

Chapter 2

Selected Topics in Cooperative Game Theory

This chapter shall provide the theoretical basis coming from cooperative game theory. We start with a survey of historical developments in the field of game theory. After this short excursion, we introduce cooperative games and their properties. As the fundamental contribution of cooperative game theory are methods to allocate cooperative profits or costs, we present the most prominent allocation methods. Thereby, we concentrate on the core and its variants as this method will be applied in the following chapters.

2.1 History of Game Theory

Let us start with answering two questions: What is game theory and where does it come from? One general definition is provided by Myerson (1991): “*Game theory* can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.” The essential point is that individual decisions will influence each other’s welfare (which may stand for profits or costs). Binmore (1992) describes that a *game* is being played whenever people interact with each other. If you and your supervisor negotiate about your last year’s bonus, it is called a game in the sense of game theory. When several small companies build up an alliance or cooperation to gain more power in handling customers or suppliers, it is a game. It becomes clear that the notion “game” is somewhat misleading because it stands for an event where people interact. Hence, Myerson (1991) suggests to use terms like “conflict analysis” or “interactive decision theory” for this stream of research. However, the term “game theory” is most common in modern literature.

Modern game theory is mostly based on the seminal book of John von Neumann and Oskar Morgenstern published in 1944 (in 2004 the fourth edition was published, see von Neumann and Morgenstern 2004). There are earlier works by Edgeworth (1881), von Neumann (1928), Cournot (1838), Bertrand (1883), and Zermelo (1913) but they are lacking a uniform theory regarding strategic behavior of interactive decisions. Thus, the book of von Neumann and Morgenstern paved the way for an extensive research in the field of game theory. First expectations of revolutionizing social sciences with insights of game theory were detached by important

applications, especially in the field of economics. Von Neumann and Morgenstern (1944) provide an abstract mathematical formulation for conflict situations. They build up an extensive theory coining notions as the normal or extensive form of games, concepts of pure and mixed strategies, coalitional games, stable sets, etc. Apart from that, they develop the utility theory.

Von Neumann and Morgenstern distinguish between two approaches to the theory of games: The first is called *strategic* or *non-cooperative* game theory, whereas the second stream is called *coalitional* or *cooperative* game theory. A non-cooperative game specifies all possible actions for every involved decision maker. Based on this knowledge, we search for a strategy that is best for each decision maker. Recall, it is assumed that the results for every party depend on the strategy of the other parties. In the case of cooperative games, von Neumann and Morgenstern deal with patterns of coalition formation under rational behavior in particular. Still today, the field of game theory is divided into those two streams of cooperative and non-cooperative game theory.

Following the work of von Neumann and Morgenstern, various contributions to the topic of game theory were published since then. Therefore, we present only selected, very important contributions in this section without claiming completeness. A scientist contributing substantially to both streams of game theory is John Nash. In Nash (1950, 1951), he succeeds in developing a general formulation of the equilibrium idea which leads to the well known notion of the *Nash equilibrium* used in non-cooperative games. The Nash equilibrium is a combination of strategies where no player could increase its expected payoff by unilaterally deviating from its strategy. In Nash (1953), he also makes a contribution to cooperative game theory. He suggests two different methods to derive solutions for a two-person cooperative game: In the first one, the cooperative game is reduced to a non-cooperative game while making the players' steps of negotiation moves in the non-cooperative model. The second approach is based on axioms that describe desirable properties and lead to a uniquely solution.

In the 1950s, the famous *prisoner's dilemma* was introduced as a special example for non-zero-sum games. Luce and Raiffa (1957) is one of the first references that discusses the prisoner's dilemma extensively.

In the stream of cooperative game theory, Gillies (1959) and Shapley (1953), provide an important basis. Gillies (1959) invents the *core* as a general solution method for cooperative n -person games. At the same time, Shapley (1953) introduces the *Shapley value* as a further solution concept. While the core in most of the cases represents a set of possible allocations with specific properties, the Shapley value provides a unique allocation following other criteria. We will study those two solution concepts and some further ones in Sect. 2.3 in detail.

In this early time of game theoretical research, most applications were found in economics, politics, military, and first steps in evolutionary biology (see Lewontin 1961). In the area of economic applications, Shubik (1962) suggests to use methods from cooperative game theory to solve cost accounting problems in a company. Following the time line, Aumann and Maschler (1964) develop a further allocation method which they call the *bargaining set*. The trend to develop more and more

different solution concepts for cooperative game theory that are based on varying properties has continued until now. None of the solution concepts appeared to be superior because all of them have their advantages and disadvantages. In contrast, the analysis of non-cooperative games is mostly based on the Nash equilibrium. However, research continues, for instance, in refining the concept of the Nash equilibrium: Selten (1965), in particular, contributes to this while introducing the notion of the *subgame perfect equilibrium*.

Most of the research today is following the classification of Harsanyi (1966) regarding *cooperative vs. non-cooperative* game theory. He defines a cooperative game as a game where commitments are fully binding and enforceable. Commitments can be, for instance, agreements, promises, or threats. A non-cooperative game has no binding commitments. Harsanyi (1966) states that any cooperative game can be replaced by a non-cooperative one. To execute this, we have to incorporate the commitments into the available strategies and the penalties into the payoff matrix when commitments are violated. However, Harsanyi (1966) does not suggest to deal only with non-cooperative games because this transformation greatly increases the size of the payoff matrix. In Harsanyi and Selten (1988), this definition is refined: The underlying assumption for a non-cooperative game is that the players are unable to make enforceable agreements except in the case where the extensive form of the game explicitly gives them an ability to model such commitments. According to that, a cooperative game contains the assumption that the players are able to make binding agreements, even if this possibility is not explicitly shown in the extensive form of the game (see Harsanyi and Selten 1988, p. 3). Such binding agreements are prevalent in many areas of economics. Almost every seller-buyer relation is characterized by some binding principles fixed in a contract – this does not only hold for one stage relations but also for multi-stage transactions. Most of these contracts include monetary penalties that discourage the partners from breaking the agreements, thus making the contract binding.

Apart from this, Harsanyi makes substantial contributions to the theory of games with incomplete information (see Harsanyi 1967, 1968a,b). A further refinement of the Nash equilibrium is developed by Selten (1975) under the name *trembling hand perfect equilibrium*. The joint work Harsanyi and Selten (1988) continues the research regarding the refinement of the Nash equilibrium and was crowned with the Nobel prize in economic sciences for Harsanyi, Nash, and Selten in 1994.

We already specified the term *game*. Let us examine some more notions that appear very often when dealing with aspects of game theory. The number of involved decision makers in a game is not bounded and we call the decision makers *players*. Myerson (1991) formulates two important assumptions regarding the properties of the players:

- **Rationality:** Each player's objective is to maximize its expected payoff (measured on some utility scale). Hence, there must be a possibility to assign utility numbers to all possible outcomes of the game (see the expected-utility maximization theorem by von Neumann and Morgenstern 2004, p. 30). This means the player's only interest is the maximization of its own payoff, no matter the other players' payoffs.

- **Intelligence:** It is assumed that a player knows everything about the game. This means the players can make the same inferences as ourselves. The same holds for cases where a theory is applied to a game. In this case, every player is aware of this knowledge.

Today, game theory is seen as an independent area of decision theory. To be more specific, game theory is an extension of classical decision theory to the case of more than one decision maker. Morgenstern (1963, p. 77), distinguishes between the case in classical decision theory, where the outcome of decisions of one player depends on (random) external events (not manipulable by the player) and the case of game theory, where two players interact; i.e., outcomes depend on the opponents and maybe additionally on (random) external events.

Until the 1970s, game theory was mostly a mathematical discipline. Later on, game theory was also applied to economical problems, like for instance antitrust analysis, monetary policy, and business problems. Any form of competition or cooperation in a company or between companies provides decision problems that can be analyzed with the help of game theory methods. Players might be companies, business units, suppliers, customers, or even employees. A groundbreaking work is done in the book of Brandenburger and Nalebuff (1996) because they use game theoretical models in the background while approaching them through real-life stories. By using game theory to explain success or failure of decisions, they summarize the lessons in checklists to make the methods applicable for non-scientists, too. Chatterjee and Samuelson (2001) and Jost (2001) provide each an extensive survey of game theoretical contributions in the field of business applications and present discussions about limits of application as well.

We do not claim to give a complete survey with the above paragraphs. To gain extensive and superior knowledge in the field of game theory, we would like to refer exemplary to the following textbooks: Myerson (1991), Fudenberg and Tirole (1991), Binmore (1992), Owen (2001), and Rasmusen (2007).

2.2 Basics in Cooperative Game Theory

This section shall give an introduction into cooperative game theory. We will learn about the definition and properties of cooperative games as a basis to analyze allocation methods in the following sections. Following Myerson (1991, p. 370), we will use the term *cooperation* in the meaning of ‘players interacting with a common purpose’. The main subjects of cooperative game theory are to analyze whether there are incentives to cooperate and to allocate costs (or profits) accrued in the cooperation among the participating players.

The contributions of John Nash coin the beginning of research in the field of cooperative game theory: Nash (1951)’s work is based on von Neumann and Morgenstern (1944)’s cooperative n -person game. Nash (1951) argues that the process of bargaining among players may result in cooperative actions. Hence, he suggests using the Nash equilibrium to analyze non-cooperative as well as

cooperative games. In the bargaining process, each player follows its individual bargaining strategy that maximizes its utility (see Myerson 1991, especially Chap. 8, for details about such a transformation). However, a consequent transformation might lead to a large set of equilibria. Apart from this transformation, Nash (1950, 1953) suggest a solution concept called the Nash bargaining solution. It is based on the assumption that the profit allocation depends only on the profits the players would expect if negotiation fails and on the set of profit allocations that are jointly feasible. It is possible to extend this approach to an n -person Nash bargaining solution but its application is questionable and unusual because the concept completely ignores the possibility of cooperation among subsets of players and their (negotiation) power. For Myerson (1991), the main assumption for differentiating between cooperative and non-cooperative game theory is that players can negotiate effectively in cooperative games. *Effective negotiation* means that in case of feasible strategy changes that would benefit all coalition members, they would agree to make this change as long as it does not contradict agreements made with players outside the coalition (see Myerson 1991, Sect. 9.1, for more details).

For substantial introductions into cooperative game theory, we refer exemplary to the following textbooks: Myerson (1991), Osborne and Rubinstein (1994), Curiel (1997), Owen (2001), and Peleg and Sudhölter (2007).

2.2.1 A Cooperative Game

A fundamental assumption in cooperative game theory is whether or not the result of a cooperation can be quantified and transferred (without gain or loss) among the players. Therefore, the assumption of *transferable utility* (TU) is often used in cooperative game theory to handle interactions in-between different coalitions. With transferable utility, we mean a commodity (like money) that can be freely transferred among the players without any third instance. While transferring, any player's utility cost/profit increases one unit for every unit of the commodity it gets (see e.g., Myerson 1991, Sects. 8.4 and 9.2). The transferable utility helps to define the characteristic function of a cooperative game. In the following course of this book, we will solely discuss games with transferable utility. For games with non-transferable utility (NTU), compare, for instance, Myerson (1991) and Peleg and Sudhölter (2007). According to the use of transferable utility, we assume that the players prefer less money to more money in the case of costs (vice versa in the case of profits) and that the players are risk averse in their preferences for money (see Maschler 1992, p. 594). Hence, it is possible to neglect individual utility functions. Literature regarding cooperative game theory mostly uses payoffs that represent the utility functions.

A *cooperative game* with transferable utilities (short: TU game) is characterized by two main factors: $N = \{1, 2, \dots, |N|\}$ is the given set of players and c is the characteristic function. Players are decision makers and we call every subset $S \subseteq N$ of cooperating players a *coalition*. If two players are not cooperating, they belong to different coalitions; i.e., there is only one occurrence of cooperation. N is called

the grand coalition. As the definition of game theory in Sect. 2.1 implies, a game emerges when two or more players interact. Analyzing possible strategies and solutions of two-player-games is manageable, but dealing with more than two decision makers increases the complexity of the problem. Obviously, this is due to the exponentially rising number of coalitions ($2^{|N|}$) that has to be taken into account when the number of players increases. For the further discussion, we assume that the set of players is finite. Games with an infinite set of players are called *nonatomic games* (see e.g., Owen 2001).

The second factor, the *characteristic function*

$$c : 2^N \rightarrow \mathbb{R}$$

assigns a cost or profit value (representing the total amount of transferable utility) to each coalition $S \subseteq N$ which determines the best outcome for the coalition S if the players in S cooperate without the players in $N \setminus S$. In general, it is assumed that for an empty set

$$c(\emptyset) = 0.$$

Assume that the players form the grand coalition and want to divide total costs $c(N)$ among themselves after some kind of bargaining process. The outcome will heavily depend on the power structure in the grand coalition. A player's power could be interpreted as its ability to help or hurt any (sub-)set of players by agreeing or refusing to cooperate (see Myerson 1991, p. 427). Hence, a characteristic function can be seen as a summary description of the power structure of the game.

In short, a game is defined by the pair

$$\Gamma := (N, c).$$

The terms *game in coalitional form* or *coalitional game* can be used instead of the notion characteristic function. It is sufficient to give the characteristic function to define a cooperative game because the player set is given implicitly via the dimension of the characteristic function. For special representations of the presented coalitional form based on offensive and defensive threats among the coalitions, see Myerson (1991, p. 423).

The characteristic function can be interpreted as profits or costs – we will speak of profit and cost games, respectively. In a profit game, players prefer a higher outcome for themselves, whereas in a cost game, they favor lower amounts. The symbol v is usually used for profit functions and c for cost functions. Most of the time, profit and cost games can be handled the same way. An exception to this is the existence of restricted cooperation (see Chap. 8 for a further discussion on restricted cooperation). For a mathematical comparison and discussion of profit and cost games, see also Bilbao and Martínez-Legaz (2002). It is also customary in game theory and its applications to convert a cost game into a cost savings game with

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S) \quad \text{for all } S \subseteq N$$

(see Young et al. 1982, p. 464, or Lemaire 1984, p. 68) or to make the transformation with the help of the so-called *dual game* ($c(S)$ is the dual game of $v(S)$)

$$c(S) = v(N) - v(N \setminus S) \quad \text{for all } S \subseteq N$$

(see Curiel 1997, p. 16, for a numerical example). Obviously, $v(\emptyset) = c(\emptyset) = 0$ holds for both transformations.

In the course of this thesis, we will deal with cost games (N, c) , unless stated otherwise, and will not use one of these transformations. Thus, positive values of the characteristic function represent costs and negative values could be interpreted as profits. Obviously, it is possible to use other quantifiable units than costs or profits, too.

π_i denotes the cost (or profit) share that will be allocated to player i . These shares should be computed in such a way that the vector $\pi = (\pi_1, \pi_2, \dots, \pi_{|N|})$ gives an allocation of the total costs for the grand coalition $c(N)$. This vector is an element of the $|N|$ -dimensional linear space R^N .

2.2.2 Properties of Cooperative Games

Some important properties to classify characteristic functions are introduced in this section. Those properties are used to classify cooperative games and to derive insights about the application of solution concepts.

A cooperative game is *monotone* if and only if the cost function does not decrease with the number of players in the coalition:

$$c(S_1) \leq c(S_2) \quad S_1 \subseteq S_2 \subseteq N. \quad (2.1)$$

The same interpretation can be used to describe monotone profit games:

$$v(S_1) \leq v(S_2) \quad S_1 \subseteq S_2 \subseteq N.$$

If an opportunity to cooperate among a set N of players exists, it is interesting to find out how the “optimal” coalition structure looks like. Structure in this sense stands for a *partition*

$$S_1, \dots, S_m \quad \text{with} \quad \bigcup_h S_h = N \quad \text{and} \quad S_h \cap S_{h'} = \emptyset \quad \text{for all } h \neq h'.$$

Let $\mathcal{P}(N)$ denote the set of all partitions of N . An “optimal” partition means a partition of players with minimum total costs ($\min \sum_h c(S_h)$). The size m of the partition is not known in advance. If we can prove, however, that the characteristic function is subadditive, then the grand coalition is the optimal partition ($m = 1$ and $S_1 = N$). Thus, *subadditivity* indicates whether two players or two disjoint coalitions have an

incentive to cooperate because they have lower costs while cooperating compared to individual activity:

$$c(S_1) + c(S_2) \geq c(S_1 \cup S_2) \quad S_1, S_2 \subseteq N, S_1 \cap S_2 = \emptyset. \quad (2.2)$$

Maschler (1992) argues that in case of a bargaining procedure to reach an allocation of the cooperative costs, players might not follow the reasoning of subadditivity which could lead to a coalition structure not containing the grand coalition (see Maschler 1992, p. 602, for a further discussion).

In the context of profit games (N, v) where payoffs are used instead of costs, the notion of *superadditivity* is used. Two disjoint subsets have an incentive to cooperate if the profits in case of cooperation are higher than the profits when acting alone:

$$v(S_1) + v(S_2) \leq v(S_1 \cup S_2) \quad S_1, S_2 \subseteq N, S_1 \cap S_2 = \emptyset.$$

From a practical point of view, sub-/superadditivity can be interpreted as economies of scale or scope. Certainly, they should compensate negative implications arising in the cooperation. Derived from this subadditivity definition, we can define the *subadditive cover* $\hat{c}$ of the game c in coalitional form that satisfies

$$\hat{c}(S) = \min \left\{ \sum_{h=1}^m c(T_h) \mid \{T_1, \dots, T_m\} \in \mathcal{P}(S) \right\} \text{ for all } S \subseteq N.$$

Thus, the total costs of a coalition S in the subadditive cover are the minimum costs that the coalition achieves when breaking up into a set of smaller disjoint coalitions (partition). This provides a possibility to transform any game in coalitional form into a corresponding subadditive game. Owen (2001) argues that the characteristic function assigns the maximin value to each $S \subseteq N$ (in case of profit games) of the two-person game played between S and $N \setminus S$. Hence, the property of superadditivity holds for every $|N|$ -person game in characteristic function form (see Owen 2001, p. 213). When a cooperative game does not satisfy the property of sub-/superadditivity, cooperation would not be advisable.

Cooperative games can be divided into essential and inessential games. An *essential game* is characterized by

$$c(N) < \sum_{i \in N} c(\{i\}).$$

$c(N) \leq \sum_{i \in N} c(\{i\})$ follows directly from subadditivity but in the case of equality, the allocation problem would be trivial because every player would get exactly its stand-alone costs.

Furthermore, we can determine whether a cooperative game is *concave*:

$$c(S_1 \cup \{i\}) - c(S_1) \geq c(S_2 \cup \{i\}) - c(S_2) \quad i \in N, S_1 \subset S_2 \subseteq N \setminus \{i\}.$$

The definition of concavity is related to subadditivity but much more severe. A concave characteristic function implies that a player causes a smaller increase in total costs while added to a subcoalition that contains more players. Some kind of “snowballing” effect occurs – the incentive for joining a coalition increases as the coalition grows. Fromen (2004, p.87) argues that this behavior might be initiated by the network effect. Sometimes, an alternative formulation for concavity is used in the literature (see e.g., Shapley 1971):

$$c(S_1) + c(S_2) \geq c(S_1 \cup S_2) + c(S_1 \cap S_2) \quad S_1, S_2 \subseteq N. \quad (2.3)$$

Note that $c(\emptyset) = 0$ in combination with the concavity definition (2.3) implies subadditivity (2.2). Convex profit games follow the same interpretation:

$$v(S_1) + v(S_2) \leq v(S_1 \cup S_2) + v(S_1 \cap S_2) \quad S_1, S_2 \subseteq N.$$

In the course of this thesis, we will deal with cooperative $|N|$ -person TU games that are subadditive. Essentiality would be desirable but cannot be assured for practical problems. Nevertheless, in most cases it will be easy to check. Concavity would be a desirable property as well, especially from an analytical point of view. However, it limits the application possibilities too much. Starting in Chap. 5, we will introduce such games with special applications in supply chain management.

2.2.3 Variants and Fundamental Applications of the Classical Cooperative Game

This section presents variants of the before presented classical game that are also studied in the literature. Furthermore, we will discuss some fundamental applications of cooperative games that are well established in the literature and have their roots in operations research. The choice may be subjective and only wants to provide an overview of what is happening in the literature next to the later on presented applications and game variants.

Cooperative games with *restricted cooperation*, see Faigle (1989), are used to represent situations where only specific and not all coalitions $S \subseteq N$ are feasible. We will study such games in detail in Chap. 8.

Another variant are *bicooperative games* introduced by Bilbao (2000). Such games have non-orthogonal coalitions; i.e., the worth of a coalition depends on the actions of its complementary coalition. Bilbao et al. (2007, 2008) continue the study of bicooperative games while defining the solution concept of the core and the Shapley value for such games.

Lehrer (2002) extends the classical definition of a cooperative game by a *temporal aspect*: At each stage, a budget is distributed among the players. He shows that specific allocation processes converge to the core.

Multiple scenario cooperative games are studied by Hinojosa et al. (2005). Additionally, they extend the solution concepts of the core, the least core, and the

nucleolus for those games. The costs of a coalition are valued in different scenarios in such games; i.e., the characteristic function is multidimensional.

Muto et al. (1989) introduce so-called *information market games*. They result from situations where intellectual property is used jointly. One player owns this property and the value of a coalition only depends on the presence of this player and its resource in the coalition.

Externality games are developed by Grafe et al. (1998). Here, each player's contribution to the value of the coalition consists of its own ability of contributing to the coalition and the second part arises from its simple presence in the coalition.

Owen (2001, p. 218) describes *weighted majority games* and *voting games* as special cases of *simple games*. Simple games have characteristic functions with values $\{0, 1\}$; i.e., a coalition either wins or loses. Closely related to voting games are *homogen games* discussed, for instance, by Maschler (1992, p. 628). Furthermore, Maschler (1992, p. 629) gives a survey of combinations of different games.

Fuzzy games represent situations where the membership of a player in coalition S can be stated in the interval $[0; 1]$ which is called participation level. The participation level of a classical game (Branzei et al. 2008a call them *crisp games*) is either 1 (if $i \in S$) or 0 (if $i \in N \setminus S$). Branzei et al. (2008a, p. 77) give an excellent survey of fuzzy games including fuzzy core variants and discussions on the convexity of fuzzy games.

Hsiao and Raghavan (1993) introduce *multichoice cooperative games* that are a generalization of a cooperative game in which the players have several activity levels at which each can choose to play. Hence, every player has an action space offering the possibility to do nothing or to work at a level k . These games were intensely studied by Hsiao and Raghavan (1992), van den Nouweland et al. (1995), Hsiao (1995a,b), Klijn et al. (1999), Calvo and Santos (2000), Peters and Zank (2005), Grabisch and Xie (2007), Hwang and Liao (2008), and Branzei et al. (2008a).

Several game variants got their names from a specific application or from classical operations research problems. Claus and Kleitman (1973) and Bird (1976) introduce (*minimum cost*) *spanning tree games*: The players are represented by nodes in a graph $G = (V, E)$ with the node set V (including a source) and $E = V \times V$ representing the edges. The graph has to be complete (every pair of nodes is connected via an edge) and the edges are undirected and weighted with costs or profits. The value of a coalition is determined by the minimal spanning tree connecting all players in the coalition and the source. Compare Borm et al. (2001, p. 152) and Curiel (1997, p. 129) for a general description and a literature survey regarding minimum cost spanning tree games.

Airport games are special network games (the fixed tree is a line graph) where the characteristic function values are determined by the “strongest” member in the coalition. The name of this game variant originates from its application: An airport operator needs to allocate the fixed capital costs for runway and terminal construction to the planes using the airport. These capital costs essentially depend on the largest plane that wants to land, therefore, the value (cost) for a coalition is determined by the largest plane/player in the coalition. These games are introduced by Littlechild and Thompson (1973) (see also Littlechild 1974; [Littlechild](#)

and Thompson 1977a; Littlechild and Thompson 1977b; Dubey 1982; Potters and Sudhölter 1999; and Owen 2001, p. 334, for a detailed description and extensions).

Alike spanning network games, *flow games* are modeled in networks. Curiel (1997) describes flow games as a special form of *linear programming games* where the restrictions of a linear optimization problem depend on the coalitions. Moghadam and Michelot (2009) present an algorithm for LP games that uses information from the final simplex tableau to allocate joint costs. Another well known variant of linear programming games are so-called *linear production games* introduced by Owen (1975). In such a game, the players try to maximize their profit via producing different products under the restriction of limited resources. Lower and upper bounds on the values of a linear production game are presented by Bjørndal and Jørnsten (2009). They determine the bounds via aggregating over the columns of the LP.

Another sort of games based on graphs is connected with the field of routing – *shortest path games*, *traveling salesman games*, and *Chinese postman games*. The main target of these games is to find a shortest path under specific conditions through a network and divide the costs or profits among the participating players along this path. Compare Borm et al. (2001, p. 155) and Curiel (1997, p. 111) for a broad survey. Estévez-Fernández et al. (2009) investigate routing games that take revenues into account as well. *Delivery games* which are based on Chinese postman games are studied by Hamers et al. (1999). The continuation of those games are vehicle routing games which are studied, for instance, by Göthe-Lundgren et al. (1996) and Engevall et al. (2004).

In *sequencing games* and *permutation games*, a number of jobs should be processed in a specific order that minimizes a cost criterion – the jobs represent the players in such a game. The difference between sequencing and permutation games is: If the members of coalition S are rearranged, it is allowed to jump over non-members in case of permutation games, but not in sequencing games (see Curiel et al. 1989 and Curiel 1997). *Assignment games* form a subclass of permutation games. In an assignment game, the player set N can be divided into two subsets such that the characteristic function value of a coalition is zero if it contains only players from one of the two subsets (see Curiel 1997, p. 55). Following Borm et al. (2001, p. 165), sequencing, permutation, and assignment games can be summarized under the topic of *scheduling games*.

2.2.4 Interval-Valued Games

We now present a special extension for the classical game that was introduced in Sect. 2.2.1. Classical cooperative game theory reaches its borders if the characteristic function values are not known with certainty. Interval-valued games are one concept to deal with uncertain values of the characteristic function, i.e., the players do not know the outcomes exactly. However, they know at least upper and lower bounds (with certainty) of the potential profits or costs generated by the coalitions,

respectively. Hence, it is possible to make decisions based on all possible realizations which belong to the intervals. Formally, an interval-valued cost game is defined by a pair (N, c^{IV}) . As before, N is the index set of players. $c^{IV} : 2^N \rightarrow I(\mathbb{R})$ is a characteristic cost function which assigns to every coalition $S \in 2^N$ a closed interval $c^{IV}(S) = [\underline{c}^{IV}(S); \bar{c}^{IV}(S)]$. The empty coalition receives $c^{IV}(\emptyset) = [0; 0]$. $I(\mathbb{R})$ is the set of all closed intervals in $\mathbb{R}$. Obviously, classical cooperative games are a special case of cooperative interval-valued games with $\underline{c}^{IV}(S) = \bar{c}^{IV}(S)$ for all coalitions $S \subseteq N$.

Cooperative interval-valued games are introduced by Branzei et al. (2003) where they apply this concept to bankruptcy situations. In such a situation, the estate is known with certainty, but for the claims exist only known intervals of real numbers. In addition to that, they develop two solution concepts related to the concept of the Shapley value. They continue their research in Branzei et al. (2004) for situations where the individual claims vary within closed intervals and in Branzei and Alparslan-Gök (2008) where they extend two allocation rules from classical bankruptcy games to the interval setting. Carpenete et al. (2005) construct coalitional interval-valued games out of games in strategic form. The interval indicates the range bounded from below by the pessimistic prediction obtained using the lower value of the associated zero-sum game and bounded from above by the optimistic prediction obtained using the upper value of that game.

Theoretical results regarding interval-valued games are presented in several papers: Alparslan-Gök et al. (2009) investigate properties of two-person cooperative games with interval uncertainty – they study balancedness, superadditivity, and solution concepts like the core and the Shapley value (see Sect. 2.3). While looking at the corresponding classical cooperative games, which are selections of the interval-valued game, they extend the solution concept of the core (see also Sect. 2.3) for cooperative interval-valued games and define it as the union of the cores of all its selections. In Alparslan-Gök et al. (2008a), several solution concepts like the interval core, the interval dominance core, and stable sets are introduced as well as the notion of $\mathcal{I}$ -balancedness. In a second paper, they focus on convex interval-valued games Alparslan-Gök et al. (2008b). Fundamental results regarding the relation between different solution concepts like the Shapley value, the Weber set, and the interval core are presented. Inspired by the classical big boss games (see Muto et al. 1988), Alparslan-Gök et al. (2008c) introduce big boss interval-valued games. Branzei et al. (2008b) continue the studies on big boss interval-valued games. How to deal with interval solutions in practical situations, is described by Branzei et al. (2008c).

On top of the application to bankruptcy games, there are further assignments to practical problems: For instance, Bauso and Timmer (2009) investigate a joint replenishment game (based on Hartman et al. 2000; Meca et al. 2003, 2004) where the demand is uncertain, but bounded by a minimum and maximum value. Furthermore, Alparslan-Gök et al. (2008d) extend the classical airport game, where costs for the runway need to be allocated, to an interval-valued game in which the costs of pieces of the runway are intervals. To summarize, Branzei et al. (2009) give a

survey about cooperative interval-valued games including theoretical results as well as applications in economic and operations research situations.

2.3 Allocating Cooperative Costs

The fundamental question in cooperative game theory is how to allocate the costs of the grand coalition among the players. We already denoted a cost allocation with the vector $\pi = (\pi_1, \pi_2, \dots, \pi_{|N|})$. π_i represents the portion of the total costs, incurred by the grand coalition N , which is allocated to player i . We will start our discussion with motivating cost allocation and a rough classification of allocation methods. As a basis for the later on defined solution concepts, we will explain desired properties of such cost allocations. Before we go into detail with game theoretical allocation approaches, we will shortly discuss the topic of cost allocation from an accounting viewpoint and show that there is a need for other allocation methods. We will concentrate our explanations regarding solution concepts on the concept of the core and its variants (including the nucleolus) as they are essential for the further content of this thesis. After this, we will shortly introduce the Shapley value in contrast to the core variants. For a more detailed description and discussion of other cooperative solution concepts like the bargaining set and the τ -value, compare, e.g., Myerson (1991, Sects. 9.4 and 9.6), Owen (2001, Chap. X and the following), Osborne and Rubinstein (1994, Part IV), Curiel (1997, Sects. 1.3 and the following), Fromen (2004, Chap. 4), or Peleg and Sudhölter (2007, Chaps. 3 and the following). For surveys on cost allocation in general see for instance, Shubik (1962), Young (1994), and Fromen (2004).

2.3.1 Motivation and Classification of Allocation Methods

Before discussing methods of cost allocation, we should analyze the motivation for allocating cooperative costs. Shubik (1962) is the first who studies cost allocation in combination with game theory. He argues that cost allocation, thus the individual cost shares, can guide individual decisions in such a way that they are preferable for the cooperation. Hence, he develops an incentive system for decentralized control (Shubik 1962, pp. 328 and 333). Moriarity (1981a) displays motives for cost allocation while dividing them in three groups: textbook rationales (e.g., external reporting of inventory values or for cost control and product pricing), rationales presented in the recent literature (encourage cooperation and discourage wasteful consumption), and rationales involving decision making (signaling optimal capacity adjustments and relative profitability of products). Demski (1981, p. 152) calls the three most important motives decomposition (decomposing complex decision problems), motivation (cooperating partners should act in the interest of the grand coalition), and coordination (coordinate actions of several partners to act in the

interest of the grand coalition). Nagarajan and Sošić (2008, p. 2) mention stability of the cooperation as the most important strategic reason for allocating costs, especially in interorganizational cooperations (see Sect. 4.1).

Barton (1988) uses a very obvious classification method for allocation concepts: *complex game theoretic approaches* and *practical accounting type solutions* – the former category uses methods from cooperative game theory as described in Sects. 2.3.4–2.3.9, the latter uses proportional values (see Sect. 2.3.3). In contrast to game theoretical approaches, the second class counts for understandability and practical applicability because game theoretical concepts have a higher complexity. Such proportional values are also called activity measures and are based, for instance, on quantity or sales figures (see Biddle and Steinberg 1984, p. 18, and Arcelus et al. 1997, p. 446). Biddle and Steinberg (1984, p. 18) distinguish a third class with *normative allocation proposals*. The normative concepts should provide a general measure for allocation but no general definition exists for this class of concepts.

Fromen (2004) develops a more sophisticated classification of the existing solution methods where concepts are sorted in four classes: “classical” concepts, bargaining concepts, proportional concepts, and others (see Fromen 2004, p. 96).

The first application of cooperative game theory can be found in Ransmeier (1942) where costs for a dam system should be allocated among the participatory users (see also Straffin and Heaney 1981 and Young et al. 1982). The work of Moriarity (1981b) presents several papers dealing with cost allocation from an accounting viewpoint, but integrates game theory methods as well. Contrary to the field of non-cooperative game theory, where the Nash equilibrium is the substantial solution method, many different solution concepts exist in cooperative game theory where no “single, all-purpose solution” is available (see Young 1994, p. 1230). Choosing the right method heavily depends on the context, organizational goals, amount of available information, etc.

2.3.2 Properties of Cost Allocations

There needs to be an incentive to cooperate, otherwise, cooperation will not occur and the outcome will be inefficient. One of the first authors discussing the importance of “fair” allocation techniques was Lemaire (1984). He describes a practical example where common accounting methods based on activity measures to allocate joint costs lead to plenty of problems and wrong incentives. Based on the explanations of Lemaire (1984) and Myerson (1991), we will now discuss important properties a cost allocation should achieve.

Theoretically, the set of solutions could contain all vectors π where at least the total costs are allocated:

$$\sum_{i \in N} \pi_i \geq c(N).$$

We call such solutions *feasible allocation vectors*. Nevertheless, the total costs of the grand coalition should be allocated completely (neither more nor less) because

if we allocate more than $c(N)$, the share of at least one player could be improved (decreased) without harming another player. *Efficiency* assures that the sum of the cost shares equals the total costs $c(N)$ of the grand coalition N :

$$\sum_{i \in N} \pi_i = c(N). \quad (2.4)$$

In some references, the terms *feasibility* or *group rationality* are used instead of efficiency (e.g., Myerson 1991, p. 427, and Maschler 1992, p. 595). Maschler (1992) calls an allocation vector that satisfies (2.4) a *preimputation*.

As described at the beginning of this chapter, cooperative game theory assumes that players act rational. Thus, it is a desirable property for a cost allocation that a single player has lower (or at least equal) costs while cooperating than when acting alone. This property is called *individual rationality*:

$$\pi_i \leq c(\{i\}) \quad i \in N. \quad (2.5)$$

An allocation is called an *imputation* if it secures efficiency (2.4) and individual rationality (2.5). No player would accept an allocation that is no imputation because he would be better off not cooperating. Therefore, most of the solution concepts in cooperative game theory are based upon the set of imputations.

We can further extend the idea of individual rationality because not only single players act rational, but also subcoalitions. A coalition $S \subset N$, $S \neq \emptyset$ of players will cooperate with the remaining players $N \setminus S$ if S cannot improve on the allocation; i.e., *coalitional rationality* holds

$$\sum_{i \in S} \pi_i \leq c(S) \quad S \subseteq N, S \neq \emptyset. \quad (2.6)$$

With constraints (2.5) and (2.6), it is possible to assure that no member of the coalition has any incentive to leave the grand coalition and form a smaller coalition S – we also speak of a *stable* allocation because a cooperation of the grand coalition would not stand in the long run if a partner would be better off not cooperating in the grand coalition. Note, restrictions (2.4) and (2.5) are special cases of (2.6) when S are the singletons or N , respectively (efficiency requires an inequality in the other direction). Instead of rationality constraint, the notion *stability* constraint is used equivalently. The principles of individual and coalitional rationality are already stated in Ransmeier (1942, p. 220).

Lemaire (1984) states that no player should be charged less than its marginal costs and called this property *collective rationality* or *marginality principle*:

$$\pi_i \geq c(N) - c(N \setminus \{i\}) \quad i \in N.$$

If a player pays less than its marginal costs $c(N) - c(N \setminus \{i\})$, then it is effectively subsidized by other players. But there is no need to continue with this restriction because it follows immediately from

Efficiency (2.4):

$$\begin{aligned}
 \sum_{i \in N} \pi_i &= c(N) \\
 \rightarrow \sum_{j \in N \setminus \{i\}} \pi_j + \pi_i &= c(N) \\
 \rightarrow \pi_i &= c(N) - \sum_{j \in N \setminus \{i\}} \pi_j \\
 &\rightarrow \pi_i \geq c(N) - c(N \setminus \{i\})
 \end{aligned}$$

Stability (2.6):

$$\begin{aligned}
 \sum_{i \in S} \pi_i &\leq c(S) \\
 \rightarrow \sum_{j \in N \setminus \{i\}} \pi_j &\leq c(N \setminus \{i\})
 \end{aligned}$$

Young et al. (1982) formulate the marginality principle also for coalitions which can be derived from (2.4) and (2.6) as well:

$$\sum_{i \in S} \pi_i \geq c(N) - c(N \setminus S) \quad S \subset N.$$

A notion coming together with desired properties of a cost allocation is *fairness* (or *equity*). Due to the wide meaning of this term, Young et al. (1982, p. 464) suggest a cost allocation where the participants agree in principle to the proposed allocation. Hence, Young et al. (1982) call an allocation fair if it fulfills (2.4) and (2.6) because it constitutes the minimum incentive to join the grand coalition for every partner. We will pick up the discussion regarding fairness later on, in particular in Sects. 2.3.6 and 3.4. For further properties that are very specific and not used in the following discussion, see e.g., Lemaire (1984), Young (1994), and Zelewski (2007).

In the following sections, we will present different solution concepts which might assure the discussed properties. A *solution concept* is a rule for allocating the total costs of the grand coalition while a *solution* gives a concrete allocation vector π . Solution concepts can be distinguished into those that yield a solution containing exactly one allocation vector (e.g., the Shapley value) and those that give a set of allocation vectors (e.g., the core).

2.3.3 Non-Game-Theoretical Cost Allocation Methods

The problem of allocating joint or common costs has its roots in the field of accounting. For a distinction between joint and common costs see Biddle and Steinberg (1984, p. 4). In the following, it is sufficient to use the term joint costs and neglect the distinction: *Joint costs* are costs for which it is impossible or would be too complex to incur the costs relating to several cost objects individually (Biddle and Steinberg 1984, p. 4).

Furthermore, Biddle and Steinberg (1984, p. 3) give an abstract definition for cost allocation: *Cost allocation* is an “efficient partitioning of a cost among a set of cost objects”. Note that in the world of accounting, a cost object is a product in

most of the cases. For contributions concerning cost allocation from an accounting viewpoint, see e.g., Pfaff (1994) and Snyder and Davenport (1997). Pfaff (1994) investigates disadvantages when joint costs are not allocated and fully allocated, respectively, and suggests an allocation mechanism which is optimal under specific conditions. A more practical discussion is presented by Snyder and Davenport (1997).

Transfer prices are related to the topic of cost allocation: Biddle and Steinberg (1984, p. 5) state that “*transfer prices* serve to allocate revenues against which allocated costs are matched”. Compare also Illner (1999, p. 21) for a detailed discussion regarding the classification of cost allocation and transfer prices.

Certainly, one of the easiest approaches to allocate costs is the *equal amount method*. As the name implies, total costs of the cooperation will be allocated in equal parts to the partners. Apparently, the input level of one partner is ignored completely. The *proportional method* uses different information about the partners in the cooperation to allocate the costs; e.g., purchasing volume, individual costs, or the willingness to pay of every partner (Young et al. 1982, p. 467, call this the *separable costs-remaining benefits method*). These concepts are working with *activity measures*. Another group of practical allocation methods is based on marginal costs: *equal or proportional allocation of the non-marginal costs*. The *equal allocation of the total gain method* employs information regarding the cost savings for allocation. A *serial cost sharing* is proposed by Moulin and Shenker (1992) where an amount is allocated to each partner depending on increasing or decreasing demand, respectively. This concept is not based on game theory but the authors show that it yields a unique Nash equilibrium. Hence, the idea of serial cost sharing finds its way into game-theoretical research, e.g., de Frutos (1998) and Albizuri et al. (2003). Compare, e.g., Lemaire (1984, p. 65), Heijboer (2003, p. 70), and Schotanus (2007, p. 163), for more details regarding the aforementioned concepts.

Normative allocation concepts are introduced in Moriarity (1975) with the main target to avoid the allocating problems when using activity measures in practical situations. Moriarity’s idea is to integrate cost savings that are yielded by the cooperation. Louderback (1976) criticizes the approach of Moriarity because it cannot be avoided in general that partners will be subsidized. Thus, he integrates inside and outside incremental costs caused by the cooperation. Balachandran and Ramakrishnan (1981) combine the two approaches with their propensity-to-contribute concept. Joint costs will be allocated according to each partner’s willingness to pay more than a minimum value. The concept of Gangolly (1981) is based on Moriarity’s allocation scheme. In contrast, Gangolly (1981) considers not only single players that benefit from cost savings in the grand coalition, but also subcoalitions.

For a comparison of game theoretical approaches and practical oriented solution concepts see e.g., Nagarajan et al. (2008, p. 10), Schotanus (2007, pp. 141 and 169), and Heijboer (2002, p. 282). Compare, furthermore, Fromen (2004, p. 43) for a general discussion of non-game-theoretical methods for cost allocation.

Young et al. (1982, p. 465) and Lemaire (1984) explain that most of the common accounting methods presented above fail to satisfy natural requirements. With the help of a practical example, Lemaire (1984, p. 64) shows that these accounting

methods have to be rejected because none of them meets the rationality restrictions. In addition to that proof, Lemaire (1984, p. 77) reviews successful applications of game theoretic approaches for solving practical cost allocation problems. However, he also describes a big drawback because game theoretic methods need more information in terms of the $2^{|N|} - 1$ cost values $c(S)$ (see also Young et al. 1982, p. 471). We will pick up this criticism in Sect. 3.2 where we will introduce a procedure to compute cost allocations which does not need explicit information for all subcoalitions $S \subset N$. Beforehand, we will present game theoretical methods to solve cost allocation problems.

2.3.4 The Core

The core is an essential and one of the most prominent solution concepts to allocate costs (or profits) in problems of cooperative game theory, especially in economic theory. It combines the first three properties mentioned in Sect. 2.3.2 (2.4)–(2.6). The notion of the core is credited to Gillies (1959). However, the idea was prefigured in the early theoretical literature on cost allocation and already discussed by Edgeworth (1881). The core is based on a very simple idea: An imputation is called stable if no coalition could form to reach a better result than in the grand coalition. This leads to the before presented property of coalitional rationality (2.6). Hence, the core combines the property of efficiency with individual and coalitional rationality.

In the literature, the core is mostly defined with the help of the term domination: The core $C(N, c)$ of a cooperative game with $|N|$ players and the characteristic cost function c is the set of all non-dominated imputations (see e.g., Owen 2001, p. 218). Following for instance Owen (2001, p. 215) the imputation π^1 dominates the imputation π^2 through the coalition S if the two following constraints hold:

1. $\pi_i^1 < \pi_i^2$ for all $i \in S$
2. $\sum_{i \in S} \pi_i^1 \geq c(S)$

π^1 dominates π^2 if there is at least one coalition S such that π^1 dominates π^2 through S . The first condition assures that every player in S is preferring π^1 to π^2 . The second restriction claims that the coalition S does not achieve less costs than they should bear while working alone without the players in the grand coalition – it assures feasibility. Hence, non-dominated imputations possess some kind of stability which directly leads to the definition of the core:

$$C(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \underbrace{\sum_{i \in N} \pi_i = c(N)}_{\text{efficiency}} \text{ and } \underbrace{\sum_{i \in S} \pi_i \leq c(S)}_{\text{rationality/stability}} \text{ for all } S \subset N, S \neq \emptyset \right\}. \quad (2.7)$$

The efficiency condition in combination with the rationality condition with $S = \{i\}$ leads to the definition of an imputation. Furthermore, assume that π^1 satisfies both

conditions and that $\pi^2 < \pi^1$ for all $i \in S$. In combination with the rationality constraints, we would get $\sum_{i \in S} \pi_i^2 < c(S)$. Hence, it is not possible that π^2 dominates π^1 (see Owen 2001, p. 218, for a further discussion).

Thus, a core allocation assures that every player is better off in the grand coalition. Otherwise, there would be some coalition S where the players in S could all do strictly better than in the grand coalition while cooperating without the players $N \setminus S$ and dividing $c(S)$ among themselves. Additionally, this is adequate to the notion of pareto optimality because there is no imputation outside the core where one player could get a better result without penalizing another player; i.e., no player or coalition will be subsidized. For a detailed discussion and derivation, see Myerson (1991, p. 427) and Owen (2001, p. 218).

Myerson (1991) describes the core as very appealing because it treats players symmetrically, includes pareto-efficient allocations, and reflects weakness or power of the players according to the characteristic function.

The efficiency constraint (2.4) defines a hyperplane H_N in R^N . Obviously, the core is a subset of this hyperplane H_N and, due to the individual rationality constraints (2.5), the core is bounded. Hence, the core is a compact convex polyhedron of at most $(|N| - 1)$ dimensions (see e.g., Shapley 1971 and Owen 2001, p. 219). That generally means, the core has more than one point; i.e., more than one outcome (allocation vector) is stable. If we know at least two core elements, we can reason, due to the convexity, that the core contains an infinite number of imputations. Hence, for choosing exactly one core element, we need some more rules. Like in Maschler et al. (1979), the core for a three player game can be illustrated graphically in a triangle where each point displays a suggested cost allocation (π_1, π_2, π_3) such that the efficiency constraint holds (see Fig. 2.1). The gray area in the center of the triangle displays the core.

Basically, values $\pi_i < 0$ are allowed. Nevertheless, all cost assignments in the core being non-negative ($\pi_i \geq 0$) holds automatically if the characteristic function is monotone; i.e., $c(S_1) \leq c(S_2)$ for all $S_1 \subseteq S_2 \subseteq N$. Compare Drechsel and Kimms (2010c) for the lemma and proof:

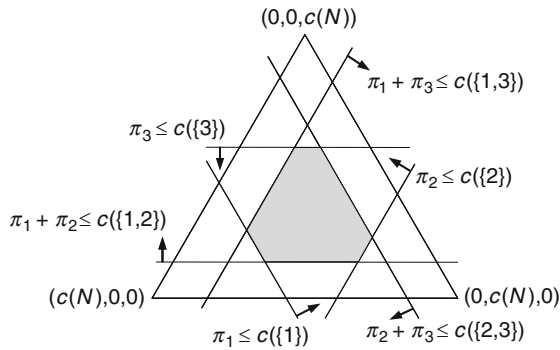


Fig. 2.1 Graphical illustration of the core for a game with three players

Table 2.1 Two numerical examples with nonempty and empty core

	$c(\{1\})$	$c(\{2\})$	$c(\{3\})$	$c(\{1, 2\})$	$c(\{1, 3\})$	$c(\{2, 3\})$	$c(\{1, 2, 3\})$
Example 1	5	5	5	7	7	7	10
Example 2	5	5	5	7	7	7	11

$$\pi_i = c(N) - \sum_{j \in N \setminus \{i\}} \pi_j \geq c(N) - c(N \setminus \{i\}) \geq 0 \quad \text{for all } i \in N.$$

The concept of the core has its drawbacks, particularly, the possible emptiness of the core. Table 2.1 presents two small examples: Both games are subadditive (see Sect. 2.2.2), however, Example 1 has a nonempty core, whereas Example 2 has an empty core. Compare the resulting core constraints:

Example 1:

$$\begin{aligned} \pi_1 + \pi_2 + \pi_3 &= 10 \\ \pi_1 &\leq 5 \\ \pi_2 &\leq 5 \\ \pi_3 &\leq 5 \\ \pi_1 + \pi_2 &\leq 7^* \\ \pi_1 + \pi_3 &\leq 7^* \\ \pi_2 + \pi_3 &\leq 7^* \end{aligned}$$

Example 2:

$$\begin{aligned} \pi_1 + \pi_2 + \pi_3 &= 11 \\ \pi_1 &\leq 5 \\ \pi_2 &\leq 5 \\ \pi_3 &\leq 5 \\ \pi_1 + \pi_2 &\leq 7^* \\ \pi_1 + \pi_3 &\leq 7^* \\ \pi_2 + \pi_3 &\leq 7^* \end{aligned}$$

The three *-marked constraints lead to $\pi_1 + \pi_2 + \pi_3 \leq 10.5$, hence, for $c(N) = 11$ the core is empty. For the second example, no matter what allocation may ultimately occur, there is always a coalition that has an incentive to leave the grand coalition. Though, the greater the economies of scale, the more likely a core allocation can be found. In general, the core of a subadditive three player game is not empty if $c(\{1, 2\}) + c(\{1, 3\}) + c(\{2, 3\}) \geq 2c(N)$ holds.

Having a game with an empty core does not imply that the players could not agree on cooperation and an allocation. There might be binding agreements that could not be broken without consent of the other players. On the other hand, if a coalition exists, that would be better off acting alone, it might hesitate to speak against the allocation because it fears further negotiations where the result is worse than in the beginning. Myerson (1991) formulates three assumptions under which a coalition might think about negotiating for a better allocation:

1. Prior agreements do not forbid such renegotiation.
2. The new agreement would be final – no more bargaining rounds follow.
3. In case of failing agreements, the allocation, valid before starting the negotiation, will be still valid.

Compare Myerson (1991, p. 431) for a further discussion.

Shapley (1971) analyzes the core of convex profit games: He proves that the core of a convex profit game is not empty. Furthermore, he studies the geometry of the core and establishes that the set of marginal worth vectors is precisely the set of vertices of the core of a convex game. It is shown by Weber (1978) that the convex hull of the set of marginal worth vectors contains the core of any game. Ichiishi (1981) relates the study of convex profit games and their marginal vectors to a special greedy algorithm for linear programming. The results can be transferred to concave cost games: A *marginal worth vector* yields a core element for a concave cost game. Let σ be a permutation on N . Define

$$P_j^\sigma \equiv \{i \in N \mid \sigma(i) < \sigma(j)\}$$

as the set of members in N which precede j with respect to the order σ . Hence, the marginal worth vector for a given permutation is defined as the vector

$$a^\sigma(c) \in \mathbb{R}^{|N|} \quad \text{with} \quad a_j^\sigma(c) = c(P_j^\sigma \cup \{j\}) - c(P_j^\sigma) \quad \text{for every } j \in N.$$

However, there exist games that are not concave and for many games it is not clear whether or not the core is empty. Consequently, many contributions to cooperative games concentrate on proving the non-emptiness of the core for specific games. Often, they do not compute core elements explicitly but they prove the balancedness of the game by using the *Bondareva–Shapley Theorem* (see Bondareva 1963 and Shapley 1967) that is based on duality theory of linear programming: The collection $\{S_1, \dots, S_k\}$ of coalitions of N is called balanced if there exist positive numbers $\lambda_1, \dots, \lambda_k$ such that

$$\sum_{j: S_j \ni i} \lambda_j = 1 \quad \text{for every } i \in N.$$

$\lambda_1, \dots, \lambda_k$ are called balancing weights.

The Bondareva–Shapley Theorem signifies that the core of the profit game v is nonempty if and only if for every minimal balanced collection $\{S_1, \dots, S_k\}$ with balancing weights $\lambda_1, \dots, \lambda_k$ the inequality $\sum_{j=1}^k \lambda_j v(S_j) \leq v(N)$ holds (see Bondareva 1963 and Shapley 1967 for the proof).

Kannai (1999) gives a broad survey on balanced games including TU and NTU games. Apart from the non-emptiness of the core, another part of literature deals with proving concavity or convexity of specific games, respectively, and other core properties. González-Díaz and Sánchez-Rodríguez (2007) discuss a particular selection from the core. They call it the core-center because this allocation rule selects a centrally located point within the core if the core is not empty. They continue the investigation of the core-center in González-Díaz and Sánchez-Rodríguez (2009). The studies about marginal vectors and convexity of profit games are advanced by van Velzen et al. (2004). They use combinatorial arguments to obtain sets of marginal vectors that characterize convexity of a game and display a formula for the minimum cardinality of sets of those marginal vectors. Hamers et al. (2002)

deal with marginal vectors as well and prove that the extreme points of the core of assignment games are marginal vectors. The properties of the core's extreme points are studied by Núñez and Rafels (1998). They prove that the extreme points have the reduced game property with respect to the Davis and Maschler reduced game (see Davis and Maschler 1965), but not the converse reduced game property. Necessary and sufficient conditions under which the core coincides with the bargaining set for superadditive profit games are proved by Solymosi (1999). Shapley and Shubik (1971) present an extensive discussion of the core regarding assignment games which gives among other things insights into the geometric structure of the core.

Due to the described drawbacks regarding existence and uniqueness of the core, some authors argue that the core is only little applicable to practical allocation problems, but should be used to categorize the set of imputations and evaluate other solution concepts (see e.g., Fromen 2004, p. 100).

In papers where core elements are computed explicitly, very specific approaches are constructed that depend highly on the underlying problem and cannot be generalized. We will discuss the computation of core elements in Chap. 3 in detail.

As a reaction to the described drawbacks, several related concepts have been developed in the literature to overcome the difficulties of nonexistence and non-uniqueness. These concepts are based on strengthening or relaxing the stability constraints that define the core.

The following presented core variants are classified according to Faigle et al. (1998b) into two groups: At first, we introduce core variants where the characteristic function values are varied by an absolute amount – *additive core variants*. The second class of core variants is marked by a relative variation of the characteristic function values – *multiplicative core variants*. Afterwards, we will introduce another two core variants which do not fit in this classification: a core variant specifically developed for non-monotone cooperative games and the interval core for the before presented interval-valued games (see Sect. 2.2.4).

2.3.5 Additive Core Variants

The ϵ -Core

The ϵ -core is introduced by Shapley and Shubik (1966). They distinguish between the *strong ϵ -core*

$$C_{\epsilon}(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \left| \sum_{i \in N} \pi_i = c(N) \text{ and } \sum_{i \in S} \pi_i \leq c(S) + \epsilon \right. \right. \\ \left. \left. \text{for all } S \subset N, S \neq \emptyset \right\}.$$

and the rarely used *weak ϵ -core*

$$C_{\epsilon}^{weak}(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \left| \sum_{i \in N} \pi_i = c(N) \text{ and } \sum_{i \in S} \pi_i \leq c(S) + |S|\epsilon \right. \right. \\ \left. \left. \text{for all } S \subset N, S \neq \emptyset \right\}.$$

ϵ is a given parameter used to enlarge the core by a small amount. Shapley and Shubik (1966) call these two variants *quasi-cores*. Faigle et al. (1998b) use this formulation under the name *additive ϵ -core*. The above formulation provides a possibility to take the costs of coalition formation into account – a fixed charge ϵ . Thus, ϵ can be interpreted as threshold for a blocking maneuver resulting from a stability constraint. In other words, if founding a coalition $S \subset N$ while quitting the grand coalition causes a cost of $\epsilon > 0$, then the grand coalition is stable even if the coalition S receives a cost share larger than $c(S)$ as long as $\sum_{i \in S} \pi_i \leq c(S) + \epsilon$ is fulfilled. The same holds if a third party offers a reward $\epsilon < 0$ for forming a coalition $S \subset N$ and leaving the grand coalition. In this case, the grand coalition is stable as long as the sum of the cost shares for the players in coalition S is smaller or equal to the coalition cost $c(S)$ less the reward.

As already mentioned, the core of a cooperative game might be empty. Obviously, if we choose a sufficiently large ϵ , the ϵ -core of the same instance is not empty. Otherwise, the ϵ -core lies in the core with an $\epsilon \leq 0$ if the core of an instance is not empty; i.e.,

$$C_{\epsilon}^{weak}(N, c) \supset C_{\epsilon}(N, c) \supset C(N, c).$$

Evidently, the core is a special case of the ϵ -core; i.e., $C_0(N, c) = C(N, c)$. In the following, we will simply use the term ϵ -core when meaning the strong ϵ -core.

Fromen (2004) argues that the ϵ -core is actually not a self-contained solution concept, but should be used particularly to analyze games with an empty core.

The Least Core

Continuing the approach of the ϵ -core, Maschler et al. (1979) develop the so-called least core $C^L(N, c)$ to ensure existence and uniqueness of the allocation vector. They define the least core as the intersection of all nonempty ϵ -cores. Hence, the least core is the ϵ -core with the smallest possible $\epsilon \in \mathbb{R}$ such that $C_{\epsilon}(N, c)$ is not empty:

$$C^L(N, c) = \bigcap_{\epsilon \in \mathbb{R}: C_{\epsilon}(N, c) \neq \emptyset} C_{\epsilon}(N, c).$$

If the core of a game is not empty, then $\epsilon \leq 0$ and the least core is centrally located within the core. By contrast, if the core is empty ($\epsilon > 0$), then the least core can be seen as revealing the “latent” position of the core. The least core can be used to

define the problem of proving non-emptiness of the core as an optimization problem that minimizes ϵ under the condition that $C_\epsilon(N, c)$ is not empty. As already mentioned, an objective function value $\epsilon^* \leq 0$ indicates that the core is not empty and the corresponding set $C_{\epsilon^*}(N, c)$ is the least core. For a nonempty core ($\epsilon^* \leq 0$), the following inclusion is straightforward:

$$C_{\epsilon^*}(N, c) = C^L(N, c) \subseteq C(N, c).$$

Obviously, the notion of the least core can be used to define the *weak least core* – compare the definition of the weak ϵ -core. For a detailed discussion and computation of the least core, see Schulz and Uhan (2007).

The Nucleolus

Schmeidler (1969) introduces the *nucleolus*. This concept is based on the idea of maximizing lexicographically the minimal satisfaction of each coalition step by step. Maschler et al. (1979) and Maschler (1992, p. 611) give illustrative descriptions to motivate the concept of the nucleolus. The measure for satisfaction is the so-called excess:

$$e(S, \pi) = c(S) - \sum_{i \in S} \pi_i.$$

Hence, the excess results from the difference between stand-alone costs (independent from the players $j \in N \setminus S$) and the costs the players $i \in S$ have to bear when cooperating in the grand coalition. The excess for the grand coalition and the empty coalition is zero, respectively:

$$\begin{aligned} e(\emptyset, \pi) &= c(\emptyset) - \sum_{i \in \emptyset} \pi_i = 0 \\ e(N, \pi) &= c(N) - \sum_{i \in N} \pi_i = 0. \end{aligned}$$

There are many variants of the nucleolus which mostly differ on the definition of the excess. A stepwise computation means that starting with the set of imputations, we first determine an imputation that maximizes the minimal excess. In the second step, an imputation that maximizes the second smallest excess is determined and so on. The procedure stops latest if all coalitions are considered.

Maschler (1992, p. 610) and Owen (2001, p. 322) give formal derivations of the nucleolus. Let $\theta(\pi)$ be a vector containing the excesses of all coalitions arranged in non-decreasing order

$$\theta(\pi) := (e(S_1, \pi), e(S_2, \pi), \dots, e(S_{2^{|N|}-1}, \pi)).$$

Note, this order depends on the allocation vector π and does not need to be unique. $\theta(\pi^1)$ is called *lexicographically smaller* than $\theta(\pi^2)$ if a positive integer q exists such that $\theta_i(\pi^1) = \theta_i(\pi^2)$ whenever $i < q$ and $\theta_q(\pi^1) < \theta_q(\pi^2)$. We denote this with $\theta(\pi^1) < \theta(\pi^2)$. Thus, the nucleolus can be defined as

$$\mathcal{N}(N, c) = \{ \pi^1 \in I(c) \mid \theta(\pi^1) \succeq \theta(\pi^2) \text{ for all } \pi^2 \in I(c) \}$$

with $I(c)$ being the set of imputations. Schmeidler (1969) proves that the nucleolus is inside the core if the core is not empty (see also Maschler 1992). That is one reason why the nucleolus is willingly used. Provided that the core exists, the nucleolus yields a unique allocation in the core. Furthermore, the nucleolus always exists. Shubik (1982) states that the nucleolus shows the location of the “latent position” of the core if the core is empty and the core’s center if it is not empty. The second statement is difficult to confirm: Maschler et al. (1979) provide two numerical examples with the same set of imputations and the same core but they have different values for the nucleolus. See Maschler et al. (1979) for an explanation regarding the geometrical position of the nucleolus inside the core.

2.3.6 Multiplicative Core Variants

The Approximate Core

In case of multiplicative core variants, there are contributions regarding the so-called *approximate core* or α -core (see e.g., Jain and Mahdian 2007, p. 389, and Schulz and Uhan 2007, p. 2). This concept allows inefficient solutions. The formal definition is

$$C_\alpha(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \alpha c(N) \leq \sum_{i \in N} \pi_i \text{ and } \sum_{i \in S} \pi_i \leq c(S) \text{ for all } S \subseteq N, S \neq \emptyset \right\}. \quad (2.8)$$

The problem of proving the non-emptiness of the core can be formulated, using the notion of the approximate core, as an optimization problem that maximizes α under the restriction that $C_\alpha(N, c)$ is not empty; i.e., that the efficiency and stability constraints following (2.8) hold. If this optimization problem yields an optimum objective function value $\alpha^* = 1$, then the core is not empty; i.e., if $\alpha^* = 1$ then

$$C_{\alpha^*}(N, c) = C(N, c).$$

The Minmax Core

Allocations in the core or the approximate core can be considered as stable because no player has an incentive to break off the grand coalition. And from this point of view, they can be seen as fair in a weak sense. But those allocations might not be

considered as inherently fair in a strong sense because players or subcoalitions in the grand coalition benefit more than others from the cost decrease. We can define benefit as the deviation of $\sum_{i \in S} \pi_i$ from $c(S)$. The ϵ -core and the least core guarantee that the absolute benefit for each coalition is at least ϵ , hence, they can be seen as fair core allocations as long as $\epsilon \leq 0$. However, if the $c(S)$ values differ a lot among the coalitions S , absolute values may not be regarded as fair. For practical use, we advise to use relative benefits. Compare Drechsel and Kimms (2010a) for some ideas of implementing fairness aspects into the notion of the classical core – these ideas will be discussed in Sect. 3.4 as well. Furthermore, Frisk et al. (2010) develop a relative measure which they call the *equal profit method*. They minimize the difference in relative savings between two participants i and j :

$$\frac{\pi_i}{c(\{i\})} - \frac{\pi_j}{c(\{j\})}.$$

However, all of these ideas consider fairness among the individual players only; i.e., the benefit of subcoalitions containing two and more players are not taken into account. Therefore, Drechsel and Kimms (2010b) propose another concept which they call the *minmax core*. The minmax core can be used to determine efficient, rational, and fair cost allocations. The principle of fairness is tackled by means of the minmax-principle. The benefit of a subcoalition S is measured as the relative benefit in percentages of $c(S)$. That means, the lower the cost assigned to a subcoalition S , the greater the benefit. The subcoalition that receives the lowest benefit (i.e., highest cost assignment in percentages) determines the fairness measure. To develop the mathematical formulation of the minmax core, we use the multiplicative ϵ -core of Faigle and Kern (1993) and Faigle et al. (1998b) (in several other papers it is called ϵ -approximation of the core or ϵ -core, see e.g., Faigle et al. 1998a; Faigle and Kern 1998). Let $(1 - \epsilon) = \eta$:

$$C_\eta(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \sum_{i \in N} \pi_i = c(N) \text{ and } \sum_{i \in S} \pi_i \leq \eta c(S) \text{ for all } S \subset N, S \neq \emptyset \right\}.$$

Drechsel and Kimms (2010b) call this the η -core. η assures that to no coalition S a cost share greater than η percentage of its stand-alone cost $c(S)$ is assigned – provided that $C_\eta(N, c)$ is not empty. This core variant suits all applications where coalition costs are changed proportional to the value (e.g., a sales tax). Faigle et al. (1998b) compare it to other “taxation models” in the literature like the weak ϵ -core defined above or the modification by Tijs and Driessen (1986) for multiplicative ϵ -tax games:

$$C_\epsilon^{\text{tax}}(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \sum_{i \in N} \pi_i = c(N) \text{ and } \sum_{i \in S} \pi_i \leq c(S) + \epsilon \left(\sum_{i \in S} c(\{i\}) - c(S) \right) \text{ for all } S \subset N, S \neq \emptyset \right\}.$$

Faigle et al. (1998b) state that this core variant equals their multiplicative ϵ -core if the game is zero-normalized (i.e., $c(\{i\}) = 0$ for all $i \in N$). Faigle et al. (1998b) do not claim to rank the different core variants but show the value of the concept of the multiplicative ϵ -core with some practical examples.

As with the other core variants before, we can use the η -core to prove non-emptiness of the core via stating it as an optimization problem which minimizes η under the condition that $C_\eta(N, c)$ is not empty. If the optimum objective function value is $\eta^* \leq 1$, then the core is not empty (the reverse might not hold for $c(S) = 0$).

Drechsel and Kimms (2010b) call $C_{\eta^*}(N, c)$ the *minmax core* $C_M(N, c)$. It is straightforward that $C_{\eta^*}(N, c) \subseteq C(N, c)$ if the core is not empty. The minmax core can be interpreted as containing cost allocations where the worst benefit over all subcoalitions is as good as possible.

We already mentioned that $\eta^* \leq 1$ indicates non-emptiness of the core. However, even though the core is not empty, η^* could be greater than 1. This might happen if a coalition $S \subseteq N$ ($S \neq \emptyset$) exists with $c(S) = 0$. To make this more obvious, assume a coalition S with $c(S) = 0$. An efficient allocation requires

$$\sum_{i \in S} \pi_i + \sum_{i \in N \setminus S} \pi_i = c(N).$$

Two of the minmax core defining stability constraints are

$$\sum_{i \in S} \pi_i \leq 0 \quad \text{and} \quad \sum_{i \in N \setminus S} \pi_i \leq \eta c(N \setminus S).$$

Consequently, we have

$$\sum_{i \in N \setminus S} \pi_i \geq c(N).$$

Hence, $c(N) \leq \eta c(N \setminus S)$ which may enforce $\eta^* > 1$. Note that this is not necessarily true, in particular, if c is not monotone.

The idea of using such relative measures has been used by only few authors so far: Apart from the above mentioned references, there is Young et al. (1982) who introduce the so-called *proportional least core* for a profit game by imposing a minimum tax t (or subsidy) on all coalitions in proportion to their cost so that the proportional least core is not empty. The mathematical definition is nearly the same as for the minmax core.

The Proportional Nucleolus

Drechsel and Kimms (2010b) remark that the idea of the minmax core can be straightforwardly enhanced: In a first step, the worst benefit shall be as good as possible. In a second step, the second worst benefit (among those solutions computed in the first step) shall be as good as possible. In the third step, the set of solutions

from step two are considered and so on. The result can be termed the *lexicographic minmax core*. This idea is somewhat similar to the nucleolus (see Sect. 2.3.5). However, the nucleolus measures the so-called excess $c(S) - \sum_{i \in S} \pi_i$ (i.e., benefit) in absolute terms while the concept of the minmax core uses η as a percentage; i.e., relative measure.

As for the core and the nucleolus, the concept of the proportional least core is enhanced to the notion of the *proportional nucleolus* by Young et al. (1982) and Lemaire (1984) which can be compared to the above described lexicographic minmax core. Since the work of Faigle et al. (1998b), this concept is also known as the *nucleon*. The excess for the nucleon is defined by

$$e(S, \pi) = \frac{c(S) - \sum_{i \in S} \pi_i}{c(S)}.$$

Lemaire (1984) describes the difference between the nucleolus and the proportional nucleolus as follows: “In one case coalitions are taxed in order to make the core exist, in the other case coalitions are subsidized in order to reduce the core to a single imputation.”

The *disruptive nucleolus* uses the following excess function (Littlechild and Vaidya 1976):

$$e^d(S, \pi) = \frac{c(N \setminus S) - \sum_{i \in N \setminus S} \pi_i}{c(S) - \sum_{i \in S} \pi_i}.$$

The propensity to disrupt for a coalition S is defined as the ratio between the amount $N \setminus S$ and S would lose if the allocation according to the disruptive nucleolus is abandoned. By construction, this variant also belongs to the core if the core is not empty.

2.3.7 The Subcoalition-Perfect Core

Assume that the characteristic function c is not monotone – however, it may be subadditive. In such a case, there exists a coalition S_1 and supercoalition $S_2 \supset S_1$ ($S_2 \subseteq N$) such that $c(S_2) < c(S_1)$. This means for $S_2 \subset N$ that the grand coalition and an arbitrary core cost allocation $\pi \in C(N, c)$ may give S_1 an incentive to leave the grand coalition even though c is subadditive; i.e., S_1 may prefer the game (S_2, c) over the game (N, c) . This may direct S_1 to force $S_2 \setminus S_1$ to form a coalition S_2 instead of the grand coalition. Drechsel and Kimms (2010c) investigate this behavior and describe it as a “force of anarchy” in the game because the cost allocation $\pi \in C(N, c)$ may not be considered as stable any longer if $c(S_2) < \sum_{i \in S_1} \pi_i \leq c(S_1)$ is true.

Look at a small example for illustration: We have a subadditive but non-monotone cooperative cost game – the values for the characteristic function are displayed in Table 2.2. Obviously, due to $c(\{2, 3\}) < c(\{3\})$, the game is not monotone. Assume $\pi = (0, -2, 9)$, player 3 would prefer the game $(\{2, 3\}, c)$ over the game (N, c) .

Table 2.2 Numerical example for the subcoalition-perfect core in a cost game

$c(1)$	$c(2)$	$c(3)$	$c(1, 2)$	$c(1, 3)$	$c(2, 3)$	$c(1, 2, 3)$
4	5	9	8	10	7	9

Based on these findings, Drechsel and Kimms (2010c) develop a new core concept which they call the *subcoalition-perfect core* to assure core cost allocations where no coalition S_1 has to carry higher costs in the grand coalition than in any other supercoalition S_2 of S_1 . Mathematically, this can be defined as

$$C^{SCP}(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \sum_{i \in N} \pi_i = c(N) \text{ and } \sum_{i \in S_1} \pi_i \leq c(S_2) \right. \\ \left. \text{for all } S_1 \subset N, S_1 \neq \emptyset \text{ and } S_1 \subseteq S_2 \subseteq N \right\}.$$

An equivalent formulation is

$$C^{SCP}(N, c) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \sum_{i \in N} \pi_i = c(N) \text{ and } \sum_{i \in S_1} \pi_i \leq \min_{S_1 \subseteq S_2 \subseteq N} c(S_2) \right. \\ \left. \text{for all } S_1 \subset N, S_1 \neq \emptyset \right\}.$$

Obviously, $C^{SCP}(N, c) \subseteq C(N, c)$ holds in general and $C^{SCP}(N, c) = C(N, c)$ is true if c is monotone. Furthermore, Drechsel and Kimms (2010c) provide a proof regarding an interesting property of the subcoalition-perfect core: The subcoalition-perfect core equals the set of non-negative core allocations

$$C^+(N, c) = \{\pi \in C(N, c) \mid \pi_i \geq 0 \text{ for all } i \in N\}.$$

The proof that $C^+(N, c) = C^{SCP}(N, c)$ can be described as follows (see Drechsel and Kimms 2010c): If the characteristic function is monotone, the proof can be derived immediately from the property described in Sect. 2.3.4 ($\pi_i \geq 0$ for all i if c is monotone) and from the fact that for all coalitions S_1

$$c(S_1) = \min_{S_1 \subseteq S_2 \subseteq N} c(S_2).$$

Hence, $C^+(N, c) = C(N, c) = C^{SCP}(N, c)$ if c is monotone. What happens if c is not monotone?

“ $C^+(N, c) \subseteq C^{SCP}(N, c)$ ”: In case of an empty set of non-negative core allocations, the proof would be trivial. Thus, assume that the set of non-negative core allocations is nonempty. Assume furthermore that there exists a non-negative core allocation $\pi \in C^+(N, c)$ that is not in the subcoalition-perfect core. Which means

that two coalitions $S_1 \subseteq S_2 \subseteq N$ exist with $c(S_2) < \sum_{i \in S_1} \pi_i \leq c(S_1)$. The chosen core allocation π fulfills

$$\sum_{i \in S_1} \pi_i + \sum_{i \in S_2 \setminus S_1} \pi_i = \sum_{i \in S_2} \pi_i \leq c(S_2) < \sum_{i \in S_1} \pi_i.$$

For this reason, there must be at least one player i in $S_2 \setminus S_1$ with a negative cost share ($\pi_i < 0$). Mathematically, this can be expressed by the following implication:

$$\exists S_1 \subset N : \exists S_1 \subset S_2 \subset N : \sum_{i \in S_1} \pi_i > c(S_2) \Rightarrow \exists i \in N : \pi_i < 0.$$

This contradicts to the before stated assumption $\pi \in C^+(N, c)$ and by contraposition we get

$$\forall i \in N : \pi_i \geq 0 \Rightarrow \forall S_1 \subset N : \forall S_1 \subset S_2 \subset N : \sum_{i \in S_1} \pi_i \leq c(S_2).$$

Hence, $\pi \in C^{SCP}(N, c)$ must hold.

“ $C^+(N, c) \supseteq C^{SCP}(N, c)$ ”: The proof would be trivial if the set of subcoalition-perfect core allocations is empty. Therefore, let this set be nonempty. Assume that there exists a subcoalition-perfect core allocation $\pi \in C^{SCP}(N, c)$ with $\pi_i < 0$ for at least one player $i \in N$; i.e., $\pi \notin C^+(N, c)$. The chosen cost allocation is efficient and, therefore, we have $c(N) = \sum_{j \in N} \pi_j$. Due to $\pi_i < 0$, we get $\sum_{j \in N} \pi_j < \sum_{j \in N \setminus \{i\}} \pi_j$. But π is an element from the subcoalition-perfect core and therefore $\sum_{j \in N \setminus \{i\}} \pi_j \leq c(N)$ must hold which leads to the conclusion that

$$c(N) = \sum_{j \in N} \pi_j < \sum_{j \in N \setminus \{i\}} \pi_j \leq c(N).$$

Hence, $\pi \in C^+(N, c)$ must be true. Summarizing the proof, $C^+(N, c) = C^{SCP}(N, c)$ holds. In Chap. 7, we will apply the subcoalition-perfect core to a practical non-monotone game.

The concept of the subcoalition-perfect core can be applied to *profit games* as well. Assume a profit game (N, v) where the characteristic function v is not monotone; i.e., there might be a coalition S_1 and a supercoalition $S_2 \supset S_1$ ($S_2 \subseteq N$) such that $v(S_2) < v(S_1)$. An allocation π would not be preferable if $v(S_2) \leq \sum_{i \in S_2} \pi_i < v(S_1) \leq \sum_{i \in S_1} \pi_i$ ($S_1 \subseteq S_2 \subseteq N$). Thus, the subcoalition-perfect core in case of profit games contains allocations where no coalition S_2 has a lower payoff in the grand coalition than in any subcoalition S_1 of S_2 .

$$C^{SCP}(N, v) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \sum_{i \in N} \pi_i = v(N) \text{ and } \sum_{i \in S_2} \pi_i \geq v(S_1) \right. \\ \left. \text{for all } S_1 \subset N, S_1 \neq \emptyset \text{ and } S_1 \subseteq S_2 \subseteq N \right\}.$$

The equivalent formulation is

$$C^{SCP}(N, v) = \left\{ \pi \in \mathbb{R}^{|N|} \mid \sum_{i \in N} \pi_i = v(N) \text{ and } \sum_{i \in S_2} \pi_i \geq \max_{S_1 \subseteq S_2 \subseteq N} v(S_1) \right. \\ \left. \text{for all } S_1 \subset N, S_1 \neq \emptyset \right\}.$$

Again, $C^{SCP}(N, v) \subseteq C(N, v)$ and if v is monotone, $C^{SCP}(N, v) = C(N, v)$ holds. Profit games behave differently in the monotonicity proof concerning positive core allocations (see p. 24):

$$\pi_i = v(N) - \sum_{j \in N \setminus \{i\}} \pi_j \leq v(N) - v(N \setminus \{i\}) \geq 0 \quad \text{for all } i \in N.$$

Hence, there is no general proof that in monotone profit games the profit shares π_i are always non-negative. But we can prove that:

“ $C^+(N, v) \subseteq C^{SCP}(N, v)$ ”: Assume that there exist non-negative core allocations and a non-negative core allocation π which is not in the subcoalition-perfect core; i.e., $S_1 \subseteq S_2 \subseteq N$ with $v(S_2) \leq \sum_{i \in S_2} \pi_i < v(S_1)$. For the chosen core allocation, π must hold

$$v(S_2) \leq \sum_{i \in S_1} \pi_i + \sum_{i \in S_2 \setminus S_1} \pi_i = \sum_{i \in S_2} \pi_i < v(S_1) \leq \sum_{i \in S_1} \pi_i.$$

Hence, there has to be at least one player in $S_2 \setminus S_1$ with a negative profit share ($\pi_i < 0$).

$$\exists S_1 \subset N : \exists S_1 \subset S_2 \subset N : \sum_{i \in S_2} \pi_i < v(S_1) \Rightarrow \exists i \in N : \pi_i < 0.$$

Hence,

$$\forall i \in N : \pi_i \geq 0 \Rightarrow \forall S_1 \subset N : \forall S_1 \subset S_2 \subset N : \sum_{i \in S_2} \pi_i \geq v(S_1).$$

Accordingly, $\pi \in C^{SCP}(N, v)$ holds.

Contrary to cost games and according to the relation between monotonicity and non-negative profit shares, $C^+(N, v) \supseteq C^{SCP}(N, v)$ does not hold: Assume that there exists an allocation in the subcoalition-perfect core and that there is an allocation in the subcoalition-perfect core with at least one $\pi_i < 0$. The efficiency constraint $v(N) = \sum_{j \in N} \pi_j$ holds. Due to $\pi_i < 0$, we have

$$v(N) = \sum_{j \in N} \pi_j < \sum_{j \in N \setminus \{i\}} \pi_j \geq v(N \setminus \{i\}).$$

Table 2.3 Numerical example for the subcoalition-perfect core in a profit game

$v(1)$	$v(2)$	$v(3)$	$v(1, 2)$	$v(1, 3)$	$v(2, 3)$	$v(1, 2, 3)$
-1	0	-1	2	-1.5	2	4

For an allocation in the subcoalition-perfect core must hold $\sum_{j \in N} \pi_j \geq v(N \setminus \{i\})$. Therefore, we might have negative cost allocations in the subcoalition-perfect core which can be supported by a small numerical example. Table 2.3 displays the characteristic function for a profit game with three players. The game is superadditive, but not monotone (e.g., $v(1) > v(1, 3)$). The following constraints define the core and the subcoalition-perfect core for this game:

$$\begin{array}{ll}
 C(N, v) : & \pi_1 + \pi_2 + \pi_3 = 4 \\
 & \pi_1 \geq -1 \\
 & \pi_2 \geq 0 \\
 & \pi_3 \geq -1 \\
 & \pi_1 + \pi_2 \geq 2 \\
 & \pi_1 + \pi_3 \geq -2 \quad \leftarrow \\
 & \pi_2 + \pi_3 \geq 2 \\
 C^{SCP}(N, v) : & \pi_1 + \pi_2 + \pi_3 = 4 \\
 & \pi_1 \geq -1 \\
 & \pi_2 \geq 0 \\
 & \pi_3 \geq -1 \\
 & \pi_1 + \pi_2 \geq 2 \\
 & \pi_1 + \pi_3 \geq -1 \quad \leftarrow \\
 & \pi_2 + \pi_3 \geq 2
 \end{array}$$

An allocation in the subcoalition-perfect core is, e.g., $\pi_1 = -1$, $\pi_2 = 5$, $\pi_3 = 0$.

Thus, we can summarize that the subcoalition-perfect core for a cost game equals the set of non-negative core allocations, but for profit games, the set of non-negative core allocations is only a subset of the subcoalition-perfect core.

2.3.8 The Interval Core

In Sect. 2.2.4, we introduced interval-valued games that can handle uncertainties of the characteristic function. To deal with practical interval-valued games, we need to define an allocation concept that can handle interval-valued characteristic functions.

We use the following notation for preparing a core variant that applies to interval-valued games. The sum of two intervals $I = [\underline{I}; \bar{I}]$ and $J = [\underline{J}; \bar{J}]$ is defined to be $I + J = [\underline{I} + \underline{J}; \bar{I} + \bar{J}]$. The interval I is called weakly better than interval J ($I \preceq J$ or $J \succeq I$), if and only if $\underline{I} \leq \underline{J}$ and $\bar{I} \leq \bar{J}$. Following for instance Alparslan-Gök et al. (2008a), the *interval core* of a cooperative interval-valued game can be defined as follows:

$$C^{IV}(N, c^{IV}) = \left\{ (I_1, \dots, I_{|N|}) \in I(\mathbb{R})^{|N|} \left| \sum_{i \in N} I_i = c^{IV}(N) \text{ and } \sum_{i \in S} I_i \preceq c^{IV}(S) \right. \right. \\
 \left. \left. \text{for all } S \subset N, S \neq \emptyset \right\}.$$

Hence, the interval core consists of vectors that aggregate the interval shares for every player in the grand coalition. These interval shares have to fulfill the known efficiency and stability constraints.

2.3.9 The Shapley Value

Apart from the core, its manifold variants, and related concepts, a lot of other solution concepts in cooperative game theory exist. In this section, we will shortly introduce the most common of them for the sake of completeness and will refer to more substantial contributions for a further discussion. Fromen (2004) gives an extensive survey of the existing solution concepts including evaluations and comparisons (see Fromen 2004, in particular Chap. 4).

Shapley (1953) aims to develop an allocation method that yields a unique solution for every game in coalitional form and follows three axioms. The axioms are derived from properties that should be satisfied by such an allocation. Before describing the axioms, we have to define a *permutation* $P : N \rightarrow N$ (a one to one mapping onto itself) of the set of players N such that, for every $j \in N$, there exists exactly one $i \in N$ that $P(i) = j$. Furthermore, we let Pc be the coalitional game such that

$$Pc(\{P(i)|i \in S\}) = c(S) \quad \text{for all } S \subseteq N.$$

That means, player i in c plays essentially the same role as player $P(i)$ in Pc . A coalition R is a *carrier* of a game if and only if

$$c(S \cap R) = c(S) \quad \text{for all } S \subseteq N.$$

Accordingly, all players $S \setminus R$ are called *dummies* in the game because their presence in the game does not change its value. The value Φ denotes a function which associates a real number with each $i \in N$ and satisfies the following axioms (see Shapley 1953):

- Symmetry: for any permutation and any player i , $P \in P(N)$, $\Phi_{P(i)}(Pc) = \Phi_i(c)$
- Efficiency: for any carrier R , $\sum_{i \in R} \Phi_i = c(R)$
- Additivity: for any two games (N, c_1) and (N, c_2) , $\Phi(c_1 + c_2) = \Phi(c_1) + \Phi(c_2)$

In other words, the *Shapley value* yields an allocation where players are handled symmetrically. A cost share is assigned to players only if they actually generate costs. Furthermore, while combining two independent games, the players' values must be added player by player. Shapley (1953) proves that no further condition is needed to define a unique value. Additionally, he shows that some more properties hold assuming the three presented axioms and derives the following formula to determine the Shapley value:

$$\Phi_i(c) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [c(S \cup \{i\}) - c(S)]. \quad (2.9)$$

Myerson (1991) gives a very clear description regarding (2.9): Imagine the players randomly lining up in a queue at the door of a big room where they should all be assembled. There exist $|N|!$ different orders for the queue. For any set S not containing i , there are $|S|!(|N| - |S| - 1)!$ permutations so that S is the set of players standing ahead of i in the queue. Hence, the relation $|S|!(|N| - |S| - 1)!/|N|!$ defines the probability when i enters the room and S is already in there. In this case, player i 's marginal contribution is $c(S \cup \{i\}) - c(S)$. Thus, the Shapley value can be interpreted as the expected marginal contribution of every player i . Until today, the Shapley value is very famous in business and mathematical literature (see e.g., Roth 1988).

The Shapley value might not be inside the core (see Lemaire 1984, p. 72, for a numerical example), but Shapley (1971) prove that in case of a convex profit game, the Shapley value always belongs to the core – it displays the center of gravity of the core's extremal points. For instance, Hamlen et al. (1980, p. 270) explain that the uniqueness of the Shapley value may not be an advantage in any case because it does not permit any flexibility (e.g., to achieve other goals set by management). They present a generalization of the Shapley allocation to restore this problem.

2.3.10 Conclusions

From the viewpoint of practical applications, the first target of cost allocation should be to ensure stability of the cooperation. No cooperation will exist a longer period of time if any player thinks that it could do better without the other participants in the cooperation. Secondly, a cost allocation can offer incentives such that the cooperating partners act in the sense of the grand coalition. Lemaire (1984, p. 63) proves with a small example that allocations where one partner is subsidized at the expense of another may not assure optimal total costs for the grand coalition. To assure stability, we introduce the so-called coalitional rationality property; i.e., no player or subcoalition should be better off when defecting from the grand coalition.

A core cost allocation guarantees these properties and displays a very intuitive and easy to comprehend concept. However, the concept of the core has two main drawbacks: The core may be empty and if it exists, it is rarely unique. The first disadvantage can be solved by using core variants like the least core or the minmax core where it is possible to violate stability constraints to a certain amount (absolute or relative). The second disadvantage can be seen as a chance to integrate other desirable properties or providing flexibility for management decisions. Neither allocation methods from accounting nor the Shapley value can assure stable cost allocations for general cooperative games.

The downside of cost allocation methods coming from cooperative game theory is that they require much more information than classical accounting approaches.

Normally, due to the coalitional rationality, we need information about the characteristic function values $c(S)$ for all subcoalitions $S \subseteq N$; i.e., data for $2^{|N|} - 1$ cost values are necessary. The application in many practical situations is therefore limited.

In the following chapter, we will tie in with these findings and present a procedure which allows to compute stable cost allocations without utilizing all characteristic function values. For more extensive discussions regarding the suitability of the different cooperative game theory approaches, compare, e.g., Young et al. (1982), Lemaire (1984), Fromen (2004), and Zelewski (2007).



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