

Chapter 1

Preliminaries

In this chapter we present the basic tools of the classical theory of filtered modules, in particular we introduce the machinery we shall use throughout this work: $\mathbb{M}$ -superficial elements and their interplay with Hilbert Functions, the Valabrega–Valla criterion, which is a basic tool for studying the depth of $\text{gr}_{\mathbb{M}}(M)$, $\mathbb{M}$ -superficial sequences for an ideal $\mathfrak{q}$ and their relevance to Sally’s machine, which is a very important device for reducing dimension in questions relating to depth properties of blowing-up rings and local rings.

1.1 Notation

Let A be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A -module. We will denote by $\lambda(\cdot)$ the length of an A -module. An (infinite) chain

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_j \supseteq \cdots$$

where the M_n are submodules of M is called a *filtration* of M , and denoted by $\mathbb{M} = \{M_n\}$. Given an ideal $\mathfrak{q}$ in A , $\mathbb{M}$ is a *$\mathfrak{q}$ -filtration* if $\mathfrak{q}M_j \subseteq M_{j+1}$ for all j , and a *good $\mathfrak{q}$ -filtration* if $M_{j+1} = \mathfrak{q}M_j$ for all sufficiently large j . Thus for example $\{\mathfrak{q}^n M\}$ is a good $\mathfrak{q}$ -filtration. In the literature a good $\mathfrak{q}$ -filtration is sometimes called a stable $\mathfrak{q}$ -filtration. We say that $\mathbb{M}$ is nilpotent if $M_n = 0$ for $n \gg 0$. Thus a good $\mathfrak{q}$ -filtration $\mathbb{M}$ is nilpotent if and only if $\mathfrak{q} \subseteq \sqrt{\text{Ann } M}$.

From now on $\mathbb{M}$ will denote always a good $\mathfrak{q}$ -filtration on the finitely generated A -module M .

We will assume that the ideal $\mathfrak{q}$ is proper. As a consequence $\bigcap_{i=0}^{\infty} M_i = \{0_M\}$. Define

$$\text{gr}_{\mathfrak{q}}(A) = \bigoplus_{j \geq 0} (\mathfrak{q}^j / \mathfrak{q}^{j+1}).$$

This is a graded ring in which the multiplication is defined as follows: if $a \in \mathfrak{q}^i$, $b \in \mathfrak{q}^j$ define $\overline{ab}$ to be $\overline{ab}$, i.e. the image of ab in $\mathfrak{q}^{i+j} / \mathfrak{q}^{i+j+1}$. This ring is the *associated graded ring* of the ideal $\mathfrak{q}$.

Similarly, if M is an A -module and $\mathbb{M} = \{M_j\}$ is a q -filtration of M , define

$$gr_{\mathbb{M}}(M) = \bigoplus_{j \geq 0} (M_j / M_{j+1})$$

which is a graded $gr_q(A)$ -module in a natural way. It is called the *associated graded module* to the q -filtration $\mathbb{M} = \{M_n\}$.

To avoid triviality we shall assume that $gr_{\mathbb{M}}(M)$ is not zero or equivalently $M \neq 0$. Each element $a \in A$ has a natural image, denoted by $a^* \in gr_q(A)$ and called initial form of a with respect to $\mathbb{M}$. If $a = 0$, then $a^* = 0$, otherwise $a^* = \bar{a} \in q^t / q^{t+1}$ where t is the unique integer such that $a \in q^t$, $a \notin q^{t+1}$.

If N is a submodule of M , by the Artin-Rees Lemma, the collection $\{N \cap M_j \mid j \geq 0\}$ is a good q -filtration of N . Since

$$(N \cap M_j) / (N \cap M_{j+1}) \simeq (N \cap M_j + M_{j+1}) / M_{j+1}$$

$gr_{\mathbb{M}}(N)$ is a graded submodule of $gr_{\mathbb{M}}(M)$.

On the other hand it is clear that $\{(N + M_j) / N \mid j \geq 0\}$ is a good q -filtration of M/N which we denote by $\mathbb{M}/N$ and we call quotient filtration. These graded modules are related by the graded isomorphism

$$gr_{\mathbb{M}/N}(M/N) \simeq gr_{\mathbb{M}}(M) / gr_{\mathbb{M}}(N).$$

If $a_1, \dots, a_r$ are elements in $q \setminus q^2$ and $I = (a_1, \dots, a_r)$, it is clear that

$$[(a_1^*, \dots, a_r^*)gr_{\mathbb{M}}(M)]_j = (IM_{j-1} + M_{j+1}) / M_{j+1}$$

for each $j \geq 1$. By the Artin-Rees Lemma one immediately gets that the following conditions are equivalent:

- (1) $gr_{\mathbb{M}/IM}(M/IM) \simeq gr_{\mathbb{M}}(M) / (a_1^*, \dots, a_r^*)gr_{\mathbb{M}}(M)$.
- (2) $gr_{\mathbb{M}}(IM) = (a_1^*, \dots, a_r^*)gr_{\mathbb{M}}(M)$.
- (3) $IM \cap M_j = IM_{j-1} \quad \forall j \geq 1$.

An interesting case in which the above equalities hold is when the elements $a_1^*, \dots, a_r^*$ form a regular sequence on $gr_{\mathbb{M}}(M)$. For example, if $r = 1$ and $I = (a)$, then $\bar{a} \in q/q^2$ is regular on $gr_{\mathbb{M}}(M)$ if and only if the map

$$M_{j-1} / M_j \xrightarrow{\bar{a}} M_j / M_{j+1}$$

is injective for every $j \geq 1$. This is equivalent to the equalities $M_{j-1} \cap (M_{j+1} : a) = M_j$ for every $j \geq 1$. An easy computation shows the following result.

Lemma 1.1. *Let $a \in q$. The following conditions are equivalent:*

- (1) $\bar{a} \in q/q^2$ is a regular element on $gr_{\mathbb{M}}(M)$.

- (2) $M_{j-1} \cap (M_{j+1} : a) = M_j$ for every $j \geq 1$.
- (3) a is a regular element on M and $aM \cap M_j = aM_{j-1}$ for every $j \geq 1$.
- (4) $M_{j+1} : a = M_j$ for every $j \geq 0$.

This result leads us to the Valabrega–Valla criterion, a tool which has been very useful in the study of the depth of blowing-up rings (see [105]).

Many authors have discussed this topic recently. For example in [53] Huckaba and Marley gave an extension of the classical result to the case of filtrations of ideals, by giving a completely new proof based on some deep investigation of a modified Koszul complex.

Instead, in [69], Puthenpurakal extended the result to the case of q -adic filtrations of a module by using the device of idealization of a module and then applying the classical result.

This would suggest that the original proof does not work in the more general setting. But, after looking at it carefully, we can say that, in order to prove the following very general statement, one does not need any new idea: a straightforward adaptation of the dear old proof does the job.

Theorem 1.1 (Valabrega–Valla). *Let $a_1, \dots, a_r$ be elements in $q \setminus q^2$ and I the ideal they generate. Then $a_1^*, \dots, a_r^*$ form a regular sequence on $gr_{\mathbb{M}}(M)$ if and only if $a_1, \dots, a_r$ form a regular sequence on M and $IM \cap M_j = IM_{j-1} \ \forall j \geq 1$.*

1.2 Superficial Elements

A fundamental tool in local algebra is the notion of superficial element. This notion goes back to P. Samuel and our methods are also related to the construction given by Zariski and Samuel [117, p. 296].

Definition 1.1. An element $a \in q$, is called $\mathbb{M}$ -superficial for q if there exists a non-negative integer c such that

$$(M_{n+1} :_M a) \cap M_c = M_n$$

for every $n \geq c$.

For every $a \in q$ and $n \geq c$, M_n is contained in $(M_{n+1} :_M a) \cap M_c$. Then it is the other inclusion that makes superficial elements special. It is clear that if $\mathbb{M}$ is nilpotent, then every element is superficial. If the length $\lambda(M/qM)$ is finite, then $\mathbb{M}$ is nilpotent if and only if $\dim M = 0$. Hence in the following, when we deal with superficial elements, we shall assume that $\dim M \geq 1$.

If this is the case, as a consequence of the definition, we deduce that $\mathbb{M}$ -superficial elements a for q have order one, that is $a \in q \setminus q^2$. With a slight modification of the given definition, superficial elements can be introduced of every order,

but in the following we need superficial elements of order 1 because they have a better behaviour for studying Hilbert functions.

Hence superficial element always means a superficial element of order 1. It is well known that superficial elements do not always exist, but their existence is guaranteed if the residue field is infinite (see [56, Proposition 8.5.7]). By passing, if needed, to the faithfully flat extension $A[x]_{\mathfrak{m}A[x]}$ (x is a variable over A) we may assume that the residue field is infinite.

If $\mathfrak{q}$ contains a regular element on M , it is easy to see that every $\mathbb{M}$ -superficial element of $\mathfrak{q}$ is regular on M .

Given A -modules M and N , let $\mathbb{M}$ and $\mathbb{N}$ be good $\mathfrak{q}$ -filtrations of M and N respectively. We define a new filtration as follows

$$\mathbb{M} \oplus \mathbb{N}: \quad M \oplus N \supseteq M_1 \oplus N_1 \supseteq \cdots \supseteq M_n \oplus N_n \supseteq \cdots$$

It is easy to see that $\mathbb{M} \oplus \mathbb{N}$ is a good $\mathfrak{q}$ -filtration on the A -module $M \oplus N$. Of course this construction can be extended to any finite number of modules.

The following remark is due to David Conti in his thesis (see [11]).

Remark 1.1. Let $\mathbb{M}_1, \dots, \mathbb{M}_n$ be $\mathfrak{q}$ -filtrations of M and let $a \in \mathfrak{q}$. Then a is $\mathbb{M}_1 \oplus \cdots \oplus \mathbb{M}_n$ -superficial for $\mathfrak{q}$ if and only if a is $\mathbb{M}_i$ -superficial for $\mathfrak{q}$ for every $i = 1, \dots, n$.

This result is an easy consequence of the good behaviour of intersection and colon of ideals with respect to direct sum of modules. As a consequence we deduce that, if the residue field is infinite, we can always find an element $a \in \mathfrak{q}$ which is superficial for a finite number of $\mathfrak{q}$ -filtrations on M . As mentioned in the introduction, we want to apply the general theory of the filtrations on a module to the study of certain blowing-up rings which are not necessarily associated graded rings to a single filtration. Since they are related to different filtrations, the above remark will be relevant in our approach.

David Conti also remarked that $\mathbb{M}$ -superficial elements of order $s \geq 1$ for $\mathfrak{q}$ can be seen as superficial elements of order 1 for a suitable filtration which is strictly related to $\mathbb{M}$. Let $\mathbb{N}$ be the $\mathfrak{q}^s$ -filtration:

$$M \oplus M_1 \oplus \cdots \oplus M_{s-1} \supseteq M_s \oplus M_{s+1} \oplus \cdots \oplus M_{2s-1} \supseteq M_{2s} \oplus M_{2s+1} \oplus \cdots \oplus M_{3s-1} \supseteq \cdots$$

Then it is easy to see that an element a is $\mathbb{M}$ -superficial of degree s for $\mathfrak{q}$ if and only if a is $\mathbb{N}$ -superficial of degree one for $\mathfrak{q}^s$. This remark could be useful in studying properties which behave well with the direct sum.

We give now equivalent conditions for an element to be $\mathbb{M}$ -superficial for $\mathfrak{q}$. Our development of the theory of superficial elements is basically the same as that given by Kirby in [60] for the case $M = A$ and $\{M_j\} = \{\mathfrak{q}^j\}$ the $\mathfrak{q}$ -adic filtration on A . If there is no confusion, we let

$$G := gr_{\mathbb{M}}(M), \quad Q := \bigoplus_{j \geq 1} (\mathfrak{q}^j / \mathfrak{q}^{j+1}).$$

Let

$$\{0_M\} = P_1 \cap \cdots \cap P_r \cap P_{r+1} \cap \cdots \cap P_s$$

be an irredundant primary decomposition of $\{0_M\}$ and let $\{\mathfrak{p}_i\} = \text{Ass}(M/P_i)$. Then $\text{Ass}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ and $\mathfrak{p}_i = \sqrt{0 : (M/P_i)}$.

We may assume $\mathfrak{q} \not\subseteq \mathfrak{p}_i$ for $i = 1, \dots, r$ and $\mathfrak{q} \subseteq \mathfrak{p}_i$ for $i = r+1, \dots, s$. Then we let

$$N := P_1 \cap \cdots \cap P_r.$$

It is clear that

$$N = \{x \in M \mid \exists n, \mathfrak{q}^n x = 0_M\}$$

and that $N \cap M_j = \{0\}$ for all large j and $\text{Ass}(M/N) = \bigcup_{i=1}^r \mathfrak{p}_i$.

Similarly we denote by H the homogeneous submodule of G consisting of the elements $\alpha \in G$ such that $Q^n \alpha = 0_G$, hence

$$H = \{\alpha \in G \mid \exists n, Q^n \alpha = 0_G\}.$$

If

$$\{0_G\} = T_1 \cap \cdots \cap T_m \cap T_{m+1} \cap \cdots \cap T_l$$

is an irredundant primary decomposition of $\{0_G\}$ and we let $\{\mathfrak{P}_i\} = \text{Ass}(G/T_i)$, then $\text{Ass}(G) = \{\mathfrak{P}_1, \dots, \mathfrak{P}_l\}$ and $\mathfrak{P}_i = \sqrt{0 : (G/T_i)}$.

Further, if we assume that $Q \not\subseteq \mathfrak{P}_i$ for $i = 1, \dots, m$ and $Q \subseteq \mathfrak{P}_i$ for $i = m+1, \dots, l$, then

$$H = T_1 \cap \cdots \cap T_m$$

and $\text{Ass}(G/H) = \bigcup_{i=1}^m \mathfrak{P}_i$.

Theorem 1.2. *Let $a \in \mathfrak{q} \setminus \mathfrak{q}^2$, the following conditions are equivalent:*

- (1) *a is $\mathbb{M}$ -superficial for $\mathfrak{q}$.*
- (2) *$a^* \notin \bigcup_{i=1}^m \mathfrak{P}_i$.*
- (3) *$H :_G a^* = H$.*
- (4) *$N : a = N$ and $M_{j+1} \cap aM = aM_j$ for all large j .*
- (5) *$(0 :_G a^*)_j = 0$ for all large j .*
- (6) *$M_{j+1} : a = M_j + (0 :_M a)$ and $M_j \cap (0 :_M a) = 0$ for all large j .*

We note with [117] that if the residue field $A/\mathfrak{m}$ is infinite, condition (2) of the above theorem ensures the existence of $\mathbb{M}$ -superficial elements for $\mathfrak{q}$ as we have already mentioned. Moreover condition (5) says that a is $\mathbb{M}$ -superficial for $\mathfrak{q}$ if and only if a^* is homogeneous filter-regular in G . We may refer to [101, 102, 104], concerning the definition and the properties of the homogeneous filter-regular elements. Filter-regular elements were also introduced in the local contest by N.T. Cuong, P. Schenzel and N.V. Trung in [19]. One of the main results in [19] says that a local ring is generalized Cohen–Macaulay if and only if every system of parameters is filter-regular. The notion of a filter-regular element in a local ring is weaker than superficial element and in general it does not behave well with Hilbert functions.

A superficial element is filter-regular, but the converse does not hold. It is enough to recall that if a is M -regular, then accordingly with [19], a is a filter-regular element, but it is not necessarily a superficial element.

A sequence of elements $a_1, \dots, a_r$ will be called a $\mathbb{M}$ -superficial sequence for $\mathfrak{q}$ if, for $i = 1, \dots, r$, a_i is an $(\mathbb{M}/(a_1, \dots, a_{i-1})M)$ -superficial element for $\mathfrak{q}$.

In order to prove properties of superficial sequences often we can argue by induction on the number of elements, the above theorem giving the first step of the induction. For example, since $\text{depth}_{\mathfrak{q}}(M) \geq 1$ implies $N = 0$, condition (4) gives the following result. Here $\text{depth}_{\mathfrak{q}}(M)$ denotes the common cardinality of all the maximal M -regular sequences of elements in $\mathfrak{q}$.

The following lemmas will be crucial devices through the whole paper.

Lemma 1.2. *Let $a_1, \dots, a_r$ be an $\mathbb{M}$ -superficial sequence for $\mathfrak{q}$. Then $a_1, \dots, a_r$ is a regular sequence on M if and only if $\text{depth}_{\mathfrak{q}}(M) \geq r$.*

In the same way, since $\text{depth}_Q(\text{gr}_{\mathbb{M}}(M)) \geq 1$ implies $H = 0$, condition 3 and Theorem 1.1 give the following result which shows the relevance of superficial elements in the study of the depth of blowing-up rings.

Lemma 1.3. *Let $a_1, \dots, a_r$ be an $\mathbb{M}$ -superficial sequence for $\mathfrak{q}$. Then $a_1^*, \dots, a_r^*$ is a regular sequence on $\text{gr}_{\mathbb{M}}(M)$ if and only if $\text{depth}_Q(\text{gr}_{\mathbb{M}}(M)) \geq r$.*

Now we come to *Sally's machine* or *Sally's descent*, a very important device for reducing dimension in questions relating to depth properties of blowing-up rings.

Lemma 1.4. (Sally's machine) *Let $a_1, \dots, a_r$ be an $\mathbb{M}$ -superficial sequence for $\mathfrak{q}$ and I the ideal they generate. Then $\text{depth}_Q(\text{gr}_{\mathbb{M}/IM}(M/IM)) \geq 1$, if and only if $\text{depth}_Q(\text{gr}_{\mathbb{M}}(M)) \geq r + 1$.*

A proof of the *if* part can be obtained by a straightforward adaptation of the original proof given by Huckaba and Marley in [53, Lemma 2.2]. The converse is an easy consequence of Lemma 1.2 and Theorem 1.1.

We will also need a property of superficial elements which seems to be neglected in the literature. It is well known that if a is M -regular, then M/aM is Cohen–Macaulay if and only if M is Cohen–Macaulay. We prove, as a consequence of the following Lemma, that the same holds for a superficial element, if the dimension of the module M is at least two.

In the following we denote by $H_{\mathfrak{q}}^i(M)$ the i -th local cohomology module of M with respect to $\mathfrak{q}$. We know that $H_{\mathfrak{q}}^0(M) := \bigcup_{j \geq 0} (0 :_M \mathfrak{q}^j) = 0 :_M \mathfrak{q}^t$ for every $t \gg 0$; further $\min\{i \mid H_{\mathfrak{q}}^i(M) \neq 0\} = \text{depth}_{\mathfrak{q}}(M)$.

Lemma 1.5. *Let a be an $\mathbb{M}$ -superficial element for $\mathfrak{q}$ and let $j \geq 1$. Then we have $\text{depth}_{\mathfrak{q}}(M) \geq j + 1$ if and only if $\text{depth}_{\mathfrak{q}}(M/aM) \geq j$.*

Proof. Let $\text{depth}_{\mathfrak{q}}(M) \geq j + 1$; then $\text{depth}_{\mathfrak{q}}(M) > 0$ so that a is M -regular. This implies $\text{depth}_{\mathfrak{q}}(M/aM) = \text{depth}_{\mathfrak{q}}(M) - 1 \geq j + 1 - 1 = j$.

Let us assume now that $\text{depth}_{\mathfrak{q}}(M/aM) \geq j$. Since $j \geq 1$, this implies $H_{\mathfrak{q}}^0(M/aM) = 0$. Hence $H_{\mathfrak{q}}^0(aM) = H_{\mathfrak{q}}^0(M)$, so that $H_{\mathfrak{q}}^0(M) \subseteq aM$. We claim that

$H_{\mathfrak{q}}^0(M) = aH_{\mathfrak{q}}^0(M)$. If this is the case, then, by Nakayama, we get $H_{\mathfrak{q}}^0(M) = 0$ which implies $\text{depth}(M) > 0$, so that a is M -regular. Hence

$$\text{depth}_{\mathfrak{q}}(M) = \text{depth}_{\mathfrak{q}}(M/aM) + 1 \geq j + 1,$$

as wanted.

Let us prove the claim. Suppose by contradiction that

$$aH_{\mathfrak{q}}^0(M) \subsetneq H_{\mathfrak{q}}^0(M) \subseteq aM,$$

and let $ax \in H_{\mathfrak{q}}^0(M), x \in M \setminus H_{\mathfrak{q}}^0(M)$. This means that for every $t \gg 0$ we have

$$\begin{cases} aq^t x = 0 \\ q^t x \neq 0 \end{cases}$$

We prove that this implies that a is not $\mathbb{M}$ -superficial for $\mathfrak{q}$. Namely, given a positive integer c , we can find an integer $t \geq c$ and an element $d \in \mathfrak{q}^t$ such that $adx = 0$ and $dx \neq 0$. Since $\cap M_i = \{0\}$, we have $dx \in M_{j-1} \setminus M_j$ for some integer j . Now, $d \in \mathfrak{q}^t$ hence $dx \in M_t \subseteq M_c$, which implies $j \geq c$. Finally we have $dx \in M_c$, $dx \notin M_j$ and $adx = 0 \in M_{j+1}$, hence

$$(M_{j+1} :_M a) \cap M_c \subsetneq M_j.$$

The claim and the Lemma are proved. $\square$

It is interesting to recall that the integral closure of the ideals generally behaves well reducing modulo a superficial element. For example S. Itoh [58, p. 648] proved that:

Proposition 1.1. *If $\mathfrak{q}$ is an $\mathfrak{m}$ -primary ideal which is integrally closed in a Cohen–Macaulay local ring $(A, \mathfrak{m})$ of dimension $r \geq 2$, then (at least after passing to a faithfully flat extension) there exists a superficial element $a \in \mathfrak{q}$ such that $\mathfrak{q}/(a)$ is integrally closed in $A/(a)$.*

This result will be useful in proofs working by induction on r . The compatibility of integrally closed ideals with specialization by generic elements can be extended to that of modules (see [48]).

However, as we will see later, superficial elements do not behave well for Ratliff–Rush closed ideals: there exist many ideals of A all of whose powers are Ratliff–Rush closed, yet for every superficial element $a \in \mathfrak{q}$, $\mathfrak{q}/(a)$ is not Ratliff–Rush closed. Examples are given by Rossi and Swanson in [78]. Recently Puthenpurakal in [71] characterized local rings and ideals for which the Ratliff–Rush filtration *behaves well* modulo a superficial element.

It is interesting to recall that Trung and Verma in [103] introduced superficial sequences with respect to a set of ideals by working in a multigraded context.

1.3 The Hilbert Function and Hilbert Coefficients

Let $\mathbb{M}$ be a good $\mathfrak{q}$ -filtration on the A -module M . From now on we shall require the assumption that the length of $M/\mathfrak{q}M$, which we denote by $\lambda(M/\mathfrak{q}M)$, is finite. In this case there exists an integer s such that $\mathfrak{m}^s M \subseteq (\mathfrak{q} + (0 :_A M))M$, hence (see [65]), the ideal $\mathfrak{q} + (0 :_A M)$ is primary for the maximal ideal $\mathfrak{m}$. Also the length of M/M_j is finite for all $j \geq 0$.

From now on $\mathfrak{q}$ will denote an $\mathfrak{m}$ -primary ideal of the local ring $(A, \mathfrak{m})$.

In this setting we can define the *Hilbert function* of the filtration $\mathbb{M}$ or simply of the filtered module M , if there is no confusion. By definition it is the function

$$H_{\mathbb{M}}(j) := \lambda(M_j/M_{j+1}).$$

It is also useful to consider the numerical function

$$H_{\mathbb{M}}^1(j) := \lambda(M/M_{j+1}) = \sum_{i=0}^j H_{\mathbb{M}}(i)$$

which is called the *Hilbert–Samuel function* of the filtration $\mathbb{M}$ or of the filtered module M .

The *Hilbert series* of the filtration $\mathbb{M}$ is the power series

$$P_{\mathbb{M}}(z) := \sum_{j \geq 0} H_{\mathbb{M}}(j) z^j.$$

The power series

$$P_{\mathbb{M}}^1(z) := \sum_{j \geq 0} H_{\mathbb{M}}^1(j) z^j$$

is called the *Hilbert–Samuel series* of $\mathbb{M}$. It is clear that

$$P_{\mathbb{M}}(z) = (1 - z)P_{\mathbb{M}}^1(z).$$

By the Hilbert–Serre theorem, see for example [6], we can write

$$P_{\mathbb{M}}(z) = \frac{h_{\mathbb{M}}(z)}{(1 - z)^r}$$

where $h_{\mathbb{M}}(z) \in \mathbb{Z}[z]$, $h_{\mathbb{M}}(1) \neq 0$ and r is the dimension of M .

The polynomial $h_{\mathbb{M}}(z)$ is called the *h-polynomial* of $\mathbb{M}$ and we clearly have

$$P_{\mathbb{M}}^1(z) = \frac{h_{\mathbb{M}}(z)}{(1 - z)^{r+1}}.$$

An easy computation shows that if we let for every $i \geq 0$

$$e_i(\mathbb{M}) := \frac{h_{\mathbb{M}}^{(i)}(1)}{i!}$$

then for $n \gg 0$ we have

$$H_{\mathbb{M}}(n) = \sum_{i=0}^{r-1} (-1)^i e_i(\mathbb{M}) \binom{n+r-i-1}{r-i-1}.$$

The polynomial

$$p_{\mathbb{M}}(X) := \sum_{i=0}^{r-1} (-1)^i e_i(\mathbb{M}) \binom{X+r-i-1}{r-i-1}$$

has rational coefficients and is called the *Hilbert polynomial* of $\mathbb{M}$; it satisfies the equality

$$H_{\mathbb{M}}(n) = p_{\mathbb{M}}(n)$$

for $n \gg 0$. The integers $e_i(\mathbb{M})$ are called the *Hilbert coefficients* of $\mathbb{M}$ or of the filtered module M with respect to the filtration $\mathbb{M}$.

In particular $e_0(\mathbb{M})$ is the *multiplicity* of $\mathbb{M}$ and, by Proposition 11.4 in [2] (also Proposition 4.6.5 in [6]) $e_0(\mathbb{M}) = e_0(\mathbb{N})$, for every pair of good $\mathfrak{q}$ -filtrations $\mathbb{M}$ and $\mathbb{N}$ of M .

Since $P_{\mathbb{M}}^1(z) = \frac{h_{\mathbb{M}}(z)}{(1-z)^{r+1}}$ the polynomial

$$p_{\mathbb{M}}^1(X) := \sum_{i=0}^r (-1)^i e_i(\mathbb{M}) \binom{X+r-i}{r-i}$$

satisfies the equality

$$H_{\mathbb{M}}^1(n) = p_{\mathbb{M}}^1(n)$$

for $n \gg 0$ and is called the *Hilbert–Samuel polynomial* of M .

In the following we will write the h -polynomial of M in the form

$$h_{\mathbb{M}}(z) = h_0(\mathbb{M}) + h_1(\mathbb{M})z + \cdots + h_s(\mathbb{M})z^s,$$

so that the integers $h_i(\mathbb{M})$ are well defined for every $i \geq 0$ and we have

$$e_i(\mathbb{M}) = \sum_{k \geq i} \binom{k}{i} h_k(\mathbb{M}). \quad (1.1)$$

Finally we remark that if M is Artinian, then $e_0(\mathbb{M}) = \lambda(M)$.

In the case of the $\mathfrak{q}$ -adic filtration on the ring A , we will denote by $H_{\mathfrak{q}}(j)$ the Hilbert function, by $P_{\mathfrak{q}}(z)$ the Hilbert series, and by $e_i(\mathfrak{q})$ the Hilbert coefficients of the $\mathfrak{q}$ -adic filtration. In the case of the $\mathfrak{q}$ -adic filtration on a module M , we will replace $\mathfrak{q}$ with $\mathfrak{q}M$ in the above notations.

The following result is called *Singh's formula* because the corresponding equality in the classical case was obtained by Singh in [97]. See also [117, Lemma 3, Chap. 8] for the corresponding equality with Hilbert polynomials.

Lemma 1.6. *Let $a \in \mathfrak{q}$; then for every $j \geq 0$ we have*

$$H_{\mathbb{M}}(j) = H_{\mathbb{M}/aM}^1(j) - \lambda(M_{j+1} : a/M_j).$$

Proof. The proof is an easy consequence of the following exact sequence:

$$0 \rightarrow (M_{j+1} : a)/M_j \rightarrow M/M_j \xrightarrow{a} M/M_{j+1} \rightarrow M/(M_{j+1} + aM) \rightarrow 0.$$

□

We remark that, by Singh's formula, for every $a \in \mathfrak{q}$ we have

$$P_{\mathbb{M}}(z) \leq P_{\mathbb{M}/aM}^1(z) \quad (1.2)$$

It is thus interesting to consider elements $a \in \mathfrak{q}$ such that $P_{\mathbb{M}}(z) = P_{\mathbb{M}/aM}^1(z)$. By the above formula and Lemma 1.1, these are exactly the elements $a \in \mathfrak{q}$ such that $\bar{a} \in \mathfrak{q}/\mathfrak{q}^2$ is regular over $\text{gr}_{\mathbb{M}}(M)$.

As a corollary of Singh's formula we get a number of useful properties of superficial elements.

Proposition 1.2. *Let a be an $\mathbb{M}$ -superficial element for $\mathfrak{q}$ and let $r = \dim M \geq 1$. Then we have:*

- (1) $\dim(M/aM) = r - 1$.
- (2) $e_j(\mathbb{M}) = e_j(\mathbb{M}/aM)$ for every $j = 0, \dots, r - 2$.
- (3) $e_{r-1}(\mathbb{M}/aM) = e_{r-1}(\mathbb{M}) + (-1)^{r-1} \lambda(0 : a)$.
- (4) *There exists an integer j such that for every $n \geq j - 1$ we have $e_r(\mathbb{M}/aM) = e_r(\mathbb{M}) + (-1)^r (\sum_{i=0}^n \lambda(M_{i+1} : a/M_i) - (n+1)\lambda(0 : a))$.*
- (5) a^* is a regular element on $\text{gr}_{\mathbb{M}}(M)$ if and only if $P_{\mathbb{M}}(z) = P_{\mathbb{M}/aM}^1(z) = \frac{P_{\mathbb{M}/aM}(z)}{1-z}$ if and only if a is M -regular and $e_r(\mathbb{M}) = e_r(\mathbb{M}/aM)$.

Proof. By Lemma 1.6 we have

$$P_{\mathbb{M}}(z) = P_{\mathbb{M}/aM}^1(z) - \sum_{i \geq 0} \lambda(M_{i+1} : a/M_i) z^i.$$

Since a is superficial, by Theorem 1.2, part 6, there exists an integer j such that for every $n \geq j$ we have

$$M_{n+1} : a/M_n = M_n + (0 : a)/M_n \simeq (0 : a)/(0 : a) \cap M_n = (0 : a).$$

Hence we can write

$$P_{\mathbb{M}}(z) = P_{\mathbb{M}/aM}^1(z) - \sum_{i=0}^{j-1} \lambda(M_{i+1} : a/M_i) z^i - \frac{\lambda(0 : a) z^j}{(1-z)}.$$

This proves (1) (actually (1) follows also from Theorem 1.2(4)) and, as a consequence, we get

$$h_{\mathbb{M}}(z) = h_{\mathbb{M}/aM}(z) - (1-z)^r \left(\sum_{i=0}^{j-1} \lambda(M_{i+1} : a/M_i) z^i \right) - (1-z)^{r-1} \lambda(0 : a) z^j.$$

This gives easily (2), (3) and (4), while (5) follows from (4) and Lemma 1.1. $\square$

As we can see in the proof of the above result, if a is an $\mathbb{M}$ -superficial element, there exists an integer j such that $M_{n+1} : a/M_n = (0 : a)$ for every $n \geq j$. Hence, Proposition 1.2 part 4 can be rewritten as follows

$$e_r(\mathbb{M}/aM) = e_r(\mathbb{M}) + (-1)^r \left(\sum_{i=0}^{j-1} \lambda(M_{i+1} : a/M_i) - j\lambda(0 : a) \right).$$

For example let $A = k[[X, Y, Z]]/(X^3, X^2Y^3, X^2Z^4) = k[[x, y, z]]$. We have $r = \dim(A) = 2$ and if we consider on $M = A$ the $\mathfrak{m}$ -adic filtration $\mathbb{M} = \{\mathfrak{m}^j\}$, then y is an $\mathbb{M}$ -superficial element for $\mathfrak{q} = \mathfrak{m}$. We have

$$P_{\mathbb{M}}(z) = \frac{1 + z + z^2 - z^5 - z^6 + z^9}{(1-z)^2}$$

so that

$$e_0(\mathbb{M}) = 2, \quad e_1(\mathbb{M}) = 1, \quad e_2(\mathbb{M}) = 12.$$

Also it is clear that

$$\lambda(\mathfrak{m}^{n+1} : y/\mathfrak{m}^n) = \begin{cases} 0 & \text{for } n = 0, \dots, 4, \\ 1 & \text{for } n = 5, \\ 2 & \text{for } n = 6, \\ 3 & \text{for } n = 7, \\ 4 = \lambda(0 : y) & \text{for } n \geq 8 \end{cases}$$

so that $j = 8$ in the above proposition. Hence, by using Proposition 1.2,

$$e_0(\mathbb{M}/(y)) = 2, \quad e_1(\mathbb{M}/(y)) = -3, \quad e_2(\mathbb{M}/(y)) = -14$$

Notice that

$$P_{\mathbb{M}/(y)}(z) = \frac{1 + z + z^2 - z^6}{1 - z}.$$

Remark 1.2. By Proposition 1.2, Theorem 1.2 and Singh's formula, if $a \in \mathfrak{q}$ is an element which is M -regular, then

$$a \text{ is } \mathbb{M}\text{-superficial for } \mathfrak{q} \iff e_i(\mathbb{M}) = e_i(\mathbb{M}/aM) \text{ for every } i = 0, \dots, r-1.$$

Thus we get a useful criterion for checking whether an element is superficial or not. From the computational point of view it reduces infinitely many conditions, coming a priori from the definition of superficial element, to the computation of the Hilbert polynomials of $\mathbb{M}$ and $\mathbb{M}/aM$ which is often available, for example, in the CoCoA system.

An $\mathbb{M}$ -superficial sequence $a_1, \dots, a_r$ for $\mathfrak{q}$ is called a *maximal $\mathbb{M}$ -superficial sequence* if $\mathbb{M}/(a_1, \dots, a_r)M$ is nilpotent, but $\mathbb{M}/(a_1, \dots, a_{r-1})M$ is not nilpotent. By Proposition 1.2, a superficial sequence is in particular a system of parameters, hence if $\dim M = r$, every $\mathbb{M}$ -superficial sequence $a_1, \dots, a_r$ for $\mathfrak{q}$ is a maximal superficial sequence.

In our approach maximal $\mathbb{M}$ -superficial sequences will play a fundamental role. We also remark that they are strictly related to minimal reductions.

Let $\mathbb{M}$ be a good $\mathfrak{q}$ -filtration of M and $J \subseteq \mathfrak{q}$ an ideal. Then J is an $\mathbb{M}$ -reduction of $\mathfrak{q}$ if $M_{n+1} = JM_n$ for $n \gg 0$. We say that J is a *minimal $\mathbb{M}$ -reduction* of $\mathfrak{q}$ if it is minimal with respect to the inclusion.

If $\mathfrak{q}$ is $\mathfrak{m}$ -primary and the residue field is infinite, there is a complete correspondence between maximal $\mathbb{M}$ -superficial sequences for $\mathfrak{q}$ and minimal $\mathbb{M}$ -reductions of $\mathfrak{q}$. Every minimal $\mathbb{M}$ -reduction J of $\mathfrak{q}$ can be generated by a maximal $\mathbb{M}$ -superficial sequence, conversely the ideal generated by a maximal $\mathbb{M}$ -superficial sequence is a minimal $\mathbb{M}$ -reduction of $\mathfrak{q}$ (see [56] and [11]).

Notice that if we consider for example $\mathfrak{q} = (x^7, x^6y, x^3y^4, x^2y^5, y^7)$ in the power series ring $k[[x, y]]$, then $J = (x^7, y^7)$ is a minimal reduction of $\mathfrak{q}$, but $\{x^7, y^7\}$ is not a superficial sequence, in particular x^7 is not superficial for $\mathfrak{q}$. Nevertheless $\{x^7 + y^7, x^7 - y^7\}$ is a minimal system of generators of J and it is a superficial sequence for $\mathfrak{q}$.

In this presentation we prefer to handle $\mathbb{M}$ -superficial sequences with respect to minimal reductions because they have a better behaviour for studying Hilbert functions and Hilbert coefficients via Proposition 1.2.

1.4 Maximal Hilbert Functions

Superficial sequences play an important role in the following result where maximal Hilbert functions are described. The result was proved in the classical case in [83, Theorem 2.2] and here is extended to the filtrations of a module which is not necessarily Cohen–Macaulay.

Theorem 1.3. *Let M be a module of dimension $r \geq 1$ and let J be the ideal generated by a maximal $\mathbb{M}$ -superficial sequence for $\mathfrak{q}$. Then*

$$P_{\mathbb{M}}(z) \leq \frac{\lambda(M/M_1) + \lambda(M_1/JM)z}{(1-z)^r}.$$

If the equality holds, then $\text{gr}_{\mathbb{M}}(M)$ is Cohen–Macaulay and hence M is Cohen–Macaulay.

Proof. We induct on r . Let $r = 1$ and $J = (a)$. We have

$$\frac{\lambda(M/M_1) + \lambda(M_1/JM)z}{(1-z)} = \lambda(M/M_1) + \lambda(M/aM) \sum_{j \geq 1} z^j$$

and $H_{\mathbb{M}}(0) = \lambda(M/M_1)$.

From the diagram

$$\begin{array}{ccc} M & \supseteq & M_n \supseteq M_{n+1} \\ || & & \cup \\ M & \supseteq & aM \supseteq aM_n \end{array}$$

we get

$$\lambda(M/aM) + \lambda(aM/aM_n) = \lambda(M/M_n) + H_{\mathbb{M}}(n) + \lambda(M_{n+1}/aM_n).$$

On the other hand, from the exact sequence

$$0 \rightarrow (0 : a + M_n)/M_n \rightarrow M/M_n \xrightarrow{a} aM/aM_n \rightarrow 0$$

we get

$$\lambda(M/M_n) = \lambda(aM/aM_n) + \lambda((0 : a + M_n)/M_n).$$

It follows that for every $n \geq 1$

$$\lambda(M/aM) = H_{\mathbb{M}}(n) + \lambda(M_{n+1}/aM_n) + \lambda((0 : a + M_n)/M_n). \quad (1.3)$$

This proves that $H_{\mathbb{M}}(n) \leq \lambda(M/aM)$ for every $n \geq 1$ and the first assertion of the theorem follows.

If we have equality above then, by (1.3), for every $n \geq 1$ we get

$$\lambda(M_{n+1}/aM_n) = \lambda((0 : a + M_n)/M_n) = 0.$$

This clearly implies that $M_{j+1} : a = M_j$ for every $j \geq 1$ so that, by Lemma 1.1, $a^* \in \mathfrak{q}/\mathfrak{q}^2$ is regular on $\text{gr}_{\mathbb{M}}(M)$ and $\text{gr}_{\mathbb{M}}(M)$ is Cohen–Macaulay.

Suppose $r \geq 2$, $J = (a_1, \dots, a_r)$ and let us consider the good $\mathfrak{q}$ -filtration $\mathbb{M}/a_1M$ on M/a_1M . We have $\dim M/a_1M = r - 1$ and we know that $a_2, \dots, a_r$ is a maximal

$\mathbb{M}/a_1M$ -superficial sequence for $\mathfrak{q}$. By the inductive assumption and since $a_1M \subseteq M_1$, we get

$$\begin{aligned} P_{\mathbb{M}/a_1M}(z) &\leq \frac{\lambda((M/a_1M)/(a_1M + M_1/a_1M)) + \lambda((a_1M + M_1/a_1M)/K(M/a_1M))z}{(1-z)^{r-1}} \\ &= \frac{\lambda(M/M_1) + \lambda(M_1/JM)z}{(1-z)^{r-1}}. \end{aligned}$$

where we let $K := (a_2, \dots, a_r)$.

By using (1.2) and since the power series $\frac{1}{1-z}$ is positive, we get

$$P_{\mathbb{M}}(z) \leq P_{\mathbb{M}/a_1M}^1(z) = \frac{P_{\mathbb{M}/a_1M}(z)}{1-z} \leq \frac{\lambda(M/M_1) + \lambda(M_1/JM)z}{(1-z)^r},$$

as wanted.

If we have equality, then

$$P_{\mathbb{M}/a_1M}(z) = \frac{\lambda(M/M_1) + \lambda(M_1/JM)z}{(1-z)^{r-1}}$$

so that $gr_{\mathbb{M}/a_1M}(M/a_1M)$ is Cohen–Macaulay and hence $gr_{\mathbb{M}}(M)$ is Cohen–Macaulay as well by Sally’s machine. In particular M is Cohen–Macaulay. $\square$

The above result says that if the h -polynomial is $h_{\mathbb{M}}(z) = \lambda(M/M_1) + \lambda(M_1/JM)z$, we may conclude that $gr_{\mathbb{M}}(M)$ is Cohen–Macaulay even if we do not assume the Cohen–Macaulayness of M . The result cannot be extended to any short h -polynomial $h_{\mathbb{M}}(z) = h_0 + h_1z$. For example if $A = k[[x, y]]/(x^2, xy, xz, y^3)$ and we consider the $\mathfrak{m}$ -adic filtration, then $P_A(z) = \frac{1+2z}{1-z}$, but $gr_{\mathfrak{m}}(A) \simeq A$ is not Cohen–Macaulay.

In the classical case of the $\mathfrak{m}$ -adic filtration on a local Cohen–Macaulay ring A , Elias and Valla in [26] proved that the h -polynomial of the form $h_{\mathfrak{m}}(z) = h_0 + h_1z + h_2z^2$ forces $gr_{\mathfrak{m}}(A)$ to be Cohen–Macaulay. We cannot extend this result to general filtrations because this is not longer true even if we consider the $\mathfrak{q}$ -adic filtration with $\mathfrak{q}$ an $\mathfrak{m}$ -primary ideal of a Cohen–Macaulay ring. The following example is due to Sally [92, Example 3.3].

Example 1.1. Let $A = k[[t^4, t^5, t^6, t^7]]$ and consider $\mathfrak{q} = (t^4, t^5, t^6)$. We have

$$P_{\mathfrak{q}}(z) = \frac{2+z+z^2}{(1-z)}$$

and $gr_{\mathfrak{q}}(A)$ is not Cohen–Macaulay because $a = t^4$ is a superficial (regular) element for $\mathfrak{q}$, but $\mathfrak{q}^2 : a \neq \mathfrak{q}$ (cf. Lemmas 1.1 and 1.3).



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