

Chapter 2

The Wetterich Equation

In this section we review the properties and derivation of the flow equation first published by Christof Wetterich in 1993 [1]. We will derive this equation from the functional integral representation of quantum field theory. The relation of the flow equation to other methods such as perturbation theory for small interaction strength is then particular clear. In principle one might also consider the flow equation as an own formulation of quantum field theory from which other formulations such as the functional integral representation or the operator formalism can be derived. For practical purposes all this different formulations have their advantages and disadvantages. While some problems are best solved with the flow equation formalism, different methods might be more suitable for other problems. Our physical insight and intuition grows if we look at physics from different perspectives. It is therefore a sensible ambition to further develop the flow equation method and learn how to apply it to various problems in modern physics.

2.1 Scale Dependent Schwinger Functional

We start with the Schwinger functional in (1.47), (1.50) which we modify by introducing an cutoff term ΔS_k

$$e^{W_k[J]} = \int D\tilde{\Phi} e^{-S[\tilde{\Phi}] - \Delta S_k[\tilde{\Phi}] + J_\alpha \tilde{\Phi}_\alpha} \quad (2.1)$$

with

$$\Delta S_k[\tilde{\Phi}] = \frac{1}{2} \tilde{\Phi}_\alpha (R_k)_{\alpha\beta} \tilde{\Phi}_\beta. \quad (2.2)$$

Again we work with an abstract index notation where e.g. α stands for both continuous and discrete variables. For simplicity we will sometimes drop this index when the meaning is clear and write for example

$$\Delta S_k[\tilde{\Phi}] = \frac{1}{2} \tilde{\Phi} R_k \tilde{\Phi}. \quad (2.3)$$

In praxis one chooses R_k to be an infrared cutoff which is diagonal in momentum space. For example, for a single complex scalar field $\tilde{\varphi}$ we use

$$\Delta S_k = \int_q \tilde{\varphi}^*(q) R_k(q) \tilde{\varphi}(q) \quad (2.4)$$

with

$$R_k(q) = A_\varphi(k^2 - \vec{q}^2) \theta(k^2 - \vec{q}^2). \quad (2.5)$$

More general the function $R_k(q)$ should have the properties

$$\begin{aligned} R_k(q) &\rightarrow \infty \quad (k \rightarrow \infty), \\ R_k(q) &\rightarrow 0 \quad (k \rightarrow 0), \\ R_k(q) &\approx k^2 \quad (q \rightarrow 0). \end{aligned} \quad (2.6)$$

The cutoff term R_k plays a similar role for the functional integral as the parameter k^2 in [Chap. 1.1](#) where we developed a flow equation to solve one-dimensional integrals. For large R_k the term $\Delta S_k[\tilde{\Phi}]$ in (2.1) suppresses the contribution of large values of $\tilde{\Phi}$ to the functional integral in (2.1). These fluctuations are included as the cutoff R_k is lowered. An additional feature to the one-dimensional case in [Chap. 1.1](#) is the q -dependence of R_k . One can choose the form of $R_k(q)$ such that it vanishes for momenta with $q^2 \gg k^2$. For a given k only the contribution of modes with $q^2 \ll k^2$ in (2.1) is then suppressed while the contribution of modes with $q^2 \gg k^2$ is not modified. We will discuss the choice of an appropriate form of the cutoff function in [Chap. 5](#).

Since the only explicitly scale-dependence of the Schwinger functional $W_k[J]$ comes from the cutoff term ΔS_k we can easily calculate its scale-derivative

$$\begin{aligned} \partial_k W_k|_J &= -\frac{1}{2} \langle \tilde{\Phi}_\alpha (\partial_k R_k)_{\alpha\beta} \tilde{\Phi}_\beta \rangle \\ &= -\frac{1}{2} (\partial_k R_k)_{\alpha\beta} \langle \tilde{\Phi}_\alpha \tilde{\Phi}_\beta \rangle_c + \Phi_\alpha \Phi_\beta. \end{aligned} \quad (2.7)$$

We can use for the connected part $\langle \tilde{\Phi}_\alpha \tilde{\Phi}_\beta \rangle_c$ the functional derivative

$$\langle \tilde{\Phi}_\alpha \tilde{\Phi}_\beta \rangle_c = \left(W_k^{(2)} \right)_{\alpha\beta} = \frac{\delta}{\delta J_\alpha} \frac{\delta}{\delta J_\beta} W_k = \frac{\delta}{\delta J_\alpha} \Phi_\beta \quad (2.8)$$

in order to write (2.7) as

$$\partial_k W_k|_J = -\frac{1}{2} \text{STr} \left\{ (\partial_k R_k) W_k^{(2)} \right\} - \frac{1}{2} \Phi (\partial_k R_k) \Phi. \quad (2.9)$$

The STr-operation sums over equal indices and includes an extra minus sign for fermionic degrees of freedom. This comes from the fact that $\langle \tilde{\Phi}_\alpha \tilde{\Phi}_\beta \rangle_c = -\langle \tilde{\Phi}_\beta \tilde{\Phi}_\alpha \rangle_c$ if $\tilde{\Phi}_\alpha$ and $\tilde{\Phi}_\beta$ are fermionic Grassmann-valued fields.

2.2 The Average Action and its Flow Equation

From the scale dependent Schwinger functional we can now go to the average action or flowing action. It is defined by subtracting from the Legendre transform

$$\tilde{\Gamma}_k[\Phi] = J_\alpha \Phi_\alpha - W_k[J] \quad \text{with} \quad \Phi_\alpha = \frac{\delta W_k}{\delta J_\alpha} \quad (2.10)$$

the cutoff term

$$\Gamma_k[\Phi] = \tilde{\Gamma}_k[\Phi] - \frac{1}{2} \Phi R_k \Phi. \quad (2.11)$$

From the definition it is immediately clear that the average action equals the quantum effective action

$$\Gamma[\Phi] = J_\alpha \Phi_\alpha - W[J] \quad \text{with} \quad \Phi_\alpha = \frac{\delta W}{\delta J_\alpha} \quad (2.12)$$

for $k \rightarrow 0$. The quantum effective action is the generating functional of the one-particle irreducible correlation functions. It is straightforward to show a number of properties of the average action. The field equation follows by taking the functional derivative

$$\frac{\delta}{\delta \Phi_\alpha} \tilde{\Gamma}_k = \pm J_\alpha. \quad (2.13)$$

The upper (lower) sign is for bosonic (fermionic) field components Φ_α . The functional derivative in (2.13) is a left derivative for Grassmann valued Φ_α . For a right-handed derivative we obtain

$$\tilde{\Gamma}_k \overleftarrow{\frac{\delta}{\delta \Phi_\alpha}} = J_\alpha. \quad (2.14)$$

The second functional derivative is

$$\left(\tilde{\Gamma}_k^{(2)} \right)_{\alpha\beta} = \frac{\delta}{\delta \Phi_\alpha} \left(\Gamma_k \overleftarrow{\frac{\delta}{\delta \Phi_\beta}} \right) = \frac{\delta}{\delta \Phi_\alpha} J_\beta. \quad (2.15)$$

Comparing this to (2.8) we find the useful relation

$$\left(\tilde{\Gamma}_k^{(2)}\right)_{\alpha\beta}\left(W_k^{(2)}\right)_{\beta\gamma}=\delta_{\alpha\gamma}. \quad (2.16)$$

or

$$W_k^{(2)}=\left(\Gamma_k^{(2)}+R_k\right)^{-1}. \quad (2.17)$$

To calculate the flow equation for Γ_k we use

$$\partial_k\tilde{\Gamma}_k|_{\Phi}=-\partial_k W_k|_J+\partial_k J_{\alpha}\left(\Phi_{\alpha}-\frac{\delta W_k}{\delta J_{\alpha}}\right)=-\partial_k W_k|_J \quad (2.18)$$

and

$$\partial_k\Gamma_k[\Phi]=\partial_k\tilde{\Gamma}_k[\Phi]-\frac{1}{2}\Phi(\partial_k R_k)\Phi. \quad (2.19)$$

Together with (2.9) and (2.17) we obtain then the central result of this chapter, the Wetterich equation [1]

$$\partial_k\Gamma_k=\frac{1}{2}\text{STr}\left\{(\partial_k R_k)\left(\Gamma_k^{(2)}+R_k\right)^{-1}\right\}. \quad (2.20)$$

2.3 Functional Integral Representation and Initial Condition

From the definition of the average action in (2.10), (2.11) and the scale dependent Schwinger functional in (2.1) we obtain the functional integral representation

$$\begin{aligned} e^{-\Gamma_k[\Phi]} &= \int D\tilde{\Phi} e^{-S[\tilde{\Phi}]-\frac{1}{2}\tilde{\Phi}R_k\tilde{\Phi}+J\tilde{\Phi}-J\Phi+\frac{1}{2}\Phi R_k\Phi} \\ &= \int D\tilde{\Phi} e^{-S[\Phi+\tilde{\Phi}]-\frac{1}{2}\tilde{\Phi}R_k\tilde{\Phi}+\frac{1}{2}\left[(\delta\Gamma_k/\delta\Phi)\tilde{\Phi}+\tilde{\Phi}\frac{\delta\Gamma_k}{\delta\Phi}\right]}. \end{aligned} \quad (2.21)$$

In the last line we performed a change of the integration measure

$$\tilde{\Phi} \rightarrow \tilde{\Phi} + \Phi \quad (2.22)$$

and used

$$\frac{1}{2}\left[(\delta\Gamma_k/\delta\Phi)\tilde{\Phi}+\tilde{\Phi}\frac{\delta\Gamma_k}{\delta\Phi}\right]=\frac{1}{2}\left[\Gamma_k\overleftarrow{\frac{\delta}{\delta\Phi}}\tilde{\Phi}+\tilde{\Phi}\overrightarrow{\frac{\delta}{\delta\Phi}}\Gamma_k\right] \quad (2.23)$$

$$\frac{1}{2} \left[(\delta \Gamma_k / \delta \Phi) \tilde{\Phi} + \tilde{\Phi} \frac{\delta \Gamma_k}{\delta \Phi} \right] = J \tilde{\Phi} - \frac{1}{2} \Phi R_k \tilde{\Phi} - \frac{1}{2} \tilde{\Phi} R_k \Phi. \quad (2.24)$$

If the cutoff is chosen such that for $k \rightarrow \infty$

$$\frac{1}{2} \tilde{\Phi} R_k \tilde{\Phi} \rightarrow \frac{1}{2} \sum_{\alpha} r_{k,\alpha} \Phi_{\alpha}^* \Phi_{\alpha} \quad \text{with } r_{k,\alpha} \rightarrow \infty, \quad (2.25)$$

we find from (2.21)

$$\lim_{k \rightarrow \infty} \Gamma_k[\Phi] = S[\Phi] + \text{const.} \quad (2.26)$$

This is a remarkable and very useful result. In the limit of large k the average action Γ_k approaches the microscopic action S . Equation (2.26) serves as an initial condition for the flow equation (2.20).

Reference

1. Wetterich C (1993) Phys Lett B 301:90

Functional Renormalization and Ultracold Quantum
Gases

Flörchinger, S.

2010, X, 202 p. 58 illus., Hardcover

ISBN: 978-3-642-14112-6