathrm{d}t \leq \alpha \qquad \text{for all} \quad u \in \mathcal{U}_T \quad \text{and} \quad T > 0. \tag{2.25}$$

For the following we assume that the Kalman condition (2.12) is satisfied.

Because of $y \neq \Theta_n$ it follows from Theorem 2.2 that for every $T > 0$ there exists some $t_T \in [0, T]$ with

$$y^T\, Y\,(t_T) \neq \Theta_m^T.$$

If we put

$$v(t)^T = y^T\, Y(t), \qquad t \in \mathrm{I\!R},$$

then there exists at least one component of v which does not vanish identically, say

$$v_1(t) = y^T\, \mathrm{e}^{-tA}\, b_1, \qquad t \in \mathrm{I\!R},$$

where b_1 is the first column of the matrix B.

Since with u also $-u$ belongs to $\mathcal{U}_T,\ T > 0$, it follows from (2.25) that

$$\left| \int_0^\infty v(t)^T\, u(t)\ \mathrm{d}t \right| \leq \alpha \qquad \text{for all} \quad u \in L^\infty\,(\mathrm{I\!R}, \mathrm{I\!R}^m) \quad \text{with}$$

$$||u(t)||_\infty \leq 1 \qquad \text{for almost all} \quad t \in (0, \infty).$$

This implies

$$\int_0^\infty ||v(t)||_1 \,\mathrm{d}t = \sum_{i=1}^n \int_0^\infty |v_i(t)| \,\mathrm{d}t \le \alpha < \infty$$

and further

$$\int_0^\infty |v_1(t)| \,\mathrm{d}t \le \alpha.$$

This implies the existence of the integrals

$$w(t) = \int_t^\infty v_1(s) \,\mathrm{d}s \qquad \text{for all} \quad t \in [0,\infty)$$

and it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = -v_1(t) \qquad \text{for all} \quad t \in [0,\infty) \quad \text{as well as} \quad \lim_{t\to\infty} w(t) = 0.$$

If we put $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$, it follows that

$$\left(D^k v_1\right)(t) = y^T(-A)^k \,\mathrm{e}^{-tA} b_1, \qquad t \in \mathbb{R}, \quad \text{for} \quad k = 0,1,2,\ldots$$

If $\psi(\lambda)$ denotes the characteristic polynomial of A, then the application of the Cayley-Hamilton-theorem implies $\psi(A) = 0$ and therefore

$$(\psi(-D)\,v_1)(t) = y^T(\psi(A))\,\mathrm{e}^{-tA} b_1 = 0 \qquad \text{for all} \quad t \in [0,\infty).$$

This implies

$$\psi(-D)(-D\,w)(t) = 0 \qquad \text{for all} \quad t \in [0,\infty)$$

and because $(-D)$ and $\psi(-D)$ are interchangeable we can conclude

$$(-D\,\psi(-D)\,w)(t) = 0 \qquad \text{for all} \quad t \in [0,\infty).$$

The characteristic equation of this linear differential equation reads

$$-\lambda\,\psi(-\lambda) = 0.$$

From $\lim_{t\to\infty} w(t) = 0$ we infer that at least one solution $-\lambda$ of the equation $\psi(-\lambda) = 0$ must have a negative real part. Therefore the matrix A must have an eigenvalue with a positive real part.

Summarizing we obtain by contraposition the

Theorem 2.4. *If the Kalman condition (2.12) is satisfied and if the matrix A has only eigenvalues with non-positive real parts, then the system (2.6) is restricted null-controllable. This means that for every initial state $x_0 \in \mathbb{R}^n$ there is some $T > 0$ and a control function $u \in \mathcal{U}_T$ (2.20) such that the unique absolutely continuous solution $x = x(t)$, $t \in \mathbb{R}$, of (2.6), (2.3) satisfies the end condition (2.18).*

The assumptions of this theorem are for instance satisfied for the system (2.15). It has been shown already that the Kalman condition (2.12) is satisfied. It is easy to show that the eigenvalues of the matrix A are given by

$$\lambda_1 = i\sqrt{\frac{g}{l}}, \quad \lambda_2 = -i\sqrt{\frac{g}{l}}, \quad \lambda_3 = \lambda_4 = 0.$$

2.1.4 Controllability of Nonlinear Systems into Rest Points

We again start with a system of the form

$$\dot{x} = f(x,u) \tag{2.1}$$

where $f \in \mathcal{C}\left(\mathrm{I\!R}^n \times \mathrm{I\!R}^m, \mathrm{I\!R}^n\right)$ and $f(\cdot,u) \in \mathcal{C}^1\left(\mathrm{I\!R}^n, \mathrm{I\!R}^m\right)$ for every $u \in \mathrm{I\!R}^m$. We make the same assumptions as at the beginning of Section 2.1.1.

Let $\widehat{x} \in \mathrm{I\!R}^n$ be a rest point of the uncontrolled system for $u \equiv \Theta_m$, i.e., a solution of the equation

$$f\left(\widehat{x}, \Theta_m\right) = \Theta_n. \tag{2.5}$$

Given an arbitrary initial state $x_0 \in \mathrm{I\!R}^n$ we then look for some time $T > 0$ and a control function $u \in \mathcal{C}\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ such that the unique solution $x \in \mathcal{C}^1\left(\mathrm{I\!R}, \mathrm{I\!R}^n\right)$ of (2.1) with

$$x(0) = x_0 \tag{2.3}$$

satisfies the end condition

$$x(T) = \widehat{x}.$$

If this problem has a solution for every $x_0 \in \mathrm{I\!R}^n$ we call the system (2.1) controllable to $\widehat{x}$.

Let us weaken this concept to a concept of local controllability. For this purpose we assume that

$$f_i \in \mathcal{C}^1\left(\mathrm{I\!R}^n \times \mathrm{I\!R}^m\right) \qquad \text{for all} \quad i = 1, 2, \ldots, n$$

and linearize the equation (2.1) in $(\widehat{x}, \Theta_m)$, i.e., we replace it by

$$\dot{x} = A\,x + B\,u \tag{2.6}$$

where

$$A = \left(f_{i x_j}\left(\widehat{x}, \Theta_m\right)\right)_{i,j=1,\ldots,n} \qquad \text{and} \qquad B = \left(f_{i u_k}\left(\widehat{x}, \Theta_m\right)\right)_{\substack{i=1,\ldots,n \\ k=1,\ldots,m}}. \tag{2.26}$$

Definition. *The system (2.1) is called locally controllable to $\widehat{x}$, if the system (2.6) with A and B by (2.26) is $\mathrm{I\!R}^m-$null-controllable, i.e., if for every $x_0 \in \mathrm{I\!R}^n$ there exists a $T > 0$ and a control function $u \in \mathcal{C}\left(\mathrm{I\!R}, \mathrm{I\!R}^m\right)$ such that the unique solution $x \in \mathcal{C}^1\left(\mathrm{I\!R}, \mathrm{I\!R}^n\right)$ of (2.6), (2.3) satisfies the end condition*

$$x(T) = \Theta_m. \tag{2.18}$$

An immediate consequence of Theorem 2.3 is the

Theorem 2.5. *If A and B given by (2.26) satisfy the Kalman condition (2.12), then the system (2.1) is locally controllable to $\widehat{x}$.*

Definition. *The system (2.1) is called locally restricted controllable to $\widehat{x}$, if the system (2.6) with A and B by (2.26) is restricted null-controllable, i.e., if for every $x_0 \in \mathbb{R}^n$ there exists a $T > 0$ and a control function $u \in \mathcal{U}_T$ (2.20) such that the unique absolutely continuous solution $x = x(t)$, $t \in \mathbb{R}$, of (2.6), (2.3) satisfies the end condition (2.18).*

An immediate consequence of Theorem 2.4 is the

Theorem 2.6. *If A and B given by (2.26) satisfy the Kalman condition (2.12) and if the matrix A has only eigenvalues with non-positive real parts, then the system (2.1) is locally restricted controllable to $\widehat{x}$.*

Let us demonstrate these two theorems by the example of a nonlinear pendulum with movable suspension point. We use the same notations as in the case of a linear pendulum in Section 2.1.2.

The movement of the pendulum is described by the differential equation

$$\ddot{\varphi}(t) = \frac{g}{l} \sin \varphi(t) - \frac{\ddot{v}(t)}{l} \cos \varphi(t), \qquad t \in \mathbb{R}. \tag{2.27}$$

If one defines functions

$$x_1(t) = \varphi(t), \quad x_2(t) = \dot{\varphi}(t), \quad x_3(t) = v(t), \quad x_4(t) = \dot{v}(t),$$

then (2.27) can be rewritten in the form

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\frac{g}{l} \sin x_1(t) - \frac{u(t)}{l} \cos x_1(t), \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= u(t) \end{aligned} \tag{2.28}$$

where $u(t) = \ddot{v}(t)$, $t \in \mathbb{R}$, is the control function. If we define

$$f(x, u) = (f_1(x, u), f_2(x, u), f_3(x, u), f_4(x, u))^T, \qquad x \in \mathbb{R}^4, \ u \in \mathbb{R},$$

where

$$\begin{aligned} f_1(x, u) &= x_2(t), \\ f_2(x, u) &= -\frac{g}{l} \sin x_1 - \frac{u}{l} \cos x_1, \\ f_3(x, u) &= x_4, \\ f_4(x, u) &= u, \end{aligned}$$

then the system (2.28) takes the form (2.1) and we have

$$f_i \in \mathcal{C}^1\left(\mathbb{R}^4, \mathbb{R}\right) \qquad \text{for} \quad i = 1, 2, 3, 4.$$

Obviously for $u \equiv 0$ the point $\Theta_4 = (0,0,0,0)^T \in \mathbb{R}^4$ is a rest point of the system (2.28) and we obtain

$$A = \left(f_{ix_j}\left(\Theta_4, 0\right)\right)_{i,j=1,2,3,4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -g/l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \left(f_{iu}\left(\Theta_4, 0\right)\right)_{i=1,2,3,4} = \begin{pmatrix} 0 \\ -1/l \\ 0 \\ 1 \end{pmatrix}.$$

We have already shown (see Section 2.1.2) that the Kalman condition (2.12) is satisfied for A and B and that the matrix A has only eigenvalues with real parts equal to zero (see Section 2.1.3).

Theorem 2.6 then implies that the system is locally restricted controllable to $\widehat{x} = \Theta_4$.

Now we return to the question of the $\widehat{x}-$controllability of the system (2.1). We again assume a set $\Omega \subseteq \mathbb{R}^n$ with $\Theta_m \in \Omega$ to be given and define, for every $T > 0$,

$$\mathcal{U}_T = \left\{u \in L^\infty\left(\mathbb{R}, \mathbb{R}^m\right) \mid u(t) \in \Omega \text{ for almost all } t \in [0, T]\right\}. \tag{2.29}$$

Further let $S\left(\widehat{x}, T\right)$, for every $T > 0$, be the set of all $x_0 \in \mathbb{R}^n$ such that there exists a $u \in \mathcal{U}_T$ such that the unique absolutely continuous solution $x = x(t),\ t \in \mathbb{R}$, of (2.1), (2.3) satisfies the end condition $x(T) = \widehat{x}$.

If we then define

$$S\left(\widehat{x}\right) = \bigcup_{T>0} S\left(\widehat{x}, T\right), \tag{2.30}$$

the set $S\left(\widehat{x}\right)$ consists of all vectors $x_0 \in \mathbb{R}^n$ such that there exists a time $T > 0$ and a control $u \in \mathcal{U}_T$ such that the unique absolutely continuous solution $x = x(t),\ t \in \mathbb{R}$, of (2.1), (2.3) satisfies the end condition $x(T) = \widehat{x}$.

With this definition we formulate the

Theorem 2.7. *Let $\widehat{x}$ be an interior point of $S\left(\widehat{x}\right)$ ($\widehat{x} \in S\left(\widehat{x}\right)$ follows from the definition of $S\left(\widehat{x}\right)$) and let $\widehat{x}$ be attractive, i.e., there exists an open set $\mathcal{U} \subseteq \mathbb{R}^n$ with $\widehat{x} \in \mathcal{U}$ such that for every $x_0 \in \mathcal{U}$ the unique solution $x \in \mathcal{C}^1\left(\mathbb{R}, \mathbb{R}^n\right)$ of $\dot{x} = f\left(x, \Theta_m\right)$ with $x(0) = x_0$ satisfies the statement $\lim_{t\to\infty} x(t) = \widehat{x}$. Then it follows that $S\left(\widehat{x}\right) \supseteq \mathcal{U}$.*

Proof. Let $x_0 \in \mathcal{U}$ be chosen arbitrarily. Then for the solution $x \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ of $\dot{x} = f(x, \Theta_m)$ with $x(0) = x_0$ it follows that $\lim_{t\to\infty} x(t) = \widehat{x}$. Since $\widehat{x}$ is an interior point of $S(\widehat{x})$, there exists some $t_1 > 0$ with $x(t_1) \in S(\widehat{x})$. This implies the existence of some time $T > 0$ and a function $u \in \mathcal{U}_T$ such that the absolutely continuous solution $\widetilde{x} = \widetilde{x}(t)$ of

$$\dot{\widetilde{x}}(t) = f(\widetilde{x}(t), u(t)), \qquad t \in \mathbb{R} \quad \text{with} \quad \widetilde{x}(0) = x(t_1)$$

satisfies the end condition $\widetilde{x}(T) = \widehat{x}$.

If we define

$$u^*(t) = \begin{cases} \Theta_m & \text{for} \quad t \le t_1, \\ u(t - t_1) & \text{for} \quad t > t_1, \end{cases}$$

then $u^* \in \mathcal{U}_T$ and it follows for the absolutely continuous solution $x^* = x^*(t),\ t \in \mathbb{R}$, of $\dot{x}^*(t) = f(x^*(t), u^*(t)),\ t \in \mathbb{R},\ x^*(0) = x(0) = x_0$ that the end condition $x^*(T + t_1) = \widetilde{x}(T) = \widehat{x}$ is satisfied which completes the proof. □

Let us apply this result to linear systems of the form (2.6). In this case we have

$$f(x, u) = A\,x + B\,u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m.$$

A rest point of the uncontrolled system

$$\dot{x} = f(x, \Theta_m) = A\,x, \qquad x \in \mathbb{R}^n, \tag{2.16}$$

is given by $\widehat{x} = \Theta_n$. This is the only one, if the matrix A is non-singular. If all the eigenvalues of A have negative real parts (which implies that A is non-singular), then $\widehat{x} = \Theta_n$ is globally attractive, i.e., for every $x_0 \in \mathbb{R}^n$ the unique solution $x \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ of (2.16) with $x(0) = x_0$ satisfies the condition $\lim_{t\to\infty} x(t) = \Theta_n$.

Theorem 2.8. *If the Kalman condition (2.12) is satisfied, then $\widehat{x} = \Theta_n$ is an interior point of $S(\Theta_n) = E$ (2.23).*

Proof. Assume that Θ_n is no interior point of E. Then Θ_n is also not an algebraically interior point of E and $K(E) = \bigcup_{\lambda \ge 0} \lambda E$ is not equal to $\mathbb{R}^n$. Since because of $E = -E$ the convex cone $K(E)$ is a linear space, the set E, as subset of $K(E)$, is contained in a hyperplane, i.e., there exists some $y \in \mathbb{R}^n$ with $y \ne \Theta_n$ such that

$$y^T e = 0 \qquad \text{for all} \quad e \in E.$$

In particular it follows for every $T > 0$ that

$$\int_0^T y^T \mathrm{e}^{tA} B\,u(t)\ \mathrm{d}t = 0 \qquad \text{for all} \quad u \in \mathcal{U}_T$$

and hence for all $u \in L^\infty([0,T], \mathbb{R}^m) = \overline{\bigcup_{\lambda \geq 0} \lambda \mathcal{U}_T}$.

This implies

$$y^T \mathrm{e}^{-tA} B = 0 \qquad \text{for all} \quad t \in [0,T]$$

and in turn $y = \Theta_n$ by Theorem 2.2 which contradicts $y \neq \Theta_n$. □

As a consequence of Theorem 2.7 and Theorem 2.8 we obtain

Theorem 2.9. *If the Kalman condition (2.12) is satisfied and if the matrix A has only eigenvalues with negative real parts, then for every $x_0 \in \mathbb{R}^n$ there exists some $T > 0$ and a control function $u \in \mathcal{U}_T$ (2.20) such that the unique absolutely continuous solution $x = x(t)$, $t \in \mathbb{R}$, of (2.6), (2.3) (which is given by (2.7)) satisfies the end condition $x(T) = \widehat{x}$ (2.18).*

This theorem is contained in Theorem 2.4.

2.1.5 Approximate Solution of the Problem of Restricted Null-Controllability

We again consider the problem of restricted null-controllability as in Section 2.1.3. We assume $T > 0$ and $x_0 \in \mathbb{R}^n$ to be given and formulate the following

Approximation Problem: Find some $u_T \in \mathcal{U}_T$ (2.20) such that

$$\left\| \int_0^T Y(t)\, u_T(t)\; \mathrm{d}t + x_0 \right\|_2 \leq \left\| \int_0^T Y(t)\, u(t)\; \mathrm{d}t + x_0 \right\|_2 \qquad \text{for all} \quad u \in \mathcal{U}_T$$

where $Y(t)$ is given by (2.8).

A solution $u_T \in \mathcal{U}_T$ of this approximation problem is considered as an approximation of a solution of the problem of restricted null-controllability.

If one defines

$$V = \left\{ y \in \mathbb{R}^n \mid y = \int_0^T Y(t)\, u(t)\; \mathrm{d}t \text{ with } u \in \mathcal{U}_T \right\},$$

one obtains a convex subset of $\mathbb{R}^n$. This set is weakly closed, i.e., the following implication holds true:

$$\left\{ \begin{array}{c} \lim\limits_{k \to \infty} y_k^T\, x = y^T x \;\; \text{for all} \quad x \in \mathbb{R}^n \\ \text{and} \quad y_k \in V \;\; \text{for all} \quad k \in \mathbb{N} \end{array} \right\} \quad \Longrightarrow \quad y \in V.$$

Proof. $y_k \in V$ implies for every $k \in \mathbb{N}$ the existence of some function

$u_k \in \{u \in L^\infty([0,T], \mathbb{R}^m) \mid ||u(t)||_\infty \leq 1 \text{ for almost all } t \in [0,T]\} = K_\infty(T)$

with

$$y_k = \int_0^T Y(t)\, u_k(t)\ \mathrm{d}t$$

which implies

$$\lim_{k\to\infty} \int_0^T u_k(t)^T\, Y(t)\, x\ \mathrm{d}t = y^T x \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Since the set $K_\infty(T)$ is weak-$*$ sequentially compact, there is some $u \in K_\infty(T)$ and a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ with

$$\lim_{i\to\infty} \int_0^T u_{k_i}(t)^T\, Y(t)\, x\ \mathrm{d}t = \int_0^T u(t)^T\, Y(t)\, x\ \mathrm{d}t$$

which implies

$$\left(y^T - \int_0^T u(t)^T\, Y(t)^T\ \mathrm{d}t\right) x = 0 \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

From this it follows that

$$y = \int_0^T Y(t)\, u(t)\ \mathrm{d}t \quad \Longleftrightarrow \quad y \in V$$

which completes the proof. □

The weak closedness of the set V implies that V is also closed.

As a consequence of a well known theorem in approximation theory we then obtain the existence of exactly one $y^* \in V$ with

$$||y^* + x_0||_2 \leq ||y + x_0||_2 \qquad \text{for all} \quad y \in V$$

which is characterized by

$$(y^* + x_0)^T\ y^* \leq (y^* + x_0)^T\ y \qquad \text{for all} \quad y \in V,$$

if $y^* + x_0 \neq \Theta_n$.

Now we distinguish two cases:

(a) $y^* + x_0 = \Theta_n$. Then every $u^* \in \mathcal{U}_T$ with

$$y^* = \int_0^T Y(t)^T\, u^*(t)\ \mathrm{d}t$$

is a solution of the problem of restricted null-controllability.

b) $y^* + x_0 \neq \Theta_n$. Then for every $u^* \in \mathcal{U}_T$ as in (a) we obtain from the above characterization of $y^* \in V$ the equation

$$-\int_0^T (y^* + x_0)^T\, Y(t)\, u^*(t)\ \mathrm{d}t = \int_0^T ||\,(y^* + x_0)^T\, Y(t)||_1\ \mathrm{d}t.$$

We try to solve this equation iteratively by starting with

$$y^0 = \Theta_n \quad \text{and} \quad u_k^0(t) = -\mathrm{sgn}\left(x_0^T\, Y(t)\right)_k \qquad \text{for} \quad k = 1, \ldots, n,\ t \in [0, T],$$

(where $\mathrm{sgn}(0) = 0$) and constructing a sequence $\left(y^N\right)_{N \in \mathbb{N}_0}$ in $\mathbb{R}^n$ and a sequence $\left(u^N\right)_{N \in \mathbb{N}_0}$ in $K_\infty(T)$ as follows:

If $y^N \in \mathbb{R}^n$ is given, then we define

$$u_k^N(t) = -\mathrm{sgn}\left(\left(y^N + x_0\right)^T\ Y(t)\right)_k \qquad \text{for} \quad k = 1, \ldots, n,\ t \in [0, T],$$

and

$$y^{N+1} = \int_0^T Y(t)\, u^N(t)\ \mathrm{d}t.$$

If the sequence $\left(u^N\right)_{N \in \mathbb{N}_0}$ converges weak-$*$ to $u^* \in K_\infty(T)$, then the sequence $\left(y^N\right)_{N \in \mathbb{N}_0}$ to $y^* = \int_0^T Y(t)\, u^*(t)\ \mathrm{d}t$ and y^* and u^* satisfy the equation in (b).

2.1.6 Time-Minimal Restricted Null-Controllability of Linear Systems

We replace the set $\mathcal{U}_T$ (2.20) in Section 2.1.3 by

$$\mathcal{U} = \left\{ u \in L^\infty\left(\mathbb{R}, \mathbb{R}^m\right) \mid \operatorname*{ess\,sup}_{t \in \mathbb{R}} ||u(t)||_2 \leq \gamma \right\} \tag{2.31}$$

where $||u(t)||_2 = \left(u(t)^T u(t)\right)^{1/2},\ t \in \mathbb{R}$.

The problem of restricted null-controllability now consists of finding a time $T > 0$ and some $u \in \mathcal{U}$ (2.31) such that

$$\int_0^T Y(t)\, u(t)\ \mathrm{d}t = -x_0 \tag{2.21}$$

where $x_0 \in \mathbb{R}^n$ is given arbitrarily.

Let us assume that the problem is solvable. Then we define a so called minimum time $T(\gamma)$ by

$$T(\gamma) = \inf\{T > 0 \mid \text{ There is some } u \in \mathcal{U} \text{ which satisfies (2.21)}\}. \tag{2.32}$$

At first we prove

Theorem 2.10. *If restricted null-controllability is possible, then time-minimal restricted null-controllability is also possible, i.e., there exists some* $u_\gamma \in \mathcal{U}$ *such that*

$$\int_0^{T(\gamma)} Y(t)\, u_\gamma(t)\ dt = -x_0. \tag{2.33}$$

Proof. By the definition of $T(\gamma)$ there exists a sequence $(T_k)_{k\in\mathbb{N}}$ of times $T_k \geq T(\gamma)$ with $\lim\limits_{k\to\infty} T_k = T(\gamma)$ and corresponding controls $u_k \in \mathcal{U}$ with

$$\int_0^{T_k} Y(t)\, u_k(t)\ \mathrm{d}t = -x_0 \qquad \text{for all} \quad k \in \mathbb{N}.$$

Since the set $\mathcal{U}$ is sequentially weak-$*$ compact, there is a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ of the sequence $(u_k)_{k\in\mathbb{N}}$ and some $u_\gamma \in \mathcal{U}$ such that

$$\lim_{i\to\infty} \int_0^{T(\gamma)} y(t)^T\, u_{k_i}(t)\ \mathrm{d}t = \int_0^{T(\gamma)} y(t)^T\, u_\gamma(t)\ \mathrm{d}t$$
$$\text{for all} \quad y \in L^1\left([0, T(\gamma)], \mathbb{R}^m\right)$$

which implies

$$\lim_{i\to\infty} \int_0^{T(\gamma)} Y(t)^T\, u_{k_i}(t)\ \mathrm{d}t = \int_0^{T(\gamma)} Y(t)\, u_\gamma(t)\ \mathrm{d}t.$$

For every $i \in \mathbb{N}$ we then have

$$\begin{aligned}&\int_0^{T(\gamma)} Y(t)\, u_\gamma(t)\ \mathrm{d}t \\ &= \underbrace{\int_0^{T(\gamma)} Y(t)\left(u_\gamma(t) - u_{k_i}(t)\right)\ \mathrm{d}t}_{\to\, \Theta_m} + \underbrace{\int_0^{T_{k_i}} Y(t)\, u_{k_i}(t)\ \mathrm{d}t}_{=\,-x_0} - \int_{T(\gamma)}^{T_{k_i}} Y(t)\, u_{k_i}(t)\ \mathrm{d}t\end{aligned}$$

From this we infer because of $\lim\limits_{i\to\infty} T_{k_i} = T(\gamma)$ and therefore

$$\lim_{i\to\infty} \int_{T(\gamma)}^{T_{k_i}} Y(t)\, u_{k_i}(t)\ \mathrm{d}t = \Theta_m$$

that (2.33) is satisfied which completes the proof. □

Every $u_\gamma \in \mathcal{U}$ which satisfies (2.33) is called a time-minimal null-control. The next step is to characterize time-minimal null-controls.

For this purpose we make use of the

Normality Condition: For every $T > 0$ and every $y \in \mathbb{R}^n$ the components of the vector function $Y(t)^T y$ only vanish on a subset of $[0, T]$ of (Lebesgue-) measure zero unless $y = \Theta_n$.

At first we consider the

Minimum Norm Problem: For $T > 0$ and $x_0 \in \mathbb{R}^n$ given determine $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ such that (2.21) is satisfied and the functional φ defined by

$$\varphi(u) = ||u||_{L^\infty([0,T],\mathbb{R}^m)} = \operatorname*{ess\ sup}_{t\in[0,T]} ||u(t)||_2 \tag{2.34}$$

is minimized.

In order to study the problem of minimizing $\varphi(u)$ defined by (2.34) on the linear manifold

$$M = \left\{ u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \,\middle|\, \int_0^T Y(t)\, u(t)\ \mathrm{d}t = -x_0 \right\} \tag{2.35}$$

we also consider the
Dual Problem: Determine $y \in \mathbb{R}^n$ such that

$$\int_0^T \left(y^T\, Y(t) Y(t)^T\, y\right)^{1/2}\ \mathrm{d}t \le 1 \tag{2.36}$$

holds true and the functional

$$\psi(y) = -x_0^T\, y \tag{2.37}$$

is minimized.

Let

$$N = \{y \in \mathbb{R}^n \mid\ (2.36) \text{ is satisfied}\}. \tag{2.38}$$

If $u \in M$ and $y \in N$, then it follows that

$$\begin{aligned}
\psi(y) = -x_0^T\, y &= \int_0^T u(t)^T\, Y(t)^T\, y\ \mathrm{d}t \\
&\le \int_0^T \left(u(t)^T u(t)\right)^{1/2} \left(y^T\, Y(t) Y(t)^T\, y\right)^{1/2}\ \mathrm{d}t \\
&\le \operatorname*{ess\ sup}_{t\in[0,T]} ||u(t)||_2 \int_0^T \left(y^T\, Y(t) Y(t)^T\, y\right)^{1/2}\ \le \varphi(u)
\end{aligned}$$

which implies

$$\sup_{y\in N} \psi(y) \le \inf_{u\in M} \varphi(u). \tag{2.39}$$

The dual problem will now be used to solve the minimum norm problem.

At first we observe that the normality condition implies that for every $T > 0$ the function

$$y \to \chi(y) = \int_0^T \left(y^T Y(t) Y(t)^T y\right)^{1/2} \, \mathrm{d}t, \qquad y \in \mathbb{R}^n,$$

is Gateaux-differentiable and its Gateaux-derivative is given by

$$D\chi(y,h) = \nabla\chi(y)^T h, \qquad h \in \mathbb{R}^n,$$

where

$$\nabla\chi(y) = \int_0^T \frac{1}{\left(y^T Y(t) Y(t)^T y\right)^{1/2}} \, Y(t) Y(t)^T y \, \mathrm{d}t.$$

Now we can prove

Theorem 2.11. *Under the normality condition the minimum norm problem has exactly one solution (on* $[0,T]$*) for every choice of* $T > 0$ *which is of the form*

$$u_T(t) = \frac{-x_0^T y_T}{\left(y_T^T Y(t) Y(t)^T y_T\right)^{1/2}} \, Y(t)^T y_T \tag{2.40}$$

for almost all $t \in [0,T]$ *and* $u_T(t) = \Theta_m$ *for all* $t \notin [0,T]$

where $y_T \in \mathbb{R}^n$ *is a solution of the dual problem and satisfies*

$$\int_0^T \left(y_T^T Y(t) Y(t)^T y_T\right)^{1/2} \, dt = 1. \tag{2.41}$$

Proof. Since the set of all $y \in \mathbb{R}^n$ which satisfy (2.36) is compact and the function $\psi(y) = -x_0^T y$ is continuous there is some $y_T \in \mathbb{R}^n$ which solves the dual problem and every solution satisfies (2.41). Hence the dual problem has the same solutions, if we assume equality to hold in (2.36).

Let $y_T \in \mathbb{R}^n$ be a solution of the dual problem. Then by the Lagrangean multiplier rule there is a multiplier $\lambda_T \in \mathbb{R}$ such that

$$\nabla\psi\,(y_T) - \lambda_T \nabla\chi\,(y_T) = \Theta_n$$

which is equivalent to

$$-x_0 = \lambda_T \int_0^T \frac{1}{\left(y_T^T Y(t) Y(t)^T y_T\right)^{1/2}} \, Y(t) Y(t)^T y_T \, \mathrm{d}t$$

and implies

$$-x_0^T y_T = \lambda_T \int_0^T \left(y_T^T Y(t) Y(t)^T y_T\right)^{1/2} \, \mathrm{d}t = \lambda_T.$$

Therefore, if we define $u_T = u_T(t)$ by (2.40), then (2.21) is satisfied for $u = u_T$. Further it follows that

$$\left(u_T(t)^T u_T(t)\right)^{1/2} = -x_0^T\, y_T = \sup_{y\in N} \psi(y) \qquad \text{for almost all} \quad t \in [0,T]$$

which implies $\varphi\left(u_T\right) = \sup_{y\in N} \psi(y)$. From (2.39) it then follows that u_T solves the minimum norm problem.

It remains to prove the uniqueness of u_T. Let $\widehat{u} \in M$ be any solution of the minimum norm problem. In order to find a solution $u \in L^\infty\left(\mathbb{R}, \mathbb{R}^m\right)$ of (2.21) we tentatively put $u(t) = Y(t)^T y$ for $t \in \mathbb{R}$. Then (2.21) reads

$$\int_0^T Y(t)Y(t)^T \;\mathrm{d}t\, y = -x_0. \tag{2.42}$$

The normality condition implies that the row vector functions $y_i^T = y_i(t)^T,\ t \in \mathbb{R},\ i = 1,\ldots,n$ of $Y = Y(t)$ are linearly independent. From this it follows that the matrix $\int_0^T Y(t)Y(t)^T \;\mathrm{d}t$ is non-singular so that for every choice of $x_0 \in \mathbb{R}^n$ the linear system (2.42) has exactly one solution $y \in \mathbb{R}^n$. Therefore the mapping

$$u \mapsto \int_0^T Y(t)\, u(t)\;\mathrm{d}t \qquad \text{from} \quad L^\infty\left(\mathbb{R}, \mathbb{R}^m\right) \quad \text{into} \quad \mathbb{R}^n$$

is surjective. This implies that there exists a multiplier $y \in \mathbb{R}^n$ such that

$$\varphi\left(\widehat{u}\right) \le \varphi(u) - y^T \left(\int_0^T Y(t)\, u(t)\;\mathrm{d}t + x_0\right) \tag{2.43}$$

for all $u \in L^\infty\left(\mathbb{R}, \mathbb{R}^m\right)$, hence

$$\varphi\left(\widehat{u}\right) + y^T x_0 \le \varphi(u) - \int_0^T y^T\, Y(t)\, u(t)\;\mathrm{d}t$$

for all $u \in L^\infty\left(\mathbb{R}, \mathbb{R}^m\right)$ which is only possible, if

$$\varphi(u) - \int_0^T y^T\, Y(t)\, u(t)\;\mathrm{d}t \ge 0 \qquad \text{for all} \quad u \in L^\infty\left(\mathbb{R}, \mathbb{R}^m\right).$$

This is equivalent to

$$\int_0^T y^T Y(t)\, u(t)\; \mathrm{d}t \le \operatorname*{ess\,sup}_{t\in[0,T]} \left(u(t)^T u(t)\right)^{1/2}$$
$$\text{for all} \quad u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$$

or

$$\int_0^T y^T Y(t)\, u(t)\; \mathrm{d}t \le 1 \text{ for all } u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \text{ with}$$
$$\operatorname*{ess\,sup}_{t\in[0,T]} \left(u(t)^T u(t)\right)^{1/2} = 1.$$

If we define

$$u(t) = \frac{1}{\left(y^T Y(t) Y(t)^T y\right)^{1/2}}\; Y(t)^T y$$
$$\text{for almost all } t \in [0,T] \text{ and } u(t) = \Theta_m \text{ for all } t \notin [0,T]$$

then it follows that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left(u(t)^T u(t)\right)^{1/2} = 1$$

and therefore

$$\int_0^T \left(y^T Y(t) Y(t)^T y\right)^{1/2} \; \mathrm{d}t = \int_0^T y^T Y(t)\, u(t)\; \mathrm{d}t \le 1.$$

Thus $y \in N$. If we choose $u \equiv \Theta_m$ in (2.43) it follows that $\varphi(\widehat{u}) \le -x_0^T y$. On the other hand we have shown (in the proof of (2.39)) that $-x_0^T y \le \varphi(\widehat{u})$. Therefore we obtain

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left(\widehat{u}(t)^T \widehat{u}(t)\right)^{1/2} = -x_0^T y = \int_0^T \widehat{u}(t)^T Y(t)^T \; \mathrm{d}t\, y$$

which is only possible, if

$$\widehat{u}(t) = \frac{-x_0^T y}{\left(y^T Y(t) Y(t)^T y\right)^{1/2}}\; Y(t)^T y \qquad \text{for almost all} \quad t \in [0,T]$$

This implies

$$\left(\widehat{u}(t)^T \widehat{u}(t)\right)^{1/2} = \varphi(\widehat{u}) = \varphi(u_T) = \left(u_T(t)^T u_T(t)\right)^{1/2}$$
$$\text{for almost all} \quad t \in [0,T].$$

Since $w(t) = \frac{1}{2}\left(\widehat{u}(t) + u_T(t)\right),\ t \in [0, T]$, is also a solution of the minimum norm problem, it follows

$$||\widehat{u}(t)||_2 = ||u_T(t)||_2 = ||w(t)||_2 \qquad \text{for almost all} \quad t \in [0, T]$$

which is only possible, if

$$\widehat{u}(t) = u_T(t) = w(t) \qquad \text{for almost all} \quad t \in [0, T].$$

This completes the proof. □

Now we make a first step towards a characterization of time-minimal null-controls by proving the

Theorem 2.12. *If restricted null-controllability holds and if the normality condition is satisfied, then the minimum time $T(\gamma)$ defined by (2.32) is also given by*

$$T(\gamma) = \inf\{T > 0 \mid v(T) \leq \gamma\} \tag{2.44}$$

where

$$v(T) = \max\left\{ -x_0^T y \,\middle|\, \int_0^T \left(y^T Y(t) Y(t)^T y\right)^{1/2} \mathrm{d}t \leq 1 \right\} \tag{2.45}$$

Proof. Let us put

$$\widehat{T}(\gamma) = \inf\{T > 0 \mid v(T) \leq \gamma\}.$$

Then it is clear that $\widehat{T}(\gamma) \leq T(\gamma)$ because of

$$\{T > 0 \mid (2.21) \text{ is satisfied for some } u \in \mathcal{U}\ (2.31)\} \subseteq \{T > 0 \mid v(T) \leq \gamma\}$$

which is a consequence of (2.39).

If $\widehat{T}(\gamma) < T(\gamma)$, then by, Theorem 2.11, for every $T \in \left(\widehat{T}(\gamma), T(\gamma)\right)$ there is some $u_T \in \mathcal{U}$ which satisfies (2.21) and for which $\varphi(u_T) = v(T) \leq \gamma$ holds true. This, however, contradicts the definition (2.32) of $T(\gamma)$ which completes the proof. □

Theorem 2.11 can be strengthened to

Theorem 2.13. *In addition to the assumption of Theorem 2.12 let the function $T \to v(t)$ (2.45) from $(0, \infty)$ into $(0, \infty)$ be continuous. Then it follows that*

$$v(T(\gamma)) = \gamma \tag{2.46}$$

and for each $u_\gamma \in \mathcal{U}$ with (2.33) it follows that $\varphi(u_\gamma) = v(T(\gamma))$, i.e., every time-minimal null-control is a minimum norm control of $[0, T(\gamma)]$ (we tacitly assume that $x_0 \neq \Theta_n$).

Proof. $x_0 \neq \Theta_n$ implies $T(\gamma) > 0$ and $\lim\limits_{T \to 0+} v(T) = \infty$.

By virtue of the continuity of $T \to v(T)$ it follows from (2.44) that $v(T(\gamma)) \leq \gamma$. Let us assume that $v(T(\gamma)) < \gamma$. By the intermediate value theorem for continuous functions then it follows that there is some $T^* \in (T, T(\gamma))$ (where $v(T) > \gamma$) with $v(T^*) = \gamma$. This implies the existence of $u_{T^*} \in \mathcal{U}$ which satisfies (2.21) and $\varphi(u_{T^*}) = v(T^*) = \gamma$. This is a contradiction to the definition (2.32) of $T(\gamma)$. Hence (2.46) must be true from which the last assertions follow immediately. □

In view of Theorem 2.11 we have the following

Corollary 2.14. *Under the assumptions of Theorem 2.13 there is exactly one time-minimal control on* $[0, T(\gamma)]$ *which is the minimum norm control* $u_{T(\gamma)}(t) \in \mathcal{U}$ *on* $[0, T(\gamma)]$ *and is given by*

$$u_{T(\gamma)}(t) = \frac{\gamma}{\left(y_{T(\gamma)}^T Y(t) Y(t)^T y_{T(\gamma)}\right)^{1/2}} \, Y(t)^T y_{T(\gamma)} \qquad \text{for almost all} \quad t \in [0, T(\gamma)] \tag{2.47}$$

where $y_{T(\gamma)} \in \mathbb{R}^n$ *is a solution of the dual problem to the minimum norm problem for* $T = T(\gamma)$.

The continuity assumption for the function $T \to v(T)$ in Theorem 2.13 can be dispensed with, since it is a consequence of the normality condition.

In order to show this we at first observe that in the case $x_0 \neq \Theta_n$, for every $T > 0$, $v(T)$ defined by (2.45) is also given by $v(T) = 1/\lambda(T)$ where

$$\lambda(T) = \min \left\{ \int_0^T \left(y^T Y(t) Y(t)^T y\right)^{1/2} \, \mathrm{d}t \;\middle|\; -x_0^T y = 1 \right\} \tag{2.48}$$

Theorem 2.15. *If the normality condition holds, then the function* $T \to \lambda(T)$ *(2.48),* $T > 0$*, is strictly increasing and continuous.*

Proof. Let $T_2 > T_1$ be given. Let

$$\lambda(T_k) = \int_0^{T_k} \left(y_{T_k}^T Y(t) Y(t)^T y_{T_k}\right)^{1/2} \, \mathrm{d}t \qquad \text{for} \quad k = 1, 2$$

with $-x_0^T y_{T_k} = 1$. For brevity we put, for every $y \in \mathbb{R}^n$,

$$||Y(t)^T y||_2 = \left(y^T Y(t) Y(t)^T y\right)^{1/2}, \qquad t \in \mathbb{R}.$$

At first we get

$$\begin{aligned}\lambda(T_1) &= \int_0^{T_1} \|Y(t)^T y_{T_1}\|_2 \,\mathrm{d}t \\ &\le \int_0^{T_1} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t \le \int_0^{T_2} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t = \lambda(T_2)\,.\end{aligned}$$

It even follows that

$$\begin{aligned}\lambda(T_2) &= \int_0^{T_2} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t \\ &= \int_0^{T_1} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t + \int_{T_1}^{T_2} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t \\ &\ge \lambda(T_1) + \int_{T_1}^{T_2} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t\end{aligned}$$

where

$$\int_{T_1}^{T_2} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t > 0,$$

since otherwise $Y(t)\,y_{T_2} = \Theta_m$ for all $t \in [T_1, T_2]$ which, by the normality condition, implies $y_{T_2} = \Theta_n$ and contradicts $-x_0^T y_{T_2} = 1$.

As a result we obtain $\lambda(T_2) > \lambda(T_1)$ which shows that $T \to \lambda(T)$ is strictly increasing. On the other hand we also have

$$\begin{aligned}\lambda(T_2) &= \int_0^{T_2} \|Y(t)^T y_{T_2}\|_2 \,\mathrm{d}t \le \int_0^{T_2} \|Y(t)^T y_{T_1}\|_2 \,\mathrm{d}t \\ &= \int_0^{T_1} \|Y(t)^T y_{T_1}\|_2 \,\mathrm{d}t + \int_{T_1}^{T_2} \|Y(t)^T y_{T_1}\|_2 \,\mathrm{d}t \\ &= \lambda(T_1) + \int_{T_1}^{T_2} \|Y(t)^T y_{T_1}\|_2 \,\mathrm{d}t,\end{aligned}$$

hence

$$0 \le \lambda(T_2) - \lambda(T_1) \le \int_{T_1}^{T_2} \|Y(t)^T y_{T_1}\|_2 \,\mathrm{d}t$$

which implies

$$\lim_{T_2 \to T_1 + 0} \lambda(T_2) = \lambda(T_1)$$

and shows the right continuity of the function $T \to \lambda(T)$.

The proof of the left continuity requires some preparation.

We choose $T_0 \in (0, T_1)$ arbitrarily and consider the function

$$y \to \int_0^{T_0} ||Y(t)^T y||_2 \, \mathrm{d}t$$

from $\mathbb{R}^n$ into $(0, \infty)$ for which

$$m_{T_0} = \inf \left\{ \int_0^{T_0} ||Y(t)^T y||_2 \, \mathrm{d}t \,\middle|\, ||y||_2 = \left(y^T y\right)^{1/2} \right\} > 0$$

holds true. This implies

$$m_{T_0} \, ||y||_2 \le \int_0^{T_0} ||Y(t)^T y||_2 \, \mathrm{d}t \qquad \text{for all} \quad y \in \mathbb{R}^n.$$

In particular we obtain

$$\begin{aligned} m_{T_0} \, ||y_{T_1}||_2 &\le \int_0^{T_0} ||Y(t)^T y_{T_1}||_2 \, \mathrm{d}t \\ &\le \int_0^{T_1} ||Y(t)^T y_{T_1}||_2 \, \mathrm{d}t = \lambda(T_1) \le \lambda(T_2) \end{aligned}$$

and further

$$\begin{aligned} 0 \le \lambda(T_2) - \lambda(T_1) &\le \int_{T_1}^{T_2} ||Y(t)^T y_{T_1}||_2 \, \mathrm{d}t \\ \le \int_{T_1}^{T_2} ||Y(t)^T||_2 \, \mathrm{d}t \, ||y_{T_1}||_2 &\le \frac{\lambda(T_2)}{m_{T_0}} \int_{T_1}^{T_2} ||Y(t)^T||_2 \, \mathrm{d}t \end{aligned}$$

with

$$||Y(t)^T||_2 = \left(\sum_{j,k=1}^{n} y_{jk}(t)^2 \right)^{1/2} .$$

From this we infer

$$\lim_{T_1 \to T_2 - 0} \lambda(T_1) = \lambda(T_2)$$

which shows the left-continuity of the function and completes the proof. □

Under the assumptions of Theorem 2.15 it follows that the function $T \to v(T) = 1/\lambda(T),\ T > 0$, is strictly decreasing and continuous.

Thus we can strengthen Theorem 2.13 to

Theorem 2.16. *If restricted null-controllability holds and if the normality condition is satisfied, then the equation (2.46) for the minimum time* $T(\gamma)$ *holds true and there is exactly one time-minimal control on* $[0, T(\gamma)]$ *which is the unique minimum norm control* $u_{T(\gamma)} \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ *on* $[0, T(\gamma)]$ *and is given by (2.47).*

2.2 The Time-Discrete Autonomous Case

2.2.1 The Problem of Fixed Point Controllability

We begin with a system of difference equations of the form

$$x(t+1) = g(x(t), u(t)), \qquad t \in \mathbb{N}_0 \tag{2.49}$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous mapping.

The functions $x : \mathbb{N}_0 \to \mathbb{R}^n$ and $u : \mathbb{N}_0 \to \mathbb{R}^m$ are considered as state and control functions, respectively. For every control function $u : \mathbb{N}_0 \to \mathbb{R}^m$ and every vector $x_0 \in \mathbb{R}^n$ there exists exactly one state function $x : \mathbb{N}_0 \to \mathbb{R}^n$ which satisfies (2.49) and

$$x(0) = x_0. \tag{2.50}$$

If we fix the control function $u : \mathbb{N}_0 \to \mathbb{R}^m$ and define

$$f_{t+1}(x) = g(x, u(t)), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{N}_0, \tag{2.51}$$

then, for every $t \in \mathbb{N}_0$, $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping and $\left(\mathbb{R}^n, (f_t)_{t\in\mathbb{N}}\right)$ is a non-autonomous time-discrete dynamical system which is controlled by the function $u : \mathbb{N}_0 \to \mathbb{R}^m$. If

$$u(t) = \Theta_m = \text{ zero vector of } \mathbb{R}^m \text{ for all } t \in \mathbb{N}_0,$$

then the system (2.49) is called uncontrolled. Let us assume that the uncontrolled system (2.49) admits fixed points $\widehat{x} \in \mathbb{R}^n$ which then solve the equation

$$\widehat{x} = g\left(\widehat{x}, \Theta_m\right). \tag{2.52}$$

Now let $\Omega \subseteq \mathbb{R}^m$ be a subset with $\Theta_m \in \Omega$. Then we define the set of admissible control functions by

$$U = \left\{u : \mathbb{N}_0 \to \mathbb{R}^m \mid u(t) \in \Omega \quad \text{for all} \quad t \in \mathbb{N}_0\right\}. \tag{2.53}$$

After these preparations we are in the position to formulate the

<u>Problem of Fixed Point Controllability:</u> Given a fixed point $\widehat{x} \in \mathbb{R}^n$ of the system

$$x(t+1) = g\left(x(t), \Theta_m\right), \qquad t \in \mathbb{N}_0, \tag{2.54}$$

i.e., a solution $\widehat{x}$ of equation (2.52) and an initial state $x_0 \in \mathbb{R}^n$ find some $N \in \mathbb{N}_0$ and a control function $u \in U$ with

$$u(t) = \Theta_m \qquad \text{for all} \quad t \geq N \tag{2.55}$$

such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.49), (2.50) satisfies the end condition

$$x(N) = \widehat{x}. \tag{2.56}$$

(This implies $x(t) = \widehat{x}$ for all $t \geq N$.) From (2.49) and (2.50) it follows that

$$x(N) = \underbrace{g\big(g\big(\cdots\big(g}_{N-\text{times}}\left(x_0, u(0)\right)\big), u(1)\big), \cdots\big), u(N-1)\big)$$
$$= G^N\left(x_0, u(0), \ldots, u(N-1)\right). \tag{2.57}$$

Let $N \in \mathbb{N}$ be given. If $u(0), \ldots, u(N-1) \in \Omega$ are solutions of the system

$$G^N\left(x_0, u(0), \ldots, u(N-1)\right) = \widehat{x} \tag{2.58}$$

and one defines

$$u(t) = \Theta_m \qquad \text{for all} \quad t \geq N,$$

then one obtains a control function $u : \mathbb{N}_0 \to \mathbb{R}^m$ which solves the problem of fixed point controllability.

From the definition (2.57) it follows that

$$G^N\left(x_0, u(0), \ldots, u(N-1)\right) = G^1\left(G^{N-1}\left(x_0, u(0), \ldots, u(N-2)\right), u(N-1)\right).$$

Conversely now let us assume that we are given a sequence $\left(G^N\right)_{N \in \mathbb{N}}$ of vector functions $G^N : \mathbb{R}^n \times \mathbb{R}^{m \cdot N} \to \mathbb{R}^n$ with this property.

Then we define, for every $t \in \mathbb{N}$,

$$x(t) = G^t\left(x_0, u(0), \ldots, u(t-1)\right)$$
$$\text{for } x_0 \in \mathbb{R}^n \text{ and } u(s) \in \mathbb{R}^m \text{ for } s = 0, \ldots, t-1$$

and conclude

$$x(t) = G^1\left(G^{t-1}\left(x_0, u(0), \ldots, u(t-2)\right), u(t-1)\right)$$
$$= G^1(x(t-1), u(t-1))$$
$$\text{for all} \quad t \in \mathbb{N},$$

if we define $G^0\left(x_0\right) = x_0$. Now let $S\left(\widehat{x}\right) \subseteq \mathbb{R}^n$ be the set of all vectors $x_0 \in \mathbb{R}^n$ such that there exists a time $N \in \mathbb{N}$ and a solution $\left(u(0)^T, \ldots, u(N-1)^T\right)^T \in \Omega^N$ of the system (2.58). Obviously, $\widehat{x} \in S\left(\widehat{x}\right)$ (choose $N = 1$ and $u(0) = \Theta_n$). A first simple sufficient condition for the solvability of the problem of fixed point controllability is then given by

Proposition 2.17. *Let $\widehat{x} \in S\left(\widehat{x}\right)$ be an interior point of $S\left(\widehat{x}\right)$ and let $\widehat{x}$ be an attractor of the uncontrolled system (2.54), i.e., there exists an open set $U \subseteq \mathbb{R}^n$ with $\widehat{x} \in U$ such that $\lim\limits_{t \to \infty} x(t) = \widehat{x}$ where $x : \mathbb{N}_0 \to \mathbb{R}^n$ is a solution of (2.54) with (2.50) for any $x_0 \in U$. Then it follows that $S\left(\widehat{x}\right) \supseteq U$ which implies that for every choice of $x_0 \in U$ the problem of fixed point controllability has a solution.*

Proof. Since $\widehat{x} \in S(\widehat{x})$ is an interior point of $S(\widehat{x})$, there is an open neighborhood $W(\widehat{x}) \subseteq \mathbb{R}^n$ of $\widehat{x}$ with $W(\widehat{x}) \subseteq S(\widehat{x})$.

Now let $x_0 \in U$ be chosen arbitrarily. Then there is some $N_1 \in \mathbb{N}$ with $x(N_1) \in W(\widehat{x})$ where $x : \mathbb{N}_0 \to \mathbb{R}^m$ is a solution of (2.54), (2.50). This implies the existence of some $N_2 \in \mathbb{N}$ and a solution $\left(u(0)^T, \ldots, u(N_2-1)^T\right)^T \in \Omega^{N_2}$ with

$$G^{N_2}(x(N_1), u(0), \ldots, u(N_2-1)) = \widehat{x}.$$

If we define

$$u^*(t) = \begin{cases} \Theta_m & \text{for} \quad t = 0, \ldots, N_1 - 1, \\ u(t - N_1) & \text{for} \quad t = N_1, \ldots, N_1 + N_2 - 1, \end{cases}$$

then it follows that

$$G^N(x_0, u^*(0), \ldots, u^*(N-1)) = \widehat{x}$$

where $N = N_1 + N_2$, i.e., $x_0 \in S(\widehat{x})$ which completes the proof. □

The essential assumption in Proposition 2.17 is that the fixed point $\widehat{x}$ of the uncontrolled system (2.54) is an interior point of the controllable set $S(\widehat{x})$.

In order to find sufficient conditions for that we assume that Ω is open and $g \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m)$ which implies $G^N \in \mathcal{C}^1\left(\mathbb{R}^n \times \mathbb{R}^{m \cdot N}\right)$ for every $N \in \mathbb{N}$ and

$$\begin{aligned} &G_x^N(x, u(0), \ldots, u(N-1)) \\ &\quad = g_x\left(G^{N-1}(x_0, u(0), \ldots, u(N-2)), u(N-1)\right) \\ &\qquad \times g_x\left(G^{N-2}(x_0, u(0), \ldots, u(N-3)), u(N-2)\right) \\ &\qquad \times \qquad\qquad\qquad \vdots \\ &\qquad \times g_x(x, u(0)), \end{aligned}$$

and

$$\begin{aligned} &G_{u(k)}^N(x, u(0), \ldots, u(N-1)) \\ &\quad = g_x\left(G^{N-1}(x_0, u(0), \ldots, u(N-2)), u(N-1)\right) \\ &\qquad \times g_x\left(G^{N-2}(x_0, u(0), \ldots, u(N-3)), u(N-2)\right) \\ &\qquad \times \qquad\qquad\qquad \vdots \\ &\qquad \times g_x\left(G^{k+1}(x, u(0), \ldots, u(k)), u(k+1)\right) \\ &\qquad \times g_{u(k)}\left(G^k(x, u(0), \ldots, u(k-1)), u(k)\right) \end{aligned}$$

for $k = 0, \ldots, N-1$.

Let us assume that $g_x(\widehat{x}, \Theta_m)$ is non-singular. Then it follows, for all $N \in \mathbb{N}$, that $G_x^N(\widehat{x}, \Theta_m, \ldots, \Theta_m)$ is also non-singular, since

$$G_x^N(\widehat{x}, \Theta_m, \ldots, \Theta_m) = g_x(\widehat{x}, \Theta_m)^N .$$

Since

$$\widehat{x} = G^N(\widehat{x}, \Theta_m, \ldots, \Theta_m),$$

there exists, by the implicit function theorem, an open set $V \subseteq \Omega^N$ with $\Theta_m^N \in V$ and a function $h : V \to \mathbb{R}^n$ with $h \in \mathcal{C}^1(V)$ such that

$$h\left(\Theta_m^N\right) = \widehat{x} \quad \text{and} \quad G^N(h(u(0), \ldots, u(N-1)), u(0), \ldots, u(N-1)) = \widehat{x}$$

for all $(u(0), \ldots, u(N-1)) \in V$.

Moreover,

$$\begin{aligned} h_{u(k)}\left(\Theta_m^N\right) &= -G_x^N(\widehat{x}, \Theta_m, \ldots, \Theta_m)^{-1}\, G_{u(k)}(\widehat{x}, \Theta_m, \ldots, \Theta_m) \\ &= -g_x(\widehat{x}, \Theta_m)^{-N}\, g_x(\widehat{x}, \Theta_m)^{N-k}\, g_{u(k)}(\widehat{x}, \Theta_m) \\ &= -g_x(\widehat{x}, \Theta_m)^{-k}\, g_{u(k)}(\widehat{x}, \Theta_m) \end{aligned}$$

for $k = 0, \ldots, N-1$.

Result. *If $g_x(\widehat{x}, \Theta_m)$ is non-singular, then, for every $N \in \mathbb{N}$, there is an open set $V_N \subseteq \Omega^N$ with $\Theta_m^N \in V_N$ and a function $h_N : V_N \to \mathbb{R}^n$ with $h_N \in \mathcal{C}^1(V_N)$ such that*

$$h_N(u(0), \ldots, u(N-1)) \in S(\widehat{x}) \qquad \textit{for all} \quad (u(0), \ldots, u(N-1)) \in V_N.$$

We next assume that, for some $N \in \mathbb{N}$,

$$\operatorname{rank}\left(h_{u(0)}\left(\Theta_m^N\right) \mid \ldots \mid h_{u(N-1)}\left(\Theta_m^N\right)\right) = n.$$

Then there are n columns in the $n \times m \cdot N-$matrix

$$\left(h_{u(0)}\left(\Theta_m^N\right) \mid \ldots \mid h_{u(N-1)}\left(\Theta_m^N\right)\right)$$

which are linearly independent.

Now let E be the $n-$dimensional subspace of $\mathbb{R}^{m \cdot N}$ consisting of all vectors whose components vanish that do not correspond to the above linearly independent columns.

If we put $U = E \cap V_N$, then U is an open subset of E and the restriction of h to U is a $\mathcal{C}^1-$function on U whose Jacobi matrix at Θ_m^N consists of the above linearly independent columns of $\left(h_{u(0)}\left(\Theta_m^N\right) \mid \ldots \mid h_{u(N-1)}\left(\Theta_m^N\right)\right)$ and is therefore invertible. By the inverse function theorem there exist open sets $\widetilde{U} \subseteq E \cap V_N$ and $\widetilde{V} \subseteq \mathbb{R}^n$ with $\Theta_m^N \in \widetilde{U}$ and $\widehat{x} \in \widetilde{V}$ such that h is homeomorphic on $\widetilde{U}$ and $h\left(\widetilde{U}\right) = \widetilde{V}$.

This implies that $\widehat{x}$ is an interior point of $S(\widehat{x})$.

Let us demonstrate this result by the predator prey model that was investigated with respect to asymptotical stability in Section 1.3.7. We consider this model as a controlled system of the form

$$\begin{aligned} x_1(t+1) &= x_1(t) + a\,x_1(t) - e\,x_1(t)^2 - b\,x_1(t)x_2(t) - x_1(t)\,u_1(t), \\ x_2(t+1) &= x_2(t) - c\,x_2(t) + d\,x_1(t)x_2(t) - x_2(t)\,u_2(t), \qquad t \in \mathbb{N}_0. \end{aligned}$$

The mapping $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ in (2.49) is therefore given by

$$\begin{pmatrix} (1+a)\,x_1 - e\,x_1^2 - b\,x_1x_2 - x_1\,u_1 \\ (1-c)\,x_2 + d\,x_1x_2 - x_2\,u_2 \end{pmatrix}, \qquad x, u \in \mathbb{R}^2.$$

We have seen in Section 1.3.7 that

$$\widehat{x} = \left(\frac{c}{d}, \frac{1}{b}\left(a - \frac{ce}{d}\right)\right)^T$$

is the only fixed point of the uncontrolled system (with $u_1 = u_2 = 0$) in $\overset{\circ}{\mathbb{R}}{}^2_+ \times \overset{\circ}{\mathbb{R}}{}^2_+$, if $a > ce/d$.

One calculates

$$g_x(\widehat{x}, \Theta_2) = \begin{pmatrix} 1 - e\,\dfrac{c}{d} & -\dfrac{bc}{d} \\ \dfrac{d}{b}\left(a - \dfrac{ce}{d}\right) & 1 \end{pmatrix}$$

which implies that $g_x(\widehat{x}, \Theta_2)$ is non-singular, if and only if

$$1 - e\,\frac{c}{d} + c\left(a - \frac{ce}{d}\right) \neq 0. \qquad (*)$$

Further we obtain

$$g_u(\widehat{x}, \Theta_2) = \begin{pmatrix} -\dfrac{c}{d} & 0 \\ 0 & -\dfrac{1}{b}\left(a - \dfrac{ce}{d}\right) \end{pmatrix}$$

which implies

$$\operatorname{rank}\left(h_{u(0)}(\Theta_2)\right) = \operatorname{rank}\left(-g_{u(0)}(\widehat{x}, \Theta_2)\right) = 2.$$

Hence $\widehat{x}$ is an interior point of $S(\widehat{x})$, if $(*)$ is satisfied.

This example is a special case of the following situation:
Let

$$g(x,u) = f(x) + F(x)\,u = f(x) + \sum_{i=1}^{m} f_i(x)\,u_i,$$
$$x \in \mathbb{R}^n,\ u_1, \ldots, u_m \in \mathbb{R}, \text{ where } f, f_i \in \mathcal{C}^1(\mathbb{R}^n),\ i = 1, \ldots, m.$$

Let $\widehat{x} \in \mathbb{R}^n$ be a fixed point of f, i.e.,

$$\widehat{x} = f(\widehat{x}) = g(\widehat{x}, \Theta_m).$$

Then,

$$g_x(\widehat{x}, \Theta_m) = f_x(\widehat{x}) \qquad \text{and} \qquad g_u(\widehat{x}, \Theta_m) = F(\widehat{x})$$

and hence

$$\begin{aligned}&\left(h_{u(0)}\left(\Theta_m^N\right) \mid \ldots \mid h_{u(N-1)}\left(\Theta_m^N\right)\right)\\ &\quad = -\left(F(\widehat{x}) \mid f_x(\widehat{x})^{-1} F(\widehat{x}) \mid \ldots \mid f_x(\widehat{x})^{-N+1} F(\widehat{x})\right).\end{aligned}$$

In the example we have $m = n = 2$ and the $2 \times 2-$matrix $F(\widehat{x})$ is non-singular. Next we come back to the solution of (2.58) which we replace by an optimization problem. For this purpose we define a cost functional $\varphi : \mathbb{R}^{m \cdot N} \to \mathbb{R}$ by putting

$$\begin{aligned}&\varphi(u(0), \ldots, u(N-1)) = \left\|G^N(x_0, u(0), \ldots, u(N-1)) - \widehat{x}\right\|_2^2\\ &\quad \text{for} \quad (u(0), \ldots, u(N-1)) \in \mathbb{R}^{m \cdot N} \qquad (\|\cdot\|_2 = \text{Euclidian norm in } \mathbb{R}^n)\end{aligned}$$

and try to find $(u(0), \ldots, u(N-1)) \in \Omega^N$ such that $\varphi(u(0), \ldots, u(N-1)) \leq \varphi(\widetilde{u}(0), \ldots, \widetilde{u}(N-1))$ for all $(\widetilde{u}(0), \ldots, \widetilde{u}(N-1)) \in \Omega^N$.

If $\varphi(u(0), \ldots, u(N-1)) = 0$, then $\varphi(u(0), \ldots, u(N-1)) \in \Omega^N$ solves the equation (2.58). Otherwise no such solution exists. We again assume that $g \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m)$. Let $\Omega \subseteq \mathbb{R}^n$ be open.

Then a necessary condition for $u(0), \ldots, u(N-1)) \in \Omega^N$ to minimize φ on Ω^N is given by

$$\begin{aligned}\varphi_{u(k)}(u(0), \ldots, u(N-1)) &= 2\, G_{u(k)}^N(x_0, u(0), \ldots, u(N-1))^T\\ &\quad \times \left(G^N(x_0, u(0), \ldots, u(N-1)) - \widehat{x}\right)\\ &= \Theta_m \qquad \text{(OC)}\end{aligned}$$

for all $k = 0, \ldots, N-1$.

For the determination of $(u(0), \ldots, u(N-1)) \in \Omega^N$ with (OC) one can apply Marquardt's algorithm: Let $(u(0), \ldots, u(N-1)) \in \Omega^N$ be chosen. If (OC) is satisfied, then $(u(0), \ldots, u(N-1))$ is taken as a solution of the optimization problem. Otherwise, for every $k \in \{0, \ldots, N-1\}$, a vector $h_\lambda(k) \in \mathbb{R}^m$ is determined as a solution of the linear system

$$\begin{aligned}&\left(2\, G_{u(k)}^N(x_0, u(0), \ldots, u(N-1))\right)^T G_{u(k)}^N(x_0, u(0), \ldots, u(N-1)) + \lambda\, I_m\, h_\lambda(k)\\ &\quad = 2\, G_{u(k)}^N(x_0, u(0), \ldots, u(N-1))^T \left(G_{u(k)}^N(x_0, u(0), \ldots, u(N-1)) - \widehat{x}\right)\end{aligned}$$

where $\lambda > 0$ and I_m is the $m \times m-$unit matrix.

Then one can show (see, for instance [22]) that for sufficiently large $\lambda > 0$ it follows that

$$(u(0) + h_\lambda(0), \dots, u(N-1) + h_\lambda(N-1)) \in \Omega$$

and

$$\varphi\left(u(0) + h_\lambda(0), \dots, u(N-1) + h_\lambda(N-1)\right) < \varphi\left(u(0), \dots, u(N-1)\right).$$

The algorithm is then continued with $(u(0) + h_\lambda(0), \dots, u(N-1) + h_\lambda(N-1))$ instead of $(u(0), \dots, u(N-1))$.

Special Case: Modelling of Conflicts

Now let us consider a special case which is motivated by a situation which occurs in the modelling of conflicts. We begin with an uncontrolled system of the form

$$\begin{aligned} x^1(t+1) &= g_1\left(x^1(t), x^2(t)\right), \\ x^2(t+1) &= g_2\left(x^1(t), x^2(t)\right), \qquad t \in \mathbb{N}_0, \end{aligned}$$

where $g_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i},\ i = 1, 2,$ are given continuous mappings and $x^i : \mathbb{N}_0 \to \mathbb{R}^{n_i},\ i = 1, 2,$ are considered as state functions.

For $t = 0$ we assume initial conditions

$$x^1(0) = x_0^1, \qquad x^2(0) = x_0^2 \tag{2.59}$$

where $x_0^1 \in \mathbb{R}^{n_1}$ and $x_0^2 \in \mathbb{R}^{n_2}$ are given vectors with

$$\Theta_{n_2} \le x_0^2 \le x_*^2$$

for some $x_*^2 \ge \Theta_{n_2}$ which is also given. We further assume that the above system admits fixed points $\left(\widehat{x}^{1\,T}, \widehat{x}^{2\,T}\right)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with

$$\Theta_{n_2} \le \widehat{x}^2 \le x_*^2$$

which are then solutions of the system

$$\widehat{x}^1 = g_1\left(\widehat{x}^1, \widehat{x}^2\right), \qquad \widehat{x}^2 = g_2\left(\widehat{x}^1, \widehat{x}^2\right).$$

Now we consider the following

Problem: Find vector functions $x^1 : N_0 \to \mathbb{R}^{n_1}$ and $x^2 : I\!N_0 \to \mathbb{R}^{n_2}$ with

$$\Theta_{n_2} \le x^2(t) \le x^2_* \qquad \text{for all} \quad t \in \mathbb{N}_0$$

which satisfy the above system equations and initial conditions and

$$x^1(t) = \widehat{x}^1, \quad x^2(t) = \widehat{x}^2 \qquad \text{for all} \quad t \ge N$$

where $N \in \mathbb{N}_0$ is a suitably chosen integer. In general this problem will not have a solution. Therefore we replace the uncontrolled system by the following controlled system

$$\begin{aligned} x^1(t+1) &= g_1\left(x^1(t), x^2(t) + u(t)\right), \\ x^2(t+1) &= g_2\left(x^1(t), x^2(t) + u(t)\right), \qquad t \in \mathbb{N}_0, \end{aligned} \tag{2.60}$$

where $u : \mathbb{N}_0 \to \mathbb{R}^{n_2}$ is a control function. Then we consider the problem of finding a control function $u : \mathbb{N}_0 \to \mathbb{R}^{n_2}$ such that the solutions $x^1 : \mathbb{N}_0 \to \mathbb{R}^{n_1}$ and $x^2 : \mathbb{N}_0 \to \mathbb{R}^{n_2}$ of (2.60) and (2.59) satisfy the conditions

$$\Theta_{n_2} \le x^2(t) + u(t) \le x^2_* \qquad \text{for all} \quad t \in \mathbb{N}_0$$

and

$$x^1(t) = \widehat{x}^1, \quad x^2(t) + u(t) = \widehat{x}^2 \qquad \text{for all} \quad t \ge N$$

where $N \in \mathbb{N}_0$ is a suitably chosen integer.

Let us assume that we can find a vector function $v : \mathbb{N}_0 \to \mathbb{R}^n$ with

$$\Theta_{n_2} \le v(t) \le x^2_* \quad \text{for} \quad t = 0, \ldots, N-1 \qquad \text{and} \qquad v(t) = \widehat{x}^2 \quad \text{for} \quad t \ge N$$

such that the solution $x^1 : \mathbb{N}_0 \to \mathbb{R}^{n_1}$ of

$$\begin{aligned} x^1(0) &= x^1_0, \\ x^1(t+1) &= g_1\left(x^1(t), v(t)\right), \qquad t \in \mathbb{N}_0 \end{aligned}$$

satisfies

$$x^1(t) = \widehat{x}^1 \qquad \text{for all} \quad t \ge N$$

where $N \in \mathbb{N}$ is a suitably chosen integer.

Then we put

$$\begin{aligned} x^2(0) &= x^2_0, \\ x^2(t+1) &= g_2\left(x^1(t), v(t)\right), \qquad t \in \mathbb{N}_0 \end{aligned}$$

and define

$$u(t) = v(t) - x^2(t) \qquad \text{for} \quad t \ge N_0.$$

With these definitions we obtain a solution of the above control problem.

Thus, in order to find such a solution, we have to find a vector function $v : \mathbb{N}_0 \to \mathbb{R}^{n_2}$ with

$$\begin{aligned} \Theta_{n_2} \le v(t) \le x_*^2 \quad & \text{for} \;\; t = 0, \ldots, N-1, \\ v(t) = \widehat{x}^2 \quad & \text{for} \;\; t \ge N \end{aligned}$$

such that the solution $x^1 : \mathbb{N}_0 \to \mathbb{R}^{n_1}$ of

$$\begin{aligned} x^1(0) &= x_0^1, \\ x^1(t+1) &= g_1\left(x^1(t), v(t)\right), \qquad t \in \mathbb{N}_0 \end{aligned}$$

satisfies

$$x^1(t) = \widehat{x}^1 \qquad \text{for all} \quad t \ge N$$

where $N \in \mathbb{N}$ is a suitably chosen integer. Let us demonstrate all this by an emission reduction model (1.65) to which we add the conditions

$$0 \le M_i(t) \le M_i^* \qquad \text{for all} \quad t \in \mathbb{N}_0 \quad \text{and} \quad i = 1, \ldots, r$$

and the initial conditions

$$E_i(0) = E_{0i} \quad \text{and} \quad M_i(0) = M_{0i} \qquad \text{for} \quad i = 1, \ldots, r$$

where $E_{0i} \in \mathbb{R}$ and $M_{0i} \in \mathbb{R}$ with $0 \le M_{0i} \le M_i^*$ for $i = 1, \ldots, r$ are given.

The corresponding controlled system (2.60) reads in this case

$$\begin{aligned} E_i(t+1) &= E_i(t) + \sum_{j=1}^{r} em_{ij}\left(M_j(t) + u_j(t)\right), \\ M_i(t+1) &= M_i(t) + u_i(t) - \lambda_i\left(M_i(t) + u_i(t)\right)\left(M_i^* - M_i(t) - u_i(t)\right) E_i(t) \end{aligned}$$

for $i = 1, \ldots, r$ and $t \in \mathbb{N}_0$.

The control functions $u_i : \mathbb{N}_0 \to \mathbb{R},\; i = 1, \ldots, r$, must satisfy the conditions

$$0 \le M_i(t) + u_i(t) \le M_i^* \qquad \text{for} \quad i = 1, \ldots, r \quad \text{and} \quad t \in \mathbb{N}_0.$$

Fixed points of the system (1.65) are of the form $\left(\widehat{E}^T, \widehat{\Theta}_r^T\right)^T$ with $\widehat{E} \in \mathbb{R}^r$ arbitrary.

We have to find a vector function $v : \mathbb{N}_0 \to \mathbb{R}^r$ with

$$\begin{aligned} \Theta_r \le v(t) \le M^* \qquad & \text{for} \ \ t = 0, \dots, N-1, \\ v(t) = \Theta_r \qquad & \text{for} \ \ t \ge N \end{aligned}$$

such that the solution $E : \mathbb{N}_0 \to \mathbb{R}^r$ of

$$\begin{aligned} E(0) &= E_0, \\ E(t+1) &= E(t) + C\, v(t), \qquad t \in \mathbb{N}_0, \ \left(C = (em_{ij})_{i,j=1,\dots,r}\right) \end{aligned}$$

satisfies

$$E(t) = \widehat{E} \qquad \text{for all} \quad t \ge N$$

where $N \in \mathbb{N}$ is a suitably chosen integer.

First of all we observe that for every $N \in \mathbb{N}$

$$E(N) = E_0 + C \left(\sum_{t=0}^{N-1} v(t) \right).$$

Let us assume that C is invertible and C^{-1} is positive. Further we assume that $\widehat{E} \ge E_0$.

Then $E(N) = \widehat{E}$, if and only if

$$\sum_{t=0}^{N-1} v(t) = C^{-1} \left(\widehat{E} - E_0 \right) \ge \Theta_r .$$

If we define

$$v(t) = \Theta_r \qquad \text{for all} \quad t \ge N,$$

then

$$E(t) = \widehat{E} \qquad \text{for all} \quad t \ge N.$$

Let us put

$$v_N = \sum_{t=0}^{N-1} v(t) = C^{-1} \left(\widehat{E} - E_0 \right).$$

If we define

$$v(t) = \frac{1}{N} v_N \qquad \text{for} \quad t = 0, \dots, N-1$$

then

$$\sum_{t=0}^{N-1} v(t) = C^{-1} \left(\widehat{E} - E_0 \right) \qquad \text{and} \qquad \Theta_r \le v(t) \le M^* \quad \text{for} \quad t = 0, \dots, N-1$$

for sufficiently large N, if $M_i^* > 0$ for all $i = 1, \dots, r$.

We finish with a numerical example: $r = 3,\ E_0 = (0,0,0)^T,\ \widehat{E} = (10,10,10)^T$, $M^* = (1,1,1)^T$, and

$$C = \begin{pmatrix} 1 & -0.8 & 0 \\ 0 & 1 & -0.8 \\ -0.1 & -0.5 & 1 \end{pmatrix}.$$

Then we have to solve the linear system

$$\begin{aligned} v_{N_1} - 0.8\, v_{N_2} \qquad\qquad &= 10, \\ v_{N_2} - 0.8\, v_{N_3} &= 10, \\ -0.1\, v_{N_1} - 0.5\, v_{N_2} + \qquad v_{N_3} &= 10. \end{aligned}$$

The solution reads

$$\begin{aligned} v_{N_1} &= 38.059701, \\ v_{N_2} &= 35.074627, \\ v_{N_3} &= 31.343284. \end{aligned}$$

We choose $N = 39$. Then we have to put

$$v(t) = \frac{1}{N}\, v_N = \begin{pmatrix} 0.9758898 \\ 0.8993494 \\ 0.803674 \end{pmatrix} \qquad \text{for} \quad t = 0, \ldots, 38.$$

2.2.2 Null-Controllability of Linear Systems

Instead of (2.49) we consider a linear system of the form

$$x(t+1) = A\,x(t) + B\,u(t), \qquad t \in \mathbb{N}_0, \tag{2.61}$$

where A is a real $n \times n-$matrix and B a real $n \times m-$matrix and where $u : \mathbb{N}_0 \to \mathbb{R}^m$ is a given control function. The corresponding uncontrolled system reads

$$x(t+1) = A\,x(t), \qquad t \in \mathbb{N}_0, \tag{2.62}$$

and admits $\widehat{x} = \Theta_n$ as a fixed point.

The problem of fixed point controllability is then equivalent to the

Problem of Null-Controllability: Given $x_0 \in \mathbb{R}^n$ find some $N \in \mathbb{N}_0$ and a control function $u \in U$ (2.53) with (2.55) such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.61), (2.50) satisfies the end condition

$$x(N) = \Theta_n \tag{2.63}$$

(which implies $x(t) = \Theta_n$ for all $t \geq N$).

From (2.61) and (2.50) it follows that

$$x(N) = A^N x_0 + \sum_{t=1}^{N} A^{N-t} B\, u(t-1) \tag{2.64}$$

so that (2.63) turns out to be equivalent to

$$\sum_{t=1}^{N} A^{N-t} B\, u(t-1) = -A^N x_0. \tag{2.65}$$

Now let A be non-singular. Then the set $S(\Theta_n)$ of all vectors $x_0 \in \mathbb{R}^n$ such that there exists a time $N \in \mathbb{N}$ and a solution $\left(u(0)^T, \ldots, u(N-1)^T\right)^T \in \Omega^N$ of the system (2.61) is given by

$$S(\Theta_n) = \bigcup_{N \in \mathbb{N}} E(N)$$

where, for every $N \in \mathbb{N}$,

$$E(N) = \left\{ x = \sum_{t=1}^{N} A^{N-t} B\, u(t-1) \;\middle|\; u \in U \;\; (2.53) \right\}.$$

Next we assume that $\Omega \subseteq \mathbb{R}^m$ is convex, has Θ_m as interior point and satisfies

$$u \in \Omega \quad \Longrightarrow \quad -u \in \Omega. \tag{2.66}$$

Then, for every $N \in \mathbb{N}$, the set $E(N)$ is convex and $E(N) = -E(N)$. This implies because of

$$E(N) \subseteq E(N+1) \qquad \text{for all} \quad N \in \mathbb{N}$$

that $S(\Theta_n)$ is also convex and $S(\Theta_n) = -S(\Theta_n)$.

Further we assume Kalman's condition, i.e., there exists some $N_0 \in \mathbb{N}$ such that

$$\operatorname{rank}\left(B \mid AB \mid \ldots \mid A^{N_0-1} B\right) = n. \tag{2.67}$$

Then we can prove

Theorem 2.18. *If A is non-singular, $\Omega \subseteq \mathbb{R}^m$ is convex, has Θ_m as interior point and satisfies (2.66) and if Kalman condition (2.67) is satisfied for some $N_0 \in \mathbb{N}$, then Θ_n is an interior point of $S(\Theta_n)$.*

Proof. Let us assume that Θ_n is not an interior point of $S(\Theta_n)$. Then $S(\Theta_n)$ must be contained in a hyperplane through Θ_n, i.e., there must exist some $y \in \mathbb{R}^n$, $y \neq \Theta_n$, with

$$y^T x = 0 \qquad \text{for all} \quad x \in S(\Theta_n).$$

This implies

$$y^T \left(\sum_{t=1}^{N} A^{N-t} B\, u(t-1) \right) = 0$$
$$\text{for all} \quad \left(u(0)^T, \ldots, u(N-1)^T\right)^T \in \mathbb{R}^{m \cdot N} \quad \text{and all} \quad N \in \mathbb{N},$$

hence

$$y^T A^{N-t} B = \Theta_m^T \qquad \text{for all} \quad N \in \mathbb{N}.$$

In particular for $N = N_0$ this implies $y = \Theta_n$ due to the Kalman condition (2.67) which contradicts $y \neq \Theta_n$. Therefore the assumption that Θ_n is not an interior point of $S(\Theta_n)$ is false. □

In addition to the assumption of Theorem 2.18 we assume that all the eigenvalues of A are less than 1 in absolute value. Then according to the Corollary following Theorem 1.6 Θ_n is a global attractor of the uncontrolled system (2.62), i.e., $\lim_{t \to \infty} x(t) = \Theta_n$ where $x : \mathbb{N}_0 \to \mathbb{R}^n$ is a solution (2.62) with (2.50) for any $x_0 \in \mathbb{R}^n$. By Proposition 2.17 therefore the problem of null-controllability has a solution for every choice of $x_0 \in \mathbb{R}^n$. If the set Ω of control vector values has the form

$$\Omega = \{u \in \mathbb{R}^m \mid ||u|| \leq \gamma\} \tag{2.68}$$

for some $\gamma > 0$ where $|| \cdot ||$ is any norm in $\mathbb{R}^m$, then this result can be strengthened to

Theorem 2.19. *Let the Kalman condition (2.67) be satisfied for some $N_0 \in \mathbb{N}$. Further let Ω be of the form (2.68). Finally let all the eigenvalues of A^T be less than or equal to one in absolute value and the corresponding eigenvectors be linearly independent.*

Then the problem of null-controllability has a solution for every choice of $x_0 \in \mathbb{R}^n$, if A is non-singular.

Proof. We have to show that for every choice of $x_0 \in \mathbb{R}^n$ there is some $N \in \mathbb{N}_0$ and a control function $u \in U$ (2.53) such that (2.65) is satisfied. Since A is non-singular, (2.65) is equivalent to

$$\sum_{t=1}^{N} A^{-t} B\, u(t-1) = -x_0.$$

For every $N \in \mathbb{N}$ we define the convex set

$$R(N) = \left\{ x = \sum_{t=1}^{N} A^{-t} B\, u(t-1) \,\middle|\, u \in U \right\}$$

and put

$$R_\infty = \bigcup_{N \in \mathbb{N}} R(N).$$

Because of

$$R(N) \subseteq R(N+1) \qquad \text{for all} \quad N \in \mathbb{N}_0$$

the set R_∞ is also convex. We have to show that $R_\infty = \mathbb{R}^n$. Let us assume that $R_\infty \neq \mathbb{R}^n$.

Then there exists some $\widetilde{x} \in \mathbb{R}^n$ with $\widetilde{x} \notin R_\infty$ which can be separated from R_∞ by a hyperplane, i.e., there exists a number $\alpha \in \mathbb{R}$ and a vector $y \in \mathbb{R}^n,\ y \neq \Theta_n$ such that

$$y^T x \leq \alpha \leq y^T \widetilde{x} \qquad \text{for all} \quad x \in R_\infty.$$

Since $\Theta_n \in R_\infty$, it follows that $\alpha \geq 0$. Further it follows from the implication $u \in \Omega \;\Rightarrow\; -u \in \Omega$ that

$$\left| \sum_{t=1}^{N} y^T A^{-t} B\, u(t-1) \right| \leq \alpha \qquad \text{for all} \quad N \in \mathbb{N} \quad \text{and all} \quad u \in U.$$

This implies

$$\sum_{t=1}^{N} \left\| \left(y^T A^{-t} B \right)^T \right\|_d \leq \alpha \qquad \text{for all} \quad N \in \mathbb{N}$$

where $||\cdot||_d$ is the norm in $\mathbb{R}^m$ which is dual to $||\cdot||$. This in turn implies

$$\lim_{t\to\infty} y^T A^{-t} B = \Theta_m^T. \tag{2.69}$$

From the Kalman condition (2.67) it follows that there exist n linearly independent vectors in $\mathbb{R}^n$ of the form

$$c_i = A^{t_i}\, b_{j_i} \qquad \text{for} \quad i = 1, \ldots, n$$

where $t_i \in \{0, \ldots, N_0 - 1\}$ and $j_i \in \{1, \ldots, m\}$ and b_{j_i} denotes the j_i-th column vector of B. From (2.69) it follows that

$$\lim_{t\to\infty} y^T A^{-t} c_i = 0 \qquad \text{for} \quad i = 1, \ldots, n.$$

This implies

$$\lim_{t\to\infty} y^T A^{-t} = \Theta_n^T$$

or, equivalently,

$$\lim_{t\to\infty} \left(A^T\right)^{-t} y = \Theta_n. \tag{2.70}$$

Now let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be eigenvalues of A^T and $y_1, \ldots, y_n \in \mathbb{C}^n$ corresponding linearly independent eigenvectors. Then there is a unique representation

$$y = \sum_{j=1}^{n} \alpha_j \, y_j \qquad \text{where not all} \quad \alpha_j \in \mathbb{C} \quad \text{are zero}$$

and

$$\left(A^T\right)^{-t} y_j = \left(\frac{1}{\lambda_j}\right)^t y_j \qquad \text{for} \quad j = 1, \ldots, n,$$

hence

$$\left(A^T\right)^{-t} y = \sum_{j=1}^{n} \alpha_j \left(\frac{1}{\lambda_j}\right)^t y_j \qquad \text{for all} \quad t \in \mathbb{N}.$$

From (2.70) we therefore infer that

$$|\lambda_j| > 1 \qquad \text{for all} \quad j \in \{1, \ldots, n\} \quad \text{with} \quad \alpha_j \neq 0.$$

This is a contradiction to

$$|\lambda_j| \leq 1 \qquad \text{for all} \quad j = 1, \ldots, n.$$

Hence the assumption $R_\infty \neq \mathbb{R}^n$ is false. □

Remark. *If we define*

$$\overline{Y} = (y_1 \mid y_2 \mid \ldots \mid y_n) \qquad \textit{and} \qquad \wedge = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

then it follows that

$$A^T \, \overline{Y} = \overline{Y} \wedge$$

which implies

$$\overline{Y}^{-1} A^T \, \overline{Y} = \wedge$$

and in turn

$$\overline{Y}^T A \left(\overline{Y}^T\right)^{-1} = \wedge \qquad \left(\textit{since} \;\; \left(A^{-1}\right)^T = \left(A^T\right)^{-1}\right)$$

from which

$$A \left(\overline{Y}^T\right)^{-1} = \left(\overline{Y}^T\right)^{-1} \wedge$$

follows.

This implies that A has the same eigenvalues as A^T (which holds for arbitrary matrices) and the eigenvectors of A are the column vectors of $\left(\overline{Y}^T\right)^{-1}$. Therefore A^T in Theorem 2.19 could be replaced by A. For the following let us assume that $\Omega = \mathbb{R}^m$.

For every $N \in \mathbb{N}$ let us define

$$\overline{Y}(N) = \left(B \mid AB \mid \ldots \mid A^{N-1}B\right).$$

Since U (2.53) consists of all functions $u : \mathbb{N}_0 \to \mathbb{R}^m$, it follows, for every $N \in \mathbb{N}$, that

$$E(N) = \left\{ x = \sum_{t=1}^{N} A^{N-t} B\, u(t-1) \,\middle|\, u : \mathbb{N}_0 \to \mathbb{R}^m \right\}.$$

Further we can prove

Proposition 2.20. *The following statements are equivalent:*

(i) rank $\overline{Y}(N) = \text{rank}\ \overline{Y}(N+1)$,
(ii) $E(N) = E(N+1)$,
(iii) $\left(A^N B\right) \mathbb{R}^m \subseteq E(N)$,
(iv) rank $\overline{Y}(N) = \text{rank}\ \overline{Y}(N+j) \qquad \textit{for all} \quad j \geq 1$.

Proof.
(i) $\Rightarrow$ (ii): This is a consequence of the fact that $E(N) \subseteq E(N+1)$.
(ii) $\Rightarrow$ (iii): This follows from $\overline{Y}(N+1) = \left(\overline{Y}(N) \mid A^N B\right)$.
(iii) $\Rightarrow$ (i): $\overline{Y}(N+1) = \left(\overline{Y}(N) \mid A^N B\right)$ shows that (iii) $\Rightarrow$ (ii) and obviously we have (ii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (iv): Since (i) implies (iii), it follows that

$$\left(A^{N+1}B\right) \mathbb{R}^m \subseteq A\, E(N) \subseteq E(N+1)$$

which implies $E(N+1) = E(N+2)$ and hence

$$\text{rank}\left(\overline{Y}(N+1)\right) = \text{rank}\left(\overline{Y}(N+2)\right).$$

(iv) $\Rightarrow$ (i): is obvious. This completes the proof. $\square$

Now let r be the smallest integer such that $I, A, \ldots, A^{r-1}$ are linearly independent in $\mathbb{R}^{n \cdot n}$ and hence there are numbers $\alpha_{r-1}, \alpha_{r-2}, \ldots, \alpha_0 \in \mathbb{R}$ such that

$$A^r + \alpha_{r-1} A^{r-1} + \ldots + \alpha_0 I = 0.$$

Defining

$$\Phi_0(\lambda) = \lambda^r + \alpha_{r-1} \lambda^{r-1} + \ldots + \alpha_0,$$

we have $\Phi_0(A) = 0$. This monic polynomial (leading coefficient 1) is the monic polynomial of least degree for which $\Phi_0(A) = 0$ and is called the minimal polynomial of A.

The polynomial

$$\Phi(\lambda) = \det(\lambda I - A) \qquad \text{with degree} \quad n$$

is called characteristic polynomial of A, and the Cayley-Hamilton Theorem states that $\Phi(A) = 0$ which implies $r \leq n$.

This leads to

Proposition 2.21. *Let s be the degree of the minimal polynomial of A ($s \leq n$). Then there is an integer $k \leq s$ such that*

$$\operatorname{rank} \overline{Y}(1) < \operatorname{rank} \overline{Y}(2) < \ldots < \operatorname{rank} \overline{Y}(k) = \operatorname{rank} \overline{Y}(k+j) \qquad \textit{for all} \quad j \in \mathbb{N}.$$

Proof. Proposition 2.20 implies the existence of such an integer k, since $\operatorname{rank} \overline{Y}(N) \leq n$ for all $N \in \mathbb{N}$. We have to show that $k \leq s$. Let $\psi(\lambda) = \lambda^s + \alpha_{s-1} \lambda^{s-1} + \ldots + \alpha_0$ be the minimal polynomial of A. Then $\psi(A) B = 0$ and $A^s B \, \mathbb{R}^m \subseteq E(s)$ which implies (by Proposition 2.20) that $\operatorname{rank} \overline{Y}(s) = \operatorname{rank} \overline{Y}(s+j)$ for all $j \in \mathbb{N}$, hence $k \leq s$. □

As a consequence of Proposition 2.21 we obtain

Proposition 2.22. *If the Kalman condition (2.67) is satisfied for some $N_0 \in \mathbb{N}$, then necessarily $N_0 \leq n$ and*

$$\operatorname{rank} \left(B \mid AB \mid \ldots \mid A^{n-1} B \right) = n,$$

hence $E(n) = \mathbb{R}^n$. Conversely, if $E(n) = \mathbb{R}^n$, then Kalman's condition (2.67) is satisfied for all $N \geq n$ and

$$E(N) = \mathbb{R}^n \qquad \textit{for all} \quad N \geq n.$$

Proposition 2.21 also implies that, if $\operatorname{rank} \overline{Y}(n) < n$, then $\operatorname{rank} \overline{Y}(N) < n$ for all $N \geq n$ and

$$E(N) = E(n) \neq \mathbb{R}^n \qquad \text{for all} \quad N \geq n.$$

If we define, for every $N \in \mathbb{N}$, the $n \times n-$matrix

$$W(N) = \overline{Y}(N) \overline{Y}(N)^T = \sum_{j=0}^{N-1} A^j B B^T \left(A^j \right)^T,$$

then it follows that

$$W(N) \, \mathbb{R}^n \subseteq E(N) \qquad \text{for every} \quad N \in \mathbb{N}.$$

Now let us assume that $\operatorname{rank} \overline{Y}(N) = n$ which is equivalent to $E(N) = \mathbb{R}^n$.

Then $W(N)$ is non-singular and

$$W(N)\,\mathbb{R}^n = \mathbb{R}^n = E(N).$$

Let $y^* \in \mathbb{R}^n$ be the unique solution of

$$W(N)\,y^* = -A^N\,x_0.$$

Further let $u \in \mathbb{R}^{m\cdot N}$ be any solution of

$$\overline{Y}(N)\,u = -A^N\,x_0 \qquad \text{(see (2.65))}.$$

If we put

$$u^* = \overline{Y}(N)^T\,y^* \qquad \left(\in \mathbb{R}^{m\cdot N}\right),$$

then

$$\overline{Y}(N)^T\,u^* = -A^N\,x_0$$

and

$$\begin{aligned} ||u^*||_2^2 &= u^{*\,T}u^* = y^{*\,T}\,\overline{Y}(N)\,u^* = y^{*\,T}W(N)y^* \\ &= -y^{*\,T}A^N\,x_0 = y^*\,\overline{Y}(N)\,u = u^{*\,T}u \le ||u^*||_2\,||u||_2 \end{aligned}$$

which implies $||u^*||_2 \le ||u||_2$.

2.2.3 A Method for Solving the Problem of Null-Controllability

Let us equip $\mathbb{R}^m$ with the Euclidian norm $||\cdot||_2$ and consider the following

Problem (P) *For a given $N \in \mathbb{N}$ find $u : \{0,\ldots,N-1\} \to \mathbb{R}^m$ such that*

$$\sum_{t=1}^{N} A^{N-t}B\,u(t-1) = -A^N\,x_0 \tag{2.71}$$

and

$$\varphi_N(u) = \max_{t=1,\ldots,N} ||u(t-1)||_2$$

is as small as possible.

If the Kalman condition (2.67) is satisfied for some $N_0 \in \mathbb{N}$ and if $N \ge N_0$, then Problem (P) has a solution $u_N \in \mathbb{R}^{m\cdot N}$. If $\varphi_N\,(u_N) \le \gamma$ (see (2.68)), then we obtain solution $u_N : \mathbb{N}_0 \to \mathbb{R}^m$ of the problem of null-controllability if we define

$$u_N(t) = \Theta_m \qquad \text{for all} \quad t \ge N.$$

If $\varphi_N\,(u_N) > \gamma$, then N must be increased.

If the matrix A is non-singular, then (2.65) is equivalent to

$$\sum_{t=1}^{N} A^{-t} B\, u(t-1) = -x_0$$

which implies

$$\varphi_{N+1}\left(u_{N+1}\right) \leq \varphi_N\left(u_N\right) \qquad \text{for all} \quad N \geq N_0.$$

Under the assumptions of Theorem 2.19 there exists, for every $\varepsilon > 0$, some $N(\varepsilon) \in \mathbb{N}$ such that

$$\varphi_{N(\varepsilon)}\left(u_{N(\varepsilon)}\right) \leq \varepsilon$$

which implies

$$\lim_{N\to\infty} \varphi_N\left(u_N\right) = 0.$$

So we can be sure, for every choice of $\gamma > 0$, to find a solution $U_{N(\gamma)} \in \mathbb{R}^{m \cdot N(\gamma)}$ of Problem (P) with $\varphi_{N(\gamma)}\left(u_{N(\gamma)}\right) \leq \gamma$ which leads to a solution $u_{N(\gamma)} : \mathbb{N}_0 \to \mathbb{R}^m$ of the problem of null-controllability, if we define

$$u_{N(\gamma)}(t) = \Theta_m \qquad \text{for all} \quad t \geq N(\gamma).$$

In order to solve Problem (P) we replace it by

Problem (D) *Minimize*

$$\chi(y) = \sum_{k=1}^{N} \left\| B^T \left(A^{N-k}\right)^T y \right\|_2, \qquad y \in \mathbb{R}^n,$$

subject to

$$c^T y = -x_0^T \left(A^T\right)^N y = 1. \tag{2.72}$$

Let $u : \{0, \ldots, N-1\} \to \mathbb{R}^m$ be a solution of (2.65) and let $y \in \mathbb{R}^n$ satisfy (2.72). Then it follows that

$$\sum_{k=1}^{N} y^T A^{N-k} B\, u(k-1) = y^T c = 1$$

which implies

$$\max_{k=1,\ldots,N} ||u(k-1)||_2 \geq \frac{1}{\chi(y)}.$$

Now let $\widehat{y} \in \mathbb{R}^n$ be a solution of Problem (D). Then there is a multiplier $\lambda \in \mathbb{R}$ such that

$$\nabla\chi\left(\widehat{y}\right) = \lambda\, c. \tag{2.73}$$

This yields

$$\nabla\chi(\widehat{y}) = \sum_{k\in I(\widehat{y})} \frac{1}{||B^T\left(A^{N-k}\right)^T \widehat{y}||_2} A^{N-k}BB^T\left(A^{N-k}\right)^T \widehat{y}$$

with

$$I(\widehat{y}) = \left\{k \,\middle|\, \left\|B^T\left(A^{N-k}\right)^T \widehat{y}\right\|_2 > 0\right\}.$$

This implies

$$\lambda = \widehat{y}^T\,\nabla\chi(\widehat{y}) = x(\widehat{y}).$$

If we define

$$u_N(k-1) = \begin{cases} \dfrac{1}{x(\widehat{y})}\,\dfrac{B^T\left(A^{N-k}\right)^T \widehat{y}}{||B^T\left(A^{N-k}\right)^T \widehat{y}||_2} & \text{if} \quad k\in I(\widehat{y}), \\ \Theta_m & \text{else}, \quad k=1,\ldots,N, \end{cases} \tag{2.74}$$

then it follows that

$$\sum_{k=1}^{N} A^{N-k}B\,u_N(k-1) = c$$

and

$$||u_N(k-1)||_2 = \frac{1}{\chi(\widehat{y})} \qquad \text{for all} \quad k\in I(\widehat{y})$$

which implies

$$\max_{k=1,\ldots,N} ||u_N(k-1)||_2 = \frac{1}{\chi(\widehat{y})}.$$

Hence $u : \{0,\ldots,N-1\} \to \mathbb{R}^m$ solves Problem (P). This result is summarized as

Theorem 2.23. *If $\widehat{y}\in\mathbb{R}^n$ solves Problem (D), then $u : \{0,\ldots,N-1\} \to \mathbb{R}^m$ defined by (2.74) solves Problem (P).*

In order to solve Problem (D) we apply the well known gradient projection method which is based on the following iteration step: Let $y^*\in\mathbb{R}^n$ with $c^T\,y^* = 1$ be given. (At the beginning we take $y^* = c/||c||_2^2$.) Then we calculate

$$h = \left(\frac{1}{||c||_2^2}\,c^T\,\nabla\chi(y^*)\right)c - \nabla\chi(y^*)$$

and see that

$$c\,h = 0$$

and

$$\nabla\chi(y^*)^T\,h = \frac{1}{||c||_2^2}\left(c^T\,\nabla\chi(y^*)\right)^2 - ||\nabla\chi(y^*)||_2^2 \le 0.$$

If $\nabla\chi\left(y^*\right)^T h = 0$, then there exists some $\lambda \geq 0$ such that (2.73) holds true which is equivalent to y^* being optimal. If $\nabla\chi\left(y^*\right)^T h < 0$, then h is a feasible direction of descent. If we determine $\widehat{\lambda} > 0$ such that

$$\chi\left(y^* + \widehat{\lambda}\, h\right) = \min_{\lambda > 0} \chi\left(y^* + \lambda\, h\right), \tag{2.75}$$

then

$$\chi\left(y^* + \widehat{\lambda}\, h\right) < \chi\left(y^*\right)$$

and $c^T\left(y^* + \widehat{\lambda}\, h\right) = 1$. The next step is then performed with $y^* + \widehat{\lambda}\, h$ instead of y^*. A necessary and sufficient condition for $\widehat{\lambda} > 0$ to satisfy (2.75) is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\chi\left(y^* + \widehat{\lambda}\, h\right) = \nabla\chi\left(y^* + \widehat{\lambda}\, h\right)^T h = 0$$

which is equivalent to

$$\sum_{k \in I\left(y^* + \widehat{\lambda}\, h\right)} \frac{h^T A^{N-k} B B^T \left(A^{N-k}\right)^T \left(y^* + \widehat{\lambda}\, h\right)}{\left\| B^T \left(A^{N-k}\right)^T \left(y^* + \widehat{\lambda}\, h\right)\right\|_2} = 0$$

and in turn to the fixed point equation

$$\widehat{\lambda} = \psi\left(\widehat{\lambda}\right)$$

with

$$\psi(\lambda) = \frac{\displaystyle\sum_{k \in I(y^* + \lambda\, h)} \frac{h^T A^{N-k} B B^T \left(A^{N-k}\right)^T y^*}{\left\| B^T (A^{N-k})^T (y^* + \lambda\, h)\right\|_2}}{\displaystyle\sum_{k \in I(y^* + \lambda\, h)} \frac{h^T A^{N-k} B B^T (A^{N-k})^T h}{\left\| B^T (A^{N-k})^T (y^* + \lambda\, h)\right\|_2}}.$$

In order to solve this equation we apply the iteration procedure

$$\lambda_{k+1} = \psi\left(\lambda_k\right), \qquad k \in \mathbb{N}_0$$

starting with $\lambda_0 = 0$. Let us return to the problem of fixed point controllability in Section 2.2.1 and let us assume that $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously Fréchet differentiable.

Then it follows that

$$
\begin{aligned}
&G^N\left(x_0, u(0), \ldots, u(N-1)\right) - \widehat{x} \\
&\quad = G^N\left(x_0, u(0), \ldots, u(N-1)\right) - G^N\left(\widehat{x}, \Theta_m, \ldots, \Theta_m\right) \\
&\quad \approx J^x_{G^N}\left(\widehat{x}, \Theta_m, \ldots, \Theta_m\right)\left(x_0 - \widehat{x}\right) + \sum_{k=1}^{N} J^{u(k-1)}_{G^N}\left(\widehat{x}, \Theta_m, \ldots, \Theta_m\right) u(k-1) \\
&\quad = J^x_g\left(\widehat{x}, \Theta_m\right)^N\left(x_0 - \widehat{x}\right) + \sum_{k=1}^{N} J^x_g\left(\widehat{x}, \Theta_m\right)^{N-k} J^u_g\left(\widehat{x}, \Theta_m\right) u(k-1)
\end{aligned}
$$

where

$$J^x_g\left(\widehat{x}, \Theta_m\right) = \left(g_{i x_j}\left(\widehat{x}, \Theta_m\right)\right)_{i,j=1,\ldots,n}$$

and

$$J^u_g\left(\widehat{x}, \Theta_m\right) = \left(g_{i u_k}\left(\widehat{x}, \Theta_m\right)\right)_{\substack{i=1,\ldots,n \\ k=1,\ldots,m}} .$$

Therefore we replace equation (2.58) by

$$\sum_{k=1}^{N} J^x_g\left(\widehat{x}, \Theta_m\right)^{N-k} J^u_g\left(\widehat{x}, \Theta_m\right) u(k-1) = -J^x_g\left(\widehat{x}, \Theta_m\right)^N\left(x_0 - \widehat{x}\right) \tag{2.76}$$

and solve the problem of finding $u : \{0, \ldots, N-1\} \rightarrow \mathbb{R}^m$ which solves (2.76) and minimizes

$$\varphi_N(u) = \max_{k=1,\ldots,N} ||u(k-1)||_2 .$$

Such a $u : \{0, \ldots, N-1\} \rightarrow \mathbb{R}^m$ is then taken as an approximate solution of (2.58).

The above problem has the form of Problem (P) at the beginning of this section and can be solved by the method described above.

Finally we consider a special case in which the problem of fixed point controllability is reduced to a sequence of such problems which can be solved more easily.

For this purpose we consider the system

$$x(t+1) = g_0(x(t)) + \sum_{j=1}^{m} g_j(x(t))\, u_j(t), \qquad t \in \mathbb{N}_0, \tag{2.77}$$

where $g_j : \mathbb{R}^n \rightarrow \mathbb{R}^n,\ j = 1, \ldots, m$, are continuous vector functions.

For every control function $u : \mathbb{N}_0 \rightarrow \mathbb{R}^m$ there is exactly one function $x : \mathbb{N}_0 \rightarrow \mathbb{R}^m$ which satisfies (2.77) and the initial condition

$$x(0) = x_0, \qquad x_0 \in \mathbb{R}^n \text{ given.} \tag{2.78}$$

We denote it by $x = x(u)$. We assume that the uncontrolled system

$$x(t+1) = g_0(x(t)), \qquad t \in \mathbb{N},$$

has a fixed point $\widehat{x} \in \mathbb{R}^n$ which then solves the system

$$\widehat{x} = g_0\left(\widehat{x}\right).$$

We again assume that the set U of admissible control functions is given by (2.53) where $\Omega \subseteq \mathbb{R}^m$ is a subset with $\Theta_m \in \Omega$.

Let us define

$$\widetilde{g}_0(x) = g_0(x) - x, \qquad x \in \mathbb{R}^n.$$

Then (2.77) can be rewritten in the form

$$x(t+1) = x(t) + \widetilde{g}_0(x(t)) + \sum_{j=1}^{m} g_j(x(t))\, u_j(t), \qquad t \in \mathbb{N}_0. \tag{2.79}$$

In order to find some $N \in \mathbb{N}_0$ and a control function $u \in U$ with (2.55) such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.77) satisfies the end condition (2.56) we apply an iterative method: Starting with some $N_0 \in \mathbb{N}_0$ and some $u^0 \in U$ (for instance $u^0(t) = \Theta_m$ for all $t \in \mathbb{N}_0$) we construct a sequence $(N_k)_{k\in\mathbb{N}}$ in $\mathbb{N}_0$ and a sequence $\left(u^k\right)_{k\in\mathbb{N}}$ in U as follows:

If $N_{k-1} \in \mathbb{N}_0$ and $u^{k-1} \in U$ are determined, we calculate $x\left(u^{k-1}\right) : \mathbb{N}_0 \to \mathbb{R}^n$ as the solution of (2.78) and (2.79) for $u = u^{k-1}$. Then we determine $N_k \in \mathbb{N}_0$ and $u^k \in U$ such that

$$u^k(t) = \Theta_m \qquad \text{for all} \quad t \geq N_k \tag*{$(2.55)_k$}$$

and the solution $x\left(u^k\right) : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.78) and

$$\begin{aligned} x\left(u^k\right)(t+1) &= x\left(u^k\right)(t) + \widetilde{g}_0\left(x\left(u^{k-1}\right)(t)\right) \\ &\quad + \sum_{j=1}^{m} g_j\left(x\left(u^{k-1}\right)(t)\right)\, u_j^{k+1}(t), \qquad t \in \mathbb{N}_0, \end{aligned} \tag*{$(2.79)_k$}$$

satisfies the end condition

$$x\left(u^k\right)(N_k) = \widehat{x}. \tag*{$(2.56)_k$}$$

If we put

$$x^k = x_0 + \sum_{t=1}^{N_k} \widetilde{g}_0\left(x\left(u^{k-1}\right)(t-1)\right)$$

and

$$B^k(t-1) = \left(g_1\left(x\left(u^{k-1}\right)(t-1)\right) \mid \ldots \mid g_m\left(x\left(u^{k-1}\right)(t-1)\right)\right),$$

then the end condition $(2.56)_k$ is equivalent to

$$\sum_{t=1}^{N} B^k(t-1)\, u^{k-1}(t-1) = \widehat{x} - x^k.$$

2.2.4 Stabilization of Controlled Systems

Let $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous mapping and let H be a family of continuous mappings $h : \mathbb{R}^n \to \mathbb{R}^m$. If we define, for every $h \in H$, the mapping $f_h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_h(x) = g(x, h(x)), \qquad x \in \mathbb{R}^n,$$

then f_h is continuous and $(\mathbb{R}^n, f_h)$ is a time-discrete autonomous dynamical system. Let $\widehat{x} \in \mathbb{R}^n$ be a fixed point of

$$f(x) = g\left(x, \Theta_m\right), \qquad x \in \mathbb{R}^n.$$

Further we assume that

$$h\left(\widehat{x}\right) = \Theta_m \qquad \text{for all} \quad h \in H$$

which implies that $\widehat{x}$ is a fixed point of all $f_h,\ h \in H$. After these preparations we can formulate the

Problem of Stabilization

Find $h \in H$ such that $\{\widehat{x}\}$ is asymptotically stable with respect to f_h.

We assume that $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and every mapping $h \in H$ are continuously Fréchet differentiable. Then every mapping $f_h : \mathbb{R}^n \to \mathbb{R}^n,\ h \in H$, is also continuously Fréchet differentiable and, for every $x \in \mathbb{R}^n$, its Jacobi matrix is given by

$$J_{f_h}(x) = J_g^x(x, h(x)) + J_g^u(x, h(x))\, J_h^x(x)$$

where

$$J_g^x(x, h(x)) = \big(g_{ix_j}(x, h(x))\big)_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$$

and

$$J_g^u(x, h(x)) = \left(g_{iu_k}(x, h(x))\right)_{\substack{i=1,\dots,n\\ k=1,\dots,m}}, \qquad J_h^x(x) = \big(h_{ix_j}(x)\big)_{\substack{i=1,\dots,m\\ j=1,\dots,n}}.$$

From the Corollary of Theorem 1.5 we then obtain the

Theorem 2.24.

(a) *Let the spectral radius $\varrho\left(J_{f_h}\left(\widehat{x}\right)\right) < 1$. Then $\widehat{x}$ is asymptotically stable with respect to f_h.*

(b) *Let $\left(J_{f_h}\left(\widehat{x}\right)\right)$ be invertible and let all the eigenvalues of $\varrho\left(J_{f_h}\left(\widehat{x}\right)\right)$ be larger than 1 in absolute value. Then $\widehat{x}$ is unstable with respect to f_h.*

Special cases:

(a) Let

$$g(x,u) = A\,x + B\,u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

where A is a real $n \times n-$matrix and B a real $n \times m-$matrix, respectively. Further let H be the family of all linear mappings $h : \mathbb{R}^n \to \mathbb{R}^m$ which are given by

$$h(x) = C\,x, \qquad x \in \mathbb{R}^n,$$

where C is an arbitrary real $m \times n-$matrix, respectively.

If we choose $\widehat{x} = \Theta_n$, then

$$f(\Theta_n) = g(\Theta_n, \Theta_m) = \Theta_n$$

and

$$h(\Theta_n) = \Theta_m \qquad \text{for all} \quad h \in H.$$

Finally we have $J_h(x) = C$,

$$J_g^x(x, h(x)) = A \qquad \text{and} \qquad J_g^u(x, h(x)) = B$$

for all $x \in \mathbb{R}^n$ and $h \in H$ which implies

$$J_{f_h}(x) = A + BC \qquad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad h \in H.$$

Thus $\widehat{x} = \Theta_n$ is asymptotically stable with respect to f_h, if

$$\varrho(A + BC) < 1,$$

and unstable with respect to f_h, if all the eigenvalues of $A+BC$ are larger than one in absolute value.

(b) Let

$$g(x,u) = F(x) + B(x)\,u, \qquad x \in X,\ u \in \mathbb{R}^m,$$

where $F : X \to X,\ X \subseteq \mathbb{R}^n$ open, is continuously Fréchet differentiable and $B(x) = (b_1(x), \ldots, b_m(x))\,,\ x \in X$, where $b_j : X \to \mathbb{R}^n,\ j = 1, \ldots, n$, are also continuously Fréchet differentiable. Let again H be the family of all linear mappings $h : \mathbb{R}^n \to \mathbb{R}^m$ which are given by

$$h(x) = C\,x, \qquad x \in \mathbb{R}^n.$$

Finally, we assume that $\Theta_n \in X$ and $F(\Theta_n) = \Theta_n$.

If we choose $\widehat{x} = \Theta_n$, then

$$f(\Theta_n) = g(\Theta_n, \Theta_m) = \Theta_n$$

and

$$h(\Theta_n) = \Theta_m \qquad \text{for all} \quad h \in H.$$

Further we obtain

$$J_h^x(x) = C,$$

$$J_g^x(x, h(x)) = J_F(x) + \sum_{j=1}^{m} J_{b_j}(x)\, h_j(x) \qquad \text{and} \qquad J_g^u(x, h(x)) = B(x)$$

for all $x \in X$ and $h \in H$ which implies

$$J_{f_h}(x) = J_F(x) + \sum_{j=1}^{m} J_{b_j}(x)\, h_j(x) + B(x)C \qquad \text{for} \quad x \in X, \ h \in H,$$

hence

$$J_{f_h}(\Theta_n) = J_F(\Theta_n) + B(\Theta_n)\, C \qquad \text{for all} \quad h \in H.$$

Thus $\widehat{x} = \Theta_n$ is asymptotically stable with respect to f_h, if

$$\varrho\,(J_F(\Theta_n) + B(\Theta_n)\, C) < 1$$

and unstable with respect to f_h, if all the eigenvalues of $J_F(\Theta_n) + B(\Theta_n)\, C$ are larger than one in absolute value.

2.2.5 Applications

a) An Emission Reduction Model

We pick up the emission reduction model that was treated as uncontrolled system in Section 1.3.6 and as controlled system in Section 2.2.1. Here we concentrate on the controlled system which we linearize at a fixed point $\left(\widehat{E}^T, \Theta_r^T\right)^T$, $\widehat{E} \in \mathbb{R}^r$, of the uncontrolled system which leads to a linear control system of the form

$$x(t+1) = A\,x(t) + B\,u(t), \qquad t \in \mathbb{N}_0,$$

with

$$A = \begin{pmatrix} I_r & C \\ 0_r & D \end{pmatrix}, \qquad B = \begin{pmatrix} C \\ D \end{pmatrix},$$

where I_r and 0_r is the $r \times r-$unit and zero-matrix, respectively, and

$$C = \begin{pmatrix} em_{11} & \cdots & em_{1r} \\ \vdots & \ddots & \vdots \\ em_{r1} & \cdots & em_{rr} \end{pmatrix}, \qquad D = \begin{pmatrix} 1 - \lambda_1 M_1^* \widehat{E}_1 & & 0 \\ & \ddots & \\ 0 & & 1 - \lambda_r M_r^* \widehat{E}_r \end{pmatrix}.$$

This implies

$$A^k B = \begin{pmatrix} C\left(I_r + D + \ldots + D^k\right) \\ D^{k+1} \end{pmatrix} \qquad \text{for all} \quad k \in \mathbb{N}_0.$$

We consider the problem of null-controllability as being discussed in Section 2.2.2. Let us assume that C and D are non-singular. Then it follows that the matrices A and

$$\begin{pmatrix} C & C\left(I_r + D\right) \\ D & D^2 \end{pmatrix}$$

are non-singular which implies that the Kalman condition (2.67) is satisfied for $N_0 = 2$. Let $d_1, \ldots, d_r$ be the diagonal elements of D. Thus the non-singularity of D is equivalent to

$$d_i \neq 0 \qquad \text{for all} \quad i = 1, \ldots, r.$$

If all $d_i \neq 1$ for $i = 1, \ldots, r$, then it follows (see Section 1.3.6) that the eigenvectors corresponding to the eigenvalues

$$\mu_i = 1 \quad \text{for} \quad i = 1, \ldots, r \qquad \text{and} \qquad \mu_{i+r} = d_i \quad \text{for} \quad i = 1, \ldots, r$$

of A are linearly independent which also holds true for A^T (which has the same eigenvalues) (see Section 2.1.2).

If

$$|d_i| \leq 1 \qquad \text{for all} \quad i = 1, \ldots, r$$

and

$$\Omega = \{u \in \mathbb{R}^r \mid ||u|| \leq \gamma\}$$

for some $\gamma > 0$ where $|| \cdot ||$ is any norm in $\mathbb{R}^r$, then by Theorem 2.19 the problem of null-controllability has a solution for every choice of $x_0 = \left(x_0^{1\,T}, x_0^{2\,T}\right)^T \in \mathbb{R}^{2r}$. This problem can be solved with the aid of Problem (P) in Section 2.2.3 which reads as follows in this case: For a given $N \in \mathbb{N}$ find $u : \{0, \ldots, N-1\} \to \mathbb{R}^r$ such that

$$\sum_{k=1}^{N} \begin{pmatrix} C\left(I_r + D + \ldots + D^k\right) \\ D^{k+1} \end{pmatrix} u(k-1) = -\begin{pmatrix} I_r & C\left(I_r + D + \ldots + D^{N-1}\right) \\ 0_r & D^N \end{pmatrix} \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$$

and

$$\varphi_N(u) = \max_{k=1,\ldots,N} ||u(k-1)||_2$$

is minimized (where $|| \cdot ||_2$ denotes the Euclidean norm in $\mathbb{R}^r$).

Finally we illustrate the method by two numerical examples.

Let $m = 3$. In both cases we choose

$$x_0^{1\,T} = x_0^{2\,T} = (1, 1, 1). \tag{2.80}$$

At first we choose

$$C = \begin{pmatrix} 0.8 & 0.5 & -0.5 \\ 0.2 & 0.2 & 0.3 \\ 0.4 & 0.3 & 0.2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

and obtain Fig. 2.1:

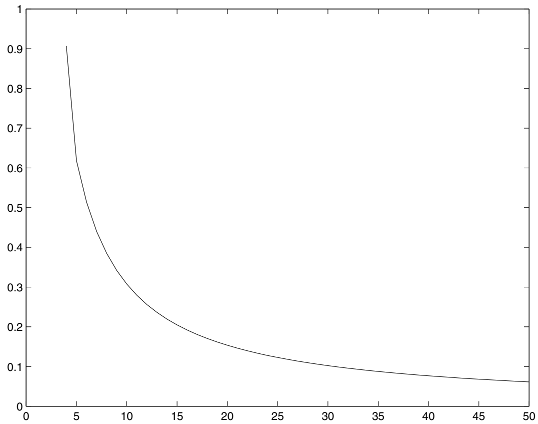


Fig. 2.1.
Ordinate, $\varphi_N(u_N) = \frac{1}{\chi(\hat{y}_N)}$
Abscissa, N

Next we choose

$$C = \begin{pmatrix} 0.8 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.6 \end{pmatrix}$$

and get Fig. 2.2.

b) A Controlled Predator-Prey Model

We pick up the predator prey model that has been discussed in Section 1.3.7 a) whose controlled version we assume to be of the form

$$\begin{aligned} x_1(t+1) &= x_1(t) + a\,x_1(t) - b\,x_1(t)x_2(t) - x_1(t)\,u_1(t), \\ x_2(t+1) &= x_2(t) - c\,x_2(t) + d\,x_1(t)x_2(t) - x_2(t)\,u_2(t) \qquad t \in \mathbb{N}_0, \end{aligned}$$

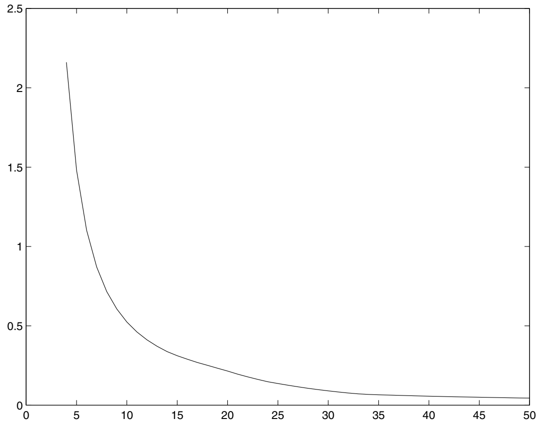


Fig. 2.2.
Ordinate, $\varphi_N(u_N) = \frac{1}{\chi(\hat{y}_N)}$
Abscissa, N

where $a > 0,\ 0 < c < 1,\ b > 0,\ d > 0,\ x_1(t)$ and $x_2(t)$ denote the density of the prey and predator population at time t, respectively, and $u_1, u_2 : \mathbb{N}_0 \to \mathbb{R}$ are control functions.

If we define

$$\widetilde{g}_0\left(x_1(t), x_2(t)\right) = \begin{pmatrix} a\,x_1(t) - b\,x_1(t)x_2(t) \\ -c\,x_1(t) + d\,x_1(t)x_2(t) \end{pmatrix}$$

and

$$g_1\left(x_1(t), x_2(t)\right) = \begin{pmatrix} -x_1(t) \\ 0 \end{pmatrix}, \qquad g_2\left(x_1(t), x_2(t)\right) = \begin{pmatrix} 0 \\ -x_2(t) \end{pmatrix},$$

then the system can be rewritten in the form

$$x(t+1) = x(t) + \widetilde{g}_0(x(t)) + \sum_{j=1}^{2} g_j(x(t))\,u_j(t), \qquad t \in \mathbb{N}_0, \tag{2.81}$$

with $x(t) = (x_1(t), x_2(t))^T$. This is exactly the system (2.79) for $m = 2$.

In addition we assume an initial condition

$$x(0) = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = x_0. \tag{2.82}$$

The uncontrolled system

$$x(t+1) = x(t) + \widetilde{g}_0(x(t)), \qquad t \in \mathbb{N}_0,$$

has $\widehat{x} = \left(\frac{c}{d}, \frac{a}{b}\right)^T$ as fixed point.

We assume the set U of admissible control functions to be given by

$$U = \left\{u : \mathbb{N}_0 \to \mathbb{R}^2 \mid ||u(t)||_2 \leq \gamma \text{ for all } t \in \mathbb{N}_0\right\}$$

where $\gamma > 0$ is a given constant and $||\cdot||_2$ is the Euclidean norm. For every $u \in U$ we denote the unique solution $x : \mathbb{N}_0 \to \mathbb{R}^2$ of (2.81) and (2.82) by $x(u)$. Our aim now consists of finding some $N \in \mathbb{N}_0$ and a control function $u \in U$ with

$$u(t) = \Theta_2 \qquad \text{for all} \quad t \geq N \tag{2.83}$$

such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^2$ of (2.81), (2.82) satisfies the end condition

$$x(N) = \widehat{x}. \tag{2.84}$$

In order to find a solution of this problem we apply the iteration method described in Section 2.2.3. In the $k-$th step of this procedure we have, for a given $u^{k-1} \in U$, to find some $N_k \in \mathbb{N}_0$ and a $u^k \in U$ with

$$u^k(t) = \Theta_2 \qquad \text{for} \quad t \geq N_k$$

such that

$$\sum_{t=1}^{N_k} \begin{pmatrix} x_1\left(u^{k-1}\right)(t-1) & 0 \\ 0 & x_2\left(u^{k-1}\right)(t-1) \end{pmatrix} \begin{pmatrix} u_1^k(t-1) \\ u_2^k(t-1) \end{pmatrix} = x^k - \widehat{x},$$

where

$$x^k = x_0 + \sum_{t=1}^{N_k} \widetilde{g}_0\left(x\left(u^{k-1}\right)(t-1)\right),$$

and

$$||u^k(t-1)||_2 \leq \gamma \qquad \text{for} \quad t = 1, \ldots, N_k.$$

For this we can apply the method developed in Section 2.2.3.

c) Control of a Planar Pendulum with Moving Suspension Point

We consider a non-linear planar pendulum of length $l(>0)$ whose movement is controlled by moving its suspension point with acceleration $u = u(t)$ along a horizontal straight line.

If we denote the deviation angle from the orthogonal position of the pendulum by $\varphi = \varphi(t)$, then the movement of the pendulum is governed by the differential equation

$$\ddot{\varphi}(t) = -\frac{g}{l} \sin \varphi(t) - \frac{u(t)}{l} \cos \varphi(t), \qquad t \in \mathbb{R}, \tag{2.85}$$

where g denotes the gravity constant.

For $t = 0$ initial conditions are given by

$$\varphi(0) = \varphi_0 \qquad \text{and} \qquad \dot{\varphi}(0) = \dot{\varphi}_0.$$

Now we discretize the differential equation by introducing a time step length $h > 0$ and replacing the second derivative $\ddot{\varphi}(t)$ by $\frac{1}{h}(\varphi(t+2h) - 2\,\varphi(t+h) + \varphi(t))$ thus obtaining the difference equation

$$\varphi(t+2h) = 2\,\varphi(t+h) - \varphi(t) - \frac{gh^2}{l} \sin \varphi(t) - \frac{u(t)\,h^2}{l} \cos \varphi(t), \qquad t \in \mathbb{R}.$$

If we define

$$y_1(t) = \varphi(t) \qquad \text{and} \qquad y_2(t) = \varphi(t+h)$$

then we obtain

$$\begin{aligned} y_1(t+h) &= y_2(t), \\ y_2(t+h) &= 2\,y_2(t) - y_1(t) - \frac{gh^2}{l} \sin y_1(t) - \frac{u(t)\,h^2}{l} \cos\ y_1(t), \qquad t \in \mathbb{R}. \end{aligned}$$

Finally we define functions $x_1 : \mathbb{N}_0 \to \mathbb{R}$ and $x_2 : \mathbb{N}_0 \to \mathbb{R}$ by putting

$$x_1(n) = y_1(n \cdot h) \quad \text{and} \quad x_2(n) = y_2(n \cdot h), \qquad n \in \mathbb{N}_0,$$

and obtain the system

$$\begin{aligned} x_1(t+1) &= x_2(t), \\ x_2(t+1) &= 2\,x_2(t) - x_1(t) - \frac{gh^2}{l} \sin x_1(t) - \frac{u(t)\,h^2}{l} \cos x_1(t), \end{aligned} \tag{2.49'}$$

$t \in \mathbb{N}_0$ which is of the form (2.79) for $m = 1$ with

$$\widetilde{g}_0\,(x_1(t), x_2(t)) = \begin{pmatrix} x_2(t) - x_1(t) \\ x_2(t) - x_1(t) - \dfrac{gh^2}{l} \sin x_1(t) \end{pmatrix},$$

$$g_1\,(x_1(t), x_2(t)) = \begin{pmatrix} 0 \\ -\dfrac{h^2}{l} \cos x_1(t) \end{pmatrix}, \qquad t \in \mathbb{N}_0.$$

In addition we assume initial conditions

$$x_1(0) = \varphi_0 \qquad \text{and} \qquad x_2(0) = \dot{\varphi}_0. \tag{2.50'}$$

The uncontrolled system

$$\begin{aligned} x_1(t+1) &= x_2(t), \\ x_2(t+1) &= 2\,x_2(t) - x_1(t) - \frac{gh^2}{l}\sin x_1(t) \end{aligned}$$

has $\widehat{x} = (0,0)$ as fixed point.

We assume the set U of admissible control functions to be given by

$$U = \{u : \mathbb{N}_0 \to \mathbb{R} \mid |u(t)| \le \gamma \text{ for all } t \in \mathbb{N}_0\}$$

where $\gamma > 0$ is a given constant.

For every $u \in U$ we denote the unique solution $x : \mathbb{N}_0 \to \mathbb{R}^2$ of (2.49') and (2.50') by $x(u)$.

Our aim consists of finding some $N \in \mathbb{N}_0$ and a control function $u \in U$ with

$$u(t) = 0 \qquad \text{for all} \quad t \ge N \tag{2.55'}$$

such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^2$ of (2.49'), (2.50') satisfies the end condition

$$x(N) = \Theta_2.$$

This condition is equivalent to

$$x_2(N) = x_2(N-1) = 0.$$

The $k-$th step of the iteration method described in Section 2.2.3 for the solution of this problem then reads as follows: Let $u^{k-1} \in U$ be given. Then we determine $N_k \in \mathbb{N}_0$ and $u^k \in U$ such that

$$u^k(t) = 0 \qquad \text{for all} \quad t \ge N_k$$

and the solution $x\left(u^k\right) : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.50') and

$$\begin{aligned} x_1\left(u^1(t+1)\right) &= x_1\left(u^k\right)(t) + x_2\left(u^{k-1}\right)(t) - x_1\left(u^{k-1}\right)(t), \\ x_2\left(u^1(t+1)\right) &= x_2\left(u^k\right)(t) + x_2\left(u^{k-1}\right)(t) - x_1\left(u^{k-1}\right)(t) \\ &\qquad -\frac{gh^2}{l}\sin x_1\left(u^{k-1}\right)(t) - \frac{h^2}{l}\cos x_1\left(u^{k-1}\right)(t)\,u^k(t), \end{aligned}$$

$t \in \mathbb{N}_0$, satisfies the end conditions

$$x_2\left(u^k\right)(N_k) = x_2\left(u^k\right)(N_k - 1) = 0.$$

These are equivalent to

$$
\begin{aligned}
&-\frac{1}{l}\sum_{t=1}^{N_k-1}\cos x_1\left(u^{k-1}\right)(t-1)\,u^k(t-1)\\
&\qquad=-\dot{\varphi}_0-x_2\left(u^{k-1}\right)(N_k-2)+\varphi_0-\frac{g}{l}\sum_{t=1}^{N_k-1}\sin x_1\left(u^{k-1}\right)(t-1)\\
&\qquad\qquad-\frac{1}{l}\cos x_1\left(u^{k-1}\right)(N_k-1)\;u^k\left(N_k-1\right)\\
&\qquad=-x_2\left(u^{k-1}\right)(N_k-1)+x_2\left(u^{k-1}\right)(N_k-2)\\
&\qquad\qquad-\frac{g}{l}\sin x_1\left(u^{k-1}\right)(N_k-1)\,.
\end{aligned}
$$

Let us consider the special case $N_k = 2$. Then it follows that

$$
\begin{aligned}
&-\frac{1}{l}\cos\varphi_0\,u^k(0)=\varphi_0-\frac{g}{l}\sin\varphi_0,\\
&-\frac{1}{l}\cos\varphi_0\,u^k(1)=-\dot{\varphi}_0+\varphi_0+\frac{gh^2}{l}\sin\varphi_0-\frac{g}{l}\sin\dot{\varphi}_0-\frac{h^2}{l}\cos\dot{\varphi}_0\,u^{k-1}(0).
\end{aligned}
$$

Let us assume that

$$
\cos\varphi_0\neq 0\qquad\text{and}\qquad\cos\dot{\varphi}_0\neq 0.
$$

Then it follows for all $k \geq 1$ that

$$
\begin{aligned}
u^k(0)&=-\frac{l\,\varphi_0}{\cos\varphi_0}+g\,\tan\varphi_0,\\
u^k(1)&=\frac{l\,(\varphi_0-\dot{\varphi}_0)}{\cos\dot{\varphi}_0}-gh^2\,\frac{\sin\varphi_0}{\cos\dot{\varphi}_0}+g\,\tan\dot{\varphi}_0+h^2\,u^{k-1}(0).
\end{aligned}
$$

2.3 The Time-Discrete Non-Autonomous Case

2.3.1 The Problem of Fixed Point Controllability

We consider a system of difference equations of the form

$$x(t+1) = g_t(x(t), u(t)), \qquad t \in \mathbb{N}_0, \tag{2.86}$$

where $(g_t)_{t\in\mathbb{N}_0}$ is a sequence of continuous vector functions $g_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $u : \mathbb{N}_0 \to \mathbb{R}^m$ is a given vector function which is called a control function. The vector function $x : \mathbb{N}_0 \to \mathbb{R}^n$ which is called a state function is uniquely defined by (2.86), if we require an initial condition

$$x(0) = x_0 \tag{2.87}$$

for some given vector $x_0 \in \mathbb{R}^n$.

If we fix the control function $u : \mathbb{N}_0 \to \mathbb{R}^m$ and define

$$f_t(x) = g_t(x, u(t)), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{N}_0, \tag{2.88}$$

then, for every $t \in \mathbb{N}_0$, $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping and $\left(\mathbb{R}^n, (f_t)_{t\in\mathbb{N}_0}\right)$ is a non-autonomous time-discrete dynamical system which is controlled by the function $u : \mathbb{N}_0 \to \mathbb{R}^m$. If

$$u(t) = \Theta_m \qquad \text{for all} \quad t \in \mathbb{N}_0,$$

then the system (2.88) is called uncontrolled. Let us assume that the uncontrolled system (2.88) admits a fixed point $\widehat{x} \in \mathbb{R}^n$ which then solves the equations

$$\widehat{x} = g_t\left(\widehat{x}, \Theta_m\right), \qquad \text{for all} \quad t \in \mathbb{N}. \tag{2.89}$$

Now let $\Omega \subseteq \mathbb{R}^m$ be a subset with $\Theta_m \in \Omega$. Then we define the set of admissible control functions by

$$U = \left\{u : \mathbb{N}_0 \to \mathbb{R}^m \mid u(t) \in \Omega \text{ for all } t \in \mathbb{N}_0\right\}. \tag{2.90}$$

After these preparations we can formulate the

Problem of Local Fixed Point Controllability

Given a fixed point $\widehat{x} \in \mathbb{R}^n$ of the system

$$x(t+1) = g_t\left(x(t), \Theta_m\right), \qquad t \in \mathbb{N}_0, \tag{2.91}$$

i.e., a solution $\widehat{x}$ of the equations (2.89) and an initial state $x_0 \in \mathbb{R}^n$, find some $N \in \mathbb{N}_0$ and a control function $u \in U$ with

$$u(t) = \Theta_m, \qquad \text{for all} \quad t \geq N \tag{2.92}$$

such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.86), (2.87) satisfies the end condition

$$x(N) = \widehat{x} \tag{2.93}$$

(which implies $x(t) = \widehat{x}$ for all $t \geq N$). Let us assume that, for every $t \in \mathbb{N}_0$, the Jacobi matrices

$$A_t = \frac{\partial g_t}{\partial x}(\widehat{x}, \Theta_m) \qquad \text{and} \qquad B_t = \frac{\partial g_t}{\partial x}(\widehat{x}, \Theta_m)$$

exist. Then it follows that, for every $t \in \mathbb{N}_0$,

$$\begin{aligned} x(t+1) - \widehat{x} &= g_t(x(t), u(t)) - g_t(\widehat{x}, \Theta_m) \\ &\approx A_t\,(x(t) - \widehat{x}) + B_t\,u(t). \end{aligned}$$

Therefore we replace the system (2.86) by

$$h(t+1) = A_t\,h(t) + B_t\,u(t), \qquad t \in \mathbb{N}_0, \tag{2.94}$$

and the initial condition (2.87) by

$$h(0) = x_0 - \widehat{x}. \tag{2.95}$$

The end condition (2.93) is replaced by

$$h(N) = \Theta_n = \text{ zero vector of } \mathbb{R}^n. \tag{2.96}$$

Now we consider the

Problem of Local Controllability

Find $N \in \mathbb{N}_0$ and $u \in U$ with

$$u(t) = \Theta_m \qquad \text{for all} \quad t \geq N$$

such that the corresponding solution $h : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.94), (2.95) satisfies the end condition (2.96) which implies

$$h(t) = \Theta_n \qquad \text{for all} \quad t \geq N.$$

From (2.94) and (2.95) we conclude that, for every $N \in \mathbb{N}$,

$$h(N) = A_{N-1} \cdots A_0\,(x_0 - \widehat{x}) + \sum_{k=1}^{N} A_{N-1} \cdots A_k B_{k-1}\,u(k-1),$$

if, for $k = N$, we put $A_{N-1} \cdots A_k = I = n \times n-$unit matrix.

Therefore the end condition (2.96) is equivalent to

$$\sum_{k=1}^{N} A_{N-1}A_{N-2}\cdots A_k B_{k-1}\, u(k-1) = -A_{N-1}A_{N-2}\cdots A_0\,(x_0 - \widehat{x})\,. \quad (2.97)$$

Let us assume that, for some $N_0 \in \mathbb{N}$,

$$\operatorname{rank}\left(B_{N_0-1} \mid A_{N_0-1}B_{N_0-2} \mid \ldots \mid A_{N_0-1}\cdots A_1 B_0\right) = n. \quad (2.98)$$

Then, for $N = N_0$, the system (2.97) has a solution $\left(u(0)^T, u(1)^T, \ldots, u\left(N_0-1\right)^T\right)^T \in \mathbb{R}^{m\cdot N_0}$ where $x_0 \in \mathbb{R}^N$ can be chosen arbitrarily. If we define

$$u(t) = \Theta_m \qquad \text{for all} \quad t \geq N_0, \quad (2.99)$$

then we obtain a control function $u : \mathbb{N}_0 \to \mathbb{R}^m$ such that the corresponding solution $h : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.94) and (2.95) satisfies the end condition (2.96) for $N = N_0$.

If A_k is non-singular for all $k \in \mathbb{N}_0$, then the assumption (2.97) implies that (2.98) holds true for all $N \geq N_0$ instead of N_0.

For instance, if we replace N_0 by $N_0 + 1$, then

$$\begin{aligned}&\left(B_{N_0} \mid A_{N_0}B_{N_0-1} \mid A_{N_0}A_{N_0-1}B_{N_0-2} \mid \ldots \mid A_{N_0}\cdots A_1B_0\right)\\ &= \underbrace{\left(B_{N_0} \mid \underbrace{A_{N_0}}_{\text{non-singular}} \underbrace{\left(B_{N_0-1} \mid A_{N_0-1}B_{N_0-2} \mid \ldots \mid A_{N_0-1}\cdots A_1B_0\right)}_{\text{rank} = n}\right)}_{\Longrightarrow\ \text{rank} = n}.\end{aligned}$$

So the system (2.97) has a solution $\left(u(0)^T, u(1)^T, \ldots, u\left(N_0-1\right)^T\right)^T$ for all $N \geq N_0$.

Next we assume that $\Omega \subseteq \mathbb{R}^m$ is convex, has Θ_m as interior point and satisfies $u \in \Omega \;\Rightarrow\; -u \in \Omega$. Let us define, for every $N \in \mathbb{N}$, the set

$$R(N) = \left\{ x = \sum_{k=1}^{N} A_{N-1}A_{N-2}\cdots A_k B_{k-1}\, u(k-1) \;\middle|\; u \in U \right\}.$$

Then we can prove (see Theorem 2.18)

Theorem 2.25. *If, for some $N_0 \in \mathbb{N}$, condition (2.98) is satisfied and if A_k is non-singular for all $k \in \mathbb{N}_0$, then Θ_n is an interior point of $R(N)$ for all $N \geq N_0$.*

Proof. Let us assume that $\Theta_n(\in R(N))$ is not an interior point of $R(N)$ for some $N \geq N_0$. Thus $R(N)$ must be contained in a hyperplane through Θ_n, i.e., there must exist some $y \in \mathbb{R}^n,\ y \neq \Theta_n$, with

$$y^T x = 0 \qquad \text{for all} \quad x \in R(N).$$

This implies

$$\sum_{k=1}^{N} y^T A_{N-1} A_{N-2} \cdots A_k B_{k-1} u^k = 0 \qquad \text{for all} \quad \left(u^{1\,T}, \ldots, u^{N\,T}\right)^T \in (\mathbb{R}^m)^N;$$

hence

$$y^T A_{N-1} A_{N-2} \cdots A_k B_{k-1} = \Theta_m^T \qquad \text{for all} \quad k = 1, \ldots, N.$$

Since (2.98) also holds true for all $N \geq N_0$ instead of N_0, it follows that $y = \Theta_n$ which contradicts $y \neq \Theta_n$. Hence the assumption is false and the proof is complete. □

As a consequence of Theorem 2.25 we obtain

Theorem 2.26. *In addition to the assumption of Theorem 2.25 let*

$$\sup_{k \in \mathbb{N}_0} ||A_k|| < 1 \qquad \textit{where } ||\cdot|| \textit{ denotes the spectral norm.} \tag{2.100}$$

Then there is some $N \in \mathbb{N}$ and some $u \in U$ such that (2.98) holds true.

Proof. Assumption (2.100) implies

$$\lim_{N \to \infty} A_{N-1} A_{N-2} \cdots A_0 \, (x_0 - \widehat{x}) = \Theta_n.$$

Hence Theorem 2.25 implies that there is some $N \in \mathbb{N}$ with $N \geq N_0$ such that

$$-A_{N-1} A_{N-2} \cdots A_0 \, (x_0 - \widehat{x}) \in R(N)$$

which completes the proof. □

2.3.2 The General Problem of Controllability

We consider the same situation as at the beginning of Section 2.3.1. However, we do not assume the existence of a fixed point $\widehat{x} \in \mathbb{R}^n$ of the uncontrolled system (2.91), i.e., a solution of (2.89). Instead we assume a vector $x_1 \in \mathbb{R}^n$ to be given and consider the general

Problem of Controllability

Find some $N \in \mathbb{N}_0$ and a control function $u \in U$ (2.90) such that the solution $x : \mathbb{N}_0 \to \mathbb{R}^n$ of (2.86), (2.87) satisfies the end condition

$$x(N) = x_1. \tag{2.101}$$

From (2.86) and (2.87) we infer

$$\begin{aligned} x(N) &= g_{N-1}\left(g_{N-2}\left(\cdots\left(g_0\left(x_0, u(0)\right), u(1)\right), \ldots\right), u(N-1)\right) \\ &= G^N\left(x_0, u(0), \ldots, u(N-1)\right). \end{aligned} \tag{2.102}$$

Hence the end condition (2.101) is equivalent to

$$G^N\left(x_0, u(0), \ldots, u(N-1)\right) = x_1. \tag{2.103}$$

So we have to find vectors $u(0), \ldots, u(N-1) \in \Omega$ such that (2.103) is satisfied.

For every $N \in \mathbb{N}$ we define the controllable set

$$\begin{aligned} S_N\left(x_1\right) = \{x \in \mathbb{R}^n \mid\ & \text{there exists some } u \in U \\ & \text{such that } G^N(x, u(0), \ldots, u(N-1)) = x_1\} \end{aligned}$$

and put

$$S\left(x_1\right) = \bigcup_{\mathbb{N} \in \mathbb{N}} S_N\left(x_1\right).$$

Now let $x_0 \in S\left(x_1\right)$. Then we ask the question under which conditions is x_0 an interior point of $S\left(x_1\right)$?

In order to find an answer to this question we assume that

$$\Omega \text{ is open} \quad \text{and} \quad g_N \in \mathcal{C}^1\left(\mathbb{R}^n \times \mathbb{R}^m\right) \qquad \text{for all} \quad N \in \mathbb{N}_0.$$

Then it follows that $G^N \in \mathcal{C}^1\left(\mathbb{R}^n \times \mathbb{R}^{m \cdot N}\right)$ for every $N \in \mathbb{N}$ and

$$\begin{aligned} & G_x^N\left(x, u(0), \ldots, u(N-1)\right) \\ & \quad = \left(g_{N-1}\right)_x\left(G^{N-1}\left(x, u(0), \ldots, u(N-2)\right), u(N-1)\right) \\ & \qquad \times\ \left(g_{N-2}\right)_x\left(G^{N-2}\left(x, u(0), \ldots, u(N-3)\right), u(N-2)\right) \\ & \qquad \times \qquad\qquad\qquad\qquad \vdots \\ & \qquad \times\ \left(g_0\right)_x\left(x, u(0)\right), \end{aligned}$$

and

$$
\begin{aligned}
&G^N_{u(k)}\left(x, u(0), \ldots, u(N-1)\right)\\
&\quad = \left(g_{N-1}\right)_x \left(G^{N-1}\left(x, u(0), \ldots, u(N-2)\right), u(N-1)\right)\\
&\qquad \times\ \left(g_{N-2}\right)_x \left(G^{N-2}\left(x, u(0), \ldots, u(N-3)\right), u(N-2)\right)\\
&\qquad \times \qquad\qquad\qquad\qquad \vdots\\
&\qquad \times\ \left(g_{k+1}\right)_x \left(G^{k+1}\left(x, u(0), \ldots, u(k)\right), u(k+1)\right)\\
&\qquad \times\ \left(g_{k}\right)_{u(k)} \left(G^{k}\left(x, u(0), \ldots, u(k-1)\right), u(k)\right)
\end{aligned}
$$

for $k = 0, \ldots, N-1,\ x \in \mathbb{R}^n$ and $u \in U$.

Let us assume that $x_0 \in S_{N_0}\left(x_1\right)$ for some $N_0 \in \mathbb{N}$, i.e.,

$$G^{N_0}\left(x_0, u_0(0), \ldots, u_0\left(N_0-1\right)\right) = x_1$$

for some $u_0 \in U$. Further let $\left(g_N\right)_x\left(x, u_0\right)$ be non-singular for all $N \in \mathbb{N}_0$, for all $x \in \mathbb{R}^n$ and all $u \in \Omega$.

Then $G^{N_0}_x\left(x_0, u_0(0), \ldots, u_0\left(N_0-1\right)\right)$ is also non-singular and, by the implicit function theorem, there exists an open set $V \subseteq \Omega^{N_0}$ with $(u_0(0), \ldots, u_0\left(N_0-1\right)) \in V$ and a function $h : V \to \mathbb{R}^n$ with $h \in \mathcal{C}^1(V)$ such that

$$h\left(u_0(0), \ldots, u_0\left(N_0-1\right)\right) = x_0$$

and

$$
\begin{gathered}
G^{N_0}\left(h\left(u_0(0), \ldots, u_0\left(N_0-1\right)\right), u(0), \ldots, u\left(N_0-1\right)\right) = x_1\\
\text{for all} \quad \left(u(0), \ldots, u\left(N_0-1\right)\right) \in V
\end{gathered}
$$

which means

$$h\left(u(0), \ldots, u\left(N_0-1\right)\right) \in S_{N_0}\left(x_1\right) \qquad \text{for all} \quad \left(u(0), \ldots, u\left(N_0-1\right)\right) \in V.$$

Moreover,

$$
\begin{aligned}
h_{u(k)}\left(u_0(0), \ldots, u_0\left(N_0-1\right)\right) = {}& -G^{N_0}_x\left(x_0, u_0(0), \ldots, u_0\left(N_0-1\right)\right)^{-1}\\
& \times G^{N_0}_{u(k)}\left(x_0, u_0(0), \ldots, u_0\left(N_0-1\right)\right).
\end{aligned}
$$

Next we assume that

$$
\begin{aligned}
\operatorname{rank}\Big(& h_{u(0)}\left(u_0(0), \ldots, u_0\left(N_0-1\right)\right) \mid \ldots\\
& \mid\ h_{u(N_0-1)}\left(u_0(0), \ldots, u_0\left(N_0-1\right)\right)\Big) = n.
\end{aligned}
$$

Then it follows with the aid of the inverse function theorem that there exists an $n-$dimensional relatively open set $\widetilde{V} \subseteq V$ with $(u_0(0), \dots, u_0(N_0 - 1)) \in \widetilde{V}$ such that the restriction of h to $\widetilde{V}$ is a homeomorphism which implies that $h\left(\widetilde{V}\right) \subseteq S_{N_0}(x_1)$ is open.

Therefore $x_0 \in h\left(\widetilde{V}\right)$ is an interior point of $S(x_1)$.

Now we consider the special case where there exists some $\widehat{x} \in \mathbb{R}^n$ with

$$g_N(\widehat{x}, \Theta_m) = \widehat{x} \qquad \text{for all} \quad N \in \mathbb{N}$$

which implies

$$G^N\left(\widehat{x}, \Theta_m^N\right) = \widehat{x} \qquad \text{for all} \quad N \in \mathbb{N}.$$

Then, for every $N \in \mathbb{N}$, it follows that $\widehat{x} \in S_N(\widehat{x})$, hence $\widehat{x} \in S(\widehat{x})$.

Let us assume that

$$(g_N)_x(\widehat{x}, \Theta_m) \text{ is non-singular} \qquad \text{for all} \quad N \in \mathbb{N}_0.$$

Then

$$G_x^N\left(\widehat{x}, \Theta_m^N\right) = (g_{N-1})_x(\widehat{x}, \Theta_m) \cdot (g_{N-2})_x(\widehat{x}, \Theta_m) \cdots (g_0)_x(\widehat{x}, \Theta_m)$$

is also non-singular for all $N \in \mathbb{N}$.

By the implicit function theorem we therefore conclude, for every $N \in \mathbb{N}$, that there exists an open set $V_N \subseteq \Omega^N$ with $\Theta_m^N \in V_N$ and a function $h_N : V_N \to \mathbb{R}^n$ with $h_N \in \mathcal{C}^1(V_N)$ such that

$$h_N\left(\Theta_m^N\right) = \widehat{x} \quad \text{and} \quad G^N\left(h_N(u(0), \dots, u(N-1)), u(0), \dots, u(N-1)\right) = \widehat{x}$$
$$\text{for all} \quad (u(0), \dots, u(N-1)) \in V_N$$

which means

$$h_N(u(0), \dots, u(N-1)) \in S_N(\widehat{x}) \qquad \text{for all} \quad (u(0), \dots, u(N-1)) \in V_N.$$

Moreover,

$$(h_N)_{u(k)}\left(\Theta_m^N\right) = -G_x^N\left(\widehat{x}, \Theta_m^N\right)^{-1} \cdot G_{u(k)}^N\left(\widehat{x}, \Theta_m^N\right).$$

Next we assume that, for some $N_0 \in \mathbb{N}$,

$$\operatorname{rank}\left((h_{N_0})_{u(0)}\left(\Theta_m^{N_0}\right) \mid \dots \mid (h_{N_0})_{u(N_0-1)}\left(\Theta_m^{N_0}\right)\right) = n.$$

Then it follows with the aid of the inverse function theorem that there exists an $n-$dimensional relatively open set $\widetilde{V}_{N_0} \subseteq V_{N_0}$ with $\Theta_m^{N_0} \in \widetilde{V}_{N_0}$ such that the restriction of h_{N_0} to $\widetilde{V}_{N_0}$ is a homeomorphism which implies that $h_{N_0}\left(\widetilde{V}_{N_0}\right) \subseteq S_{N_0}(\widehat{x})$ is open. Therefore $\widehat{x} \in h_{N_0}\left(\widetilde{V}_{N_0}\right)$ is an interior point of $S(\widehat{x})$. This result is a generalization of Theorem 2.25, if Ω in addition is open.

2.3.3 Stabilization of Controlled Systems

Let $(g_t)_{t\in\mathbb{N}}$ be a sequence of continuous mappings $g_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and let $\mathcal{H}$ be a family of continuous mappings $h : \mathbb{R}^n \to \mathbb{R}^m$. If we define, for every $h \in \mathcal{H}$ and $t \in \mathbb{N}$, the mapping $f_t^h : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_t^h(x) = g_t(x, h(x)), \qquad x \in \mathbb{R},$$

then we obtain a non-autonomous time-discrete dynamical system $\left(\mathbb{R}^n, \left(f_t^h\right)_{t\in\mathbb{N}}\right)$. The dynamics in this system is defined by the sequence $F^h = \left(F_t^h\right)_{t\in\mathbb{N}}$ of mappings $F_t^h : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F_t^h(x) = f_t^h \circ f_{t-1}^h \circ \ldots \circ f_1^h(x) \qquad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad t \in \mathbb{N}$$

and

$$F_0^h(x) = x \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

We also obtain the dynamical system $\left(\mathbb{R}^n, \left(f_t^h\right)_{t\in\mathbb{N}}\right)$, if we replace the control function $u : \mathbb{N}_0 \to \mathbb{R}^m$ in the system (2.86) by the feedback controls $h(x) : \mathbb{N}_0 \to \mathbb{R}^m,\ x \in \mathbb{R}^n$.

The problem of stabilization of the controlled system (2.86) by the feedback controls $h(x),\ x \in \mathbb{R}^n$, then reads as follows: Given $x_0 \in \mathbb{R}^n$ such that the limit set $L_{F^h}(x_0)$ defined by (1.70) (see Section 1.3.8) is non-empty and compact for all $h \in \mathcal{H}$.

Find a mapping $h \in \mathcal{H}$ such that $L_{F^h}(x_0)$ is stable, an attractor or asymptotically stable with respect to $\left(f_t^h\right)_{t\in\mathbb{N}}$.

Let us consider the special case

$$g_t(x, u) = A_t(x)\, x + B_t(x)\, u \qquad \text{for} \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{2.104}$$

where $(A_t(x))_{t\in\mathbb{N}}$ and $(B_t(x))_{t\in\mathbb{N}}$ is a sequence of real, continuous $n \times n-$ and $n \times m-$matrix functions on $\mathbb{R}^n$, respectively.

Let $\mathcal{H}$ be the family of all linear mappings $h : \mathbb{R}^n \to \mathbb{R}^m$ (which are automatically continuous). Every $h \in \mathcal{H}$ is then representable in the form

$$h(x) = C^h\, x, \qquad x \in \mathbb{R}^n,$$

where C^h is a real $m \times n-$matrix. For every $t \in \mathbb{N}$ and $h \in \mathcal{H}$ we therefore obtain

$$f_t^h(x) = \left(A_t(x) + B_t(x)\, C^h\right) x, \qquad x \in \mathbb{R}^n. \tag{2.105}$$

Let us put

$$D_t^h(x) = A_t(x) + B_t(x)\, C^h \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

If we choose $x_0 \in \Theta_n =$ zero vector of $\mathbb{R}^n$, then we conclude

$$F_t^h(x_0) = x_0 \qquad \text{for all} \quad t \in \mathbb{N}_0,\ h \in \mathcal{H},$$

and therefore $L_{F^h}(x_0) = \{x_0\}$.

The problem of stabilization of the controlled system (2.86) with $g_t,\ t \in \mathbb{N}$, given by (2.104) in this situation consists of finding an $m \times n-$matrix C^h such that $\{x_0 = \Theta_n\}$ is stable, an attractor or asymptotically stable with respect to $\left(f_t^h\right)_{t \in \mathbb{N}}$ with f_t^h given by (2.105).

Now let us assume that

$$||D_t^h(x)|| \leq 1 \qquad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad t \in \mathbb{N} \tag{2.106}$$

where $||\cdot||$ denotes the spectral norm.

Let $U \subseteq \mathbb{R}^n$ be a relatively compact open set with $x_0 = \Theta_n \in U$. Then there is some $r > 0$ such that

$$B_U = \{x \in \mathbb{R}^n \mid ||x||_2 < r\} \subseteq U.$$

Hence B_U is open, $x_0 \in B_U$, and assumption (2.106) implies

$$f_t^h(B_U) \subseteq B_U \qquad \text{for all} \quad t \in \mathbb{N}.$$

If we define

$$V(x) = ||x||_2^2 = x^T x \quad \text{for} \quad x \in \mathbb{R}^n,$$

then

$$V(x) \geq 0 \qquad \text{for all} \quad x \in \mathbb{R}^n \qquad \text{and} \qquad (V(x) = 0 \ \Leftrightarrow\ x = x_0 = \Theta_n)$$

and

$$\begin{aligned} V\left(f_t^h(x)\right) - V(x) &= x^T D_t^h(x)^T D_t^h(x)\, x - x^T x \\ &= ||D_t^h(x)\, x||_2^2 - ||x||_2^2 \leq \left(||D_t^h(x)\, x|| - 1\right) ||x||_2^2 \leq 0 \end{aligned}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{N}$.

This shows that V is a Lyapunov function with respect to $\left(f_t^h\right)_{t \in \mathbb{N}}$ on $G = \mathbb{R}^n$ which is positive definite with respect to $\{x_0 = \Theta_n\}$. By Theorem 1.9 we therefore conclude that $\{x_0 = \Theta_n\}$ is stable with respect to $\left(f_t^h\right)_{t \in \mathbb{N}}$ with f_t^h given by (2.105).

Next we assume that

$$\sup_{t \in \mathbb{N}} ||D_t^h(x)|| < 1 \qquad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.107}$$

Then it follows from

$$\begin{aligned}
V\left(F_t^h(x)\right) &= x^T\, D_1^h(x)^T \cdots D_t^h\left(F_{t-1}^h(x)\right)^T D_t^h\left(F_{t-1}^h(x)\right) \cdots D_1^h(x)\, x \\
&= ||D_t^h\left(F_{t-1}^h(x)\right) \cdots D_1^h(x)\, x||_2^2 \\
&\leq ||D_t^h\left(F_{t-1}^h(x)\right)\, x||^2 \cdots ||D_1^h(x)\, x||^2\, ||x||^2
\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{N}$ that

$$\lim_{t\to\infty} V\left(F_t^h(x)\right) = 0 \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

This implies

$$\lim_{t\to\infty} F_t^h(x) = \Theta_n \qquad \text{for all} \quad x \in \mathbb{R}^n$$

and shows that $\{x_0 = \Theta_n\}$ is an attractor with respect to $\left(f_t^h\right)_{t\in\mathbb{N}}$ with f_t^h given by (2.105).

Result. *Under the assumption (2.107) the set* $\{x_0 = \Theta_n\}$ *is asymptotically stable with respect to* $\left(f_t^h\right)_{t\in\mathbb{N}}$ *with* f_t^h *given by (2.105).*

2.3.4 The Problem of Reachability

We again consider the situation at the beginning of Section 2.3.1 without necessarily assuming the existence of a fixed point $\widehat{x} \in \mathbb{R}^n$ of the uncontrolled system (2.91). Let $\Omega \subseteq \mathbb{R}^m$ be a non-empty subset. For a given $x_0 \in \mathbb{R}^n$ we then define the set of states that are reachable from x_0 in $N \in \mathbb{N}$ steps by

$$\begin{aligned}
R_N\left(x_0\right) = \Big\{ x = G^N\left(x_0, u(0), \ldots, u(N-1)\right) \\
\Big|\; u(k) \in \Omega,\ k = 0, \ldots, N-1 \Big\}
\end{aligned} \tag{2.108}$$

where the map $G^N : \mathbb{R}^{m\cdot N} \to \mathbb{R}^n$ is defined by (2.102).

Further we define the set of states reachable from x_0 in a suitable number of steps by

$$R\left(x_0\right) = \bigcup_{N\in\mathbb{N}} R_N\left(x_0\right). \tag{2.109}$$

The question we are interested in now is: Under which conditions does $R\left(x_0\right)$ have a non-empty interior?

A simple answer to this question gives

Theorem 2.27. *Let Ω be open. If there is some $N \in \mathbb{N}$ and there exist $u(0), \dots, u(N-1) \in \Omega$ such that*

$$\operatorname{rank}\Big(G^N_{u(0)}\left(x_0, u(0), \dots, u(N-1)\right) \mid \dots \\ \mid G^N_{u(N-1)}\left(x_0, u(0), \dots, u(N-1)\right)\Big) = n, \qquad (2.110)$$

then $R_N(x_0)$ has a non-empty interior and therefore also $R(x_0)$.

Proof. Condition (2.110) implies that the $n \times N \cdot m-$matrix

$$\Big(G^N_{u(0)}\left(x_0, u(0), \dots, u(N-1)\right) \mid \dots \mid G^N_{u(N-1)}\left(x_0, u(0), \dots, u(N-1)\right)\Big)$$

has n linearly independent column vectors. Let E be the $n-$dimensional subset of Ω^N consisting of all vectors whose components which do not correspond to these linearly independent column vectors are equal to the ones of $\left(u(0)^T, \dots, u(N-1)^T\right)^T$. If we restrict the mapping G^N to E, then the Jacobi matrix of this restriction consists of these linearly independent column vectors and is therefore non-singular.

By the inverse function theorem therefore there exists an open set (with respect to E) $U \subseteq \Omega^N$ with $\left(u(0)^T, \dots, u(N-1)^T\right)^T \in U$ which is mapped homeomorphically by G^N on an open set $V \subseteq R_N(x_0)$ with $G^N(x_0, u(0), \dots, u(N-1)) \in V$. This completes the proof. □

Next we consider the linear case where

$$g_t(x, u) = A_t\, x + B_t\, u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

with $n \times n-$matrices A_t and B_t, respectively, for every $t \in \mathbb{N}_0$.

Then, for every $N \in \mathbb{N}$ and every $x_0 \in \mathbb{R}^n$, we obtain

$$G^N\left(x_0, u(0), \dots, u(N-1)\right) = A_{N-1} \cdots A_0\, x_0 \\ + \sum_{k=1}^{N} A_{N-1} \cdots A_k B_{k-1}\, u(k-1)$$

where for $k = N$ we put $A_{N-1} \cdots A_k = I = n \times n-$unit matrix. Further we have, for every $N \in \mathbb{N}$ and every $x_0 \in \mathbb{R}^n$,

$$R_N(x_0) = \Big\{ x = A_{N-1} \cdots A_0\, x_0 + \sum_{k=1}^{N} A_{N-1} \cdots A_k B_{k-1}\, u(k-1) \\ \Big|\ u(k) \in \Omega,\ k = 0, \dots, N-1 \Big\}.$$

Because of

$$G^N_{u(k-1)}(x_0, u(0), \dots, u(N-1)) = A_{N-1} \cdots A_k B_{k-1} \qquad \text{for} \quad k = 1, \dots, N$$

it follows that the condition (2.110) for $N = N_0$ coincides with the condition (2.98).

If this is satisfied, then by Theorem 2.27 the set $R(x_0)$ (2.109) of states reachable from x_0 has a non-empty interior, if Ω is open.

If $\Omega = \mathbb{R}^m$, it follows in addition that $R(x_0) = \mathbb{R}^n$ for all $x_0 \in \mathbb{R}^n$.

Proof. Let $x, x_0 \in \mathbb{R}^n$ be given arbitrarily. Then condition (2.98) implies the existence of $u(k) \in \mathbb{R}^m$ for $k = 0, \dots, N_0 - 1$ such that

$$x - A_{N_0-1} \cdots A_0 x_0 = \sum_{k=1}^{N_0} A_{N_0-1} \cdots A_k B_{k-1} u(k-1)$$

holds true which shows that $x \in R_{N_0}(x_0) \subseteq R(x_0)$. □

For every $k = 1, \dots, N$ let us define an $n \times m-$matrix C^k by

$$C^k = A_{N-1} \cdots A_k B_{k-1} \qquad \text{for} \quad k = 1, \dots, N-1$$

and

$$C^N = B_{N-1}.$$

The condition (2.98) implies the existence of n column vectors

$$\begin{pmatrix} c^{k_l}_{1 j_{k_l}} \\ \vdots \\ c^{k_l}_{n j_{k_l}} \end{pmatrix} \qquad \text{for} \quad l = 1, \dots, n \quad \text{which are linearly independent.}$$

Let us define the $n \times n-$matrix C and a vector $u \in \mathbb{R}^n$ by

$$C = \begin{pmatrix} c^{k_1}_{1 j_{k_1}} & \cdots & c^{k_n}_{1 j_{k_n}} \\ & \vdots & \\ c^{k_1}_{n j_{k_1}} & \cdots & c^{k_n}_{n j_{k_n}} \end{pmatrix} \qquad \text{and} \qquad u = \begin{pmatrix} u_{j k_l}(k_1 - 1) \\ \vdots \\ u_{j k_n}(k_n - 1) \end{pmatrix}, \text{ respectively,}$$

and put

$$u_j(k-1) = 0 \qquad \text{for} \quad k \neq k_l,\ j \neq j_{k_l},\ l = 1, \dots, n.$$

Then we obtain

$$G^N\left(x_0, u(0), \ldots, u(N-1)\right) = A_{N-1} \cdots A_0\, x_0 + C\, u$$

which implies

$$u = C^{-1}\Big(\underbrace{G^N\left(x_0, u(0), \ldots, u(N-1)\right)}_{=x} - A_{N-1} \cdots A_0\, x_0\Big).$$

Now let

$$E = \Big\{u = (u(0), \ldots, u(N-1)) \in \mathbb{R}^{m \cdot N} \\ \Big|\ u_j(k-1) = 0 \text{ for } k \neq k_l \text{ and } j \neq j_{k_l},\ l = 1, \ldots, n\Big\}.$$

Then $G^N(x_0, \cdot)$ is a linear isomorphism from E on $\mathbb{R}^n$.

Therefore

$$G^N(x_0, u) = x \qquad \text{for some} \quad u \in E$$

implies

$$u = G^N(x_0, \cdot)^{-1}(x),$$
$$G^N(x_0, u) = A_{N-1} \cdots A_0\, x_0 + C \cdot G^N(x_0, \cdot)^{-1}(x).$$

If all $A_k,\ k \in \mathbb{N}_0$ are invertible it follows that

$$x_0 = A_0^{-1} \cdot A_{N-1}^{-1}\, x - A_0^{-1} \cdot A_{N-1}^{-1} C \cdot G^N(x_0, \cdot)^{-1}(x).$$

In the nonlinear case we have the following situation:

If the condition (2.110) is satisfied, there exists an $n-$dimensional subset E of Ω^N and a set $U \subseteq E$ which is open with respect to E and contains $\left(u(0)^T, \ldots, u(N-1)^T\right)^T$ and which is mapped homeomorphically on an open $V \subseteq R_N(x_0)$ by the restriction of $G^N(x_0, \cdot)$ to E. If

$$x = G^N\left(x_0, u(0), \ldots, u(N-1)\right),$$

then

$$\left(u(0)^T, \ldots, u(N-1)^T\right)^T = G^N(x_0, \cdot)^{-1}(x).$$

If in addition $G_x^N\left(x_0, u(0), \ldots, u(N-1)\right)$ is non-singular, then by the implicit function theorem there exists an open set $W \subseteq \Omega^N$ which contains $\left(u(0)^T, \ldots, u(N-1)^T\right)^T$ and a function $h : W \to \mathbb{R}^n$ with $h \in \mathcal{C}^1(W)$ such that

$$h(u(0), \ldots, u(N-1)) = x_0$$

and

$$G^N\left(h\left(\widetilde{u}(0),\ldots,\widetilde{u}(N-1)\right),\widetilde{u}(0),\ldots,\widetilde{u}(N-1)\right)=x$$
$$\text{for all}\quad \left(\widetilde{u}(0)^T,\ldots,\widetilde{u}(N-1)^T\right)^T\in W.$$

This implies

$$x_0=h\left(G^N\left(x_0,\cdot\right)^{-1}(x)\right).$$

Since

$$\begin{aligned}h_{u(k)}\left(u(0),\ldots,u(N-1)\right)=-G_x\left(x_0,u(0),\ldots,u(N-1)\right)^{-1}&\\ \times G^{N_0}_{u(k)}\left(x_0,u(0),\ldots,u(N-1)\right)&\\ \text{for}\quad k=0,\ldots,N-1,&\end{aligned}$$

it follows from (2.110) that

$$\operatorname{rank}\left(h_{u(0)}\left(u(0),\ldots,u(N-1)\right)\mid\ldots\mid h_{u(N-1)}\left(u(0),\ldots,u(N-1)\right)\right)=n$$

which implies that h maps $U\cap W$ homeomorphically onto an open set $\widetilde{V}\subseteq R_N\left(x_0\right)$ which contains x. Therefore $h\circ G^N\left(x_0,\cdot\right)^{-1}$ maps $V\cap\widetilde{V}$ homeomorphically on an open set $\widetilde{\widetilde{V}}$ which contains x_0 and is contained in

$$\begin{aligned}S_N(x)=\{\widetilde{x}\in\mathbb{R}^n\mid\ &\text{there exists some } \widetilde{u}\in U\ (2.53)\\ &\text{with } G^N\left(\widetilde{x},\widetilde{u}(0),\ldots,\widetilde{u}(N-1)\right)=x\}.\end{aligned}$$

Finally let us assume (as in Theorem 2.27) that Ω is open and (for given $x_0\in\mathbb{R}^n$) there exists some $N\in\mathbb{N}$ such that the condition (2.110) is satisfied for all $\left(u(0)^T,\ldots,u(N-1)^T\right)^T\in\Omega^N$. Then it follows from the proof of Theorem 2.27 that every $x\in R_N\left(x_0\right)$ is an interior point of $R_N\left(x_0\right)$, i.e., $R_N\left(x_0\right)$ is open.

This implies in the linear case with Ω being an open subset of $\mathbb{R}^m$ that $R_N\left(x_0\right)$ is open for every $x_0\in\mathbb{R}^n$, if the condition (2.98) is satisfied.

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