Max–Stable Processes: Representations, Ergodic Properties and Statistical Applications

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Abstract Max–stable processes arise as limits in distribution of component–wise maxima of independent processes, under suitable centering and normalization. Therefore, the class of max–stable processes plays a central role in the study and modeling of extreme value phenomena. This chapter starts with a review of classical and some recent results on the representations of max–stable processes. Recent results on necessary and sufficient conditions for the ergodicity and mixing of stationary max–stable processes are then presented. These results readily yield the consistency of many statistics for max–stable processes. As an example, a new estimator of the extremal index for a stationary max–stable process is introduced and shown to be consistent.

1 Introduction

In the past 30 years, the structure of max–stable random vectors and processes has been vigorously explored. A number of seminal papers such as Balkema and Resnick [1], de Haan [4, 5], de Haan and Pickands [9], Gine, Hahn and Vatan [13], Resnick and Roy [20], just to name a few, have lead to an essentially complete picture of the dependence structure of max–stable processes. Complete accounts of the state-of-the-art can be found in the books of Resnick [18], de Haan and Ferreira [6], and Resnick [19], and the references therein.

The stochastic process X is said to be max–stable if all its finite–dimensional distributions are max–stable. Recall that a random vector $\mathbf{Y} = (Y(j))_{1\le j\le d}$ in $\mathbb{R}^d$ is said to be *max–stable* if, for all $n \in \mathbb{N}$, there exist $\mathbf{a}_n > 0$ and $\mathbf{a}_n,\ \mathbf{b}_n \in \mathbb{R}^d$, such that

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$$\bigvee_{1\le i\le n} \mathbf{Y}_i \stackrel{d}{=} \mathbf{a}_n \mathbf{Y} + \mathbf{b}_n.$$

Here $\mathbf{Y}_i = (Y_i(j))_{1\le j\le d},\ i = 1,\dots,n$ are independent copies of $\mathbf{Y}$ and the above inequalities, vector multiplications, additions and maxima are taken coordinate–wise.

The importance of max–stable processes in applications stems from the fact that they appear as limits of component–wise maxima. Suppose that $\xi_i = \{\xi_i(t)\}_{t\in T}$ are independent and identically distributed stochastic processes, where T is an arbitrary index set. Consider the coordinate–wise maximum process

$$M_n(t) := \bigvee_{1\le i\le n} \xi_i(t) \equiv \max_{1\le i\le n} \xi_i(t),\ \ t\in T.$$

Suppose that for suitable non–random sequences $a_n(t) > 0$ and $b_n(t),\ t\in T$, we have

$$\Big\{\frac{1}{a_n(t)} M_n(t) - b_n(t)\Big\}_{t\in T} \stackrel{f.d.d.}{\longrightarrow} \{X(t)\}_{t\in T}, \tag{1}$$

as $n\to\infty$, for some *non–degenerate* limit process $X = \{X(t)\}_{t\in T}$, where $\stackrel{f.d.d.}{\longrightarrow}$ denotes convergence of the finite–dimensional distributions. The processes that appear in the limit of (1) are *max–stable* (see e.g. Proposition 5.9 in [18]). The classical results of Fisher & Tippett and Gnedenko indicate that the marginal distributions of X are one of three types of *extreme value distributions*: Fréchet, Gumbel, or reversed Weibul. The dependence structure of the limit X, however, can be quite intricate. Our main focus here is on the study of various aspects of the dependence structure of max–stable processes.

Max–stable processes have a peculiar property, namely their dependence structure is in a sense invariant to the type of their marginals. More precisely, consider a process $X = \{X(t)\}_{t\in T}$ and its transformed version $h\circ X = \{h_t(X_t)\}_{t\in T}$, where $h = \{h_t(\cdot)\}_{t\in T}$ is a collection of deterministic functions, strictly increasing on their domains. It turns out that if X is a max–stable process and if the marginals of $h\circ X$ are extreme value distributions, then the transformed process $h\circ X$ is also max–stable (see e.g. Proposition 5.10 in [18]). That is, one does not encounter max–stable processes with more rich dependence structures if one allows for the marginal distributions of X to be of different types. Thus, for convenience and without loss of generality, we shall focus here on max–stable processes $X = \{X(t)\}_{t\in T}$ with $\alpha-$Fréchet marginals. A random variable ξ is said to have the $\alpha-$Fréchet distribution if:

$$\mathscr{P}\{\xi\le x\} = \exp\{-\sigma^{\alpha} x^{-\alpha}\},\ \ (x>0),$$

for some $\sigma>0$ and $\alpha>0$. The parameter $\sigma>0$ plays the role of a scale coefficient, and thus, by analogy with the convention for sum–stable processes, we shall use the notation

$$\|\xi\|_{\alpha} := \sigma.$$

Note that here $\|\cdot\|_{\alpha}$ is not the usual $L^{\alpha}-$norm but we have $\|c\xi\|_{\alpha} = c\|\xi\|_{\alpha}$, for all $c>0$. The $\alpha-$Fréchet laws have heavy Pareto–like tails with tail exponent $\alpha>0$, that is,

$$\mathscr{P}\{\xi > x\} \sim \|\xi\|_\alpha^\alpha x^{-\alpha}, \quad \text{as } x \to \infty.$$

Therefore, the $p-$moment $(p > 0)$, $\mathbb{E}\xi^p < \infty$ is finite if and only if $p < \alpha$.

It is convenient to introduce the notion of an $\alpha-$Fréchet process. Namely, the process $X = \{X(t)\}_{t\in T}$ is said to be an $\alpha-$Fréchet process if all (positive) max–linear combinations of $X(t)$'s:

$$\bigvee_{1\le i\le k} a_i X(t_i), \;\; a_i \ge 0,\, t_i \in T,$$

are $\alpha-$Fréchet random variables.

It turns out that the max–stable processes with $\alpha-$Fréchet marginals are precisely the *$\alpha-$Fréchet processes* (see, de Haan [4]). Therefore, in the sequel we shall use the terms *Fréchet processes* and *max–stable processes with Fréchet marginals* interchangeably.

Let now X be an $\alpha-$Fréchet process. The structure of the finite–dimensional distributions of X is already known, In fact, we have the following explicit formula of the finite–dimensional distributions of X:

$$\mathscr{P}\{X_{t_i} \le x_i,\, 1 \le i \le k\} = \exp\Big\{-\int_0^1 \bigvee_{1\le i\le k} \Big(\frac{f_{t_i}(u)}{x_i}\Big)^\alpha du\Big\},\;\; (x_i > 0,\, 1\le i\le k), \quad (2)$$

where $f_{t_i}(u) \ge 0$ are suitable Borel functions, such that $\int_0^1 f_{t_i}^\alpha(u)du < \infty$, for $1 \le i \le k$. The $f_{t_i}(u)$'s are known as *spectral functions* of the max–stable vector $(X_{t_i})_{1\le i\le k}$ and even though they are not unique, they will play an important role in our representations of max–stable processes. Observe for example, that (2) yields

$$\mathscr{P}\Big\{\bigvee_{1\le i\le k} a_i X_{t_i} \le x\Big\} = \exp\Big\{-\int_0^1 \Big(\bigvee_{1\le i\le k} a_i f_{t_i}(u)\Big)^\alpha du\, x^{-\alpha}\Big\},$$

and therefore $\vee_{1\le i\le k} a_i X_{t_i}$ is an $\alpha-$Fréchet variable with scale coefficient

$$\Big\|\bigvee_{1\le i\le k} a_i X_{t_i}\Big\|_\alpha^\alpha = \int_0^1 \Big(\bigvee_{1\le i\le k} a_i f_{t_i}(u)\Big)^\alpha du.$$

Thus, the knowledge of the spectral functions $\{f_t(u)\}_{t\in T} \subset L_+^\alpha([0,1],du)$ allows us to handle all finite–dimensional distributions of the process X.

One can alternatively express the finite–dimensional distributions in (2) by using the *spectral measure* of the vector $(X_{t_i})_{1\le i\le k}$. Namely, consider an arbitrary norm $\|\cdot\|$ in $\mathbb{R}^k$ and let $\mathbb{S}_+ := \{w = (w_i)_{1\le i\le k} : w_i \ge 0, \|w\| = 1\}$ be the *non–negative* unit sphere in $\mathbb{R}^k$. We then have

$$\mathscr{P}\{X_{t_i} \le x_i,\, 1 \le i \le k\} = \exp\Big\{-\int_{\mathbb{S}_+} \bigvee_{1\le i\le k} \frac{w_i}{x_i^\alpha} \nu_{\mathbb{S}_+}(dw)\Big\}, \quad (3)$$

where $\nu_{\mathbb{S}_+}(dw)$ is a finite measure on $\mathbb{S}_+$.

The two types of representations in (2) and (3) have both advantages and disadvantages depending on the particular setting. The finite measure $\nu_{\mathbb{S}_+}$ associated with the max–stable vector $(X_{t_i})_{1\le i\le k}$ is said to be its *spectral measure* and it is uniquely determined. Thus, when handling max–stable random vectors of fixed dimension, the spectral measure is a natural object to use and estimate statistically. On the other hand, when handling stochastic processes, one encounters spectral measures defined on spaces of different dimensions, which may be hard to reconcile. In such a setting, it may be more natural to use representations based on a set of spectral functions $\{f_t\}_{t\in T}$, which are ultimately defined on the same measure space.

More details and the derivations of (2) and (3) can be found in [18]. Novel perspectives to spectral measures on 'infinite dimensional' spaces are adopted in Gine, Hahn and Vatan [13] , de Haan and Lin [7, 8] and de Haan and Ferreira [6]. Hult and Lindskog [14] develop powerful new tools based on the related notion of regular variation in infinite–dimensional function spaces.

Let now $X = \{X(t)\}_{t\in T}$ with $T = \mathbb{R}$ or $\mathbb{Z}$ be a stationary $\alpha-$Fréchet process. From statistical perspective, it is important to know whether the process X is ergodic, mixing, or non–ergodic. Despite the abundance of literature on max–stable processes, the problem of ergodicity had not been explored until recently. To the best of our knowledge only Weintraub in [27] addressed it indirectly by introducing mixing conditions through certain measures of dependence. Recently, in [25], by following the seminal work of [3], we obtained necessary and sufficient conditions for the process X to be ergodic or mixing. In the case of mixing, these conditions take a simple form and are easy to check for many particular cases of max–stable processes.

The goal of this chapter is to primarily review results estabilished in [24, 25]. This is done in Sections 2 and 3 below. These results are then illustrated and applied to some statistical problems in Seciton 4. Section 4.2 contains new results on the consistency of extremal index estimators for stationary max–stable processes.

2 Representations of Max–Stable Processes

Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be an $\alpha-$Fréchet process ($\alpha > 0$) indexed by $\mathbb{R}$. As indicated above all finite–dimensional distributions of X can be expressed in terms of a family of *spectral functions* in

$$L^\alpha_+([0,1],du) = \{f : [0,1] \to \mathbb{R}_+ \,:\, \int_{[0,1]} f^\alpha(u)du < \infty\}.$$

The seminal paper of de Haan [5] shows that provided $X = \{X(t)\}_{\mathbb{R}}$ is continuous in probability, there exists a family of *spectral functions* $\{f_t(u)\}_{t\in\mathbb{R}} \subset L^\alpha_+(du)$ indexed by $\mathbb{R}$, which yield (2). This was done from the appealing perspective of Poisson point processes. Namely, let $X = \{X(t)\}_{t\in\mathbb{R}}$ be continuous in probability. Then, there exist a collection of non–negative functions $\{f_t(u)\}_{t\in\mathbb{R}} \subset L_+^\alpha([0,1],du)$, such that

$$\{X(t)\}_{t\in T} \stackrel{d}{=} \Big\{ \bigvee_{i\in\mathbb{N}} \frac{f_t(U_i)}{\varepsilon_i^{1/\alpha}} \Big\}_{t\in T}, \tag{4}$$

where $\{(U_i,\varepsilon_i)\}_{i\in\mathbb{N}}$ is a Poisson point process on $[0,1]\times[0,\infty]$ with intensity $du \times ds$, and where $\stackrel{d}{=}$ means equality in the sense of finite–dimensional distributions.

We now present an alternative but ultimately equivalent approach to representing $\alpha-$Fréchet max–stable processes developed in Stoev and Taqqu [24]. It is based on the notion of extremal integrals with respect to $\alpha-$Fréchet sup–measures.

Definition 0.1. Let $\alpha > 0$ and let $(E,\mathscr{E},\mu)$ be a measure space. A random set–function M_α, defined on $\mathscr{E}$, is said to be an $\alpha-$Fréchet random sup–measure with control measure μ if the following conditions hold:

(i) For all disjoint $A_j \in \mathscr{E},\ 1 \le j \le n$, the random variables $M_\alpha(A_j),\ 1 \le j \le n$ are independent.

(ii) For all $A \in \mathscr{E}$, the random variable $M_\alpha(A)$ is $\alpha-$Fréchet, with scale coefficient $\|M_\alpha(A)\|_\alpha = \mu(A)^{1/\alpha}$, i.e.

$$\mathscr{P}\{M_\alpha(A) \le x\} = \exp\{-\mu(A)x^{-\alpha}\},\ \ x > 0. \tag{5}$$

(iii) For all disjoint $A_j \in \mathscr{E},\ j \in \mathbb{N}$,

$$M_\alpha(\cup_{j\in\mathbb{N}} A_j) = \bigvee_{j\in\mathbb{N}} M_\alpha(A_j), \quad \text{almost surely.} \tag{6}$$

By convention, we set $M_\alpha(A) = \infty$ if $\mu(A) = \infty$.

Condition *(i)* in the above definition means that the random measure is *independently scattered* i.e. it assigns independent random variables to disjoint sets and Condition *(ii)* shows that the scale of $M_\alpha(A)$ is governed by the deterministic *control measure* μ of M_α. Relation (6), on the other hand, indicates that the random measure M_α is sup–additive, rather than additive. This is the fundamental difference between the usual additive random measures and the sup–measures. For more general studies of sup–measures see [26]. The important work of Norberg [17] unveils the connections between random sup–measures, the theory of random sets, and random capacities. Here, the focus is on the concrete and simple case of $\alpha-$Fréchet sup–measures, most relevant to the study of max–stable processes.

As shown in Proposition 2.1 of [24] (by using the Kolmogorov's extension theorem) for any measure space $(E,\mathscr{E},\mu)$ one can construct an $\alpha-$Fréchet random sup–measure M_α with control measure μ, on a sufficiently rich probability space. Given such a random measure M_α on $(E,\mathscr{E},\mu)$, one can then define the *extremal integral* of a non–negative deterministic function with respect to M_α as follows. Consider first a non–negative simple function $f(u) = \sum_{i=1}^n a_i 1_{A_i}(u)$, $a_i \ge 0$ with disjoint A_i's and define the extremal integral of f as

$$I(f) \equiv \int^{e}_{E} f dM_\alpha := \bigvee_{i=1}^{n} a_i M_\alpha(A_i),$$

i.e. the sum in the typical definition of an integral is replaced by a maximum. Since the $M_\alpha(A_i)$'s are independent and $\alpha-$Fréchet, Relation (5) implies

$$\mathscr{P}\{I(f) \le x\} = \exp\Big\{ - \int_E f^\alpha d\mu \; x^{-\alpha} \Big\}, \;\; x > 0.$$

The following properties are immediate (see e.g. Proposition 2.2 in [24]):

Properties:

- For all non–negative simple functions f, the extremal integral $\int^{e}_{E} f dM_\alpha$ is an $\alpha-$Fréchet random variable with scale coefficient

$$\Big\| \int^{e}_{E} f dM_\alpha \Big\|_\alpha = \Big(\int_E f^\alpha d\mu \Big)^{1/\alpha}. \tag{7}$$

- *(max–linearity)* For all $a, b \ge 0$ and all non–negative simple functions f and g, we have

$$\int^{e}_{E} (af \vee bg) dM_\alpha = a \int^{e}_{E} f dM_\alpha \vee b \int^{e}_{E} g dM_\alpha, \quad \text{almost surely.} \tag{8}$$

- *(independence)* For all simple functions f and g, $\int^{e}_{E} f dM_\alpha$ and $\int^{e}_{E} g dM_\alpha$ are independent *if and only if* $fg = 0$, $\mu-$almost everywhere.

Relation (8) shows that the extremal integrals are *max–linear*. Note that for any collection of non–negative simple functions f_i and $a_i \ge 0, \; 1 \le i \le n$, we have that

$$\bigvee_{1 \le i \le n} a_i \int^{e}_{E} f_i dM_\alpha = \int^{e}_{E} \Big(\bigvee_{1 \le i \le n} a_i f_i \Big) dM_\alpha$$

is $\alpha-$Fréchet. This shows that the set of extremal integrals of non–negative simple functions is *jointly $\alpha-$Fréchet*, i.e. the distribution of $(I(f_i))_{1 \le i \le n}$ is multivariate max–stable. It turns out that one can metrize the convergence in probability in the spaces of jointly $\alpha-$Fréchet random variables by using the following metric:

$$\rho_\alpha(\xi, \eta) := 2\|\xi \vee \eta\|_\alpha^\alpha - \|\xi\|_\alpha^\alpha - \|\eta\|_\alpha^\alpha. \tag{9}$$

If now $\xi = \int^{e}_{E} f dM_\alpha$ and $\eta = \int^{e}_{E} g dM_\alpha$, for some simple functions $f \ge 0$ and $g \ge 0$, we obtain

$$\rho_\alpha(\xi, \eta) = 2 \int_E (f^\alpha \vee g^\alpha) d\mu - \int_E (f^\alpha \vee g^\alpha) d\mu \int_E (f^\alpha \vee g^\alpha) d\mu \equiv \int_E |f^\alpha - g^\alpha| d\mu. \tag{10}$$

By using this relationship one can extend the definition of the extremal integral $\int^{e}_{E} f dM_\alpha$ to integrands in the space $L_+^\alpha(\mu) \equiv L_+^\alpha(E, \mathscr{E}, \mu)$ of all non–negative deterministic f's with $\int_E f^\alpha d\mu < \infty$. Moreover, the above properties of the extremal integrals remain valid for all such integrands.

To complete the picture, consider the space

$$\mathscr{M}_\alpha = \overline{\vee - \mathrm{span}}^P \{M_\alpha(A) \,:\, A \in \mathscr{E}\}$$

of jointly $\alpha-$Fréchet variables containing all max–linear combinations of $M_\alpha(A)$'s and their limits in probability. One can show that $(\mathscr{M}_\alpha, \rho_\alpha)$ is a complete metric space and ρ_α in (9), as indicated above, metrizes the convergence in probability. Let also $L_+^\alpha(\mu)$ be equipped with the metric

$$\rho_\alpha(f,g) := \int_E |f^\alpha - g^\alpha| d\mu. \tag{11}$$

Then, relation (10) implies that the extremal integral

$$I : L_+^\alpha(\mu) \to \mathscr{M}_\alpha$$

is a *max–linear isometry* between the metric spaces $(L_+^\alpha(\mu), \rho_\alpha)$ and $(\mathscr{M}_\alpha, \rho_\alpha)$, which is one-to-one and onto. Thus, in particular if $\xi_n := {\int^e}_E f_n dM_\alpha$ and $\xi = {\int^e}_E f dM_\alpha$, $f_n, f \in L_+^\alpha(\mu)$, we have that

$$\xi_n \xrightarrow{P} \xi, \quad \text{as } n \to \infty, \ \textit{if and only if} \ \ \rho_\alpha(f_n, f) = \int_E |f_n^\alpha - f^\alpha| d\mu \longrightarrow 0, \ \text{as } n \to \infty.$$

For more details see Stoev and Taqqu [24].

The so developed extremal integrals provide us with tools to construct and handle max–stable processes. Indeed, for any collection of deterministic integrands $\{f_t\}_{t\in T} \subset L_+^\alpha(\mu)$, one can define

$$X(t) := {\int^e}_E f_t dM_\alpha, \ \ t \in T. \tag{12}$$

The resulting process $X = \{X(t)\}_{t\in T}$ is $\alpha-$Fréchet and in view of (7) and (8), we obtain

$$\Big\| \bigvee_{1\le i\le k} a_i X(t_i) \Big\|_\alpha = \Big(\int_E \bigvee_{1\le i \le k} a_i^\alpha f_{t_i}^\alpha d\mu \Big)^{1/\alpha},$$

where $a_i \ge 0$. Therefore, with $a_i := 1/x_i, \ 1 \le i \le k$, we obtain

$$\mathscr{P}\{X(t_i) \le x_i, \ 1 \le i \le k\} = \exp\Big\{ - \int_E \bigvee_{1\le i\le k} \frac{f_{t_i}^\alpha}{x_i^\alpha} d\mu \Big\}.$$

This shows that the f_t's play the role of the spectral functions of the max–stable process X as in (2) but now these functions can be defined over an arbitrary measure space $(E, \mathscr{E}, \mu)$. Thus, by choosing suitable families of integrands (kernels) f_t's, one can explicitly model and manipulate a variety of max–stable processes. For example, if $E \equiv \mathbb{R}$ is the real line equipped with the Lebesgue measure, one can define the *moving maxima* processes:

$$X(t) := {\int^{e}}_{\mathbb{R}} f(t-u) M_\alpha(du), \ \ t \in \mathbb{R}, \tag{13}$$

where $f \geq 0$, $\int_{\mathbb{R}} f^\alpha(u)du < \infty$, and where M_α is an $\alpha-$Fréchet random sup–measure with the Lebesgue control measure. More generally, we define a *mixed moving maxima* process or field as follows:

$$X(t) \equiv X(t_1,\cdots,t_d) := {\int^{e}}_{\mathbb{R}^d \times V} f(t-u,v) M_\alpha(du,dv), \ \ t = (t_i)_{i=1}^d \in \mathbb{R}^d, \tag{14}$$

where $f \geq 0$, $\int_{\mathbb{R}^d \times V} f^\alpha(u,v)du\nu(dv) < \infty$ and where now the random sup–measure M_α is defined on the product space $\mathbb{R}^d \times V$ and has control measure $du \times \nu(dv)$, for some measure $\nu(dv)$ on the set V.

Further, interesting classes of processes are obtained when the measure space $(E,\mathscr{E},\mu)$ is viewed as another *probability space* and the collection of deterministic integrands $\{f_t\}_{t\in T}$ is then interpreted as a stochastic process on this probability space. This leads to certain *doubly stochastic* max–stable processes, whose dependence structure is closely related to the stochastic properties of the integrands f_t's. For more details, see Section 4.1 below.

Let $X = \{X(t)\}_{t\in T}$ be an $\alpha-$Fréchet process. As shown in [24], the representation in (4) (or equivalently in (12) with $(E,\mathscr{E},\mu) \equiv ((0,1),\mathscr{B}_{(0,1)},dx))$ is possible if and only if the process X is *separable in probability*. The max–stable process X is said to be separable in probability if, there exists a countable set $\mathscr{I} \subset T$, such that for all $t \in T$, the random variable X_t is a limit in probability of max–linear combinations of the type $\max_{1\leq i\leq n} a_i X_{s_i}$, with $s_i \in \mathscr{I}$ and $a_i \geq 0, \ 1 \leq i \leq n$. Clearly, if $T \equiv \mathbb{R}$ and X is continuous in probability, then it is also separable in probability and therefore it has the representation (4) with suitable f_t's (see Theorem 3 in [5]). On the other hand, even if X is not separable in probability, it may still be possible to express as in (12) provided that the measure space $(E,\mathscr{E},\mu)$ is sufficiently rich.

Remarks:

1. The representation (4) is similar in spirit to the Le Page, Woodroofe & Zinn's series representation for sum–stable processes. Namely, let $X = \{X(t)\}_{t\in\mathbb{R}}$ be an $\alpha-$stable process, which is separable in probability. For simplicity, suppose that X is totally skewed to the right and such that $0 < \alpha < 1$. Then, by Theorems 3.10.1 and 13.2.1 in Samorodnitsky and Taqqu [22], we have

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \Big\{ \sum_{i\in\mathbb{N}} \frac{f_t(U_i)}{\varepsilon_i^{1/\alpha}} \Big\}_{t\in\mathbb{R}}, \tag{15}$$

   where $\{f_t(u)\}_{t\in\mathbb{R}} \subset L^\alpha([0,1],du)$, and $\{(U_i,\varepsilon_i)\}_{i\in\mathbb{N}}$ is a standard Poisson point process on $[0,1]\times[0,\infty]$. Relation (15) is analogous to (4) where the sum is replaced by a maximum and only non–negative spectral functions $f_t(\cdot)$'s are considered.

2. The representation (4) is particularly convenient when studying the path properties of max–stable processes. It was used in [20] to establish necessary and sufficient conditions for the continuity of the paths of max–stable processes.
3. The moving maxima (M2) (in discrete time) were first considered by Deheuvels [11]. Zhang and Smith [29] studied further the discrete–time multivariate mixed moving maxima (M4 processes) generated by sequences of independent $\alpha-$Fréchet variables.

3 Ergodic Properties of Stationary Max–stable Processes

Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a (strictly) stationary $\alpha-$Fréchet process as in (12). To be able to discuss ergodicity in continuous time, we shall suppose that X is measurable. This is not a tall requirement since any continuous in probability process has a measurable modification. All results are valid and in fact have simpler versions in discrete time. We first recall the definitions of ergodicity and mixing in our context.

One can introduce a group of *shift operators* $S_\tau,\ \tau\in\mathbb{R}$, which acts on all random variables, measurable with respect to $\{X(t)\}_{t\in\mathbb{R}}$. Namely, for all $\xi = g(X_{t_1},\cdots,X_{t_k})$, we define

$$S_\tau(\xi) := g(X_{\tau+t_1},\cdots,X_{\tau+t_k}),$$

where $g:\mathbb{R}^k\to\mathbb{R}$ is a Borel function. The definition of the S_τ's can be extended to the class of all $\{X_t\}_{t\in\mathbb{R}}-$measurable random variables. Note also that $S_t\circ S_s = S_{t+s},\ t,s\in\mathbb{R}$. Clearly, the shift operators map indicator functions to indicator functions and therefore one can define $S_\tau(A) := \{S_\tau(1_A)=1\}$, for all events $A\in\sigma\{X_t,\ t\in\mathbb{R}\}$. These mappings are well–defined and unique up to equality almost surely (for more details, see e.g. Ch. IV in [21]).

The stationarity of the process X implies that the shifts S_τ's are *measure preserving*, i.e.

$$\mathscr{P}(S_\tau(A)) = \mathscr{P}(A),\quad \text{for all } A\in\sigma\{X_t,\ t\in\mathbb{R}\}.$$

Let now $\mathscr{F}_{inv}$ denote the $\sigma-$algebra of *shift–invariant* sets, namely, the collection of all $A\in\sigma\{X_t,\ t\in\mathbb{R}\}$ such that $\mathscr{P}(A\Delta S_\tau(A))=0$ for all $\tau\in\mathbb{R}$.

Recall that the process X is said to be *ergodic* if the shift–invariant $\sigma-$algebra $\mathscr{F}_{inv}$ is trivial, i.e. for all $A\in\mathscr{F}_{inv}$, we have that either $\mathscr{P}(A)=0$ or $\mathscr{P}(A)=1$. On the other hand, X is said to be *mixing* if

$$\mathscr{P}(A\cap S_\tau(B)) \longrightarrow \mathscr{P}(A)\mathscr{P}(B),\quad \text{as } \tau\to\infty,$$

for all $A,\ B\in\sigma\{X_t,\ t\in\mathbb{R}\}$.

It is easy to show that mixing implies ergodicity. Furthermore, ergodicity has important statistical implications. Indeed, fix $t_i\in\mathbb{R},\ 1\le i\le k$ and let $h:\mathbb{R}^k\to\mathbb{R}$ be a Borel measurable function such that $\mathbb{E}|h(X(t_1),\cdots,X(t_k))|<\infty$. The Birgkhoff's ergodic theorem implies that, as $T\to\infty$,

$$\frac{1}{T}\int_0^T h(X(\tau+t_1),\cdots,X(\tau+t_k))d\tau \longrightarrow \xi,$$

almost surely and in the L^1-sense, where $\mathbb{E}\xi = \mathbb{E}h(X(t_1),\cdots,X(t_k)$. The limit ξ is shift–invariant, that is $S_\tau(\xi) = \xi$, almost surely, for all $\tau > 0$, and therefore ξ is measurable with respect to $\mathscr{F}_{inv}$. Hence, if the process X is ergodic, then the limit ξ is constant, and we have the following strong law of large numbers:

$$\frac{1}{T}\int_0^T h(X(\tau+t_1),\cdots,X(\tau+t_k))d\tau \stackrel{a.s.\ \&\ L^1}{\longrightarrow} \mathbb{E}h(X(t_1),\cdots,X(t_k)), \quad \text{as } T\to\infty. \tag{16}$$

In fact, one can show that X is ergodic if and only if Relation (16) holds, for all such Borel functions h and all $k\in\mathbb{N}$. For more details on ergodicity and mixing, see e.g. [21].

Relation (16) indicates the importance of knowing whether a process X is ergodic or not. Ergodicity implies the strong consistency of a wide range of statistics based on the empirical time–averages in (16).

Our goal in this section is to review necessary and sufficient conditions for the ergodicity or mixing of the process X. These conditions will be formulated in terms of the deterministic integrands $\{f_t\}_{t\in\mathbb{R}} \subset L_+^\alpha(\mu)$ and the important notion of *max–linear isometry*.

Definition 0.2. A mapping $U : L_+^\alpha(\mu) \to L_+^\alpha(\mu)$ is said to be a max–linear isometry, if

(i) For all $f,g \in L_+^\alpha(\mu)$ and $a,b \geq 0$,

$$U(af \vee bg) = aU(f) \vee bU(g), \quad \mu-\text{a.e.}$$

(ii) For all $f \in L_+^\alpha(\mu)$,

$$\|U(f)\|_{L_+^\alpha(\nu)} = \|f\|_{L_+^\alpha(\mu)}.$$

Consider a collection of max–linear isometries $U_t : L_+^\alpha(\mu) \to L_+^\alpha(\mu)$, which forms a group with respect to composition, indexed by $t\in\mathbb{R}$, i.e. $U_t \circ U_s = U_{t+s},\ t,s\in\mathbb{R}$ and $U_0 \equiv \mathrm{id}_E$.

Now, fix $f_0 \in L_+^\alpha(\mu)$, let $f_t := U_t(f_0)$, $t\in\mathbb{R}$, and consider the $\alpha-$Fréchet process

$$X(t) := \int_E^{e} U_t(f_0)dM_\alpha,\ \ t\in\mathbb{R}. \tag{17}$$

Definition 0.2 and the group structure of the U_t's readily implies that $X = \{X(t)\}_{t\in\mathbb{R}}$ is stationary. Indeed,

$$\mathscr{P}\{X(\tau+t_i)\leq x_i,\ 1\leq i\leq k\} = \exp\Big\{-\int_E \bigvee_{1\leq i\leq k} \frac{U_\tau(f_{t_i})^\alpha}{x_i^\alpha} d\mu\Big\}$$

$$= \exp\Big\{-\int_E U_\tau\Big(\bigvee_{1\leq i\leq k}\frac{f_{t_i}a}{x_i^\alpha}\Big)^\alpha d\mu\Big\} = \mathscr{P}\{X(t_i)\leq x_i,\ 1\leq i\leq k\}.$$

For example, in the particular the case of moving maxima defined in (13), we have that (17) holds, where $U_t(g)(u) = g(t+u)$ is the simple translation in time and $f_0(u) = f(-u)$, for all $u \in \mathbb{R}$.

The representation in (17) is valid for a large class of stationary max–stable processes. In fact, as shown in Stoev [25], the above defined max–linear isometries are precisely the *pistons* of de Haan and Pickands [9]. Thus, by Theorem 6.1 in de Haan and Pickands [9], Relation (17) holds for all *continuous in probability* $\alpha-$Fréchet processes.

The following two results, established in Stoev [25], provide necessary and sufficient conditions for the ergodicity and mixing of the process X, respectively.

Theorem 3.1 (Theorem 3.2 in [25]). *Let X be a measurable $\alpha-$Fréchet process, defined by* (17)*. The process X is ergodic, if and only if, for some (any) $p > 0$,*

$$\frac{1}{T}\int_0^T \|U_\tau g \wedge g\|_{L^\alpha(\mu)}^p d\tau \longrightarrow 0, \tag{18}$$

as $T \to \infty$, for all $g \in F_U(f_0)$, where $a \wedge b = \min\{a,b\}$. Here

$$F_U(f_0) := \overline{\vee\text{-span}}\{U_t(f_0),\ t \in \mathbb{R}\},$$

is the set of all max–linear combinations of the $U_t(f_0)$'s, closed with respect to the metric ρ_α in (11)*.*

The corresponding necessary and sufficient condition for mixing is as follows

Theorem 3.2 (Theorem 3.3 in [25]). *Let X be a measurable $\alpha-$Fréchet process, defined by* (17)*. The process X is mixing, if and only if,*

$$\|U_\tau h \wedge g\|_{L^\alpha(\mu)} \longrightarrow 0, \quad \text{as } \tau \to \infty, \tag{19}$$

for all $g \in F_U^-(f_0) := \overline{\vee\text{-span}}\{U_t(f_0),\ t \le 0\}$ and $h \in F_U^+(f_0) := \overline{\vee\text{-span}}\{U_t(f_0),\ t \ge 0\}$.

Although these results provide complete characterization of the ergodic and/or mixing $\alpha-$Fréchet processes, they are hard to use in practice. This is because the conditions (18) and/or (19) should be verified for arbitrary elements g and/or h in the max–linear spaces $F_U(f_0)$ and/or $F_U^{\pm}(f_0)$. Fortunately, in the case of mixing, these conditions can be formulated simpler in terms of a natural *measure of dependence*. Namely, for any $\xi = \int^e_E f dM_\alpha$ and $\eta = \int^e_E g dM_\alpha$, $f,g \in L_+^\alpha(\mu)$ define

$$d(\xi,\eta) := \|\xi\|_\alpha^\alpha + \|\eta\|_\alpha^\alpha - \|\xi \vee \eta\|_\alpha^\alpha.$$

Observe that since $\|\xi\|_\alpha^\alpha = \int_E f^\alpha d\mu$ and $\|\eta\|_\alpha^\alpha = \int_E g^\alpha d\mu$, we have

$$d(\xi,\eta) = \int_E \Big(f^\alpha + g^\alpha - f^\alpha \vee g^\alpha\Big) d\mu \equiv \int_E f^\alpha \wedge g^\alpha d\mu. \tag{20}$$

Note that $d(\xi,\eta) = 0$ if and only if the random variables ξ and η are independent. This observation and the intuition about extremal integrals, suggest that the quantity $d(\xi,\eta)$ can be interpreted as a measure of dependence between ξ and η. The following result established in Stoev [25] shows that $d(\xi,\eta)$ indeed plays such a role.

Theorem 3.3 (Theorem 3.4 in [25]). *Let X be a stationary and continuous in probability α−Fréchet process. The process X is mixing if and only if $d_\alpha(X(\tau),X(0)) \to 0$, as $\tau \to \infty$.*

Remarks:

1. Observe that by Theorem 3.2 and Relation (20), the condition $d(X_\tau,X_0) \to 0,\ \tau \to \infty$ is necessary for X to be mixing. Surprisingly, Theorem 3.3 implies that this condition is also sufficient. In many situations it is easy to check whether the dependence coefficient $d(X_\tau,X_0)$ vanishes as the lag τ tends to infinity. The explicit knowledge of the max–linear isometries U_t in (17) is not necessary.
2. The recent monograph of Dedecker *et al.* [10] provides many classes of remarkably flexible measures of dependence. To the best of my knowledge, these measures of dependence have not yet been studied in the context of max–stable processes. The knowledge of sharp inequalities involving these measures of dependence could lead to many interesting statistical results.

In the following section we will illustrate further the above results with concrete examples and applications.

4 Examples and Statistical Applications

4.1 Ergodic Properties of Some Max–Stable Processes

• *(Mixed Moving Maxima)* It is easy to show that all *moving maxima* and *mixed moving maxima processes* defined in (13) and (14) are mixing. Indeed, let

$$X(t) := {\int^{e}}_{\mathbb{R}\times V} f(t-u,v)M_\alpha(du,dv),\ \ t\in\mathbb{R},$$

for some $f \in L^\alpha_+(du,\nu(dv)),\ \alpha > 0$ and observe that

$$\begin{aligned} d(X(t),X(0)) &= \int_{\mathbb{R}\times V} f(t+u,v)^\alpha \wedge f(u,v)^\alpha du\nu(dv) \\ &\le 2\int_{|u|\ge t/2}\Big(\int_V f(u,v)^\alpha \nu(dv)\Big)du. \end{aligned} \tag{21}$$

The last inequality follows from the fact that for all $u\in\mathbb{R}$, and $t>0$, either $|u|\ge t/2$ or $|t+u|\ge t/2$ and therefore,

$$f(t+u,v)^{\alpha} \wedge f(u,v)^{\alpha} \le f(t+u,v)^{\alpha} 1_{\{|t+u| \ge t/2\}} + f(u,v)^{\alpha} 1_{\{|u| \ge t/2\}}.$$

The inequality (21) and the integrability of f^{α} imply that $d(X(t),X(0)) \to 0$, as $t \to \infty$. This, in view of Theorem 3.3, implies that the mixed moving maxima process X is mixing.

• *(Doubly Stochastic Processes)* As in the theory of sum–stable processes (see e.g. the monograph of Samorodnitsky and Taqqu [22]), we can associate a max–stable $\alpha-$Fréchet processes with any positive stochastic process $\xi = \{\xi(t)\}_{t \in T}$ with $\mathbb{E}\xi(t)^{\alpha} < \infty$. Namely, suppose that M_{α} is a random sup–measure on a measure space $(E, \mathscr{E}, \mu)$, where the control measure μ is now a *probability measure* (i.e. $\mu(E) = 1$). Any collection of spectral functions $\{f(t,u)\}_{t \in T} \subset L^{\alpha}_{+}(E, \mu(du))$ may be viewed as a stochastic process, defined on the probability space (E, μ). Conversely, a *non–negative* stochastic process $\xi = \{\xi(t)\}_{t \in T}$, defined on $(E, \mathscr{E}, \mu)$, and such that $\mathbb{E}_{\mu}\xi(t)^{\alpha} = \int_E \xi(t,u)^{\alpha} \mu(du) < \infty$ may be used to define an $\alpha-$Fréchet process as follows:

$$X(t) := {\int_E}^{e} \xi(t,u) M_{\alpha}(du), \ \ t \in T. \tag{22}$$

The $\alpha-$Fréchet process $X = \{X(t)\}_{t \in T}$ will be called *doubly stochastic*. Note that from the perspective of the random sup–measure M_{α}, the integrands $\xi(t)$'s are *non–random* since they 'live' on a different probability space. The main benefit from this new way of defining a max–stable process X is that one can use the properties of the stochastic process $\xi = \{\xi(t)\}_{t \in T}$ to establish the properties of the $\alpha-$Fréchet process X.

For example, let $\xi = \{\xi(t)\}_{t \in \mathbb{R}}$ be a strictly stationary, non–negative process on $(E, \mathscr{E}, \mu)$ such that $\mathbb{E}_{\mu}\xi(t)^{\alpha} < \infty$. We then have that X in (22) is also stationary. Indeed, for all $t_i \in \mathbb{R}$, $x_i > 0$, $1 \le i \le n$, and $h \in \mathbb{R}$, we have

$$\mathscr{P}\{X(t_i + h) \le x_i, \ 1 \le i \le n\} = \exp\Big\{ - \mathbb{E}_{\mu}\Big(\bigvee_{1 \le i \le n} \xi(t_i + h)/x_i \Big)^{\alpha} \Big\}$$
$$= \exp\Big\{ - \mathbb{E}_{\mu}\Big(\bigvee_{1 \le i \le n} \xi(t_i)/x_i \Big)^{\alpha} \Big\} = \mathscr{P}\{X(t_i) \le x_i, \ 1 \le i \le n\},$$

where in the second equality above we used the stationarity of ξ. Borrowing terminology from theory of sum–stable processes (see e.g. [3]), if the process ξ is stationary, we call the $\alpha-$Fréchet process X *doubly stationary*. The following result shows the perhaps surprising fact that if the process ξ is *mixing*, then the doubly stationary process X is *non–ergodic*.

Proposition 0.1. *Let $X = \{X(t)\}_{t \in \mathbb{R}}$ be a doubly stationary process defined as in* (22) *with non–zero $\xi(t)$'s. If the stationary process $\xi = \{\xi(t)\}_{t \in \mathbb{R}}$ is mixing, then X is non–ergodic.*

Proof. Consider the quantity

$$d(X(t),X(0)) = \int_E \Big(\xi(t,u) \wedge \xi(0,u) \Big)^{\alpha} \mu(du) \equiv \mathbb{E}_{\mu}(\xi(t)^{\alpha} \wedge \xi(0)^{\alpha}).$$

We will show that $\liminf_{t\to\infty} d(X(t),X(0)) = c > 0$. This would then imply that the time–averages in Theorem 3.1 do not vanish, and hence X is not ergodic.

Observe that since ξ is mixing, for all Borel sets $A,\ B \subset \mathbb{R}$, we have

$$\mathscr{P}\{\xi(t) \in A,\ \xi(0) \in B\} \longrightarrow \mathscr{P}\{\xi(t) \in A\}\mathscr{P}\{\xi(0) \in B\}, \quad \text{as } t \to \infty.$$

Consider the intervals $A = B = (\varepsilon^{1/\alpha},\infty)$, for some $\varepsilon > 0$, and note that the last relation is equivalent to

$$\mathscr{P}\{\xi(t)^\alpha \wedge \xi(0)^\alpha > \varepsilon\} \longrightarrow \mathscr{P}\{\xi(t)^\alpha > \varepsilon\}\mathscr{P}\{\xi(0)^\alpha > \varepsilon\}, \quad \text{as } t \to \infty. \tag{23}$$

Since the $\xi(t)$'s are not identically zero, there exists an $\varepsilon > 0$, such that $\mathscr{P}\{\xi(t)^\alpha > \varepsilon\} \equiv \mathscr{P}\{\xi(0)^\alpha > \varepsilon\} > 0$. Now, note that

$$\mathbb{E}(\xi(t)^\alpha \wedge \xi(0)^\alpha) \geq \varepsilon \mathscr{P}\{\xi(t)^\alpha \wedge \xi(0)^\alpha > \varepsilon\}.$$

This, in view of the convergence in (23) implies that

$$\liminf_{t\to\infty} \mathbb{E}(\xi(t)^\alpha \wedge \xi(0)^\alpha) > 0,$$

which as argued above, implies that the process X is non–ergodic. $\square$

The above result suggests that most doubly stochastic $\alpha-$Fréchet processes are non–ergodic. This fact can be intuitively explained by the conceptual difference between the independence in the $\xi(t)$'s and the independence of their extremal integrals $X(t)$'s. Indeed, for $X(t)$ and $X(s)$ to be independent, one must have $\xi(t)\xi(s) = 0$, $\mu-$almost surely. The latter, unless the process ξ is trivial, implies that $\xi(t)$ and $\xi(s)$ are dependent. The following example shows, however, that one can have *ergodic* and in fact *mixing* doubly stochastic processes. These processes will be stationary but *not* doubly stationary.

• *(Brown–Resnick Processes)* Let now $w = \{w(t)\}_{t\in\mathbb{R}}$ be a standard Brownian motion, defined on the probability space $(E,\mathscr{E},\mu)$, i.e. $\{w(-t)\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$ are two independent standard Brownian motions. Introduce the non–negative process $\xi(t) := e^{w(t)/\alpha-|t|/2\alpha}$, $t \in \mathbb{R}$ and observe that $\mathbb{E}_\mu \xi(t)^\alpha = 1$. for all $t \in \mathbb{R}$.

The following doubly stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ is said to be a *Brown–Resnick process*:

$$X(t) := \int_E^{e} \xi(t,u) M_\alpha(du) \equiv \int_E^{e} e^{w(t,u)/\alpha-|t|/2\alpha} M_\alpha(du),\ \ t \in \mathbb{R}. \tag{24}$$

The max–stable process $\{\log X(t)\}_{t\geq 0}$ with $\alpha = 1$ and Gumbel marginals was first introduced by Brown and Resnick [2] as a limit involving extremes of Brownian motions. Surprisingly, the resulting max–stable process $X = \{X(t)\}_{t\in\mathbb{R}}$ is stationary. The one–sided stationarity of X is easy to show, by using the fact that $\{w(t)\}_{t\geq 0}$ has stationary and independent increments (see e.g. [25]).

Recently, Kabluchko, Schlather and de Haan [15] studied general *doubly stochastic* processes of Brown–Resnick type. They established necessary and sufficient conditions for the stationarity of such max stable processes. The two–sided stationarity of the classical Brown–Resnick process X above follows from their general results.

We now focus on the Brown–Resnick process in (24) and show that it is mixing. Indeed, the continuity in probability of X follows from the $L^{\alpha}-$continuity of $\xi(t) = e^{w(t)/\alpha-|t|/2\alpha}$. Therefore, by Theorem 3.3, to prove that X is mixing, it is enough to show that $d(X(t),X(0)) \to 0$, as $t \to \infty$. We have that, for all $t > 0$,

$$d(X(t),X(0)) = \mathbb{E}_{\mu}\left(e^{w(t)-t/2} \wedge e^{w(0)}\right) = \mathbb{E}_{\mu}\left(e^{\sqrt{t}Z-t/2} \wedge 1\right),$$

where Z is a standard Normal random variable under μ. The last expectation is bounded above by:

$$\mathscr{P}\{Z > \sqrt{t}/2\} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{t}/2} e^{\sqrt{t}z-t/2}e^{-z^2/2}dz =$$
$$\Phi(-\sqrt{t}/2) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{t}/2} e^{-(z-\sqrt{t})^2/2}dz,$$

which equals $2\Phi(-\sqrt{t}/2)$, where $\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} e^{-x^2/2}dx$. Therefore,

$$d(X(t),X(0)) \le 2\Phi(-\sqrt{t}/2) \le \frac{2}{\sqrt{2\pi}}e^{-t^2/2} \longrightarrow 0, \quad \text{as } t \to \infty.$$

This implies that the Brown–Resnick process X is mixing.

In [25], the ergodicity of more general Brown–Resnick type processes was established where the process w in (24) is replaced by certain infinitely divisible Lévy processes. It would be interesting to define and study other classes of doubly stochastic processes by using different types of integrands.

4.2 *Estimation of the Extremal Index*

The extremal index is an important statistical quantity that can be used to measure the asymptotic *dependence* of stationary sequences. Here, we will briefly review the definition of the extremal index and discuss some estimators for the special case of max–stable time series.

Let $Y = \{Y_k\}_{k\in\mathbb{Z}}$ be strictly stationary time series, which is not necessarily max–stable. Associate with Y a sequence of independent and identically distributed variables $Y^* = \{Y_k^*\}_{k\in\mathbb{R}}$, with the *same marginal* distribution as the Y_k's. Consider the running maxima $M_n := \max_{1\le j\le n} Y_j$ and $M_n^* := \max_{1\le j\le n} Y_j^*$ and suppose that

$$\mathscr{P}\left\{\frac{1}{a_n}M_n^* - b_n \le x\right\} \stackrel{w}{\longrightarrow} G(x), \quad \text{as } n \to \infty, \tag{25}$$

where G is one of the three extreme value distribution functions. If the last convergence holds, for suitable normalization and centering constants a_n and b_n, then we say that the distribution of the Y_k's belongs to the maximum domain of attraction of G (for more details, see e.g. [18]).

Definition 0.3. Suppose that (25) holds for the maxima of independent Y_k^*'s. We say that the time series Y has an *extremal index* θ, if

$$\mathscr{P}\Big\{\frac{1}{a_n}M_n - b_n \le x\Big\} \stackrel{w}{\longrightarrow} G^{\theta}(x), \quad \text{as } n \to \infty, \tag{26}$$

where the a_n's and b_n's are as in (25).

It turns out that if the time series $Y = \{Y_k\}_{k\in\mathbb{Z}}$ has an extremal index θ, then it necessarily follows that

$$0 \le \theta \le 1.$$

Observe that if the Y_k's are independent and belong to the maximum domain of attraction of an extreme value distribution, then trivially, Y has extremal index $\theta = 1$. The converse however, is not true, that is, $\theta = 1$ does not imply the independence of the Y_k's, in general. It is important that the centering and normalization sequences in (25) and (26) be the same. For more details, see the monograph of Leadbetter, Lindgren and Rootzén [16].

A number of statistics have been proposed for the estimation of the extremal index (see e.g. [23], [28], [12]). Here, our goal is to merely illustrate the use of some new estimation techniques for the extremal index in the special case of max–stable $\alpha-$Fréchet processes. We propose a method to construct asymptotically consistent upper bounds for the extremal index, if it exists. The detailed analysis of these methods for the case of general time series is beyond the scope of this work.

Suppose now that $Y = \{Y_k\}_{k\in\mathbb{Z}}$ is a stationary, max–stable time series with the following extremal integral representation

$$Y_k = {\int^{e}}_{E} f_k(u) M_\alpha(du), \;\; k \in \mathbb{Z}, \tag{27}$$

The extremal index of $Y = \{Y_k\}_{k\in\mathbb{Z}}$, if it exists, can be expressed simply as follows. By the max–linearity of the extremal integrals, we have with M_n as in (26), that:

$$\mathscr{P}\Big\{\frac{1}{n^{1/\alpha}}M_n \le x\Big\} = \exp\Big\{-\frac{1}{n}{\int^{e}}_{E}\Big(\bigvee_{1\le k\le n} f_k^{\alpha}\Big)d\mu x^{-\alpha}\Big\}.$$

On the other hand, by the independence of the Y_k^*'s, we have

$$\mathscr{P}\Big\{\frac{1}{n^{1/\alpha}}M_n^* \le x\Big\} = \exp\{-\|Y_1\|_\alpha^\alpha x^{-\alpha}\},$$

where $\|Y_1\|_\alpha^\alpha = \int_E f_1^\alpha d\mu$.

Thus, the extremal index of Y exists and equals θ, if and only if, the following limit exists:

$$\theta := \frac{1}{\|Y_1\|_\alpha^\alpha} \lim_{n\to\infty} \frac{1}{n} \int_E \Big(\bigvee_{1\le k\le n} f_k^\alpha \Big) d\mu. \tag{28}$$

This fact suggests simple and intuitive ways of expressing the extremal index of Y. To do so, let $r \in \mathbb{N}$ and consider the time series $Y(r) := \{Y_k(r)\}_{k\in\mathbb{Z}}$ of non–overlapping block–maxima of size r:

$$Y_k(r) := \max_{1\le i\le r} Y_{i+(k-1)r},\ \ k \in \mathbb{Z}.$$

Observe that $Y = Y(1)$ is the original time series. We shall denote the extremal index of $Y(r)$ as $\theta(r)$, when it exists. The following result yields a simple relationship between the $\theta(r)$'s.

Proposition 0.2. *Let $Y = \{Y_k\}_{k\in\mathbb{Z}}$ be as in* (27)*. If Y has a positive extremal index $\theta = \theta(1)$, then the time series $Y(r) = \{Y_k(r)\}_{k\in\mathbb{Z}}$ also has an extremal index $\theta(r)$ equal to:*

$$\theta(r) = \frac{1}{\theta_r(1)}\theta(1),$$

where

$$\theta_r(1) = \frac{\|Y_1\|_\alpha^{-\alpha}}{r} \int_E \Big(\bigvee_{1\le k\le r} f_k^\alpha \Big) d\mu,\ \ r \in \mathbb{N}. \tag{29}$$

Moreover, for all $r \in \mathbb{N}$, we have $\theta_r(1) \in (0,1]$ and hence

$$\theta(1) \le \theta(r) \quad and \quad \theta(1) \le \theta_r(1). \tag{30}$$

We will see in the sequel that, for fixed r, the quantity $\theta_r(1)$ can be consistently estimated from the data, provided that the underlying time series Y is ergodic.

Proof (Proposition 0.2). In view of (28), we have that

$$\begin{aligned}\theta(r) &= \|Y_1 \vee \cdots \vee Y_r\|_\alpha^{-\alpha} \lim_{n\to\infty} \frac{1}{n} \int_E \Big(\bigvee_{1\le k\le n} \bigvee_{1\le i\le r} f_{i+(k-1)r}^\alpha \Big) d\mu \\ &= r\|Y_1 \vee \cdots \vee Y_r\|_\alpha^{-\alpha} \lim_{n\to\infty} \frac{1}{rn} \int_E \Big(\bigvee_{1\le j\le nr} f_j^\alpha \Big) d\mu,\end{aligned} \tag{31}$$

where in the last relation we multiplied and divided by the constant r.
By assumption, the limit in the right–hand side of (31) exists and equals $\|Y_1\|_\alpha^\alpha \theta(1)$, which implies that the time series $Y(r)$ has an extremal index $\theta(r)$. This, fact since

$$\|Y_1 \vee \cdots \vee Y_r\|_\alpha^\alpha = \int_E \Big(\bigvee_{1\le k\le r} f_k^\alpha \Big) d\mu$$

and in view of (29) also implies that $\theta(r) = \theta(1)/\theta_r(1)$.

Finally, the inequalities in (30) follow from the fact that

$$\frac{1}{r}\int_E \Big(\bigvee_{1\le k\le r} f_k^\alpha\Big) d\mu \le \frac{1}{r}\sum_{k=1}^r \int_E f_k^\alpha d\mu = \|Y_1\|_\alpha^\alpha,$$

which yields that $0 < \theta_r(1) \le 1$. □

We now focus on the estimation of the parameter $\theta_r(1)$ for a given fixed value of r, from observations $\{Y_k,\ 1\le k\le n\}$ of the time series Y. Suppose that the time series Y is ergodic. Note that $\mathbb{E}Y_k^p = \Gamma(1-p/\alpha)\|Y_1\|_\alpha^p$ is finite, for all $p<\alpha$. Therefore, the ergodicity implies that

$$\widehat{m}_p(1) := \frac{1}{n}\sum_{k=1}^n Y_k^p \xrightarrow{a.s.} \Gamma(1-p/\alpha)\|Y_1\|_\alpha^p, \tag{32}$$

as $n\to\infty$. For the block–maxima time series, we also have that

$$\widehat{m}_p(r) := \frac{1}{[n/r]}\sum_{k=1}^{[n/r]} Y_k(r)^p \xrightarrow{a.s.} \Gamma(1-p/\alpha)\|Y_1\vee\cdots\vee Y_r\|_\alpha^p. \tag{33}$$

Note that here we have only $[n/r]$ observations from the block–maxima time series $\{Y_k(r),\ 1\le k\le [n/r]\}$ available from the original data set.

Relation (29) and the convergences in (32) and (33) suggest the following estimator for the parameter $\theta_r(1)$:

$$\widehat{\theta}_r(1;p,n) := \frac{1}{r}\Big(\widehat{m}_p(r)/\widehat{m}_p(1)\Big)^{\alpha/p}. \tag{34}$$

Proposition 0.3. *Suppose that $\{Y_k,\ 1\le k\le n\}$ is a sample from an ergodic stationary $\alpha-$Fréchet time series. Then, for all $p<\alpha$, we have*

$$\widehat{\theta}_r(1;p,n) \xrightarrow{a.s.} \theta_r(1), \quad \textit{as } n\to\infty.$$

This result shows the strong consistency of the estimator in (34). The proof of this proposition is an immediate consequence from Relations (32) and (33).

Note that one can use also *overlapping* block–maxima to estimate the quantity $\|Y_1(r)\|_\alpha^\alpha = \|Y_1\vee\cdots\vee Y_r\|_\alpha^\alpha$. Indeed, for an ergodic time series Y, we also have

$$\widehat{m}_{p,\mathrm{ovlp}}(1;p,n) := \frac{1}{n-r+1}\sum_{k=1}^{n-r+1}(Y_k\vee Y_{k+1}\vee\cdots\vee Y_{k+r-1})^p \xrightarrow{a.s.} \Gamma(1-p/\alpha)\|Y_1(r)\|_\alpha^p,$$

as $n\to\infty$. This suggests another flavor of an estimator for $\theta_r(1)$:

$$\widehat{\theta}_{r,\mathrm{ovlp}}(1;p,n) := \frac{1}{r}\Big(\widehat{m}_{p,\mathrm{ovlp}}(r)/\widehat{m}_p(1)\Big)^{\alpha/p}.$$

Clearly, as for the estimator $\widehat{\theta}_r$, we also have *strong consistency*:

$$\widehat{\theta}_{r,\mathrm{ovlp}}(1;p,n) \xrightarrow{a.s.} \theta_r(1), \quad \text{as } n \to \infty.$$

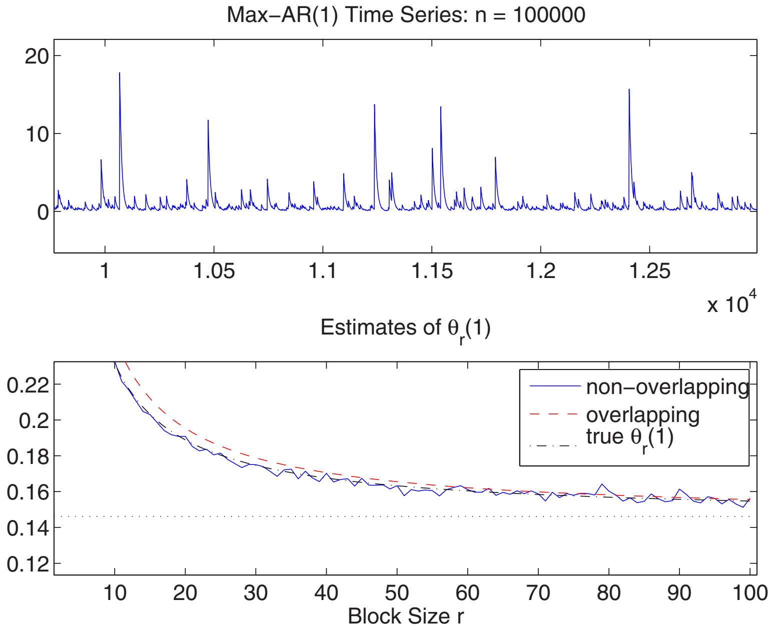

Fig. 1 *Top panel*: Simulated max–autoregressive $\alpha-$Fréchet time series Y_k, $1 \le k \le n$ defined as in (35) with $n = 100000$, $\phi = 0.9$ and $\alpha = 1.5$. The theoretical value of $\theta = \theta(1)$ equals $(1-\phi^\alpha) = 0.1462$. *Bottom panel:* Estimates $\widehat{\theta}_r(1)$ (solid line) and $\widehat{\theta}_{r,\mathrm{ovlp}}(1)$ (dashed line) as a function of the block–size r. The true value of $\theta(1)$ is indicated by the dotted line.

Figure 1 illustrates the performance of the estimators $\widehat{\theta}_r(1)$ and $\widehat{\theta}_{r,\mathrm{ovlp}}(1)$ over a simulated max–autoregressive time series. The time series Y is defined as:

$$Y_k = \phi Y_{k-1} \vee (1-\phi) Z_k = (1-\phi) \bigvee_{j=0}^{\infty} \phi^j Z_{k-j}, \tag{35}$$

where $\phi \in [0,1]$, and where the Z_j's are independent standard $\alpha-$Fréchet variables. We used parameter values $\alpha = 1.5$, $\phi = 0.9$ and $p = 0.6$.

One can show that $\|Y_1\|_\alpha^\alpha = (1-\phi)^\alpha/(1-\phi^\alpha)$, and that

$$Y_1 \vee \cdots \vee Y_r = Y_1 \vee (1-\phi)(\max_{2\le k\le r} Z_k).$$

Thus, for $\theta_r(1)$ and $\theta(1)$ we obtain:

$$\theta_r(1) = (1-\phi^\alpha)\frac{(r-1)}{r} + \frac{1}{r} \quad \text{and} \quad \theta(1) = \lim_{r\to\infty} \theta_r(1) = (1-\phi^\alpha).$$

The true value of $\theta_r(1)$, indicated by the dot–dashed line is nearly covered by the solid line, indicating the realizations of $\widehat{\theta}_r(1)$. The 'overlapping blocks' estimator $\widehat{\theta}_{r,\text{ovlp}}(1)$, on the other hand, shows a small but systematic bias, which decreases as the block–size r grows. This limited simulation experiment indicates that $\widehat{\theta}_r(1)$ and $\widehat{\theta}_{r,\text{ovlp}}(1)$ accurately estimate $\theta_r(1)$. Furthermore, in this setting, $\widehat{\theta}_r(1)$ is more accurate for small values of r and $\widehat{\theta}_{r,\text{ovlp}}(1)$ is competitive and likely to be more accurate for large values of r. Note also that since $\theta_r(1)$ converges to $\theta(1)$, as $r\to\infty$, both $\widehat{\theta}_r(1)$ and $\widehat{\theta}_{r,\text{ovlp}}(1)$ can be used to estimate $\theta(1)$ in practice, when sufficiently large values of r are chosen.

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