Chapter 4
Invariant Fiber Bundles

In a broad range of qualitative studies on nonlinear dynamical systems, invariant manifolds are omnipresent and play a crucial role for local as well as global questions: For instance, local stable and unstable manifolds dictate the saddle-point behavior in the vicinity of hyperbolic solutions (or surfaces) of a system. As illustrated by the celebrated reduction principle of Pliss, center manifolds are a paramount tool to simplify given dynamical systems in terms of a reduction of their state space dimension. Concerning a more global perspective, stable manifolds serve as separatrix between different domains of attractions and allow a classification of solutions with a specific asymptotic behavior. Systems with a gradient structure possess global attractors consisting of unstable manifolds (and equilibria). Finally, so-called inertial manifolds are global versions of the classical center-unstable manifolds and yield a global reduction principle for typically infinite-dimensional dissipative equations.

The invariant fiber bundles introduced in this chapter generalize invariant manifolds from the well-known autonomous dynamical systems to nonautonomous difference equations. Precisely, we call a nonautonomous set $\mathcal{W}$ in the extended state space $\mathcal{X}$ a *(forward) invariant fiber bundle*[1] of (D), if it is *(forward) invariant* and each fiber $\mathcal{W}(k)$ is a submanifold of a linear space X_k for $k \in \mathbb{I}$.

The contents of this chapter can be summarized as follows:

- From a technical perspective it is advantageous to initially work with semilinear implicit difference equations. For this type of systems, we provide an existence and uniqueness criterion for forward and backward solutions, as well as assumptions guaranteeing the existence of a nontrivial global attractor.
- In the following section, we present and discuss a fairly general version of an existence theorem for invariant fiber bundles of semilinear equations. It applies to non-invertible implicit nonautonomous difference equations, whose linear part can be pseudo-hyperbolic, i.e., associated to an arbitrary spectral splitting. More detailed, each gap in an exponential splitting (see Fig. 3.4) gives rise to two

[1] We refer to [1, p. 184, Definition 3.4.27] for the general notion of a *fiber bundle* in differential topology. In this sense, our fiber bundles $\mathcal{W}$ are trivial with the discrete interval $\mathbb{I}$ as base space and submanifolds $\mathcal{W}(k)$ as fibers.

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invariant fiber bundles intersection along a complete (exponentially) bounded solution. These fiber bundles consist of solutions with a particular exponential growth behavior in forward resp. backward time – thus, our approach is based on the Lyapunov–Perron method. Using a less functional analytical argument, we also construct nontrivial intersections of invariant fiber bundles yielding an extended hierarchy.

- Whereas in Sect. 4.2 we construct invariant fiber bundles, we next investigate the asymptotic behavior of solutions which are not contained in these bundles. Indeed, due to our general pseudo-hyperbolic framework, attractivity properties of invariant fiber bundles need to be generalized to exponential boundedness of solutions approaching the bundle. Here, we work with invariant foliations which are equivalence classes of solutions converging towards a given solution at an exponential rate. In order to obtain an asymptotic phase property, we track a particular solution starting on a fiber bundle.
- Besides existence we next tackle the smoothness of invariant fiber bundles, which is of fundamental importance in applications. In particular, an elementary yet lengthy proof for the differentiability is presented: Fundamentally based on the classical contraction mapping principle only, none of the classical approaches, i.e., Banach space scale techniques, a Henry-type lemma or the fiber contraction theorem, is involved.
- As prerequisite for persistence under perturbation or discretization, we also establish the normal hyperbolicity of invariant fiber bundles.
- The subsequent sections present two applications of this flexible framework. First, we weaken the global assumptions and obtain (pseudo-) stable and unstable fiber bundles, which are related to given (pseudo-) hyperbolic reference solutions and describe the local saddle-point structure around them. W.r.t. the aspects given above, the corresponding Theorem 4.6.4 extends stable manifold theorems commonly found in the literature. Intersections of these pseudo-stable and -unstable bundles yield center-like bundles and in particular the classical hierarchy of the stable, center-stable, center, center-unstable and the unstable bundle. The center-unstable fiber bundle's asymptotic phase enables us to derive a nonautonomous version of Pliss' reduction principle. It states that stability properties of nonhyperbolic solutions are determined by their behavior on the center-unstable fiber bundle. In order to apply it, we address local approximation issues of invariant fiber bundles by means of Taylor series. Differing from the autonomous situation, the time-dependent Taylor coefficients are bounded solutions of a linear difference equation rather than solutions of a linear algebraic problem.
- Furthermore, discrete versions of inertial manifolds are constructed. Despite not having an asymptotic phase, they still possess the beneficial property of being asymptotically complete. Beyond the situation of classical center-unstable manifolds, inertial fiber bundles allow a global reduction principle guaranteeing that the essential dynamics of a possibly infinite-dimensional problem is given by a finite-dimensional difference equation. In particular, inertial bundles contain the global attractor of dissipative equations.

- Last but not least, we discuss an approximation method for the invariant fiber bundles. It is based on fixed point iteration for the Lyapunov–Perron operator. A corresponding error estimate justifies that one can pass over to the so-called truncated Lyapunov–Perron operator, which involves only finite sums. In a nutshell, this means that invariant fiber bundles can be approximated by solving nonlinear systems of algebraic equations, which can be efficiently and successfully achieved using, e.g., Newton methods from numerical analysis.
- Our theoretical results from Sect. 4.1 apply to full discretizations of semilinear FDEs. Using a simple example we show that under corresponding assumptions, attractors of discretized semilinear DDEs can be nontrivial. The following subsection on time-discretized abstract evolutionary equations shows how global integral manifolds can be constructed using an appropriate discretization from Sect. 2.6.2; moreover, we provide criteria for the existence of a global attractor. Using two examples, we illustrate how hierarchies of invariant fiber bundles can be constructed for temporal discretizations of parabolic evolution equation. For full discretizations of scalar RDEs and the complex Ginzburg–Landau equation we prove the existence of inertial fiber bundles – here, our results are quantitative and we obtain explicit dimension estimates. The algorithm from Sect. 4.8 is exemplified in order to approximate the inertial manifold of a scalar RDE of Chafee–Infante type.

Throughout the chapter, we suppose that $\mathbb{I}$ is an unbounded discrete interval, the extended state space $\mathcal{X}$ consists of Banach spaces and $\mathcal{Y}$ of linear spaces.

4.1 Semilinear Difference Equations

During this opening section and beyond, nonautonomous equations of the form

$$B_{k+1}x' = A_k x + f_k(x, x') \tag{S}$$

are in the center of our interest. As opposed to (S_g) studied in Sect. 3.5, we do not suppose that (S) admits the trivial solution.

One denotes (S) as *semilinear*, when it is studied using perturbation techniques on the basis of an established linear theory applicable to

$$B_{k+1}x' = A_k x, \tag{L_0}$$

where A_k, B_k are as in Definition 3.1.1. For instance, this is possible for globally Lipschitzian or linearly bounded nonlinearities f_k. Note that for a *linearly implicit equation* the functions f_k do not depend on their second argument x'.

Hypothesis 4.1.1. *Suppose that the linear homogeneous equation* (L_0) *satisfies* (3.1a) *and* $f_k : X_k \times X_{k+1} \to Y_{k+1}$ *fulfills* $f_k(X_k, X_{k+1}) \subseteq \operatorname{im} B_{k+1}$,

$$\operatorname{lip}_2 B_{k+1}^{-1} f_k < 1 \quad \textit{for all } k \in \mathbb{I}'. \tag{4.1a}$$

Remark 4.1.2. The global Lipschitz condition (4.1a) trivially holds for linearly implicit equation (S). In various discretizations it can be fulfilled for small temporal stepsizes. Especially for the θ-method from Example 2.1.4 or the 2-stage θ-method in Example 2.1.6, small values of $\theta \in [0,1]$ ensure (4.1a).

Proposition 4.1.3. *Let $m \in \mathbb{N}$. Under Hypothesis 4.1.1 the following holds:*

(a) The general forward solution φ to (S) *exists on $\mathcal{X}$.*
(b) If $B_{k+1}^{-1}A_k \in L(X_k, X_{k+1})$ and $B_{k+1}^{-1}f_k(\cdot, x') : X_k \to X_{k+1}$, $x' \in X_{k+1}$, is continuous for all $k \in \mathbb{I}'$, then also φ is continuous.
(c) If $B_{k+1}^{-1}A_k \in L(X_k, X_{k+1})$ and $B_{k+1}^{-1}f_k \in C^m(X_k \times X_{k+1}, X_{k+1})$ for all $k \in \mathbb{I}'$, then also $\varphi(k;\kappa,\cdot) \in C^m(X_\kappa, X_k)$ for all $\kappa \le k$.

Proof. The claims (a) and (b) are an immediate consequence of Theorem 2.3.6, while assertion (c) follows from Theorem 2.3.9. □

Next we formulate a dual version of Proposition 4.1.3 for backward solutions.

Proposition 4.1.4. *Let $m \in \mathbb{N}$ and suppose that* (L_0) *satisfies* (3.1a) *with $B_{k+1}^{-1}A_k \in GL(X_k, X_{k+1})$, $k \in \mathbb{I}'$. If $f_k : X_k \times X_{k+1} \to Y_{k+1}$ fulfills*

$$f_k(X_k, X_{k+1}) \subseteq \operatorname{im} B_{k+1}, \quad \left\|(B_{k+1}^{-1}A_k)^{-1}\right\|_{L(X_{k+1},X_k)} \operatorname{lip}_1 B_{k+1}^{-1} f_k < 1$$

for all $k \in \mathbb{I}'$, then the following holds:

(a) The general backward solution φ to (S) *exists on $\mathcal{X}$.*
(b) In case $B_{k+1}^{-1}f_k(x,) : X_{k+1} \to X_{k+1}$, $x \in X_k$, is continuous for all $k \in \mathbb{I}'$, then φ is continuous.
(c) In case $B_{k+1}^{-1}f_k \in C^m(X_k \times X_{k+1}, X_{k+1})$ for all $k \in \mathbb{I}'$, then one has the inclusion $\varphi(k;\kappa,\cdot) \in C^m(X_\kappa, X_k)$ for all $k \le \kappa$.

Proof. In order to construct backward solutions, we proceed as in Proposition 4.1.3 using Theorem B.1.1 for (a), (b), resp. Theorem B.1.5 for (c) applied to the fixed point problem $B_{k+1}^{-1}A_k\left[x' - B_{k+1}^{-1}f_k(x,x')\right] = x$ for all $k \in \mathbb{I}'$, $x' \in X_{k+1}$. □

Having the variation of constants formula from Theorem 3.1.16 available, we can prove dissipativity results for semilinear equations (S). While the Lipschitz conditions assumed in Theorem 3.5.8 implied a condition for exponential stability (see Remark 3.5.9(1)), we now are interested in boundedness properties:

Proposition 4.1.5. *Suppose that beyond Hypothesis 4.1.1 the following holds:*

(i) $B_{k+1}^{-1}A_k \in L(X_k, X_{k+1})$, $k \in \mathbb{I}'$, and there exist $K \ge 1$ and $a : \mathbb{I} \to (0,\infty)$ with

$$\|\Phi(k,l)\|_{L(X_l,X_k)} \le K e_a(k,l) \quad \textit{for all } l \le k. \tag{4.1b}$$

(ii) There exist sequences $\beta_k, \gamma_k \ge 0$ and $\delta \in [0, 1/K)$ such that

$$\left\|B_{k+1}^{-1}f_k(x,x')\right\|_{X_{k+1}} \le \beta_k + \max\left\{\gamma_k \|x\|_{X_k}, \delta \|x'\|_{X_{k+1}}\right\} \tag{4.1c}$$

for all $k \in \mathbb{I}'$, $x \in X_k$ and $x' \in X_{k+1}$.

Then the general forward solution of (S) *satisfies for all $\kappa \le k$, $\xi \in X_\kappa$ that*

$$\|\varphi(k;\kappa,\xi)\|_{X_k} \le \frac{K}{1-\delta K}\left(e_{\frac{a+\gamma K}{1-\delta K}}(k,\kappa)\,\|\xi\|_{X_\kappa} + \sum_{l=\kappa}^{k-1} e_{\frac{a+\gamma K}{1-\delta K}}(k,l+1)\beta_l\right).$$

Proof. Thanks to Corollary 3.1.18(a) the forward evolution operator Φ for (L_0) exists and is bounded. From Proposition 4.1.3(a) we know that the general forward solution φ to (S) exists. For a fixed $(\kappa,\xi) \in \mathcal{X}$ we abbreviate $\varphi(k) := \varphi(k;\kappa,\xi)$. Thus, according to Theorem 3.1.16(a) the sequence φ satisfies

$$\varphi(k) \overset{(3.1h)}{=} \Phi(k,\kappa)\xi + \sum_{l=\kappa}^{k-1} \Phi(k,l+1)B_{l+1}^{-1} f_l(\underline{\varphi(l)})$$

for all $\kappa \le k$, where we have abbreviated $f_l(\underline{\varphi(l)} := f_l(\varphi(l),\varphi'(l))$ (see p. 54). Passing over to the norms, we obtain

$$\begin{aligned}
\|\varphi(k)\| &\overset{(4.1b)}{\le} K e_a(k,\kappa)\,\|\xi\| + K\sum_{l=\kappa}^{k-1} e_a(k,l+1)\left\|B_{l+1}^{-1} f_l(\underline{\varphi(l)})\right\| \\
&\overset{(4.1c)}{\le} K e_a(k,\kappa)\,\|\xi\| + K\sum_{l=\kappa}^{k-1} e_a(k,l+1)\beta_l \\
&\quad + K\sum_{l=\kappa}^{k-1} e_a(k,l+1)\gamma_l\,\|\varphi(l)\| + \delta K \sum_{l=\kappa}^{k-1} e_a(k,l+1)\,\|\varphi'(l)\|,
\end{aligned}$$

abbreviating $u(k) := e_a(\kappa,k)\,\|\varphi(k)\|$ yields

$$u(k) \le K\,\|\xi\| + K\sum_{l=\kappa}^{k-1} e_a(\kappa,l+1)\beta_l + K\sum_{l=\kappa}^{k-1}\frac{\gamma_l}{a(l)}u(l) + \delta K\sum_{l=\kappa}^{k-1} u'(l)$$

and from this we finally infer

$$u(k) \le \frac{K}{1-\delta K}\left(\|\xi\| + \sum_{l=\kappa}^{k-1} e_a(\kappa,l+1)\beta_l\right) + \frac{K}{1-\delta K}\sum_{l=\kappa}^{k-1}\left(\frac{\gamma_l}{a(l)}+\delta\right)u(l)$$

for all $\kappa \le k$. The Gronwall lemma in Proposition A.2.1(a) implies

$$u(k) \overset{(A.2b)}{\le} \frac{K}{1-\delta K} e_{1+b}(k,\kappa)\,\|\xi\| + \frac{K}{1-\delta K}\sum_{l=\kappa}^{k-1} e_{1+b}(k,l+1)e_a(\kappa,l+1)\beta_l$$

with $b(k) := \frac{K}{1-\delta K}\left(\frac{\gamma_k}{a(k)} + \delta\right)$ and consequently by Proposition A.1.2(b)

$$\|\varphi(k)\| \overset{\text{(A.1b)}}{\leq} \frac{K}{1-\delta K} e_{a+ab}(k,\kappa)\,\|\xi\| + \frac{K}{1-\delta K}\sum_{l=\kappa}^{k-1} e_{a+ab}(k,l+1)\beta_l$$

for all $\kappa \leq k$, which was our claim. $\square$

Corollary 4.1.6. *Suppose $\varepsilon > 0$ is fixed and $\mathbb{I}$ is unbounded below. If the assumptions of Proposition 4.1.5 hold with the* summability condition

$$\rho_k := \sum_{l=-\infty}^{k-1} e_{\frac{a+\gamma K}{1-\delta K}}(k,l+1)\beta_l < \infty \quad \textit{for all } k \in \mathbb{I},$$

then the nonautonomous set $\mathcal{A} := \left\{(k,x) \in \mathcal{X} : \|x\|_{X_k} \leq \varepsilon + \frac{K\rho_k}{1-\delta K}\right\}$ is $\hat{\mathcal{B}}$-absorbing for every absorption universe

$$\hat{\mathcal{B}} \subseteq \left\{\mathcal{B} \subseteq \mathcal{X} : \lim_{n\to\infty} e_{\frac{a+\gamma K}{1-\delta K}}(k,k-n)\,\left|\mathcal{B}(k-n)\right| = 0 \textit{ for all } k \in \mathbb{I}\right\},$$

and $\hat{\mathcal{B}}$-uniformly absorbing for every absorption universe

$$\hat{\mathcal{B}} \subseteq \left\{\mathcal{B} \subseteq \mathcal{X} : \lim_{n\to\infty} \sup_{k\in\mathbb{I}} e_{\frac{a+\gamma K}{1-\delta K}}(k,k-n)\,\left|\mathcal{B}(k-n)\right| = 0\right\}.$$

Proof. Using Proposition 4.1.5 this can be shown as Corollary 2.3.19. $\square$

Corollary 4.1.7. *Let $\mathbb{I}$ be unbounded below. If beyond Hypothesis 4.1.1 the mappings $B_{k+1}^{-1}A_k$, $B_{k+1}^{-1}f_k$ satisfy* (4.1c) *and a Darbo condition with*

$$q(k) := \operatorname{dar} B_{k+1}^{-1}A_k + \operatorname{dar} B_{k+1}^{-1}f_k \in [0,1] \quad \textit{for all } k \in \mathbb{I}',$$

then the semilinear difference equation (S) *is:*

(a) $\hat{\mathcal{B}}$-contracting for every family

$$\hat{\mathcal{B}} \subseteq \left\{\mathcal{B} \subseteq \mathcal{S} : \lim_{n\to\infty} e_q(k,k-n)\chi_{k-n}(\mathcal{B}(k-n)) = 0 \quad \textit{for all } k \in \mathbb{I}\right\}.$$

(b) $\hat{\mathcal{B}}$-uniformly contracting, provided $\lim_{n\to\infty}\sup_{k\in\mathbb{I}} e_q(k,k-n) = 0$.

Proof. Let $k \in \mathbb{I}'$. By Proposition 4.1.3 the general forward solution φ of (S) exists on $\mathcal{X}$. Referring to the proof of Theorem 2.3.6 one constructs the generator $\hat{\varphi}_k$ of φ using the fixed point equation (2.3e), which in the present situation reads as

$$x = B_{k+1}^{-1}A_k\xi + B_{k+1}^{-1}f_k(\xi,x) =: T_k(\xi,x) \quad \text{for all } (k,\xi) \in \mathcal{X}.$$

Due to the assumed inequality (4.1c) its right-hand side $T_k(\cdot, x)$ is bounded for every fixed $x \in X_{k+1}$ and [35, p. 39, Proposition 5.3(c)] guarantees

$$\operatorname{dar} T_k \le \operatorname{dar} B_{k+1}^{-1} A_k + \operatorname{dar} B_{k+1}^{-1} f_k = q(k) \le 1 \quad \text{for all } \kappa \in \mathbb{I}'.$$

Hence, we can apply Corollary 2.3.8 yielding the claim. □

This brings us to the main result of this section:

Theorem 4.1.8. *Let $q \in [0, 1)$, suppose that $\mathbb{I}$ is unbounded below, the family $\hat{\mathcal{B}}$ consists of all uniformly bounded nonautonomous sets and that beyond Hypothesis 4.1.1 the following holds for all $k \in \mathbb{I}'$:*

(i) $B_{k+1}^{-1} A_k \in L(X_k, X_{k+1})$ *satisfies* (4.1b).
(ii) $B_{k+1}^{-1} f_k(\cdot, x') : X_k \to X_{k+1}$, $x' \in X_{k+1}$, *is continuous with* (4.1c).
(iii) $\sup_{l \in \mathbb{I}'} \beta_l < \infty$ *and one has the estimate* $\frac{a(k)+K\gamma_k}{1-K\delta} \in [0, q]$.

If the semilinear equation (S) *is $\hat{\mathcal{B}}$-contracting, then it possesses a uniformly bounded global attractor $\mathcal{A}^*$, which additionally satisfies*

$$\mathcal{A}^*(k) \subseteq B_{\frac{K \sup_{l \in \mathbb{I}'} \beta_l}{(1-q)(1-K\delta)}}(0, X_k) \quad \textit{for all } k \in \mathbb{I}.$$

Proof. First of all, by Proposition 4.1.3 the general forward solution φ of (S) exists and is continuous. Choose $\mathcal{B} \in \hat{\mathcal{B}}$ and by assumption (iii) with Proposition A.1.2(d) we get

$$e_{\frac{a+\gamma K}{1-\delta K}}(k, k-n) \left|\mathcal{B}(k-n)\right| \le q^n \left|\mathcal{B}(k-n)\right| \xrightarrow[n\to\infty]{} 0 \quad \text{uniformly in } k \in \mathbb{I},$$

as well as $\sum_{l=-\infty}^{k-1} e_{\frac{a+\gamma K}{1-\delta K}}(k, l+1)\beta_l \le \frac{\sup_{l\in\mathbb{I}} \beta_l}{1-q}$ for all $k \in \mathbb{I}$. So Corollary 4.1.6 ensures that φ has a closed $\hat{\mathcal{B}}$-uniformly absorbing set $\mathcal{A} \in \hat{\mathcal{B}}$, i.e., φ is uniformly bounded dissipative. We assumed that φ is $\hat{\mathcal{B}}$-contracting. Because $\mathcal{A}$ is $\hat{\mathcal{B}}$-uniformly absorbing, for each $\mathcal{B} \in \hat{\mathcal{B}}$ there exists an $N = N(\mathcal{B}) \ge 0$ such that (cf. Definition 1.3.6(b))

$$\gamma_{\mathcal{B}}^N(k) = \bigcup_{n \ge N} \varphi(k; k-n, \mathcal{B}(k-n)) \subseteq \mathcal{A}(k) \quad \text{for all } k \in \mathbb{I}$$

and $\gamma_{\mathcal{B}}^N \subseteq \mathcal{A} \in \hat{\mathcal{B}}$. Thus, Proposition 1.2.30 implies that φ is $\hat{\mathcal{B}}$-asymptotically compact. With this we have verified the assumptions of Theorem 1.3.9 and (S) admits a global attractor $\mathcal{A}^*$, which is uniformly bounded; in particular Theorem 1.3.9(b) implies the claimed bound on the fibers $\mathcal{A}^*(k)$. □

4.2 Existence of Invariant Fiber Bundles

Every five years or so, if not more often, someone 'discovers' the theorem of Hadamard and Perron, proving it by Hadamard's method of proof, or by Perron's.
D.V. Anosov (cf. [11])

For linear difference equations admitting an exponential splitting we have a solid understanding of their dynamical behavior. Indeed, by virtue of Remark 3.4.16 it was possible to characterize the set of $c^{\pm}$-bounded solutions using kernels and ranges of associated invariant projectors – the invariant vector bundles form the skeleton of the extended state space. Moreover, for autonomous equations these vector bundles become generalized eigenspaces and can be determined using purely algebraic methods.

In this section, we aim to extend the above observation to nonlinear equations, yielding invariant fiber bundles. As explicated in the introduction to this chapter, applications for invariant manifolds or fiber bundles cover local questions (behavior near reference solutions in Sect. 4.6, linearization in Chap. 5) as well as global ones (dynamics of dissipative equations in Sect. 4.7). For this reason, we need flexible existence theorems which apply to both cases and carry most of the technical preparations. It is the goal of the present section to tackle this problem for a sufficiently wide class of equations, namely semilinear equations as already considered above.

According to this, our intent is centered around semilinear equations

$$B_{k+1}x' = A_k x + f_k(x, x'), \tag{S}$$

which are understood as a perturbation of the linear homogeneous system

$$B_{k+1}x' = A_k x. \tag{L$_0$}$$

Thus, it will be a standing assumption throughout the remaining chapter that (L_0) has an exponential splitting. The alert reader surely remembers that for autonomous or periodic linear equations (L_0), one can formulate the following crucial hypothesis in terms of spectral properties for A_k, B_k (cf. Theorems 3.4.28 and 3.4.31).

Hypothesis 4.2.1. *Suppose that $A_k \in \operatorname{Hom}(X_k, Y_{k+1})$, $B_k \in \operatorname{Hom}(X_k, Y_k)$ has an inverse with $B_{k+1}^{-1}A_k \in L(X_k, X_{k+1})$, $k \in \mathbb{I}'$, and that the linear equation* (L_0) *admits a strongly regular exponential N-splitting on $\mathbb{I}$ with $N > 1$, namely*

$$S(A, B; P) = \bigcup_{i=0}^{N-1} (b_{i+1}, a_i),$$

where the sequences b_i are bounded above. The associated Green's functions are abbreviated by $G_i := G_{P_1^i}$, $1 \leq i < N$.

Remark 4.2.2. If a linear equation (L_0) is $\hat{\mathcal{B}}$-contracting, $\hat{\mathcal{B}}$ denoting the family of uniformly bounded subsets of $\mathcal{X}$, then the pseudo-unstable vector bundles $\mathcal{P}_1^i$ are finite-dimensional, as soon as $b_i \geq 1$ (cf. Proposition 3.4.24).

In contrast to Sect. 4.1, we additionally impose that the nonlinearity f_k satisfies a global Lipschitz condition, as opposed to the linear bound in (4.1c). It will be demonstrated in the upcoming Sects. 4.6–4.7 how such a restrictive assumption can be weakened when it comes to applications. Yet, we try to make our results as explicit as possible, since quantitative results pay off when it comes to domain estimates of locally invariant fiber bundles (see Sect. 4.6) or dimension estimates for inertial fiber bundles (see Sect. 4.7).

Conditions for solutions of implicit difference equations to exist have been given throughout Sects. 2.2 and 2.3; moreover, the particular situation of semilinear equations (S) is addressed in Proposition 4.1.3. We, however, explicitly suppose the existence of forward solutions in order to remain flexible:

Hypothesis 4.2.3. *Let the general forward solution φ of equation* (S) *exist on $\mathcal{X}$. Suppose that $f_k : X_k \times X_{k+1} \to Y_{k+1}$ fulfills $f_k(X_k, X_{k+1}) \subseteq \operatorname{im} B_{k+1}$ for all $k \in \mathbb{I}'$ and that we have the global Lipschitz estimates*

$$L_j := \sup_{k \in \mathbb{I}'} \operatorname{lip}_j B_{k+1}^{-1} f_k < \infty \quad \text{for } j = 1, 2. \tag{4.2a}$$

Remark 4.2.4 (*spectral gap condition*). For an integer $1 \leq i < N$ we require the *spectral gap condition* that there exists a $\varsigma_i \in \left(0, \frac{\lfloor b_i - a_i \rfloor}{2}\right)$ such that

$$\frac{\max\left\{K_i^-, K_i^+\right\}(L_1 + \lceil b_i \rceil L_2)}{1 + \max\left\{K_i^-, K_i^+\right\} L_2} < \varsigma_i, \tag{G_i}$$

choose a fixed real number $\varsigma \in \left(\max\left\{K_i^-, K_i^+\right\}(L_1 + \lceil b_i - \varsigma_i \rceil L_2), \varsigma_i\right)$ and define intervals $\bar{\Gamma}_i := [a_i + \varsigma, b_i - \varsigma]$.

(1) The gap condition (G_i) guarantees that neither the real interval for ς nor $\bar{\Gamma}_i$ itself is empty. If we introduce the function $g : [0, \infty) \to \mathbb{R}$ by $g(t) := \frac{t(L_1 + \lceil b_i \rceil L_2)}{1 + tL_2}$, then (G_i) can be written as $g(\max\left\{K_i^-, K_i^+\right\}) < \varsigma_i$. For later use we point out that g is strictly increasing on $[0, \infty)$ from 0 to ∞. Thus, if $g(t^*) < \varsigma_i$ holds for one $t^* > 0$, one surely has $g(t) \leq \varsigma_i$ for all $t \in (0, t^*]$.

(2) For semi-implicit equations (S) the gap condition (G_i) simplifies to

$$\max\left\{K_i^-, K_i^+\right\} L_1 < \varsigma_i.$$

(3) In order to give an intuition for the crucial condition (G_i) we observe the following: Assume a more classical situation in which the linear part (L_0) is autonomous and generates a bounded discrete semigroup $((B^{-1}A)^k)_{k \in \mathbb{Z}_0^+}$ on a common space $X = X_k$. Referring to Theorem 3.4.28, an exponential dichotomy holds, provided the spectrum $\sigma(A, B)$ allows a decomposition $\sigma(A, B) = \sigma_+ \dot{\cup} \sigma_-$ into disjoint spectral sets $\sigma_+, \sigma_- \subseteq \mathbb{C}$ such that $\max_{z \in \sigma_-} |z| < a_i < b_i < \inf_{z \in \sigma_+} |z|$ with positive reals a_i, b_i (see Fig. 3.2). Moreover, we suppose (S) is linearly implicit, i.e., one has $L_2 = 0$ and (G_i) reduces to the above inequality. Hence, we are able

to fulfill the spectral gap condition (G_i), if one of the following two conditions is satisfied:

(i) For a given spectral gap $b_i - a_i$ and $\varsigma_i \in \left(0, \frac{b_i - a_i}{2}\right)$ the nonlinear perturbation f_k is so weak that its Lipschitz constant $L_1 > 0$ fulfills (G_i).
(ii) Given a fixed value for $L_1 > 0$, the spectral gap $b_i - a_i > 0$ has to be sufficiently large so that there exists a $\varsigma_i \in \left(0, \frac{b_i - a_i}{2}\right)$ satisfying (G_i).

Which of these perspectives (i) or (ii) is favorable, depends on the application. When dealing with local questions, i.e., in the context of invariant fiber bundles associated to fixed reference solutions (see Sect. 4.6), the first interpretation applies. For inertial fiber bundles (see Sect. 4.7), which are global in nature, the second one is crucial. Admittedly, the situation of (ii) changes for implicit nonlinearities, i.e., $L_2 > 0$. In fact, we still have to require a smallness condition on L_2, no matter how large the gap $b_i - a_i$ in the spectrum is.

Remark 4.2.5 (*growth condition).* For $1 \leq i < N$ we require the *growth conditions*

$$\exists \kappa \in \mathbb{I}: \varGamma_\kappa^+(i) := \sup_{k \in \mathbb{Z}_\kappa^+} \left\| B_{k+1}^{-1} f_k(0,0) \right\|_{X_{k+1}} e_{a_i}(\kappa, k) < \infty, \qquad (\varGamma_i^+)$$

$$\exists \kappa \in \mathbb{I}: \varGamma_\kappa^-(i) := \sup_{k \in \mathbb{Z}_\kappa^-} \left\| B_{k+1}^{-1} f_k(0,0) \right\|_{X_{k+1}} e_{b_i}(\kappa, k) < \infty, \qquad (\varGamma_i^-)$$

provided the discrete interval $\mathbb{I}$ is unbounded above resp. below:

(1) If a constant $\varGamma_\kappa^\pm(i)$ exist for one $\kappa \in \mathbb{I}$, then it exists for all $\kappa \in \mathbb{I}$.

(2) Besides in Sect. 4.6, we do not assume that the nonlinearity $f_k(0,0)$ vanishes identically on $\mathbb{I}'$. Rather, we weaken this frequently made assumption to an exponential boundedness, i.e., the finite existence of $\varGamma_\kappa^\pm(i)$. More detailed, the condition $(\varGamma_i^+)$ is equivalent to the inclusion $f_\cdot(0,0) \in \mathcal{X}_{\kappa,a_i,B}^+$, i.e., the sequence $B_{\cdot+1}^{-1} f_\cdot(0,0)$ is a_i^+-bounded and dually $(\varGamma_i^-)$ means $f_\cdot(0,0) \in \mathcal{X}_{\kappa,b_i,B}^-$, i.e., the sequence $B_{\cdot+1}^{-1} f_\cdot(0,0)$ is b_i^--bounded. By Lemma 3.3.26 this yields the implications

$$(\varGamma_i^+) \Rightarrow (\varGamma_{i-1}^+), \qquad (\varGamma_{i-1}^-) \Rightarrow (\varGamma_i^-) \quad \text{for all } 2 \leq i < N.$$

We construct invariant fiber bundles of (S) using a functional analytical approach. For this, let $(\kappa, \xi) \in \mathcal{X}$, $c : \mathbb{I} \to (0, \infty)$ and suppose the discrete interval $\mathbb{I}$ is unbounded below. We aim to characterize the solutions of (S) which exist in backward time and are c^--bounded. Choose a fixed $1 \leq i < N$. For given $\phi \in \mathcal{X}_{\kappa,c}^-$, we formally define a sequence-valued mapping – the so-called *Lyapunov–Perron operator*

$$T_\kappa^-(\phi; \xi) := \varPhi_{P_1^i}^-(\cdot, \kappa) P_1^i(\kappa) \xi + \sum_{n=-\infty}^{\kappa-1} G_i(\cdot, n+1) B_{n+1}^{-1} f_n(\underline{\phi(n)}) \qquad (4.2b)$$

resembling the Lyapunov–Perron sums in Theorem 3.5.3(b). The backward evolution operators $\varPhi_{P_1^i}^-$ of (L_0) exist on $\mathcal{P}_1^i$ by the strongly regular splitting from Hypothesis 4.2.1.

The next lemma establishes a solid part of our notation:

Lemma 4.2.6. *Assume Hypotheses 4.2.1 and 4.2.3. If* (Γ_i^-) *holds and* $c, d : \mathbb{I} \to (0, \infty)$ *satisfy*

$$c \in (a_i, b_i), \quad 0 \ll d \le c \quad \textit{for one } 1 \le i < N, \tag{4.2c}$$

then the mapping $T_\kappa^- : \mathcal{X}_{\kappa,c}^- \times X_\kappa \to \mathcal{X}_{\kappa,d}^-$ *is well-defined with*

$$\begin{aligned} \left\| T_\kappa^-(\phi;\xi) \right\|_{\kappa,d}^- &\le K_i^- \left\| P_1^i(\kappa)\xi \right\|_{X_\kappa} + C_i(c)\left(\Gamma_\kappa^-(i) + L(c)\left\|\phi\right\|_{\kappa,c}^- \right), \\ \left\| Q_1^i(\kappa) T_\kappa^-(\kappa,\phi;\xi) \right\|_{X_\kappa} &\le \frac{K_i^+ \Gamma_\kappa^-(i)}{\lfloor c - a_i \rfloor} + \ell_i^+(c)\left\|\phi\right\|_{\kappa,c}^- \end{aligned} \tag{4.2d}$$

for all $(\kappa, \xi) \in \mathcal{X}$, $\phi \in \mathcal{X}_{\kappa,c}^-$ *and we have Lipschitz estimates*

$$\operatorname{lip}_1 Q_1^i(\kappa) T_\kappa^-(\kappa, \cdot) \le \ell_i^+(c), \qquad \operatorname{lip}_1 T_\kappa^- \le \ell_i(c), \qquad \operatorname{lip}_2 T_\kappa^- \le K_i^- \tag{4.2e}$$

with the constants $C_i(c)$ *from Theorem 3.5.3 and*

$$\ell_i(c) := C_i(c)(L_1 + \lceil c \rceil L_2), \qquad L(c) := L_1 + \lceil c \rceil L_2,$$
$$\ell_i^+(c) := \frac{K_i^+}{\lfloor c - a_i \rfloor}(L_1 + \lceil c \rceil L_2), \qquad \ell_i^-(c) := \frac{K_i^-}{\lfloor b_i - c \rfloor}(L_1 + \lceil c \rceil L_2).$$

Proof. Let $(\kappa, \xi) \in \mathcal{X}$ be given and choose growth rates $c, d : \mathbb{I} \to (0, \infty)$ as required in (4.2c). We begin with preparatory estimates. For a sequence $\phi \in \mathcal{X}_{\kappa,c}^-$, using the triangle inequality and (Γ_i^-), one has

$$\left\| B_{n+1}^{-1} f_n(\underline{\phi(n)}) \right\| \stackrel{(4.2a)}{\le} \left(\Gamma_\kappa^-(i) + L(c)\left\|\phi\right\|_{\kappa,c}^- \right) e_c(n,\kappa) \quad \text{for all } n \in \mathbb{Z}_\kappa^-$$

as in the proof of Theorem 3.5.8 (cf. (3.5h)). Using the splitting estimates (3.4g), we obtain almost identically to the proof of Theorem 3.5.3(b) that

$$\left\| P_1^i(k) T_\kappa^-(k,\phi;\xi) \right\| e_d(\kappa,k) \le K_i^- \left\| P_1^i(\kappa)\xi \right\| + \frac{K_i^-}{\lfloor b_i - c \rfloor}\left(\Gamma_\kappa^-(i) + L(c)\left\|\phi\right\|_{\kappa,c}^- \right)$$
$$\left\| Q_1^i(k) T_\kappa^-(k,\phi;\xi) \right\| e_d(\kappa,k) \le \frac{K_i^+}{\lfloor c - a_i \rfloor}\left(\Gamma_\kappa^-(i) + L(c)\left\|\phi\right\|_{\kappa,c}^- \right)$$

for all $k \in \mathbb{Z}_\kappa^-$ (cf. Proposition A.1.2). Combining these two estimates, one deduces the inclusion $T_\kappa^-(\phi;\xi) \in \mathcal{X}_{\kappa,d}^-$ as well as the first estimate (4.2d). The second relation (4.2d) follows from the latter above estimate by setting $k = \kappa$. To prove the Lipschitz estimates in (4.2e), let $\phi, \bar\phi \in \mathcal{X}_{\kappa,c}^-$ and $\xi, \bar\xi \in X_\kappa$. We get from (3.4g) that

$$\left\| P_1^i(k)\left[T_\kappa^-(k,\phi;\xi) - T_\kappa^-(k,\bar\phi;\xi) \right] \right\| \stackrel{(4.2a)}{\le} \frac{K_i^- L(c)}{\lfloor b_i - c \rfloor} \left\| \phi - \bar\phi \right\|_{\kappa,c}^- e_c(k,\kappa)$$

for all $k \in \mathbb{Z}_\kappa^-$, and similarly

$$\left\|Q_1^i(k)\left[T_\kappa^-(k,\phi;\xi) - T_\kappa^-(k,\bar{\phi};\xi)\right]\right\| \overset{(4.2a)}{\le} \frac{K_i^+ L(c)}{\lfloor c - a_i \rfloor}\left\|\phi - \bar{\phi}\right\|_{\kappa,c}^- e_c(k,\kappa)$$

for all $k \in \mathbb{Z}_\kappa^-$. Setting $k = \kappa$ gives us the first relation in (4.2e). Multiplying both above estimates with $e_d(\kappa,k)$, the definition of the norm $\|\cdot\|_{\kappa,d}^-$ immediately yields the Lipschitz condition for $T_\kappa^-(\cdot;\xi)$, i.e., the middle relation of (4.2e). Finally, using (4.2b), (3.4g), the remaining Lipschitz estimate in (4.2e) follows from

$$\left\|T_\kappa^-(k,\phi;\xi) - T_\kappa^-(k,\phi;\bar{\xi})\right\| e_d(\kappa,k) \le K_i^- \left\|\xi - \bar{\xi}\right\| \quad \text{for all } k \in \mathbb{Z}_\kappa^-.$$

This establishes Lemma 4.2.6. □

By virtue of the Lyapunov–Perron operator T_κ^- from (4.2b), we will characterize the exponentially bounded solutions of (S) as its fixed points, and solve the corresponding problem using the contraction mapping principle.

Lemma 4.2.7. *Let $(\kappa,\xi) \in \mathcal{X}$ and assume Hypotheses 4.2.1 and 4.2.3 are fulfilled. If (Γ_i^-) holds, a sequence $c : \mathbb{I} \to (0,\infty)$ satisfies (4.2c) and $\phi \in \mathfrak{X}_{\kappa,c}^-$, then for the mapping $T_\kappa^-(\cdot;\xi) : \mathfrak{X}_{\kappa,c}^- \to \mathfrak{X}_{\kappa,c}^-$ the following statements are equivalent:*

(a) ϕ solves the difference equation (S) with $P_1^i(\kappa)\phi(\kappa) = P_1^i(\kappa)\xi$.
(b) ϕ is a solution of the fixed point equation

$$\phi = T_\kappa^-(\phi;\xi). \tag{4.2f}$$

Proof. Let $(\kappa,\xi) \in \mathcal{X}$ and define a sequence $g_k := f_k(\underline{\phi(k)})$ in $\mathcal{Y}$ for $k \in \mathbb{I}'$. As in the above proof of Lemma 4.2.6 one has the estimate

$$\left\|B_{k+1}^{-1} g_k\right\| \le \left(\Gamma_\kappa^-(i) + L(c)\left\|\phi\right\|_{\kappa,c}^-\right) e_c(k,\kappa) \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

with the constant $L(c)$ from Lemma 4.2.6, and consequently $g \in \mathfrak{X}_{\kappa,c,B}^-$.

$(a) \Rightarrow (b)$ Let $\phi : \mathbb{Z}_\kappa^- \to \mathfrak{X}$ be a c^--bounded solution of (S) satisfying $P_1^i(\kappa)\phi(\kappa) = P_1^i(\kappa)\xi$. Then ϕ also solves the linear inhomogeneous difference equation $B_{k+1}x' = A_k x + g_k$ and Theorem 3.5.3(b) implies assertion (b).

$(b) \Rightarrow (a)$ Conversely, a fixed point of $T_\kappa^-(\cdot,\xi)$ is a solution of the above linear inhomogeneous equation, and thus of the semilinear equation (S) satisfying the relation $P_1^i(\kappa)\phi(\kappa) = P_1^i(\kappa)\xi$. □

Lemma 4.2.8. *Let $(\kappa,\xi) \in \mathcal{X}$ and assume Hypotheses 4.2.1 and 4.2.3. If (G_i), (Γ_i^-) hold and $c \in \bar{\Gamma}_i$, then $T_\kappa^-(\cdot;\xi) : \mathfrak{X}_{\kappa,c}^- \to \mathfrak{X}_{\kappa,c}^-$ has a unique fixed point $\phi_\kappa(\xi) \in \mathfrak{X}_{\kappa,c}^-$. The fixed point mapping $\phi_\kappa : X_\kappa \to \mathfrak{X}_{\kappa,c}^-$ satisfies $\phi_\kappa(\xi) = \phi_\kappa(P_1^i(\kappa)\xi)$ and:*

(a) $\phi_\kappa : X_\kappa \to \mathcal{X}^-_{\kappa,c}$ is linearly bounded, i.e., for $\xi \in X_\kappa$ it is

$$\begin{aligned}\|\phi_\kappa(\xi)\|^-_{\kappa,c} &\le \frac{K_i^-}{1-\ell_i(c)}\left\|P_1^i(\kappa)\xi\right\|_{X_\kappa} + \frac{C_i(c)\Gamma_\kappa^-(i)}{1-\ell_i(c)},\\ \left\|Q_1^i(\kappa)\phi_\kappa(\kappa,\xi)\right\|_{X_\kappa} &\le \frac{K_i^+\Gamma_\kappa^-(i)}{\lfloor c-a_i\rfloor} + \frac{\ell_i^+(c)}{1-\ell_i(c)}\cdot\\ &\quad\cdot\left(K_i^-\left\|P_1^i(\kappa)\xi\right\|_{X_\kappa} + C_i(c)\Gamma_\kappa^-(i)\right),\end{aligned} \tag{4.2g}$$

(b) ϕ_κ is globally Lipschitzian with

$$\operatorname{lip}\phi_\kappa \le \frac{K_i^-}{1-\ell_i(c)}, \qquad \operatorname{lip}Q_1^i(\kappa)\phi_\kappa \le \frac{K_i^-\ell_i^+(c)}{1-\ell_i(c)}, \tag{4.2h}$$

where the constants $C_i(c), \ell_i(c) \in [0,1)$, $\ell_i^+(c)$ are defined in Lemma 4.2.6.

Proof. Let $(\kappa,\xi) \in \mathcal{X}$ be given. The spectral gap condition (G_i) implies

$$\ell_i(c) = \max\left\{\frac{K_i^+}{\lfloor c-a_i\rfloor}, \frac{K_i^-}{\lfloor b_i-c\rfloor}\right\}L(c) \le \frac{\max\left\{K_i^+,K_i^-\right\}}{\varsigma}L(c) < 1 \tag{4.2i}$$

for all $c \in \bar{\Gamma}_i$. Thus, from the middle estimate (4.2e) in Lemma 4.2.6 we know that $T_\kappa^-(\cdot;\xi)$ is a contraction on the Banach space $\mathcal{X}^-_{\kappa,c}$ (cf. Lemma 3.3.25(a)) and the contraction mapping theorem (see, for example, [295, p. 361, Lemma 1.1]) implies the existence of a unique fixed point $\phi_\kappa(\xi) \in \mathcal{X}^-_{\kappa,c}$. Furthermore, the claimed relation $\phi_\kappa(\xi) = \phi_\kappa(P_1^i(\kappa)\xi)$ follows from the fact $T_\kappa^-(\cdot;\xi) = T_\kappa^-(\cdot;P_1^i(\kappa)\xi)$, and consequently the fixed points of the two contractions coincide.

(a) Thanks to $\ell_i(c) < 1$ (see (4.2i)), the first estimate (4.2g) follows from

$$\begin{aligned}\|\phi_\kappa(\xi)\|^-_{\kappa,c} &\overset{(4.2f)}{=} \left\|T_\kappa^-(\phi_\kappa(\xi);\xi)\right\|^-_{\kappa,c}\\ &\overset{(4.2d)}{\le} K_i^-\left\|P_1^i(\kappa)\xi\right\| + C_i(c)\left(\Gamma_\kappa^-(i) + L(c)\|\phi_\kappa(\xi)\|^-_{\kappa,c}\right)\end{aligned}$$

and the second estimate (4.2g) is a consequence of the above inequality and

$$\begin{aligned}\left\|Q_1^i(\kappa)\phi_\kappa(\kappa,\xi)\right\| &\overset{(4.2f)}{=} \left\|Q_1^i(\kappa)T_\kappa^-(\kappa,\phi_\kappa(\xi);\xi)\right\|\\ &\overset{(4.2d)}{\le} \frac{K_i^+\Gamma_\kappa^-(i)}{\lfloor c-a_i\rfloor} + \ell_i^+(c)\|\phi_\kappa(\xi)\|^-_{\kappa,c} \quad \text{for all } \xi \in X_\kappa.\end{aligned}$$

(b) Next we derive the Lipschitz estimates in (4.2h). Let $\xi, \bar{\xi} \in X_\kappa$ be given and from (4.2f), (4.2e) we obtain using the triangle inequality

$$\left\|\phi_\kappa(\xi) - \phi_\kappa(\bar{\xi})\right\|^-_{\kappa,c} \le K_i^-\left\|\xi-\bar{\xi}\right\| + \ell_i(c)\left\|\phi_\kappa(\xi)-\phi_\kappa(\bar{\xi})\right\|^-_{\kappa,c},$$

which yields the left relation in (4.2h). From (4.2f), (4.2b), (4.2e) one has

$$\left\|Q_1^i(\kappa)\left[\phi_\kappa(\kappa,\xi)-\phi_\kappa(\kappa,\bar{\xi})\right]\right\| \leq \ell_i^+(c)\left\|\phi_\kappa(\xi)-\phi_\kappa(\bar{\xi})\right\|_{\kappa,c}^-,$$

and this in connection with the left estimate for ϕ_κ in (4.2h) established above, leads to the remaining estimate claimed in (4.2h). □

After these preparations we formulate and prove the main existence theorem for invariant fiber bundles. It states that the vector bundles $\mathcal{Q}_1^i$ and $\mathcal{P}_1^i$ guaranteed by Hypothesis 4.2.1 and Remark 3.4.17 resp. Remark 3.4.18 persist globally as invariant fiber bundles $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$ over $\mathcal{Q}_1^i$ and $\mathcal{P}_1^i$, resp., under nonlinear perturbations as in Hypothesis 4.2.3. This includes the dynamical characterization from (3.4k) and (3.4l).

Theorem 4.2.9 (of Hadamard–Perron). *Assume Hypotheses 4.2.1 and 4.2.3 are satisfied and that (G_i) holds for one $1 \leq i < N$.*

(a) If $\mathbb{I}$ is unbounded above and (Γ_i^+) holds, then the nonautonomous set

$$\mathcal{W}_i^+ := \left\{(\kappa,\xi) \in \mathcal{X} : \varphi(\cdot;\kappa,\xi) \in \mathfrak{X}_{\kappa,c}^+\right\}$$

is a forward invariant fiber bundle of (S)*, which is independent of $c \in \bar{\Gamma}_i$ and possesses the representation as graph*

$$\mathcal{W}_i^+ = \left\{(\kappa, \eta + w_i^+(\kappa,\eta)) \in \mathcal{X} : (\kappa,\eta) \in \mathcal{Q}_1^i\right\}$$

of a uniquely determined mapping $w_i^+ : \mathcal{X} \to \mathfrak{X}$ with

$$w_i^+(\kappa,\xi) = w_i^+(\kappa, Q_1^i(\kappa)\xi) \in \mathcal{P}_1^i(\kappa) \quad \text{for all } (\kappa,\xi) \in \mathcal{X} \tag{4.2j}$$

and satisfying the invariance equation

$$\begin{aligned} w_i^+(\kappa+1,\eta_1) &= B_{\kappa+1}^{-1}A_\kappa w_i^+(\kappa,\eta) \\ &\quad + P_1^i(\kappa+1)B_{\kappa+1}^{-1}f_\kappa(\eta + w_i^+(\kappa,\eta), \eta_1 + w_i^+(\kappa+1,\eta_1)), \\ \eta_1 &= B_{\kappa+1}^{-1}A_\kappa \eta + B_{\kappa+1}^{-1}f_\kappa(\eta + w_i^+(\kappa,\eta), \eta_1 + w_i^+(\kappa+1,\eta_1)) \end{aligned} \tag{4.2k}$$

for all $(\kappa,\eta) \in \mathcal{Q}_1^i$, $\eta_1 \in \mathcal{Q}_1^i(\kappa+1)$. Furthermore, for all $c \in \bar{\Gamma}_i$ it holds:

(a_1) *$w_i^+ : \mathcal{X} \to \mathfrak{X}$ is linearly bounded, i.e., for $(\kappa,\xi) \in \mathcal{X}$ one has*

$$\left\|w_i^+(\kappa,\xi)\right\|_{X_\kappa} \leq \frac{K_i^- \Gamma_\kappa^+(i)}{\lfloor b_i - c\rfloor} + \frac{\ell_i^-(c)}{1-\ell_i(c)}\left(K_i^+\left\|Q_1^i(\kappa)\xi\right\|_{X_\kappa} + C_i(c)\Gamma_\kappa^+(i)\right),$$

(a_2) *$w_i^+(\kappa,\cdot)$ is globally Lipschitzian with $\operatorname{lip}_2 w_i^+ \leq \tilde{\ell}_i^+(c)$.*

(b) If $\mathbb{I}$ *is unbounded below and* (Γ_i^-) *holds, then the nonautonomous set*

$$\mathcal{W}_i^- := \left\{ (\kappa, \xi) \in \mathcal{X} \,\middle|\, \begin{array}{l} \textit{there exists a solution } \phi : \mathbb{I} \to \mathfrak{X} \textit{ of } (\mathrm{S}) \\ \textit{with } \phi(\kappa) = \xi \in X_\kappa \textit{ and } \phi|_{\mathbb{Z}_\kappa^-} \in \mathfrak{X}_{\kappa,c}^- \end{array} \right\}$$

is an invariant fiber bundle of (S)*, which is independent of* $c \in \bar{\Gamma}_i$ *and possesses the representation as graph*

$$\mathcal{W}_i^- = \left\{ (\kappa, \eta + w_i^-(\kappa, \eta)) \in \mathcal{X} : \, (\kappa, \eta) \in \mathcal{P}_1^i \right\} \tag{4.2l}$$

of a uniquely determined mapping $w_i^- : \mathcal{X} \to \mathfrak{X}$ *with*

$$w_i^-(\kappa, \xi) = w_i^-(\kappa, P_1^i(\kappa)\xi) \in \mathcal{Q}_1^i(\kappa) \quad \textit{for all } (\kappa, \xi) \in \mathcal{X}$$

and satisfying the invariance equation

$$\begin{aligned} w_i^-(\kappa+1, \eta_1) &= B_{\kappa+1}^{-1} A_\kappa w_i^-(\kappa, \eta) \\ &\quad + Q_1^i(\kappa+1) B_{\kappa+1}^{-1} f_\kappa(\eta + w_i^-(\kappa, \eta), \eta_1 + w_i^-(\kappa+1, \eta_1)), \\ \eta_1 &= B_{\kappa+1}^{-1} A_\kappa \eta + B_{\kappa+1}^{-1} f_\kappa(\eta + w_i^-(\kappa, \eta), \eta_1 + w_i^-(\kappa+1, \eta_1)) \end{aligned} \tag{4.2m}$$

for all $(\kappa, \eta) \in \mathcal{P}_1^i$*,* $\eta_1 \in \mathcal{P}_1^i(\kappa+1)$*. Furthermore, for all* $c \in \bar{\Gamma}_i$ *it holds:*

(b_1) $w_i^- : \mathcal{X} \to \mathfrak{X}$ *is linearly bounded, i.e., for* $(\kappa, \xi) \in \mathcal{X}$ *one has*

$$\left\| w_i^-(\kappa, \xi) \right\|_{X_\kappa} \leq \frac{K_i^+ \Gamma_\kappa^-(i)}{\lfloor c - a_i \rfloor} + \frac{\ell_i^+(c)}{1 - \ell_i(c)} \left(K_i^- \left\| P_1^i(\kappa)\xi \right\|_{X_\kappa} + C_i(c) \Gamma_\kappa^-(i) \right), \tag{4.2n}$$

(b_2) $w_i^-(\kappa, \cdot)$ *is globally Lipschitzian with* $\mathrm{lip}_2\, w_i^- \leq \tilde{\ell}_i^-(c)$*,*

where the constants $C_i(c)$*,* $\ell_i^\pm(c), \ell_i(c) \in [0,1)$ *are defined in Lemma 4.2.6 and* $\tilde{\ell}_i^\pm(c) := \frac{K_i^\pm \ell_i^\mp(c)}{1 - \ell_i(c)}$.

Remark 4.2.10. (1) The (forward) invariance of $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$ implies for all $\kappa \leq k$ the relations

$$\begin{aligned} P_1^i(k)\varphi(k; \kappa, \xi) &= w_i^+(k, Q_1^i(k)\varphi(k; \kappa, \xi)) \quad \text{for all } (\kappa, \xi) \in \mathcal{W}_i^+, \\ Q_1^i(k)\varphi(k; \kappa, \xi) &= w_i^-(k, P_1^i(k)\varphi(k; \kappa, \xi)) \quad \text{for all } (\kappa, \xi) \in \mathcal{W}_i^-. \end{aligned} \tag{4.2o}$$

(2) $\mathcal{W}_i^+$ is an invariant fiber bundle, if the general solution to (S) exists on $\mathcal{X}$.

(3) The fiber bundle $\mathcal{W}_i^+$ can be considered as set of all c^+-bounded forward solutions, while $\mathcal{W}_i^-$ consists of c^--bounded backward solutions for (S). In detail, given $(\kappa_1, \xi_1), (\kappa_2, \xi_2) \in \mathcal{X}$ we introduce the following equivalence relations on $\mathcal{X}$:

- With $\kappa := \max\{\kappa_1, \kappa_2\}$ define (cf. Remark 3.4.17)

$$(\kappa_1, \xi_1) \sim_i^+ (\kappa_2, \xi_2) \;:\Leftrightarrow\; \begin{cases} \text{there exist solutions } \phi_j : \mathbb{Z}_{\kappa_j}^+ \to \mathfrak{X} \text{ to (S) with} \\ \phi_j(\kappa_j) = \xi_j \text{ and } \phi_1 - \phi_2 \in \mathfrak{X}_{\kappa,c}^+ \text{ for all } c \in \bar{\Gamma}_i. \end{cases}$$

- With $\kappa := \min\{\kappa_1, \kappa_2\}$ define (cf. Remark 3.4.18)

$$(\kappa_1, \xi_1) \sim_i^- (\kappa_2, \xi_2) \;:\Leftrightarrow \begin{cases} \text{there exist solutions } \phi_j : \mathbb{Z}_{\kappa_j}^- \to \mathcal{X} \text{ to (S) with} \\ \phi_j(\kappa_j) = \xi_j \text{ and } \phi_1 - \phi_2 \in \mathcal{X}_{\kappa,c}^- \text{ for all } c \in \bar{\Gamma}_i. \end{cases}$$

Provided we have a $c^{\pm}$-bounded solution $\phi_* : \mathbb{I} \to \mathcal{X}$ of (S) with $c \in \bar{\Gamma}_i$, the corresponding equivalence classes fulfill

$$[(\kappa, \phi_*(\kappa))]_i^+ = \mathcal{W}_i^+, \qquad [(\kappa, \phi_*(\kappa))]_i^- = \mathcal{W}_i^-.$$

(4) It is possible to apply Theorem 4.2.9 in case of a linear function f_k. In this sense, Theorem 4.2.9 resembles our previous roughness result Theorem 3.6.5 for exponential splittings. However, the gap condition (G_i) appears to be weaker than (3.6j).

Proof. Let $(\kappa, \xi) \in \mathcal{X}$, we choose a fixed $1 \le i < N$, $c \in \bar{\Gamma}_i$ and abbreviate

$$\begin{aligned} P_+(k) &:= I_{X_k} - P_1^i(k), & P_-(k) &:= P_1^i(k), \\ \mathcal{P}_+ &:= \mathcal{Q}_i^i, & \mathcal{P}_- &:= \mathcal{P}_i^i, \\ w^{\pm}(\kappa, x) &:= w_i^{\pm}(\kappa, x), & \mathcal{W}_{\pm} &:= \mathcal{W}_i^{\pm} \end{aligned} \tag{4.2p}$$

for all $k, \kappa \in \mathbb{I}$, $x \in X_\kappa$. This brief and intuitive notation will be useful later as well.

(a) Since the present part (a) of Theorem 4.2.9 can be proved along the same lines as part (b) we present only a sketch of the proof. Analogously to Lemma 4.2.7, the c^+-bounded solutions of (S) can be characterized as fixed points of the *Lyapunov–Perron operator* $T_\kappa^+ : \mathcal{X}_{\kappa,c}^+ \times X_\kappa \to \mathcal{X}_{\kappa,c}^+$,

$$T_\kappa^+(\phi; \xi) := \Phi(\cdot, \kappa) P_+(\kappa)\xi + \sum_{n=\kappa}^{\infty} G_i(\cdot, n+1) B_{n+1}^{-1} f_n(\underline{\phi(n)}) \tag{4.2q}$$

(cf. Theorem 3.5.3(a)). Here, the forward evolution operator Φ for (L_0) exists due to Lemma 3.3.6(a). In particular, under (Γ_i^+) corresponding counterparts to our preparatory Lemmata 4.2.6, 4.2.7 and 4.2.8 hold true in the Banach space $\mathcal{X}_{\kappa,c}^+$, where the proof of Lemma 4.2.7 relies on Theorem 3.5.3(a). It follows from the spectral gap condition (G_i) that $T_\kappa^+(\cdot, \xi)$, $\xi \in X_\kappa$, is a contraction on the Banach space $\mathcal{X}_{\kappa,c}^+$ (cf. Lemma 3.3.25(a)) and we denote its unique fixed point by $\phi_\kappa^+(\xi) \in \mathcal{X}_{\kappa,c}^+$. Then the function $w^+(\kappa, \cdot) : X_\kappa \to X_\kappa$ is defined by $w^+(\kappa, \xi) := P_-(\kappa)\phi_\kappa^+(\kappa, \xi)$.

(b) We want to show first that $\mathcal{W}_-$ is an invariant fiber bundle of (S). By definition, for each pair of initial values $(\kappa, \xi_0) \in \mathcal{W}_-$ there exists a solution $\phi \in \mathcal{X}_{\kappa,c}^-$ of (S) with $\phi(\kappa) = \xi_0$. Due to the uniqueness of forward solutions guaranteed by Hypothesis 4.2.3, we have $\phi = \varphi(\cdot; l, \phi(l))$; accordingly $\varphi(\cdot; l, \phi(l))$ is a c^--bounded solution and this yields the inclusion $\varphi(l; \kappa, \xi) \in \mathcal{W}_-(l)$ for all $l \in \mathbb{Z}_\kappa^+$. Conversely, for $\xi_1 \in \mathcal{W}_-'(\kappa)$ there exists a c^--bounded solution $\phi : \mathbb{I} \to \mathcal{X}$

of (S) with $\phi'(\kappa) = \xi_1$. Obviously, $\xi_0 := \phi(\kappa) \in \mathcal{W}_-(\kappa)$ and since the general forward solutions exists, $\xi_1 = \varphi(\kappa + 1; \kappa, \xi_0)$, i.e., we have the inclusion $\xi_1 \in \varphi(\kappa + 1; \kappa, \mathcal{W}_-(\kappa))$.

Our Lemma 4.2.8 implies that the mapping $T_\kappa^-(\cdot;\xi) : \mathcal{X}_{\kappa,c}^- \to \mathcal{X}_{\kappa,c}^-$ has a unique fixed point $\phi_\kappa(\xi) \in \mathcal{X}_{\kappa,c}^-$. This fixed point is independent of the growth rate $c \in \bar{\Gamma}_i$ because one has the inclusion $\mathcal{X}_{\kappa,b_i-\varsigma}^- \subseteq \mathcal{X}_{\kappa,c}^-$ (cf. Lemma 3.3.26) and every $T_\kappa^-(\cdot;\xi) : \mathcal{X}_{\kappa,c}^- \to \mathcal{X}_{\kappa,c}^-$ possesses the same fixed point as the restriction $T_\kappa^-(\cdot;\xi)|_{\mathcal{X}_{\kappa,b_i-\varsigma}^-}$. Furthermore, the fixed point $\phi_\kappa(\xi)$ is a solution of (S) on $\mathbb{Z}_\kappa^-$ satisfying $P_-(\kappa)\phi_\kappa(\kappa,\xi) = P_-(\kappa)\xi$ (cf. Lemma 4.2.7). Now we define

$$w^-(\kappa,\xi) := P_+(\kappa)\phi_\kappa(\kappa,\xi) \tag{4.2r}$$

and immediately conclude $w^-(\kappa,\xi) \in \mathcal{P}_+(\kappa)$. In addition, (4.2b) and the relation $\phi_\kappa(\xi) = \phi_\kappa(P_-(\kappa)\xi)$ in Lemma 4.2.8 imply

$$w^-(\kappa,\xi) \stackrel{(4.2r)}{=} P_+(\kappa)\phi_\kappa(\kappa,\xi) = P_+(\kappa)\phi_\kappa(\kappa,P_-(\kappa)\xi) \stackrel{(4.2r)}{=} w^-(\kappa,P_-(\kappa)\xi).$$

We now verify the representation (4.2l) and the invariance equation (4.2m).

($\subseteq$) Let $(\kappa,x_0) \in \mathcal{W}_-$, i.e., there exists a c^--bounded solution $\phi : \mathbb{I} \to \mathcal{X}$ of (S) with $\phi(\kappa) = x_0$. Then ϕ satisfies $P_-(\kappa)\phi(\kappa) = P_-(\kappa)x_0$ and is consequently the unique fixed point of $T_\kappa^-(\cdot;x_0)$, i.e., $\phi = \phi_\kappa(x_0)$ (see Lemma 4.2.7). This, and $\phi_\kappa(\xi) = \phi_\kappa(P_-(\kappa)\xi)$ (cf. Lemma 4.2.8), implies

$$x_0 = P_-(\kappa)\phi_\kappa(\kappa,x_0) + P_+(\kappa)\phi_\kappa(\kappa,x_0) = P_-(\kappa)x_0 + P_+(\kappa)\phi_\kappa(\kappa,P_-(\kappa)x_0).$$

So, setting $\xi := P_-(\kappa)x_0$, we have $x_0 = \xi + P_+(\kappa)\phi_\kappa(\kappa,\xi) = \xi + w^-(\kappa,\xi)$ by (4.2r) and finally the first inclusion of (4.2l) is verified.

($\supseteq$) On the other hand, let $x_0 \in X_\kappa$ be of the form $x_0 = \xi + w^-(\kappa,\xi)$ for some $\xi \in \mathcal{P}_-(\kappa)$. Then

$$x_0 \stackrel{(4.2r)}{=} \xi + P_+(\kappa)\phi_\kappa(\kappa,\xi) = P_-(\kappa)\phi_\kappa(\kappa,\xi) + P_+(\kappa)\phi_\kappa(\kappa,\xi) = \phi_\kappa(\kappa,\xi)$$

and therefore, $\phi = \phi_\kappa(\xi)$ is a c^--bounded solution of (S) with $\phi(\kappa) = x_0$.

With points $(\kappa,\xi_0) \in \mathcal{W}_-$, $\kappa \in \mathbb{I}'$, the invariance of $\mathcal{W}_-$ (i.e., (4.2o)) implies the relation $\varphi(k;\kappa,\xi_0) = P_-(k)\varphi(k;\kappa,\xi_0) + w^-(k,P_-(k)\varphi(k;\kappa,\xi_0))$ and multiplication with $P_+(k)$ yields $P_+(k)\varphi(k;\kappa,\xi_0) = w^-(k,P_-(k)\varphi(k;\kappa,\xi_0))$ for $k \in \mathbb{Z}_\kappa^-$. We set $k = \kappa + 1$ and the solution identity for (S) eventually implies the invariance equation (4.2m).

(b_1) We obtain (4.2n) from Lemma 4.2.8(a), since (4.2r) yields

$$\left\|w^-(\kappa,\xi)\right\| \stackrel{(4.2g)}{\leq} \tilde{\ell}_i^-(c)\left\|P_-(\kappa)\xi\right\| + \frac{K_i^+\Gamma_\kappa^-(i)}{\lfloor c - a_i \rfloor} + \frac{C_i(c)\Gamma_\kappa^-(i)\ell_i^+(c)}{1 - \ell_i(c)}.$$

(b_2) To prove the claimed Lipschitz estimate, consider $\xi, \bar{\xi} \in X_\kappa$ and corresponding fixed points $\phi_\kappa(\xi), \phi_\kappa(\bar{\xi}) \in \mathcal{X}^-_{\kappa,c}$ of $T^-_\kappa(\cdot;\xi)$ and $T^-_\kappa(\cdot;\bar{\xi})$, respectively. One obtains from Lemma 4.2.8(b) that

$$\left\|w^-(\kappa,\xi) - w^-(\kappa,\bar{\xi})\right\| \stackrel{(4.2r)}{=} \left\|P_+(\kappa)\left[\phi_\kappa(\kappa,\xi) - \phi_\kappa(\kappa,\bar{\xi})\right]\right\| \stackrel{(4.2h)}{\leq} \tilde{\ell}^-_i(c)\left\|\xi - \bar{\xi}\right\|$$

and this finishes the present proof of Theorem 4.2.9. □

It can be seen that the gap condition (G_i) is optimal in the following sense:

Example 4.2.11 (*optimal spectral gap*). Let reals $0 < \alpha < 1 < \beta$ and ε be given. For $\mathcal{X} = \mathbb{Z} \times \mathbb{R}^2$ we consider an autonomous explicit difference equation (S) with

$$A_k :\equiv \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad B_k :\equiv I_{\mathbb{R}^2}, \qquad f_k(x) := \varepsilon \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

The linear part (L_0) is hyperbolic on $\mathbb{Z}$ and for the nonlinearity we obtain the Lipschitz constant $\operatorname{lip} f_k = |\varepsilon|$. The right-hand side of $x' = A_k x + f_k(x)$ can be written as linear system $x' = \left(\begin{smallmatrix} \alpha & \varepsilon \\ \varepsilon & \beta \end{smallmatrix}\right)x$, whose coefficient matrix has the two eigenvalues $\frac{1}{2}\left(\alpha + \beta \pm \sqrt{(\beta-\alpha)^2 - 4\varepsilon^2}\right)$. Thus, the origin is a saddle, precisely as long as $|\varepsilon| < \frac{\beta-\alpha}{2}$ and this is equivalent to the spectral gap condition (G_i). For nonlinearities with $|\varepsilon| \geq \frac{\beta-\alpha}{2}$ the origin becomes a sink or a source (depending on the value $\frac{\alpha+\beta}{2}$) and we do not have invariant fiber bundles given by graphs over images of nontrivial projectors as postulated in Theorem 4.2.9.

Our next observation is that the invariant fiber bundles $\mathcal{W}^\pm_i$ from Theorem 4.2.9 are nested, which means they are ordered w.r.t. the set inclusion.

Corollary 4.2.12. *Let $1 \leq i_* < N$ and $S(A, B; P)$ be minimal.*

(a) In case $\mathbb{I}$ is unbounded above, $(\Gamma^+_{i_*})$ and (G_i) for $1 \leq i \leq i_*$ hold, we obtain the pseudo-stable hierarchy *of forward invariant fiber bundles,*

$$\mathcal{W}^+_{i_*} \subset \mathcal{W}^+_{i_*-1} \subset \ldots \subset \mathcal{W}^+_1 \subset \mathcal{X}.$$

(b) In case $\mathbb{I}$ is unbounded below, $(\Gamma^-_{i_*})$ and (G_i) for $i_* \leq i < N$ hold, we obtain the pseudo-unstable hierarchy *of invariant fiber bundles,*

$$\mathcal{W}^-_{i_*} \subset \mathcal{W}^-_{i_*+1} \subset \ldots \subset \mathcal{X}.$$

Proof. From Remark 4.2.5(2) we see that the growth condition $(\Gamma^-_{i_*})$ implies (Γ^+_i) for all $1 \leq i \leq i_*$. Thus, by Theorem 4.2.9(a) there exist forward invariant fiber bundles $\mathcal{W}^+_i$ consisting of c^+_i-bounded forward solutions. The growth rates $c_i \in \bar{\Gamma}_i$ fulfill $c_{i+1} \ll c_i$ and therefore claim (a) follows from the embeddings in Lemma 3.3.26. The pseudo-unstable hierarchy in (b) can be established analogously. □

Generally speaking, through a given point $(\kappa, \xi) \in \mathcal{X}$ there may exist no or a multiple number of backward solutions of (S). The next result ensures that for $(\kappa, \xi) \in \mathcal{W}_i^-$, exactly one of them lies on the fiber bundle $\mathcal{W}_i^-$ and that $\varphi(k; \kappa, \cdot)$ is Lipschitzian between the fibers of $\mathcal{W}_i^-$. Moreover, it relates the dynamics of equation (S) to a reduced equation (the so-called $\mathcal{W}_i^-$*-reduced equation*), which is finite-dimensional provided $\dim \mathcal{P}_1^i < \infty$.

Corollary 4.2.13 ($\mathcal{W}_i^-$-reduced equation). *The general solution φ of* (S) *exists on $\mathcal{W}_i^-$. Moreover, the so-called $\mathcal{W}_i^-$*-reduced equation

$$B_{k+1}x' = A_k x + B_{k+1}P_1^i(k+1)B_{k+1}^{-1} f_k(x + w_i^-(k, x), x' + w_i^-(k+1, x')) \quad (4.2s)$$

in the pseudo-unstable vector bundle $\mathcal{P}_1^i$ has the following properties:

(a) Its general solution $\hat{\varphi}$ exists on $\mathcal{P}_1^i$ and φ is related to $\hat{\varphi}$ by virtue of

$$\hat{\varphi}(k; \kappa, \xi) = P_1^i(k)\varphi(k; \kappa, \xi + w_i^-(\kappa, \xi)) \quad \textit{for all } (k, \kappa, \xi) \in \mathbb{I} \times \mathcal{P}_1^i.$$

(b) Under the condition $K_i^- \max\left\{1, \tilde{\ell}_i^-(c)\right\}\left(\frac{L_1}{b_i(k)} + L_2\right) < 1$ for all $k \in \mathbb{I}$ and a fixed $c \in \bar{\Gamma}_i$, one has the following Lipschitz estimate for all $k \in \mathbb{Z}_\kappa^-$,

$$\operatorname{lip} \varphi(k; \kappa, \cdot)|_{\mathcal{W}_i^-(\kappa)} \le K_i^- \max\left\{1, \tilde{\ell}_i^-(c)\right\}\left(1 + L_2 \max\left\{1, \tilde{\ell}_i^-(c)\right\}\right) e_{\hat{b}_i}(k, \kappa), \quad (4.2t)$$

where $\hat{b}_i(k) := b_i(k) - K_i^+ \max\left\{1, \tilde{\ell}_i^-(c)\right\}(L_1 + b_i(k)L_2)$.

Proof. Let $\kappa \in \mathbb{I}$ be given and choose $c \in \bar{\Gamma}_i$. First of all, we prove that the general solution φ of (S) exists on $\mathcal{W}_i^-$. Due to the invariance of $\mathcal{W}_i^-$ we know that the mapping $\varphi(\kappa + 1; \kappa, \cdot) : \mathcal{W}_i^-(\kappa) \to \mathcal{W}_i^-(\kappa + 1)$ is onto. Let us show now that the inverse of this mapping is given by $\xi \mapsto \phi_{\kappa+1}(\kappa, \xi)$, with the $\phi_{\kappa+1}(\xi)$ from the proof of Theorem 4.2.9(b). Indeed, for each $\xi \in \mathcal{W}_i^-(\kappa + 1)$ there exists a c^--bounded solution of (S), given by $\phi_{\kappa+1}(\xi) : \mathbb{Z}_{\kappa+1}^- \to \mathfrak{X}$, and Lemma 4.2.7 yields

$$\begin{aligned} \xi &= P_1^i(\kappa + 1)\xi + w_i^-(\kappa + 1, P_1^i(\kappa + 1)\xi) \\ &\overset{(4.2r)}{=} P_1^i(\kappa + 1)\xi + Q_1^i(\kappa + 1)\phi_{\kappa+1}(\kappa + 1, \xi) = \phi_{\kappa+1}(\kappa + 1, \xi). \end{aligned}$$

Since the sequence $\phi_{\kappa+1}(\xi)$ is a solution of (S), one has $\varphi(\kappa + 1; \kappa, \phi_{\kappa+1}(\kappa, \xi)) = \phi_{\kappa+1}(\kappa+1, \xi) = \xi$. It remains to derive the relation $\phi_{\kappa+1}(\kappa, \varphi(\kappa+1, \kappa, \xi)) = \xi$ for all $\xi \in \mathcal{W}_i^-(\kappa)$. For this purpose, we define $\psi(\kappa+1) := \varphi(\kappa+1; \kappa, \xi)$ and $\psi(k) := \phi_{\kappa+1}(k + 1, \xi)$ for $k \le \kappa$. Then $\psi : \mathbb{Z}_{\kappa+1}^- \to \mathfrak{X}$ is a c^--bounded solution of (S) and again Lemma 4.2.7 implies $\psi(\kappa) = \phi_{\kappa+1}(\kappa, \psi'(\kappa)) = \phi_{\kappa+1}(\kappa, \varphi(\kappa; \kappa+1, \xi))$. Noting that $\psi(\kappa) = \phi_{\kappa+1}(\kappa + 1, \xi) = \xi$ holds we are done. Hence, each mapping $\varphi(\kappa + 1; \kappa, \cdot) : \mathcal{W}_i^-(\kappa) \to \mathcal{W}_i^-(\kappa + 1)$ is bijective and therefore φ exists on $\mathcal{W}_i^-$.

(a) By multiplying the solution identity for φ with $P_1^i(k + 1)$, using (3.3h) and the invariance of $\mathcal{W}_i^-$ given in (4.2o), it is easily seen that the relation between φ and $\hat{\varphi}$ holds and that $\hat{\varphi}$ is defined on $\mathcal{P}_1^i$, yielding the assertion (a).

(b) Finally, it remains to prove the Lipschitz estimate (4.2t). We pick two points $\xi_1, \xi_2 \in \mathcal{W}_i^-(\kappa)$ and the invariance of $\mathcal{W}_i^-$ implies

$$\varphi(k;\kappa,\xi_j) \overset{(4.2o)}{=} \hat{\varphi}(k;\kappa,P_1^i(\kappa)\xi_j) + w_i^-(k,\hat{\varphi}(k;\kappa,P_1^i(\kappa)\xi_j)) \quad \text{for all } k \in \mathbb{I}$$

and $j = 1, 2$, which, in turn, using Theorem 4.2.9(b_2) yields the estimate

$$\begin{aligned} &|\varphi(k;\kappa,\xi_1) - \varphi(k;\kappa,\xi_2)|_i \\ &\quad \le \max\left\{1, \operatorname{lip}_2 w_i^-\right\} \left\|\hat{\varphi}(k;\kappa,P_1^i(\kappa)\xi_1) - \hat{\varphi}(k;\kappa,P_1^i(\kappa)\xi_2)\right\| \end{aligned}$$

for all $k \in \mathbb{I}$. To obtain an estimate for the difference $\hat{\varphi}(k;\kappa,\bar{\xi}_1) - \hat{\varphi}(k;\kappa,\bar{\xi}_2)$ with $\bar{\xi}_1, \bar{\xi}_2 \in \mathcal{P}_1^i(\kappa)$, we remark that $\Phi(k,l)$ is an isomorphism between the fibers of $\mathcal{P}_1^i$ (cf. Lemma 3.3.6(b)). If we abbreviate $\hat{\varphi}_j(k) := \hat{\varphi}(k;\kappa,\bar{\xi}_j)$, then the variation of constants formula from Theorem 3.1.16(b) and Remark 3.1.17(1) yields

$$\begin{aligned} \hat{\varphi}_j(k) = \Phi(k,\kappa)\bar{\xi}_j - \sum_{n=k}^{\kappa-1} \Phi(k,n+1)P_1^i(n+1)B_{n+1}^{-1} \cdot \\ \cdot f_n\big(\hat{\varphi}_j(n) + w_i^-(n,\hat{\varphi}_j(n)), \hat{\varphi}_j(n+1) + w_i^-(n+1,\hat{\varphi}_j(n+1))\big) \end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$ and $j = 1, 2$. Thus, setting $u(k) := \|\hat{\varphi}_1(k) - \hat{\varphi}_2(k)\|$, the relations (3.4g), (4.2a) imply the inequality

$$\begin{aligned} u(k)e_{b_i}(\kappa,k) &\le K_i^- \left\|\bar{\xi}_1 - \bar{\xi}_2\right\| + K_i^- L_1 \max\left\{1, \operatorname{lip}_2 w_i^-\right\} \sum_{n=k}^{\kappa-1} e_{b_i}(\kappa,n+1)u(n) \\ &\quad + K_i^- L_2 \max\left\{1, \operatorname{lip}_2 w_i^-\right\} \sum_{n=k}^{\kappa-1} e_{b_i}(\kappa,n+1)u'(n) \\ &\le K_i^- \left[1 + L_2 \max\left\{1, \operatorname{lip}_2 w_i^-\right\}\right] \left\|\bar{\xi}_1 - \bar{\xi}_2\right\| \\ &\quad + K_i^- \max\left\{1, \operatorname{lip}_2 w_i^-\right\} \sum_{n=k}^{\kappa-1} \left(\frac{L_1}{b_i(n)} + L_2\right) e_{b_i}(\kappa,n)u(n) \end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$, so that our assumption allows us to apply Gronwall's lemma in backward time (cf. Proposition A.2.1(b)), which leads to

$$u(k) \le K_i^- \left[1 + L_2 \max\left\{1, \operatorname{lip}_2 w_i^-\right\}\right] e_{\hat{b}_i}(k,\kappa) \left\|\bar{\xi}_1 - \bar{\xi}_2\right\|$$

for all $k \le \kappa$. Thanks to Theorem 4.2.9(b_2), this finally implies (4.2t). $\square$

Before proceeding, we need a technical result for later purpose in Sect. 4.7. It states that the general forward solution φ of (S) satisfies a Lipschitz estimate, if the linear part of the $\mathcal{W}_i^-$-reduced equation (4.2s) has bounded forward growth. Note that the required estimate becomes void for explicit equations.

Corollary 4.2.14. *Let* $\mathbb{I} = \mathbb{Z}$, $K_i \geq 1$ *and* $\bar{a}_i : \mathbb{Z} \to (0, \infty)$. *If*

$$\left\| \Phi(k,l) P_1^i(l) \right\|_{L(\mathcal{P}_1^i(l), \mathcal{P}_1^i(k))} \leq K_i e_{\bar{a}_i}(k,l) \quad \textit{for all } l \leq k$$

and $K_i L_2 \max\left\{1, \tilde{\ell}_i^-(c)\right\} < 1$ *for a fixed* $c \in \bar{\Gamma}_i$, *then the Lipschitz estimate*

$$\operatorname{lip} \varphi(k; \kappa, \cdot)|_{\mathcal{W}_i^-(\kappa)} \leq \frac{K_i \max\left\{1, \tilde{\ell}_i^-(c)\right\}}{1 - K_i L_2 \max\left\{1, \tilde{\ell}_i^-(c)\right\}} e_{\hat{a}_i}(k, \kappa) \quad \textit{for all } k \in \mathbb{Z}_\kappa^+$$

holds, where $\hat{a}_i(k) := \bar{a}_i(k) + \left(1 + \frac{K_i \max\left\{1, \tilde{\ell}_i^-(c)\right\}}{1 - K_i L_2 \max\left\{1, \tilde{\ell}_i^-(c)\right\}}\right)(L_1 + \bar{a}_i(k) L_2)$.

Proof. The argument follows along the lines to infer (4.2t): The difference is to apply results in forward time $k \in \mathbb{Z}_\kappa^+$, namely the variation of constants formula for (4.2s) from Theorem 3.1.16(a) and the Gronwall's lemma in Proposition A.2.1(a). □

The bundles $\mathcal{W}_i^-$ and $\mathcal{W}_i^+$ intersect along exponentially bounded solutions.

Corollary 4.2.15. *Let* $\mathbb{I} = \mathbb{Z}$, $1 \leq i < N$ *and assume* $c, d : \mathbb{Z} \to (0, \infty)$ *satisfy* $c, d \in \bar{\Gamma}_i$. *If beyond* (Γ_i^+) *and* (Γ_i^-) *also the strengthened gap condition*

$$\frac{2 \max\left\{K_i^-, K_i^+\right\} (L_1 + \lceil b_i \rceil L_2)}{1 + 2 \max\left\{K_i^-, K_i^+\right\} L_2} < \varsigma_i \tag{4.2u}$$

holds, then (S) *has a unique* c, d*-bounded solution* $\phi_* : \mathbb{Z} \to \mathcal{X}$ *and beyond* $[(\kappa, \phi_*(\kappa))]_i^\pm = \mathcal{W}_i^\pm$, $\kappa \in \mathbb{Z}$, *one has* $\mathcal{W}_i^+ \cap \mathcal{W}_i^- = \phi_*$.

Remark 4.2.16. (1) From Remark 4.2.4(1) one sees that (4.2u) implies (G_i).

(2) Under the assumption $f_k(0,0) \equiv 0$ on $\mathbb{Z}$, the fiber bundles $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$ intersect along the trivial solution, i.e., $\mathcal{W}_i^+ \cap \mathcal{W}_i^- = \mathbb{Z} \times \{0\}$.

Proof. Given $c, d \in \bar{\Gamma}_i$ we proceed in two steps:

(I) We first show that Theorem 3.5.10 is applicable to equation (S). For this, we observe that $g_k := f_k(0,0)$ satisfies the inclusion $g \in \mathcal{X}_{a_i, b_i, B} \subseteq \mathcal{X}_{c,d,B}$ (cf. Lemma 3.3.26) due to $(\Gamma_i^\pm)$. Moreover, the constant $D_i(c,d)$ defined in Theorem 3.5.4 fulfills the estimate $D_i(c,d) \leq \frac{2}{\varsigma} \max\left\{K_i^+, K_i^-\right\}$ and from

$$(L_1 + L_2 \max\{\lceil c \rceil, \lceil d \rceil\}) D_i(c,d) \leq (L_1 + L_2 \lceil b_i \rceil - L_2 \varsigma_i) D_i(c,d) \frac{2 \max\left\{K_i^-, K_i^+\right\}}{\varsigma_i}$$

we see that (4.2u) implies the condition (3.5j). Thus, our semilinear equation (S) admits a unique c, d-bounded solution $\phi_* : \mathbb{Z} \to \mathcal{X}$.

(II) Let $\kappa \in \mathbb{Z}$ be arbitrary. Then the above Theorem 4.2.9 immediately yields that the c, d-bounded complete solution ϕ_* must satisfy $\phi_*(\kappa) \in \mathcal{W}_i^-(\kappa) \cap \mathcal{W}_i^+(\kappa)$. Conversely, a point in the intersection $\xi \in \mathcal{W}_i^-(\kappa) \cap \mathcal{W}_i^+(\kappa)$ is an initial value for a c, d-bounded solution to (S). Since such solutions are uniquely determined by step (I), we have $\phi_*(\kappa) = \xi$ and thus the assertion follows. □

Our next intention is to have a look at nontrivial intersections and to show that also the extended hierarchy from Remark 3.4.19 persists under weak perturbations:

Proposition 4.2.17 (intersection of invariant fiber bundles). *Let $\mathbb{I} = \mathbb{Z}$ and assume Hypotheses 4.2.1, 4.2.3 hold. For pairs (i, j) with $1 < i \leq j < N$ satisfying (Γ_{i-1}^+), (Γ_j^-) and the strengthened spectral gap conditions*

$$\frac{(K_n^+ K_n^- + \max\{K_n^-, K_n^+\})(L_1 + \lceil b_n \rceil L_2)}{1 + (K_n^+ K_n^- + \max\{K_n^-, K_n^+\}) L_2} < \bar{\varsigma}_n \qquad (\tilde{G}_n)$$

for $n \in \{i-1, j\}$, the nonautonomous set

$$\mathcal{W}_i^j := \left\{ (\kappa, \xi) \in \mathcal{X} \,\middle|\, \begin{array}{l} \textit{there exists a solution } \phi : \mathbb{Z} \to \mathfrak{X} \textit{ of} \\ \text{(S) } \textit{with } \phi(\kappa) = \xi \in X_\kappa \textit{ and } \phi \in \mathfrak{X}_{c,d} \end{array} \right\}$$

is a forward invariant fiber bundle for (S)*, which is independent of $c \in \bar{\Gamma}_{i-1}$, $d \in \bar{\Gamma}_j$ and possesses the representation as graph*

$$\mathcal{W}_i^j = \mathcal{W}_{i-1}^+ \cap \mathcal{W}_j^- = \left\{ (\kappa, \eta + w_i^j(\kappa, \eta)) \in \mathcal{X} : (\kappa, \eta) \in \mathcal{U}_i^j \right\} \qquad (4.2\text{v})$$

of a uniquely determined mapping $w_i^j : \mathcal{X} \to \mathfrak{X}$ with

$$w_i^j(\kappa, \xi) = w_i^j(\kappa, P_i^j(\kappa)\xi) \in \mathcal{Q}_i^j(\kappa) \quad \textit{for all } (\kappa, \xi) \in \mathcal{X}.$$

Furthermore, it holds:

(a) $w_i^j : \mathcal{X} \to \mathfrak{X}$ is linearly bounded, i.e., for all $(\kappa, \xi) \in \mathcal{X}$ one has

$$\begin{aligned} \left\| w_i^j(\kappa, \xi) \right\|_{X_\kappa} \leq{} & \frac{2\ell_{ij}(c, d)}{1 - \ell_{ij}(c, d)} \left\| P_i^j(\kappa)\xi \right\|_{X_\kappa} \\ & + \frac{2}{1 - \ell_{ij}(c, d)} \max \left\{ \left(\frac{K_{i-1}^-}{\lfloor b_{i-1} - d \rfloor} + \frac{\ell_{i-1}^-(d) C_{i-1}(d)}{1 - \ell_{i-1}(d)} \right) \Gamma_\kappa^+(i-1), \right. \\ & \left. \left(\frac{K_j^+}{\lfloor c - a_j \rfloor} + \frac{\ell_j^+(c) C_j(c)}{1 - \ell_j(c)} \right) \Gamma_\kappa^-(j) \right\}. \end{aligned}$$

(b) $w_i^j(\kappa,\cdot)$ *is globally Lipschitzian with* $\operatorname{lip}_2 w_i^j \le \frac{2\ell_{ij}(c,d)}{1-\ell_{ij}(c,d)}$,

where the constants $C_i(c)$, $\ell_i^{\pm}(c), \ell_i(c) \in [0,1)$ *are defined in Lemma 4.2.6,* $\tilde{\ell}_i^{\pm}(c)$ *is given in Theorem 4.2.9 and* $\ell_{ij}(c,d) := \max\left\{\tilde{\ell}_{i-1}^{+}(c), \tilde{\ell}_j^{-}(d)\right\} \in [0,1)$.

Remark 4.2.18. (1) Under the conventions $\mathcal{W}_0^+ = \mathcal{Q}_1^0 = \mathcal{X}$, $\mathcal{W}_N^- = \mathcal{P}_1^N = \mathcal{X}$ (provided $N < \infty$), the intersection (4.2v) extends to the cases $i = 1$, $j = N$ and one obtains $\mathcal{W}_1^j = \mathcal{W}_j^-$, $\mathcal{W}_i^N = \mathcal{W}_{i-1}^+$ (provided $N < \infty$).

(2) By Remark 4.2.4(1) one sees that $(\tilde{G}_n)$ for $n \in \{i-1, j\}$ implies both the spectral gap conditions (G_{i-1}) and (G_j). Moreover, $(\tilde{G}_i)$ is sufficient for (4.2u).

(3) The fiber bundle $\mathcal{W}_i^j$ consists of all c,d-bounded complete solutions to (S). More detailed, given a complete solution $\phi_* \in \mathfrak{X}_{c,d}$ with $c \in \bar{\Gamma}_{i-1}$, $d \in \bar{\Gamma}_j$, as in Remark 3.4.19 we define the equivalence relation

$$(\kappa_1,\xi_1) \sim_i^j (\kappa_2,\xi_2) \quad :\Leftrightarrow \quad \begin{cases} \text{there exist complete solutions } \phi_j : \mathbb{Z} \to \mathfrak{X} \text{ to} \\ \text{(S) with } \phi_j(\kappa_j) = \xi_j \text{ and } \phi_2 - \phi_1 \in \mathfrak{X}_{c,d} \text{ for} \\ \text{all } a_{i-1} \ll c \le b_{i-1} \text{ and } a_j \ll d \le b_j, \end{cases}$$

whose equivalence classes $[\cdot]_i^j$ satisfy $\mathcal{W}_i^j = [(\kappa,\phi_*(\kappa))]_i^j$ for all $\kappa \in \mathbb{Z}$.

(4) The (forward) invariance of $\mathcal{W}_i^j$ guarantees

$$Q_i^j(k)\varphi(k;\kappa,\xi) = w_i^j(k, P_i^j(k)\varphi(k;\kappa,\xi)) \quad \text{for all } (\kappa,\xi) \in \mathcal{W}_i^j,\ \kappa \le k.$$

(5) Another possibility to construct the desired mapping $w_i^j : \mathcal{X} \to \mathfrak{X}$ is using the *Lyapunov–Perron operator* $T_\kappa^{\pm} : \mathfrak{X}_{c,d} \times X_\kappa \to \mathfrak{X}_{c,d}$,

$$\begin{aligned} T_\kappa^{\pm}(\phi,\eta) := {} & \Phi(\cdot,\kappa)P_i^j(\kappa)\eta + \sum_{n=-\infty}^{k-1} \Phi(k,n+1)Q_1^{j-1}(n+1)B_{n+1}^{-1} f_n(\underline{\phi(n)}) \\ & + \sum_{n=\kappa}^{k-1} \Phi(k,n+1)P_i^j(n+1)B_{n+1}^{-1} f_n(\underline{\phi(n)}) \\ & - \sum_{n=\kappa}^{\infty} \Phi(k,n+1)P_1^{i-1}(n+1)B_{n+1}^{-1} f_n(\underline{\phi(n)}). \end{aligned}$$

Applying methods as previously in this section, $T_\kappa^{\pm}(\cdot,\eta)$ turns out to be a contraction on $\mathfrak{X}_{c,d}$. Having the unique fixed point $\phi_\kappa(\eta) \in \mathfrak{X}_{c,d}$ at hand, $\mathcal{W}_i^j$ is the graph of the unique mapping $w_i^j(\kappa,\eta) := \left[Q_1^{j-1}(\kappa) + P_1^{i-1}(\kappa)\right]\phi_\kappa(\eta)$ over $\mathcal{P}_i^j$. However, our construction of the fiber bundle $\mathcal{W}_i^j$ is more geometrical in nature.

Proof. We suppose $1 < i \le j < N$ and subdivide the proof into several steps:

(I) Our Theorem 4.2.9 guarantees the existence of two fiber bundles $\mathcal{W}_{i-1}^{+}$ and $\mathcal{W}_{j}^{-}$. Here, thanks to our assumption $(\tilde{G}_n)$, one sees from Theorem 4.2.9(a_2) and (b_2) that the corresponding functions w_{i-1}^{+} and w_j^{-} satisfy

$$\operatorname{lip}_2 w_{i-1}^{+} \le \tilde{\ell}_{i-1}^{+}(c) < 1, \qquad \operatorname{lip}_2 w_j^{-} \le \tilde{\ell}_j^{-}(d) < 1$$

and consequently $\ell_{ij}(c,d) < 1$ for all $c \in \bar{\varGamma}_{i-1}$, $d \in \bar{\varGamma}_j$. Having this at our disposal, for every $\kappa \in \mathbb{Z}$ we define the operator $T_\kappa : X_\kappa^2 \times X_\kappa \to X_\kappa^2$ by

$$T_\kappa(x,z;y) := \Big(w_{i-1}^{+}(\kappa, z+y), w_j^{-}(\kappa, x+y)\Big). \tag{4.2w}$$

Considering $y \in X_\kappa$ as a fixed parameter, thanks to the estimate

$$\begin{aligned}\|T_\kappa(x,z;y) - T_\kappa(\bar{x},\bar{z};y)\| &\overset{(??)}{=} \max\Big\{\big\|w_{i-1}^{+}(\kappa, z+y) - w_{i-1}^{+}(\kappa, \bar{z}+y)\big\|,\\ &\qquad \big\|w_j^{-}(\kappa, x+y) - w_j^{-}(\kappa, \bar{x}+y)\big\|\Big\}\\ &\le \ell_{ij}(c,d)\,\big\|\big(x-\bar{x}, z-\bar{z}\big)\big\| \quad \text{for all } x,\bar{x},z,\bar{z} \in X_\kappa,\end{aligned}$$

the operator $T_\kappa(\cdot,y) : X_\kappa^2 \to X_\kappa^2$ is a uniform contraction in $y \in X_\kappa$. Similarly, we deduce from Theorem 4.2.9(a_2) and (b_2) that $\operatorname{lip}_3 T_\kappa \le \ell_{ij}(c,d)$ and the uniform contraction principle in Theorem B.1.1 ensures that there exists a unique fixed point $\varUpsilon_{i,j}(\kappa,y) = (\varUpsilon_{i,j}^{+}, \varUpsilon_{i,j}^{-})(\kappa,y) \in X_\kappa^2$ of $T_\kappa(\cdot,y)$.

(II) Now we infer the representation (4.2v) of $\mathcal{W}_i^j$ as graph of a function w_i^j over $\mathcal{P}_i^j$. From Theorem 4.2.9(a) we know that a point $(\kappa,x_0) \in \mathcal{X}$ is contained in $\mathcal{W}_{i-1}^{+}$, if and only if there exists a $\xi_0 \in \mathcal{Q}_1^{i-1}(\kappa)$ such that $x_0 = \xi_0 + w_{i-1}^{+}(\kappa,\xi_0)$ and accordingly $Q_1^{i-1}(\kappa)x_0 = \xi_0 + Q_1^{i-1}(\kappa)w_{i-1}^{+}(\kappa,x_0) = \xi_0$ (cf. (4.2j)). This yields $(\kappa,x_0) \in \mathcal{W}_{i-1}^{+}$ if and only if $x_0 = Q_1^{i-1}(\kappa)x_0 + w_{i-1}^{+}(\kappa, Q_1^{i-1}(\kappa)x_0)$. Analogously from Theorem 4.2.9(b) we have the inclusion $(\kappa,x_0) \in \mathcal{W}_j^{-}$ if and only if $x_0 = P_1^j(\kappa)x_0 + w_j^{-}(\kappa, P_1^j(\kappa)x_0)$. The unique decomposition $x_0 = \xi+\eta+\zeta$ into $\xi \in \mathcal{Q}_1^{i-1}(\kappa)$, $\eta \in \mathcal{P}_i^j(\kappa)$, $\zeta \in \mathcal{P}_1^j(\kappa)$ leads to the equivalence

$$\begin{aligned}(\kappa,x_0) \in \mathcal{W}_i^j \quad &\Leftrightarrow \quad x_0 = Q_1^{i-1}(\kappa)x_0 + w_{i-1}^{+}(\kappa, Q_1^{i-1}(\kappa)x_0) \text{ and}\\ &\qquad\; x_0 = P_1^j(\kappa)x_0 + w_j^{-}(\kappa, P_1^j(\kappa)x_0)\\ &\Leftrightarrow \quad \zeta = w_{i-1}^{+}(\kappa, \xi+\eta) \text{ and } \xi = w_j^{-}(\kappa, \eta+\zeta)\\ &\overset{(4.2w)}{\Leftrightarrow} \quad (\xi,\zeta) = T_\kappa(\xi,\zeta;\eta),\end{aligned}$$

i.e., the pair $(\xi, \zeta) \in \mathcal{Q}_1^{i-1}(\kappa) \times \mathcal{P}_1^j(\kappa)$ is a fixed point of $T_\kappa(\cdot;\eta)$; from the above step (I) it is uniquely determined by $\Upsilon_{i,j}(\kappa,\eta)$. As a result, if we define $w_i^j(\kappa, x_0) := \Upsilon_{i,j}^+(\kappa, P_i^j(\kappa)x_0) + \Upsilon_{i,j}^-(\kappa, P_i^j(\kappa)x_0)$ for $(\kappa, x_0) \in \mathcal{X}$, then the representation (4.2v) holds. Moreover, by construction one has

$$w_i^j(\kappa, P_i^j(\kappa)x_0) = w_i^j(\kappa, x_0) = w_{i-1}^+(\kappa, x_0) + w_j^-(\kappa, x_0) \in \mathcal{Q}_i^j(\kappa).$$

The fiber bundle $\mathcal{W}_i^j$ is forward invariant, because for $(\kappa, x_0) \in \mathcal{W}_i^j$ we obtain $\varphi(\cdot;\kappa, x_0) \in \mathfrak{X}_{\kappa,c}^+$, as well as the existence of a d^--bounded solution $\phi : \mathbb{Z} \to \mathfrak{X}$ of (S) with $\phi(\kappa) = x_0$ for all $c \in \bar{\Gamma}_{i-1}$, $d \in \bar{\Gamma}_j$. Then the semigroup property (2.3a) implies that $\varphi(\cdot; k_0, \varphi(k_0;\kappa, x_0))$, $k_0 \in \mathbb{Z}_\kappa^+$, is also c^+- and d^--bounded (in the sense above), and therefore $(k_0, \varphi(k_0;\kappa, x_0)) \in \mathcal{W}_i^j$.

(a) Let $(\kappa, x) \in \mathcal{X}$. In order to establish the linear bound for $w_i^j(\kappa, x)$ we point out that the inclusions $w_{i-1}^+(\kappa, x) \in \mathcal{P}_1^{i-1}(\kappa)$ and $w_j^-(\kappa, x) \in \mathcal{Q}_1^j(\kappa)$ readily imply the inclusion $\Upsilon_{i,j}^+(\kappa, x) \in \mathcal{P}_1^{i-1}(\kappa)$ resp. $\Upsilon_{i,j}^-(\kappa, x) \in \mathcal{Q}_1^j(\kappa)$. Therefore, the fixed point relation for $\Upsilon_{i,j}^+(\kappa, x)$ and Theorem 4.2.9(a_1) yields the estimate

$$\begin{aligned}
\left\|\Upsilon_{i,j}^+(\kappa, y)\right\| &\overset{(4.2w)}{=} \left\|w_{i-1}^+(\kappa, \Upsilon_{i,j}^-(\kappa, y) + y)\right\| \\
&\le \frac{K_{i-1}^- \Gamma_\kappa^+(i-1)}{\lfloor b_{i-1} - c\rfloor} + \frac{\ell_{i-1}^-(c)}{1 - \ell_{i-1}(c)}\Big[K_{i-1}^+ \left\|Q_1^{i-1}(\kappa)\left(\Upsilon_{i,j}^-(\kappa, y) + y\right)\right\| \\
&\qquad + C_{i-1}(c)\Gamma_\kappa^+(i-1)\Big], \\
&\le \frac{K_{i-1}^- \Gamma_\kappa^+(i-1)}{\lfloor b_{i-1} - c\rfloor} + \frac{\ell_{i-1}^-(c)}{1 - \ell_{i-1}(c)}\Big[K_{i-1}^+ \left(\left\|Q_1^{i-1}(\kappa)y\right\| + \left\|\Upsilon_{i,j}^-(\kappa, y)\right\|\right) \\
&\qquad + C_{i-1}(c)\Gamma_\kappa^+(i-1)\Big],
\end{aligned}$$

while correspondingly Theorem 4.2.9(b_1) leads to

$$\begin{aligned}
\left\|\Upsilon_{i,j}^-(\kappa, y)\right\| &\overset{(4.2w)}{=} \left\|w_j^-(\kappa, \Upsilon_{i,j}^+(\kappa, y) + y)\right\| \\
&\le \frac{K_j^+ \Gamma_\kappa^-(j)}{\lfloor d - a_j\rfloor} + \frac{\ell_j^+(d)}{1 - \ell_j(d)}\Big[K_j^- \left\|P_1^j(\kappa)\left(\Upsilon_{i,j}^+(\kappa, y) + y\right)\right\| \\
&\qquad + C_j(d)\Gamma_\kappa^-(j)\Big] \\
&\le \frac{K_j^+ \Gamma_\kappa^-(j)}{\lfloor d - a_j\rfloor} + \frac{\ell_j^+(d)}{1 - \ell_j(d)}\Big[K_j^- \left(\left\|P_1^j(\kappa)y\right\| + \left\|\Upsilon_{i,j}^+(\kappa, y)\right\|\right) \\
&\qquad + C_j(d)\Gamma_\kappa^-(j)\Big]
\end{aligned}$$

for all $y \in X_\kappa$. Using (**??**), this equips us with the relation

$$\begin{aligned}
&\left\| \left(\Upsilon_{i,j}^{+}(\kappa, y), \Upsilon_{i,j}^{-}(\kappa, y) \right) \right\| \\
&\quad \leq \max \left\{ \tfrac{K_{i-1}^{-} \Gamma_\kappa^{+}(i-1)}{\lfloor b_{i-1} - c \rfloor} + \tfrac{\ell_{i-1}^{-}(c)}{1-\ell_{i-1}(c)} \left[K_{i-1}^{+} \left\| Q_1^{i-1}(\kappa) y \right\| + C_{i-1}(c) \Gamma_\kappa^{+}(i-1) \right], \right. \\
&\qquad\qquad \left. \tfrac{K_j^{+} \Gamma_\kappa^{-}(j)}{\lfloor d - a_j \rfloor} + \tfrac{\ell_j^{+}(d)}{1-\ell_j(d)} \left[K_j^{-} \left\| P_1^{j}(\kappa) y \right\| + C_j(d) \Gamma_\kappa^{-}(j) \right] \right\} \\
&\quad + \ell_{ij}(c,d) \left\| \left(\Upsilon_{i,j}^{+}(\kappa, y), \Upsilon_{i,j}^{-}(\kappa, y) \right) \right\| \quad \text{for all } y \in X_\kappa
\end{aligned}$$

and by definition of w_i^j we can use the elementary inequality

$$a + b \leq 2 \max \{a, b\} \quad \text{for all } a, b \geq 0, \tag{4.2x}$$

in order to deduce the estimate claimed in assertion (a).

(b) From Theorem B.1.1(b) we know that $\Upsilon_{i,j}(\kappa, \cdot) : X_\kappa \to X_\kappa^2$ fulfills the Lipschitz estimate $\operatorname{lip}_2 \Upsilon_{i,j} \leq \frac{\ell_{ij}(c,d)}{1-\ell_{ij}(c,d)}$ and as a result of (4.2x) the assertion (b) yields. $\square$

For the sake of completeness we also state that the extended hierarchy of invariant vector bundles from Corollary 3.3.18 persists under nonlinear perturbations yielding the promised $\frac{(N+2)(N-1)}{2}$ nontrivial invariant fiber bundles of (S). Using the conventions explained in Remark 4.2.18(1) we consequently have

Corollary 4.2.19 (hierarchy of invariant fiber bundles). *Let $S(A, B; P)$ be minimal. If $(\Gamma_n^{\pm})$ and the strengthened spectral gap condition $(\tilde{G}_n)$ hold for $1 \leq n < N$, then:*

(a) One has the inclusions $\mathcal{W}_{i-1}^{j} \supset \mathcal{W}_i^j \subset \mathcal{W}_i^{j+1}$ for all $1 < i \leq j < N$.
(b) In case $N < \infty$ one has the extended hierarchy

$$\begin{array}{ccccccccc}
\mathcal{W}_1^1 & \subset & \mathcal{W}_1^2 & \subset \ldots \subset & \mathcal{W}_1^{N-1} & \subset & \mathcal{X} \\
 & & \cup & & \cup & & \cup \\
 & & \mathcal{W}_2^2 & \subset \ldots \subset & \mathcal{W}_2^{N-1} & \subset & \mathcal{W}_2^N \\
 & & & & \cup & & \cup \\
 & & & \ddots & \vdots & & \vdots \\
 & & & & \cup & & \cup \\
 & & & & \mathcal{W}_{N-1}^{N-1} & \subset & \mathcal{W}_{N-1}^N \\
 & & & & & & \cup \\
 & & & & & & \mathcal{W}_N^N
\end{array}$$

Proof. Referring to the notation from Remark 4.2.18(1), the cases $i = 1$ and $j = N$ have already been shown in Corollary 4.2.12. We thus restrict to indices $1 < i \leq j < N$. Above all, we choose growth rates $c \in \bar{\Gamma}_{i-1}$, $d \in \bar{\Gamma}_j$ and point out that the sets $\mathcal{W}_i^j$ are dynamically characterized using c, d-bounded solutions. A growth rate $\bar{d} \in \bar{\Gamma}_{j+1}$ satisfies $\bar{d} \ll d$ and Lemma 3.3.26 yields the inclusion $\mathfrak{X}_{c,d} \subseteq \mathfrak{X}_{c,\bar{d}}$ guaranteeing $\mathcal{W}_i^j \subset \mathcal{W}_i^{j+1}$. Analogously, for growth rates $\bar{c} \in \bar{\Gamma}_{i-2}$ one has $c \ll \bar{c}$, $\mathfrak{X}_{c,d} \subset \mathfrak{X}_{\bar{c},d}$ by Lemma 3.3.26 and thus $\mathcal{W}_i^j \subset \mathcal{W}_{i-1}^j$. $\square$

Corollary 4.2.20. *Under the assumption $f_k(0,0) \equiv 0$ on $\mathbb{I}$ one has for $\kappa \in \mathbb{I}$:*

(a) If $\mathbb{I}$ is unbounded above, then $w_i^+(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and $[(\kappa, 0)]_i^+ \cong \mathcal{W}_i^+$.
(b) If $\mathbb{I}$ is unbounded below, then $w_i^-(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and $[(\kappa, 0)]_i^- \cong \mathcal{W}_i^-$.
(c) If $\mathbb{I} = \mathbb{Z}$ and $(\tilde{G}_n)$, $n \in \{i-1, j\}$, holds, then $w_i^j(\kappa, 0) \equiv 0$ on $\mathbb{Z}$ and one also has $[(\kappa, 0)]_i^j = \mathcal{W}_i^j$.

Proof. The assumption $f_k(0,0) \equiv 0$ on $\mathbb{I}$ yields $\Gamma_\kappa^\pm(i) = 0$ for every $\kappa \in \mathbb{I}$. Both the claims (a) and (b) follow from the respective assertions (a_1) and (b_1) of Theorem 4.2.9, whereas (c) is a consequence of Proposition 4.2.17(a). $\square$

The following corollary deals with the case where (S) is periodic in time or even autonomous. In this case the fibers of the bundles $\mathcal{W}_i^\pm$ and $\mathcal{W}_i^j$ repeat periodically; in the autonomous case they are identical copies of each other.

Corollary 4.2.21. *Let $p \in \mathbb{N}$.*

(a) If (S) *is p-periodic, then one has for all $(\kappa, \xi) \in \mathcal{X}$ that*

$$w_i^\pm(\kappa + p, \xi) = w_i^\pm(\kappa, \xi), \qquad w_i^j(\kappa + p, \xi) = w_i^j(\kappa, \xi),$$

i.e., the mappings $w_i^\pm, w_i^j$ are also p-periodic in their first argument.
(b) If (S) *is autonomous, then the mappings w_i^+, w_i^-, w_i^j do not depend on their first argument, i.e., the constant fibers $\mathcal{W}_i^\pm(\kappa)$, $\mathcal{W}_i^j(\kappa)$, $\kappa \in \mathbb{Z}$, are* invariant manifolds *of* (S).

Proof. Let $\kappa \in \mathbb{Z}$. The assertions of (b) readily follow from (a) and we restrict to the proof for the mapping w_i^+. We choose a growth rate $c \in \bar{\Gamma}_i$ and an arbitrary point $\xi_0 \in \mathcal{P}_1^i(\kappa)$. By Theorem 4.2.9(a) the solution $\phi := \varphi(\cdot; \kappa, \xi_0 + w_i^+(\kappa, \xi_0))$ of (S) is c^+-bounded and due to the p-periodicity of (S) we know that also $\psi : \mathbb{Z}_{\kappa+p}^+ \to \mathfrak{X}$, $\psi(k) := \phi(k - p)$ is a c^+-bounded solution (cf. Proposition 2.5.3). Hence, we have the inclusion $(\kappa + p, \psi(\kappa)) \in \mathcal{W}_i^+$ and consequently, using our convention that periodic equations have periodic invariant projectors (cf. Corollary 3.4.25),

$$\begin{aligned} w_i^+(\kappa + p, \xi_0) &\overset{(4.2j)}{=} w_i^+(\kappa + p, P_1^i(\kappa)\phi(\kappa + p - p)) \\ &= w_i^+(\kappa + p, P_1^i(\kappa + p)\psi(\kappa + p)) \overset{(4.2o)}{=} P_1^i(\kappa + p)\psi(\kappa + p) \overset{(4.2j)}{=} w_i^+(\kappa, \xi_0), \end{aligned}$$

i.e., we have established the p-periodicity of $w_i^+(\cdot, \xi_0)$ in case $\xi_0 \in \mathcal{P}_1^i(\kappa)$. With this the p-periodicity of $w_i^+(\cdot, x_0)$ for general $x_0 \in \mathcal{X}$ follows from (4.2j). $\square$

4.3 Invariant Foliations and Asymptotic Phase

As starting point, consider a linear homogenous equation

$$B_{k+1}x' = A_k x \tag{L_0}$$

for which we have an exponential N-splitting as in Hypothesis 4.2.1. Given a fixed reference solution $\phi_* : \mathbb{I} \to \mathfrak{X}$ to (L_0), we are interested in the nonautonomous set $\mathcal{V}_{\phi_*} \subseteq \mathcal{X}$ consisting of all initial pairs $(\kappa, \xi) \in \mathcal{X}$ such that the difference $\Phi(\cdot, \kappa)\xi - \phi_*$ is c^+-bounded. This enables us to group points in $\mathcal{X}$ according to the above asymptotic behavior as equivalence classes.

The latter are easily characterized, since we have the equivalence

$$\Phi(\cdot, \kappa)[\xi - \phi_*(\kappa)] \in \mathfrak{X}^+_{\kappa, c_i} \quad \overset{(3.4k)}{\Leftrightarrow} \quad \xi = \phi_*(\kappa) + \mathcal{Q}^i_1(\kappa) \quad \text{for all } \kappa \in \mathbb{I}$$

and $1 \leq i < N$, yielding $\mathcal{V}_{\phi_*} = \phi_* + \mathcal{Q}^i_1$. We expect that this observation persists when passing over from the linear equation (L_0) to (S). Accordingly, in this section we investigate attraction properties of the invariant fiber bundles $\mathcal{W}^\pm_i$ from Theorem 4.2.9 in the generalized framework of $c^\pm$-boundedness. For this, our main tools will be certain invariant fibers, which serve as leaves for an invariant foliation of the extended state space $\mathcal{X}$. Equivalently, given $(\kappa, \xi) \in \mathcal{X}$ we aim to characterize the equivalence classes $[(\kappa, \xi)]^\pm_i$ defined in Remark 4.2.10(3).

This enables us to construct an asymptotic phase property for each $\mathcal{W}^\pm_i$ by choosing $(\kappa, \xi) \in \mathcal{W}^\pm_i$. This means that $\mathcal{W}^\pm_i$ is not only exponentially attracting in forward resp. backward time, but solutions are also in phase with corresponding solutions on the invariant set $\mathcal{W}^\pm_i$.

Our strategy in the first part of this section is parallel to the previous one. Nonetheless, the present assumptions are stronger than in Sect. 4.2, in a sense that continuity of the general forward solution $\varphi(k; \kappa, \cdot) : X_\kappa \to X_k$ to

$$B_{k+1}x' = A_k x + f_k(x, x'), \tag{S}$$

will play a crucial role. Such issues were addressed in Theorem 2.3.6 for general implicit equations and in Proposition 4.1.3 for semilinear problems (S).

Hypothesis 4.3.1. *Let the general forward solution φ of* (S) *exist as a continuous mapping. Suppose $f_k : X_k \times X_{k+1} \to Y_{k+1}$ with $f_k(X_k, X_{k+1}) \subseteq \operatorname{im} B_{k+1}$ for all $k \in \mathbb{I}'$ and that we have the global Lipschitz estimates* (4.2a)*.*

Next, we introduce an appropriate *Lyapunov–Perron operator* to construct invariant fibers. Choose a fixed $1 \leq i < N$ and suppose $\mathbb{I}$ is unbounded above. Let $(\kappa, \eta, \xi) \in \mathcal{Q}^i_1 \times \mathcal{X}$ and $c : \mathbb{I} \to (0, \infty)$. Since the general forward solution φ of (S) exists, for $\psi \in \mathfrak{X}^+_{\kappa, c}$, we can formally define the mapping

$$S_\kappa^+(\psi;\eta,\xi) := \Phi(\cdot,\kappa)\left[\eta - Q_1^i(\kappa)\xi\right] + \sum_{n=\kappa}^{\infty} G_i(\cdot,n+1)B_{n+1}^{-1}\cdot \\ \cdot\left[f_n(\underline{\psi(n)+\varphi(n;\kappa,\xi)}) - f_n(\underline{\varphi(n;\kappa,\xi)})\right].$$

Note that the forward evolution operator Φ and Green's function G_i for (L_0) exist by Lemma 3.3.6(a). Due to the particular difference structure of S_κ^+, the growth conditions $(\Gamma_i^\pm)$ will be of minor importance in the following considerations.

Lemma 4.3.2. *Assume Hypotheses 4.2.1 and 4.3.1. If $c,d : \mathbb{I} \to (0,\infty)$ satisfy (4.2c), then the mapping $S_\kappa^+ : \mathcal{X}_{\kappa,c}^+ \times \mathcal{Q}_1^i(\kappa) \times X_\kappa \to \mathcal{X}_{\kappa,d}^+$ is well-defined with*

$$\begin{aligned} \left\|S_\kappa^+(\psi;\eta,\xi)\right\|_{\kappa,d}^+ &\le K_i^+ \left\|\eta - Q_1^i(\kappa)\xi\right\|_{X_\kappa} + \ell_i(c)\left\|\psi\right\|_{\kappa,c}^+, \\ \left\|P_1^i(\kappa)S_\kappa^+(\kappa,\psi;\eta,\xi)\right\|_{X_\kappa} &\le \ell_i^-(c)\left\|\psi\right\|_{\kappa,c}^+ \end{aligned} \tag{4.3a}$$

for all $(\kappa,\eta,\xi) \in \mathcal{Q}_1^i \times \mathcal{X}$, $\psi \in \mathcal{X}_{\kappa,d}^+$ and we have the Lipschitz estimates

$$\operatorname{lip}_1 P_1^i(\kappa)S_\kappa^+(\kappa,\cdot) \le \ell_i^-(c), \qquad \operatorname{lip}_1 S_\kappa^+ \le \ell_i(c), \qquad \operatorname{lip}_2 S_\kappa^+ \le K_i^+, \tag{4.3b}$$

where the constants $\ell_i(c), \ell_i^-(c)$ are defined in Lemma 4.2.6.

Proof. Let $c \in (a_i,b_i)$, $\psi \in \mathcal{X}_{\kappa,c}^+$ and $(\kappa,\eta,\xi) \in \mathcal{Q}_1^i \times \mathcal{X}$ be given. First, we show that the sequence $S_\kappa^+(\psi;\eta,\xi)$ is d^+-bounded for $c \le d$. For this, from (3.5a), (3.4g), (4.2a) one has using Lemma A.1.5(a) that

$$\begin{aligned} &\left\|P_1^i(k)S_\kappa^+(k,\psi;\xi,\eta)\right\| e_d(\kappa,k) \\ &\le K_i^- L(c)\sum_{n=k}^{\infty} e_{b_i}(k,n+1)e_c(n,\kappa)\left\|\psi\right\|_{\kappa,c}^+ e_d(\kappa,k) \overset{\text{(A.1d)}}{\le} \ell_i^-(c)\left\|\psi\right\|_{\kappa,c}^+ \end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$, where $L(c)$ is defined in Lemma 4.2.6. Accordingly, (3.5a), (3.4g) and (4.2a) also imply with Lemma A.1.5(b) the dual inequality

$$\begin{aligned} \left\|Q_1^i(k)S_\kappa^+(k,\psi;\eta,\xi)\right\| e_d(\kappa,k) &\le K_i^+ e_{\frac{a_i}{d}}(k,\kappa)\left\|\eta - Q_1^i(\kappa)\xi\right\| \\ &\quad + K_i^+ L(c)\sum_{n=\kappa}^{k-1} e_{a_i}(k,n+1)e_c(n,\kappa)\left\|\psi\right\|_{\kappa,c}^+ e_d(\kappa,k) \\ &\overset{\text{(A.1e)}}{\le} K_i^+\left\|\eta - Q_1^i(\kappa)\xi\right\| + \ell_i^+(c)\left\|\psi\right\|_{\kappa,c}^+ \end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$. Combining these estimates, Lemma 3.3.22 leads to

$$\left|S_\kappa^+(k,\psi;\eta,\xi)\right|_i e_d(\kappa,k) \le K_i^+\left\|\eta - Q_1^i(\kappa)\xi\right\| + \ell_i(c)\left\|\psi\right\|_{\kappa,c}^+ \quad \text{for all } k \in \mathbb{Z}_\kappa^+.$$

This implies $S_\kappa^+(\psi;\eta,\xi)\in\mathcal{X}_{\kappa,d}^+$, as well as the first estimate (4.3a). Moreover, if we set $k=\kappa$, then the second relation in (4.3a) is a consequence of the above. Next we derive the Lipschitz estimates (4.3b). Let $\psi,\bar\psi\in\mathcal{X}_{\kappa,c}^+$, $\eta,\bar\eta\in\mathcal{Q}_1^i(\kappa)$ and fix $\xi\in X_\kappa$. We obtain from (3.5a), (3.4g), (4.2a) and Lemma A.1.5(a),

$$\begin{aligned}&\left\|P_1^i(k)\left[S_\kappa^+(k,\psi;\eta,\xi)-S_\kappa^+(k,\bar\psi;\eta,\xi)\right]\right\|e_d(\kappa,k)\\&\le K_i^-L(c)\sum_{n=k}^{\infty}e_{b_i}(k,n+1)e_c(n,\kappa)\left\|\psi-\bar\psi\right\|_{\kappa,c}^+e_d(\kappa,k)\overset{(\text{A.1d})}{\le}\ell_i^-(c)\left\|\psi-\bar\psi\right\|_{\kappa,c}^+\end{aligned}\tag{4.3c}$$

for all $k\in\mathbb{Z}_\kappa^+$, and dually (3.5a), (4.2a) have the consequence

$$\begin{aligned}&\left\|Q_1^i(k)\left[S_\kappa^+(k,\psi;\eta,\xi)-S_\kappa^+(k,\bar\psi;\eta,\xi)\right]\right\|e_d(\kappa,k)\\&\le K_i^+L(c)\sum_{n=\kappa}^{k-1}e_{a_i}(k,n+1)e_c(n,\kappa)\left\|\psi-\bar\psi\right\|_{\kappa,c}^+e_d(\kappa,k)\overset{(\text{A.1e})}{\le}\ell_i^+(c)\left\|\psi-\bar\psi\right\|_{\kappa,c}^+\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^+$. Again, combining the last two inequalities in order to estimate the difference $S_\kappa^+(\psi;\eta,\xi)-S_\kappa^+(\bar\psi;\eta,\xi)$, we are using the norm $|\cdot|_i$ from Lemma 3.3.22. This implies the middle relation in (4.3b); moreover, setting here $k=\kappa$ leads to the first estimate in (4.3b). Finally, the remaining right Lipschitz estimate in (4.3b) follows from $\|S_\kappa^+(k,\psi;\eta,\xi)-S_\kappa^+(k,\psi;\bar\eta,\xi)\|\,e_d(\kappa,k)\le K_i^+\|\eta-\bar\eta\|$ for every $k\in\mathbb{Z}_\kappa^+$, which we get from (4.2b), (3.4g). Thus, we are done. □

The next lemma provides a dynamical interpretation of the operator S_κ^+.

Lemma 4.3.3. *Let $(\kappa,\eta,\xi)\in\mathcal{Q}_1^i\times\mathcal{X}$ and assume Hypotheses 4.2.1 and 4.3.1. If one chooses $c:\mathbb{I}\to(0,\infty)$ according to (4.2c) and $\psi\in\mathcal{X}_{\kappa,c}^+$, then the following statements are equivalent for $S_\kappa^+(\cdot;\eta,\xi):\mathcal{X}_{\kappa,c}^+\to\mathcal{X}_{\kappa,c}^+$:*

(a) There exists a $\zeta\in X_\kappa$ such that $\psi=\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)\in\mathcal{X}_{\kappa,c}^+$ and

$$Q_1^i(\kappa)\psi(\kappa)=\eta-Q_1^i(\kappa)\xi.\tag{4.3d}$$

(b) ψ is a solution of the fixed point equation

$$\psi=S_\kappa^+(\psi;\eta,\xi).\tag{4.3e}$$

Proof. Let $(\kappa,\xi)\in\mathcal{X}$ and assume $\psi\in\mathcal{X}_{\kappa,c}^+$. We define the inhomogeneity

$$g_k:=f_k(\underline{\psi(k)+\varphi(k;\kappa,\xi)})-f_k(\underline{\varphi(k;\kappa,\xi)}),$$

which clearly satisfies $g_k\in Y_{k+1}$, $k\in\mathbb{I}$. In addition, due to the estimate

$$\left\|B_{k+1}^{-1}g_k\right\|\overset{(4.2\text{a})}{\le}(L_1+\lceil c\rceil L_2)\left\|\psi\right\|_{\kappa,c}^+e_c(k,\kappa)\quad\text{for all }k\in\mathbb{Z}_\kappa^+$$

one obtains the inclusion $g \in \mathfrak{X}^+_{\kappa,c,B}$. From these preparations we get:

$(a) \Rightarrow (b)$ Let $\eta \in \mathcal{Q}_1^i(\kappa)$ and assume there exists a $\zeta \in X_\kappa$ such that the difference $\psi = \varphi(\cdot;\kappa,\zeta) - \varphi(\cdot;\kappa,\xi)$ is c^+-bounded and $Q_1^i(\kappa)\psi(\kappa) = \eta - Q_1^i(\kappa)\xi$. Then ψ is a c^+-bounded solution of the inhomogeneous equation $B_{k+1}x' = A_k x + g_k$ and Theorem 3.5.3(a) implies that ψ is a fixed point of $S_\kappa^+(\cdot;\xi,\eta)$.

$(b) \Rightarrow (a)$ Conversely, assume that $\psi \in \mathfrak{X}^+_{\kappa,c}$ satisfies (4.3e) for some $\eta \in \mathcal{Q}_1^i(\kappa)$, $\xi \in X_\kappa$, define $\zeta := Q_1^i(\kappa)\,[\xi + \psi(\kappa)] + \eta$ and set $\phi := \psi + \varphi(\cdot;\kappa,\xi)$. Hence,

$$\begin{aligned}\phi(\kappa) = \psi(\kappa) + \xi &\overset{(4.3e)}{=} P_1^i(\kappa)\psi(\kappa) + Q_1^i(\kappa)S_\kappa^+(\kappa,\psi;\eta,\xi)(\kappa) + \xi \\ &= P_1^i(\kappa)\psi(\kappa) + \eta - Q_1^i(\kappa)\xi + \xi = P_1^i(\kappa)\,[\psi(\kappa)+\xi] + \eta = \zeta\end{aligned}$$

and ϕ also solves (S). Due to the uniqueness of forward solutions guaranteed by Hypothesis 4.3.1, this gives us $\phi = \varphi(\cdot;\kappa,\zeta)$, i.e., $\psi = \varphi(\cdot;\kappa,\zeta) - \varphi(\cdot;\kappa,\xi)$. Finally, one has $Q_1^i(\kappa)\psi(\kappa) = Q_1^i(\kappa)\,[\zeta - \xi] = Q_1^i(\kappa)\,[\eta - \xi] = \eta - Q_1^i(\kappa)\xi$. □

Lemma 4.3.4. *Let $(\kappa,\eta,\xi) \in \mathcal{Q}_1^i \times \mathcal{X}$ and assume Hypotheses 4.2.1 and 4.3.1. If (G_i) holds and $c \in \bar{\Gamma}_i$, then the mapping $S_\kappa^+(\cdot;\eta,\xi) : \mathfrak{X}^+_{\kappa,c} \to \mathfrak{X}^+_{\kappa,c}$ has a unique fixed point $\psi_\kappa(\eta,\xi) \in \mathfrak{X}^+_{\kappa,c}$. Moreover, the fixed point mapping $\psi_\kappa : \mathcal{Q}_1^i(\kappa) \times X_\kappa \to \mathfrak{X}^+_{\kappa,c}$ satisfies $\psi_\kappa(\eta,\xi) = \psi_\kappa(Q_1^i(\kappa)\eta,\xi)$ and one has:*

(a) $\psi_\kappa : \mathcal{Q}_1^i(\kappa) \times X_\kappa \to \mathfrak{X}^+_{\kappa,c}$ is linearly bounded, i.e., it is

$$\begin{aligned}\|\psi_\kappa(\eta,\xi)\|^+_{\kappa,c} &\le \frac{K_i^+}{1-\ell_i(c)}\left\|\eta - Q_1^i(\kappa)\xi\right\|_{X_\kappa}, \\ \left\|P_1^i(\kappa)\psi_\kappa(\kappa,\eta,\xi)\right\|_{X_\kappa} &\le \tilde{\ell}_i^+(c)\left\|\eta - Q_1^i(\kappa)\xi\right\|_{X_\kappa}.\end{aligned} \tag{4.3f}$$

(b) One has the Lipschitz estimates

$$\operatorname{lip}_1 \psi_\kappa \le \frac{K_i^+}{1-\ell_i(c)}, \qquad \operatorname{lip}_1 P_1^i(\kappa)\psi_\kappa(\kappa,\cdot) \le \tilde{\ell}_i^+(c) \tag{4.3g}$$

and $\psi_\kappa : \mathcal{Q}_1^i(\kappa) \times X_\kappa \to \mathfrak{X}^+_{\kappa,c}$ is continuous for $c \in (a_i + \varsigma, b_i - \varsigma]$,

where the constants $\ell_i(c) \in [0,1)$, $\tilde{\ell}_i^+(c)$ are defined in Lemma 4.2.6.

Proof. Let $(\kappa,\eta,\xi) \in \mathcal{Q}_1^i \times \mathcal{X}$. From the middle estimate (4.3b) in Lemma 4.3.2 and (4.2i) we know that $S_\kappa^+(\cdot;\eta,\xi)$ is a contraction on $\mathfrak{X}^+_{\kappa,c}$ and Banach's theorem (see, for instance, [295, p. 361, Lemma 1.1]) implies the existence of a unique fixed point denoted by $\psi_\kappa(\eta,\xi) \in \mathfrak{X}^+_{\kappa,c}$.

(a) Thanks to $\ell_i(c) < 1$ (cf. (4.2i)), the first estimate (4.3f) follows from

$$\begin{aligned}\|\psi_\kappa(\eta,\xi)\|^+_{\kappa,c} &\overset{(4.3e)}{=} \left\|S_\kappa^+(\psi_\kappa(\eta,\xi);\eta,\xi)\right\|^+_{\kappa,c} \\ &\overset{(4.3a)}{\le} K_i^+\left\|\eta - Q_1^i(\kappa)\xi\right\| + \ell_i(c)\,\|\psi_\kappa(\eta,\xi)\|^+_{\kappa,c},\end{aligned}$$

and similarly we get

$$\left\|P_1^i(\kappa)\psi_\kappa(\kappa,\eta,\xi)\right\| \stackrel{(4.3e)}{=} \left\|P_1^i(\kappa)S_\kappa^+(\kappa,\psi_\kappa(\eta,\xi);\eta,\xi)\right\| \stackrel{(4.3a)}{\leq} \ell_i^-(c)\left\|\psi_\kappa(\eta,\xi)\right\|_{\kappa,c}^+;$$

by the inequality shown before, this implies (4.3f).

(b) Next we derive the Lipschitz estimates in (4.3g). For this, let $\eta,\bar{\eta}\in\mathcal{Q}_1^i(\kappa)$, fix $\xi\in X_\kappa$, and from (4.3e) we obtain

$$\left\|\psi_\kappa(\eta,\xi)-\psi_\kappa(\bar{\eta},\xi)\right\|_{\kappa,c}^+ \stackrel{(4.3b)}{\leq} K_i^+\left\|\eta-\bar{\eta}\right\|+\ell_i(c)\left\|\psi_\kappa(\eta,\xi)-\psi_\kappa(\bar{\eta},\xi)\right\|_{\kappa,c}^+,$$

yielding the left relation in (4.3g). Similarly, using (4.3e), (4.3b) one has

$$\left\|P_1^i(\kappa)\left[\psi_\kappa(\kappa,\eta,\xi)-\psi_\kappa(\kappa,\bar{\eta},\xi)\right]\right\|\leq\ell_i^-(c)\left\|\psi_\kappa(\eta,\xi)-\psi_\kappa(\bar{\eta},\xi)\right\|_{\kappa,c}^+$$

leading to the remaining right relation in (4.3g). To complete the proof of (b), one has to show the continuity of $\psi_\kappa:\mathcal{Q}_1^i(\kappa)\times X_\kappa\to\mathfrak{X}_{\kappa,c}^+$ for $c\in(a_i+\varsigma,b_i-\varsigma]$. Here, our strategy serves as a prototype for various related continuity proofs in the following. Due to (4.3e), in order to prove the continuity of ψ_κ, it suffices to show for arbitrary fixed $(\kappa,\eta_0)\in\mathcal{Q}_1^i$ the following limit relation:

$$\lim_{\xi\to\xi_0}\left\|\psi_\kappa(\eta_0,\xi)-\psi_\kappa(\eta_0,\xi_0)\right\|_{\kappa,c}^+=0 \tag{4.3h}$$

(cf. Lemma B.1.3). To arrive at a short notation, we suppress the dependence on $\eta_0\in\mathcal{Q}_1^i(\kappa)$ from now on and define mappings $H_k:X_k\times X_{k+1}\times X_\kappa\to X_{k+1}$,

$$H_k(x,y,\xi):=B_{k+1}^{-1}[f_k(x+\varphi(k;\kappa,\xi),y+\varphi(k+1;\kappa,\xi))-f_k(\underline{\varphi(k;\kappa,\xi)})]$$

and $\bar{H}_k(\zeta,\xi):=B_{k+1}^{-1}H_k(\underline{\psi_\kappa(k,\zeta)},\xi)$ for $k\in\mathbb{Z}_\kappa^+$. Note that H_k and $\bar{H}_k(\zeta,\cdot)$ are continuous due to Hypothesis 4.3.1. For any parameter $\xi_0\in X_\kappa$ we obtain from (4.3e), similarly to (4.3c), the estimate

$$\begin{aligned}\left\|Q_1^i(\kappa)\left[\psi_\kappa(k;\xi)-\psi_\kappa(k;\xi_0)\right]\right\|\leq&\left\|\Phi(k,\kappa)Q_1^i(\kappa)\right\|\left\|\xi-\xi_0\right\|\\&+K_i^+\sum_{n=\kappa}^{k-1}e_{a_i}(k,n+1)\left\|\bar{H}_n(\xi,\xi)-\bar{H}_n(\xi_0,\xi_0)\right\|,\\\left\|\psi_\kappa(k;\xi)-\psi_\kappa(k;\xi_0)\right\|\leq&K_i^-\sum_{n=k}^{\infty}e_{b_i}(k,n+1)\left\|\bar{H}_n(\xi,\xi)-\bar{H}_n(\xi_0,\xi_0)\right\|\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^+$, using the dichotomy estimates (3.4g). Subtraction and addition of $\bar{H}_n(\xi_0,\xi)$ in the corresponding norms in connection with Lemma 3.3.22 leads to

$$\left|\psi_\kappa(k;\xi)-\psi_\kappa(k;\xi_0)\right|_i\leq\max\left\{S_0+S_2+S_4,S_1+S_3\right\}\quad\text{for all } k\in\mathbb{Z}_\kappa^+,$$

where (cf. (3.4g) and (4.2a)) $S_0 := K_i^+ e_{a_i}(k,\kappa)\,\|\xi-\xi_0\|$,

$$\begin{aligned}
S_1 &:= K_i^- L_1 \sum_{n=k}^{\infty} e_{b_i}(k,n+1)\,\|\psi_\kappa(n,\xi)-\psi_\kappa(n,\xi_0)\| \\
&\quad + K_i^- L_2 \sum_{n=k}^{\infty} e_{b_i}(k,n+1)\,\|\psi_\kappa(n+1,\xi)-\psi_\kappa(n+1,\xi_0)\|\,, \\
S_2 &:= K_i^+ L_1 \sum_{n=\kappa}^{k-1} e_{a_i}(k,n+1)\,\|\psi_\kappa(n,\xi)-\psi_\kappa(n,\xi_0)\| \\
&\quad + K_i^+ L_2 \sum_{n=\kappa}^{k-1} e_{a_i}(k,n+1)\,\|\psi_\kappa(n+1,\xi)-\psi_\kappa(n+1,\xi_0)\|\,, \\
S_3 &:= K_i^- \sum_{n=k}^{\infty} e_{b_i}(k,n+1)\,\left\|\bar{H}_n(\xi_0,\xi)-\bar{H}_n(\xi_0,\xi_0)\right\|\,, \\
S_4 &:= K_i^+ \sum_{n=\kappa}^{k-1} e_{a_i}(k,n+1)\,\left\|\bar{H}_n(\xi_0,\xi)-\bar{H}_n(\xi_0,\xi_0)\right\|\,.
\end{aligned}$$

Using well-known arguments we get $S_l e_c(\kappa,k) \le \ell_i(c)\,\|\psi_\kappa(\xi)-\psi_\kappa(\xi_0)\|_{\kappa,c}^+$ for every $l\in\{1,2\}$ and $k\in\mathbb{Z}_\kappa^+$. Therefore, we obtain the estimate

$$\begin{aligned}
|\psi_\kappa(k;\xi)-\psi_\kappa(k;\xi_0)|_i\, e_c(\kappa,k) \le \max\left\{K_i^+\,\|\xi-\xi_0\| + S_3 e_c(\kappa,k), S_4 e_c(\kappa,k)\right\} \\
+\, \ell_i(c)\,\|\psi_\kappa(\xi)-\psi_\kappa(\xi_0)\|_{\kappa,c}^+
\end{aligned}$$

and by passing over to the least upper bound for $k\in\mathbb{Z}_\kappa^+$, we get

$$\|\psi_\kappa(\xi)-\psi_\kappa(\xi_0)\|_{\kappa,c}^+ \le \frac{K_i^+}{1-\ell_i(c)}\,\|\xi-\xi_0\| + \frac{\max\left\{K_i^-,K_i^+\right\}}{1-\ell_i(c)} \sup_{k\in\mathbb{Z}_\kappa^+} U(k,\xi)$$

with the mapping

$$\begin{aligned}
U(k,\xi) &:= e_c(\kappa,k) \sum_{n=k}^{\infty} e_{b_i}(k,n+1)\,\left\|\bar{H}_n(\xi_0,\xi)-\bar{H}_n(\xi_0,\xi_0)\right\| \\
&\quad + e_c(\kappa,k) \sum_{n=\kappa}^{k-1} e_{a_i}(k,n+1)\,\left\|\bar{H}_n(\xi_0,\xi)-\bar{H}_n(\xi_0,\xi_0)\right\|\,.
\end{aligned}$$

Therefore, it suffices to prove the limit relation

$$\lim_{\xi\to\xi_0} \sup_{k\in\mathbb{Z}_\kappa^+} U(k,\xi) = 0 \tag{4.3i}$$

in order to establish (4.3h). We proceed indirectly. If (4.3i) does not hold, then there exists an $\varepsilon > 0$ and a sequence $(\xi_j)_{j\in\mathbb{N}}$ in X_κ with $\lim_{j\to\infty}\xi_j = \xi_0$ and $\sup_{k\in\mathbb{Z}_\kappa^+} U(k,\xi_j) > \varepsilon$ for all $j \in \mathbb{N}$. This implies the existence of a sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ such that

$$U(k_j,\xi_j) > \varepsilon \quad \text{for all } j \in \mathbb{N}. \tag{4.3j}$$

From now on assume $a_i + \varsigma \ll c$, choose a fixed growth rate $d \in (a_i + \varsigma, c)$ and remark that the inequality $d \ll c$ will play an important role below. Because of Hypothesis 4.3.1 and the inclusion $\psi_\kappa(\xi) \in \mathcal{X}_{\kappa,d}^+$ we arrive at $\left\|\bar{H}_n(\xi_0,\xi)\right\| \leq L(c)\left\|\psi_\kappa(\xi_0)\right\|_{\kappa,d}^+ e_d(n,\kappa)$ for all $n \in \mathbb{Z}_\kappa^+$ (cf. (4.2a)) and Lemma A.1.5 leads to

$$U(k,\xi) \leq \left\|\psi_\kappa(\xi_0)\right\|_{\kappa,d}^+ \left(\frac{L(c)}{\lfloor b_i - d\rfloor} + \frac{L(c)}{\lfloor d - a_i\rfloor}\right) e_{\frac{d}{c}}(k,\kappa) \quad \text{for all } k \in \mathbb{Z}_\kappa^+.$$

Because of $\frac{d}{c} \ll 1$, passing over to the limit $k \to \infty$ yields $\lim_{k\to\infty} U(k,\xi) = 0$ uniformly in $\xi \in X_\kappa$, and taking into account (4.3j) the sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ has to be bounded above, i.e., there exists an integer $K > \kappa$ with $k_j \leq K$ for all $j \in \mathbb{N}$. Hence, we can infer using Proposition A.1.2(d) that

$$\begin{aligned}
U(k,\xi_j) &\leq e_{\frac{b_i}{c}}(k,\kappa)\sum_{n=\kappa}^{\infty} e_{b_i}(\kappa,n+1)\left\|\bar{H}_n(\xi_0,\xi_j) - \bar{H}_n(\xi_0,\xi_0)\right\| \\
&\qquad + \underbrace{e_{\frac{a_i}{c}}(k,\kappa)}_{\leq 1}\sum_{n=\kappa}^{K} e_{a_i}(\kappa,n+1)\left\|\bar{H}_n(\xi_0,\xi_j) - \bar{H}_n(\xi_0,\xi_0)\right\| \\
&\leq e_{\frac{b_i}{c}}(K,\kappa)\sum_{n=\kappa}^{\infty} e_{b_i}(\kappa,n+1)\left\|\bar{H}_n(\xi_0,\xi_j) - \bar{H}_n(\xi_0,\xi_0)\right\| \\
&\qquad + \sum_{n=\kappa}^{K} e_{a_i}(\kappa,n+1)\left\|\bar{H}_n(\xi_0,\xi_j) - \bar{H}_n(\xi_0,\xi_0)\right\| \quad \text{for all } j \in \mathbb{N},
\end{aligned}$$

where the finite sum tend to zero for $j \to \infty$ by the continuity of H_n. Continuity properties of H_n also imply $\lim_{j\to\infty}\bar{H}_n(\xi_0,\xi_j) = \bar{H}_n(\xi_0,\xi_0)$ and with the dominated convergence theorem of Lebesgue[2] the infinite sum tends to zero in the limit $j \to \infty$. Thus, we derived the relation $\lim_{j\to\infty} U(k_j,\xi_j) = 0$, which obviously contradicts (4.3j). Consequently, we have shown the continuity of the fixed point mapping $\psi_\kappa(\eta_0,\cdot) : X_\kappa \to \mathcal{X}_{\kappa,c}^+$ and the proof of (b) is finished. □

Our preparations yield the first basic result in this section. It guarantees the existence of invariant fiber bundles through given solutions, i.e., invariant fibers:

[2] In order to apply this result from integration theory (see, e.g., [295, p. 141, Theorem 5.8]), one has to write the infinite sum as an integral over piecewise-constant functions and use the Lipschitz estimate on H_n, which is implied by (4.2a), to get an integrable majorant.

Proposition 4.3.5 (existence of invariant fibers)**.** *Let* $(\kappa, \xi) \in \mathcal{X}$*. Assume that Hypothesi 4.2.1, 4.3.1 are satisfied and that* (G_i) *holds for one* $1 \leq i < N$*.*

(a) If $\mathbb{I}$ *is unbounded above, then the* forward fiber through (κ, ξ)*, given by*

$$\mathcal{V}_i^+(\kappa, \xi) := \left\{\zeta \in X_\kappa : \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi) \in \mathfrak{X}_{\kappa,c}^+\right\}$$

is independent of $c \in \bar{\Gamma}_i$*, forward invariant w.r.t.* (S)*, i.e.*

$$\varphi(k; \kappa, \mathcal{V}_i^+(\kappa, \xi)) \subseteq \mathcal{V}_i^+(k, \varphi(k; \kappa, \xi)) \quad \text{for all } k \in \mathbb{Z}_\kappa^+ \tag{4.3k}$$

and possesses the representation

$$\mathcal{V}_i^+(\xi) = \left\{(\kappa, \eta + v_i^+(\kappa, \eta, \xi)) : (\kappa, \eta) \in \mathcal{Q}_1^i\right\} \tag{4.3l}$$

as graph of a unique mapping $v_i^+ : \mathcal{X} \times \mathcal{X} \to \mathfrak{X}$ *satisfying*

$$v_i^+(\kappa, \eta, \xi) \in \mathcal{P}_1^i(\kappa) \quad \text{for all } (\kappa, \eta, \xi) \in \mathcal{Q}_1^i \times \mathcal{X} \tag{4.3m}$$

and the invariance equation

$$\begin{aligned} v_i^+(\kappa+1, \eta_1, \xi_1) = {} & B_{\kappa+1}^{-1} A_\kappa v_i^+(\kappa, \eta, \xi) \\ & + Q_1^i(\kappa+1) f_\kappa(\eta + v_i^+(\kappa, \eta, \xi), \eta_1 \\ & + v_i^+(\kappa+1, \eta_1, \xi_1)), \end{aligned} \tag{4.3n}$$

$$\begin{aligned} \eta_1 &= B_{\kappa+1}^{-1} A_\kappa \eta + Q_1^i(\kappa+1) f_\kappa(\eta + v_i^+(\kappa, \eta, \xi), \eta_1 + v_i^+(\kappa+1, \eta_1, \xi)), \\ \xi_1 &= B_{\kappa+1}^{-1} A_\kappa \xi + B_{\kappa+1}^{-1} f_\kappa(\xi, \xi_1) \quad \text{for all } (\kappa, \eta, \xi) \in \mathcal{Q}_1^i \times \mathcal{X}. \end{aligned}$$

Furthermore, for all $c \in \bar{\Gamma}_i$ *it holds:*

(a_1) $v_i^+ : \mathcal{Q}_1^i \times \mathcal{X} \to \mathfrak{X}$ *is continuous and linearly bounded, i.e., for all triple* $(\kappa, \eta, \xi) \in \mathcal{Q}_1^i \times \mathcal{X}$ *one has*

$$\left\|v_i^+(\kappa, \eta, \xi)\right\|_{X_\kappa} \leq \left\|P_1^i(\kappa)\xi\right\|_{X_\kappa} + \tilde{\ell}_i^+(c) \left\|\eta - Q_1^i(\kappa)\xi\right\|_{X_\kappa}. \tag{4.3o}$$

(a_2) $v_i^+(\kappa, \cdot, \xi)$ *is globally Lipschitzian with* $\operatorname{lip}_2 v_i^+ \leq \tilde{\ell}_i^+(c)$*.*

(b) If $\mathbb{I}$ *is unbounded below and the general solution* φ *of* (S) *exists on* $\mathcal{X}$ *as continuous mapping, then the* backward fiber through (κ, ξ)*, given by*

$$\mathcal{V}_i^-(\kappa, \xi) := \left\{\zeta \in X_\kappa : \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi) \in \mathfrak{X}_{\kappa,c}^-\right\}$$

is independent of $c \in \bar{\Gamma}_i$*, invariant w.r.t.* (S)*, i.e.*

$$\varphi(k; \kappa, \mathcal{V}_i^-(\kappa, \xi)) = \mathcal{V}_i^-(k, \varphi(k; \kappa, \xi)) \quad \text{for all } k \in \mathbb{Z}_\kappa^+$$

and possesses the representation

$$\mathcal{V}_i^-(\xi) = \left\{(\kappa, \eta + v_i^-(\kappa, \eta, \xi)) : (\kappa, \eta) \in \mathcal{P}_1^i\right\}$$

as graph of a unique mapping $v_i^- : \mathcal{X} \times \mathcal{X} \to \mathfrak{X}$ *satisfying*

$$v_i^-(\kappa, \eta, \xi) \in \mathcal{Q}_1^i(\kappa) \quad \textit{for all } (\kappa, \eta, \xi) \in \mathcal{P}_1^i \times \mathcal{X}$$

and the invariance equation

$$\begin{aligned} v_i^-(\kappa+1, \eta_1, \xi_1) = {} & B_{\kappa+1}^{-1} A_\kappa v_i^-(\kappa, \eta, \xi) \\ & + P_1^i(\kappa+1) f_\kappa(\eta + v_i^-(\kappa, \eta, \xi), \eta_1 + v_i^-(\kappa+1, \eta_1, \xi_1)), \end{aligned}$$

$$\begin{aligned} \eta_1 &= B_{\kappa+1}^{-1} A_\kappa \eta + P_1^i(\kappa+1) f_\kappa(\eta + v_i^-(\kappa, \eta, \xi), \eta_1 + v_i^-(\kappa+1, \eta_1, \xi)), \\ \xi_1 &= B_{\kappa+1}^{-1} A_\kappa \xi + B_{\kappa+1}^{-1} f_\kappa(\xi, \xi_1) \quad \textit{for all } (\kappa, \eta, \xi) \in \mathcal{P}_1^i \times \mathcal{X}. \end{aligned}$$

Furthermore, for all $c \in \bar{\Gamma}_i$ *it holds:*

(b_1) $v_i^- : \mathcal{P}_1^i \times \mathcal{X} \to \mathfrak{X}$ *is continuous and linearly bounded, i.e., for all triple* $(\kappa, \eta, \xi) \in \mathcal{P}_1^i \times \mathcal{X}$ *one has*

$$\left\|v_i^-(\kappa, \eta, \xi)\right\|_{X_\kappa} \leq \left\|Q_1^i(\kappa)\xi\right\|_{X_\kappa} + \tilde{\ell}_i^-(c) \left\|\eta - P_1^i(\kappa)\xi\right\|_{X_\kappa}.$$

(b_2) $v_i^-(\kappa, \cdot, \xi)$ *is globally Lipschitzian with* $\operatorname{lip}_2 v_i^- \leq \tilde{\ell}_i^-(c)$,

where the constants $\tilde{\ell}_i^\pm(c)$ *are defined in Theorem 4.2.9.*

Remark 4.3.6. (1) In case $f_k(0,0) \equiv 0$ on $\mathbb{I}$ holds, one has $\mathcal{W}_i^\pm = \mathcal{V}_i^\pm(0)$ and in this setting, Theorem 4.2.9 can be seen as a special case of Proposition 4.3.5 in the sense that the c, d-bounded solution ϕ_* from Corollary 4.2.15 is the trivial one.

(2) Given a pair $(\kappa, \xi) \in \mathcal{X}$, we have the relation $[(\kappa, \xi)]_i^\pm = \mathcal{V}_i^\pm(\xi)$ with the equivalence classes from Remark 4.2.10(3).

(3) For the existence of a function $v_i^- : \mathcal{X} \times \mathcal{X} \to \mathfrak{X}$ parametrizing $\mathcal{V}_i^-(\xi)$ and satisfying both the assertions (b_1) and (b_2) it is sufficient to assume that the general backward solution of (S) exists as a continuous mapping.

(4) If the general solution to (S) exists on $\mathcal{X}$, then the fibers $\mathcal{V}_i^+(\xi)$ are invariant w.r.t. (S), i.e., the inclusion (4.3k) can be strengthened to

$$\varphi(k; \kappa, \mathcal{V}_i^\pm(\kappa, \xi)) = \mathcal{V}_i^\pm(k, \varphi(k; \kappa, \xi)) \quad \text{for all } k \in \mathbb{I}. \tag{4.3p}$$

For instance, a combination of the assumptions for Propositions 4.1.3 and 4.1.4 yields the existence of the general solution φ on $\mathcal{X}$.

Proof. Let $c \in \bar{\Gamma}_i$ be given.

(a) Let $(\kappa, \eta, \xi) \in \mathcal{Q}_1^i \times \mathcal{X}$. First, we show the invariance assertion for the forward fiber $\mathcal{V}_i^+(\xi)$. Let $x_0 \in \varphi(k; \kappa, \mathcal{V}_i^+(\kappa, \xi))$ for some $k \in \mathbb{Z}_\kappa^+$, and by definition this is equivalent to the existence of a $\zeta \in X_\kappa$ with $x_0 = \varphi(k; \kappa, \zeta)$ guaranteeing a difference $\varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi) \in \mathfrak{X}_{\kappa,c}^+$. Therefore,

$$\begin{aligned} \varphi(\cdot; k, x_0) - \varphi(\cdot; k, \varphi(k; \kappa, \xi)) &= \varphi(\cdot; k, \varphi(k; \kappa, \zeta)) - \varphi(\cdot; k, \varphi(k; \kappa, \xi)) \\ &\overset{(2.3a)}{=} \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi), \end{aligned}$$

i.e., $x_0 \in \mathcal{V}_i^+(k, \varphi(k; \kappa, \xi))$ for all $k \in \mathbb{Z}_\kappa^+$.

Due to the spectral gap condition (G_i) one has (4.2i) and the middle estimate (4.3b) in Lemma 4.3.2 shows that $S_\kappa^+(\cdot; \eta, \xi) : \mathfrak{X}_{\kappa,c}^+ \to \mathfrak{X}_{\kappa,c}^+$ is a contraction. Hence, Lemma 4.3.4 implies that $S_\kappa^+(\cdot; \eta, \xi)$ has a unique fixed point $\psi_\kappa(\eta, \xi) \in \mathfrak{X}_{\kappa,c}^+$, which is independent of $c \in \bar{\Gamma}_i$, because due to Lemma 3.3.26 one has the inclusion $\mathfrak{X}_{\kappa,a_i+\varsigma}^+ \subseteq \mathfrak{X}_{\kappa,c}^+$ and every mapping $S_\kappa^+(\cdot; \eta, \xi) : \mathfrak{X}_{\kappa,c}^+ \to \mathfrak{X}_{\kappa,c}^+$ possesses the same fixed point as the restriction $S_\kappa^+(\cdot; \eta, \xi)\big|_{\mathfrak{X}_{\kappa,a_i+\varsigma}^+}$. Furthermore, the fixed point is of the form $\psi_\kappa(\eta, \xi) = \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi)$ with $\zeta \in X_\kappa$ (cf. Lemma 4.3.3). Having this available, we define

$$v_i^+(\kappa, \eta, \xi) := P_1^i(\kappa) \left[\xi + \psi_\kappa(\kappa, \eta, \xi)\right] \tag{4.3q}$$

and evidently get $v_i^+(\kappa, x_0) \in \mathcal{P}_1^i(\kappa)$. Let us verify the representation (4.3l).
$(\subseteq)$ Let $\zeta \in \mathcal{V}_i^+(\kappa, \xi)$, i.e., $\psi = \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi) \in \mathfrak{X}_{\kappa,c}^+$. By Lemma 4.3.3,

$$\begin{aligned} \zeta &= \psi(\kappa) + \xi = P_1^i(\kappa)\psi(\kappa) + Q_1^i(\kappa)\psi(\kappa) + \xi \\ &\overset{(4.3d)}{=} P_1^i(\kappa)\psi(\kappa) + \eta - Q_1^i(\kappa)\xi + \xi = P_1^i(\kappa)\psi(\kappa) + \eta + P_1^i(\kappa)\xi, \end{aligned}$$

hence $Q_1^i(\kappa)\zeta = \eta$, and $\zeta = Q_1^i(\kappa)\zeta + P_1^i(\kappa)\left[\xi + \psi_\kappa(\kappa, \eta, \xi)\right]$. Thus, ζ is contained in the graph of $v_i^+(\kappa, \cdot, \xi)$ over $\mathcal{Q}_1^i(\kappa)$.
$(\supseteq)$ On the other hand, suppose that $\zeta \in X_\kappa$ is of the form $\zeta = \eta + v_i^+(\kappa, \eta, \xi)$ with some given $\eta \in \mathcal{Q}_1^i(\kappa)$. Then (4.3e) implies $Q_1^i(\kappa)\psi_\kappa(\eta, \xi) = \eta - Q_1^i(\kappa)\xi$, which yields $\zeta = \eta + P_1^i(\kappa)\left[\xi + \psi_\kappa(\kappa, \eta, \xi)\right] = \xi + \psi_\kappa(\kappa, \eta, \xi)$, and consequently the inclusion $\varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi) \in \mathfrak{X}_{\kappa,c}^+$, i.e., $\zeta \in \mathcal{V}_i^+(\kappa, \xi)$.

To establish our invariance equation (4.3n), we observe that (4.3l) and the forward invariance of $\mathcal{V}_i^+(\xi)$ imply

$$\begin{aligned} \varphi(k; \kappa, \eta + v_i^+(\kappa, \eta, \xi)) = {} & Q_1^i(k)\varphi(k; \kappa, \eta + v_i^+(\kappa, \eta, \xi)) \\ & + v_i^+(k, Q_1^i(k)\varphi(k; \kappa, \eta + v_i^+(\kappa, \eta, \xi)), \varphi(k; \kappa, \xi)) \end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$ and $\eta \in \mathcal{Q}_1^i(\kappa)$. Multiplying this relation with the projection $P_1^i(k)$, setting $k = \kappa + 1$, and keeping the inclusion (4.3m) in mind, this yields (4.3n) using the solution identity for (S).

(a_1) We obtain (4.3o) from Lemma 4.3.4, since (4.3q), (4.3f) imply

$$\left\|v_i^+(\kappa,\eta,\xi)\right\| \le \left\|P_1^i(\kappa)\xi\right\| + \tilde{\ell}_i^+(c)\left\|\eta - Q_1^i(\kappa)\xi\right\|.$$

(a_2) To prove the claimed Lipschitz estimate, consider $\eta,\bar{\eta} \in \mathcal{Q}_1^i(\kappa)$, $\xi \in X_\kappa$ and the corresponding fixed points $\psi_\kappa(\eta,\xi), \psi_\kappa(\bar{\eta},\xi) \in \mathfrak{X}_{\kappa,c}^+$ of $S_\kappa^+(\cdot;\eta,\xi)$ and $S_\kappa^+(\cdot;\bar{\eta},\xi)$, respectively. One gets from Lemma 4.3.4(b) that

$$\begin{aligned}\left\|v_i^+(\kappa,\eta,\xi) - v_i^+(\kappa,\bar{\eta},\xi)\right\| &\overset{(4.3q)}{=} \left\|P_1^i(\kappa)\left[\psi_\kappa(\kappa,\eta,\xi) - \psi_\kappa(\kappa,\bar{\eta},\xi)\right]\right\| \\ &\overset{(4.3g)}{\le} \frac{K_i^- \ell_i^-(c)}{1-\ell_i(c)}\left\|\eta - \bar{\eta}\right\|.\end{aligned}$$

From Hypothesis 4.3.1 we deduce by Lemma 4.3.4(b) that $\psi_\kappa : \mathcal{Q}_1^i(\kappa) \times X_\kappa \to \mathfrak{X}_{\kappa,c}^+$ is continuous, and by definition in (4.3q) we get the continuity of $v_i^+(\kappa,\cdot)$.

(b) The proof of assertion (b) is dual to (a) and we merely present a sketch. Above all, since the general (backward) solution φ to (S) exists on $\mathcal{X}$, we can define the *Lyapunov–Perron operator* $S_\kappa^-(\cdot;\eta,\xi) : \mathfrak{X}_{\kappa,c}^- \to \mathfrak{X}_{\kappa,c}^-$,

$$\begin{aligned}S_\kappa^-(\psi;\eta,\xi) := \Phi_{P_1^i}^-(\cdot,\kappa)\left[\eta - Q_1^i(\kappa)\xi\right] + \sum_{n=-\infty}^{\kappa-1} G_i(\cdot,n+1)B_{n+1}^{-1} \cdot \\ \cdot\left[f_n\big(\underline{\psi(n)+\varphi(n;\kappa,\xi)}\big) - f_n\big(\underline{\varphi(n;\kappa,\xi)}\big)\right]\end{aligned}$$

for all $(\kappa,\eta,\xi) \in \mathcal{P}_1^i \times \mathcal{X}$. Using dual versions of Lemmata 4.3.2, 4.3.3 and 4.3.4 in the linear spaces $\mathfrak{X}_{\kappa,c}^-$, one shows that $S_\kappa^-(\cdot;\eta,\xi)$ is a contraction on $\mathfrak{X}_{\kappa,c}^-$, uniformly in the parameters η,ξ. With its unique fixed point $\psi_\kappa^-(\eta,\xi) \in \mathfrak{X}_{\kappa,c}^-$ at hand, we define $v_i^-(\kappa,\eta,\xi) := Q_1^i(\kappa)\left[\xi + \psi_\kappa^-(\kappa,\eta,\xi)\right]$ and proceed as above. □

In a descriptive way, the subsequent asymptotic phase property is also referred as "exponential tracking" of the fiber bundle $\mathcal{W}_i^-$. It states that convergence to $\mathcal{W}_i^-$ is actually "in phase" with solutions on the invariant fiber bundle $\mathcal{W}_i^-$, and for that reason we speak of an *asymptotic phase* (see Fig. 4.1). The proof relies on a geometric argument, which demands a stronger spectral gap condition.

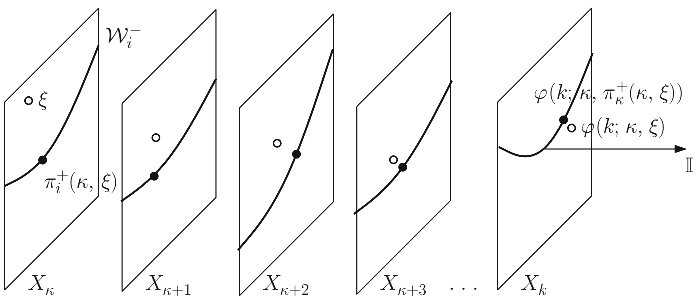

Fig. 4.1 Asymptotic forward phase of $\mathcal{W}_i^-$ and decaying solutions $\varphi(\cdot;\kappa,\xi) - \varphi(\cdot;\kappa,\pi_\kappa^+(\kappa,\xi))$

Theorem 4.3.7 (existence of an asymptotic phase). *Let $\mathbb{I} = \mathbb{Z}$ and $(\kappa, \xi) \in \mathcal{X}$. Assume Hypotheses 4.2.1 and 4.3.1 are satisfied and define*

$$\hat{K}_i := \begin{cases} \frac{\sqrt{K_i^+ K_i^-} - 1}{K_i^+ K_i^- - 1} & \text{if } K_i^+ K_i^- > 1, \\ \frac{1}{2} & \text{if } K_i^+ K_i^- = 1. \end{cases}$$

Moreover, suppose the strengthened spectral gap condition

$$\frac{\max\left\{K_i^+, K_i^-\right\}(L_1 + \lceil b_i \rceil L_2)}{\hat{K}_i + \max\left\{K_i^+, K_i^-\right\} L_2} < \varsigma_i \tag{$\hat{G}_i$}$$

holds for some $1 \leq i < N$ and choose $c \in \bar{\Gamma}_i$.

(a) If the growth condition (Γ_i^-) is satisfied, then the invariant fiber bundle $\mathcal{W}_i^-$ from Theorem 4.2.9(b) possesses an asymptotic forward phase*, i.e., there exists a mapping $\pi_i^+ : \mathcal{X} \to \mathfrak{X}$ with the property that for all $k \in \mathbb{Z}_\kappa^+$,*

$$\left\|\varphi(k; \kappa, \xi) - \varphi(k; \kappa, \pi_i^+(\kappa, \xi))\right\|_{X_k} \leq \frac{K_i^+ e_c(k, \kappa)}{1 - \ell_i(c)} \left(\left\|Q_1^i(\kappa)\xi\right\|_{X_\kappa} + \frac{\tilde{C}_\kappa^+(\xi, c)}{1 - \tilde{\ell}_i(c)}\right). \tag{4.3r}$$

Geometrically, $\pi_i^+(\kappa, \xi)$ is given as unique intersection

$$\mathcal{W}_i^-(\kappa) \cap \mathcal{V}_i^+(\kappa, \xi) = \left\{\pi_i^+(\kappa, \xi)\right\} \quad \text{for all } (\kappa, \xi) \in \mathcal{X} \tag{4.3s}$$

and one has:

(a_1) *$\pi_i^+(\kappa, \cdot) : X_\kappa \to \mathcal{W}_i^-(\kappa)$ is a continuous retraction onto the κ-fiber $\mathcal{W}_i^-(\kappa)$, linearly bounded, i.e.*

$$\left\|\pi_i^+(\kappa, \xi)\right\|_{X_\kappa} \leq \left(1 + \tilde{\ell}_i^+(c)\right)\left(\left\|Q_1^i(\kappa)\xi\right\|_{X_\kappa} + \frac{\tilde{C}_\kappa^+(\xi, c)}{1 - \tilde{\ell}_i(c)}\right)$$

and, thus, maps bounded subsets of X_κ on bounded subsets of $\mathcal{W}_i^-(\kappa)$,

(a_2) *$\varphi(k; \kappa, \cdot) \circ \pi_i^+(\kappa, \cdot) = \pi_i^+(k, \cdot) \circ \varphi(k; \kappa, \cdot)$ for all $k \in \mathbb{Z}_\kappa^+$.*

(b) If the growth condition (Γ_i^+) is satisfied and the general solution φ of (S) *exists on $\mathcal{X}$ as a continuous mapping, then the invariant fiber bundle $\mathcal{W}_i^+$ from Theorem 4.2.9(a) possesses an* asymptotic backward phase*, i.e., there exists a mapping $\pi_i^- : \mathcal{X} \to \mathfrak{X}$ with the property that for all $k \in \mathbb{Z}_\kappa^-$,*

$$\left\|\varphi(k; \kappa, \xi) - \varphi(k; \kappa, \pi_i^-(\kappa, \xi))\right\|_{X_k} \leq \frac{K_i^- e_c(k, \kappa)}{1 - \ell_i(c)} \left(\left\|P_1^i(\kappa)\xi\right\|_{X_\kappa} + \frac{\tilde{C}_\kappa^-(\xi, c)}{1 - \tilde{\ell}_i(c)}\right).$$

Geometrically, $\pi_i^-(\kappa,\xi)$ is given as unique intersection

$$\mathcal{W}_i^+(\kappa)\cap\mathcal{V}_i^-(\kappa,\xi)=\left\{\pi_i^-(\kappa,\xi)\right\}\quad \textit{for all } (\kappa,\xi)\in\mathcal{X}$$

and one has:

(b_1) *$\pi_i^-(\kappa,\cdot):X_\kappa\to\mathcal{W}_i^+(\kappa)$ is a continuous retraction onto the κ-fiber $\mathcal{W}_i^+(\kappa)$, linearly bounded, i.e.*

$$\left\|\pi_i^-(\kappa,\xi)\right\|_{X_\kappa}\le\left(1+\tilde\ell_i^-(c)\right)\left(\left\|P_1^i(\kappa)\xi\right\|_{X_\kappa}+\frac{\tilde C_\kappa^-(\xi,c)}{1-\tilde\ell_i(c)}\right)$$

and, thus, maps bounded subsets of X_κ on bounded subsets of $\mathcal{W}_i^+(\kappa)$,

(b_2) *$\varphi(k;\kappa,\cdot)\circ\pi_i^-(\kappa,\cdot)=\pi_i^-(k,\cdot)\circ\varphi(k;\kappa,\cdot)$ for all $k\in\mathbb{Z}_\kappa^+$,*

where the constants $\ell_i(c)\in[0,1)$ are defined in Lemma 4.2.6, $\tilde\ell_i^\pm(c)$ are given in Theorem 4.2.9 and $\tilde\ell_i(c):=\tilde\ell_i^-(c)\tilde\ell_i^+(c)\in[0,1)$,

$$\tilde C_\kappa^+(\xi,c):=\left(\tfrac{K_i^+\Gamma_\kappa^-(i)}{\lfloor c-a_i\rfloor}+\tfrac{\ell_i^+(c)C_i(c)\Gamma_\kappa^-(i)}{1-\ell_i(c)}\right)+\tilde\ell_i^-(c)\left(1+\tilde\ell_i^+(c)\right)\left\|Q_1^i(\kappa)\xi\right\|_{X_\kappa},$$
$$\tilde C_\kappa^-(\xi,c):=\left(\tfrac{K_i^-\Gamma_\kappa^+(i)}{\lfloor b_i-c\rfloor}+\tfrac{\ell_i^-(c)C_i(c)\Gamma_\kappa^+(i)}{1-\ell_i(c)}\right)+\tilde\ell_i^+(c)\left(1+\tilde\ell_i^-(c)\right)\left\|P_1^i(\kappa)\xi\right\|_{X_\kappa}.$$

Remark 4.3.8 (spectral gap condition). Thanks to $\hat K_i\le 1$ one sees that $(\hat G_i)$ implies (G_i). Indeed, our different gap conditions are related as follows:

$$(\tilde G_i)\quad\Rightarrow\quad(4.2\text{u})\quad\Rightarrow\quad(G_i)\quad\Leftarrow\quad(\hat G_i).$$

Moreover, $(\hat G_i)$ can be replaced by $(\tilde G_i)$ in Theorem 4.3.7. Actually, the consequences of our three different spectral gap conditions are as follows:

- Condition (G_i) guarantees $\ell_i(c)<1$ (cf. (4.2i)) and therefore the operators $T_\kappa^\pm$ from the proof of Theorem 4.2.9 and $S_\kappa^\pm$ needed to prove Proposition 4.3.5 satisfy

$$\operatorname{lip}_1 T_\kappa^\pm<1,\qquad\qquad \operatorname{lip}_1 S_\kappa^\pm<1.$$

- Condition $(\tilde G_i)$ yields $\tilde\ell_i^\pm(c)<1$ and so the mappings $w_i^\pm$ from Theorem 4.2.9 and $v_i^\mp$ in Proposition 4.3.5 satisfy

$$\operatorname{lip}_2 w_i^\pm<1,\qquad\qquad \operatorname{lip}_2 v_i^\mp<1.$$

- Condition $(\hat G_i)$ finally ensures $\operatorname{lip}_2 w_i^\pm\cdot\operatorname{lip}_2 v_i^\mp<1$ (see (4.3t) below).

Proof. Let $c\in\bar\Gamma_i$ and fix a pair $(\kappa,\xi)\in\mathcal{X}$. We begin with a preliminary remark illustrating the consequences of $(\hat G_i)$. From $(\hat G_i)$ one easily deduces the inequalities $\ell_i^+(c)\le\frac{K_i^+}{\varsigma}L(b_i-\varsigma)<\hat K_i$ and $\ell_i^-(c)\le\frac{K_i^-}{\varsigma}L(b_i-\varsigma)<\hat K_i$, which guarantee

$$K_i^+K_i^-\ell_i^+(c)^2<(1-\ell_i^+(c))^2,\qquad K_i^+K_i^-\ell_i^-(c)^2<(1-\ell_i^-(c))^2$$

and these two relations imply:

- If $\ell_i^-(c) \le \ell_i^+(c)$, then

$$K_i^+ K_i^- \ell_i^+(c)\ell_i^-(c) \le K_i^+ K_i^- \ell_i^+(c)^2 < (1-\ell_i^+(c))^2 = (1-\ell_i(c))^2,$$

- if $\ell_i^+(c) \le \ell_i^-(c)$, then

$$K_i^+ K_i^- \ell_i^+(c)\ell_i^-(c) \le K_i^+ K_i^- \ell_i^-(c)^2 < (1-\ell_i^-(c))^2 = (1-\ell_i(c))^2.$$

We can therefore conclude that the functions $w_i^\pm : \mathcal{X} \to \mathfrak{X}$ from Theorem 4.2.9, as well as $v_i^\mp : \mathcal{X} \times \mathcal{X} \to \mathfrak{X}$ from Proposition 4.3.5 satisfy

$$\operatorname{lip}_2 w_i^\pm \cdot \operatorname{lip}_2 v_i^\mp \le \frac{K_i^+ K_i^- \ell_i^+(c)\ell_i^+(c)}{(1-\ell_i(c))^2} = \tilde{\ell}_i(c) < 1. \tag{4.3t}$$

(a) We show that there exists one and only one $\zeta \in \mathcal{W}_i^-(\kappa) \cap \mathcal{V}_i^+(\kappa, \xi)$. For this, note that $\zeta \in \mathcal{W}_i^-(\kappa) \cap \mathcal{V}_i^+(\kappa, \xi)$ holds if and only if $\zeta = P_1^i(\kappa)\zeta + w_i^-(\kappa, P_1^i(\kappa)\zeta)$ and $\zeta = Q_1^i(\kappa)\zeta + v_i^+(\kappa, Q_1^i(\kappa)\zeta, \xi)$, which is equivalent to

$$Q_1^i(\kappa)\zeta = w_i^-(\kappa, P_1^i(\kappa)\zeta) \quad \text{and} \quad P_1^i(\kappa)\zeta = v_i^+(\kappa, Q_1^i(\kappa)\zeta, \xi).$$

Due to the contraction condition (4.3t) we can apply Corollary B.1.4(a) to the above equations. Thus, there exist two uniquely determined functions $q_\kappa : X_\kappa \to \mathcal{Q}_1^i(\kappa)$, $p_\kappa : X_\kappa \to \mathcal{P}_1^i(\kappa)$ satisfying the identities

$$q_\kappa(\xi) \equiv w_i^-(\kappa, p_\kappa(\xi)) \quad \text{and} \quad p_\kappa(\xi) \equiv v_i^+(\kappa, q_\kappa(\xi), \xi) \quad \text{on } X_\kappa. \tag{4.3u}$$

Therefore, $\pi_i^+(\kappa, \xi) := p_\kappa(\xi) + q_\kappa(\xi)$ is the unique element in the intersection $\mathcal{W}_i^-(\kappa) \cap \mathcal{V}_i^+(\kappa, \xi)$. We now derive the estimate (4.3r). From (4.3u) we get

$$\begin{aligned}
\|q_\kappa(\xi)\| &\overset{(4.2n)}{\le} \frac{K_i^+ \varGamma_\kappa^-(i)}{\lfloor c - a_i \rfloor} + \frac{\ell_i^+(c)}{1-\ell_i(c)} \left(K_i^- \left\| v_i^+(\kappa, q_\kappa(\xi), \xi) \right\| + C_i(c)\varGamma_\kappa^-(i) \right) \\
&\overset{(4.3o)}{\le} \frac{K_i^+ \varGamma_\kappa^-(i)}{\lfloor c - a_i \rfloor} + \frac{\ell_i^+(c) C_i(c) \varGamma_\kappa(i)}{1-\ell_i(c)} \\
&\quad + \tilde{\ell}_i^-(c) \left(\left\| Q_1^i(\kappa)\xi \right\| + K_i^+ \frac{\ell_i^-(c)}{1-\ell_i(c)} \left\| q_\kappa(\xi) - Q_1^i(\kappa)\xi \right\| \right)
\end{aligned}$$

and the triangle inequality together with (4.3t) yields

$$\|q_\kappa(\xi)\| \le \frac{\tilde{C}_\kappa^+(\xi, c)}{1-\tilde{\ell}_i(c)}. \tag{4.3v}$$

Since by construction one has the inclusion $\pi_i^+(\kappa,\xi) \in \mathcal{V}_i^+(\kappa,\xi)$ for all $\xi \in X_\kappa$, it follows from Lemma 4.3.3 that $\varphi(\cdot;\kappa,\xi) - \varphi(\cdot;\kappa,\pi_i^+(\kappa,\xi)) = \psi_\kappa(Q_1^i(\kappa)\pi_i^+(\kappa,\xi),\xi)$ and Lemma 4.3.4 together with (4.3f) implies

$$\left\|\varphi(k;\kappa,\xi) - \varphi(k;\kappa,\pi_i^+(\kappa,\xi))\right\|_{\kappa,c}^+ \leq \frac{K_i^+}{1-\ell_i(c)}\left(\|q_\kappa(\xi)\| + \left\|Q_1^i(\kappa)\xi\right\|\right).$$

Thanks to (4.3v) this gives us (4.3r).

(a_1) Proposition 4.3.5(a_1) yields the continuity of $v_i^+(\kappa,\cdot) : \mathcal{Q}_1^i(\kappa) \times X_\kappa \to \mathcal{P}_1^i(\kappa)$ and accordingly Corollary B.1.4(b) implies that also $\pi_i^+(\kappa,\cdot) : X_\kappa \to X_\kappa$ is continuous. It remains to estimate the norm of $\pi_i^+(\kappa,\xi)$. From (4.3u) we get

$$\begin{aligned}\|p_\kappa(\xi)\| &\overset{(4.3o)}{\leq} \left\|Q_1^i(\kappa)\xi\right\| + \tilde{\ell}_i^+(c)\left\|q_\kappa(\xi) - Q_1^i(\kappa)\xi\right\| \\ &\overset{(4.3v)}{\leq} \left(1 + \tilde{\ell}_i^+(c)\right)\left\|Q_1^i(\kappa)\xi\right\| + \tilde{\ell}_i^+(c)\frac{\tilde{C}_\kappa^+(\xi,c)}{1-\tilde{\ell}_i(c)}\end{aligned}$$

and together with (4.3v) this yields the claimed inequality.

(a_2) The (forward) invariance of $\mathcal{W}_i^-$ and $\mathcal{V}_i^+(\kappa,\xi)$ implies for all $k \in \mathbb{Z}_\kappa^+$,

$$\begin{aligned}\varphi(k;\kappa,\pi_i^+(\kappa,\xi)) &\overset{(4.3s)}{\in} \varphi(k;\kappa,\mathcal{W}_i^-(\kappa) \cap \mathcal{V}_i^+(\kappa,\xi)) \\ &\subseteq \varphi(k;\kappa,\mathcal{W}_i^-(\kappa)) \cap \varphi(k;\kappa,\mathcal{V}_i^+(\kappa,\xi)) \\ &\subseteq \mathcal{W}_i^-(k) \cap \mathcal{V}_i^+(k,\varphi(k;\kappa,\xi)) \overset{(4.3s)}{=} \left\{\pi_i^+(k,\varphi(k;\kappa,\xi))\right\}.\end{aligned}$$

(b) In order to construct the asymptotic backward phase π_i^- one proceeds analogously as in (a). For this, one again uses Corollary B.1.4 in order to obtain unique functions $p_\kappa : X_\kappa \to \mathcal{P}_1^i(\kappa)$, $q_\kappa : X_\kappa \to \mathcal{Q}_1^i(\kappa)$ such that

$$p_\kappa(\xi) \equiv w_i^+(\kappa,q_\kappa(\xi)) \quad \text{and} \quad q_\kappa(\xi) \equiv v_i^-(\kappa,p_\kappa(\xi),\xi) \quad \text{on } X_\kappa,$$

where the mappings w_i^+ and v_i^- had been constructed in Theorem 4.2.9(a) resp. in Proposition 4.3.5(b). Then $\pi_i^-(\kappa,\xi) := p_\kappa(\xi) + q_\kappa(\xi)$ fulfills the above assertions. □

As consequence of Proposition 4.3.5 and Theorem 4.3.7 we obtain that for arbitrary pairs $(\kappa,\xi) \in \mathcal{W}_i^\mp$ the fibers $\mathcal{V}_i^\pm(\kappa,\xi)$ are mutually disjoint. In conclusion, the nonautonomous sets $\mathcal{V}_i^\pm(\xi)$ form a foliation of the extended state space $\mathcal{X}$.

Corollary 4.3.9 (invariant foliation over $\mathcal{W}_i^\pm$). *The nonautonomous sets $\mathcal{V}_i^\pm(\xi)$ from Proposition 4.3.5 are leaves of a* forward invariant foliation *over each fiber of the bundle $\mathcal{W}_i^\mp$ from Theorem 4.2.9, i.e., for $\kappa \in \mathbb{Z}$ and $\xi_1,\xi_2 \in \mathcal{W}_i^\mp(\kappa)$, $\xi_1 \neq \xi_2$ we have*

$$X_\kappa = \bigcup_{\xi \in \mathcal{W}_i^\mp(\kappa)} \mathcal{V}_i^\pm(\kappa,\xi), \qquad \mathcal{V}_i^\pm(\kappa,\xi_1) \cap \mathcal{V}_i^\pm(\kappa,\xi_2) = \emptyset. \tag{4.3w}$$

Remark 4.3.10. The fibers $\mathcal{V}_i^+(\kappa,\xi)$, $\xi \in \mathcal{W}_i^-(\kappa)$, constitute the *pseudo-stable foliation* over the invariant fiber bundle $\mathcal{W}_i^-$ of (S). A dual result holds in the sense that $\mathcal{V}_i^-(\kappa,\xi)$, $\xi \in \mathcal{W}_i^+(\kappa)$, is the *pseudo-unstable foliation* over the fiber bundle $\mathcal{W}_i^+$.

Proof. Let $(\kappa,\xi) \in \mathcal{X}$ be given. The forward invariance of $\mathcal{V}_i^\pm(\kappa,\xi)$ has been established in Proposition 4.3.5. By relation (4.3r) we get $\varphi(\cdot;\kappa,\xi) - \varphi(\cdot;\kappa,\pi_i^\pm(\kappa,\xi)) \in \mathcal{X}_{\kappa,c}^\pm$ and thus Proposition 4.3.5 implies $\xi \in \mathcal{V}_i^\pm(\kappa,\pi_i^\pm(\kappa,\xi))$. Since $\xi \in X_\kappa$ was arbitrary, we established the left relation in (4.3w). The pair-wise disjointness in (4.3w) follows from $\emptyset = \{\xi_1\} \cap \{\xi_2\} = \mathcal{V}_i^\pm(\kappa,\xi_1) \cap \mathcal{V}_i^\pm(\kappa,\xi_2)$ for all $\xi_1,\xi_2 \in \mathcal{W}_i^\mp(\kappa)$ with $\xi_1 \neq \xi_2$. □

In case $a_i + \varsigma \ll 1$, the asymptotic forward phase π_κ^+ from Theorem 4.3.7(a) implies forward convergence of every solution to the nonautonomous set $\mathcal{W}_i^-$, but it does not instantly imply the convergence to a specific fiber $\mathcal{W}_i^-(k)$. In order to achieve this, one needs to start "progressively earlier" leading to the concept of attraction discussed in Chap. 1. Under an additional assumption we can prove such an attraction property of the invariant fiber bundle $\mathcal{W}_i^-$.

Corollary 4.3.11 (attraction). *Assume $a_i + \varsigma \ll 1$. If the sequence $(\Gamma_\kappa^-(i))_{\kappa\in\mathbb{Z}}$ from (Γ_i^-) is backward tempered, then the invariant fiber bundle $\mathcal{W}_i^-$ from Theorem 4.2.9(b) is exponentially $\hat{\mathcal{B}}$*-attracting, *i.e., for all $\mathcal{B} \subseteq \mathcal{X}$ one has exponential convergence*

$$\lim_{n\to\infty} h_{X_k}(\varphi(k;k-n,\mathcal{B}(k-n)),\mathcal{W}_i^-(k)) = 0 \quad \text{for all } k \in \mathbb{Z},$$

with an attraction universe $\hat{\mathcal{B}}$ consisting of uniformly bounded subsets of $\mathcal{X}$.

Proof. By assumption there exists a real $\gamma \in (0,1)$ with $\gamma \in \bar{\Gamma}_i$. Let $\mathcal{B} \subseteq \mathcal{X}$ be uniformly bounded and w.l.o.g. we can assume $\mathcal{B} \subseteq \mathcal{B}_R$ for some $R > 0$. For an arbitrary $k \in \mathbb{Z}$ we choose a sequence $\xi_n \in \mathcal{B}(k-n)$, $n \in \mathbb{N}$. The dichotomy estimates (3.4g) imply that the sequences $(\|P_1^i(k-n)\xi_n\|)_{n\in\mathbb{N}}$, $(\|Q_1^i(k-n)\xi_n\|)_{n\in\mathbb{N}}$ are bounded by K_i^+R resp. K_i^-R. Hence, they are backward tempered. Moreover, the assumption on $(\Gamma_\kappa^-(i))_{\kappa\in\mathbb{Z}}$ ensures that the sequence $(\tilde{C}_{k-n}^+(\xi_n,\gamma))_{n\geq 0}$, where $\tilde{C}_\kappa^+(\xi,c)$ is given in Theorem 4.3.7, is backward tempered uniformly in $\xi_n \in B_R(0)$. Thus, if we choose $\epsilon \in \left(1,\frac{1}{\gamma}\right)$ there exists an integer $K = K(\gamma,\epsilon,R)$ such that

$$\frac{\tilde{C}_\kappa^+(\xi,\gamma)}{1-\tilde{\ell}_i(\gamma)} \leq \epsilon^{-\kappa} \quad \text{for all } \kappa \leq K,\ \xi \in B_R(0). \tag{4.3x}$$

For each $\xi_n \in \mathcal{B}(k-n)$ the invariance of $\mathcal{W}_i^-$ and (4.3r) imply

$$\begin{aligned}
\operatorname{dist}(\varphi(k;k-n,\xi_n),\mathcal{W}_i^-(k)) &= \operatorname{dist}\big(\varphi(k;k-n,\xi_n),\varphi(k;k-n,\mathcal{W}_i^-(k-n))\big) \\
&\leq \big\|\varphi(k;k-n,\xi_n) - \varphi(k;k-n,\pi_i^+(k-n,\xi))\big\| \\
&\overset{(4.3r)}{\leq} \frac{K_i^+}{1-\ell_i(\gamma)}\left(K_i^+R + \frac{\tilde{C}_{k-n}^+(\xi_n,c)}{1-\tilde{\ell}_i(\gamma)}\right)\gamma^n
\end{aligned}$$

for all $n \in \mathbb{Z}_0^+$, and together with (4.3x) this guarantees

$$\operatorname{dist}(\varphi(k; k-n, \xi_n), \mathcal{W}_i^-(k)) \leq \frac{K_i^+}{1-\ell_i(\gamma)} \left(K_i^+ R\gamma^n + \varepsilon^{-k}(\epsilon\gamma)^n\right)$$

for all $\xi_n \in \mathcal{B}(k-n)$ and $n \geq k-K$. Since the right-hand side of this estimate does not depend on ξ_n we get

$$\operatorname{dist}(\varphi(k; k-n, \xi_n), \mathcal{W}_i^-(k)) \xrightarrow[n\to\infty]{} 0 \quad \text{for all } k \in \mathbb{Z},$$

where the choice of γ implies convergence at an exponential rate. □

Corollary 4.3.12. *Let $p \in \mathbb{N}$.*

(a) If (S) *is p-periodic, then one has $\pi_i^\pm(\kappa+p,\xi) = \pi_i^\pm(\kappa,\xi)$ for all $(\kappa,\xi) \in \mathcal{X}$, i.e., the mappings π_i^+, π_i^- are also p-periodic in their first argument.*
(b) If (S) *is autonomous, then the mappings π_i^+, π_i^- do not depend on their first argument.*

Proof. Let $(\kappa,\xi) \in \mathcal{X}$ and choose a growth rate $c \in \bar{\Gamma}_i$. By construction, the solution $\phi : \mathbb{Z}_\kappa^+ \to \mathfrak{X}$, $\phi(k) := \varphi(k;\kappa,\pi_i^+(\kappa,\xi))$ fulfills $\phi - \varphi(\cdot;\kappa,\xi) \in \mathfrak{X}_{\kappa,c}^+$. Since our equation (S) is p-periodic, also the shifted sequence $\psi := \phi(\cdot - p) : \mathbb{Z}_{\kappa+p}^+ \to \mathfrak{X}$ solves (S) and the difference $\psi - \varphi(\cdot - p;\kappa,\xi)$ is c^+-bounded. The p-periodicity of (S) implies $\varphi(k-p;\kappa,\xi) = \varphi(k;\kappa+p,\xi)$ for all $k \in \mathbb{Z}_{\kappa+p}^+$ (cf. Proposition 2.5.3) and

$$\mathcal{W}_i^-(\kappa+p) \cap \mathcal{V}_i^+(\kappa+p,\xi) = \{\psi(\kappa+p)\}\,.$$

This yields $\pi_i^+(\kappa+p,\xi) = \psi(\kappa+p) = \phi(\kappa) = \pi_i^+(\kappa,\xi)$ and thus the sequence $\pi_i^+(\cdot,\xi)$ is p-periodic. The claim for the asymptotic backward phase π_i^- can be shown similarly. Finally, assertion (b) is an immediate consequence of (a). □

4.4 Smoothness of Fiber Bundles and Foliations

At first glance it seems to be a straight forward task to derive continuous differentiability of the invariant fiber bundles $\mathcal{W}_i^\pm$ constructed in the Hadamard–Perron Theorem 4.2.9, provided the nonlinearities are sufficiently smooth. One is tempted to apply the uniform C^m-contraction principle from Theorem B.1.5 to the fixed point equation (4.2f) with $\xi \in X_\kappa$ as parameter. In fact, this approach is successful for the fiber bundles associated to a hyperbolic splitting of the linear part (L_0) – and also for weakly nonhyperbolic situations (cf. [26]).

Yet, for arbitrary splittings the situation is different, since exponentially bounded sequences need not to be bounded in the classical way. As a result, substitution operators on spaces of such sequences need not to be differentiable (see [26, Examples 4.7 and 4.9]) and thus Theorem B.1.5 is unfortunately not applicable. This necessitates another more flexible proof strategy and various other techniques have been developed, where we refer to Sect. 4.10 for a survey.

We carefully try to give a clear and accessible "ad hoc" proof for the maximal smoothness class of invariant fiber bundles. Moreover, we give an example which shows that our necessary gap conditions are sharp. Instead of applying Theorem B.1.5 directly, we formally differentiate the fixed point equation (4.2f), obtain a new fixed point relation and show that its unique solutions are the desired derivatives of $w_i^{\pm}$. This approach needs no technical tools beyond the contraction mapping principle and Lebesgue's theorem. The C^m-smoothness of invariant fiber bundles is proved by induction over m. The induction over the smoothness class m is the key for understanding the structure of the problem. Our focus is not to hide the core of the proof by omitting the technical induction argument as it is frequently done in the literature. To our understanding this is one of the reasons why the "Hadamard–Perron-Theorem" has been reproven by so many authors for similar situations over the years. The induction argument of the proof is crucial because it is needed to rigorously compute the higher order derivatives of compositions of maps, the so-called "derivative tree". It turned out to be advantageous to use two different representations of the derivative tree: First, a "totally unfolded derivative tree" to show that a fixed point operator is well-defined and to compute explicit global bounds for the higher order derivatives of the fiber bundles. Second, a "partially unfolded derivative tree" to elaborate the induction argument in a recursive way.

After this foretaste for the things to come, we again deal with semilinear equations

$$B_{k+1}x' = A_k x + f_k(x, x') \tag{S}$$

and begin our analysis with a technical lemma valid in the setting from Hypothesis 4.2.3 of only Lipschitzian nonlinearities.

Lemma 4.4.1. *Let $\kappa \in \mathbb{I}$, $\phi, \bar{\phi} : \mathbb{Z}_\kappa^{\pm} \to \mathcal{X}$ be solutions to* (S) *and suppose Hypothesis 4.2.1 and 4.2.3 hold true. If $c \in (a_i, b_i)$ and $\ell_i(c) < 1$ for one $1 \le i < N$, then:*

(a) For $\mathbb{I}$ unbounded above and $\phi - \bar{\phi} \in \mathcal{X}_{\kappa,c}^{+}$ one has

$$\left|\phi(k) - \bar{\phi}(k)\right|_i \le \frac{K_i^+}{1 - \ell_i(c)} \left|\phi(\kappa) - \bar{\phi}(\kappa)\right|_i e_c(k, \kappa) \quad \text{for all } k \in \mathbb{Z}_\kappa^+.$$

(b) For $\mathbb{I}$ unbounded below and $\phi - \bar{\phi} \in \mathcal{X}_{\kappa,c}^{-}$ one has

$$\left|\phi(k) - \bar{\phi}(k)\right|_i \le \frac{K_i^-}{1 - \ell_i(c)} \left|\phi(\kappa) - \bar{\phi}(\kappa)\right|_i e_c(k, \kappa) \quad \text{for all } k \in \mathbb{Z}_\kappa^-,$$

where the constant $\ell_i(c)$ is given in Lemma 4.2.6.

Proof. Using Theorem 3.5.3(a) the present assertion (a) can be shown similarly to the following proof of (b). Above all, we remark that $\phi - \bar{\phi} \in \mathcal{X}_{\kappa,c}^-$ solves the linear inhomogeneous equation $B_{k+1}x' = A_k x + g_k$ with $g_k := f_k(\underline{\phi(k)}) - f_k(\underline{\bar{\phi}(k)})$. By the Lipschitz conditions (4.2a) one gets $\left\|B_{k+1}^{-1}g_k\right\| \leq L(c)\left\|\phi - \bar{\phi}\right\|_{\kappa,c}^- e_c(k,\kappa)$ for all $k \in \mathbb{Z}_\kappa^-$ and therefore the inclusion $g \in \mathcal{X}_{\kappa,c,B}^-$ with $\|g\|_{\kappa,c,B}^- \leq L(c)\left\|\phi - \bar{\phi}\right\|_{\kappa,c}^-$. Consequently, we can apply Theorem 3.5.3(b) in order to infer the estimate

$$\left\|\phi - \bar{\phi}\right\|_{\kappa,c}^- \leq K_i^- \left|\phi(\kappa) - \bar{\phi}(\kappa)\right|_i + C_i(c)L(c)\left\|\phi - \bar{\phi}\right\|_{\kappa,c}^-,$$

where we made use of Remark 3.5.9(3). Thus, our assumption $C_i(c)L(c) = \ell_i(c) < 1$ (cf. (4.2i)) finally implies the claimed inequality. □

From now on we strengthen Hypothesis 4.2.3 by imposing globally bounded Fréchet-differentiable nonlinearities $B_{k+1}^{-1}f_k$.

Hypothesis 4.4.2. *Let $m \in \mathbb{N}$. Suppose that $B_{k+1}^{-1}f_k : X_k \times X_{k+1} \to X_{k+1}$, $k \in \mathbb{I}'$, are of class C^m and for all $1 \leq n \leq m$ one has*

$$\begin{aligned} \delta_i^+(n) &:= \sup_{(k,x,x') \in \mathcal{X} \times \mathcal{X}'} \left\|Q_1^i(k)D^n B_{k+1}^{-1}f_k(x,x')\right\|_{L_n(X_k \times X_{k+1};X_{k+1})} < \infty, \\ \delta_i^-(n) &:= \sup_{(k,x,x') \in \mathcal{X} \times \mathcal{X}'} \left\|P_1^i(k)D^n B_{k+1}^{-1}f_k(x,x')\right\|_{L_n(X_k \times X_{k+1};X_{k+1})} < \infty. \end{aligned} \tag{4.4a}$$

Remark 4.4.3. Under Hypothesis 4.2.3 it follows using Proposition C.1.1 that

$$\sup_{(k,x,x') \in \mathcal{X} \times \mathcal{X}'} \left\|DB_{k+1}^{-1}f_k(x,x')\right\|_{L(X_k \times X_{k+1};X_{k+1})} \leq L_1 + L_2 \tag{4.4b}$$

and the constants $\delta_1^{\pm}(i)$ exist per se. Furthermore, if we suppose Hypothesis 4.1.1 and $B_{k+1}^{-1}A_k \in L(X_k, X_{k+1})$, $k \in \mathbb{I}'$, then Proposition 4.1.3(c) guarantees that the general forward solution of (S) exists with $\varphi(k;\kappa,\cdot) \in C^m(X_\kappa, X_k)$, $\kappa \leq k$. A similar statement holds for the general backward solution under the assumptions of Proposition 4.1.4.

Remark 4.4.4 (spectral gap condition). For an integer $1 \leq i < N$ we define

$$\begin{aligned} \varsigma_i^+(m) &:= \min\left\{\frac{\lfloor b_i - a_i \rfloor}{2}, \left\lfloor a_i \sqrt[m]{\frac{a_i + b_i}{a_i + a_i^m}} - a_i \right\rfloor\right\}, \\ \varsigma_i^-(m) &:= \min\left\{\frac{\lfloor b_i - a_i \rfloor}{2}, \left\lfloor b_i - b_i \sqrt[m]{\frac{a_i + b_i}{b_i + b_i^m}} \right\rfloor\right\} \end{aligned}$$

and strengthen (G_i) to the *spectral gap condition*

$$a_i^m \ll b_i, \qquad \exists \varsigma_i \in \left(0, \varsigma_i^+(m)\right) : (G_i) \text{ holds}, \tag{$G_{i,m}^+$}$$
$$a_i \ll b_i^m, \qquad \exists \varsigma_i \in \left(0, \varsigma_i^-(m)\right) : (G_i) \text{ holds}, \tag{$G_{i,m}^-$}$$

choose a fixed real number $\varsigma \in \left(\max\left\{K_i^-, K_i^+\right\}(L_1 + \lceil b_i - \varsigma_i \rceil L_2), \varsigma_i\right)$ and define intervals $\bar{\Gamma}_i := [a_i + \varsigma, b_i - \varsigma]$. The condition $a_i^m \ll b_i$ guarantees $\varsigma_i^+(m) > 0$, while $a_i \ll b_i^m$ ensures that $\varsigma_i^-(m) > 0$. In addition, one has $\varsigma_i^\pm(1) = \frac{\lfloor b_i - a_i \rfloor}{2}$ and the conditions $(G_{i,m}^+)$ and $(G_{i,m}^-)$ coincide for $m = 1$.

In order not to interrupt our later argument, we insert the elementary

Lemma 4.4.5. *Let $\alpha, \beta, \varsigma > 0$ and $m \in \mathbb{N}$:*

(a) If $\alpha^m < \beta$ and $\varsigma < \alpha \sqrt[m]{\frac{\alpha+\beta}{\alpha+\alpha^m}} - \alpha$, then $(\alpha+\varsigma)^m < \beta - \varsigma$.
(b) If $\alpha < \beta^m$ and $\varsigma < \beta - \sqrt[m]{\frac{\alpha+\beta}{\beta+\beta^m}}$, then $\alpha + \varsigma < (\beta - \varsigma)^m$.

Proof. The assumption $\alpha^m < \beta$ implies $0 < \alpha \sqrt[m]{\frac{\alpha+\beta}{\alpha+\alpha^m}} - \alpha$, which in turn is equivalent to $\alpha + \varsigma < (\beta^m - \beta)\left(1 - \frac{\varsigma}{\beta}\right)^m$ and consequently

$$\begin{aligned} \alpha + \beta &< (\beta^m + \beta)\left(1 - \frac{\varsigma}{\beta}\right)^m \leq \beta^m \left(1 - \frac{\varsigma}{\beta}\right)^m + \beta\left(1 - \frac{\varsigma}{\beta}\right) \\ &= (\beta - \varsigma)^m + \beta - \varsigma, \end{aligned}$$

i.e., $\alpha + \varsigma < (\beta - \varsigma)^m$. The assertion (b) can be shown along the same lines. □

Theorem 4.4.6 (smoothness of invariant fiber bundles). *Assume Hypotheses 4.2.1, 4.2.3 and 4.4.2 are satisfied and choose $1 \leq i < N$, $c \in \bar{\Gamma}_i$.*

(a) If $\mathbb{I}$ is unbounded above and (Γ_i^+), $(G_{i,m}^+)$ hold, then the map $w_i^+(\kappa, \cdot) : X_\kappa \to \mathcal{P}_1^i(\kappa)$ from Theorem 4.2.9(a) is of class C^m with globally bounded derivatives

$$\sup_{(\kappa,\xi)\in\mathcal{X}} \left\| D_2^n w_i^+(\kappa, \xi) \right\|_{L_n(X_\kappa)} \leq C_n \quad \textit{for all } 1 \leq n \leq m,$$

where in particular $C_1 := \tilde{\ell}_i^+(c)$.
(b) If $\mathbb{I}$ is unbounded below and (Γ_i^-), $(G_{i,m}^-)$ hold, then the map $w_i^-(\kappa, \cdot) : X_\kappa \to \mathcal{Q}_1^i(\kappa)$ from Theorem 4.2.9(b) is of class C^m with globally bounded derivatives

$$\sup_{(\kappa,\xi)\in\mathcal{X}} \left\| D_2^n w_i^-(\kappa, \xi) \right\|_{L_n(X_\kappa)} \leq C_n \quad \textit{for all } 1 \leq n \leq m,$$

where in particular $C_1 := \tilde{\ell}_i^-(c)$.

(c) The global bounds $C_2, \dots, C_m \geq 0$ *can be determined recursively using*

$$C_n := \max\left\{ \frac{K_i^-}{\varsigma(1-\ell_i(c))} \sum_{j=2}^n \delta_i^-(j) \sum_{(N_1,\dots,N_j)\in P_j^<(n)} \max\{1, \lceil b_i - \varsigma\rceil\}^j \prod_{\nu=1}^j C_{\#N_\nu}, \right.$$
$$\left. \frac{K_i^+}{\varsigma(1-\ell_i(c))} \sum_{j=2}^n \delta_i^+(j) \sum_{(N_1,\dots,N_j)\in P_j^<(n)} \max\{1, \lceil b_i - \varsigma\rceil\}^j \prod_{\nu=1}^j C_{\#N_\nu} \right\} \tag{4.4c}$$

for all $2 \leq n \leq m$,

where the constants $\ell_i(c) \in [0,1)$ *are defined in Lemma 4.2.6 and* $\tilde{\ell}_i^{\pm}(c)$ *is given in Theorem 4.2.9.*

Remark 4.4.7. In case $a_i \leq 1$ or $1 \leq b_i$ it makes sense to investigate the behavior of the conditions $(G_{i,m}^+)$ resp. $(G_{i,m}^-)$ for arbitrarily large values of m. Having constant rates $a_i(k) \equiv \alpha_i$, $b_i(k) \equiv \beta_i$ on $\mathbb{I}$, the asymptotic behavior of the sequences $\varsigma_i^{\pm}(m)$ is as follows:

$$\lim_{m\to\infty} \varsigma_i^+(m) = 0, \text{ if } \alpha_i \leq 1, \qquad \lim_{m\to\infty} \varsigma_i^-(m) = \beta_i - 1, \text{ if } \beta_i \geq 1.$$

Hence, the spectral gap condition $(G_{i,m}^+)$ for $\mathcal{W}_i^+$ becomes increasingly restrictive for growing $m \in \mathbb{N}$.

Proof. Let $(\kappa, \xi) \in \mathcal{X}$ and $c \in \bar{\Gamma}_i$ for a fixed $1 \leq i < N$.

Above all, we remark that the assumptions of Theorem 4.2.9 are fulfilled and we use the brief notation introduced in (4.2p). So there exist invariant fiber bundles $\mathcal{W}_\pm$ which are graphs of globally Lipschitzian mappings $w^\pm : \mathcal{X} \to \mathfrak{X}$ over the vector bundles $\mathcal{P}_\pm$. These mappings are given by the relation

$$w^\pm(\kappa,\xi) = P_\mp(\kappa)\phi_\kappa^\pm(\kappa,\xi), \tag{4.4d}$$

where $\phi_\kappa^\pm(\xi) \in \mathfrak{X}_{\kappa,c}^\pm$ is the unique fixed point of the Lyapunov–Perron operators T_κ^+ and T_κ^- defined in (4.2q) resp. (4.2b). Here, thanks to Lemma 4.2.6 the operators $T_\kappa^\pm : \mathfrak{X}_{\kappa,c}^\pm \times X_\kappa \to \mathfrak{X}_{\kappa,c}^\pm$ are uniform contractions in their first argument with

$$\operatorname{lip}_1 T_\kappa^\pm \overset{(4.2e)}{\leq} \ell_i(c) \overset{(4.2i)}{<} 1. \tag{4.4e}$$

(a) Since the arguments for the operator T_κ^+ are analogous, we only sketch the higher order smoothness case. We formally differentiate the fixed point identity for the operator T_κ^+ defined in (4.2q) w.r.t. $\xi \in X_\kappa$ and obtain another fixed point equation $\phi_\kappa^l(\xi) = T_\kappa^{l,+}(\phi_\kappa^l(\xi);\xi)$ with the right-hand side

$$T_\kappa^{l,+}(\phi^l;\xi) := \sum_{n=\kappa}^\infty G_i(\cdot, n+1)\left[D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi^l(n,\xi)} + R_n^l(\xi)\right] \tag{4.4f}$$

for all $l \in \{2, \dots, m\}$. The remainder R_n^l allows representations analogous to (4.4p) and (4.4q) below. We refer to the following for further details.

(b) Our induction argument is involved and subdivided into two main steps. First, we address the case of continuous differentiability in step (I) and show the general C^m-situation on the foundation of induction over $l \in \{1, \dots, m\}$ in step (II).

(I) We conveniently abbreviate $\hat{f}_k := B_{k+1}^{-1} f_k$, $k \in \mathbb{I}'$. By formal differentiation of the fixed point equation (cf. (4.2f) and (4.2b))

$$\phi_\kappa(k, \xi) = \Phi_{P_-}^-(k, \kappa) P_-(\kappa)\xi + \sum_{n=-\infty}^{\kappa-1} G_i(k, n+1) \hat{f}_n(\underline{\phi_\kappa(n, \xi)})$$

for all $k \in \mathbb{Z}_\kappa^-$, w.r.t. the parameter $\xi \in X_\kappa$ we obtain another fixed point equation

$$\phi_\kappa^1(\xi) = T_\kappa^{1,-}(\phi_\kappa^1(\xi); \xi) \quad \text{for all } \xi \in X_\kappa \tag{4.4g}$$

for the formal derivative ϕ_κ^1 of the fixed point mapping $\phi_\kappa : X_\kappa \to \mathfrak{X}_{\kappa,c}^-$ from Lemma 4.2.8, where the right-hand side of (4.4g) is given by

$$T_\kappa^{1,-}(\phi^1; \xi) := \Phi_{P_-}^-(\cdot, \kappa) P_-(\kappa) + \sum_{n=-\infty}^{\kappa-1} G_i(\cdot, n+1) D\hat{f}_n(\underline{\phi_\kappa(n, \xi)})\overline{\phi^1(n)}.$$

Here, the sequence $\phi^1(k)$, $k \in \mathbb{Z}_\kappa^-$, has values in $L(X_\kappa, X_k)$ and in the following we investigate this operator $T_\kappa^{1,-}$.

(I_1) Claim: *For every $c \in \bar{\Gamma}_i$ the operator $T_\kappa^{1,-} : \mathfrak{X}_{\kappa,c}^{1,-} \times X_\kappa \to \mathfrak{X}_{\kappa,c}^{1,-}$ is well-defined and satisfies the estimate*

$$\left\| T_\kappa^{1,-}(\phi^1; \xi) \right\|_{\kappa,c}^- \leq K_i^- + \ell_i(c) \left\| \phi^1 \right\|_{\kappa,c}^- \quad \textit{for all } \phi^1 \in \mathfrak{X}_{\kappa,c}^{1,-}, \xi \in X_\kappa. \tag{4.4h}$$

Choose $\phi^1 \in \mathfrak{X}_{\kappa,c}^{1,-}$. Using Lemma A.1.5 we argue as in the proof of Theorem 3.5.3 in order to show the two inequalities

$$\left\| P_-(k) T_\kappa^{1,-}(k, \phi^1; \xi) \right\| \leq K_i^- e_{b_i}(k, \kappa) + \frac{K_i^- L(c)}{\lfloor b_i - c \rfloor} \left\| \phi^1 \right\|_{\kappa,c}^-,$$

$$\left\| P_+(k) T_\kappa^{1,-}(k, \phi^1; \xi) \right\| \leq \frac{K_i^+ L(c)}{\lfloor c - a_i \rfloor} \left\| \phi^1 \right\|_{\kappa,c}^- \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

and with the norm from Lemma 3.3.22 this yields

$$\left| T_\kappa^{1,-}(k, \phi^1; \xi) \right|_i e_c(\kappa, k) \leq K_i^- + \ell_i(c) \left\| \phi^1 \right\|_{\kappa,c}^- \quad \text{for all } k \in \mathbb{Z}_\kappa^-.$$

Consequently, we have the inclusion $T_\kappa^{1,-}(\phi^1; \xi) \in \mathfrak{X}_{\kappa,c}^{1,-}$ and passing over to the least upper bound over $k \in \mathbb{Z}_\kappa^-$ implies the linear bound (4.4h).

(I$_2$) Claim: *For every $c \in \bar{\Gamma}_i$ the operator $T_\kappa^{1,-}(\cdot;\xi) : \mathcal{X}_{\kappa,c}^{1,-} \to \mathcal{X}_{\kappa,c}^{1,-}$ is a uniform contraction in $\xi \in X_\kappa$; moreover, the unique fixed point $\phi_\kappa^1(\xi) \in \mathcal{X}_{\kappa,c}^{1,-}$ does not depend on $c \in \bar{\Gamma}_i$ and satisfies*

$$\left\|\phi^1(\xi)\right\|_{\kappa,c}^- \le \frac{K_i^-}{1-\ell_i(c)} \quad \textit{for all } \xi \in X_\kappa. \tag{4.4i}$$

Analogous to the estimate deduced in step (I$_1$) we obtain using Lemma A.1.5,

$$\left\|T_\kappa^{1,-}(\phi^1;\xi) - T_\kappa^{1,-}(\bar{\phi}^1;\xi)\right\|_{\kappa,c}^- \le \ell_i(c) \left\|\phi^1 - \bar{\phi}^1\right\|_{\kappa,c}^- \quad \text{for all } \phi^1, \bar{\phi}^1 \in \mathcal{X}_{\kappa,c}^{1,-}.$$

Taking the estimate (4.4e) into account, Banach's fixed point theorem (cf. [295, p. 361, Lemma 1.1]) guarantees the unique existence of a fixed point $\phi_\kappa^1(\xi) \in \mathcal{X}_{\kappa,c}^{1,-}$ for $T_\kappa^{1,-}(\cdot;\xi) : \mathcal{X}_{\kappa,c}^{1,-} \to \mathcal{X}_{\kappa,c}^{1,-}$. This fixed point $\phi_\kappa^1(\xi)$ is independent of the growth rate $c \in \bar{\Gamma}_i$ since with Lemmata 3.3.26 and 3.3.27 we have $\mathcal{X}_{\kappa,b_i-\varsigma}^{1,-} \subseteq \mathcal{X}_{\kappa,c}^{1,-}$ and thus every mapping $T_\kappa^{1,-}(\cdot;\xi) : \mathcal{X}_{\kappa,c}^{1,-} \to \mathcal{X}_{\kappa,c}^{1,-}$ has the same fixed point as the restriction $T_\kappa^{1,-}(\cdot;\xi)|_{\mathcal{X}_{\kappa,b_i-\varsigma}^{1,-}}$. Finally, the fixed point property (4.4g) together with (4.4h) imply the global bound for $\phi_\kappa^1(\xi)$.

(I$_3$) Claim: *For every $c \in [a_i + \varsigma, b_i - \varsigma)$ the mapping $\phi_\kappa : X_\kappa \to \mathcal{X}_{\kappa,c}^+$ is differentiable with derivative*

$$D\phi_\kappa = \phi_\kappa^1 : X_\kappa \to \mathcal{X}_{\kappa,c}^{1,-}. \tag{4.4j}$$

In relation (4.4j), as well as in all subsequent considerations, we use the isomorphism between the Banach spaces $\mathcal{X}_{\kappa,c}^{1,-}$ and $L(X_\kappa, \mathcal{X}_{\kappa,c}^-)$ from Lemma 3.3.27 and identify them. To show the differentiability assertion, we derive the quotients

$$\Delta\phi(k,h) := \frac{1}{|h|_i}\left(\phi_\kappa(k,\xi+h) - \phi_\kappa(k,\xi) - \phi_\kappa^1(k,\xi)h\right) \quad \text{for all } \xi \in X_\kappa,$$

$$\Delta\hat{f}_k(x,y;h,\bar{h}) := \frac{\hat{f}_k(x+h,y+\bar{h}) - \hat{f}_k(x,y) - D\hat{f}_k(x,y)\begin{pmatrix} h \\ \bar{h}\end{pmatrix}}{\|(h_1,h_2)\|}$$

for all $k \in \mathbb{I}$, $h, x \in X_\kappa$ and $\bar{h}, y \in X_{\kappa+1}$, where $h, \bar{h} \neq 0$. Thereby, the inclusion $\Delta\phi(\cdot,h) \in \mathcal{X}_{\kappa,c}^-$ holds due to (I$_2$) and Lemma 4.2.8. To prove differentiability, we have to show the limit relation $\lim_{h\to 0}\Delta\phi(\cdot,h) = 0$ in $\mathcal{X}_{\kappa,c}^-$. For this, consider growth rates $c \ll b_i - \varsigma$, $d \in (c, b_i - \varsigma)$ and from Lemma 4.4.1(b) we obtain

$$\frac{1}{|h|_i}\left|\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi)\right|_i \le \frac{K_i^-}{1-\ell_i(c)} e_d(n,\kappa) \quad \text{for all } n \in \mathbb{Z}_\kappa^-. \tag{4.4k}$$

Using the fixed point relation (4.2f) for $\phi_\kappa(\xi)$ and (4.4g) for $\phi_\kappa^1(\xi)$ it results

$$\begin{aligned}\Delta\phi_\kappa(k,h) = \frac{1}{|h|_i}\sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)\\ \cdot\left[\hat f_n(\underline{\phi_\kappa(n,\xi+h)}) - \hat f_n(\underline{\phi_\kappa(n,\xi)}) - D\hat f_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^1(n,\xi)h}\right]\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$, where subtraction and addition of the expression

$$D\hat f_n(\underline{\phi_\kappa(n,\xi)})\left(\overline{\phi_\kappa(n,\xi+h)} - \overline{\phi_\kappa(n,\xi)}\right)$$

in the above parenthesis implies the estimate

$$\begin{aligned}&\|P_-(k)\Delta\phi(k,h)\|\\ &\overset{(3.4g)}{\leq} K_i^-\sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)\left\|\Delta\hat f_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+k)-\phi_\kappa(n,\xi)})\right\|\\ &\quad\cdot\frac{1}{|h|_i}\left\|\overline{\phi_\kappa(n,\xi+h)-\phi_\kappa(n,\xi)}\right\|\\ &\quad + K_i^- L(d)\sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)\,|\Delta\phi(n,h)|_i \quad \text{for all } k\in\mathbb{Z}_\kappa^-\end{aligned}$$

and together with (4.4k) we infer

$$\begin{aligned}\|P_-(k)\Delta\phi_\kappa(k,h)\| \ &\leq K_i^-\frac{K_i^-\max\{1,\lceil d\rceil\}}{1-\ell_i(d)}\sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)\\ &\quad\cdot\left\|\Delta\hat f_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+k)-\phi_\kappa(n,\xi)})\right\|\\ &\quad + K_i^- L(d)\sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)\,|\Delta\phi(n,h)|_i \quad \text{for all } k\in\mathbb{Z}_\kappa^-.\end{aligned}$$

Analogously, also using (4.4k) we can derive a similar estimate

$$\begin{aligned}\|P_+(k)\Delta\phi_\kappa(k,h)\| \ &\leq K_i^+\frac{K_i^-\max\{1,\lceil d\rceil\}}{1-\ell_i(d)}\sum_{n=-\infty}^{k-1} e_{a_i}(k,n+1)\\ &\quad\cdot\left\|\Delta\hat f_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+k)-\phi_\kappa(n,\xi)})\right\|\\ &\quad + K_i^+ L(d)\sum_{n=-\infty}^{k-1} e_{a_i}(k,n+1)\,|\Delta\phi(n,h)|_i \quad \text{for all } k\in\mathbb{Z}_\kappa^-\end{aligned}$$

and thanks to the norm from Lemma 3.3.22 we obtain

$$|\Delta\phi_\kappa(k,h)|_i \leq \max\{S_1+S_2, S_3+S_4\} \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

with the abbreviations

$$\begin{aligned}
S_1 &:= \frac{(K_i^-)^2 \max\{1,\lceil d\rceil\}}{1-\ell_i(d)} \sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)e_d(n,\kappa) \\
&\quad \cdot \left\| \Delta\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h)-\phi_k(n,\xi)}) \right\|, \\
S_2 &:= K_i^- L(d) \sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)\, |\Delta\phi_\kappa(n,k)|_i\,, \\
S_3 &:= \frac{K_i^- K_i^+ \max\{1,\lceil d\rceil\}}{1-\ell_i(d)} \sum_{n=-\infty}^{k-1} e_{a_i}(k,n+1)e_d(n,\kappa) \\
&\quad \cdot \left\| \Delta\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h)-\phi_k(n,\xi)}) \right\|, \\
S_4 &:= K_i^+ L(d) \sum_{n=-\infty}^{k-1} e_{a_i}(k,n+1)\, |\Delta\phi_\kappa(n,k)|_i\,.
\end{aligned}$$

The elementary estimate $\max\{S_1+S_2, S_3+S_4\} \leq S_1+S_3+\max\{S_2,S_4\}$ together with Lemma A.1.5 yields

$$\begin{aligned}
|\Delta\phi_\kappa(k,h)|_i\, e_c(\kappa,k) &\leq (S_1+S_3)e_c(\kappa,k) \\
&\quad + L(d)\max\left\{\frac{K_i^-}{\lfloor b_i-d\rfloor}, \frac{K_i^+}{\lfloor d-a_i\rfloor}\right\} \|\Delta\phi_\kappa(h)\|_{\kappa,c}^- \quad \text{for all } k\in\mathbb{Z}_\kappa^-
\end{aligned}$$

and passing over to the supremum over $k \in \mathbb{Z}_\kappa^-$ ensures (cf. (4.4e))

$$\|\Delta\phi_\kappa(h)\|_{\kappa,c}^- \leq \frac{K_i^- \max\{K_i^-, K_i^+\}\max\{1,\lceil d\rceil\}}{(1-\ell_i(d))^2} \sup_{k\in\mathbb{Z}_\kappa^-} V(k,h)$$

with

$$\begin{aligned}
V(k,h) &:= e_c(\kappa,k) \sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)e_d(n,\kappa) \\
&\qquad \cdot \left\| \Delta\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h)-\phi_k(n,\xi)}) \right\|, \\
&\qquad + e_c(\kappa,k) \sum_{n=-\infty}^{k-1} e_{b_i}(k,n+1)e_d(n,\kappa) \\
&\qquad \cdot \left\| \Delta\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h)-\phi_k(n,\xi)}) \right\| \quad \text{for all } k\in\mathbb{Z}_\kappa^-.
\end{aligned}$$

Thus, in order to prove claim (I_3), we only have to show the limit relation

$$\lim_{h \to 0} \sup_{k \in \mathbb{Z}_\kappa^-} V(k,h) = 0, \tag{4.4l}$$

which will be done indirectly. Supposing (4.4l) is not true, there exists an $\varepsilon > 0$ and a sequence $(h_j)_{j \in \mathbb{N}}$ in X_κ with limit 0 such that $\sup_{k \in \mathbb{Z}_\kappa^-} V(k,h_j) > \varepsilon$ for all $j \in \mathbb{N}$. This, in turn, implies the existence of a further sequence $(k_j)_{j \in \mathbb{N}}$ in $\mathbb{Z}_\kappa^-$ with

$$V(k_j,h_j) > \varepsilon \quad \text{for all } j \in \mathbb{N}. \tag{4.4m}$$

Using the crude estimate $\left\| \Delta \hat{f}_n(x,y,h_1,h_2) \right\| \leq 2(L_1 + L_2)$, which results from (4.2a) and (4.4b), it follows using Lemma A.1.5 that

$$V(k,h) \leq \left(\frac{L_1 + L_2}{\lfloor d - a_i \rfloor} + \frac{L_1 + L_2}{\lfloor b_i - d \rfloor} \right) e_{\frac{d}{c}}(k,\kappa) \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

and due to Lemma A.1.3(b) the right-hand side of this estimate converges to 0 for $k \to \infty$, i.e., we have $\lim_{k \to \infty} V(k,h) = 0$ uniformly in $h \in X_\kappa$. Because of (4.4m) the sequence $(k_j)_{j \in \mathbb{N}}$ has to be bounded in $\mathbb{Z}_\kappa^-$, i.e., there exists an integer $K \leq \kappa$ with $k_j \in [K,\kappa]_{\mathbb{Z}}$ for all $j \in \mathbb{N}$. We consequently obtain from Proposition A.1.2(a) that

$$\begin{aligned} V(k_j,h_j) \leq & \underbrace{e_{\frac{c}{b_i}}(\kappa,k)}_{\leq 1} \sum_{n=K}^{\kappa-1} e_{b_i}(\kappa,n+1) e_d(n,\kappa) \\ & \cdot \left\| \Delta \hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h_j) - \phi_k(n,\xi)}) \right\|, \\ & + e_c(\kappa,K) \sum_{n=-\infty}^{\kappa-1} e_{a_i}(K,n+1) e_d(n,\kappa) \\ & \cdot \left\| \Delta \hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h_j) - \phi_k(n,\xi)}) \right\| \quad \text{for all } j \in \mathbb{N} \end{aligned} \tag{4.4n}$$

and due to the continuity of the fixed point mapping $\phi_\kappa(n,\cdot) : X_\kappa \to X_n$ guaranteed by both Lemmata 4.3.4(b) and 3.3.28,

$$\lim_{j \to \infty} \phi_\kappa(n,\xi + h_j) = \phi_\kappa(n,\xi) \quad \text{for all } n \in \mathbb{Z}_\kappa^-,$$

as well as using the differentiability of $\hat{f}_n$, required in Hypothesis 4.4.2,

$$\lim_{(h,\bar{h}) \to (0,0)} \left\| \Delta \hat{f}_n(x_1,x_2,h,\bar{h}) \right\| = 0,$$

which leads to the limit relation

$$\lim_{j\to\infty}\left\|\Delta\hat{f}_n(\underline{\phi_\kappa(n,\xi)},\underline{\phi_\kappa(n,\xi+h_j)-\phi_\kappa(n,\xi)}\right\| = 0 \quad \text{for all } n\in\mathbb{Z}_\kappa^- .$$

We can conclude that the finite sum in (4.4n) tends to 0 in the limit $j\to\infty$. As in the proof of Lemma 4.3.4(b), Lebesgue's theorem ensures that also the infinite sum in (4.4n) converge to 0 for $j\to\infty$. In conclusion, $\lim_{j\to\infty} V(k_j,h_j)=0$, which contradicts (4.4m). Hence, the claim (I_3) is true, where (4.4j) follows by the uniqueness of Fréchet derivatives.

(I_4) Claim: *For every $c\in[a_i+\varsigma,b_i-\varsigma)$ the mappings $D\phi_\kappa : X_\kappa\to\mathcal{X}_{\kappa,c}^{1,-}$ and $D_2w^-(\kappa,\cdot):X_\kappa\to L(X_\kappa)$ are continuous.*
With a view to (4.4j) it is sufficient to show the continuity of the fixed point mapping $\phi_\kappa^1 : X_\kappa\to\mathcal{X}_{\kappa,c}^{1,-}$. In order to do this, fix $\xi_0\in X_\kappa$ and choose $\xi\in X_\kappa$. Using the fixed point equation (4.4g) for ϕ_κ^1, we can estimate the norm $\left\|\phi_\kappa^1(\xi)-\phi_\kappa^1(\xi_0)\right\|_{\kappa,c}^-$ and as in the continuity proof of Lemma 4.3.4(b) it follows $\lim_{\xi\to\xi_0}\phi_\kappa^1(\xi)=\phi_\kappa^1(\xi_0)$ in $\mathcal{X}_{\kappa,c}^{1,-}$. By the identity (4.4d) and Lemma 3.3.28 also $D_2w^-(\kappa,\cdot)$ is continuous. Hence, we have shown the assertion (b) for $m=1$, where the given bound C_1 is a consequence of Theorem 4.2.9(b_2) interplaying with Proposition C.1.1.

(II) Now let $m\geq 2$. By formal differentiation of the fixed point equation (4.2f) w.r.t. $\xi\in X_\kappa$, using the higher order chain rule from Theorem C.1.3, we obtain another fixed point equation

$$\phi_\kappa^l(\xi) = T_\kappa^{l,-}(\phi_\kappa^l(\xi);\xi) \tag{4.4o}$$

for the formal derivative ϕ_κ^l of $\phi_\kappa : X_\kappa\to\mathcal{X}_{\kappa,c}^+$ of order $l\in\{2,\dots,m\}$, where the right-hand side of (4.4o) is given by

$$T_\kappa^{l,-}(\phi^l;\xi) := \sum_{n=-\infty}^{\kappa-1} G_i(\cdot,n+1)\left[D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi^l(n,\xi)}+R_n^l(\xi)\right].$$

Here, $\phi^l(k)\in L_l(X_\kappa,X_k)$, $k\in\mathbb{Z}_\kappa^-$, and the remainder R_n^l has the representations:

- As partially unfolded derivative tree

$$R_n^l(\xi) \stackrel{\text{(C.1a)}}{=} \sum_{j=1}^{l-1}\binom{l-1}{j}\frac{d^j}{d\xi^j}D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi^l(n,\xi)}, \tag{4.4p}$$

  which is appropriate for the induction in the subsequent step,
- and as totally unfolded derivative tree

$$R_n^l(\xi) \stackrel{\text{(C.1b)}}{=} \sum_{j=2}^{l}\sum_{(N_1,\dots,N_j)\in P_j^<(l)} D^j\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi^{\#N_1}(n,\xi)}\cdots\overline{\phi^{\#N_j}(n,\xi)}, \tag{4.4q}$$

  which enables us to get explicit global bounds for higher order derivatives.

For our forthcoming considerations it is crucial that R_n^l does not depend on ϕ_κ^l. In the next steps, we will solve the fixed point equation (4.4o) for the operator $T_\kappa^{l,-}$. As preparation, for every $l \in \{1, \ldots, m\}$ we introduce the growth rates

$$c_l(k) := \begin{cases} b_i(k) - \varsigma & \text{if } b_i(k) - \varsigma \geq 1, \\ (b_i(k) - \varsigma)^l & \text{if } b_i(k) - \varsigma < 1 \end{cases} \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

and $c_1, \ldots, c_l \in (a_i + \varsigma, b_i - \varsigma]$ holds, which in case $b_i(k) - \varsigma \geq 1$ follows from the relation $\varsigma < \frac{\lfloor b_i - a_i \rfloor}{2}$ and otherwise results from $a_i + \varsigma \ll (b_i - \varsigma)^m$ (cf. Lemma 4.4.5(b)). We formulate for $\bar{m} \in \{1, \ldots, m\}$ the induction hypothesis:

$$A(\bar{m}) : \begin{cases} \text{For every } l \in \{1, \ldots, \bar{m}\} \text{ and growth rates } c \in (a_i + \varsigma, c_l] \text{ the operator} \\ T_\kappa^l : \mathcal{X}_{\kappa,c}^{l,-} \times X_\kappa \to \mathcal{X}_{\kappa,c}^{l,-} \text{ satisfies:} \\ \text{(a) It is well-defined.} \\ \text{(b) } T_\kappa^{l,-}(\cdot;\xi) \text{ is a uniform contraction in } \xi \in X_\kappa. \\ \text{(c) The unique fixed point } \phi_\kappa^l(\xi) \text{ of } T_\kappa^{l,-}(\cdot;\xi) \text{ is globally bounded in the} \\ \quad c_l^+\text{-norm } \left\|\phi_\kappa^l(n,\xi)\right\| \leq C_l e_{c_l}(n,\kappa) \text{ for all } n \in \mathbb{Z}_\kappa^-, \xi \in X_\kappa \text{ with the} \\ \quad \text{constants } C_l \geq 0 \text{ given in (4.4c).} \\ \text{(d) If } c \ll c_l \text{ then } \phi_\kappa^{l-1} : X_\kappa \to \mathcal{X}_{\kappa,c}^{l,-} \text{ is continuously differentiable} \\ \quad \text{w.r.t. } \xi \in X_\kappa \text{ and derivative } D\phi_\kappa^{l-1} = \phi_\kappa^l : X_\kappa \to \mathcal{X}_{\kappa,c}^{l,-}. \end{cases}$$

For $\bar{m} = 1$ our step (I) establishes the induction hypothesis $A(1)$ with $C_1 = \frac{K_i^-}{1-\ell_i(c)}$ (cf. (4.4i)). Actually, thanks to Theorem 4.2.9(b_2) one can even choose $C_1 = \tilde{\ell}_i^-(c)$. Now we assume $A(\bar{m}-1)$ holds true for some $\bar{m} \in \{2, \ldots, m\}$ and we are going to prove $A(\bar{m})$ in the following steps:

(II$_1$) Claim: *For every $c \in (a_i+\varsigma, c_{\bar{m}}]$ the operator $T_\kappa^{\bar{m},-} : \mathcal{X}_{\kappa,c}^{\bar{m},-} \times X_\kappa \to \mathcal{X}_{\kappa,c}^{\bar{m},-}$ is well-defined and satisfies the estimate*

$$\left\|T_\kappa^{\bar{m},-}(\phi^{\bar{m}};\xi)\right\|_{\kappa,c}^- \leq \ell_i(c) \left\|\phi^{\bar{m}}\right\|_{\kappa,c}^- + \bar{C}_{\bar{m}}, \tag{4.4r}$$

with the constant

$$\bar{C}_{\bar{m}} := \max\left\{ \frac{K_i^-}{\varsigma} \sum_{j=2}^{\bar{m}} \delta_i^-(j) \sum_{(N_1,\ldots,N_j)\in P_j^<(\bar{m})} \max\{1, \lceil b_i - \varsigma \rceil\}^j \prod_{\nu=1}^j C_{\#N_\nu}, \right.$$
$$\left. \frac{K_i^+}{\varsigma} \sum_{j=2}^{\bar{m}} \delta_i^+(j) \sum_{(N_1,\ldots,N_j)\in P_j^<(\bar{m})} \max\{1, \lceil b_i - \varsigma \rceil\}^j \prod_{\nu=1}^j C_{\#N_\nu} \right\},$$

i.e., the assertion $A(\bar{m})(a)$ holds true.

Let $l \in \{2, \dots, \bar{m}\}$ and choose $c \in (a_i + \varsigma, c_l]$. Using the estimate $c_{\#N_1} \cdot \ldots \cdot c_{\#N_j} \geq c_l$ for any ordered partition $(N_1, \dots, N_l) \in P_j^<(l)$ of length $j \in \{2, \dots, l\}$, from (3.4g), (4.4a), (4.4q) and $A(\bar{m}-1)(c)$, we obtain the inequalities

$$\begin{aligned}
&\left\| P_-(k) \sum_{n=k}^{\kappa-1} \Phi_{P_-}^-(k, n+1) R_n^l(\xi) \right\| \\
&\overset{\text{(A.1d)}}{\leq} \frac{K_i^- e_c(k,\kappa)}{\lfloor b_i - c_l \rfloor} \sum_{j=2}^{l} \delta_i^-(j) \sum_{(N_1,\dots,N_l)\in P_j^<(l)} \prod_{\nu=1}^{j} C_{\#N_\nu} \max\{1, \lceil c_{\#N_\nu} \rceil\}
\end{aligned}$$

using Lemma A.1.5(b) and analogously using Lemma A.1.5(a) one has

$$\begin{aligned}
&\left\| P_+(k) \sum_{n=-\infty}^{k-1} \Phi(k, n+1) R_n^l(\xi) \right\| \\
&\overset{\text{(A.1e)}}{\leq} \frac{K_i^+ e_c(k,\kappa)}{\lfloor c_l - a_i \rfloor} \sum_{j=2}^{l} \delta_i^+(j) \sum_{(N_1,\dots,N_l)\in P_j^<(l)} \prod_{\nu=1}^{j} C_{\#N_\nu} \max\{1, \lceil c_{\#N_\nu} \rceil\}
\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$. Given $\phi^{\bar{m}} \in \mathfrak{X}_{\kappa,c}^{\bar{m},-}$ the above estimates yield

$$\left| T_\kappa^{\bar{m},-}(k, \phi^{\bar{m}}; \xi) \right|_i e_c(\kappa, k) \leq \ell_i(c) \left\| \phi^{\bar{m}} \right\|_{\kappa,c}^- + \bar{C}_{\bar{m}} \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

with the constants $\bar{C}_{\bar{m}} \geq 0$ defined above. As usual, passing over to the supremum for $k \in \mathbb{Z}_\kappa^-$ implies $T_\kappa^{\bar{m},-}(\phi^{\bar{m}}; \xi) \in \mathfrak{X}_{\kappa,c}^{\bar{m},-}$. In particular, the estimate (4.4r) follows due to the inclusion $c \in (a_i + \varsigma, b_i - \varsigma]$.

(II_2) Claim: *For every $c \in (a_i + \varsigma, c_{\bar{m}}]$ the operator $T_\kappa^{\bar{m},-}(\cdot; \xi) : \mathfrak{X}_{\kappa,c}^{\bar{m},-} \to \mathfrak{X}_{\kappa,c}^{\bar{m},-}$ is a uniform contraction in $\xi \in X_\kappa$; moreover, the unique fixed point $\phi_\kappa^{\bar{m}}(\xi) \in \mathfrak{X}_{\kappa,c}^{\bar{m},-}$ does not depend on $c \in (a_i + \varsigma, c_{\bar{m}}]$ and satisfies*

$$\left\| \phi_\kappa^{\bar{m}}(\xi) \right\|_{\kappa,c}^- \leq C_{\bar{m}} \quad \textit{for all } \xi \in X_\kappa, \tag{4.4s}$$

i.e., the assertions $A(\bar{m})(b)$ and $A(\bar{m})(c)$ hold true.

Choose $c \in (a_i + \varsigma, c_{\bar{m}}]$ and let $\phi^{\bar{m}}, \bar{\phi}^{\bar{m}} \in \mathfrak{X}_{\kappa,c}^{\bar{m},-}$. Keeping in mind that the remainder in (4.4p) and (4.4q) does not depend on $\phi^{\bar{m}}, \bar{\phi}^{\bar{m}}$, resp., from (3.4g) and (4.2a) we obtain the Lipschitz estimates

$$\begin{aligned}
&\left| T_\kappa^{\bar{m},-}(k, \phi^{\bar{m}}; \xi) - T_\kappa^{\bar{m},-}(k, \bar{\phi}^{\bar{m}}; \xi) \right|_i e_c(\kappa, k) \\
&\quad \leq L(c) \max\left\{ K_i^- \sum_{n=k}^{\kappa-1} e_{b_i}(k, n+1) e_c(n, \kappa), \right. \\
&\qquad \left. K_i^+ \sum_{n=-\infty}^{k-1} e_{a_i}(k, n+1) e_c(n, \kappa) \right\} \left\| \phi^{\bar{m}} - \bar{\phi}^{\bar{m}} \right\|_{\kappa,c}^- \leq \ell_i(c) \left\| \phi^{\bar{m}} - \bar{\phi}^{\bar{m}} \right\|_{\kappa,c}^-
\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$ using Lemma A.1.5. Passing over to the supremum over $k \in \mathbb{Z}_\kappa^-$ with (4.4e) implies the contraction property for $T_\kappa^{\bar{m},-}(\cdot;\xi)$ and, e.g., [295, p. 361, Lemma 1.1]) implies the existence of a unique fixed point $\phi_\kappa^{\bar{m}}(\xi) \in \mathfrak{X}_{\kappa,c}^{\bar{m},-}$. It can be seen along the same lines as in (I_2) that $\phi_\kappa^{\bar{m}}(\xi)$ does not depend on $c \in (a_i + \varsigma, c_{\bar{m}}]$. The fixed point property (4.4o) with (4.4r) implies the bound (4.4s).

(II_3) Claim: *For every $c \in (a_i + \varsigma, c_{\bar{m}})$ the mapping $\phi_\kappa^{\bar{m}-1} : X_\kappa \to \mathfrak{X}_{\kappa,c}^{\bar{m},-}$ is differentiable with derivative*

$$D\phi_\kappa^{\bar{m}-1} = \phi_\kappa^{\bar{m}} : X_\kappa \to \mathfrak{X}_{\kappa,c}^{\bar{m},-}. \tag{4.4t}$$

Let $c \in (a_i + \varsigma, c_{\bar{m}})$ be fixed. First we show that $\phi_\kappa^{\bar{m}-1}$ is differentiable and then we prove that the derivative is given by $\phi_\kappa^{\bar{m}} : X_\kappa \to L(X_\kappa, \mathfrak{X}_{\kappa,c}^{\bar{m}-1,-}) \cong \mathfrak{X}_{\kappa,c}^{\bar{m},-}$ (cf. Lemma 3.3.27). Thereto choose $\xi \in X_\kappa$ arbitrarily, but fixed. Using the fixed point equation (4.4o) for $\phi_\kappa^{\bar{m}-1}$ we get for $h \in X_\kappa$ the identity

$$\begin{aligned}
&\phi_\kappa^{\bar{m}-1}(k;\xi+h) - \phi_\kappa^{\bar{m}-1}(k;\xi) \\
&\quad = \sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)\left[D\hat{f}_n(\underline{\phi_\kappa(n,\xi+h)})\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)} + R_n^{\bar{m}-1}(\xi+h)\right] \\
&\qquad - \sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)\left[D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^{\bar{m}-1}(n,\xi)} + R_n^{\bar{m}-1}(\xi)\right]
\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$. This leads to

$$\begin{aligned}
&\phi_\kappa^{\bar{m}-1}(k;\xi+h) - \phi_\kappa^{\bar{m}-1}(k;\xi) \\
&\quad - \sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)D\hat{f}_n(\underline{\phi_\kappa(n,\xi+h)})\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h) - \phi_\kappa^{\bar{m}-1}(n,\xi)} \\
&= \sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)\left[D\hat{f}_n(\underline{\phi_\kappa(n,\xi+h)}) - D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\right]\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)} \\
&\quad + \sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)\left[R_n^{\bar{m}-1}(\xi+h) - R_n^{\bar{m}-1}(\xi)\right]
\end{aligned} \tag{4.4u}$$

for all $k \in \mathbb{Z}_\kappa^-$. With sequences $\phi^{\bar{m}-1} \in \mathfrak{X}_{\kappa,c}^{\bar{m}-1,-}$ and $h \in X_\kappa$ we define the operators $H \in L\left(\mathfrak{X}_{\kappa,c}^{\bar{m}-1,-}\right)$, $E \in L\left(X_\kappa, \mathfrak{X}_{\kappa,c}^{\bar{m}-1,-}\right)$, $J : X_\kappa \to \mathfrak{X}_{\kappa,c}^{\bar{m}-1,-}$ as follows

$$\begin{aligned}
H\phi^{\bar{m}-1} &:= \sum_{n=-\infty}^{\kappa-1} G_i(\cdot,n+1)D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi^{\bar{m}-1}(n)}, \\
Eh &:= \sum_{n=-\infty}^{\kappa-1} G_i(\cdot,n+1)R_1^{\bar{m}}(n,\xi)h
\end{aligned}$$

and

$$J(h) := \sum_{n=-\infty}^{\kappa-1} G_i(\cdot, n+1)\Big\{ \Big[D\hat{f}_n(\underline{\phi_\kappa(n,\xi+h)}) - D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\Big] \\ \cdot\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)} + R_n^{\bar{m}-1}(\xi+h) - R_n^{\bar{m}-1}(\xi) - R_n^{\bar{m}}(\xi)h\Big\} \tag{4.4v}$$

for all $k \in \mathbb{Z}_\kappa^-$. In the subsequent lines we will show that H, E and J are well-defined. Using (3.4g) and (4.4a) it is easy to see that $H : \mathcal{X}_{\kappa,c}^{\bar{m}-1,-} \to \mathcal{X}_{\kappa,c}^{\bar{m}-1,-}$ is linear and satisfies $\left\|H\phi^{\bar{m}-1}\right\|_{\kappa,c}^- \leq \ell_i(c)\left\|\phi^{\bar{m}-1}\right\|_{\kappa,c}^-$, which in turn gives us

$$\|H\|_{L(\mathcal{X}_{\kappa,c}^{\bar{m}-1,-})} \overset{(4.4e)}{<} 1. \tag{4.4w}$$

Keeping in mind that $Eh = T_\kappa^{\bar{m}-1,-}(0;\xi)h$, our Step (II$_1$) yields $Eh \in \mathcal{X}_{\kappa,c}^{\bar{m}-1,-}$, while E is obviously linear and continuous, hence $E \in L(X_\kappa, \mathcal{X}_{\kappa,c}^{\bar{m}-1,-})$. Arguments similar to those in Step (II$_1$) lead to the inclusion $J(h) \in \mathcal{X}_{\kappa,c}^{\bar{m}-1,-}$ for any $h \in X_\kappa$. Because of (4.4u) we obtain

$$\left[\phi_\kappa^{\bar{m}-1}(\xi+h) - \phi_\kappa^{\bar{m}-1}(\xi)\right] - H\left[\phi_\kappa^{\bar{m}-1}(\xi+h) - \phi_\kappa^{\bar{m}-1}(\xi)\right] = Eh + J(h)$$

for all $h \in X_\kappa$. Using the Neumann series (cf., e.g., [295, p. 74, Theorem 2.1] or Theorem B.3.1) and (4.4w), the linear mapping $I_{\mathcal{X}_{\kappa,c}^{\bar{m}-1,-}} - H \in L(\mathcal{X}_{\kappa,c}^{\bar{m}-1,-})$ is invertible and this implies

$$\phi_\kappa^{\bar{m}-1}(\xi+h) - \phi_\kappa^{\bar{m}-1}(\xi) = \left[I_{\mathcal{X}_{\kappa,c}^{\bar{m}-1,-}} - H\right]^{-1}[Eh + J(h)] \quad \text{for all } h \in X_\kappa.$$

Thus, it remains to show $\lim_{h\to 0}\frac{J(h)}{\|h\|} = 0$ in $\mathcal{X}_{\kappa,c}^{\bar{m}-1,-}$, because then one gets

$$\lim_{h\to 0}\frac{1}{\|h\|}\left\|\phi_\kappa^{\bar{m}-1}(\xi+h) - \phi_\kappa^{\bar{m}-1}(\xi) - \left[I_{\mathcal{X}_{\kappa,c}^{\bar{m}-1,-}} - H\right]^{-1}Eh\right\|_{\kappa,c}^- = 0,$$

i.e., the claim of the present Step (II$_3$) follows. Nevertheless the proof of the limit relation $\lim_{h\to 0}\frac{\|J(h)\|_{\kappa,c}^-}{\|h\|} = 0$ needs a certain technical effort. Thereto we use the fact that due to the induction hypothesis $A(\bar{m}-1)(d)$ the remainder

$$R_n^{\bar{m}-1}(\xi) \overset{(4.4p)}{=} \sum_{j=1}^{\bar{m}-2}\binom{\bar{m}-2}{j}\frac{\partial^j}{\partial\xi^j}\left[D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\right]\overline{\phi_\kappa^{\bar{m}-1-j}(n,\xi)}$$

is differentiable w.r.t. $\xi \in X_\kappa$, where the derivative is given by the product rule (cf. [295, p. 336]) as

$$DR_n^{\bar{m}-1}(\xi) \overset{(4.4p)}{=} R_n^{\bar{m}}(\xi) - D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^1(n,\xi)}\overline{\phi_\kappa^{\bar{m}-1}(n,\xi)}.$$

Using the abbreviation

$$\Delta R_n^{\bar{m}-1}(\xi,h) := \frac{1}{\|h\|}\Big\{R_n^{\bar{m}-1}(\xi+h) - R_n^{\bar{m}-1}(\xi) - \Big[R_n^{\bar{m}}(\xi) - D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^1(n,\xi)}\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi)}\Big]\,h\Big\}$$

we obtain $\lim_{h\to 0}\Delta R_n^{\bar{m}-1}(\xi,h) = 0$ for $n \in \mathbb{Z}_\kappa^-$. Now we prove estimates for the components J_- and J_+ of J in $\mathcal{P}_-$ resp. $\mathcal{P}_+$, separately. For $k \in \mathbb{Z}_\kappa^-$ we get

$$\begin{aligned} &J_-(k,h) \\ &\overset{(4.4\mathrm{v})}{=} \sum_{n=k}^{\kappa-1}\Phi_{P_-}^-(k,n+1)\Big\{\Big[D\hat{f}_n(\underline{\phi_\kappa(n,\xi+h)}) - D\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\Big]\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)} \\ &\quad - D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^1(n,\xi)}\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi)}h + \Delta R_n^{\bar{m}-1}(\xi,h)\,\|h\|\Big\}, \end{aligned}$$

where subtraction and addition of the expression

$$D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi) - \phi_\kappa^1(n,\xi)h}\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)}$$

leads to

$$\begin{aligned} J_-(k,h) = \sum_{n=k}^{\kappa-1}\Phi_{P_-}^-(k,n+1)\Big\{&\Big[D\hat{f}_n(\underline{\phi_\kappa(n,\xi+h)}) - D\hat{f}_n(\underline{\phi_\kappa(n,\xi)}) \\ &- D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi)}\Big]\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)} \\ &+ D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi) - \phi_\kappa^1(n,\xi)h}\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)} \\ &+ D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^1(n,\xi)}\,\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h) - \phi_\kappa^{\bar{m}-1}(n,\xi)}h \\ &+ \Delta R_n^{\bar{m}-1}(\xi,h)\,\|h\|\Big\} \quad \text{for all } k \in \mathbb{Z}_\kappa^-. \end{aligned}$$

Using the quotient

$$\Delta D\hat{f}_n(x,y,h,\bar{h}) := \frac{D\hat{f}_n(x+h,y+\bar{h}) - D\hat{f}_n(x,y) - D^2\hat{f}_n(x,y)\begin{pmatrix} h \\ \bar{h}\end{pmatrix}}{\left\|(h,\bar{h})\right\|}$$

for all $n \in \mathbb{Z}_\kappa^-$, $x \in X_\kappa$, $y \in \mathcal{Y}$, $h \in X_\kappa \setminus \{0\}$ and $\bar{h} \in X_{\kappa+1} \setminus \{0\}$, we obtain

$$\begin{aligned}\|J_-(k,h)\| \leq \sum_{n=k}^{\kappa-1} \left\|\Phi_{P_-}^-(k,n+1)\right\| \Big[\Big\|\Delta D\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h)-\phi_\kappa(n,\xi)})\Big\| \\ \cdot \left\|\overline{\phi_\kappa(n,\xi+h)-\phi_\kappa(n,\xi)}\right\| \left\|\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)}\right\| + \left\|D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\right\| \\ \cdot \left\|\overline{\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi) - \phi_\kappa^1(n,\xi)h}\right\| \left\|\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h)}\right\| \\ + \left\|D^2\hat{f}_n(\underline{\phi_\kappa(n,\xi)})\right\| \left\|\overline{\phi_\kappa^1(n,\xi)}\right\| \left\|\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h) - \phi_\kappa^{\bar{m}-1}(n,\xi)h}\right\| \\ + \left\|\Delta R_1^{\bar{m}-1}(n,\xi,h)\right\| \|h\| \Big] \quad \text{for all } k \in \mathbb{Z}_\kappa^-.\end{aligned}$$

With Hypotheses 4.2.1 and 4.4.2 (cf. (3.4g), (4.2a)) and $A(\bar{m}-1)(c)$, we therefore get

$$\begin{aligned}\|J_-(k,h)\| \leq K_i^- \sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1) \Big[\Big\|\Delta D\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h)-\phi_\kappa(n,\xi)})\Big\| \\ \cdot \frac{C_{\bar{m}-1}}{\|h\|} \left\|\overline{\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi)}\right\| e_{c_{\bar{m}-1}}(n,\kappa) \\ + C_{\bar{m}-1}\delta_i^-(2) \left\|\overline{\Delta\phi_\kappa(n,h)}\right\| e_{c_{\bar{m}-1}}(n,\kappa) \\ + C_1\delta_i^-(2) \left\|\overline{\phi_\kappa^{\bar{m}-1}(n,\xi+h) - \phi_\kappa^{\bar{m}-1}(n,\xi)}\right\| e_{c_1}(n,\kappa) \\ + \left\|\Delta R_n^{\bar{m}-1}(\xi,h)\right\| \Big] \|h\|\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^-$. Rewriting this estimate and using Lemma 4.4.1(b) we obtain

$$\begin{aligned}\frac{\|J_-(h)\|_{\kappa,c}^-}{\|h\|} \leq \frac{C_{\bar{m}-1}(K_i^-)^2 \max\{1, \lceil c \rceil\}}{1-\ell_i(c)} \sup_{k \in \mathbb{Z}_\kappa^-} V_1(k,h) \\ + C_{\bar{m}-1}K_i^-\delta_i^-(2) \sup_{k \in \mathbb{Z}_\kappa^-} V_2(k,h) \\ + K_i^- C_1 \delta_i^-(2) \sup_{k \in \mathbb{Z}_\kappa^-} V_3(k,h) + K_i^- \sup_{k \in \mathbb{Z}_\kappa^-} V_4(k,h)\end{aligned}$$

with

$$\begin{aligned}V_1(k,h) := e_c(\kappa,k) \sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1) e_{c_{\bar{m}-1}}(n,\kappa) \\ \cdot \left\|\Delta D\hat{f}_n(\underline{\phi_\kappa(n,\xi)}, \underline{\phi_\kappa(n,\xi+h) - \phi_\kappa(n,\xi)})\right\|, \\ V_2(k,h) := e_c(\kappa,k) \sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1) e_{c_{\bar{m}-1}}(n,\kappa) \left\|\overline{\Delta\phi(n,h)}\right\|,\end{aligned}$$

$$V_3(k,h) := e_c(\kappa,k)\sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)e_{c_1}(n,\kappa)\left\|\overline{\phi_\kappa^{\bar m-1}(n,\xi+h)-\phi_\kappa^{\bar m-1}(n,\xi)}\right\|,$$

$$V_4(k,h) := e_c(\kappa,k)\sum_{n=k}^{\kappa-1} e_{b_i}(k,n+1)\left\|\Delta R_n^{\bar m-1}(\xi,h)\right\|.$$

Similarly to Step (I_4) we deduce $\lim_{h\to0}\sup_{k\in\mathbb{Z}_\kappa^-} V_l(k,h)=0$ for $l\in\{1,\dots,4\}$, proving the limit relation $\lim_{h\to0}\frac{\|J_-(h)\|_{\kappa,c}^-}{\|h\|}=0$. Completely analogous one shows the relation $\lim_{h\to0}\frac{\|J_+(h)\|_{\kappa,c}^-}{\|h\|}=0$ and therefore we have verified the differentiability of the mapping $\phi_\kappa^{\bar m-1}: X_\kappa\to\mathfrak{X}_{\kappa,c}^{\bar m-1,-}$. Finally we derive that the derivative

$$D\phi_\kappa^{\bar m-1}: X_\kappa\to L(X_\kappa,\mathfrak{X}_{\kappa,c}^{\bar m-1,-})\cong\mathfrak{X}_{\kappa,c}^{\bar m,-}$$

is the fixed point mapping $\phi_\kappa^{\bar m}: X_\kappa\to\mathfrak{X}_{\kappa,c}^{\bar m,-}$ of $T_\kappa^{\bar m-1,-}$. From the fixed point equation (4.4o) for $\phi_\kappa^{\bar m-1}$ we obtain by differentiation w.r.t. $\xi\in X_\kappa$ the identity

$$\begin{aligned}D_2\phi_\kappa^{\bar m-1}(k;\xi) = &\sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)D\hat f_n(\underline{\phi_\kappa(n,\xi)})\overline{D_2\phi_\kappa^{\bar m-1}(n,\xi)}\\ &+\sum_{n=-\infty}^{\kappa-1} G_i(k,n+1)R_n^{\bar m}(\xi)\quad\text{for all } k\in\mathbb{Z}_\kappa^-.\end{aligned}$$

Hence, the derivative $D\phi_\kappa^{\bar m-1}(\xi)\in L(X_\kappa,\mathfrak{X}_{\kappa,c}^{\bar m-1,-})\cong\mathfrak{X}_{\kappa,c}^{\bar m,-}$ (cf. Lemma 3.3.27) is a fixed point of $\mathcal{T}_\kappa^{\bar m}(\cdot;\xi)$, which in turn is unique by Step (II_2), and so (4.4t) holds.

(II_4) Claim: *For every $c\in(a_i+\varsigma,c_{\bar m})$ the mappings $D^{\bar m}\phi_\kappa: X_\kappa\to\mathfrak{X}_{\kappa,c}^{\bar m,-}$ and $D_2^{\bar m}w^-(\kappa,\cdot): X_\kappa\to L_{\bar m}(X_\kappa)$ are continuous, i.e., also $A(\bar m)(d)$ holds.*
Due to the relation (4.4t) it suffices to prove the continuity of $\phi_\kappa^{\bar m}: X_\kappa\to\mathfrak{X}_{\kappa,c}^{\bar m,-}$ and this is analogous to Step (I_4) by adding and subtracting the expressions $D\hat f_n(\underline{\phi_\kappa(n,\xi)})\overline{\phi_\kappa^{\bar m}(n,\xi_0)}$ in the corresponding estimates. We established $A(\bar m)$.

(II_5) In the preceding four steps we saw that $\phi_\kappa: X_\kappa\to\mathfrak{X}_{\kappa,c}^-$ is m_s-times continuously differentiable. With the identity $w_i^-(\kappa,\xi)=P_1^i(\kappa)\phi_\kappa(\kappa,\xi)$ (see (4.4d)) the claim follows from properties of the evaluation map (see Lemma 3.3.28) and the global bound for the derivatives can be obtained using the fact

$$\left\|D_2^n w_i^-(\kappa,\xi)\right\|=\left\|D^n P_1^i(\kappa)\phi_\kappa(\kappa,\xi)\right\|\le\left\|P_1^i(\kappa)\phi_\kappa^n(\xi)\right\|_{\kappa,c}^-\overset{(4.4s)}{\le}C_n$$

for all $1\le n\le m_s$. The expression for C_1 is a consequence of Theorem 4.2.9(b_2).

(c) The given recursion for the global bounds $C_n\ge0$ of the partial derivatives $\left\|D_2^n w_i^-(\kappa,\xi)\right\|$ for $n\in\{2,\dots,m\}$ in (4.4c) is a consequence of the estimate (4.4i) from step (II_2) in the present proof of (b). A dual argument shows that the solution of the fixed point equation for (4.4f) is globally bounded by C_n as well, and an estimate analogous to (4.4r) gives us the global bounds for the derivatives of w_i^+. Hence, we have shown assertion (c) and the proof of Theorem 4.4.6 is finished. $\square$

For our nest result we impose strengthened spectral gap conditions

$$a_i^m \ll b_i, \qquad \exists \varsigma_i \in \big(0, \varsigma_i^+(m)\big) : (\tilde{G}_i) \text{ holds}, \qquad (\tilde{G}_{i,m}^+)$$

$$a_i \ll b_i^m, \qquad \exists \varsigma_i \in \big(0, \varsigma_i^-(m)\big) : (\tilde{G}_i) \text{ holds}. \qquad (\tilde{G}_{i,m}^-)$$

Proposition 4.4.8 (smooth intersection of invariant fiber bundles). *Assume $\mathbb{I} = \mathbb{Z}$ and that Hypotheses 4.2.1, 4.2.3 and 4.4.2 holds If pairs (i,j) with $1 < i \leq j < N$ satisfy (Γ_{i-1}^+) and (Γ_j^-), as well as the strengthened spectral gap conditions $(\tilde{G}_{i-1,m}^+)$ and $(\tilde{G}_{j,m}^-)$, then the function $w_i^j : \mathcal{P}_i^j \to \mathcal{X}$ from Proposition 4.2.17 is of class C^m with globally bounded partial derivatives $D_2^n w_i^j(\kappa, \cdot) : X_\kappa \to L_n(X_\kappa)$ for $1 \leq n \leq m$.*

Proof. Let $\kappa \in \mathbb{Z}$ be fixed. Referring Theorem 4.4.6, the map $T_\kappa : X_\kappa^2 \times X_\kappa \to X_\kappa^2$ introduced in (4.2w) is m-times continuously differentiable and fulfills the contraction condition $\operatorname{lip}_1 T_\kappa \leq \ell_{ij}(c,d) < 1$ for all $c \in \bar{\Gamma}_{i-1}$, $d \in \bar{\Gamma}_j$. Thus, one can show as in Proposition 4.2.17 that T_κ satisfies the assumptions of the uniform C^m-contraction principle in Theorem B.1.5. We conclude that $T_\kappa(\cdot, y)$, $y \in X_\kappa$, has a unique fixed point $\Upsilon_{ij}(y) \in X_\kappa^2$ and the fixed point mapping $\Upsilon_{ij} : X_\kappa \to X_\kappa^2$ is of class C^m – for the sake of a convenient notation we suppressed the dependence of Υ_{ij} on $\kappa \in \mathbb{Z}$. By construction, the smoothness of Υ_{ij} carries over to $w_i^j(\kappa, \cdot) : X_\kappa \to X_\kappa$.

It remains to show that w_i^j has globally bounded partial derivatives. From Theorem 4.4.6 we see that $T_\kappa : X_\kappa^2 \times X_\kappa \to X_\kappa^2$ has globally bounded derivatives up to order m, where

$$\|D^n T_\kappa(x_1, x_2; y)\| \overset{(4.2w)}{\leq} C_n \quad \text{for all } n \in [1,m]_{\mathbb{Z}}, \, x_1, x_2, y \in X_\kappa.$$

In order to show that this property carries over to Υ_{ij}, we proceed by induction. For $m = 1$ we differentiate the fixed point identity $T_\kappa(\Upsilon_{ij}(y), y) \equiv \Upsilon_{ij}(y)$ on X_κ and obtain using the chain rule (cf. Theorem C.1.3)

$$D\Upsilon_{ij}(y) \equiv D_1 T_\kappa(\Upsilon_{ij}(y), y) D\Upsilon_{ij}(y) + D_2 T_\kappa(\Upsilon_{ij}(y), y) \quad \text{on } X_\kappa.$$

Since Proposition C.1.1 guarantees $\|D_1 T_\kappa(\Upsilon_{ij}(y), y)\| \leq \ell_{ij}(c,d)$ for $y \in X_\kappa$, we get the estimate $\|D\Upsilon_{ij}(y)\| \leq \frac{\ell_{ij}(c,d)}{1-\ell_{ij}(c,d)}$ from (4.2w). Now suppose that $m > 1$ and choose $n \in [2,m]_{\mathbb{Z}}$. Our induction hypothesis is that there exist reals $K_1, \ldots, K_{n-1} \geq 0$ such that $\left\|D^l \Upsilon_{ij}(y)\right\| \leq K_l$ for all $y \in X_\kappa$, $l \in [1,n)_{\mathbb{Z}}$. With this we can define C^m-mappings $\bar{\Upsilon}_{ij} : X_\kappa \to X_\kappa$, $\bar{\Upsilon}_{ij}(y) := (y, \Upsilon_{ij}(y))$, whose derivatives

$$D^l \bar{\Upsilon}_{ij}(y) y_1 \cdots y_l = \begin{cases} (y_1, D\Upsilon_{ij}(y) y_1) & \text{for } l = 1, \\ (0, D^l \Upsilon_{ij}(y) y_1 \cdots y_l) & \text{for } l > 1 \end{cases}$$

for all $y_1, \dots, y_l \in X_\kappa$ satisfy (cf. (**??**) and our induction hypothesis)

$$\left\|D\bar{\Upsilon}_{ij}(y)\right\| \le \max\left\{1, \frac{\ell_{ij}(c,d)}{1-\ell_{ij}(c,d)}\right\}, \quad \left\|D^l\bar{\Upsilon}_{ij}(y)\right\| \le K_l \quad \text{for all } l \in [2,n)_{\mathbb{Z}}\,.$$

Thus, relation (C.1b) in the higher order chain rule from Theorem C.1.3 implies

$$\|D^n\Upsilon_{ij}(y)\| \le \ell_{ij}(c,d)\left\|\Upsilon^n_{ij}(y)\right\| + 2\sum_{l=2}^{n} C_l \sum_{(N_1,\dots,N_l)\in P_l^<(n)} \prod_{\nu=1}^{l} \left\|D^{\#N_\nu}\bar{\Upsilon}_{ij}(y)\right\|$$

and since components $N_\nu \subseteq \{1,\dots,n\}$ of a partition $(N_1,\dots,N_l) \in P_l^<(n)$, $l \in [2,n]_{\mathbb{Z}}$, have a smaller cardinality than n, we conclude from our hypothesis

$$\begin{aligned}\|D^n\Upsilon_{ij}(y)\| \le{}& \frac{2}{1-\ell_{ij}(c,d)}\\ &\cdot\sum_{l=2}^{n} C_l \sum_{(N_1,\dots,N_l)\in P_l^<(n)} \prod_{\nu=1}^{l} \begin{cases}\max\left\{1, \frac{\ell_{ij}(c,d)}{1-\ell_{ij}(c,d)}\right\} & \text{for } \#N_\nu = 1,\\ K_\nu & \text{for } \#N_\nu > 1.\end{cases}\end{aligned}$$

Therefore, the mapping $D^n\Upsilon_{ij} : X_\kappa \to L_n(X_\kappa; X_\kappa^2)$ is globally bounded and by definition of w_i^j, this implies our assertion. □

An even more delicate question is the smooth dependence of the invariant fibers $\mathcal{V}_i^\pm(\xi) \subseteq \mathcal{X}$ on the initial point $\xi \in X_\kappa$. Here, the C^m-smoothness of the 2-parameter semigroup generated by (S) carries over to the mappings $v_i^\pm$ from Proposition 4.3.5 only for $m = 0$. For higher order differentiability also the growth behavior of $D_3^n\varphi(\cdot;\kappa,\xi)$, $(\kappa,\xi) \in \mathcal{X}$ and $1 \le n \le m$ plays an important rule and due to the resulting technical complexity we waive a corresponding statement and proof. Yet, corresponding references can be found in Sect. 4.10:

Proposition 4.4.9 (smoothness of invariant fibers). *Let $(\kappa,\xi) \in \mathcal{X}$ and for given $1 \le i < N$ choose $c \in \bar{\Gamma}_i$. Assume that Hypotheses 4.2.1, 4.3.1 and 4.4.2 hold.*

(a) If $\mathbb{I}$ is unbounded above and $(G^+_{i,m})$ holds, then the mapping $v_i^+(\kappa,\cdot,\xi) : X_\kappa \to \mathcal{P}_1^i(\kappa)$ from Proposition 4.3.5(a) is m-times differentiable with continuous partial derivatives $D_2^n v_i^+(\kappa,\cdot) : X_\kappa^2 \to L_n(X_\kappa)$ and the global bounds

$$\sup_{(\kappa,\eta,\xi)\in\mathcal{X}\times\mathcal{X}} \left\|D_2^n v_i^+(\kappa,\eta,\xi)\right\|_{L_n(X_\kappa)} \le C_n \quad \text{for all } 1 \le n \le m.$$

(b) If $\mathbb{I}$ is unbounded below, the general solution φ of (S) exists on $\mathcal{X}$ as a continuous mapping and $(G^-_{i,m})$ holds, then the mapping $v_i^-(\kappa,\cdot,\xi) : X_\kappa \to \mathcal{Q}_1^i(\kappa)$ from Proposition 4.3.5(b) is m-times differentiable with continuous partial derivatives $D_2^n v_i^-(\kappa,\cdot) : X_\kappa^2 \to L_n(X_\kappa)$ and the global bounds

$$\sup_{(\kappa,\eta,\xi)\in\mathcal{X}\times\mathcal{X}} \left\|D_2^n v_i^-(\kappa,\eta,\xi)\right\|_{L_n(X_\kappa)} \le C_n \quad \text{for all } 1 \le n \le m.$$

The constants $C_n \ge 0$ are recursively given by (4.4c) in Theorem 4.4.6(c).

Proof. Let the pair $(\kappa,\xi)\in\mathcal{X}$ be fixed.

(a) In order to show the continuous differentiability of $v_i^+(\kappa,\cdot,\xi):X_\kappa\to X_\kappa$, we proceed as follows: Again abbreviate $\hat{f}_k := B_{k+1}^{-1}f_k$, $k\in\mathbb{I}'$, and formally differentiate the fixed point equation (cf. (4.3e))

$$\begin{aligned}\psi_\kappa(k,\eta,\xi) &= \Phi(k,\kappa)[\eta - Q_1^i(\kappa)\xi] \\ &\quad + \sum_{n=\kappa}^{\infty} G_i(k,n+1)\big[\hat{f}_n(\underline{\psi_\kappa(n,\eta,\xi)+\varphi(n;\kappa,\xi)}) - \hat{f}_n(\underline{\varphi(n;\kappa,\xi)})\big]\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^+$ w.r.t. the variable $\eta\in X_\kappa$. Suppressing the dependence on $\xi\in X_\kappa$ from now on, this leads to a further fixed point equation

$$\psi_\kappa^1(\eta) = S_\kappa^{1,+}(\psi_\kappa^1(\eta);\eta) \quad \text{for all } \eta\in\mathcal{Q}_1^i(\kappa), \tag{4.4x}$$

yielding the formal derivative ψ_κ^1 w.r.t. $\eta\in X_\kappa$ for the mapping $\psi_\kappa:\mathcal{Q}_1^i(\kappa)\to\mathcal{X}_{\kappa,c}^+$ from Lemma 4.3.4. Precisely, the operator $S_\kappa^{1,+}$ is given by

$$\begin{aligned}S_\kappa^{1,+}(\psi^1;\eta,\xi) := \Phi(\cdot,\kappa) + \sum_{n=\kappa}^{\infty} G_i(k,n+1)\Big[& D\hat{f}_n(\underline{\psi_\kappa(n,\eta)+\varphi(n;\kappa,\xi)}) \\ & \cdot \overline{\psi^1(n,\eta)+D_3\varphi(n;\kappa,\xi)}\Big],\end{aligned}$$

where the variable $\psi^1(k)$, $k\in\mathbb{I}$, is a sequence with values in $L(X_\kappa,X_k)$. The analysis of this operator $S_\kappa^{1,+}$ strongly resembles the one for $T_\kappa^{1,-}$ given in the above proof of Theorem 4.4.6. We consequently omit the corresponding further details.

(b) Since the argument is analogous to (a), we skip the details. □

4.5 Normal Hyperbolicity

To motivate our further considerations, we return to the linear equation (L_0). As we have seen, the invariant fiber bundle $\mathcal{W}_i^-$ for (S), as formulated in Theorem 4.2.9(b), is a perturbation of the pseudo-unstable bundle $\mathcal{P}_1^i$, and the linear spectral gap condition $a_i \ll b_i$ implies that $\mathcal{P}_1^i$ is normally hyperbolic in the sense that (L_0) possesses an exponential dichotomy. Now we tackle the problem whether this normal hyperbolicity persists under nonlinear perturbations.

As important result from the previous section, we know that the invariant fiber bundle $\mathcal{W}_i^-$ and its invariant foliation $\mathcal{V}_i^+(\xi)$ are of class C^1, if (S) has this property. Hence, for each $(\kappa,x,y)\in\mathcal{W}_i^-\times\mathcal{X}$ we can define the *tangent bundles*

$$T_x\mathcal{W}_i^- := \left\{(\kappa, \xi + D_2 w_i^-(\kappa, x)\xi) \in \mathcal{X} : \xi \in \mathcal{P}_1^i(\kappa)\right\},$$
$$T_y\mathcal{V}_i^+(x) := \left\{(\kappa, \eta + D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)\eta) \in \mathcal{X} : \eta \in \mathcal{Q}_1^i(\kappa)\right\}$$

to $\mathcal{W}_i^-$ resp. $\mathcal{V}_i^+$. For simplicity reasons, we assume for the remaining section that (S) is semi-implicit, i.e., instead of (S) we consider

$$B_{k+1}x' = A_k x + f_k(x). \tag{S$'$}$$

As a consequence, under (3.1a) the general forward solution to (S$'$) exists, for instance the strengthened gap condition $(\tilde{G}_i)$ simplifies to

$$\exists \varsigma_i \in \left(0, \tfrac{\lfloor b_i - a_i \rfloor}{2}\right) : \left(K_i^+ K_i^- + \max\left\{K_i^+, K_i^-\right\}\right) L_1 < \varsigma_i \tag{4.5a}$$

and, in particular, both the invariance equations (4.2m), (4.3n) become easier to handle. The sets $\bar{\Gamma}_i$ are defined as in the previous sections.

The subsequent lemma roughly states that the two tangential bundles defined above provide a splitting of each fiber X_κ of the extended state space $\mathcal{X}$. Before delving into preparations, let us point out that we focus on the invariant fiber bundles $\mathcal{W}_i^-$ in this section and that we suppose $\mathbb{I} = \mathbb{Z}$ from now on.

Lemma 4.5.1. *Assume Hypotheses 4.2.1, 4.3.1, 4.4.2 with $m = 1$ and (Γ_i^-) and (4.5a) hold for one $1 \leq i < N$. Then for each $\kappa \in \mathbb{Z}$ we have the decomposition*

$$X_\kappa = T_x\mathcal{W}_i^-(\kappa) \oplus T_y\mathcal{V}_i^+(\kappa, x) \quad \text{for all } x \in \mathcal{W}_i^-(\kappa),\ y \in X_\kappa \tag{4.5b}$$

and the splitting is continuous in $(x, y) \in \mathcal{W}_i^-(\kappa) \times X_\kappa$.

Proof. Fix a triple $(\kappa, x, y) \in \mathcal{W}_i^- \times \mathcal{X}$. In order to prove that the tangent spaces $T_x\mathcal{W}_i^-(\kappa)$ and $T_y\mathcal{V}_i^+(\kappa, x)$ satisfy (4.5b) we show that each $\zeta \in X_\kappa$ possesses the representation $\zeta = \bar{\xi} + \bar{\eta}$ with unique $\bar{\xi} \in T_x\mathcal{W}_i^-(\kappa)$ and $\bar{\eta} \in T_y\mathcal{V}_i^+(\kappa, x)$. This is equivalent to the unique existence of points $\xi \in \mathcal{P}_1^i(\kappa)$, $\eta \in \mathcal{Q}_1^i(\kappa)$ such that $\zeta = \xi + D_2 w_i^-(\kappa, x)\xi + \eta + D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)\eta$, which holds if and only if

$$P_1^i(\kappa)\zeta = \xi + D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)\eta, \qquad Q_1^i(\kappa)\zeta = \eta + D_2 w_i^-(\kappa, x)\xi$$

and this, in turn, is equivalent to

$$\xi = P_1^i(\kappa)\zeta - D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)Q_1^i(\kappa)\zeta + D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)D_2 w_i^-(\kappa, x)\xi,$$
$$\eta = Q_1^i(\kappa)\zeta - D_2 w_i^-(\kappa, x)P_1^i(\kappa)\zeta + D_2 w_i^-(\kappa, x)D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)\eta.$$

By Theorem 4.2.9(b_2) and Proposition 4.3.5(a_2) the Lipschitz constants $\operatorname{lip}_2 w_i^-$, $\operatorname{lip}_2 v_i^+$, respectively, exist and their product is less than 1 (cf. (4.3t)), so that the operators $I - D_2 v_i^+(\kappa, Q_1^i(\kappa)y, x)D_2 w_i^-(\kappa, x)$ and $I - D_2 w_i^-(\kappa, x)v_i^+(\kappa, Q_1^i(\kappa)y, x)$ are invertible in the Banach algebra $L(X_\kappa)$ (cf. [295, p. 74, Theorem 2.1] or

Theorem B.3.1). Therefore, one can represent $\zeta \in X_\kappa$ uniquely as $\zeta = \tilde{P}_1^i(\kappa,x,y)\zeta + \tilde{Q}_1^i(\kappa,x,y)\zeta$, where $\tilde{P}_1^i(\kappa,x,y) \in L(X_\kappa)$ is the projection of X_κ onto $T_x\mathcal{W}_i^-(\kappa)$ along $T_y\mathcal{V}_i^+(\kappa,x)$,

$$\begin{aligned}\tilde{Q}_1^i(\kappa,x,y) := \left[I - D_2v_i^+(\kappa, Q_1^i(\kappa)y, x)D_2w_i^-(\kappa,x)\right]^{-1} \cdot \\ \cdot \left[P_1^i(\kappa) - D_2v_i^+(\kappa, Q_1^i(\kappa)y, x)\right],\end{aligned}$$

and accordingly $\tilde{Q}_1^i(\kappa,x,y) \in L(X_\kappa)$ is the projection of X_κ onto $T_y\mathcal{V}_i^+(\kappa,x)$ along $T_x\mathcal{W}_i^-(\kappa)$ given by

$$\tilde{P}_1^i(\kappa,x,y) := \left[I - D_2w_i^-(\kappa,x)D_2v_i^+(\kappa, Q_1^i(\kappa)y, x)\right]^{-1}\left[Q_1^i(\kappa) - D_2w_i^-(\kappa,x)\right].$$

Due to both our Theorem 4.4.6(b), Proposition 4.4.9(a) and the fact that the inversion operator $\cdot^{-1} : L(X_\kappa) \to L(X_\kappa)$ is of class C^∞ (cf. [1, p. 117, Lemma 2.5.5]), we see that $\tilde{P}_1^i(\kappa,x,y)$, $\tilde{Q}_1^i(\kappa,x,y)$ depend continuously on $(x,y) \in \mathcal{W}_i^-(\kappa) \times X_\kappa$. Thus, the splitting (4.5b) is continuous. □

Consider the difference equation in $\mathcal{X} \times \mathcal{X}$ given by (S) and the corresponding variational equation

$$\begin{cases} B_{k+1}x' = A_kx + f_k(x) \\ B_{k+1}z' = [A_k + Df_k(x)]\,z \end{cases}; \tag{4.5c}$$

its general solution will be denoted by (φ,ϕ) and exists due to (3.1a). In the following it is our aim to show that the invariant fiber bundle $\mathcal{W}_i^-$ is normally hyperbolic; that is to say that the tangential and normal bundle for $\mathcal{W}_i^-$ are invariant under (4.5c), and that we have an exponential dichotomy w.r.t. these bundles. To be more precise, we have

Lemma 4.5.2 (tangent bundle). *Assume Hypotheses 4.2.1, 4.3.1, 4.4.2 with $m = 1$ and (Γ_i^-) and (4.5a) hold for one $1 \le i < N$. If*

$$K_i^-L_1\left(1 + \tilde{\ell}_i^-(c)\right) < b_i(k) \quad \textit{for all } k \in \mathbb{Z} \tag{4.5d}$$

is satisfied for one $c \in \bar{\Gamma}_i$, then the tangent bundle

$$T\mathcal{W}_i^- := \left\{(\kappa,\xi,\zeta) \in \mathcal{X} \times \mathcal{X} : (\kappa,\xi) \in \mathcal{W}_i^-, \zeta \in T_\xi\mathcal{W}_i^-(\kappa)\right\}$$

is invariant w.r.t. (4.5c), the general solution (φ,ϕ) of (4.5c) exists on $T\mathcal{W}_i^-$ and one has the backward estimate

$$\|\phi(k;\kappa,\xi,\zeta)\|_{X_k} \le K_i^- e_{\hat{b}_i}(k,\kappa)\left\|P_1^i(\kappa)\zeta\right\|_{X_\kappa} \quad \textit{for all } k \in \mathbb{Z}_\kappa^- \tag{4.5e}$$

and $(\kappa,\xi,\zeta) \in T\mathcal{W}_i^-$, where $\hat{b}_i(k) := b_i(k) - K_iL_1\left(1 + \tilde{\ell}_i^-(c)\right)$.

Proof. First, we abbreviate $\hat{f}_k := B_{k+1}^{-1} f_k$. Choose any triple $(\kappa, \xi, \zeta) \in T\mathcal{W}_i^-$ and consequently we have representations $\xi = \xi_0 + w_i^-(\kappa, \xi_0)$, $\zeta = \zeta_0 + D_2 w_i^-(\kappa, \xi_0)\zeta_0$ for some points $\xi_0, \zeta_0 \in \mathcal{P}_1^i(\kappa)$. Then Corollary 4.2.13(a) implies that the general solution $\hat{\varphi}$ of the $\mathcal{W}_i^-$-reduced equation (4.2s) is defined on $\mathcal{P}_i^-$. The further proof is subdivided into four steps:

(I) Claim: *The general solution $\tilde{\varphi}$ of the variational equation for* (4.2s),

$$B_{k+1}x' = A_k x + P_1^i(k+1) D\hat{f}_k\big(\tilde{\varphi}(k;\kappa,\xi_0) + w_i^-(k, \tilde{\varphi}(k;\kappa,\xi_0))\big) \cdot \big[x + D_2 w_i^-(k, \tilde{\varphi}(k;\kappa,\xi_0))x\big] \tag{4.5f}$$

is defined on $\mathcal{P}_1^i$.

We differentiate the solution identity for $\hat{\varphi}$ w.r.t. ξ_0. Then $D_3\hat{\varphi}(\cdot;\kappa,\xi_0)$ is an operator solution of (4.5f) satisfying the initial condition $x(\kappa) = I$, and $\tilde{\varphi}(k;\kappa,\xi_0) := D_3\hat{\varphi}(k;\kappa,\xi_0)\xi_0$ defines the general solution of (4.5f) for $k \in \mathbb{Z}$, $(\kappa, \xi_0) \in \mathcal{P}_1^i$.

(II) Claim: *The tangent bundle $T\mathcal{W}_i^-$ is forward invariant w.r.t.* (4.5c)*.* Define the sequence $\psi_1 : \mathbb{Z}_\kappa^+ \to \mathcal{X}$, $\psi_1(k) := \hat{\varphi}(k;\kappa,\xi_0) + w_i^-(k, \hat{\varphi}(k;\kappa,\xi_0))$ and due to the inclusion $\hat{\varphi}(k;\kappa,\xi_0) \in \mathcal{P}_1^i(k)$ one obviously has $\psi_1(k) \in \mathcal{W}_i^-(k)$ for all $k \in \mathbb{Z}_\kappa^+$. In addition, from the invariance equation (4.2m) we see that ψ_1 is a solution of the first equation in (4.5c) with $\psi_1(\kappa) = \xi$ and this yields $\varphi(k;\kappa,\xi) = \psi_1(k) \in \mathcal{W}_i^-(k)$ for all $k \in \mathbb{Z}_\kappa^+$. Next we define the sequence $\psi_2 : \mathbb{Z}_\kappa^+ \to \mathcal{X}$, $\psi_2(k) = \tilde{\varphi}(k;\kappa,\zeta_0) + D_2 w_i^-(k, \varphi(k;\kappa,\xi))\tilde{\varphi}(k;\kappa,\zeta_0)$. Observing the inclusion $\tilde{\varphi}(k;\kappa,\zeta_0) \in \mathcal{P}_1^i(k)$ one has $\psi_2(k) \in T_{\varphi(k;\kappa,\xi)}\mathcal{W}_i^-(k)$ for all $k \in \mathbb{Z}_\kappa^+$. Using an identity obtained by differentiating the invariance equation (4.2m) w.r.t. the variable in $\mathcal{P}_1^i(k)$, one verifies that ψ_2 solves the second equation in (4.5c) and satisfies $\psi_2(\kappa) = \zeta_0 + D_2 w_i^-(k, \xi)\zeta_0$. Hence, $\phi(k;\kappa,\xi,\zeta) = \psi_2(k) \in T_{\varphi(k;\kappa,\xi)}\mathcal{W}_i^-(k)$ and the tangent bundle $T\mathcal{W}_i^-$ is forward invariant.

(III) The fact that φ is defined on $\mathcal{W}_i^-$ is given in Corollary 4.2.13(a) and we will show that the second component ϕ is defined on $T\mathcal{W}_i^-$. For this, let $k \in \mathbb{Z}$. From Step (II) we have $\phi(k+1;k,\cdot) : T_{\varphi(k;\kappa,\xi)}\mathcal{W}_i^-(k) \to T_{\varphi(k+1;\kappa,\xi)}\mathcal{W}_i^-(k+1)$ is well-defined and it suffices to show that this mapping is bijective. Let $\eta \in T_{\varphi(k+1;\kappa,\xi)}\mathcal{W}_i^-(k+1)$, i.e., $\eta = \eta_1 + D_2 w_i^-(k+1, \varphi(k+1;k,\xi))\eta_1$ for some $\eta_1 \in \mathcal{P}_1^i(k+1)$; note that $\operatorname{lip}_2 w_i^- < 1$ (cf. Remark 4.3.8) and consequently η_1 uniquely determines the point η. We show that the endomorphism

$$B_{k+1}^{-1}A_k + P_1^i(k+1)D\hat{f}_k(\varphi(k;\kappa,\xi))\left[I + D_2 w_i^-(k, \varphi(k;\kappa,\xi))\right]$$

is actually an isomorphism between the linear spaces $\mathcal{P}_1^i(k)$ and $\mathcal{P}_1^i(k+1)$. We abbreviate $\Phi_k := P_1^i(k+1)D\hat{f}_k(\varphi(k;\kappa,\xi))\left[I + D_2 w_i^-(k, \varphi(k;\kappa,\xi))\right]$ and from Theorem 4.2.9(b_2) with (4.2a) one derives $\|\Phi_k\| \le L_1\left(1 + \tilde{\ell}_i^-(c)\right)$ for all $k \in \mathbb{Z}$. On the other hand, from Hypothesis 4.2.1 we know that the inverse $B_{k+1}^{-1}A_k|_{\mathcal{P}_1^i(k)}^{-1}$ exists (see Lemma 3.3.6(b)) and (3.4g) implies $\left\|B_{k+1}^{-1}A_k|_{\mathcal{P}_1^i(k)}^{-1}\right\| \le \frac{K_i^-}{b_i(k-1)}$ for all $k \in \mathbb{Z}$. Then our assumption (4.5d) and Theorem B.3.1

shows the invertibility of $B_{k+1}^{-1}A_k + \Phi_k \in L(\mathcal{P}_1^i(k), \mathcal{P}_1^i(k+1))$, and $\eta_0 := \left[B_{k+1}^{-1}A_k + \Phi_k\right]^{-1}\eta_1$ is the unique point in $\mathcal{P}_1^i(k)$ satisfying the relation $\phi(k+1; k, \xi, \eta_0 + D_2 w_i^-(k, \varphi(k;\kappa,\xi))\eta_0) = \eta$. In particular, (4.5d) ensures that the Gronwall estimate Proposition A.2.1(b) can be applied.

(IV) Referring to Step (I) we know that the general solution $\tilde{\varphi}(k;\kappa,\cdot)$ of the variational equation (4.5f) exists for $k \in \mathbb{Z}_\kappa^-$. So, the variation of constants formula in backward time from Theorem 3.1.16(b) (see also Remark 3.1.17(1)) implies the relation

$$
\begin{aligned}
\tilde{\varphi}(k;\kappa,\zeta_0) = \Phi_{P_1^i}^-(k,\kappa)\zeta_0 - \sum_{n=k}^{\kappa-1} & \Phi_{P_1^i}^-(k,n+1)P_1^i(n+1) \\
& \cdot D\hat{f}_n\big(\tilde{\varphi}(n;\kappa,\xi_0) + w_i^-(n,\tilde{\varphi}(n;\kappa,\xi_0))\big) \\
& \cdot \left[I + D_2 w_i^-(n,\tilde{\varphi}(n;\kappa,\xi_0))\right]\tilde{\varphi}(n;\kappa,\zeta_0) \quad \text{for all } k \in \mathbb{Z}_\kappa^-
\end{aligned}
$$

and analogous to the proof of (4.2t) in Corollary 4.2.13 one gets the estimate (4.5e). □

Lemma 4.5.3 (normal bundle). *Assume Hypotheses 4.2.1, 4.3.1, 4.4.2 with* $m = 1$ *and* (Γ_i^-) *and* (4.5a) *hold for one* $1 \le i < N$. *Then the* normal bundle

$$
N\mathcal{W}_i^- := \left\{(\kappa,\xi,\zeta) \in \mathcal{X} \times \mathcal{X} : (\kappa,\xi) \in \mathcal{W}_i^-, \zeta \in T_\xi \mathcal{V}_i^+(\kappa,\xi)\right\}
$$

is forward invariant w.r.t. (4.5c)*, and one has the forward estimate*

$$
\|\phi(k;\kappa,\xi,\zeta)\|_{X_k} \le K_i^+ \left(1 + \tilde{\ell}_i^+(c)\right) e_{\hat{a}_i}(k,\kappa) \left\|Q_1^i(\kappa)\zeta\right\|_{X_\kappa} \quad \textit{for all } k \in \mathbb{Z}_\kappa^+
$$

and $(\kappa,\xi,\zeta) \in N\mathcal{W}_i^-$, *where* $\hat{a}_i(k) := a_i(k) + K_i^+ L_1 \left(1 + \tilde{\ell}_i^+(c)\right)$.

Proof. Let $(\kappa,\xi,\zeta) \in N\mathcal{W}_i^-$ and $\hat{f}_k := B_{k+1}^{-1} f_k$. We proceed in two steps:

(I) To show the forward invariance of $N\mathcal{W}_i^-$ we choose an arbitrary $\eta \in \mathcal{W}_i^-(\kappa)$ and let $\eta_0 \in \mathcal{Q}_1^i(\kappa)$ be such that $\eta = \eta_0 + v_i^+(\kappa,\eta_0,\xi)$. From the forward invariance of $\mathcal{V}_i^+(\xi)$ guaranteed by Proposition 4.3.5(a) we know that there exists a sequence of points $\psi_1(k) \in \mathcal{Q}_1^i(k)$ satisfying

$$
\varphi(k;\kappa,\eta) = \psi_1(k) + v_i^+(k,\psi_1(k),\varphi(k)) \quad \text{for all } k \in \mathbb{Z}_\kappa^+, \tag{4.5g}
$$

where we abbreviate $\varphi(k) = \varphi(k;\kappa,\xi)$ from now on, since $\xi \in \mathcal{W}_i^-(\kappa)$ remains fixed. If we multiply the solution identity for φ with $Q_1^i(k+1)$, we see that the sequence $\psi_1 = Q_1^i(\cdot)\varphi(\cdot;\kappa,\eta) : \mathbb{Z}_\kappa^+ \to \mathfrak{X}$ solves the equation

$$
x' = B_{k+1}^{-1}A_k x + Q_1^i(k+1)\hat{f}_k\big(x + v_i^+(k,x,\varphi(k))\big). \tag{4.5h}
$$

Let ψ denote the general solution of (4.5h). Then the partial derivative $D_3\psi$ exists and $D_3\psi(\cdot;\kappa,\eta_0)Q_1^i(\kappa)\zeta$ is a solution of the variational equation (cf. (4.5g))

$$\begin{aligned} x' = {} & B_{k+1}^{-1}A_k x + Q_1^i(k+1)D\hat{f}_k\big(\psi(k;\kappa,\eta_0) + v_i^+(k,\psi(k;\kappa,\eta_0),\varphi(k;\kappa,\eta))\big) \\ & \cdot\big[x + D_2v_i^+(k,Q_1^i(k)\varphi(k;\kappa,\eta),\varphi(k))x\big] \end{aligned} \tag{4.5i}$$

satisfying the initial condition $x(\kappa) = Q_1^i(\kappa)\zeta$. On the other hand, the invariance equation (4.3n) yields the identity

$$\begin{aligned} v_i^+(k+1,\psi(k+1;\kappa,\eta_0),\varphi'(k)) \equiv {} & B_{k+1}^{-1}A_k v_i^+(k;\psi(k;\kappa,\eta_0),\varphi(k)) \\ & + Q_1^i(k+1)\hat{f}_k\big(\psi(k;\kappa,\eta_0) + v_i^+(k,\psi(k;\kappa,\eta_0),\varphi(k))\big) \quad \text{on } \mathbb{Z}_\kappa^+ \end{aligned}$$

and if we differentiate this identity w.r.t. η_0 and apply $Q_1^i(\kappa)\zeta$ one gets

$$\begin{aligned} & D_2v_i^+(k+1,\psi(k+1;\kappa,\eta_0),\varphi'(k))D_3\psi(k+1;\kappa,\eta_0)Q_1^i(\kappa)\zeta \\ & \quad \equiv B_{k+1}^{-1}A_kD_2v_i^+(k;\psi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta \\ & \qquad +Q_1^i(k+1)D\hat{f}_k\big(\psi(k;\kappa,\eta_0) + v_i^+(k,\psi(k;\kappa,\eta_0),\varphi(k))\big) \\ & \qquad \cdot\big[D_3\psi(k;\kappa,\eta_0) + D_2v_i^+(k,\psi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)\big]\,Q_1^i(\kappa)\zeta \end{aligned}$$

on $\mathbb{Z}_\kappa^+$. From this, and the solution identity for $D_3\psi(\cdot;\kappa,\eta_0)Q_1^i(\kappa)\zeta$ (cf. (4.5i)) we see that the sum

$$\begin{aligned} \sigma(k) :&= D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta + D_2v_i^+(k;\psi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta \\ &= D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta \\ & \quad +D_2v_i^+(k;Q_1^i(k)\varphi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta \\ & \in T_{\varphi(k;\kappa,\eta)}\mathcal{V}_i^-(k,\varphi(k)) \quad \text{for all } k\in\mathbb{Z}_\kappa^+ \end{aligned}$$

is a solution of the linear difference equation $B_{k+1}x' = A_kx + Df_k(\varphi(k;\kappa,\eta))x$ satisfying $\sigma(\kappa) = Q_1^i(\kappa)\zeta + D_2v_i^+(\kappa,Q_1^i(\kappa)\eta,\zeta)Q_1^i(\kappa)\zeta$. Since $\eta\in\mathcal{V}_i^+(\kappa,\xi)$ was arbitrary, we can choose $\eta = \pi_i^+(\kappa,\xi)$ now, and $\xi\in\mathcal{W}_i^-(\kappa)$ yields $\eta = \pi_i^+(\kappa,\xi) = \xi$ (cf. Theorem 4.3.7(a)). Hence, $\sigma(k)\in T_{\varphi(k)}\mathcal{V}_i^+(k,\varphi(k))$ for all $k\in\mathbb{Z}_\kappa^+$ and $\sigma(\kappa) = \zeta$. The uniqueness of forward solutions implies $\phi(k;\kappa,\xi,\zeta) = \sigma(k)$, i.e.

$$\begin{aligned} \phi(k;\kappa,\xi,\zeta) = {} & D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta \\ & +D_2v_i^+(k;Q_1^i(k)\varphi(k;\kappa,\xi),\varphi(k))D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta \end{aligned} \tag{4.5j}$$

and due to the invariance of $\mathcal{W}_i^-$ we have $(\varphi,\phi)(k;\kappa,\xi,\zeta)\in N\mathcal{W}_i^-(k)$, $k\in\mathbb{Z}_\kappa^+$.

(II) It remains to deduce the claimed forward estimate for ϕ. The variation of constants formula from Theorem 3.1.16(a), applied to (4.5i), gives us

$$\begin{aligned}D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta &= \varPhi(k,\kappa)Q_1^i(\kappa)\zeta \\ &+ \sum_{n=\kappa}^{k-1}\varPhi(k,n+1)Q_1^i(n+1)D\hat{f}_n\big(\psi(n;\kappa,\eta_0)+v_i^+(n,\psi(k;\kappa,\eta_0),\varphi(n;\kappa,\eta))\big) \\ &\cdot\left[D_3\psi(k;\kappa,\eta_0)+D_2v_i^+(k,Q_1^i(k)\varphi(k;\kappa,\eta),\varphi(k))D_3\psi(k;\kappa,\eta_0)\right]Q_1^i(\kappa)\zeta\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$, and from (3.4g), (3.5a) and Proposition 4.3.5(a_2) we get

$$\begin{aligned}&\left\|D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta\right\| e_{a_i}(\kappa,k) \\ &\le K_i^+\left\|Q_1^i(\kappa)\zeta\right\| + K_i^+\left(1+\tilde{\ell}_i^+(c)\right)L_1\sum_{n=\kappa}^{k-1}\frac{e_{a_i}(\kappa,n)}{a_i(n)}\left\|D_3\psi(n;\kappa,\eta_0)Q_1^i(\kappa)\zeta\right\|\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$. The Gronwall lemma from Theorem A.2.1(a) implies

$$\left\|D_3\psi(k;\kappa,\eta_0)Q_1^i(\kappa)\zeta\right\| \le K_i^+ e_{\hat{a}_i}(k,\kappa)\left\|Q_1^i(\kappa)\zeta\right\| \quad \text{for all } k \in \mathbb{Z}_\kappa^+$$

and (4.5j) together with Proposition 4.3.5(a_2) leads to our assertion. □

Theorem 4.5.4 (normal hyperbolicity). *Assume Hypotheses 4.2.1, 4.3.1, 4.4.2 with $m = 1$ and (Γ_i^-) and (4.5a) hold for one $1 \le i < N$. If (4.5d) is satisfied for one $c \in \bar{\Gamma}_i$, then the invariant fiber bundle $\mathcal{W}_i^-$ is* normally hyperbolic*:*

(a) One has the Whitney sum $\mathcal{X} \times \mathcal{X} = T\mathcal{W}_i^- \oplus N\mathcal{W}_i^-$*, where the splitting is continuous in each fiber.*
(b) The nonautonomous sets $T\mathcal{W}_i^-$ and $N\mathcal{W}_i^-$ possess the properties stated in Lemma 4.5.2 and Lemma 4.5.3, resp.
(c) In particular, the pseudo-contraction in the normal direction of $\mathcal{W}_i^-$ is stronger than in the tangential direction.

Proof. The claim follows readily from the above Lemmata 4.5.1, 4.5.2 and 4.5.3. Here, thanks to the strengthened spectral gap condition $(\tilde{G}_i)$ we have

$$(K_i^+ + K_i^-)L_1 \le \left(K_i^+K_i^- + \max\left\{K_i^+, K_i^-\right\}\right)L_1 \overset{(4.5a)}{<} \varsigma_i < \frac{\lfloor b_i - a_i\rfloor}{2}$$

and from this one easily derives $\hat{a}_i \ll \hat{b}_i$, i.e., the normal pseudo-contraction rate $\hat{a}_i$ is stronger than the corresponding tangential rate $\hat{b}_i$. □

4.6 Pseudo-stable and Pseudo-unstable Fiber Bundles

In this section, we make the first attempt to weaken the global assumptions in form of Hypotheses 4.2.3, 4.3.1 or 4.4.2. Indeed, we return to general equations

$$H_{k+1}(x') = F_k(x,x') \tag{D}$$

as in Definition 2.1.1, where $\mathcal{X}$ consists of Banach spaces. We are interested in the local behavior of (D) near a fixed reference solution $\phi_* : \mathbb{I} \to \mathfrak{X}$, which, for instance, might be a constant, a periodic or a general bounded solution. In particular, we want to provide a local description of the *stable set* corresponding to ϕ_*,

$$\mathcal{W}_{\phi_*}^+ := \left\{ (\kappa, \xi) \in \mathcal{X} \,\middle|\, \begin{array}{l} \text{there exists a solution } \phi : \mathbb{Z}_\kappa^+ \to \mathfrak{X} \text{ of (D) with} \\ \phi(\kappa) = \xi \in X_\kappa \text{ and } \lim_{k \to \infty} \|\phi(k) - \phi_*(k)\|_{X_k} = 0 \end{array} \right\},$$

when $\mathbb{I}$ is unbounded above, as well as of the *unstable set* corresponding to ϕ_*,

$$\mathcal{W}_{\phi_*}^- := \left\{ (\kappa, \xi) \in \mathcal{X} \,\middle|\, \begin{array}{l} \text{there exists a solution } \phi : \mathbb{Z}_\kappa^- \to \mathfrak{X} \text{ of (D) with} \\ \phi(\kappa) = \xi \in X_\kappa \text{ and } \lim_{k \to -\infty} \|\phi(k) - \phi_*(k)\|_{X_k} = 0 \end{array} \right\},$$

provided $\mathbb{I}$ is unbounded below.

Let us suppose the difference equation (D) is defined on a nonautonomous set $\mathcal{S}$ containing a convex neighborhood of the reference solution ϕ_*, i.e., there exists a $\rho_0 > 0$ such that $\mathcal{B}_{\rho_0}(\phi_*) \subseteq \mathcal{S}$. It is advantageous to subtract the solution identity $H_{k+1}(\phi_*'(k)) \equiv F_k(\phi_*(k), \phi_*'(k))$ on $\mathbb{I}'$ for ϕ_* from the equation of ϕ_*-perturbed motion $(D)_{\phi_*}$ yielding the equation

$$H_{k+1}(x' + \phi_*'(k)) - H_{k+1}(\phi_*'(k)) = F_k(x + \phi_*(k), x' + \phi_*'(k)) - F_k(\phi_*(k), \phi_*'(k))$$

with the nonautonomous set $\mathcal{S} - \phi_*$ as state space. This equation has the trivial solution and under appropriate assumptions on the mappings H_{k+1}, F_k, $k \in \mathbb{I}'$, we can write it in the form (S). Indeed, this is possible in each of the settings:

- Provided H_{k+1}, F_k are continuously differentiable, one introduces

$$\begin{aligned} B_{k+1} &:= DH_{k+1}(\phi_*'(k)) - D_2 F_k(\phi_*(k), \phi_*'(k)), \\ A_k &:= D_1 F_k(\phi_*(k), \phi_*'(k)), \\ f_k(x, x') &:= -H_{k+1}(x' + \phi_*'(k)) + H_{k+1}(\phi_*'(k)) + DH_{k+1}(\phi_*'(k))x' \\ &\quad + F_k(x + \phi_*(k), x' + \phi_*'(k)) - F_k(\phi_*(k), \phi_*'(k)) \\ &\quad - D_1 F_k(\phi_*(k), \phi_*'(k))x - D_2 F_k(\phi_*(k), \phi_*'(k))x' \end{aligned}$$

  and the linear part (L_0) is the linearization of (D) along ϕ_*.
- Provided H_{k+1} satisfies the invertibility condition (2.2c) and the composition $g_k := H_{k+1}^{-1} \circ F_k$ is of class C^1, one defines

$$\begin{aligned} B_{k+1} &:= I_{X_{k+1}} - D_2 g_k(\phi_*(k), \phi_*'(k)), \\ A_k &:= D_1 g_k(\phi_*(k), \phi_*'(k)), \\ f_k(x, x') &:= g_k(x + \phi_*(k), x' + \phi_*'(k)) - D_1 g_k(\phi_*(k), \phi_*'(k))x \\ &\quad - g_k(\phi_*(k), \phi_*'(k)) \end{aligned}$$

  and the linear part (L_0) is the variational equation as in Corollary 2.3.11.

In conclusion, in order to describe the above sets $\mathcal{W}^+_{\phi_*}$ and $\mathcal{W}^-_{\phi_*}$ locally, it is possible to formulate (D) in the familiar form

$$B_{k+1}x' = A_k x + f_k(x, x'). \tag{S}$$

Yet, differing from our previous analysis, the nonlinearities f_k are not assumed to be globally Lipschitzian or to possess globally bounded derivatives as required for instance in Theorem 4.2.9 resp. Theorem 4.4.6.

Hypothesis 4.6.1. *Let $\rho_0 > 0$ and let the general forward solution φ of* (S) *exist on $\mathcal{B}_{\rho_0}$. Suppose that $f_k : X_k \times X_{k+1} \to Y_{k+1}$ with $f_k(X_k, X_{k+1}) \subseteq \operatorname{im} B_{k+1}$, $k \in \mathbb{I}'$, and:*

(i) $f_k(0,0) \equiv 0$ *on* $\mathbb{I}$.
(ii) The following limit relations hold

$$\lim_{r \searrow 0} \sup_{k \in \mathbb{I}'} \operatorname{lip}_j B^{-1}_{k+1} f_k|_{B_r(0,X_k) \times B_r(0,X_{k+1})} = 0 \quad \textit{for } j = 1, 2. \tag{4.6a}$$

When interested in differentiability results, it is reasonable to demand a smooth right-hand side of (S). However, the use of cut-off functions in order to derive local results from global ones additionally requires smooth norms and the concept of a C^m-Banach space. For a survey on such results we refer to Sect. C.2.

Hypothesis 4.6.2. *Let $m \in \mathbb{N}$. Suppose that $\mathcal{X}$ consists of C^m-Banach spaces, the mappings $B^{-1}_{k+1} f_k : X_k \times X_{k+1} \to X_{k+1}$ are of class C^m for all $k \in \mathbb{I}'$ and that the derivatives $D^n B^{-1}_{k+1} f_k : X_k \times X_{k+1} \to L_n(X_k \times X_{k+1}; X_{k+1})$ are* uniformly bounded, *i.e., for each uniformly bounded $\mathcal{B} \subseteq \mathcal{X}$ one has*

$$\sup_{k \in \mathbb{I}'} \sup_{\substack{x \in \mathcal{B}(k) \\ y \in \mathcal{B}'(k)}} \left\| D^n B^{-1}_{k+1} f_k(x,y) \right\|_{L_n(X_k \times X_{k+1}; X_{k+1})} < \infty.$$

Remark 4.6.3. From Proposition C.1.1 and (4.6a) we obtain the limit relation

$$\lim_{(x,y) \to (0,0)} D B^{-1}_{k+1} f_k(x,y) = 0 \quad \text{uniformly in } k \in \mathbb{I}. \tag{4.6b}$$

Before we formulate our first result, a weaker version of the invariance notion established Definition 1.2.1 is due, which is tailor-made for fiber bundles. Given a vector bundle $\mathcal{X}_0 \subseteq \mathcal{X}$ and an open neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of 0, we say the graph

$$\mathcal{W} := \{(\kappa, \xi + w(\kappa, \xi)) \in \mathcal{X} : \xi \in \mathcal{X}_0(\kappa) \cap \mathcal{U}(\kappa)\}$$

of a given mapping $w : \mathcal{X}_0 \cap \mathcal{U} \to \mathcal{X}$ is a *locally forward invariant fiber bundle* of (S), if the implication

$$(k_0, x_0) \in \mathcal{W} \quad \Rightarrow \quad (k, \varphi(k; k_0, x_0)) \in \mathcal{W}$$

holds for all $k \geq k_0$ as long as $\varphi(k; k_0, x_0) \in \mathcal{U}(k)$. Accordingly, one speaks of a *locally invariant fiber bundle* $\mathcal{W}$, if it is locally forward invariant and for each initial pair $(k_0, x_0) \in \mathcal{W}$ the solution $\varphi(\cdot; k_0, x_0)$ has a backward continuation in $\mathcal{W}$ as long as $(k, \varphi(k; k_0, x_0)) \in \mathcal{U}$. In this context, in case $\mathcal{U} = \mathcal{X}$ we say $\mathcal{W}$ is a *global (forward) invariant fiber bundle* of (S), if the above conditions holds for all $k \geq k_0$ resp. all $k \in \mathbb{I}$. One speaks of a C^m*-fiber bundle* of (S), provided the partial derivatives $D_2^n w$ exist and are continuous for $n \in \{1, \ldots, m\}$.

Theorem 4.6.4 (**pseudo-stable and -unstable fiber bundles**). *Let $m \in \mathbb{N}$. If both Hypotheses 4.2.1 and 4.6.1 are satisfied for some $1 \leq i < N$, then there exist reals $\rho \in (0, \rho_0)$, $\gamma_0, \ldots, \gamma_m \geq 0$ such that the following holds:*

(a) For $\mathbb{I}$ unbounded above there exists a locally forward invariant bundle

$$\mathcal{W}_i^+ := \left\{(\kappa, \eta + w_i^+(\kappa, \eta)) \in \mathcal{X} : (\kappa, \eta) \in \mathcal{B}_\rho\right\}$$

of (S), where $w_i^+ : \mathcal{B}_\rho \to \mathfrak{X}$ is a Lipschitzian mapping with

$$w_i^+(\kappa, \xi) = w_i^+(\kappa, Q_1^i(\kappa)\xi) \in \mathcal{P}_1^i(\kappa) \quad \textit{for all } (\kappa, \xi) \in \mathcal{B}_\rho$$

which satisfies the invariance equation (4.2k) for all $(\kappa, \eta) \in \mathcal{B}_\rho \subseteq \mathcal{Q}_1^i$ and also $\eta_1 \in B_\rho(0, X_{\kappa+1}) \subseteq \mathcal{Q}_1^i(\kappa + 1)$. Moreover, one has

(a_1) *$w_i^+(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and $\left\|w_i^+(\kappa, \xi)\right\|_{X_\kappa} \leq \rho$ for all $(\kappa, \xi) \in \mathcal{B}_\rho$,*
(a_2) *$\operatorname{lip}_2 w_i^+ < 1$ and $\lim_{r \searrow 0} \operatorname{lip}_2 w_i^+|_{\mathcal{B}_r} = 0$,*
(a_3) *if additionally Hypothesis 4.6.2 holds and*

$$a_i^m \ll b_i, \tag{4.6c}$$

then the nonautonomous set $\mathcal{W}_i^+$ is a C^m-fiber bundle, i.e., $w_i^+ : \mathcal{B}_\rho \to \mathfrak{X}$ is of class C^m in the second variable, $D_2 w_i^+(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and

$$\left\|D_2^n w_i^+(k, x)\right\| \leq \gamma_n \quad \textit{for all } (k, x) \in \mathcal{B}_\rho,\ 0 \leq n \leq m. \tag{4.6d}$$

One denotes $\mathcal{W}_i^+$ as pseudo-stable fiber bundle *of (S).*

(b) For $\mathbb{I}$ unbounded below there exists a locally invariant bundle

$$\mathcal{W}_i^- := \left\{(\kappa, \eta + w_i^-(\kappa, \eta)) \in \mathcal{X} : (\kappa, \eta) \in \mathcal{B}_\rho\right\}$$

of (S), where $w_i^- : \mathcal{B}_\rho \to \mathfrak{X}$ is a Lipschitzian mapping with

$$w_i^-(\kappa, \xi) = w_i^-(\kappa, P_1^i(\kappa)\xi) \in \mathcal{Q}_1^i(\kappa) \quad \textit{for all } (\kappa, \xi) \in \mathcal{B}_\rho \tag{4.6e}$$

which satisfies the invariance equation (4.2m) for all $(\kappa, \eta) \in \mathcal{B}_\rho \subseteq \mathcal{P}_1^i$ and also $\eta_1 \in B_\rho(0, X_{\kappa+1}) \subseteq \mathcal{P}_1^i(\kappa + 1)$. Moreover, one has:

(b_1) *$w_i^-(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and $\left\|w_i^-(\kappa, \xi)\right\|_{X_\kappa} \leq \rho$ for all $(\kappa, \xi) \in \mathcal{B}_\rho$.*

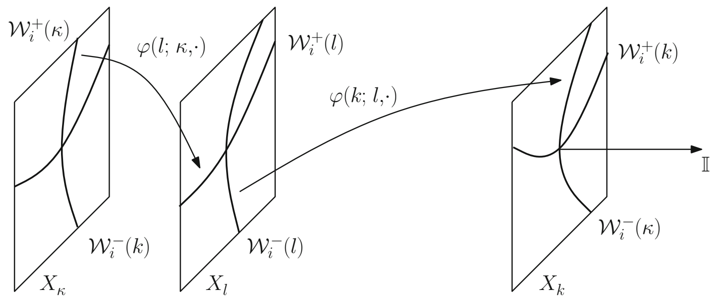

Fig. 4.2 Pseudo-stable and -unstable fiber bundles $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$

(b_2) $\operatorname{lip}_2 w_i^- < 1$ *and* $\lim_{r\searrow 0} \operatorname{lip}_2 w_i^-|_{\mathcal{B}_r} = 0$.
(b_3) *If additionally Hypothesis 4.6.2 holds and*

$$a_i \ll b_i^m, \tag{4.6f}$$

then the nonautonomous set $\mathcal{W}_i^-$ is a C^m-fiber bundle, i.e., $w_i^- : \mathcal{B}_\rho \to \mathcal{X}$ is of class C^m in the second variable, $D_2 w_i^-(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and

$$\left\| D_2^n w_i^-(k,x) \right\| \le \gamma_n \quad \textit{for all } (k,x) \in \mathcal{B}_\rho,\ 0 \le n \le m.$$

One denotes $\mathcal{W}_i^-$ as pseudo-unstable fiber bundle *of* (S).
(c) For $\mathbb{I} = \mathbb{Z}$ one has $\mathcal{W}_i^+ \cap \mathcal{W}_i^- = \mathbb{Z} \times \{0\}$.

The pseudo-stable and -unstable fiber bundles $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$ intersecting along the trivial solution are illustrated in Fig. 4.2. In the C^1-case, by $D_2 w_i^\pm(\kappa, 0) \equiv 0$ on $\mathbb{I}$, they are tangential to the invariant vector bundles $\mathcal{Q}_1^i$ resp. $\mathcal{P}_1^i$.

Remark 4.6.5. (1) If the condition $a_{i_*} \ll 1$ holds for a minimal $1 \le i_* < N$, then $\mathcal{W}_s := \mathcal{W}_{i_*}^+$ is called *stable fiber bundle* of (S) and every fiber bundle $\mathcal{W}_i^-$, $i_* < i$, in the *stable hierarchy*

$$\mathbb{I} \times \{0\} \subset \ldots \subset \mathcal{W}_{i_*+1}^+ \subset \mathcal{W}_{i_*}^+ = \mathcal{W}_s$$

is denoted as a *strongly stable fiber bundle*. One speaks of a *center-stable fiber bundle* $\mathcal{W}_{cs} := \mathcal{W}_{i_*}^+$, provided $1 \le a_{i_*}$ and thus $1 \ll b_{i_*}$. Under our Hypothesis 4.6.2 with $a_{i_*}^m \ll b_{i_*}$, all members $\mathcal{W}_i^+$, $i_* \le i$, fulfill (4.6c) and are C^m-fiber bundles.

(2) The dual situation occurs, if there exists a maximal index j_* with $1 \ll b_{j_*}$. Then $\mathcal{W}_u := \mathcal{W}_{j_*}^-$ is the *unstable fiber bundle* of (S) and every fiber bundle of the unstable hierarchy

$$\mathbb{I} \times \{0\} \subset \ldots \subset \mathcal{W}_{j_*-1}^- \subset \mathcal{W}_{j_*}^- = \mathcal{W}_u$$

is a *strongly unstable fiber bundle*. A *center-unstable fiber bundle* $\mathcal{W}_{cu} := \mathcal{W}_{j_*}^-$ occurs for a spectral gap with $b_{j_*} \le 1$ and hence $a_{j_*} \ll 1$. Under Hypothesis 4.6.2 with $a_{j_*} \ll b_{j_*}^m$, all members $\mathcal{W}_j^-$, $j \le j_*$, fulfill (4.6f) and are C^m-fiber bundles.

(3) The above fiber bundles $\mathcal{W}_i^{\pm}$ are associated to the trivial solution of (S). Consequently, the nonautonomous sets $\phi_* + \mathcal{W}_i^+$, $\phi_* + \mathcal{W}_i^-$ are denoted as *pseudo-stable* resp. *-unstable* fiber bundle of the solution ϕ_*.

(4) For a p-periodic equation (S) the fiber bundles $\mathcal{W}_i^{\pm}$ are also p-periodic. In particular, for autonomous equations (S) the fibers are constant and one calls $\mathcal{W}_i^+(\kappa)$ a *pseudo-stable manifold* and every $\mathcal{W}_i^-(\kappa)$ a *pseudo-unstable manifold*.

(5) In a differentiable setting of Hypothesis 4.6.2 one has explicit constants $\gamma_0 = \rho$ and $\gamma_1 = 1$. In general, the radius $\rho > 0$ depends on $m \in \mathbb{N}$ as well. This is due to the fact that $\varsigma_i^+(m)$ might decay to 0 as m increases (cf. Remark 4.4.7), which makes the spectral gap condition $(G_{i,m}^+)$ increasingly restrictive.

Proof. (I) Above all, let $r_{X_k} : X_k \to \bar{B}_1(0)$ denote the radial retraction on X_k, $k \in \mathbb{I}$ (cf. Lemma C.2.1). In order to obtain Lipschitzian extensions we define the constant $r_{\mathcal{X}}^* := \sup_{k \in \mathbb{I}} \operatorname{lip} r_{X_k}$ and remark that Lemma C.2.1 guarantees the relation $r_{\mathcal{X}}^* \in [1, 2]$. For $\rho \in (0, \rho_0)$ we define the Lipschitz constants

$$L_i(\rho) := \sup_{k \in \mathbb{I}} \operatorname{lip}_i B_{k+1}^{-1} f_k|_{B_\rho(0,X_k) \times B_\rho(0,X_{k+1})} \stackrel{(4.6a)}{<} \infty \quad \text{for all } i = 1, 2$$

and observe from Hypothesis 4.6.1(ii) that the limit relations $\lim_{r \searrow 0} L_i(r) = 0$ hold. We consider the modified difference equation

$$B_{k+1}x' = A_k x + f_k^\rho(x, x'), \tag{$\tilde{S}$}$$

where the nonlinearities $B_{k+1}^{-1} f_k^\rho$ are globally Lipschitzian extensions of $B_{k+1}^{-1} f_k$ as provided in Proposition C.2.5. Due to $f_k^\rho(0, 0) \equiv f_k(0, 0) \equiv 0$ on $\mathbb{I}'$ the growth conditions $(\varGamma_i^{\pm})$ are trivially fulfilled for all $1 \leq i < N$. Furthermore, it is possible to choose $\rho > 0$ so small that $L_2(\rho) < 1$ holds and Proposition 4.1.3 guarantees that the general forward solution $\tilde{\varphi}$ to $(\tilde{S})$ exists on $\mathcal{X}$. Since (S) and $(\tilde{S})$ coincide on the set $\mathcal{B}_\rho$, one has $\tilde{\varphi}(k; \kappa, \xi) = \varphi(k; \kappa, \xi)$ as long as $\tilde{\varphi}(k; \kappa, \xi) \in B_\rho(0)$, where $(\kappa, \xi) \in \mathcal{B}_\rho$.

(II) Let $1 \leq i < N$ and choose $\varsigma_i \in \left(0, \frac{\lfloor b_i - a_i \rfloor}{2}\right)$, $c \in \bar{\varGamma}_i$ as defined in Hypothesis 4.2.3. Via a further downsizing of $\rho > 0$ we can enforce

$$r_{\mathcal{X}}^* \frac{2 \max\left\{K_i^-, K_i^+\right\} (L_1(\rho) + \lceil b_i \rceil L_2(\rho))}{1 + 2 r_{\mathcal{X}}^* \max\left\{K_i^-, K_i^+\right\} L_2(\rho)} < \varsigma_i, \qquad \tilde{\ell}_i^{\pm}(c) < 1;$$

note here that in the definition of $\ell_i^{\mp}(c), \ell_i(c)$ in Lemma 4.2.6 one has to replace the constants L_1, L_2 by $L_1(\rho), L_2(\rho)$, respectively. Firstly, this ensures that $(\tilde{S})$ satisfies the spectral gap condition (G_i); actually it fulfills even the strengthened spectral gap condition (4.2u). In conclusion, all the assumptions of Theorem 4.2.9 are satisfied for $(\tilde{S})$ and there exist (forward) invariant fiber bundles $\tilde{\mathcal{W}}_i^{\pm}$ given as graph of a mapping $\tilde{w}_i^{\pm}$ over the vector bundles $\mathcal{P}_1^i$ resp. $\mathcal{Q}_1^i$. We now show that the mapping $w_i^{\pm} := \tilde{w}_i^{\pm}|_{\mathcal{B}_\rho}$ fulfills the assertions claimed in Theorem 4.6.4:

Indeed, since the two equations (S) and $(\tilde{S})$ coincide on $\mathcal{B}_\rho$, the nonautonomous sets $\mathcal{W}_i^\pm$ are locally (forward) invariant and the invariance equations (4.2k) resp. (4.2m) hold near the trivial solution. From Corollary 4.2.20 one deduces $w_i^\pm(\kappa,0) \equiv 0$ on $\mathbb{I}$. In case $\mathbb{I} = \mathbb{Z}$, the assertion (c) follows directly using Corollary 4.2.15. Since both the mappings $w_i^\pm$ and $\tilde{w}_i^\pm$ share the same Lipschitz constant $\tilde{\ell}_i(c) < 1$ (cf. Theorem 4.2.9 (a_2) and (b_2)), we infer from $\lim_{r\searrow 0} L_i(r) = 0$ that the limit relations in assertion (a_2) and (b_2) hold true. In addition, the estimate

$$\left\|w_i^\pm(\kappa,\xi)\right\| \leq \tilde{\ell}_i^\pm(c)\,\|\xi\| \leq \rho \quad \text{for all } (\kappa,\xi) \in \mathcal{B}_\rho$$

implies that $w_i^\pm(\kappa,0)$ has values in $B_\rho(0)$.

(III) Instead of using Proposition C.2.5 in order to modify the nonlinearity $B_{k+1}^{-1} f_k$, under Hypothesis 4.6.1 we obtain a C^m-smooth extension via Proposition C.2.17. Then $(\tilde{S})$ satisfies the assumptions of Theorem 4.4.6, provided $\rho > 0$ is sufficiently small. More precisely, we have to choose $\varsigma_i \in \left(0, \varsigma_i^\pm(m)\right)$, $c \in \bar{\Gamma}_i$, and $\rho > 0$ so small that

$$3\frac{\max\left\{K_i^-, K_i^+\right\}\left(L_1(\rho) + \lceil b_i \rceil\, L_2(\rho)\right)}{1 + 3\max\left\{K_i^-, K_i^+\right\} L_2(\rho)} < \varsigma_i$$

and this relation also holds with ρ replaced by $s\rho$ with $s > 1$ close to 1. □

The following example shows that the gap condition (4.6c) is sharp, i.e., the invariant fiber bundle $\mathcal{W}_i^+$ from Theorem 4.6.4(a) is not of class C^m in general, even if the nonlinearity f_k is a C^∞-function.

Example 4.6.6. Let $\mathcal{X} = \mathbb{Z} \times \mathbb{R}^2$ and $e = \exp(1)$. Given an integer $m \geq 2$, let us consider the planar autonomous difference equation

$$\begin{cases} x' = ex \\ y' = e^m y + e^m x^m \end{cases}, \tag{4.6g}$$

satisfying the assumptions of Theorem 4.6.4(a) in form of an exponential 2-splitting with $a_1 = e$, $b_1 = e^m$ and $K_1^\pm = 1$. Thus, there exists a pseudo-stable fiber bundle $\mathcal{W}_1^+ \subseteq \mathbb{Z} \times \mathbb{R}^2$ given as graph of a function $w_1^+ : \mathbb{Z} \times B_\rho(0) \to \mathbb{R}^2$ for some $\rho > 0$. On the other hand, for every $\gamma \in \mathbb{R}$ the sets

$$W_\gamma := \left\{(\xi,\eta) \in B_\rho(0) \setminus \{0\} : \eta = \tfrac{\xi^m}{2} \ln \xi^2 + \gamma \xi^m\right\} \cup \{0\}$$

contain the origin and are (locally) forward invariant w.r.t. (4.6g), i.e., $\mathbb{Z} \times W_\gamma$ is a forward invariant fiber bundle. Additionally, each point $(\xi,\eta) \in B_\rho(0)$, $\xi \neq 0$, is contained in exactly one of the sets W_γ, namely for $\gamma = \frac{\eta}{\xi^m} - \frac{\ln \xi^2}{2}$. Thus, the pseudo-stable fiber bundle $\mathcal{W}_1^+$ from Theorem 4.6.4(a) has the form $\mathbb{Z} \times W_{\gamma^*}$ for some $\gamma^* \in \mathbb{R}$ (see Fig. 4.3). Every fiber W_γ is graph of a C^{m-1}-function $w_\gamma(\xi) = \eta$, but w_γ fails to be m-times continuously differentiable. Note that in the present example the gap condition $a_1 < b_1^{m_s}$ is only fulfilled for $1 \leq m_s < m$.

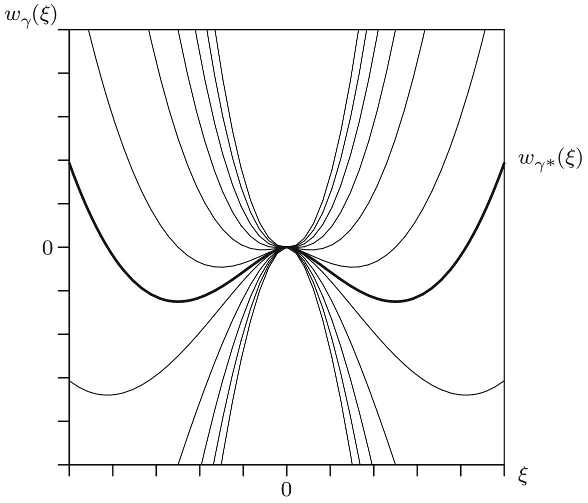

Fig. 4.3 Graphs of the functions w_γ from Example 4.6.7

Next we illustrate that center-unstable fiber bundles as postulated above in Theorem 4.6.4 need not to be uniquely determined.

Example 4.6.7. For $\mathcal{X} = \mathbb{Z} \times \mathbb{R}^2$ consider the two-dimensional autonomous equation

$$\begin{cases} x' = e^{-1}x, \\ y' = y + \frac{y^2}{1-y}, \end{cases} \tag{4.6h}$$

satisfying the assumptions of Theorem 4.6.4(b) with an exponential 2-splitting, where $K_1^+ = K_1^- = 1$, $a_1 = e^{-1}$ and $b_1 = 1$. It is easy to verify that

$$\mathcal{W}_\gamma := \left\{ (\kappa, \xi, \eta) \in \mathbb{Z} \times \mathbb{R} \times (-\infty, 1) : \xi = \gamma e^{1/\eta} \text{ for } \eta < 0 \text{ and } \xi = 0 \text{ for } \eta \geq 0 \right\}$$

is a center-unstable fiber bundle of (S) for any parameter $\gamma \in \mathbb{R}$ in the sense that $\mathcal{W}_\gamma$ is a locally invariant graph containing the zero solution and being tangential to the center-unstable vector bundle.

Our following result shows that compact 2-parameter semigroups induce unstable fiber bundles of finite-dimension. More precisely, one has

Corollary 4.6.8. *Suppose that Hypothesis 4.6.2 holds and let $\hat{\mathcal{B}}$ be the family of all uniformly bounded subsets of $\mathcal{X}$. Provided φ is $\hat{\mathcal{B}}$-contracting with*

$$q(k) := \operatorname{dar} \hat{\varphi}_k, \qquad \lim_{n \to \infty} e_q(k, k-n) = 0 \quad \text{for all } k \in \mathbb{I}',$$

then the pseudo-unstable fiber bundles $\mathcal{W}_i^-$ are finite-dimensional, if $b_i \geq 1$.

Proof. Thanks to Hypothesis 4.6.1(i) and (4.6b) the general forward solution φ to (S) and the evolution operator of (L_0) are related by $D_3\varphi(k;\kappa,0) = \Phi(k,\kappa)$, $\kappa \leq k$; for the corresponding generators this means $D\hat{\varphi}_k = \hat{\Phi}_k$, $k \in \mathbb{I}'$. Hence, our assumptions and Proposition C.1.2 imply $\operatorname{dar}\hat{\Phi}_k \leq q(k)$, $k \in \mathbb{I}'$ and consequently Corollary 1.2.29(a) guarantees that (L_0) is $\hat{\mathcal{B}}$-contracting. Thus, Proposition 3.4.24(b) implies $\dim \mathcal{P}_1^i < \infty$. □

We continue with an asymptotic description of the stable and center-stable fiber bundles, as well as of their unstable counterparts. This requires

Hypothesis 4.6.9. *Let $\mathbb{I}$ be a discrete interval, $\rho > 0$ as in Theorem 4.6.4 and suppose that $\phi_* : \mathbb{I} \to \mathcal{X}$ is a reference solution of* (D) *such that the corresponding equation (D_{ϕ_*}) can be brought into the form* (S) *satisfying Hypotheses 4.2.1 and 4.6.1.*

Corollary 4.6.10. *If Hypothesis 4.6.9 holds and $\phi : \mathbb{I} \to \mathcal{X}$ is a solution of* (D)*, then:*

(a) For $\mathbb{I}$ unbounded above:

- (a_1) *If $\phi - \phi_*$ decays exponentially in forward time, then there exists a $k_0 \in \mathbb{I}$ such that $(k, \phi(k)) \in \phi_* + \mathcal{W}_s$ for all $k \geq k_0$.*
- (a_2) *There exists a $\rho_1 \in (0, \rho)$ such that every forward solution of* (D) *starting in $\phi_* + (\mathcal{W}_s \cap \mathcal{B}_{\rho_1})$ decays exponentially in forward time.*

(b) For $\mathbb{I}$ unbounded below:

- (b_1) *If $\phi - \phi_*$ decays exponentially in backward time, then there exists a $k_0 \in \mathbb{I}$ so that $(k, \phi(k)) \in \phi_* + \mathcal{W}_u$ for all $k \leq k_0$.*
- (b_2) *There exists a $\rho_1 \in (0, \rho)$ such that every backward solution of* (D) *starting in $\phi_* + (\mathcal{W}_u \cap \mathcal{B}_{\rho_1})$ decays exponentially in backward time.*

Proof. W.l.o.g. we can assume that ϕ_* is the trivial solution of (S).

(a) We choose $1 \leq i < N$ minimal with $a_i \ll 1 \leq b_i$ and growth rates a, b with

$$a_i \ll a \ll b \ll b_i, \qquad \frac{a+b}{2} \ll 1.$$

Thus, (L_0) admits an exponential 2-splitting $\Sigma(A, B; P) = (0, a) \cup (b, \infty)$ and as in the proof of Theorem 4.6.4(a) there exists a forward invariant fiber bundle $\tilde{\mathcal{W}}^+$ of ($\tilde{S}$), consisting of forward solutions to ($\tilde{S}$) in $\mathcal{X}^+_{\kappa,c}$ with $c \ll \frac{a+b}{2}$.

(a_1) Since the solution ϕ is exponentially decaying, there exists a positive sequence $d \ll 1$ such that $\phi \in \mathcal{X}^+_{\kappa,d}$, $\kappa \in \mathbb{I}$; by an appropriate choice of a, b one has $d \leq c$. Consequently, there exists an entry time $k_0 \in \mathbb{I}$ such that $\phi(k) \in B_\rho(0)$ for $k \geq k_0$. Because the stable fiber bundle $\mathcal{W}_s$ of (S) and $\tilde{\mathcal{W}}^+$ coincide on $\mathcal{B}_\rho$, one has $(k, \phi(k)) \in \mathcal{W}_s$ for all $k \geq k_0$.

(a_2) Every initial pair $(\kappa, \xi) \in \mathcal{W}_s \cap \mathcal{B}_\rho$ is contained in a fiber bundle $\tilde{\mathcal{W}}^+$ of the modified equation ($\tilde{S}$) and moreover yields a c^+-bounded solution $\tilde{\varphi}(\cdot;\kappa,\xi)$ of ($\tilde{S}$). Due to $c \ll 1$ this solution decays exponentially in forward time. Accordingly, for sufficiently small $\rho_1 \in (0, \rho)$ one has $(k, \tilde{\varphi}(k; \kappa, \xi)) \in \mathcal{W}_s \cap \mathcal{B}_\rho$ and $\tilde{\varphi}(\cdot; \kappa, \xi)$ coincides with a solution of (S) starting in (κ, ξ).

(b) For $1 \leq i < N$ minimal with $a_i \leq 1 \ll b_i$ this can be shown analogously. □

Corollary 4.6.11. *If Hypothesis 4.6.9 holds and $\phi : \mathbb{I} \to \mathcal{X}$ is a solution of* (D)*, then:*

(a) If $\mathbb{I}$ is unbounded above and there exists a $k_0 \in \mathbb{I}$ with $(k, \phi(k)) \in \mathcal{B}_\rho(\phi_*)$ for all $k_0 \leq k$, then $(k, \phi(k)) \in \phi_* + \mathcal{W}_{cs}$ for all $k_0 \leq k$.
(b) If $\mathbb{I}$ is unbounded below and there exists a $k_0 \in \mathbb{I}$ with $(k, \phi(k)) \in \mathcal{B}_\rho(\phi_*)$ for all $k \leq k_0$, then $(k, \phi(k)) \in \phi_* + \mathcal{W}_{cu}$ for all $k \leq k_0$.

Proof. W.l.o.g. we again suppose that ϕ_* is the trivial solution of (S).

(a) First, choose $1 \leq i < N$ minimal with $1 \leq a_i$ and growth rates a, b such that $a_i \ll a \ll b \ll b_i$. Then the exponential 2-splitting $\Sigma(A, B; P) := (0, a) \cup (b, \infty)$ and the proof of Theorem 4.6.4(a) guarantees an invariant fiber bundle $\tilde{\mathcal{W}}^+$ of the modified system $(\tilde{S})$. We know that $\tilde{\mathcal{W}}^+$ consists of c^+-bounded solutions for some $1 \ll c$. If a solution $\phi : \mathbb{Z}_\kappa^+ \to \mathcal{X}$ of (S) stays in $\mathcal{B}_\rho$ for all $k \geq k_0$, then it also solves $(\tilde{S})$ and c^+-bounded (cf. Lemma 3.3.26). Hence, the solution is contained in $\tilde{\mathcal{W}}^+$ for $k \geq k_0$ and therefore on $\mathcal{W}_{cs} = \tilde{\mathcal{W}}^+ \cap \mathcal{B}_\rho$.

(b) One proceeds analogously with a minimal $1 \leq i < N$ with $b_i \leq 1$. □

Proposition 4.6.12 (pseudo-center fiber bundles). *Let $m \in \mathbb{N}$ and $\mathbb{I} = \mathbb{Z}$. Assume that Hypotheses 4.2.1 and 4.6.1 are satisfied. If (i, j) is a pair satisfying $1 < i \leq j < N$, then there exists a $\rho \in (0, \rho_0)$ such that the intersection*

$$\mathcal{W}_i^j := \mathcal{W}_{i-1}^+ \cap \mathcal{W}_j^-$$

is a locally forward invariant fiber bundle of (S)*, representable as graph*

$$\mathcal{W}_i^j = \left\{ (\kappa, \eta + w_i^j(\kappa, \eta)) \in \mathcal{X} : (\kappa, \eta) \in \mathcal{B}_\rho \right\}$$

of a Lipschitzian mapping $w_i^j : \mathcal{B}_\rho \to \mathcal{X}$ with

$$w_i^j(\kappa, \xi) = w_i^j(\kappa, P_i^j(\kappa)\xi) \in \mathcal{Q}_i^j(\kappa) \quad \textit{for all } (\kappa, \xi) \in \mathcal{B}_\rho.$$

Furthermore, it holds:

(a) $w_i^j(\kappa, 0) = 0$ on $\mathbb{I}$ and $\left\| w_i^j(\kappa, \xi) \right\|_{X_\kappa} \leq \rho$ for all $(\kappa, \xi) \in \mathcal{B}_\rho$.
(b) $\operatorname{lip}_2 w_i^j < 1$ and $\lim_{r \searrow 0} \operatorname{lip}_2 w_i^j|_{\mathcal{B}_r} = 0$.
(c) If additionally Hypothesis 4.6.2 and the conditions $a_{i-1}^m \ll b_{i-1}$, $a_j \ll b_j^m$ hold, then $w_i^j : \mathcal{B}_\rho \to \mathcal{X}$ is a C^m-mapping in the second argument, $D_2 w_i^j(\kappa, 0) \equiv 0$ on $\mathbb{I}$ and the derivatives $D_2^n w_i^j(\kappa, \cdot) : B_\rho(0, X_\kappa) \to L_n(X_\kappa)$ are globally bounded for $n \in \{2, \dots, m\}$ (uniformly in $\kappa \in \mathbb{Z}$).

One denotes $\mathcal{W}_i^j$ as pseudo-center fiber bundle *of* (S).

Remark 4.6.13 (*classical hierarchy).* For (S) possessing a linear part admitting an exponential splitting with $b_{i_*} \leq 1 \leq a_{i_*-1}$ and growth rates $a_n \ll c_n \ll b_n$, we get the following *classical invariant fiber bundles*. As long as solutions stay in $\mathcal{B}_\rho$ one can describe them asymptotically as follows:

- *Stable fiber bundle* $\mathcal{W}_s = \mathcal{W}_{i_*+1}^{i_*+1}$: Because of $c_{i_*} \ll b_{i_*}$ and the dynamical characterization all solutions on $\mathcal{W}_s$ converge to 0 exponentially for $k \to \infty$. Due to $a_{i_*}^m \ll b_{i_*}$ it is of class C^m.
- *Center-stable fiber bundle* $\mathcal{W}_{cs} = \mathcal{W}_{i_*}^{i_*+1}$: All solutions which are not growing too fast as $k \to \infty$ (in the sense that they are $c_{i_*-1}^+$-bounded with $c_{i_*-1} \leq b_{i_*-1}$) are contained in $\mathcal{W}_{cs}$, like e.g., solutions bounded in forward time.
- *Center-unstable fiber bundle* $\mathcal{W}_{cu} = \mathcal{W}_1^{i_*}$: All solutions which exist and are not growing too strong as $k \to -\infty$ (in the sense of $c_{i_*}^-$-boundedness with $a_{i_*} \leq c_{i_*}$) lie on $\mathcal{W}_{cu}$, like e.g., solutions bounded in backward time.
- *Unstable fiber bundle* $\mathcal{W}_u = \mathcal{W}_1^{i_*-1}$: All solutions on the unstable fiber bundle exist in backward time and converge exponentially to 0 as $k \to -\infty$. It is of class C^m, since $a_{i_*-1} \ll b_{i_*-1}^m$ holds.
- *Center fiber bundle* $\mathcal{W}_c := \mathcal{W}_{i_*}^{i_*}$: The center fiber bundle consists of those solutions which are contained both in the center-stable and the center-unstable fiber bundle. Particularly, bounded complete solutions in $\mathcal{B}_\rho$ lie on $\mathcal{W}_c$.

These invariant fiber bundles form the *classical hierarchy* depicted in Fig. 4.4. If a_{i_*-1} can be chosen close to 1, then the center-stable bundle $\mathcal{W}_{cs}$ is of class C^m. The same holds for the center-unstable bundle $\mathcal{W}_{cu}$ if b_{i_*} is near 1. In this sense, the classical hierarchy inherits its smoothness from (S).

Remark 4.6.14. For a p-periodic equation (S) the fiber bundles $\mathcal{W}_i^j$ are also p-periodic. In particular, for autonomous equations (S) one speaks of *invariant manifolds* and the above classical special cases are denoted as *stable*, *center-stable*, *center-unstable*, *unstable* resp. *center manifold*.

$$
\begin{array}{ccccc}
\mathcal{W}_u \subset & \mathcal{W}_{cu} & \subset & \mathcal{X} & \\
 & \cup & & \cup & \\
 & \mathcal{W}_c & \subset & \mathcal{W}_{cs} & \\
 & & & \cup & \\
 & & & \mathcal{W}_s &
\end{array}
$$

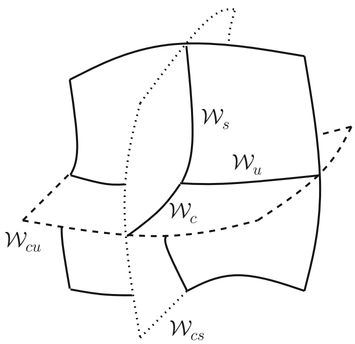

Fig. 4.4 Classical hierarchy of invariant fiber bundles (*left*) and classical invariant manifolds $\mathcal{W}_{cs}$ (*dotted*), $\mathcal{W}_{cu}$ (*dashed*) and $\mathcal{W}_u, \mathcal{W}_s, \mathcal{W}_c$ (*right*)

Proof. The argument is parallel to the proof of Theorem 4.6.4. The conditions (Γ_{i-1}^{-}) and (Γ_j^{+}) are trivially fulfilled and instead of Theorem 4.2.9 one applies the global result Theorem 4.2.17 to the modified equation $(\tilde{S})$. Here, $\rho \in (0, \rho_0)$ has to be chosen so small that in particular $(\tilde{G}_n)$ holds for $n \in \{i-1, j\}$. We omit further details. □

In the remaining section, we discuss an application of locally invariant fiber bundles to stability theory. The simplest situation is given for solutions ϕ_* admitting a hyperbolic variational equation. In absence of an unstable vector bundle, the principle of linearized stability indicated in Remark 3.5.9(2) yields (exponential) stability of ϕ_*. Conversely, if there is an unstable vector bundle, Theorem 4.6.4(b) guarantees an unstable fiber bundle and consequently the instability of ϕ_*. Between these two cases is the situation of a nonhyperbolic variational equation, where a center-unstable vector bundle exists. Then stability properties are determined by the behavior on the center-unstable fiber bundle $\mathcal{W}_{i_*}^{-}$ and therefore by the lower-dimensional $\mathcal{W}_{i_*}^{-}$-reduced equation (4.2s).

Theorem 4.6.15 (reduction principle). *Let $\mathbb{I} = \mathbb{Z}$. Suppose that Hypotheses 4.2.1 and 4.6.1 are satisfied for some $1 \le i_* < N$ with $a_{i_*} \ll 1$. The zero solution of* (S) *is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable, pullback stable, uniformly pullback stable, asymptotically pullback stable, uniformly asymptotically pullback stable, or unstable), if and only if the zero solution of the $\mathcal{W}_{i_*}^{-}$-reduced equation* (4.2s) *has the respective stability property.*

Remark 4.6.16. For difference equations (D) generating a compact general forward solution and $1 \le b_{i_*}$, we know from Corollary 4.6.8 that the $\mathcal{W}_{i_*}^{-}$-reduced equation (4.2s) is finite-dimensional.

Proof. Our assumptions guarantee that one can choose a growth rate $c \in \bar{\Gamma}_{i_*}^{-}$ with $c \ll 1$. In addition, there exists a pseudo-unstable fiber bundle $\mathcal{W}_{i_*}^{-}$ associated to the trivial solution of (S); it is graph of a function $w_{i_*}^{-}$ defined on a neighborhood $\mathcal{B}_\rho$ in $\mathcal{P}_1^{i_*}$. By construction (cf. the proof of Theorem 4.6.4(b)), $\mathcal{W}_{i_*}^{-}$ is the restriction of global fiber bundle $\tilde{\mathcal{W}}_{i_*}^{-}$ for the modified equation $(\tilde{S})$ as in the proof of Theorem 4.6.4, which is graph of a mapping $\tilde{w}_{i_*}^{-}$ and $w_{i_*}^{-} = \tilde{w}_{i_*}^{-}|_{\mathcal{B}_\rho}$. Thanks to Theorem 4.3.7(a) the invariant fiber bundle $\mathcal{W}_{i_*}^{-}$ has an asymptotic forward phase satisfying (4.3r), where our assumptions yield that $\tilde{C}_\kappa^{+}(\xi, c)$ simplifies to

$$\tilde{C}_\kappa^{+}(\xi, c) = K_{i_*}(c) \left\| Q_1^{i_*}(\kappa)\xi \right\|, \qquad K_{i_*}(c) := \tilde{\ell}_{i_*}^{-}(c)\left(1 + \tilde{\ell}_{i_*}^{+}(c)\right)$$

which, due to $\left\| Q_1^{i_*}(\kappa) \right\| \le K_{i_*}^{+}$, does not depend on $\kappa \in \mathbb{Z}$.

($\Rightarrow$) By virtue of Corollary 4.2.13, the reduced equation (4.2s) describes the dynamics of (S) on the locally invariant pseudo-unstable fiber bundle $\mathcal{W}_{i_*}^{-}$. This local invariance yields that stability properties of the zero solution for (S) carry over to (4.2s).

($\Leftarrow$) Conversely, if the zero solution of the reduced equation (4.2s) is unstable, then by invariance of $\mathcal{W}_{i_*}^-$, also the zero solution of (S) is unstable (cf. Corollary 4.2.13).

Now, let $\varepsilon > 0$, $\kappa \in \mathbb{Z}$ be given. We suppose the zero solution of (4.2s) is stable, i.e., by Definition 2.4.11 there exists a $\delta \in (0, \rho)$ so that

$$\|\phi_0(k)\| < \varepsilon \quad \text{for all } k \in \mathbb{Z}_\kappa^+ \tag{4.6i}$$

and any solution $\phi_0 : \mathbb{Z}_\kappa^+ \to \mathcal{X}$ of (4.2s) with $\phi_0(\kappa) \in B_\delta(0, \mathcal{P}_1^{i_*}(\kappa))$. In the following, let $\phi : \mathbb{Z}_\kappa^+ \to \mathcal{X}$ be an arbitrary solution of (S) with

$$\|\phi(\kappa)\| < \min\left\{\frac{\varepsilon}{3C_1}, \frac{\delta}{2C_2}\right\},$$

where $C_1 := \frac{(K_i^+)^2}{1-\ell_i(c)}\left(1 + \frac{K_i(c)}{1-\tilde{\ell}_i(c)}\right)$ and $C_2 := \frac{K_i^+ K_i(c)}{1-\tilde{\ell}_i(c)}$. Due to the asymptotic forward phase from Theorem 4.3.7(a), we establish that there exists a corresponding solution $\tilde{\phi}_0 : \mathbb{Z}_\kappa^+ \to \mathcal{X}$ of the global equation

$$B_{k+1}x' = A_k x + B_{k+1}P_1^{i_*}(k+1)B_{k+1}^{-1}f_k^\rho(x + \tilde{w}_{i_*}^-(k, x), x' + \tilde{w}_{i_*}^-(k+1, x'))$$

(cf. (4.2s)) in the pseudo-unstable vector bundle $\mathcal{P}_1^{i_*}$ with

$$\left\|\tilde{\varphi}(k; \kappa, \phi(\kappa)) - \tilde{\varphi}\big(k; \kappa, \tilde{\phi}_0(\kappa) + \tilde{w}_{i_*}^-(\kappa, \tilde{\phi}_0(\kappa))\big)\right\| \overset{(4.3r)}{\leq} C_1 \|\phi(\kappa)\| e_c(k, \kappa)$$

for all $k \in \mathbb{Z}_\kappa^+$, where $\tilde{\varphi}$ is the general solution of $(\tilde{S})$. We have from Theorem 4.3.7(a),

$$\left\|\tilde{\phi}_0(\kappa)\right\| = \left\|Q_1^{i_*}(\kappa)\pi_{i_*}^+(\kappa, \phi(\kappa))\right\| \overset{(4.3v)}{\leq} C_2 \|\phi(\kappa)\| < \delta$$

and thus (4.6i) gives us $\left\|\tilde{\phi}_0(k)\right\| < \frac{\varepsilon}{2\left(1+\tilde{\ell}_{i_*}^-(c)\right)}$ for all $k \in \mathbb{Z}_\kappa^+$. But this yields (note $e_c(k, \kappa) \leq 1$ for $k \in \mathbb{Z}_\kappa^+$) with the triangle inequality and Theorem 4.2.9(b_2),

$$\begin{aligned}
&\|\tilde{\varphi}(k; \kappa, \phi(\kappa))\| \\
&\quad \leq \left\|\tilde{\varphi}(k; \kappa, \phi(\kappa)) - \tilde{\varphi}\big(k; \kappa, \pi_{i_*}^+(\kappa, \phi(\kappa))\big)\right\| + \left\|\tilde{\varphi}\big(k; \kappa, \pi_{i_*}^+(\kappa, \phi(\kappa))\big)\right\| \\
&\quad \leq C_1 \|\phi(\kappa)\| e_c(k, \kappa) + \left\|\tilde{\phi}_0(k) + \tilde{w}_{i_*}^-(k, \tilde{\phi}_0(k))\right\| \\
&\quad \leq C_1 \|\phi(\kappa)\| + \left(1 + \tilde{\ell}_{i_*}^-(c)\right)\left\|\tilde{\phi}_0(k)\right\| < \varepsilon \quad \text{for all } k \in \mathbb{Z}_\kappa^+
\end{aligned}$$

and 0 is a stable solution of $(\tilde{S})$. However, since the systems (S) and $(\tilde{S})$ coincide on the ball $\mathcal{B}_\rho$, and due to $\tilde{\varphi}(k; \kappa, \phi(\kappa)) \in B_\rho(0)$ for all $k \in \mathbb{Z}_\kappa^+$, it is $\phi = \tilde{\varphi}(\cdot; \kappa, \phi(\kappa))$.

Thus, the zero solution is also stable w.r.t. (S). Keeping in mind that $\mathcal{W}_{i_*}^-$ is uniformly exponentially attracting (cf. (4.3r)) with constants independent of $\kappa \in \mathbb{Z}$, a similar reasoning gives us the assertion on the remaining stability properties. □

Taylor Approximation of Invariant Fiber Bundles

The striking advantage of Theorem 4.6.15 is that stability investigations can be performed using the lower-dimensional $\mathcal{W}_{i_*}^-$-reduced equation (4.2s), which for compact semigroups is even finite-dimensional (cf. Corollary 4.6.8). Yet, since its linear part is critical, stability depends on the nonlinearity, which in turn involves the center-unstable fiber bundle $\mathcal{W}_{i_*}^-$. In fact, it suffices to know the Taylor coefficients of the corresponding mapping $w_{i_*}^-$ up to a certain order.

For the remaining, we tackle this problem and describe a procedure to compute Taylor approximations of locally invariant C^m-fiber bundles and in particular of center-unstable bundles. Here, a convenient and compact notation is advisable and as in Sect. 4.6 we restrict to the case where (S) is semi-implicit

$$B_{k+1}x' = A_k x + f_k(x). \tag{S$'$}$$

Since Taylor approximations only make sense for smooth functions, we suppose that beyond Hypotheses 4.2.1, 4.6.1 also Hypothesis 4.6.2 is satisfied for (S′). In particular, its linear part admits an exponential splitting and thus the strong regularity condition (3.3j) holds. Hence, (S′) is equivalent to the explicit problem

$$x' = C_k x + \hat{f}_k(x) \tag{S$_{\hat{f}}{}'$}$$

with $C_k := B_{k+1}^{-1}A_k \in L(X_k, X_{k+1})$ and $\hat{f}_k := B_{k+1}^{-1}f_k : X_k \to X_{k+1}$, $k \in \mathbb{I}'$.

In the subsequent considerations we choose a fixed $1 \le i_* < N$ and use the brief notation introduced in (4.2p). The existence of the locally forward invariant fiber bundles $\mathcal{W}_\pm$ for (S′) or (S$_{\hat{f}}{}'$) is guaranteed by Theorem 4.6.4. It furthermore yields that the mappings $w^\pm : \mathcal{U} \to \mathfrak{X}$ are defined on an open convex neighborhood $\mathcal{U}$ of 0, are continuously differentiable in the second argument and satisfy

$$\begin{aligned} w^\pm(k,0) &\equiv 0 \quad \text{on } \mathbb{I}, \\ \lim_{x\to 0} D_2 w^\pm(k,x) &= 0 \quad \text{uniformly in } k \in \mathbb{I}, \\ w^\pm(k,x) = w^\pm(k, P_\pm(k)x) &\in \mathcal{P}_\mp(k) \quad \text{for all } (k,x) \in \mathcal{U}. \end{aligned} \tag{4.6j}$$

In our present semi-implicit setting the invariance equations (4.2k) and (4.2m) for $w^\pm$ postulated in Theorem 4.6.4 simplify to

$$\begin{aligned} &C_k w^\pm(k,\xi) + P_\mp'(k)\hat{f}_k(\xi + w^\pm(\xi,k)) \\ &\qquad = w^\pm\big(k+1, C_k\xi + P_\pm'(k)\hat{f}_k(\xi + w^\pm(\xi,k))\big) \end{aligned} \tag{4.6k}$$

for all $(k,\xi) \in \mathcal{P}_\pm \cap \mathcal{U}$ so that $C_k\xi + P_\pm'(k)\hat{f}_k(\xi + w^\pm(k,\xi)) \in \mathcal{U}(k)$.

As demonstrated in Example 4.6.7, locally invariant fiber bundles are not unique in general. However, they can be obtained as restrictions of uniquely determined global fiber bundles of appropriately modified difference equations, and calculated using Taylor expansions. We will show this under the mild assumption

$$\sup_{k\in\mathbb{I}} \left\|B_{k+1}^{-1}A_k\right\|_{L(X_k,X_{k+1})} < \infty. \tag{4.6l}$$

Proposition 4.6.17. *Suppose that Hypotheses 4.2.1, 4.6.1 and 4.6.2 with (4.6l) hold. If $\mathcal{W}_\pm$ denotes a locally forward invariant C^m-fiber bundle of* (S′)*, where the corresponding mapping $w^\pm : \mathcal{U} \to \mathfrak{X}$ possesses uniformly bounded derivatives $D_2^n w^\pm$ and (4.6c) (when $\mathcal{W}_+$ is considered) resp. (4.6f) (when $\mathcal{W}_-$ is considered) holds, then there exists a $\rho > 0$ and mappings $\bar{g}^\rho : \mathcal{X} \to \mathfrak{X}$, $w_\rho^\pm : \mathcal{X} \to \mathfrak{X}$ such that:*

(a) A global forward invariant C^m-fiber bundle is given by the graph

$$\mathcal{W}_\pm^\rho := \left\{(\kappa, \xi + w_\rho^\pm(\kappa,\xi)) \in \mathcal{X} : \xi \in \operatorname{im} P_\pm(\kappa)\right\}.$$

(b) $\bar{g}^\rho(k,x) = \hat{f}_k(x)$ for all $(k,x) \in \mathcal{B}_\rho$.
(c) $w_\rho^\pm(k,x) = w^\pm(k,x)$ for $(k,x) \in \mathcal{B}_\rho$ and $\mathcal{W}_\pm^\rho \cap \mathcal{B}_\rho = \mathcal{W}_\pm \cap \mathcal{B}_\rho$.

Proof. See [383, Proposition 3.3]. □

Our next theorem states that all locally forward invariant C^m-fiber bundles $\mathcal{W}_\pm$ of (S′) have the same Taylor series up to order m. Moreover, it enables us to calculate them using solutions of the invariance equation (4.6k).

Theorem 4.6.18 (Taylor expansion). *Suppose that Hypotheses 4.2.1, 4.6.1, 4.6.2 and (4.6l) hold. Assume that $\mathcal{W}_\pm$ denotes a locally forward invariant C^m-fiber bundle of* (S′)*, where the corresponding mapping $w^\pm : \mathcal{U} \to \mathfrak{X}$ possesses uniformly bounded derivatives $D_2^n w^\pm$ and one has (4.6c) (when $\mathcal{W}_+$ is considered) resp. (4.6f) (when $\mathcal{W}_-$ is considered). If a mapping $\omega : \mathcal{X} \to \mathfrak{X}$ is m-times continuously differentiable in the second variable and satisfies:*

(i) $\omega(k,0) \equiv 0$ on $\mathbb{I}$, $\lim_{x\to 0} D_2\omega(k,x) = 0$ uniformly in $k \in \mathbb{I}$, $D_2^n\omega$ are uniformly bounded and $\omega(k,x) = \omega(k, P_\pm(k)x) \in \mathcal{P}_\mp(k)$ for $(k,x) \in \mathcal{X}$.
(ii) With $r > 0$ so small that $x + \omega(k,x) \in \mathcal{B}_{\rho_0}(k)$ holds for all $(k,x) \in \mathcal{B}_r$, the mapping $\upsilon_k\omega : B_r(0) \to X_{k+1}$,

$$\begin{aligned}(\upsilon_k\omega)(x) := {} & C_k\omega(k,x) + P_\mp'(k)\hat{f}_k(x+\omega(k,x)) \\ & - \omega\big(k+1, C_kP_\pm(k)x + P_\pm'(k)\hat{f}_k(x+\omega(k,x))\big)\end{aligned}$$

satisfies

$$D^n(\upsilon_k\omega)(0) = 0 \quad \text{for all } n \in \{1,\dots,m\},\ k \in \mathbb{I}, \tag{4.6m}$$

then we have $D_2^n\omega(k,0) = D_2^n w^\pm(k,0)$ for all $k \in \mathbb{I}$, $n \in \{1,\dots,m\}$.

Remark 4.6.19. The assumption (i) of Theorem 4.6.18 holds for polynomials

$$\omega(k,x)=\sum_{n=2}^{m}\omega_n(k)_{P_\pm(k)}x^n$$

with bounded coefficient sequences $\omega_n : \mathbb{I} \to L_n(X_\kappa)$ satisfying the inclusion $\omega_n(k) \in L_n\big(X_k;\operatorname{im} P_\mp(k)\big)$ for all $n \in \{2,\dots,m\}$, $k \in \mathbb{I}$.

Proof. Define a C^m-diffeomorphism $\Psi_k : X_k \to X_k$, $k \in \mathbb{I}$, by $\Psi_k(x) := x - \omega(k,x)$ and the change of variables $x \mapsto \Psi_k(x)$ transforms (S′) into (S'_F) with

$$\begin{aligned}F_k(x) &:= C_k\omega(k,x)+\hat{f}_k(x+\omega(k,x))\\ &\quad -\omega\big(k+1, C_kP_\pm(k)x+P_\pm(k+1)\hat{f}_k(x+\omega(k,x))\big).\end{aligned}$$

From our assumption (i) we have $F_k(0) \equiv 0$ on $\mathbb{I}$, and a consequence of (4.6b) with (4.6l) is $\lim_{x\to 0} DF_k(x) = 0$ uniformly in $k \in \mathbb{I}$. Moreover, it follows from (3.5a) that $P'_\mp(k)F_k(x)_{P_\pm(k)} = (\upsilon_k\omega)(x)$ and (4.6m) yields

$$P'_\mp(k)D^nF_k(0) \equiv 0 \quad \text{on } \mathbb{I} \text{ and for all } n \in \{1,\dots,m\}\,.$$

Also the graph $\{(\kappa, \xi + (w^\pm - \omega)(\kappa,\xi)) \in \mathcal{X} : \xi \in \operatorname{im} P_\pm(\kappa)\}$ is a locally invariant fiber bundle for (S'_F). An application of Proposition 4.6.17 to (S'_F) then guarantees the existence of a $\rho > 0$ and a mapping $w_\rho^\pm : \mathcal{X} \to \mathfrak{X}$ with $w_\rho^\pm(k,x) \equiv (w^\pm - \sigma)(k,x)$ on the ball $\mathcal{B}_\rho$. The construction of the mapping $w_\rho^\pm$ in Theorems 4.2.9 and 4.4.6 in connection with the above identity implies $D_2^n(w^\pm - \sigma)(k,0) \equiv D_2^n w_\rho^\pm(k,0) \equiv 0$ on $\mathbb{I}$ for $n \in \{2,\dots,m\}$. This proves the assertion. □

We are interested in local approximations of a mapping $w^\pm : \mathcal{U} \to \mathfrak{X}$ defining a forward invariant C^m-fiber bundle for (S′). For this purpose, Taylor's Theorem (cf. [295, p. 350]) together with (4.6j) implies the representation

$$w^\pm(k,x)=\sum_{n=2}^{m}\frac{1}{n!}w_n^\pm(k)x^n+R_m^\pm(k,x) \tag{4.6n}$$

with coefficient functions $w_n^\pm(k) \in L_n(X_k)$ given by $w_n^\pm(k) := D_2^n w^\pm(k,0)$ and a remainder $R_m^\pm$ satisfying $\lim_{x\to 0}\frac{R_m^\pm(k,x)}{\|x\|^m} = 0$. Theorem 4.6.18 guarantees that $w_n^\pm(k)$ is uniquely determined by the mapping from Theorem 4.6.4. In addition, the latter result yields that the sequences $w_n^\pm$ are bounded, i.e., one has $\|w_n^\pm(k)\| \le \gamma_n$ for all $k \in \mathbb{I}$, $n \in \{2,\dots,m\}$ with reals $\gamma_2,\dots,\gamma_m \ge 0$. The following notation is helpful:

- We introduce $W^\pm : \mathcal{U} \to \mathfrak{X}$, $W^\pm(k,x) := P_\pm(k)x + w^\pm(k,x)$, satisfying

$$D_2W^\pm(k,0) \overset{(4.6j)}{=} P_\pm(k), \qquad D_2^nW^\pm(k,0) = D_2^n w^\pm(k,0) \tag{4.6o}$$

for all $k \in \mathbb{I}$ and $n \in \{2,\dots,m\}$. Hence, for the corresponding derivatives $W_n^{\pm}(k) := D_2^n W^{\pm}(k,0)$ we have the estimates

$$\left\|W_1^{\pm}(k)\right\| \stackrel{(3.4g)}{\leq} K_{\pm}, \quad \left\|W_n^{\pm}(k)\right\| \stackrel{(4.6d)}{\leq} \gamma_n \quad \text{for all } n \in \{2,\dots,m\}. \quad (4.6p)$$

- We abbreviate $\Gamma^{\pm}(k,x) := P_{\pm}'(k)\left[C_k x + \hat{f}_k(P_{\pm}(k)x + w^{\pm}(k,x))\right]$ and the chain rule from Theorem C.1.3 yields that the corresponding partial derivatives $\Gamma_n^{\pm}(k) := D_2^n \Gamma^{\pm}(k,0)$ are given by (cf. (4.6b) and (4.6j))

$$\begin{aligned}
&\Gamma_1^{\pm}(k)x_1 \stackrel{(3.5a)}{=} C_k P_{\pm}(k)x_1,\\
&\Gamma_n^{\pm}(k)x_1\cdots x_n\\
&=\sum_{l=2}^{n}\sum_{(N_1,\dots,N_l)\in P_l^<(n)} P_{\pm}'(k)D^l\hat{f}_k(0)W_{\#N_1}^{\pm}(k)_{P_{\pm}(k)}x_{N_1}\cdots W_{\#N_l}^{\pm}(k)_{P_{\pm}(k)}x_{N_l}
\end{aligned}$$

  for all $x_1,\dots,x_n \in X_k$ and $n \in \{2,\dots,m\}$. Moreover, the uniform boundedness assumption for $D^l\hat{f}_k$ (cf. Hypothesis 4.6.2) and the estimates (**??**), (3.4g), (4.6p) imply that $\Gamma_n^{\pm}(k) \in L_n(X_k;X_{k+1})$ are bounded sequences for $n \in \{2,\dots,m\}$.

Note that both the mappings $W^{\pm}$ and $\Gamma^{\pm}$ satisfy (cf. (4.6j))

$$W^{\pm}(k,x) = W^{\pm}(k,P_{\pm}(k)x), \quad \Gamma^{\pm}(k,x) = \Gamma^{\pm}(k,P_{\pm}(k)x) \quad \text{for all } (k,x) \in \mathcal{B}_r,$$

where $r > 0$ is chosen so small that $W^{\pm}(k,x) \in B_\rho(0)$, $\Gamma^{\pm}(k,x) \in \mathcal{U}(k)$ for every $(k,x) \in \mathcal{B}_r$. From the invariance equation (4.6k) and (4.6j),

$$\begin{aligned}
&C_k w^{\pm}(k,x) + P_{\mp}'(k)\hat{f}_k(P_{\pm}(k)x + w^{\pm}(k,x))\\
&\qquad = w^{\pm}(k+1, C_k P_{\pm}(k)x + P_{\pm}'(k)\hat{f}_k(P_{\pm}(k)x + w^{\pm}(k,x)))
\end{aligned}$$

and using the notation introduced above, this reads as

$$C_k w^{\pm}(k,x) + P_{\mp}'(k)\hat{f}_k(W^{\pm}(k,x)) \equiv w^{\pm}(k+1,\Gamma^{\pm}(k,x)) \quad \text{on } \mathcal{B}_r.$$

If we differentiate this identity using Theorem C.1.3 and set $x = 0$, one gets

$$\begin{aligned}
&w_n^{\pm}(k+1)_{C_kP_{\pm}(k)}x_1\cdots x_n\\
&\quad+\sum_{l=2}^{n-1}\sum_{(N_1,\dots,N_l)\in P_l^<(n)} w_l^{\pm}(k+1)\Gamma_{\#N_1}^{\pm}(k)_{P_{\pm}(k)}x_{N_1}\cdots\Gamma_{\#N_l}^{\pm}(k)_{P_{\pm}(k)}x_{N_l}\\
&\stackrel{(C.1b)}{=} C_k w_n^{\pm}(k)_{P_{\pm}(k)}x_1\cdots x_n + P_{\mp}(k+1)\big[D^n\hat{f}_k(0)_{P_{\pm}(k)}x_1\cdots x_n\\
&\quad+\sum_{l=2}^{n-1}\sum_{(N_1,\dots,N_l)\in P_l^<(n)} D^l\hat{f}_k(0)W_{\#N_1}^{\pm}(k)_{P_{\pm}(k)}x_{N_1}\cdots W_{\#N_l}^{\pm}(k)_{P_{\pm}(k)}x_{N_l}\big]
\end{aligned}$$

for every $n \in \{2, \dots, m\}$ and $x_1, \dots, x_n \in X_k$. Therefore, we see that the Taylor coefficient $w_n^\pm : \mathbb{I} \to L_n(X_\kappa)$ is a solution of the linear difference equation

$$X'_{C_k P_\pm(k)} = C_k X_{P_\pm(k)} + H_n^\pm(k)_{P_\pm(k)}, \tag{4.6q}$$

denoted as *homological equation* for $\mathcal{W}_\pm$ with $H_n^\pm : \mathbb{I} \to L_n(X_\kappa)$ defined by

$$\begin{aligned} H_n^\pm(k) x_1 \cdots x_n :=& P_\mp(k+1) \Big[D^n \hat{f}_k(0)_{P^\pm(k)} x_1 \cdots x_n \\ &+ \sum_{l=2}^{n-1} \sum_{(N_1,\dots,N_l) \in P_l^<(n)} \big(D^l \hat{f}_k(0) W_{\#N_1}^\pm(k)_{P_\pm(k)} x_{N_1} \cdots W_{\#N_l}^\pm(k)_{P_\pm(k)} x_{N_l} \\ &- w_l^\pm(k+1) \Gamma_{\#N_1}^\pm(k)_{P_\pm(k)} x_{N_1} \cdots \Gamma_{\#N_l}^\pm(k)_{P_\pm(k)} x_{N_l} \big) \Big]. \end{aligned} \tag{4.6r}$$

Obviously, one has $H_2^\pm(k) = P'_\mp(k) D^2 \hat{f}_k(0)_{P_\pm(k)}$ and for $n \in \{3, \dots, m\}$ the values $H_n^\pm(k)$ only depend on $w_2^\pm, \dots, w_{n-1}^\pm$. This leads to the following

Theorem 4.6.20. *Suppose that Hypotheses 4.2.1, 4.6.1 and 4.6.2 with* (4.6l) *are satisfied. If* $w^\pm : \mathcal{U} \to \mathcal{X}$ *is a mapping as in Theorem 4.6.4, then one has:*

(a) For $\mathbb{I}$ *unbounded above, the coefficients* $w_n^+ : \mathbb{I} \to L_n(X_\kappa)$ *in the Taylor expansion* (4.6n) *of the mapping* $w^+ : \mathcal{U} \to \mathcal{X}$ *can be determined recursively from the* Lyapunov–Perron sums

$$w_n^+(k) = -\sum_{j=k}^{\infty} \Phi_{P_-}^-(k, j+1) B_{j+1}^{-1} H_n^+(j)_{\Phi(j,k)P_+(k)} \quad \textit{for all } 2 \le n \le m.$$

(b) For $\mathbb{I}$ *unbounded below, the coefficients* $w_n^- : \mathbb{I} \to L_n(X_\kappa)$ *in the Taylor expansion* (4.6n) *of the mapping* $w^- : \mathcal{U} \to \mathcal{X}$ *can be determined recursively from the* Lyapunov–Perron sums

$$w_n^-(k) = \sum_{j=-\infty}^{k-1} \Phi(k, j+1) B_{j+1}^{-1} H_n^-(j)_{\Phi_{P_-}^-(j,k)P_-(k)} \quad \textit{for all } 2 \le n \le m.$$

Remark 4.6.21. If (S′) is autonomous, the above Lyapunov–Perron sums are constant and the stationary solutions of the homological equation (4.6q).

Proof. In the explanations preceding Theorem 4.6.20 we have seen that the sequence $w_n^\pm : \mathbb{I} \to L_n(X_\kappa)$ is a bounded solution of the homological equation (4.6q). It follows recursively from Hypothesis 4.6.2, (4.6p), (3.4g) and (4.6r) that each inhomogeneity $H_n^\pm$ is bounded, i.e., 1-bounded. Consequently, due to the gap conditions (4.6c) and (4.6f), it yields from Lemma 3.5.12 that $w_n^\pm$ has the claimed appearance. □

4.7 Inertial Fiber Bundles

In the first instance, the goal of this section is to provide a discrete counterpart for the concept of an inertial manifold, which is applicable to equations

$$B_{k+1}x' = A_k x + f_k(x, x') \tag{S}$$

as studied above. Regarding this, our approach so far lacks certain features. Thus, let us reconsider the theory developed in this chapter from an applied point of view. In this context, two aspects need to be addressed:

- At least in the autonomous or periodic situation, classical spectral or Floquet theory provides sufficient criteria that the linear part of (S) meets the exponential splitting assumption Hypothesis 4.2.1. The global Lipschitz condition Hypothesis 4.2.3 on the nonlinearity f_k, however, will hardly be satisfied in relevant applications. More often the nonlinear term f_k is only Lipschitzian on bounded sets.
- The existence of inertial manifolds relies on a certain kind of dissipativity. Hence, we need appropriate counterparts of notions like absorbing sets or attractors in our nonautonomous framework. Here, the concept of pullback convergence as discussed throughout Chap. 1, will serve as the right tool.

To incorporate these two points into our theory, we weaken Hypothesis 4.2.3 or 4.3.1 by imposing the following

Hypothesis 4.7.1. *Let $L_1, L_2 : [0,\infty) \to [0,\infty)$ denote nondecreasing upper-semicontinuous functions, suppose the general forward solution φ of* (S) *exists as a continuous mapping, $f_k : X_k \times X_{k+1} \to Y_{k+1}$ fulfills $f_k(X_k, X_{k+1}) \subseteq \operatorname{im} B_{k+1}$ for all $k \in \mathbb{Z}$ and that the following local Lipschitz estimates are satisfied: For each $r > 0$ one has*

$$L_j(r) := \sup_{k\in\mathbb{Z}} \operatorname{lip}_j B_{k+1}^{-1} f_k|_{B_r(0,X_k)\times B_r(0,X_{k+1})} < \infty \quad \text{for } j = 1, 2. \tag{4.7a}$$

Next we suppose (S) is uniformly bounded dissipative as follows:

Hypothesis 4.7.2. *Let $\rho > 0$ and $\hat{\mathcal{B}}$ be the family of uniformly bounded subsets of $\mathcal{X}$. Suppose that* (S) *has a $\hat{\mathcal{B}}$-uniformly absorbing set $\mathcal{A} \subseteq \mathcal{X}$, i.e., for every $\mathcal{B} \in \hat{\mathcal{B}}$ there exists an $M = M(\mathcal{B}) \in \mathbb{Z}_0^+$ with*

$$\varphi(k; k-n, \mathcal{B}(k-n)) \subseteq \mathcal{A}(k) \quad \text{for all } k \in \mathbb{Z},\ n \geq M.$$

In order to obtain Lipschitzian extensions we define the constants

$$r_{\mathcal{X}}^* := \sup_{k\in\mathbb{Z}} \operatorname{lip} r_{X_k}$$

and remark that Lemma C.2.1 guarantees $r_{\mathcal{X}}^* \in [1, 2]$.

Theorem 4.7.3 (inertial fiber bundles). *Let $\mathbb{I} = \mathbb{Z}$. Assume that Hypotheses 4.2.1, 4.7.1 and 4.7.2 are satisfied for some $1 \le i < N$ and that the boundedness condition $\sup_{k\in\mathbb{Z}} \left\| B_{k+1}^{-1} A_k P_1^i(k) \right\|_{L(X_k, X_{k+1})} < \infty$ holds. Beyond that suppose*

$$r_{\mathcal{X}}^* L_2(\rho) < 1, \tag{4.7b}$$

$$r_{\mathcal{X}}^* K_i^- \left(L_1(\rho) + b_i(k) L_2(\rho)\right) < b_i(k) \quad \text{for all } k \in \mathbb{Z}, \tag{4.7c}$$

and the following strengthened spectral gap condition

$$r_{\mathcal{X}}^* \frac{\left(K_i^+ K_i^- + \max\left\{K_i^+, K_i^-\right\}\right)\left(L_1(\rho) + \lceil b_i \rceil L_2(\rho)\right)}{1 + r_{\mathcal{X}}^* \left(K_i^+ K_i^- + \max\left\{K_i^+, K_i^-\right\}\right) L_2(\rho)} < \varsigma_i. \tag{4.7d}$$

If (Γ_i^-) holds, then there exists a nonautonomous set $\mathcal{W}_i \subseteq \mathcal{X}$, which is forward invariant w.r.t. (S)*, and possesses the following properties:*

(a) $\mathcal{W}_i$ is graph of a mapping $w_i : \mathcal{O} \to \mathfrak{X}$ over a nonempty open set $\mathcal{O} \subseteq \mathcal{P}_1^i$, i.e., $\mathcal{W}_i = \{(\kappa, \eta + w_i(\kappa, \eta)) : (\kappa, \eta) \in \mathcal{O}\}$, $w_i(\kappa, \cdot) : \mathcal{O}(\kappa) \to \mathcal{Q}_1^i(\kappa)$ is well-defined, globally Lipschitzian with $\operatorname{lip}_2 w_i < 1$ and the invariance equation (4.2m) *holds for all $(\kappa, \eta) \in \mathcal{O}$ such that $\eta_1 \in \mathcal{O}'(\kappa)$.*

(b) The nonautonomous set $\mathcal{W}_i$ is asymptotically complete*, i.e., for every initial value pair $(\kappa, \xi) \in \mathcal{X}$ there exists a point $(\kappa_0, \eta) \in \mathcal{W}_i$ with $\kappa \le \kappa_0$ such that*

$$\|\varphi(k; \kappa, \xi) - \varphi(k; \kappa_0, \eta)\|_{X_k} \le C_i(\kappa, \xi) e_c(k, \kappa) \quad \text{for all } k \in \mathbb{Z}_{\kappa_0}^+,$$

where the constant $C_i(\kappa, \xi) \ge 0$ depends boundedly on κ, ξ and $c \in \bar{\Gamma}_i$.

Under the assumption $\dim \mathcal{P}_1^i < \infty$ we call $\mathcal{W}_i$ inertial fiber bundle *of* (S)*.*

Remark 4.7.4. (1) Note that the condition (4.7b) becomes void for semi-implicit equations (S′) and (4.7c), (4.7d) degenerate in this case to

$$r_{\mathcal{X}}^* K_i^- L_1(\rho) < b_i(k), \qquad r_{\mathcal{X}}^* \left(K_i^+ K_i^- + \max\left\{K_i^-, K_i^+\right\}\right) L_1(\rho) < \varsigma_i,$$

respectively. If $\mathcal{X}$ consists of Hilbert spaces, one has $r_{\mathcal{X}}^* = 1$ (cf. Lemma C.2.1).

(2) A typical situation, where the pseudo-unstable vector bundle $\mathcal{P}_1^i$ is finite-dimensional, arises when the linear part (L_0) is $\hat{\mathcal{B}}$-contracting, $b_i \ge 1$ and the family $\hat{\mathcal{B}}$ is as in Hypothesis 4.7.2. This is a consequence of Proposition 3.4.24(b).

(3) The inertial fiber bundle $\mathcal{W}_i$ can be seen as a nonautonomous discrete counterpart of an *inertial manifold* (cf. [432, p. 569ff]). If $\dim \mathcal{P}_1^i(\kappa) < \infty$ for one $\kappa \in \mathbb{Z}$, then Lemma 3.3.6(b) guarantees that all fibers $\mathcal{W}_i(k)$, $k \in \mathbb{Z}$, possess the same finite dimension, they are Lipschitzian, $\mathcal{W}_i$ is forward invariant w.r.t. (S) and in case $a_i + \varsigma \ll 1$ also exponentially attractive. In conclusion, the forward dynamics of (S) is equivalent to the finite-dimensional $\mathcal{W}_i$-reduced equation (4.2s), which is called *inertial form* in this context.

The usual procedure to prove Theorem 4.7.3 is to replace (S) by an appropriately modified difference equation and to apply our previous global results from both Sects. 4.2 and 4.3 to the modified equation. It then remains to show that this modification does not affect the long term dynamics.

Proof. Let $\hat{\mathcal{B}}$ be the family of uniformly bounded subsets of $\mathcal{X}$ and $\rho > 0$ be the radius of the ball $\mathcal{B}_\rho$ containing the absorbing set required in Hypothesis 4.7.2. Above all, we use Proposition C.2.5 in order to obtain a globally Lipschitzian extension $B_{k+1}^{-1} f_k^\rho$ of $B_{k+1}^{-1} f_k$, where both functions coincide on $\mathcal{B}_\rho \times \mathcal{B}_\rho$. By Hypothesis 4.7.1 this yields

$$\sup_{k\in\mathbb{Z}} \operatorname{lip}_1 B_{k+1}^{-1} f_k^\rho \le r_{\mathcal{X}}^* L_1(\rho), \qquad \sup_{k\in\mathbb{Z}} \operatorname{lip}_2 B_{k+1}^{-1} f_k^\rho \le r_{\mathcal{X}}^* L_2(\rho).$$

Having this at hand, we can focus on the modified equation

$$B_{k+1} x' = A_k x + f_k^\rho(x, x'), \tag{$\tilde{S}$}$$

which fulfills Hypothesis 4.2.1 and the Lipschitz conditions required in Hypothesis 4.2.3. In addition, by Proposition 4.1.3 the general forward solution $\tilde{\varphi}$ to ($\tilde{S}$) exists as a continuous mapping, since (4.7b) holds. This ensures Hypothesis 4.3.1 as well.

Our next goal is to apply Theorem 4.2.9(b) to ($\tilde{S}$). Indeed, by the gap condition (4.7d), Theorem 4.2.9(b) and Theorem 4.3.7(a) apply and there exists an invariant fiber bundle $\tilde{\mathcal{W}}_i^-$ of the modified equation ($\tilde{S}$), which is graph of a mapping $\tilde{w}_i^-$ over $\mathcal{P}_1^i$ with an asymptotic forward phase π_i^+. In particular, (4.7d) yields $\operatorname{lip}_2 \tilde{w}_i^- < 1$.

We now demonstrate how to derive from $\tilde{\mathcal{W}}_i^-$ a forward invariant nonautonomous set $\mathcal{W}_i$ for the initial equation (S). Since $\mathcal{A}$ is $\hat{\mathcal{B}}$-uniformly absorbing, there exists an $M = M(\mathcal{B}_\rho) \in \mathbb{N}$ such that

$$\varphi(k; k-n, \mathcal{B}_\rho(k-n)) \subseteq \mathcal{A}(k) \quad \text{for all } k \in \mathbb{Z},\ n \ge M \tag{4.7e}$$

and we define the nonautonomous set $\mathcal{B}_* \subseteq \mathcal{X}$ by its fibers

$$\mathcal{B}_*(k) := \bigcup_{n\ge M} \varphi(k; k-n, \mathcal{B}_\rho(k-n)) \quad \text{for all } k \in \mathbb{Z}.$$

Then (4.7e) implies $\mathcal{B}_* \subseteq \mathcal{A}$, $\operatorname{cl} \mathcal{B}_* \subseteq \mathcal{B}_\rho$ and the inclusion

$$\begin{aligned}
\varphi(k; l, \mathcal{B}_*(l)) &= \varphi\Big(k; l, \bigcup_{n\ge M} \varphi(l; l-n, \mathcal{B}_\rho(l-n))\Big) \\
&\overset{(2.3a)}{\subseteq} \bigcup_{n\ge M} \varphi(k; l-n, \mathcal{B}_\rho(l-n)) \\
&= \bigcup_{n\ge M+k-l} \varphi(k; k-n, \mathcal{B}_\rho(k-n)) \subseteq \mathcal{B}_*(k) \quad \text{for all } l \le k,
\end{aligned} \tag{4.7f}$$

which yields $\varphi(k;l,\cdot)|_{\mathcal{B}_*(l)} = \tilde{\varphi}(k;l,\cdot)|_{\mathcal{B}_*(l)}$ for all $l \leq k$, and $\mathcal{B}_*$ is also attracting for the initial equation (S). Now define $\mathcal{W}_i^* := \tilde{\mathcal{W}}_i^- \cap \mathcal{B}_*$ and we obtain

$$\begin{aligned}\varphi(k;\kappa,\mathcal{W}_i^*(\kappa)) &= \tilde{\varphi}(k;\kappa,\mathcal{W}_i^*(\kappa)) \subseteq \tilde{\varphi}(k;\kappa,\tilde{\mathcal{W}}_i^-(\kappa)) \cap \tilde{\varphi}(k;\kappa,\mathcal{B}_*(\kappa)) \\ &\subseteq \tilde{\mathcal{W}}_i^-(k) \cap \mathcal{B}_*(k) = \mathcal{W}_i^*(k) \quad \text{for all } k \in \mathbb{Z}_\kappa^+,\end{aligned}$$

so that $\mathcal{W}_i^*$ is forward invariant w.r.t. the initial equation (S), as well as $(\tilde{S})$.

Choose $\varepsilon > 0$ so small that the open ε-neighborhood $\mathcal{B}_\varepsilon(\mathcal{B}_*)$ of $\mathcal{B}_*$ is contained in $\mathcal{B}_\rho$ and set $\mathcal{W}_i^\varepsilon := \tilde{\mathcal{W}}_i^- \cap \mathcal{B}_\varepsilon(\mathcal{B}_*)$. Then $\mathcal{W}_i^\varepsilon$ is an open neighborhood of $\mathcal{W}_i^*$ in $\tilde{\mathcal{W}}_i^-$ and due to the uniform continuity of $\tilde{\varphi}(k;\kappa,\cdot)$ in $k-\kappa \leq M$ (see the Lipschitz estimate (4.2t) in Corollary 4.2.13, which can be applied due to (4.7c)), we obtain a $\delta > 0$ such that the open δ-neighborhood $\mathcal{W}_i^\delta$ of $\mathcal{W}_i^*$ in $\tilde{\mathcal{W}}_i^-$ satisfies

$$\tilde{\varphi}(k;\kappa,\mathcal{W}_i^\delta(\kappa)) \subseteq \mathcal{W}_i^\varepsilon(k) \quad \text{for all } k-\kappa \leq M.$$

Thus, using the above inclusion (4.7e) we obtain $\varphi(k;\kappa,\mathcal{W}_i^\delta(\kappa)) \subseteq \mathcal{W}_i^\varepsilon(k)$ and $\varphi(k;\kappa,\mathcal{W}_i^\delta(\kappa)) = \tilde{\varphi}(k;\kappa,\mathcal{W}_i^\delta(\kappa))$ for all $k \in \mathbb{Z}_\kappa^+$. Let us show that $\mathcal{W}_i$, defined by

$$\mathcal{W}_i(k) := \bigcup_{n \geq 0} \varphi(k;k-n,\mathcal{W}_i^\delta(k-n)) \quad \text{for all } k \in \mathbb{Z}$$

is the desired forward invariant nonautonomous set for (S). By definition, we readily see the inclusion $\varphi(k;\kappa,\mathcal{W}_i(\kappa)) \subseteq \mathcal{W}_i(k)$ for all $k \in \mathbb{Z}_\kappa^+$, i.e., $\mathcal{W}_i$ is forward invariant w.r.t. (S).

(a) Thanks to Corollary 4.2.13 and Corollary 4.2.14, which apply due to (4.7c), we are able to deduce that the restriction $\tilde{\varphi}(k;\kappa,\cdot)|_{\tilde{\mathcal{W}}_i^-(\kappa)} : \tilde{\mathcal{W}}_i^-(\kappa) \to \tilde{\mathcal{W}}_i^-(k)$ is a homeomorphism (indeed a Lipeomorphism), so that it sends open subsets of $\tilde{\mathcal{W}}_i^-(\kappa)$ into open sets of $\tilde{\mathcal{W}}_i^-(k)$. Thus, $\varphi(k;\kappa,\mathcal{W}_i^\delta(\kappa)) = \tilde{\varphi}(k;\kappa,\mathcal{W}_i^\delta(\kappa))$ is open in $\mathcal{W}_i^\delta(k)$ for $k \in \mathbb{Z}_\kappa^+$, and therefore $\mathcal{W}_i(k)$ and $\mathcal{W}_i$ are open in $\tilde{\mathcal{W}}_i^-(k)$ and $\tilde{\mathcal{W}}_i^-$, respectively. Due to the fact that $I + \tilde{w}_i^-(k,\cdot) : \mathcal{P}_1^i(k) \to \tilde{\mathcal{W}}_i^-(k)$ is a homeomorphism (see Theorem B.3.1 and note $\operatorname{lip}_2 \tilde{w}_i^- < 1$), also the set $\mathcal{O} \subseteq \mathcal{X}$, fiber-wise given by

$$\mathcal{O}(k) := \left[I + \tilde{w}_i^-(k,\cdot)\right]^{-1}(\mathcal{W}_i(k)) \quad \text{for all } k \in \mathbb{Z}$$

is open in $\mathcal{P}_1^i$. Now, if we define $w_i := \tilde{w}_i^-|_{\mathcal{O}}$, then $w_i(\kappa,\cdot) : \mathcal{O}(\kappa) \to \mathcal{Q}_1^i(\kappa)$ satisfies $\operatorname{lip}_2 w_i < 1$ and also the further claims instantly follows from the corresponding properties for $\tilde{w}_i^-$ guaranteed by Theorem 4.2.9(b).

(b) Let $(\kappa,\xi) \in \mathcal{X}$ and $c \in \bar{\Gamma}_i$. Choose $\mathcal{B} \in \hat{\mathcal{B}}$ such that $(\kappa,\xi) \in \mathcal{B}$ and we see that there exists a $N_1 = N_1(\mathcal{B}) \in \mathbb{Z}_0^+$ with $\varphi(k;k-n,\mathcal{B}(k-n)) \subseteq \mathcal{B}_*(k)$ for all $k \in \mathbb{Z}$, $n \geq N_1$. In particular, this yields $\xi_0 := \varphi(\kappa+N_1;\kappa,\xi) \in \mathcal{B}_*(\kappa+N_1)$, thanks to (4.7f) one has $\varphi(k;\kappa+N_1,\xi_0) = \tilde{\varphi}(k;\kappa+N_1,\xi_0)$,

$$\varphi(k;\kappa,\xi) \overset{(2.3a)}{=} \varphi(k;\kappa+N_1,\xi_0) = \tilde{\varphi}(k;\kappa+N_1,\xi_0) \quad \text{for all } k \geq \kappa+N_1. \tag{4.7g}$$

Due to the asymptotic forward phase of $\tilde{\mathcal{W}}_i^-$ (cf. Theorem 4.3.7(a)) there exists a point $\eta_0 \in \tilde{\mathcal{W}}_i^-(\kappa + N_1)$ such that

$$\|\tilde{\varphi}(k;\kappa+N_1,\xi_0) - \tilde{\varphi}(k;\kappa+N_1,\eta_0)\|_{X_k} \le Ce_c(k,\kappa) \tag{4.7h}$$

for all $k \ge \kappa + N_1$, where the real constant $C \ge 0$ depends boundedly on κ, ξ, as well as c. Now we choose another set $\mathcal{C} \in \hat{\mathcal{B}}$ such that $(\kappa + N_1, \eta_0) \in \mathcal{C}$. Again, there exists $N_2 = N_2(\mathcal{B}) \in \mathbb{Z}_0^+$ with $\varphi(k; k-n, \mathcal{C}(k-n)) \subseteq \mathcal{B}_*(k)$ for all $k \in \mathbb{Z}$, $n \ge N_2$, and in particular $\eta := \varphi(\kappa+N_1+N_2, \eta_0) \in \mathcal{B}_*(\kappa+N_1+N_2)$. Then, the forward invariance of $\mathcal{B}_*$ from (4.7f) implies $\varphi(k;\kappa+N_1,\eta_0) = \tilde{\varphi}(k;\kappa+N_1,\eta_0)$ and therefore

$$\varphi(k;\kappa+N_1+N_2,\eta_0) \overset{(2.3a)}{=} \varphi(k;\kappa+N_1,\eta_0) = \tilde{\varphi}(k;\kappa+N_1,\eta_0) \tag{4.7i}$$

for all $k \ge \kappa + N_1 + N_2$. Setting $\kappa_0 := \kappa + N_1 + N_2$, inserting (4.7g) and (4.7i) into the estimate (4.7h) gives us the claim (b). This finishes the proof. □

Another important feature of inertial fiber bundles is that they contain the attractor of a dissipative equation. As we know from Chap. 1, under compactness assumptions on φ, the existence of an attractor is implied by a more easily determinable absorbing set. As we will see next, this feature fits well into our theory.

Corollary 4.7.5 (attractors). *Assume $a_i + \varsigma \ll 1$. If the sequence $(\Gamma_\kappa^-(i))_{\kappa\in\mathbb{Z}}$ from (Γ_i^-) is backward tempered, then every w.r.t.* (S) *invariant nonautonomous set $\mathcal{B} \in \hat{\mathcal{B}}$ with $\mathcal{B}(k) \subseteq \bigcup_{n\ge M} \varphi(k; k-n, \mathcal{B}_\rho(k-n))$ for all $k \in \mathbb{Z}$ satisfies $\mathcal{B} \subseteq \mathcal{W}_i$. In particular, the fiber bundle $\mathcal{W}_i$ contains the global attractor $\mathcal{A}^*$ of* (S), *i.e., $\mathcal{A}^* \subseteq \mathcal{W}_i$.*

Proof. Let $k \in \mathbb{Z}$ be arbitrary. Then $\mathcal{B} \subseteq \mathcal{B}_*$ and the invariance of $\mathcal{B}$ leads to

$$\begin{aligned} h(\mathcal{B}(k), \tilde{\mathcal{W}}_i^-(k)) &= h(\varphi(k;k-n,\mathcal{B}(k-n)), \tilde{\mathcal{W}}_i^-(k)) \\ &= h(\tilde{\varphi}(k;k-n,\mathcal{B}(k-n)), \tilde{\mathcal{W}}_i^-(k)) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

due to Corollary 4.3.11. Consequently, $\mathcal{B}(k) \subseteq \operatorname{cl} \tilde{\mathcal{W}}_i^-(k)$, but $\mathcal{B} \subseteq \mathcal{B}_*$ and the inclusion $\mathcal{W}_i \supseteq \tilde{\mathcal{W}}_i^- \cap \mathcal{B}_\delta(\mathcal{B}_*)$ implies the desired relation $\mathcal{B}(k) \subseteq \mathcal{W}_i(k)$. Obviously, this holds for the special case $\mathcal{B} = \mathcal{A}^*$. □

Corollary 4.7.6 (smoothness of inertial fiber bundles). *Suppose that Hypothesis 4.6.2 holds. Under the strengthened spectral gap condition* (4.7d) *replaced by $a_i \ll b_i^m$,*

$$3\frac{\left(K_i^+K_i^- + \max\left\{K_i^+, K_i^-\right\}\right)\left(L_1(\rho) + \lceil b_i \rceil L_2(\rho)\right)}{1 + 3\left(K_i^+K_i^- + \max\left\{K_i^+, K_i^-\right\}\right)L_2(\rho)} < \varsigma_i^-(m) \tag{4.7j}$$

the mapping $w_i(\kappa,\cdot) : \mathcal{O}(\kappa) \to \mathcal{Q}_1^i(\kappa)$ is of class C^m with globally bounded derivatives (uniformly in $\kappa \in \mathbb{Z}$).

Proof. Let $s > 1$ and $\rho > 0$ as in the proof of Theorem 4.7.3. Instead of Proposition C.2.5 we use it differentiable version Proposition C.2.17 in order to obtain a C^m-smooth modification $B_{k+1}^{-1} f_k^{\rho}$ of $B_{k+1}^{-1} f_k$. It satisfies the global Lipschitz conditions

$$\sup_{k \in \mathbb{Z}} \operatorname{lip}_1 B_{k+1}^{-1} f_k^{\rho} \leq (1+2s) L_1(s\rho), \qquad \sup_{k \in \mathbb{Z}} \operatorname{lip}_2 B_{k+1}^{-1} f_k^{\rho} \leq (1+2s) L_2(s\rho)$$

and the modified equation $(\tilde{S})$ fulfills Hypotheses 4.2.1, 4.2.3, and the global smoothness assumption Hypothesis 4.4.2. Thanks to (4.7j) there exists a $s > 1$ close to 1 such that

$$(1+2s) \frac{\left(K_i^+ K_i^- + \max\left\{K_i^+, K_i^-\right\}\right)\left(L_1(s\rho) + \lceil b_i \rceil L_2(s\rho)\right)}{1 + (1+2s)\left(K_i^+ K_i^- + \max\left\{K_i^+, K_i^-\right\}\right) L_2(s\rho)} < \varsigma_i^-(m)$$

holds and we can apply Theorem 4.4.6(b). It yields a mapping $\tilde{w}_i^- : \mathcal{O} \to \mathfrak{X}$ as in the proof of Theorem 4.7.3, but now $\tilde{w}_i^-(\kappa, \cdot)$ is m-times continuously differentiable with globally bounded derivatives (uniformly in $\kappa \in \mathbb{Z}$). As above, we see that the restriction $w_i := \tilde{w}_i^-|_{\mathcal{O}}$ is the desired C^m-mapping. $\square$

4.8 Approximation of Invariant Fiber Bundles

The reduction principle from Theorem 4.6.15 is local in nature and a Taylor approximation of the center-unstable bundle is sufficient in order to apply it. The inertial fiber bundles constructed in the previous Sect. 4.7, however, are global objects having a dynamical meaning on the whole absorbing set of a semilinear equation

$$B_{k+1} x' = A_k x + f_k(x, x'). \tag{S}$$

Therefore, and for various other reasons mentioned in the introduction to this chapter, it is important to provide more global approximation techniques for invariant fiber bundles, which also work for merely Lipschitzian nonlinearities f_k. In order to meet the requirements of our nonautonomous framework, we provide an approach based on the Lyapunov–Perron method.

For given $\kappa \in \mathbb{I}$ and $K \in \mathbb{Z}_0^+ \cup \{\infty\}$ define discrete intervals

$$\mathbb{I}_\kappa^+(K) := \begin{cases} [\kappa, \kappa + K]_{\mathbb{Z}} & \text{for } K < \infty, \\ [\kappa, \infty)_{\mathbb{Z}} & \text{for } K = \infty, \end{cases} \qquad \mathbb{I}_\kappa^-(K) := \begin{cases} [\kappa - K, \kappa]_{\mathbb{Z}} & \text{for } K < \infty, \\ (-\infty, \kappa]_{\mathbb{Z}} & \text{for } K = \infty \end{cases}$$

and impose our standing assumptions for this section:

Hypothesis 4.8.1. *Suppose that the linear part* (L_0) *satisfies Hypothesis 4.2.1, for the nonlinearity* $f_k : X_k \times X_{k+1} \to Y_{k+1}$ *we require Hypothesis 4.7.1 and we assume the growth condition* $(\Gamma_i^{\pm})$ *holds for one* $1 \leq i < N$.

For a convenient notation, we keep $1 \leq i < N$ fixed as required in Hypothesis 4.8.1 and use the abbreviations established in (4.2p). Having this at hand, we can establish our necessary functional analytical framework. Given a sequence $c : \mathbb{I} \to (0, \infty)$, $\kappa \in \mathbb{I}$ and $K \in (0, \infty]_{\mathbb{Z}}$ such that $\kappa - K \in \mathbb{I}$ or $\kappa + K \in \mathbb{I}$, respectively, it is not difficult to see that the following spaces of exponentially bounded sequences

$$\mathcal{X}^{\pm}_{\kappa,c}(K) := \left\{ \phi : \mathbb{I}^{\pm}_{\kappa}(K) \to \mathcal{X} \middle| \sup_{k \in \mathbb{I}^{\pm}_{\kappa}(K)} e_c(\kappa, k) \left\| \phi(k) \right\|_{X_k} < \infty \right\} \tag{4.8a}$$

in $\mathcal{X}$ become Banach spaces w.r.t. the respective norms

$$\|\phi\|^{\pm}_{\kappa,c} := \sup_{k \in \mathbb{I}^{\pm}_{\kappa}(K)} e_c(\kappa, k) \max\left\{ \|P_-(k)\phi(k)\|_{X_k}, \|P_+(k)\phi(k)\|_{X_k} \right\}.$$

Indeed, for $K = \infty$ we can briefly write $\mathcal{X}^{\pm}_{\kappa,c} := \mathcal{X}^{\pm}_{\kappa,c}(\infty)$ in correspondence with Definition 3.3.19. Clearly, the condition $\sup_{k \in \mathbb{I}^{\pm}_{\kappa}(K)} e_c(\kappa, k) \left\| \phi(k) \right\|_{X_k} < \infty$ is always fulfilled for finite $K < \infty$.

Our idea is to find fixed points of the Lyapunov–Perron operators (4.2q) and (4.2b) in the product space $\mathcal{X}^{\pm}_{\kappa,c}(K) = \bigtimes_{k \in \mathbb{I}^{\pm}_{\kappa}(K)} X_k$, after we passed over to finite sums. This yields a problem in $\bigtimes_{k \in \mathbb{I}^{\pm}_{\kappa}(K)} X_k$ instead of in the sequence space $\mathcal{X}^{\pm}_{\kappa,c}$. As we will see in Proposition 4.8.3, a spectral gap condition still guarantees that the Lyapunov–Perron operators are contractions when passing over to finite sums.

Of central importance for our approximation purposes are the following *truncated Lyapunov–Perron operators* $T^{\pm}_{\kappa,K} : \mathcal{X}^{\pm}_{\kappa,c}(K) \times X_\kappa \to \mathcal{X}^{\pm}_{\kappa,c}(K)$, which, for a given pair $(\kappa, \xi) \in \mathcal{X}$, read as

$$T^{+}_{\kappa,K}(\phi, \xi) = \Phi(\cdot, \kappa) P_+(\kappa)\xi + \sum_{l=\kappa}^{\kappa+K-1} G_i(\cdot, l+1) B^{-1}_{l+1} f_l(\underline{\phi(l)}),$$

$$T^{-}_{\kappa,K}(\phi, \xi) = \Phi^{-}_{P_-}(\cdot, \kappa) P_-(\kappa)\xi - \sum_{l=\kappa-K}^{\kappa-1} G_i(\cdot, l+1) B^{-1}_{l+1} f_l(\underline{\phi(l)}),$$

respectively. Note that the case $K = \infty$ is explicitly allowed here and $T^{+}_{\kappa,\infty}$ is the operator T^{+}_{κ} defined in (4.2q), whereas $T^{-}_{\kappa,\infty}$ has been denoted as T^{-}_{κ} in (4.2b). The corresponding respective fixed point problems

$$\phi = \Phi(\cdot, \kappa) P_+(\kappa)\xi + \sum_{l=\kappa}^{\kappa+K-1} G_i(\cdot, l+1) B^{-1}_{l+1} f_l(\underline{\phi(l)}), \tag{LP^{+}_{K}}$$

$$\phi = \Phi^{-}_{P_-}(\cdot, \kappa) P_-(\kappa)\xi - \sum_{l=\kappa-K}^{\kappa-1} G_i(\cdot, l+1) B^{-1}_{l+1} f_l(\underline{\phi(l)}) \tag{LP^{-}_{K}}$$

in $\mathcal{X}_{\kappa,c}^{+}(K)$ resp. $\mathcal{X}_{\kappa,c}^{-}(K)$, are denoted as *truncated Lyapunov–Perron equations*. Their relation to the dynamical behavior of (S) is described in the following counterpart to Lemma 4.2.7:

Lemma 4.8.2. *Let $(\kappa, \xi) \in \mathcal{X}$ and suppose Hypothesis 4.8.1. If $\phi \in \mathcal{X}_{\kappa,c}^{\pm}$, $a_i \ll c \ll b_i$, is a sequence satisfying*

$$L_1\Big(\sup_{k \in \mathbb{I}_\kappa^\pm} \|\phi(k)\|_{X_k}\Big) < \infty, \qquad L_2\Big(\sup_{k \in \mathbb{I}_\kappa^\pm} \|\phi(k)\|_{X_k}\Big) < \infty,$$

then the following assertions are equivalent:

(a) ϕ solves the difference equation (S) *with $P_\pm(\kappa)\phi(\kappa) = \xi$.*
(b) ϕ is a fixed point of the Lyapunov–Perron equation $(LP_\infty^\pm)$.

Proof. Referring to (4.7a), we assumed that $B_{k+1}^{-1} f_k$ fulfills a Lipschitz condition in a ball containing ϕ. Hence, the proof is essentially identical to the one of Lemma 4.2.7, since the sequence g defined there satisfies $g \in \mathcal{X}_{\kappa,c,B}^{\pm}$. $\square$

Under stronger global conditions, we can establish the existence of unique solutions for the Lyapunov–Perron equations:

Proposition 4.8.3. *Let $(\kappa, \xi) \in \mathcal{X}$, $K \in \mathbb{N}$ and assume Hypothesis 4.8.1 holds with*

$$l_j := \sup_{r \geq 0} L_j(r) < \infty \quad \text{for } j = 1, 2. \tag{4.8b}$$

If the spectral gap condition

$$\exists \bar{\varsigma} \in \left(0, \frac{\lfloor b_i - a_i \rfloor}{2}\right) : \frac{\max\{K_-, K_+\}(l_1 + \lceil b_i \rceil l_2)}{1 + \max\{K_-, K_+\} l_2} < \bar{\varsigma}, \tag{4.8c}$$

is fulfilled and we have chosen a real $\varsigma \in (\max\{K_-, K_+\}(l_1 + \lceil b_i - \bar{\varsigma} \rceil l_2), \bar{\varsigma})$, then for all $c \in \bar{\Gamma}_i$ one has:

(a) The truncated Lyapunov–Perron equations $(LP_K^\pm)$ have a uniquely determined solution $\phi_{\kappa,K}^\pm(\xi) \in \mathcal{X}_{\kappa,c}^\pm(K)$, which moreover satisfy

$$\begin{aligned}\left\|\phi_{\kappa,K}^\pm(\xi)\right\|_{\kappa,c}^\pm \leq{} & \frac{\varsigma K_\pm}{\varsigma - \max\{K_-, K_+\} L(b_i - \varsigma)} \|P_\pm(\kappa)\xi\|_{X_\kappa} \\ & + \frac{\max\{K_-, K_+\} \Gamma_\kappa^\pm(i)}{\varsigma - \max\{K_-, K_+\} L(b_i - \varsigma)}.\end{aligned}$$

(b) For every initial sequence $\psi_0^\pm \in \mathcal{X}_{\kappa,c}^\pm(K)$ and $n \geq 1$ the recursively defined sequence $\psi_n^\pm := T_{\kappa,K}^\pm(\psi_{n-1}^\pm, \xi) \in \mathcal{X}_{\kappa,K}^\pm(K)$ satisfies

$$\left\|\psi_n^\pm - \phi_{\kappa,K}^\pm(\xi)\right\|_{\kappa,c}^\pm \leq \frac{q^n}{1-q} \left\|\psi_0^\pm - T_{\kappa,K}^\pm(\psi_0^\pm, \xi)\right\|_{\kappa,c}^\pm, \tag{4.8d}$$

where $q := \frac{\max\{K_-, K_+\}}{\varsigma} L(b_i - \varsigma) \in (0,1)$,
and the sequences $\phi_{\kappa,K}^\pm(\xi)$ do not depend on c, where $L(c) := l_1 + \lceil c \rceil l_2$.

Proof. Let $(\kappa,\xi)\in\mathcal{X}$ and $K\in\mathbb{N}$.

(a) We only sketch the proof, since it resembles the one of Lemmata 4.2.6 and 4.2.8. For this, the spectral gap condition (G_i) holds due to (4.8c). Consider the truncated Lyapunov–Perron operator $T^{\pm}_{\kappa,K}:\mathfrak{X}^{\pm}_{\kappa,c}(K)\times X_\kappa\to\mathfrak{X}^{\pm}_{\kappa,c}(K)$. It can be verified as (4.2h) that $T^{\pm}_{\kappa,K}$ is well-defined and satisfies the two Lipschitz estimates

$$\operatorname{lip}_1 T^{\pm}_{\kappa,K}\overset{(4.2i)}{\leq}\ell_i(c)\leq q<1,\qquad\qquad \operatorname{lip}_2 T^{\pm}_{\kappa,K}\leq K_{\pm}. \tag{4.8e}$$

By the first inequality in (4.8e) we get that $T^{\pm}_{\kappa,K}(\cdot,\xi)$ is a contraction on $\mathfrak{X}^{\pm}_{\kappa,c}(K)$, uniformly in $\xi\in X_\kappa$, and Banach's theorem (see, e.g., [295, p. 361, Lemma 1.1]) implies that there exists a unique fixed point $\phi^{\pm}_{\kappa,K}(\xi)\in\mathfrak{X}^{\pm}_{\kappa,c}(K)$. Moreover, the second inequality in (4.8e) yields the bound on $\phi^{\pm}_{\kappa,K}(\xi)$.

(b) The stated inequality (4.8d) is just the standard a priori error estimate (cf., e.g., [465, p. 17, Theorem 1.A]) for the successive iterates in Banach's fixed point theorem applied to the contraction $T^{\pm}_{\kappa,K}(\cdot,\xi):\mathfrak{X}^{\pm}_{\kappa,c}(K)\to\mathfrak{X}^{\pm}_{\kappa,c}(K)$. □

The relations (cf. (4.2r))

$$w^+(\kappa,\xi)=P_-(\kappa)\phi^+_\kappa(\kappa,\xi),\quad w^-(\kappa,\xi)=P_+(\kappa)\phi^-_\kappa(\kappa,\xi)\quad\text{for all }(\kappa,\xi)\in\mathcal{X} \tag{4.8f}$$

are central in our approach to compute the invariant fiber bundles $\mathcal{W}_\pm\subseteq\mathcal{X}$. Actually, in order to compute the functions $w^\pm$ defining $\mathcal{W}_\pm$, we solve the Lyapunov–Perron equations $(LP^{\pm}_K)$ for $K<\infty$. The corresponding error estimate for the distance between the fixed points $\phi^{\pm}_{\kappa,K}(\xi)$ and $\phi^{\pm}_\kappa(\xi)$ is deduced in

Proposition 4.8.4. *Let $(\kappa,\xi)\in\mathcal{X}$, $K\in\mathbb{N}$, suppose Hypothesis 4.8.1 holds with (4.8c) and choose ς as in Proposition 4.8.3. Then the mapping $w^\pm:\mathcal{X}\to\mathfrak{X}$ defining the fiber bundle $\mathcal{W}_\pm$ satisfies*

$$\left\|w^\pm(\kappa,\xi)-P_\mp(\kappa)\phi^{\pm}_{\kappa,K}(\kappa,\xi)\right\|_{X_\kappa}\leq C(\kappa,\xi)e_{\frac{a_i+\varsigma}{b_i-\varsigma}}(\kappa,\kappa-K), \tag{4.8g}$$

where the constant $C(\kappa,\xi)$ is linearly bounded in $\|P_\pm(\kappa)\xi\|$ and $\Gamma^{\pm}_i(\kappa)$.

Remark 4.8.5 (*spectral ratio condition*). Since the Lipschitz constant $\ell_i(c)$ from (4.8e) is supposed to be small, one can choose ς close to 0 and the decay rate $\frac{a_i+\varsigma}{b_i-\varsigma}$ in (4.8g) basically depends on the ratio $\frac{a_i}{b_i}$. Thus, we get a good approximation for small values of $K>0$ in (4.8g), if $\frac{a_i}{b_i}$ is near 0. In the autonomous situation, this means that consecutive spectral points have moduli with small quotients.

Proof. To avoid redundancy, we only prove the assertion in the pseudo-unstable situation of w^- and $\phi^-_{\kappa,K}$. We choose $(\kappa,\xi)\in\mathcal{X}$ fixed, a finite integer $K>0$ and $c\in(a_i+\varsigma,b_i-\varsigma]$. Thanks to $\varsigma<\frac{\lfloor b_i-a_i\rfloor}{2}$, we can select a $d\in[a_i+\varsigma,c)$. Suppressing the dependence on ξ, let $\phi^-_{\kappa,K}\in\mathfrak{X}^-_{\kappa,c}(K)$, $\phi^-_\kappa\in\mathfrak{X}^-_{\kappa,c}$ be the unique solutions of the respective Lyapunov–Perron equations (LP^-_K) and (LP^-_∞). Then, on the finite interval $[\kappa - K, \kappa]_{\mathbb{Z}}$, one evidently has $\phi^-_{\kappa,K}, \phi^-_\kappa|_{\mathbb{I}^-_\kappa(K)} \in \mathcal{X}^-_{\kappa,d}(K)$, and we derive two preparatory estimates. First, using the triangle inequality it is

$$\begin{aligned}
&\left\|\sum_{n=-\infty}^{\kappa-1-K} \Phi(k,n+1)P'_+(n)B^{-1}_{n+1} f_n(\underline{\phi^-_\kappa(n)})\right\| e_d(\kappa,k) \\
\leq\ &\left\|\sum_{n=-\infty}^{\kappa-1-K} \Phi(k,n+1)P'_+(n)B^{-1}_{n+1}\left[f_n(\underline{\phi^-_\kappa(n)}) - f_n(0,0)\right]\right\| e_d(\kappa,k) \\
&+\left\|\sum_{n=-\infty}^{\kappa-1-K} \Phi(k,n+1)P'_+(n)B^{-1}_{n+1} f_n(0,0)\right\| e_d(\kappa,k) \\
\stackrel{(3.4g)}{\leq}\ &K_+ \sum_{n=-\infty}^{\kappa-1-K} e_{a_i}(k,n+1)\left\|B^{-1}_{n+1}\left[f_n(\underline{\phi^-_\kappa(n)}) - f_n(0,0)\right]\right\| e_d(\kappa,k) \\
&+K_+ \sum_{n=-\infty}^{\kappa-1-K} e_{a_i}(k,n+1)\left\|B^{-1}_{n+1} f_n(0,0)\right\| e_d(\kappa,k) \\
\stackrel{(4.8b)}{\leq}\ &K_+ L(c) \sum_{n=-\infty}^{\kappa-1-K} e_{a_i}(k,n+1)\left\|\phi^-_\kappa(n)\right\| e_d(\kappa,k) \\
&+K_+ \Gamma^-_\kappa(i) \sum_{n=-\infty}^{\kappa-1-K} e_{a_i}(k,n+1) e_{b_i}(n,\kappa) e_d(\kappa,k) \\
\leq\ &K_+ L(c)\left\|\phi^-_\kappa\right\|^-_{\kappa,c} \sum_{n=-\infty}^{\kappa-1-K} e_{a_i}(k,n+1) e_c(n,\kappa) e_d(\kappa,k) \\
&+K_+ \Gamma^-_\kappa(i) \sum_{n=-\infty}^{\kappa-1-K} e_{a_i}(k,n+1) e_c(n,\kappa) e_d(\kappa,k) \\
\stackrel{(A.1d)}{\leq}\ &\frac{K_+}{\lfloor c - a_i \rfloor}\left(L(c)\left\|\phi^-_\kappa\right\|^-_{\kappa,c} + \Gamma^-_\kappa(i)\right) e_{\frac{d}{c}}(\kappa,\kappa - K) \quad \text{for all } k \in [\kappa - K, \kappa]_{\mathbb{Z}}
\end{aligned}$$

and second, also using (4.8b) one has

$$\begin{aligned}
&\left\|\sum_{n=\kappa-K}^{k-1} \Phi(k,n+1)P'_+(n)B^{-1}_{n+1}\left[f_n(\underline{\phi^-_\kappa(n)}) - f_n(\underline{\phi^-_{\kappa,K}(n)})\right]\right\| e_d(\kappa,k) \\
&\stackrel{(3.4g)}{\leq} K_+ L(d) \sum_{n=\kappa-K}^{k-1} e_{a_i}(k,n+1)\left\|\phi^-_\kappa(n) - \phi^-_{\kappa,K}(n)\right\| e_d(\kappa,k) \\
&\stackrel{(A.1d)}{\leq} \frac{K_+ L(d)}{\lfloor d - a_i \rfloor}\left\|\phi^-_\kappa - \phi^-_{\kappa,K}\right\|^-_{\kappa,d} \quad \text{for all } k \in [\kappa - K, \kappa]_{\mathbb{Z}},
\end{aligned}$$

where the sums have been evaluated using Lemma A.1.5(a). Having these two estimates at hand, we can conclude

$$\begin{aligned}
&\left\|P_+(k)\left[\phi_\kappa^-(k)-\phi_{\kappa,K}^-(k)\right]\right\| e_d(\kappa,k) \leq \\
\leq &\left\|\sum_{n=-\infty}^{\kappa-1-K}\Phi(k,n+1)P_+'(n)B_{n+1}^{-1}f_n(\underline{\phi_\kappa^-(n)})\right\| e_d(\kappa,k) \\
&+\left\|\sum_{n=\kappa-K}^{k-1}\Phi(k,n+1)P_+'(n)B_{n+1}^{-1}\cdot\left[f_n(\underline{\phi_\kappa^-(n)})-f_n(\underline{\phi_{\kappa,K}^-(n)})\right]\right\| e_d(\kappa,k) \\
\leq &\frac{K_+}{\lfloor c-a_i\rfloor}\left(L(c)\left\|\phi_\kappa^-\right\|_{\kappa,c}^- + \Gamma_\kappa^-(i)\right)e_{\frac{d}{c}}(\kappa,\kappa-K)+\frac{K_+L(d)}{\lfloor d-a_i\rfloor}\left\|\phi_\kappa^- - \phi_{\kappa,K}^-\right\|_{\kappa,d}^-
\end{aligned}$$

for all $k\in[\kappa-K,\kappa]_{\mathbb{Z}}$, and similarly by (3.4g) and (4.8b) we get

$$\left\|P_-(k)\left[\phi_\kappa^-(k)-\phi_{\kappa,K}^-(k)\right]\right\| e_d(\kappa,k) \leq \frac{K_-L(d)}{\lfloor b_i-d\rfloor}\left\|\phi_\kappa^- - \phi_{\kappa,K}^-\right\|_{\kappa,d}^-$$

for all $k\in[\kappa-K,\kappa]_{\mathbb{Z}}$. By definition of the $\|\cdot\|_{\kappa,d}^-$-norm and due to the inclusion $c,d\in\bar{\Gamma}_i$, we arrive at

$$\begin{aligned}
\left\|\phi_{\kappa,K}^- - \phi_\kappa^-\right\|_{\kappa,d}^- \leq &\frac{K_+}{\varsigma}\left(L(c)\left\|\phi_\kappa^-\right\|_{\kappa,c}^- + \Gamma_\kappa^-(i)\right)e_{\frac{d}{c}}(\kappa,\kappa-K) \\
&+\frac{L(d)}{\varsigma}\max\{K_-,K_+\}\left\|\phi_\kappa^- - \phi_{\kappa,K}^-\right\|_{\kappa,d}^-
\end{aligned}$$

and consequently (note the inequality $L(d)\max\{K_-,K_+\}<\varsigma$),

$$\begin{aligned}
&\left\|P_+(k)\left[\phi_\kappa^-(k)-\phi_{\kappa,K}^-(k)\right]\right\| e_d(\kappa,k) \\
&\leq \frac{K_+}{\varsigma-L(d)\max\{K_-,K_+\}}\left(L(c)\left\|\phi_\kappa^-\right\|_{\kappa,c}^- + \Gamma_\kappa^-(i)\right)e_{\frac{d}{c}}(\kappa,\kappa-K)
\end{aligned}$$

for all $k\in[\kappa-K,\kappa]_{\mathbb{Z}}$. Therefore, the claim follows from Lemma 4.2.8, if we use (4.2g), (4.8f) and set $k=\kappa$, $d=a_i+\varsigma$, $c=b_i-\varsigma$ in the above estimate. □

Having these error estimates at hand, we are in a position to solve the truncated fixed point equations $(LP_K^\pm)$ instead of $(LP_\infty^\pm)$ for some fixed length $K>0$ and an initial pair $(\kappa,\xi)\in\mathcal{X}$. So we reduce the infinite-dimensional problem $(LP_\infty^\pm)$ to a nonlinear algebraic equation $T_{\kappa,K}^\pm(\psi,\xi)=\psi$ in $\bigtimes_{k\in\mathbb{I}_\kappa^\pm(K)}X_k$. For Lipschitzian nonlinearities this can be done using successive iterations $\psi_n=T_{\kappa,K}^\pm(\psi_{n-1},\xi)$ for $n\in\mathbb{N}$ and an arbitrary starting sequence $\psi_0\in\mathcal{X}_{\kappa,c}^\pm(K)$. Hence, a combination of the error estimates in Propositions 4.8.3(b) and 4.8.4 yields the following

Algorithm 4.8.6 (approximation of $w^{\pm}(\kappa,\xi)$). *Choose a desired accuracy $\varepsilon > 0$, $(\kappa,\xi) \in \mathcal{P}_{\pm}$, a value ς as in Proposition 4.8.3 and $\vartheta \in (0,1)$.*

(1) Set $n := 0$, $\psi_0 := 0 \in \mathfrak{X}^{\pm}_{\kappa,c}(K)$ with an integer $K > 0$ so large that

$$C(\kappa,\xi)e_{\frac{a_i+\varsigma}{b_i-\varsigma}}(\kappa,\kappa-K) < \vartheta\varepsilon.$$

(2) Set $q := \frac{\max\{K_-,K_+\}}{\varsigma}L(b_i-\varsigma) \in (0,1)$ and choose $n_* \in \mathbb{N}$ so large that

$$\frac{K_{\pm}q^{n_*}}{1-q}\left\|T^{\pm}_{\kappa,K}(\psi_0,\xi)\right\|^{\pm}_{\kappa,c} < (1-\vartheta)\varepsilon.$$

(3) For $n = 1,\dots,n_*$ compute $\psi_n := T^{\pm}_{\kappa,K}(\psi_{n-1},\xi)$.

Remark 4.8.7. (1) By construction of this algorithm, the distance between the approximate invariant fiber bundle $P_{\mp}(\kappa)\psi_{n_*}(\kappa)$ and $w^{\pm}(\kappa,\xi)$ satisfies

$$\left\|w^{\pm}(\kappa,\xi) - P_{\mp}(\kappa)\psi_{n_*}(\kappa)\right\| < \varepsilon. \tag{4.8h}$$

For small values of the accuracy $\varepsilon > 0$, the constant K becomes large and therefore one has to iterate the truncated Lyapunov–Perron operator in a high-dimensional product space $\bigtimes_{k\in\mathbb{I}^{\pm}_{\kappa}(K)} X_k$. This might make our approach numerically difficult. Hence, we have introduced the parameter $\vartheta \in (0,1)$ in order to balance between the iteration depth n_* and the dimension of the problem.

(2) A further possible strategy to pick the initial sequence $\psi_0 \in \mathfrak{X}^{\pm}_{\kappa,c}(K)$ is as follows: One considers the nonlinear problem (S) as perturbation of the linear equation (L_0) and starts the iteration with the exact solution of the unperturbed equation. This offers one of the respective choices:

$$\psi_0(k) = \Phi(k,\kappa)\xi \quad \text{for all } k \in \mathbb{I}^{+}_{\kappa}(K), \quad \psi_0(k) = \Phi^{-}_{P_-}(k,\kappa)\xi \quad \text{for all } k \in \mathbb{I}^{-}_{\kappa}(K).$$

When it comes to concrete implementations on a computer, further simplifications are advisable. First, in order to tackle (S) numerically, $\mathcal{X}$ has to consist of finite-dimensional spaces and an initial spatial discretization is indispensable. Second, multiplying the Lyapunov–Perron equation (LP^{-}_{K}) with projections $P_+(k)$ and $P_-(k)$ implies

$$\begin{aligned}
\psi_+(k) &= \sum_{n=\kappa-K}^{k-1} \Phi(k,n+1)P'_+(n)B^{-1}_{n+1}f_n(\underline{\psi_+(n)+\psi_-(n)}),\\
\psi_-(k) &= \Phi(k,\kappa)P_-(\kappa)\xi\\
&\quad - \sum_{n=k}^{\kappa-1} \Phi(k,n+1)P'_-(n)B^{-1}_{n+1}f_n(\underline{\psi_+(n)+\psi_-(n)}),
\end{aligned}$$

resp., where we abbreviated $\psi_\pm(k)=P_\pm(k)\phi^-_{\kappa,K}(k,\xi)$. In particular, we have the relation $\psi_-(\kappa) = P_-(\kappa)\xi$. The variation of constants formula from Theorem 3.1.16(b) guarantees that ψ_- is a backward solution of the equation

$$B_{k+1}x' = A_k P_-(k)x + P'_-(k)f_k(x+\psi_+(k), x'+\psi'_+(k)),$$

and we simplified (LP^-_K) to the following system of nonlinear equations

$$\begin{aligned} \psi_+(k) &= \sum_{n=\kappa-K}^{k-1} \Phi(k,n+1)P'_+(n)B^{-1}_{n+1}f_n(\underline{\psi_+(n)+\psi_-(n)}) = 0 \\ &\qquad \text{for all } k \in [\kappa-K,\kappa]_{\mathbb{Z}}\,, \\ B_{k+1}\psi'_-(k) &= A_k\psi_-(k) + P'_-(k)f_k(\underline{\psi_+(k)+\psi_-(k)}) = 0 \\ &\qquad \text{for all } k \in [\kappa-K,\kappa-1]_{\mathbb{Z}}\,, \\ \psi_-(\kappa) &= P_-(\kappa)\xi. \end{aligned} \tag{4.8i}$$

The first equation in (4.8i) degenerates into $\psi_+(\kappa-K)=0$ for $k=\kappa-K$, which causes no confusion, since (4.8i) is used to obtain $P_\pm(k)\phi^-_\kappa(k,\xi)$ only for $k=\kappa$.

For the corresponding dual approximation method of the fiber bundle $\mathcal{W}_+$, we set $\psi_\pm(k)=P_\pm(k)\phi^+_{\kappa,K}(k,\xi)$, and using Theorem 3.1.16(a) the Lyapunov–Perron equation (LP^+_K) reduces to

$$\begin{aligned} \psi_+(\kappa) &= P_+(\kappa)\xi, \\ B_{k+1}\psi'_+(k) &= A_k\psi_+(k) + P'_+(k)f_k(\underline{\psi_+(k)+\psi_-(k)}) \\ &\qquad \text{for all } k \in [\kappa,\kappa+K-1]_{\mathbb{Z}}\,, \\ \psi_-(k) &= -\sum_{n=k}^{\kappa+K-1} \Phi(k,n+1)P'_-(n)B^{-1}_{n+1}f_n(\underline{\psi_+(n)+\psi_-(n)}) \\ &\qquad \text{for all } k \in [\kappa,\kappa+K]_{\mathbb{Z}}\,. \end{aligned} \tag{4.8j}$$

Both the (4.8i) resp. (4.8j) are nonlinear systems of algebraic equations depending on the parameter $(\kappa,\xi) \in \mathcal{P}_\pm$. Hence, they can be solved using various methods from numerical analysis:

- For merely Lipschitzian nonlinearities $B^{-1}_{k+1}f_k$, successive iteration is practicable and yields linear convergence (cf., e.g., [465, p. 17, Theorem 1.A]).
- Newton methods lead to quadratic convergence, provided the mappings $B^{-1}_{k+1}f_k$ are of class C^2. However, since the algebraic equations (4.8i) and (4.8j) are typically high-dimensional, we made better experiences using Quasi-Newton methods (cf. [384]) without an update of the Jacobian in every step.

4.9 Applications

As before, our starting point is a discretization mesh $(t_k)_{k\in\mathbb{I}}$ satisfying

$$h_k := t_{k+1} - t_k \in [\varpi T, T] \quad \text{for all } k \in \mathbb{I}'$$

and a given stepsize bound $T > 0$ and the balancing factor $\varpi \in (0, 1]$.

4.9.1 Discretized Functional Differential Equations

Let $r > 0$ and $d \in \mathbb{N}$. This subsection deals with semilinear FDEs

$$\dot{u}(t) = L(t)u_t + f(t, u_t), \tag{4.9a}$$

whose linear part $L(t) : C_r \to \mathbb{R}^d$, $t \in \mathbb{R}$, fulfills the assumptions made in Sect. 3.7.1. For the nonlinearity we suppose

Hypothesis 4.9.1. *Let $b, c \geq 0$ be reals and suppose that $f : \mathbb{R} \times C_r \to \mathbb{R}^d$ is continuous and linearly bounded $\|f(t, \psi)\| \leq b + c\,|\psi|_r$ for all $t \in \mathbb{R}$, $\psi \in C_r$.*

Next we apply the full discretization scheme established in both Sects. 2.6.1 and 3.7.1 to (4.9a). Given $\theta \in [0, 1]$ and $h = \frac{r}{N}$, this immediately yields a semilinear difference equation

$$x' = A_k(\theta, h)x + f_k(x, x') \tag{4.9b}$$

with the operators $A_k(\theta, h) \in L(C_{r,N})$ given in Sect. 3.7.1 and inducing an evolution operator $\Phi(k, l)$ on $C_{r,N}$. Moreover, the nonlinearity $f_k : C^2_{r,N} \to C_{r,N}$ is defined as in (2.6d). Since (4.9b) is an equation in $\mathcal{S} = \mathbb{I} \times C_{r,N}$ this leads us to

Proposition 4.9.2. *Let $q \in [0, 1)$ and suppose that $\mathbb{I}$ is unbounded below. If beyond Hypothesis 4.9.1 the following holds for all $k \in \mathbb{I}'$:*

(i) The general forward solution of (4.9b) exists as a continuous mapping.
(ii) There exist $\alpha \in (0, 1)$, $K \geq 1$ such that $|\Phi(k, l)| \leq K\alpha^{k-l}$ for all $l \leq k$.
(iii) $Kh\theta c < \frac{1}{2}$ and one has the estimate $\frac{\alpha+2h(1-\theta)Kc}{1-2h\theta Kc} \in [0, q]$,

then the semilinear equation (4.9b) possesses a uniformly bounded global attractor $\mathcal{A}^* \subseteq \mathbb{I} \times C_{r,N}$, which also satisfies $\mathcal{A}^*(k) \subseteq B_{\frac{hKb}{(1-q)(1-2h\theta Kc)}}(0, C_{r,N})$, $k \in \mathbb{I}$.

Remark 4.9.3. (1) Proposition 2.6.1 yields conditions for assumption (i) to hold.

(2) Since (4.9b) is uniformly bounded dissipative (see the proof below), we can prove the existence of an inertial fiber bundle along the lines of Theorem 4.7.3, provided there exists a gap in the dichotomy spectrum of (LΔE) and the function $f : \mathbb{R} \times C_r \to \mathbb{R}^d$ is Lipschitzian in the second argument with sufficiently small constant. Caused by similar investigations in the following Sects. 4.9.4 and 4.9.5, we neglect the technical details.

Proof. Let the family $\hat{\mathcal{B}}$ consists of all uniformly bounded nonautonomous sets in $\mathcal{S}$. Since $\mathcal{S}$ consists of finite-dimensional spaces, (4.9b) is $\hat{\mathcal{B}}$-contracting. From Hypothesis 4.9.1 we see for all $x, x' \in C_{r,N}$ that

$$\begin{aligned}\|f_k(x,x')\| &\le h(1-\theta)b + h\theta b + hc(1-\theta)\|x\| + hc\theta\|x'\| \\ &\le hb + 2h\max\{(1-\theta)c\|x\|, c\theta\|x'\|\} \quad \text{for all } k \in \mathbb{I}'\end{aligned}$$

and thus the condition (4.1c) holds. We conclude our assertion from Theorem 4.1.8, since the estimate (4.1a) in Hypothesis 4.1.1 was only required to obtain a continuous general forward solution. □

We conclude the subsection with an example illustrating that the assumptions of Theorem 4.1.8 do not enforce a trivial attractor for (S).

Example 4.9.4 (*discrete Krisztin–Walther equation*). Let $a \in (0,1), h > 0, N \in \mathbb{Z}_0^+$ and suppose $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing odd C^1-function fulfilling the limit relation $\lim_{x\to\pm\infty} g'(x) = 0$. We consider a scalar delay difference equation

$$x_{k+1} = ax_k + hg(x_{k-N}), \tag{4.9c}$$

which can be interpreted as an explicit Euler discretization of the Krisztin–Walther equation $\dot{x}(t) = -\alpha x(t) + g(x(t-r))$ (choose $a = 1 - h\alpha$ and h, N according to $hN = r$ for positive delays $r > 0$ and $h\alpha \in (0,1)$). Above all, we suppose $g'(0) > \frac{1-a}{h}$, which ensures that (4.9c) admits three equilibria $-x^* < 0 < x^*$. Moreover, there exists a $\xi > 0$ such that $hg'(\xi) = a^N \frac{1-a}{2}$ and we infer

$$h|g(x)| \le hg(\xi) - a^N \tfrac{1-a}{2}\xi + a^N \tfrac{1-a}{2}|x| \le hg(\xi) + a^N \tfrac{1-a}{2}|x| \quad \text{for all } x \in \mathbb{R}$$

(cf. Fig. 4.5). Following the procedure from Example 2.1.10, we write (4.9c) as an explicit autonomous difference equation

$$x' = Ax + f(x) \tag{4.9d}$$

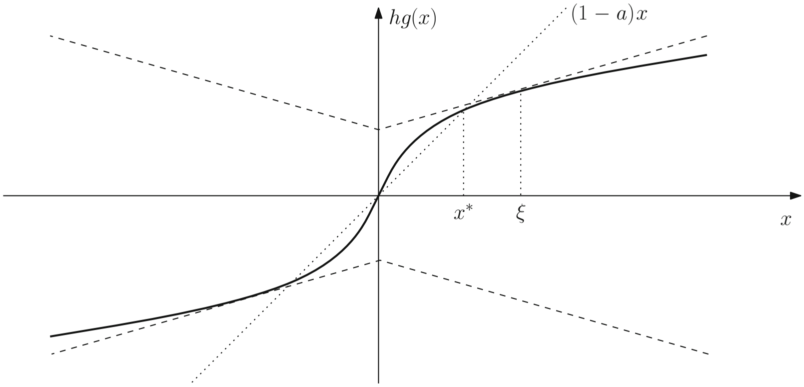

Fig. 4.5 Graph of the function g in Example 4.9.4

in the space $\mathbb{R}^{N+1}$ with

$$A := \begin{pmatrix} a & 0 & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}, \qquad f(x) := h \begin{pmatrix} g(x_{N+1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and verify the assumptions of Theorem 4.1.8, where $\mathbb{R}^{N+1}$ is equipped with the $\max$-norm. It is not difficult to show that $\|A^n\| \le a^{-N} a^n$ for all $n \in \mathbb{Z}_0^+$ and consequently the evolution operator for $x' = Ax$ satisfies $\|\Phi(k,l)\| \le a^{-N} a^{k-l}$ for all $l \le k$. For the nonlinearity one deduces

$$\|f(x)\| \le h\,|g(x)| \le hg(\xi) + a^N \tfrac{1-a}{2}\,|x_{N+1}| \quad \text{for all } x \in \mathbb{R}^{N+1}$$

and we apply Theorem 4.1.8 with $K = a^{-N}$, $\beta = hg(\xi)$ and $\gamma = a^N \frac{1-a}{2}$. Due to $a + K\gamma = \frac{1+a}{2} < 1$ we infer that (4.9d) is uniformly bounded dissipative with an absorbing set of radius $a^{-N} \frac{2h}{1-a} g(\xi)$. Hence, there exists a global attractor for (4.9d) and (4.9c), which is nontrivial, since it contains the three equilibria of (4.9c).

4.9.2 Time-Discretized Abstract Evolution Equations

Assume here that X, Y are Banach spaces with $X \subseteq Y$. We consider a temporal discretization for abstract nonautonomous evolutionary equations

$$u_t + B(t)u = f(t,u) \tag{AE}$$

in the space Y as discussed at the end of Sect. 2.6.2. Being based on mild solutions, we suppose that Hypothesis 1.5.4 is satisfied throughout. As explained in Sect. 1.5.2, for each pair $(t_0, u_0) \in \mathbb{R} \times X$ there exists a unique mild solution $u(\cdot; t_0, u_0) : [t_0, \infty) \to X$ of (AE) and $u : \{(t,s,x) \in \mathbb{R}^2 \times X : s \le t\} \to X$ is continuous. The linear part of (AE) yields an evolution family $(U(t,s))_{s \le t}$ on X.

Motivated by Sect. 2.6.2 let us investigate the explicit equation

$$x' = A_k x + f_k(x) \tag{A$\varDelta$E}$$

in $\mathcal{X} = \mathbb{I} \times X$ with mappings $A_k := U(t_{k+1}, t_k) \in L(X)$ and a nonlinearity

$$f_k : X \to X, \quad f_k(x) := \int_{t_k}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, x))\, ds \quad \text{for all } k \in \mathbb{I}'.$$

Working with semilinear equations, we first apply results from Sect. 4.1 to (A$\varDelta$E).

Lemma 4.9.5. *If Hypothesis 1.5.4 holds, then the function $f_k : X \to X$ is continuous and fulfills the linear growth bound* (4.1c) *with $\delta = 0$,*

$$\beta_k := \frac{Kb}{1-r}\left(1 + \frac{KcT^{1-r}}{1-r}E_{1-r}(\mu T)\right)h_k^{1-r}, \quad \gamma_k := \frac{K^2c}{1-r}E_{1-r}(\mu T)h_k^{1-r}$$

for all $k \in \mathbb{I}'$, where $\mu > 0$ is the constant from Lemma 1.5.5(a).

Proof. The proof is based on Lemma 1.5.5(a) and a direct estimate for $f_k : X \to X$ using the linear growth of $f : \mathbb{R} \times X \to Y$. Details are left to the reader. □

Proposition 4.9.6. *Let $q \in [0, 1)$ and suppose that $\mathbb{I}$ is unbounded below. If beyond Hypothesis 1.5.4 with $\omega < 0$ also*

(i) for every bounded $S \subseteq X$ one has $\lim_{s\to\infty} \chi(u(t; t-s, S)) = 0$, $t \in \mathbb{R}$,
(ii) the real $c \geq 0$ is so small that $e^{\omega h_k} + K\gamma_k \in (0, q]$ for all $k \in \mathbb{I}'$

hold, then the semilinear difference equation (AΔE) *has a uniformly bounded global attractor $\mathcal{A}^* \subseteq \mathbb{I} \times X$, which additionally satisfies*

$$\mathcal{A}^*(k) \subseteq B_{\frac{K \sup_{k\in\mathbb{I}'} \beta_k}{(1-q)}}(0, X) \quad \textit{for all } k \in \mathbb{I},$$

where the sequences $\beta_k, \gamma_k \geq 0$ are defined in Lemma 4.9.5.

Proof. We will successively verify the assumptions of Theorem 4.1.8. Above all, our Corollary 3.7.6 implies that (4.1b) holds with $a(k) = e^{\omega h_k}$ and Lemma 4.9.5 guarantees the linear growth bound (4.1c), where the sequence $(\beta_k)_{k\in\mathbb{I}}$ is clearly bounded. Thanks to (1.5e) and Lemma 1.5.5(a) we also see that φ is $\hat{\mathcal{B}}$-contracting, where $\hat{\mathcal{B}}$ is the family of all uniformly bounded nonautonomous sets in $\mathcal{X}$. □

Corollary 4.9.7 (parabolic case). *The assumption (i) in Proposition 4.9.6 can be replaced by $U(t, s) \in L(X)$ is compact for all $s < t$.*

Proof. We show that the assumptions of Corollary 4.1.7 are fulfilled and Proposition 4.9.6 can be applied. First, $A_k = U(t_{k+1}, t_k) \in L(X)$, $k \in \mathbb{I}'$, is compact and thus dar $A_k = 0$. Next we show that also $f_k : X \to X$ is compact. For this purpose, if $B \subseteq X$ is bounded, then Lemma 1.5.5(a) yields that $u(t; s, \cdot)$ is a bounded mapping; more precisely, there exists a $C(B) \geq 0$ (we suppress the dependence on T) with $\|u(t; s, x)\|_X \leq C(T)$ for all $x \in B$ and reals $s \leq t$ with $t - s \leq T$. From the decomposition

$$\begin{aligned} f_k(x) =& U(t_{k+1}, t_k - \varepsilon) \int_{t_k}^{t_{k+1}-\varepsilon} U(t_{k+1} - \varepsilon, s) f(s, u(s; t_k, x))\, ds \\ &+ \int_{t_{k+1}-\varepsilon}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, x))\, ds \end{aligned}$$

one has $f_k(B) \subseteq U(t_{k+1}, t_k - \varepsilon)B_1 + B_2$, where $B_1, B_2 \subseteq X$ are defined by

$$B_1 := \left\{ \int_{t_k}^{t_{k+1}-\varepsilon} U(t_{k+1} - \varepsilon, s) f(s, u(s; t_k, x))\, ds \in X : x \in B \right\},$$
$$B_2 := \left\{ \int_{t_{k+1}-\varepsilon}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, x))\, ds \in X : x \in B \right\}$$

and $\varepsilon > 0$ is arbitrary satisfying $\varepsilon < h_k$. Similarly to the argument below, one can show that $B_1 \subseteq X$ is bounded. For the set B_2 we deduce from

$$\begin{aligned}&\left\| \int_{t_{k+1}-\varepsilon}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, x))\, ds \right\|_X \\ &\quad\leq K(b+cC(B)) \int_{t_{k+1}-\varepsilon}^{t_{k+1}} e^{\omega(t_{k+1}-s)} (t_{k+1}-s)^{-r}\, ds \leq \frac{K(b + cC(B))}{1-r} \varepsilon^{1-r}\end{aligned}$$

for all $x \in B$ that $\operatorname{diam} B_2 \leq 2\frac{K(b+cC(B))}{1-r}\varepsilon^{1-r}$. We equip the Banach space X with a measure of noncompactness χ satisfying $\chi(B_2) \leq \operatorname{diam} B_2$ (cf. (B.0b)) and

$$\chi(f_k(B)) \leq \chi(U(t_{k+1}, t_k - \varepsilon)B_1) + \chi(B_2) = \chi(B_2) \leq 2\frac{K(b + cC(B))}{1-r}\varepsilon^{1-r}.$$

In the one-sided limit $\varepsilon \searrow 0$ this implies $\chi(f_k(B)) = 0$ and thus $\operatorname{dar} f_k = 0$. Consequently, Corollary 4.1.7 yields that φ is $\hat{\mathcal{B}}$-contracting. □

Example 4.9.8 (*sectorial evolutionary equation*). A typical example where the above Corollary 4.9.7 can be applied, are sectorial evolutionary equation (SE), when B has a compact resolvent. Here, X is a fractional power space $X = Y^r$.

In the remaining subsection, we illustrate how a discrete equation (AΔE) can be used to construct integral manifolds of the nonautonomous differential equation (AE). To shorten our explanations, we restrict to the pseudo-stable situation.

Theorem 4.9.9. *Let $\mathbb{I}$ be unbounded above, suppose that Hypotheses 1.5.4 and 3.7.4 are satisfied with*

$$L := \sup_{\rho > 0} \ell(\rho), \quad \sup_{k \in \mathbb{I}} \left\| \int_{t_k}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, 0))\, ds \right\|_X e^{\alpha h_k} < \infty \quad (4.9e)$$

and that $\max\left\{e^{\omega T}, E_{1-r}(\mu T), \frac{bT^{1-r}}{1-r}\right\} \leq \sqrt{2}$. *If the spectral gap condition*

$$\frac{\max\{K^+, K^-\} K^2}{1-r} \frac{E_{1-r}(\bar{\mu}T) \max\left\{e^{\omega\varpi T}, e^{\omega T}\right\} T^{1-r}}{\min\{\varpi T e^{\alpha\varpi T}, T e^{\alpha T}\}} L < \frac{\beta - \alpha}{2}$$

holds and we choose a fixed

$$\varsigma \in \left(\max\left\{K^+, K^-\right\} K^2 L E_{1-r}(\bar{\mu}T) \max\left\{e^{\omega\varpi T}, e^{\omega T}\right\} \tfrac{T^{1-r}}{1-r}, \inf_{k\in\mathbb{I}'} \tfrac{e^{\beta h_k} - e^{\alpha h_k}}{2} \right),$$

then the nonautonomous set $\mathcal{W}^+ := \left\{(\kappa,\xi) \in \mathcal{X} : \varphi(\cdot;\kappa,\xi) \in \mathfrak{X}^+_{\kappa,c}\right\}$ *is a forward invariant fiber bundle of* (AΔE)*, which is independent of* $c \in [a_1 + \varsigma, b_1 - \varsigma]$ *and has the representation* $\mathcal{W}^+ = \left\{(\kappa, \eta + w^+(\kappa,\eta)) \in \mathcal{X} : \eta \in \ker \bar{P}(t_\kappa)\right\}$ *as graph of a unique mapping* $w^+ : \mathcal{X} \to \mathfrak{X}$*, globally Lipschitzian in the second argument with* $w^+(\kappa,\xi) = w^+(\kappa, [I - \bar{P}(t_\kappa)]\xi) \in \operatorname{im} \bar{P}(t_\kappa)$ *for all* $(\kappa,\xi) \in \mathcal{X}$*.*

Here, one has $\bar{\mu} := (KL\Gamma(1-r))^{1/(1-r)}$*,* μ *is from Lemma 1.5.5 and the growth rates* a_1, b_1 *are defined in Lemma 3.7.5.*

Proof. We aim to verify the assumptions of the Hadamard–Perron Theorem 4.2.9(a). Above all, due to Lemma 3.7.5 the linear part of (AΔE) admits a strongly regular 2-splitting with $a_1(k) = e^{\alpha h_k}$, $b_1(k) := e^{\beta h_k}$ and constants K^+, K^-. Therefore, Hypothesis 4.2.1 is satisfied. In order to show that also Hypothesis 4.2.3 holds, we choose $k \in \mathbb{I}'$ and $x, \bar{x} \in X$. Then Lemma 1.5.5(b) guarantees

$$\begin{aligned}
&\|f_k(x) - f_k(\bar{x})\| \\
&\leq KL \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{-r} e^{\omega(t_{k+1}-s)} \|u(s;t_k,x) - u(s;t_k,\bar{x})\| \, ds \\
&\leq K^2 L E_{1-r}(\bar{\mu}T) e^{\omega h_k} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{-r} \, ds \, \|x - \bar{x}\|
\end{aligned}$$

for all $k \in \mathbb{I}'$ and consequently $f_k : X \to X$ satisfies a global Lipschitz condition (4.2a) with constant $L_1 = K^2 L E_{1-r}(\bar{\mu}T) \max\left\{e^{\omega\varpi T}, e^{\omega T}\right\} \frac{T^{1-r}}{1-r}$.

ad (Γ_1^+): Due to (4.9e) we have the growth condition.

ad (G_1): Since (AΔE) is an explicit equation, the gap condition simplifies to

$$\exists \varsigma \in \left(0, \tfrac{\lfloor b_1 - a_1 \rfloor}{2}\right) : \max\left\{K^+, K^-\right\} L_1 < \varsigma,$$

which we establish as follows: Using the elementary mean value theorem for scalar functions one has

$$b_1(k) - a_1(k) = e^{\beta h_k} - e^{\alpha h_k} \geq e^{\alpha h_k} h_k(\beta - \alpha) \geq \min\left\{\varpi T e^{\alpha\varpi T}, T e^{\alpha T}\right\}(\beta - \alpha)$$

for all $k \in \mathbb{I}'$ and consequently $\lfloor b_1 - a_1 \rfloor \geq \min\left\{\varpi T e^{\alpha\varpi T}, T e^{\alpha T}\right\}(\beta - \alpha)$. Thus, our assumptions imply that (G_1) holds and Theorem 4.2.9(a) yields the assertion. $\square$

For simplicity we suppose next that (AE) has the trivial solution:

Hypothesis 4.9.10. *Suppose that* $f(t,0) \equiv 0$ *holds on* $\mathbb{R}$.

Corollary 4.9.11. *Let* $\gamma := \frac{\alpha+\beta}{2}$*. If Hypothesis 4.9.10 holds, then*

$$\mathcal{W}^+ = \left\{(\tau,\xi) \in \mathcal{X} : \sup_{\tau \le t} \|u(t;\tau,\xi)\|_X \, e^{\gamma(\tau-t)} < \infty\right\}.$$

Proof. We define $c(k) := e^{\gamma h_k}$ and obtain $c \in \bar{\Gamma}_1$. Abbreviating the right-hand side of the claimed set equality by $\mathcal{W}^+_{\mathbb{R}}$, we need to show two inclusions:

($\subseteq$) Let $(\tau,x) \in \mathcal{W}^+_{\mathbb{R}}$ with $\tau = t_\kappa$ for some $\kappa \in \mathbb{I}$. Then the relation

$$\|\varphi(k;\kappa,x)\| \, e_c(\kappa,k) \overset{(1.5e)}{\le} \|u(t_k;\tau,x)\| \, e^{\gamma(t_\kappa - t_k)} \le \sup_{\tau\le t} \|u(t;\tau,x)\| \, e^{\gamma(\tau-t)} < \infty$$

for all $k \in \mathbb{Z}^+_\kappa$ implies that $\varphi(\cdot;\kappa,x)$ is c^+-bounded and the dynamical characterization of $\mathcal{W}^+$ yields the inclusion $(\tau,x) \in \mathcal{W}^+$.

($\supseteq$) Conversely, let $(\kappa,x) \in \mathcal{W}^+$ and so $\varphi(\cdot;\kappa,x)$ must be c^+-bounded. Thanks to Lemma 1.5.5(a) and the identity $f(t,0) \equiv 0$ there exists a constant $C(T) \ge 0$ with $\|u(t;s,u_0)\| \le C(T)\|u_0\|$ for all $u_0 \in X$ and $s \le t$ with $t-s \le T$. We define $\tau = t_\kappa$ and thus, given $t \ge \tau$ for $k \in \mathbb{I}$ maximal with $t_k \le t$, we obtain

$$\begin{aligned}
&\|u(t;\tau,x)\| \, e^{\gamma(\tau-t)} \\
&\quad = \|u(t;t_k,u(t_k,\tau,x))\| \, e^{\gamma(\tau-t)} \le C(T) e^{\gamma(t_k-t)} \|u(t_k,\tau,x)\| \, e^{\gamma(\tau-t_k)} \\
&\quad \le C(T) \max\left\{e^{\gamma\varpi T}, e^{\gamma T}\right\} \|\varphi(k;\kappa,x)\| \, e_c(\kappa,k) \\
&\quad \le C(T) \max\left\{e^{\gamma\varpi T}, e^{\gamma T}\right\} \|\varphi(\cdot;\kappa,x)\|^+_{\kappa,c} \quad \text{for all } \tau \le t.
\end{aligned}$$

Therefore, we arrive at the inclusion $(\tau,x) \in \mathcal{W}^+_{\mathbb{R}}$. □

4.9.3 Time-Discretized Parabolic Evolution Equations

Our set-up is as follows: Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitzian boundary. We consider a sectorial evolution equation

$$u_t + Bu = f(t,u), \tag{SE}$$

where the sectorial operator B on the Banach space $Y = L^2(\Omega)$ is a symmetric uniformly elliptic differential operator B as in Sect. 3.7.3; under Dirichlet boundary conditions we obtain the domain $D(B) = H^2(\Omega) \cap H^1_0(\Omega)$ and the fractional power space $X = Y^{1/2} = H^1_0(\Omega)$. Concerning the nonlinearity, we suppose Hypothesis 4.9.10 and that the mapping $f : \mathbb{R} \times X \to Y$ is m-times, $m \in \mathbb{N}$, continuously Frechét differentiable in the second argument such that the derivatives map bounded subsets of X into bounded sets – uniformly in $t \in \mathbb{R}$.

If B has compact resolvent, then Example 4.9.8 and Corollary 4.9.7 apply. However, our focus is different and using two examples, we briefly illustrate how Theorem 4.6.4 can be employed in order to construct a pseudo-stable and -unstable hierarchy of invariant fiber bundles for temporal discretizations of (SE).

Our first simplifying assumption is that $A :\equiv D_2 f(t,0) \in L(X,Y)$ is independent of $t \in \mathbb{R}$. We need this in order to obtain information on an exponential splitting for $u_t + (B-A)u = 0$ on the basis of the spectrum $\sigma(B-A)$ alone, instead of using roughness arguments (cf. [201, pp. 237–238, Theorem 7.6.10] or [432, p. 216ff]).

As a second simplification we restrict to a constant stepsize T, linearly implicit Euler method applied to (SE), which yields a semi-implicit difference equation

$$B_k x' = A_k x + f_k(x) \tag{S'}$$

with $A_k x := x$, $B_k := I_Y + T(B-A)$ and $f_k(x) := T\left[f(t_k,x) - D_2 f(t_k,0)x\right]$. Instead of deriving a general corollary from Theorem 4.6.4, we rather focus on two examples where the above setting is applicable:

Example 4.9.12 (*Chafee–Infante equation*). Given $\Omega = (-a,a)$, $a, \alpha_1 > 0$ and $\delta > 0$, as before in Example 1.5.8 we consider a nonautonomous *Chafee–Infante equation*

$$u_t - \delta u_{xx} = u(\alpha_1 - \alpha_2(t)u^2)$$

subject to the boundary conditions $u(-a) = u(a) = 0$, with a continuous bounded reaction function $\alpha_2 : \mathbb{R} \to (0,\infty)$ which is uniformly bounded away from 0. Using the above notation we obtain $(B-A)u := -\alpha_1 u - \delta u_{xx}$ and $f(t,u) := -\alpha_2(t)u^3$.

On the one hand, Example 3.7.7 implies the discrete spectrum

$$\sigma(B_{k+1}^{-1}A_k) = \{0\} \cup \left\{ \frac{4a^2}{4a^2 - T(4a^2\alpha_1 + \delta\pi^2 n^2)} \right\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \quad \text{for all } k \in \mathbb{I}',$$

whose only accumulation point is 0 and outside every neighborhood of 0 are only finitely many eigenvalues. Every interval $(\bar{\alpha}_i, \bar{\beta}_i)$ with reals $0 < \bar{\alpha}_1 < \bar{\beta}_i$ and disjoint from $\sigma(B_{k+1}^{-1}A_k)$ yields an exponential dichotomy; thus, one employs Theorem 3.4.30 in order to deduce an exponential N-splitting for the linear part $B_{k+1}x' = A_k x$, whose pseudo-unstable vector bundles are finite-dimensional and independent of k; hence, Hypothesis 4.2.1 holds.

On the other hand, $H_0^1(-a,a)$ is a Hilbert space and thus a C^∞-Banach space (cf. Proposition C.2.10). Following the reasoning of [432, p. 270ff] one shows that $B_{k+1}^{-1}f_k : H_0^1(-a,a) \to H_0^1(-a,a)$, $B_{k+1}^{-1}f_k(u) = T\left[I_Y + T(B-A)\right]^{-1} u^3$ does define a C^2-mapping satisfying Hypotheses 4.6.1 and 4.6.2 with $m = 2$. Thus, the linearly implicit Euler discretization (S′) of the above Chafee–Infante equation has C^1-smooth pseudo-stable and unstable-hierarchies of invariant fiber bundles associated to the trivial solution. Provided the respective conditions (4.6c) or (4.6f) hold for $m = 2$ also, one obtains fiber bundles of class C^2.

Example 4.9.13 (*scalar Ginzburg–Landau equation*). Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2\}$, be a bounded domain as above. We consider a nonautonomous complex *Ginzburg–Landau equation* with cubic nonlinearity

$$u_t - \mu_1 u - (1 + i\nu)\Delta u + (1 + i\mu_2(t)) |u|^2 u = 0 \quad \text{in } (t_0, \infty) \times \Omega \tag{GL}$$

under Dirichlet or periodic boundary conditions (cf. [86, p. 118]), and we assume $\nu, \mu_1 \in \mathbb{R}$ and that $\mu_2 : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. In the present setting it is $(B - A)u = -\mu_1 u - (1 + i\nu)\Delta u$ and $f(t, u) = -(1 + i\mu_2(t))|u|^2 u$.

If $\lambda_n \geq 0$ denote the eigenvalues of the Laplacian Δ arranged in decreasing order of magnitude, we obtain the spectrum

$$\sigma(B_{k+1}^{-1} A_k) = \{0\} \cup \left\{ \frac{1}{1 - T\left[(1 + i\nu)\lambda_n + \mu_1\right]} \right\}_{n\in\mathbb{N}} \qquad \text{for all } k \in \mathbb{I}',$$

which induces an exponential splitting for $B_{k+1}x' = A_k x$ by virtue of Theorem 3.4.30, since every annulus in $\mathbb{C}$ centered in 0 and disjoint from $\sigma(B_{k+1}^{-1} A_k)$ yields an exponential dichotomy. As in Example 4.9.12 one deduces both pseudo-stable and pseudo-unstable hierarchies of invariant C^1-fiber bundles for the linearly implicit Euler discretization of (GL).

4.9.4 Fully Discretized Reaction-Diffusion Equations

Let $\mathbb{I}$ be a discrete interval unbounded below. In this subsection, we continue our investigations begun in Sect. 2.6.3 on scalar nonautonomous parabolic initial-boundary value problems

$$\begin{aligned} u_t - \delta(t)\Delta u &= f(t, x, u) && \text{for } t > t_0,\ x \in \Omega, \\ u(t, x) &= 0 && \text{for } t \geq t_0,\ x \in \operatorname{bd}\Omega, \\ u(t_0, x) &= u_0(x) && \text{for } x \in \Omega \end{aligned} \tag{RDE}$$

equipped with homogeneous Dirichlet boundary conditions. The corresponding spatial discretization (2.6g) is supposed to fulfill Hypothesis 2.6.6 and for the sake of a full discretization we use the θ-method

$$M\frac{v' - v}{h_k} + \delta(t_k^\theta)A\left[(1 - \theta)v + \theta v'\right] = MF(t_k^\theta, (1 - \theta)v + \theta v') \tag{4.9f}$$

with $t_k^\theta := (1 - \theta)t_k + \theta t_{k+1}$, $k \in \mathbb{I}'$, as in (2.6j). Under appropriate stepsize restrictions, we know from Proposition 2.6.11(b) that the general forward solution φ to (4.9f) exists as a C^m-function. Moreover, by Proposition 2.6.12 it is uniformly bounded dissipative (if $\theta \in (\frac{1}{2}, 1]$) and Proposition 2.6.14 yields a global attractor, which is uniformly bounded in the discrete Lebesgue space L^2_N. Additionally to Hypothesis 2.6.6 we impose

Hypothesis 4.9.14. *Suppose that we have:*

(i) $\sigma(A, M) = \{\lambda_1, \dots, \lambda_N\}$ *with* $0 \le \lambda_n < \lambda_{n+1}$ *for all* $1 \le n < N$*, where* $\phi_n \in \mathbb{R}^N$ *are the eigenvectors corresponding to* λ_n *forming an orthonormal basis of* L^2_N*.*
(ii) $\sup_{t\in\mathbb{R}} \|F(t, 0)\| < \infty$ *and for every* $r \ge 0$ *there exists a real* $L(r) \ge 0$ *with*

$$\sup_{(t,u)\in\mathbb{R}\times\bar{B}_r(0,L^2_N)} \|D_2F(t, u)\|_{L(L^2_N)} \le L(r).$$

Remark 4.9.15. Explicit eigenvalues λ_n, $1 \le n \le N$, and eigenvectors ϕ_n for various spatial discretizations schemes have been given in Sect. 3.7.4.

For the purpose of this section it is offered to write (4.9f) as semilinear equation

$$B_{k+1}x' = A_k x + f_k(x, x') \tag{S}$$

in L^2_N, with the mappings

$$A_k := I_{L^2_N} - (1-\theta)h_k\delta(t^\theta_k)M^{-1}A, \qquad B_{k+1} := I_{L^2_N} + \theta h_k\delta(t^\theta_k)M^{-1}A$$

and the nonlinearity $f_k(x, x') := h_k F(t^\theta_k, (1-\theta)x + \theta x')$ for all $k \in \mathbb{I}'$.

Lemma 4.9.16. *If Hypothesis 4.9.14 holds, then* $B_{k+1}, B^{-1}_{k+1}A_k \in \mathbb{R}^{N\times N}$*,* $k \in \mathbb{I}'$*, are invertible and with complementary orthogonal projections* $P^n_1, Q^n_1 \in \mathbb{R}^{N\times N}$*,*

$$Q^n_1 x := \sum_{j=n+1}^{N} \langle x, \phi_j\rangle \phi_j, \qquad P^n_1 x := x - P_n x,$$

one has the following properties for all integers $k \in \mathbb{I}'$*,* $n = 1, \dots, N-1$*:*

(a) $B^{-1}_{k+1}A_k P^n_1 = P^n_1 B^{-1}_{k+1}A_k$*,*
(b) $\left\|B^{-1}_{k+1}A_k Q^n_1\right\|_{L(L^2_N)} \le \frac{1-(1-\theta)h_k\delta(t^\theta_k)\lambda_{n+1}}{1+\theta h_k\delta(t^\theta_k)\lambda_{n+1}}$*,*
(c) $\left\|B^{-1}_{k+1}A_k P^n_1\right\|_{L(L^2_N)} \ge \frac{1-(1-\theta)h_k\delta(t^\theta_k)\lambda_{n}}{1+\theta h_k\delta(t^\theta_k)\lambda_{n}}$*,*

where $\phi_1, \dots, \phi_N \in L^2_N$ *are the orthonormal eigenvectors from Hypothesis 4.9.14.*

Proof. Let $k \in \mathbb{I}'$ be fixed, choose $n \in [1, N)_\mathbb{Z}$ and define the strictly decreasing function $\psi : [0, \infty) \to (0, 1]$, $\psi(x) := \frac{1-(1-\theta)x}{1+\theta x}$. Using the spectral mapping theorem (cf., e.g., [96, p. 204]) one derives the explicit relations

$$\sigma(B_{k+1}) = \left\{1 + \theta h_k\delta(t^\theta_k)\lambda_j > 0 : j = 1, \dots, N\right\},$$
$$\sigma(B^{-1}_{k+1}A_k) = \{\upsilon_j(k) \in \mathbb{R} : j = 1, \dots, N\}$$

with eigenvalues $\upsilon_j(k) := \psi(h_k\delta(t_k^\theta)\lambda_j)$, which are strictly decreasing in j. In particular, both B_{k+1}, $B_{k+1}^{-1}A_k$ are invertible. For an arbitrary $x \in \mathbb{R}^N$ with representation $x = \sum_{j=1}^N x_j\phi_j$ and $x_j = \langle x, \phi_j\rangle$ we get claim (a) from

$$\begin{aligned} B_{k+1}^{-1}A_kP_1^n x &= \sum_{j=1}^n x_j B_{k+1}^{-1}A_k\phi_j = \sum_{j=1}^n \upsilon_j(k)x_j\phi_j = P_1^n\sum_{j=1}^N x_j\upsilon_j(k)\phi_j \\ &= P_1^nB_{k+1}^{-1}A_k\sum_{j=1}^N x_j\phi_j = P_1^nB_{k+1}^{-1}A_kx \quad \text{for all } k \in \mathbb{I}'. \end{aligned}$$

As in the proof of Theorem 3.4.30 one shows the estimates in (b) and (c). □

Lemma 4.9.17. *If Hypothesis 4.9.14 holds, then for every* $r > 0$ *and* $u, v \in \bar{B}_r(0)$ *the nonlinearity* $f_k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies for all* $k \in \mathbb{I}'$ *and* $\theta \in [0,1]$,

$$\begin{aligned} \operatorname{lip}_1 f_k|_{\bar{B}_r(0,L_N^2)\times\bar{B}_r(0,L_N^2)} &\le (1-\theta)h_kL(r), \\ \operatorname{lip}_2 f_k|_{\bar{B}_r(0,L_N^2)\times\bar{B}_r(0,L_N^2)} &\le \theta h_kL(r). \end{aligned}$$

Proof. Let $r > 0$ and $x_1, x_2, \bar{x}_1, \bar{x}_2 \in \bar{B}_r(0)$. Referring to the convexity of the L_N^2-ball $\bar{B}_r(0)$ also the convex combinations $x_i^\theta := (1-\theta)x_i + \theta x_i'$ are contained in $\bar{B}_r(0)$, as well as $x_1^\theta + h(x_2^\theta - x_1^\theta) \in \bar{B}_r(0)$ for $\theta, h \in [0,1]$. With this, the mean value theorem (see [295, p. 341, Theorem 4.2]) implies

$$\begin{aligned} \|f_k(x_1,x_1') - f_k(x_2,x_1')\| &= h_k\left\|F(t_k^\theta,x_1^\theta) - F(t_k^\theta,x_1^\theta)\right\| \\ &\le h_k(1-\theta)\left\|\int_0^1 D_2F(t_k^\theta, x_1^\theta + h(x_2^\theta - x_1^\theta))\,dh\right\|\|x_1 - x_2\| \end{aligned}$$

for all $k \in \mathbb{I}'$ and thus the first Lipschitz estimate follows. Analogously we show the second claimed inequality. □

Theorem 4.9.18 (fully discretized RDEs). *Let* $\mathbb{I} = \mathbb{Z}$. *Suppose that Hypotheses 2.6.6 and 4.9.14 hold true, choose* $\omega \in (0,1)$, $\theta \in \left(\frac{1}{2}, 1\right]$ *and* $\rho > \rho_0$, *where* ρ_0 *is the radius of the absorbing ball from Remark 2.6.13. If there exists an* $n \in [1,N)_\mathbb{Z}$ *with*

$$\frac{2L(\rho)}{\omega\varpi\inf_{t\in\mathbb{R}}\delta(t)} < \lambda_{n+1} - \lambda_n \tag{4.9g}$$

and if the stepsizes h_k *satisfy the conditions* (2.6k),

$$\begin{gathered} \theta L(\rho)T < 1, \qquad\qquad 0 < \inf_{k\in\mathbb{Z}} h_k(b + \delta(t_k^\theta)\lambda_1), \\ \left[(1-\theta) + \theta b_n(k)\right]L(\rho) < b_n(k), \quad \theta h_k\delta(t_k^\theta)\lambda_{n+1} < \frac{1-\omega}{\omega} \quad \textit{for all } k \in \mathbb{Z}, \end{gathered}$$

then the following holds:

(a) *The full discretization* (4.9f) *of* (RDE) *has an n-dimensional inertial fiber bundle $\mathcal{W} \subseteq \mathbb{I} \times \bar{B}_\rho(0, L_N^2)$ as in Theorem 4.7.3.*
(b) *There exists a unique global attractor $\mathcal{A}^*$ for* (4.9f) *with $\mathcal{A}^* \subseteq \mathcal{W}$.*

Proof. Let $\theta \in \left(\frac{1}{2}, 1\right]$, choose $\rho > \rho_0$ as required above and let $\hat{\mathcal{B}}$ be the family of all uniformly bounded subsets of $\mathbb{Z} \times L_N^2$. We have to verify successively the assumptions of Theorem 4.7.3.

ad Hypothesis 4.2.1: We deduce from Lemma 4.9.16 that the linear part of (4.9f) resp. (S) admits a strongly regular N-splitting. More precisely, for each $n \in [1, N)_\mathbb{Z}$ we deduce an exponential dichotomy on $\mathbb{Z}$ with constant projectors P_1^n, constants $K_n^\pm = 1$ and growth rates

$$a_n(k) := \psi(h_k, \delta(t_k^\theta)\lambda_{n+1}), \qquad b_n(k) := \psi(h_k, \delta(t_k^\theta)\lambda_n)$$

with the strictly decreasing function $\psi : [0, \infty) \to (0, 1]$ already defined in the proof of Lemma 4.9.16. We get $a_n \ll b_n$ and Lemma 4.9.16 yields the estimates

$$\|\Phi(k,l)Q_1^n\|_{L(L_N^2)} \leq \prod_{j=l}^{k-1} \left\|B_{j+1}^{-1} A_j Q_1^n\right\|_{L(L_N^2)} \leq e_{a_n}(k,l) \quad \text{for all } l \leq k$$

and also $\|\Phi(k,l)P_1^n\|_{L(L_N^2)} \leq e_{b_n}(k,l)$ for all $k \leq l$.

ad Hypothesis 4.7.1: We have established in Proposition 2.6.11(b) that the general forward solution of (4.9f) exists as a C^m-function. Moreover, from Lemma 4.9.17 we deduce the local Lipschitz constants

$$L_1(r) := (1-\theta)TL(r), \qquad L_2(r) := \theta TL(r).$$

ad Hypothesis 4.7.2: From Proposition 2.6.12 and Remark 2.6.13 we know that $\mathbb{Z} \times \bar{B}_\rho(0, L_N^2)$ is a $\hat{\mathcal{B}}$-uniformly absorbing set. Since L_N^2 is a Hilbert space, the Lipschitz constant of the associated radial retraction is $\lim r_{L_N^2}^* = 1$ (cf. Lemma C.2.1). Thus, our stepsize assumptions guarantee that (4.7b) and (4.7c) are fulfilled. It remains to verify both the growth and the spectral gap condition.

ad $(\Gamma_\kappa^-(n))$: Thanks to Hypothesis 4.9.14(ii) and $b_n(k) \leq 1$ we obtain that the growth condition holds.

ad $(\hat{G}_n)$: In the present setting the spectral gap condition simplifies to the relation (cf. Lemma 4.9.17)

$$\exists \varsigma \in \left(0, \tfrac{\lfloor b_n - a_n \rfloor}{2}\right) : \frac{2[(1-\theta) + \lceil b_n \rceil \theta]TL(\rho)}{1 + 2T\theta L(\rho)} < \varsigma. \tag{4.9h}$$

In order to verify (4.9h), we abbreviate $\alpha := h_k\delta(t_k^\theta\lambda_{n+1})$, $\beta := h_k\delta(t_k^\theta\lambda_n)$ and observe that our stepsize restrictions imply $(1+\theta\alpha)^{-1}, (1+\theta\beta)^{-1} \geq \omega$. Hence,

$$b_n(k) - a_n(k) = \psi(\beta) - \psi(\alpha) = \frac{\alpha - \beta}{(1+\theta\beta)(1+\theta\alpha)} \geq \omega^2 h_k \delta(t_k^\theta)(\lambda_{n+1} - \lambda_n)$$

for all $k \in \mathbb{Z}$, and on the other hand, due to $\lceil b_n \rceil \leq 1$ we have

$$\frac{2[(1-\theta)+\lceil b_n \rceil \theta]TL(\rho)}{1+2T\theta L(\rho)} \leq 2[(1-\theta)+\lceil b_n \rceil \theta]TL(\rho) \leq 2L(\rho)T.$$

Consequently, our assumption (4.9g) ensures that (4.9h) holds true.

(a) Since we have verified all assumptions of Theorem 4.7.3 we know that the implicit difference equation (S) in $\mathbb{Z} \times L_N^2$ admits an inertial fiber bundle $\mathcal{W}$. Every fiber $\mathcal{W}(k)$, $k \in \mathbb{Z}$, is a graph over $P_1^n L_N^2$ and by definition of the projector P_1^n in Lemma 4.9.16 can conclude $\dim \mathcal{W} = \dim P_1^n L_N^2 = n$.

(b) By Proposition 2.6.14 there exists a of unique global attractor $\mathcal{A}^* \subseteq \mathbb{Z} \times B_\rho(0, L_N^2)$ and due to $b_n \ll 1$ we can apply Corollary 4.7.5 yielding $\mathcal{A}^* \subseteq \mathcal{W}$. $\square$

Our final comprehensive example in this subsection illustrates how to approximate an inertial manifold of a scalar nonautonomous RDE. For this, we apply the algorithm from Sect. 4.8 to a finite-dimensional difference equation, which has been obtained from the original evolutionary PDE by a spectral Galerkin method for spatial, and a linearly implicit Euler scheme for temporal discretization. Error estimates for such full discretizations have been obtained in [121, 236].

As special case of (RDE) we now study the following nonautonomous problem

$$u_t - u_{xx} = f(t,u), \tag{4.9i}$$

subject to homogeneous Dirichlet boundary conditions $u(t,0) = u(t,\pi) = 0$ and an initial condition $u(\tau,x) = u_0(x)$ for given data $\tau \in \mathbb{R}$, $u_0 \in L^2(0,\pi)$. This problem fits into the framework of Sect. 1.5.3 with $N = d = 1$, $\Omega = (0,\pi)$, if $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, the partial derivative $D_2^2 f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ exists as a continuous mapping and that there exist reals $C_1, C_2, C_3, \gamma > 0$, $p \geq 2$ such that

$$f(t,v)v \leq C_1 - \gamma |v|^p, \quad |f(t,v)|^{\frac{p}{p-1}} \leq C_2(1+|v|^p), \quad D_2 f(t,v) \leq C_3 \tag{4.9j}$$

for all $t, v \in \mathbb{R}$ (cf. Hypothesis 1.5.6). Moreover, choose $K_1, K_2 : [0,\infty) \to \mathbb{R}$ such that

$$K_i(r) \geq \sup_{t \in \mathbb{R}} \sup_{|v| \leq \sqrt{\pi} r} \left| D_2^i f(t,v) \right| \quad \text{for all } i = 1,2,\ r \geq 0.$$

In Sect. 1.5.3 we have seen that (4.9i) generates a dissipative 2-parameter semiflow on the space $L^2(0,\pi)$. On the other hand, following [432, Sect. 5.1], we can formulate (4.9i) as abstract nonautonomous evolutionary equation

$$\dot{u} + Bu = g(t,u) \tag{4.9k}$$

with linear part $B := -D_{xx}$ and substitution operator $g(t,u)(x) := f(t,u(x))$. Referring to [432, p. 272, Theorem 51.1], the mild solutions of (4.9k) generate a dissipative 2-parameter semiflow on $H_0^1(0,\pi)$, and in [86, p. 290, Proposition 3.5] it is shown that the radius of the associate absorbing set in $H_0^1(0,\pi)$ is bounded by

$$r_0 := 2\sqrt{2C_1C_3}.$$

Thanks to Example 3.7.7 the eigenvalues of B equipped with zero boundary conditions $u(0) = u(\pi) = 0$ are $\lambda_n = n^2$, $n \in \mathbb{N}$, with pair-wise L^2-orthonormal eigenfunctions $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ for $n \in \mathbb{N}$. Let $P_i : L^2(0,\pi) \to L^2(0,\pi)$ be the orthogonal projection onto the i-dimensional space $\operatorname{span}\{\phi_1,\dots,\phi_i\}$ and $Q_i := I - P_i$ be the complementary projector. Under our above assumptions and provided $i \in \mathbb{N}$ satisfies

$$i > \sqrt{2}L(r_0) - 1/2, \qquad L(r) := \sqrt{2K_1(r)^2 + r^2K_2(r)^2}, \tag{4.9l}$$

the RDE (4.9i) has an i-dimensional inertial manifold

$$\mathcal{W}_{\mathbb{R}}^- = \left\{(\tau, \xi + w_{\mathbb{R}}^-(\tau,\xi)) \in \mathbb{R} \times H_0^1(0,\pi) : \xi \in \operatorname{im} P_i\right\}$$

with a smooth function $w_{\mathbb{R}}^- : \mathbb{R} \times \operatorname{im} P_i \to \operatorname{im} Q_i$ (cf. [384, Proposition 4]).

Now we describe our discretization strategy for (4.9i). First, the spatial approximation with N Fourier modes, $N \geq 1$, is obtained by inserting the ansatz

$$u(t,x) = \sum_{i=1}^{N} v_i(t)\phi_i(x)$$

into (4.9i) and taking the L^2-inner product with ϕ_j, $j \in [1,N]_{\mathbb{Z}}$, leads to an initial value problem in $\operatorname{im} P_N$. We canonically identify this linear space with $\mathbb{R}^N$ and arrive at the N-dimensional ODE

$$\dot{v}_j = -j^2 v_j + f_j(t,v) \quad \text{for all } j \in [1,N]_{\mathbb{Z}} \tag{4.9m}$$

with the nonlinearities $f_j : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$,

$$f_j(t,v) = \int_0^\pi f\left(t, \sum_{i=1}^{N} v_i(t)\phi_i(x)\right)\phi_j(x)\,dx \tag{4.9n}$$

and initial condition $v(\tau) = \eta$, $\eta_j = \int_0^\pi u_0(x)\phi_j(x)dx$. Respecting the stiffness of the matrix $-\operatorname{diag}(j^2)_{j=1}^N$, we apply a linearly implicit Euler discretization (with stepsize $T > 0$) to (4.9m) and arrive at the nonautonomous difference equation

$$v' = A_T v + F_T(k,v) \tag{4.9o}$$

with linear part $A_T := \operatorname{diag}\big(\frac{1}{1+Tj^2}\big)_{j=1}^N$ and a nonlinearity $F_T : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$, whose components are given by

$$F_T(k,v)_j := \frac{T}{1+Tj^2} f_j(\tau + Tk, v) \quad \text{for all } j \in [1,N]_{\mathbb{Z}}.$$

Henceforth, we deduce the existence of an attractive invariant fiber bundle for the difference equation (4.9o). Choosing an integer i according to (4.9l), the linear part of (4.9o) satisfies Hypothesis 4.2.1 with constants $K_i^{\pm} = 1$, growth rates

$$a_i(k) :\equiv \frac{1}{1+T(i+1)^2}, \qquad b_i(k) :\equiv \frac{1}{1+Ti^2} \quad \text{on } \mathbb{Z}$$

and projectors $P_1^i = \operatorname{diag}(1,\dots,1,0,\dots,0)$. Moreover, one can verify Hypothesis 4.7.1 and we employ the methods from Sect. 4.8 to approximate the invariant fiber bundle

$$\mathcal{W}_{T,N}^- = \left\{(k, \xi + w_{T,N}^-(t_k,\xi)) \in \mathbb{Z} \times \mathbb{R}^N : \, k \in \mathbb{Z}, \, \xi \in \operatorname{im} P_1^i\right\}$$

of the discretization (4.9o). An error estimate relating the inertial manifold $\mathcal{W}_{\mathbb{R}}^-$ of the full reaction-diffusion equation (4.9i) to the finite-dimensional invariant fiber bundles $\mathcal{W}_{T,N}^-$, can be found in [236, Theorem 5.3] and is of the form

$$\left\| w_{T,N}^-(k,\xi) - w_{\mathbb{R}}^-(t_k,\xi) \right\| \le \bar{K}_1 \lambda_N T + \bar{K}_2 \sqrt{\frac{\lambda_{i+1}}{\lambda_{N+1}}} \tag{4.9p}$$

with constants $\bar{K}_1, \bar{K}_2 > 0$, sufficiently large N and small T (cf. also [121]).

Example 4.9.19 (*Chafee–Infante equation*). We intent to compute fibers of the nonautonomous set $\mathcal{W}_{T,N}^-$. As in Example 1.5.8, we retreat to a Chafee–Infante equation with time-dependent coefficients

$$u_t - u_{xx} = u(\alpha_1(t) - \alpha_2(t)u^2), \tag{4.9q}$$

under the above initial-boundary conditions. For continuous bounded functions $\alpha_1, \alpha_2 : \mathbb{R} \to (0,\infty)$ we know from Example 1.5.8 that (4.9q) fulfills Hypothesis 1.5.6 with

$$C_1 := \frac{1}{2} \sup_{t\in\mathbb{R}} \frac{\alpha_1(t)^2}{\alpha_2(t)}, \qquad C_3 := \sup_{t\in\mathbb{R}} \alpha_1(t) < \infty.$$

Furthermore, we can choose the functions $K_1, K_2 : [0,\infty) \to \mathbb{R}$ as

$$K_1(r) := \max\left\{C_3, 3\pi r^2 \sup_{t\in\mathbb{R}} \alpha_2(t) - \inf_{t\in\mathbb{R}} \alpha_1(t)\right\}, \quad K_2(r) := 6\sqrt{\pi} r \sup_{t\in\mathbb{R}} \alpha_2(t).$$

In the next step we compute a Galerkin approximation for (4.9q). Unfortunately, the constants $\bar{K}_1, \bar{K}_2 > 0$ in the mentioned error estimate (4.9p) from [236, Theorem 5.3] are not immediately accessible. For this reason, we heuristically choose a spatial approximation of order $N = 6$. With help of some computer algebra to evaluate the integrals (4.9n), the resulting nonlinearities $f_1, \dots, f_6$ read as follows:

$$
\begin{aligned}
f_1(t,v) = \frac{\alpha_2(t)}{2\pi}\big(& -6v_2v_3v_4 + 3v_2^2v_5 - 6v_1v_5^2 - 6v_1v_6^2 - 6v_2v_4v_5 - 6v_1v_3^2 \\
& - 3v_3^2v_5 + 6v_1v_3v_5 - 3v_2^2v_3 + 6v_2v_3v_6 + 6v_1v_4v_6 - 6v_2v_5v_6 \\
& - 6v_1v_4^2 - 6v_1v_2^2 + 6v_1v_2v_4 - 3v_1^3 - 6v_3v_4v_6 + 3v_1^2v_3\big) \\
+\alpha_1(t)v_1&, \\
f_2(t,v) = \frac{\alpha_2(t)}{2\pi}\big(& -6v_3v_4v_5 - 6v_1v_2v_3 + 3v_1^2v_4 + 6v_1v_3v_6 - 6v_1v_4v_5 - 3v_2^3 \\
& - 6v_1^2v_2 - 6v_1v_5v_6 - 3v_4^2v_6 - 6v_1v_3v_4 - 3v_3^2v_4 - 6v_2v_4^2 \\
& + 3v_2^2v_6 - 6v_2v_6^2 - 6v_3v_5v_6 - 6v_2v_5^2 + 6v_1v_2v_5 - 6v_2v_3^2\big) \\
+\alpha_1(t)v_2&, \\
f_3(t,v) = \frac{\alpha_2(t)}{2\pi}\big(& v_1^3 - 3v_3^3 - 6v_2v_5v_6 - 3v_1v_2^2 - 6v_1v_3v_5 - 6v_1v_2v_4 \\
& + 6v_1v_2v_6 - 6v_1v_4v_6 - 6v_2v_4v_5 - 6v_4v_5v_6 - 6v_2v_3v_4 \\
& - 6v_3v_4^2 + 3v_1^2v_5 - 6v_3v_5^2 - 6v_3v_6^2 - 6v_1^2v_3 - 3v_4^2v_5 - 6v_2^2v_3\big) \\
+\alpha_1(t)v_3&, \\
f_4(t,v) = \frac{\alpha_2(t)}{2\pi}\big(& -6v_3v_4v_5 - 3v_2v_3^2 - 6v_4v_5^2 - 3v_4^3 - 6v_2^2v_4 - 6v_2v_4v_6 \\
& + 3v_1^2v_6 - 6v_3^2v_4 - 6v_1v_3v_6 + 3v_1^2v_2 - 6v_1v_2v_3 - 6v_3v_5v_6 \\
& - 6v_1v_2v_5 - 6v_2v_3v_5 - 6v_1^2v_4 - 3v_5^2v_6 - 6v_4v_6^2\big) + \alpha_1(t)v_4, \\
f_5(t,v) = \frac{\alpha_2(t)}{2\pi}\big(& -6v_1v_2v_6 - 6v_4v_5v_6 - 3v_3v_4^2 - 6v_4^2v_5 - 3v_1v_3^2 + 3v_1^2v_3 \\
& - 6v_1v_2v_4 - 6v_3v_4v_6 + 3v_1v_2^2 - 6v_2^2v_5 - 6v_2v_3v_6 - 6v_5v_6^2 \\
& - 6v_2v_3v_4 - 6v_1^2v_5 - 3v_5^3 - 6v_3^2v_5\big) + \alpha_1(t)v_5, \\
f_6(t,v) = \frac{\alpha_2(t)}{2\pi}\big(& -6v_1v_3v_4 - 3v_6^3 - 3v_4v_5^2 + 6v_1v_2v_3 - 3v_2v_4^2 - 6v_2^2v_6 + v_2^3 \\
& + 3v_1^2v_4 - 6v_5^2v_6 - 6v_1v_2v_5 - 6v_1^2v_6 - 6v_3^2v_6 - 6v_4^2v_6 \\
& - 6v_3v_4v_5 - 6v_2v_3v_5\big) + \alpha_1(t)v_6.
\end{aligned}
$$

To perform actual computations, we choose α_2 constant and define $\alpha_1 : \mathbb{R} \to \mathbb{R}$ by

$$\alpha_1(t) := \alpha_2\left(\tfrac{\pi}{2} + \sin t\right).$$

Hence, the radius of the absorbing set for the RDE (4.9q) is bounded above by $r_0 = 2\pi^{5/2}\alpha_2$. Consequently, (4.9q) admits a nonautonomous inertial manifold $\mathcal{W}_{\mathbb{R}}^-$, whose dimension is the minimal integer $i \geq 0$ satisfying

$$i > 2\pi\alpha_2\sqrt{\max\left\{12\pi^3\alpha_2^2, 1\right\}^2 + 288\pi^5\alpha_2^4} - 1/2.$$

We are fixing the parameter value $\alpha_2 = \frac{471}{5000}$ and the evolutionary equation (4.9i) admits a two-dimensional inertial manifold, i.e., we can choose $d = 2$ and also obtain a two-dimensional invariant fiber bundle $\mathcal{W}_{T,6}^-$ for the spectral Galerkin Euler discretization (4.9o). In particular, this nonautonomous set $\mathcal{W}_{T,6}^-$ is given as graph of a function $w_{T,6}^- : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^4$.

We used Algorithm 4.8.6 to approximate $w_{T,6}^-$ over the square $[-1, 1] \times [-1, 1]$ with a uniform grid of 21×21 points, for an Euler stepsize $T = 0.1$, a truncation length $K = 15$ and accuracy $\varepsilon = 10^{-5}$. Yet, the nonlinear equations (4.8i) have been solved numerically by the inexact Newton method `K_NSoli1` (see [384] for details). The results of this computation are visualized: Fig. 4.6 depicts how the second component $w_{T,6}^-(k, \xi)_2$ changes under varying fibers, while Fig. 4.7 shows all four components of $w_{T,6}(k, \xi)$ at the fixed instant $k = 20$.

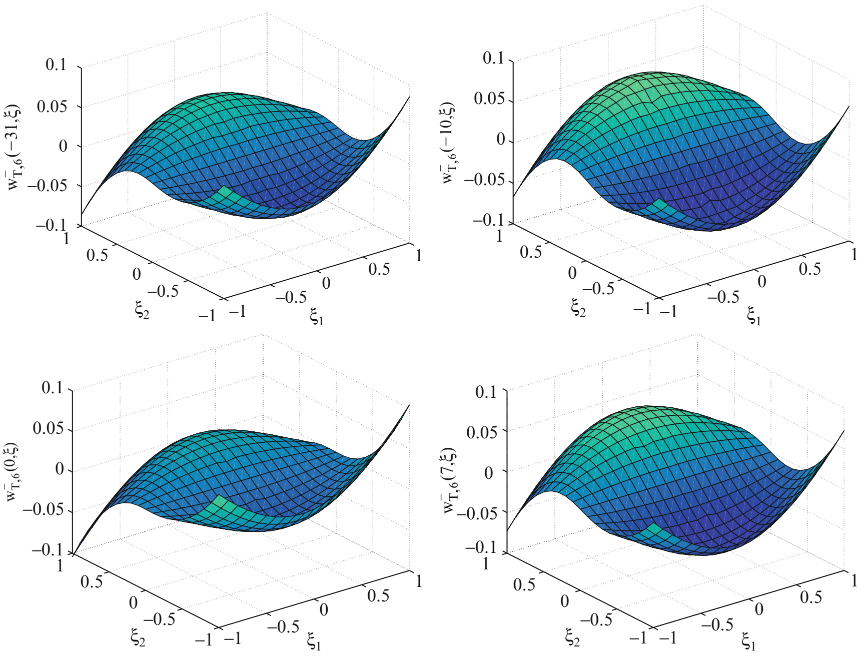

Fig. 4.6 Graphs of $w_{T,6}^-(k, \xi)_2$ for $k \in \{-31, -10, 0, 7\}$

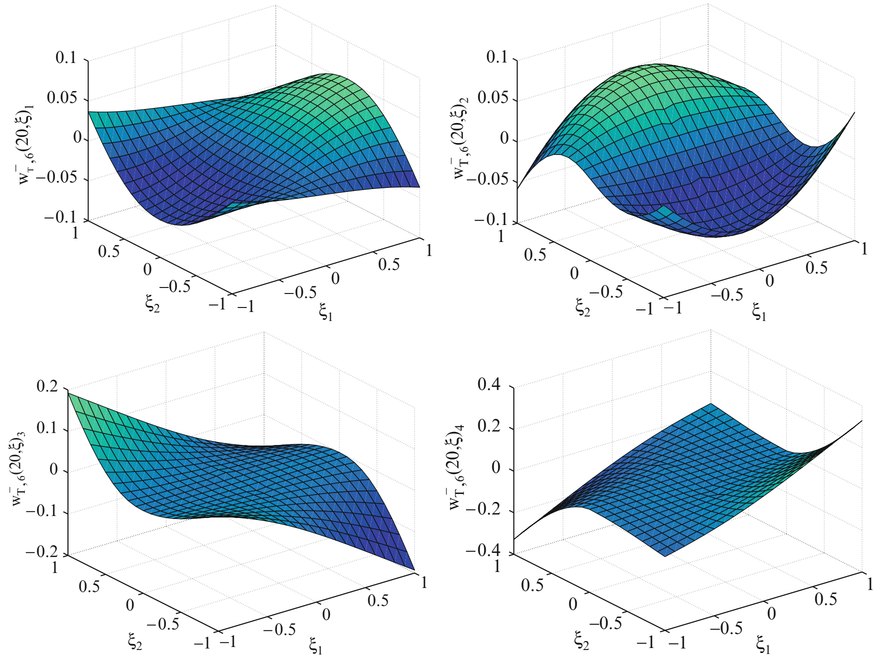

Fig. 4.7 Graphs of $w^-_{T,6}(20,\xi)_i$ for $i \in \{1,2,3,4\}$

4.9.5 Fully Discretized Finite Difference Ginzburg–Landau Equation

Let $\mathbb{I}$ be a discrete interval unbounded below. We now return to the Ginzburg–Landau equation (GL) previously considered in Example 1.5.9 and Sect. 2.6.4, and rely on the notation introduced there. As full discretization of (GL) we investigated the implicit difference equation

$$\frac{x' - x}{h_k} + \tilde{\Delta}_h x' = F(t_{k+1}, x')$$

in $\mathbb{I} \times \mathbb{C}^N$, but differing from Sect. 2.6.4 where it was advantageous to consider it as one-step method, we now write it in the form

$$x' = A_k x + f_k(x') \qquad (\Delta\mathrm{GL})$$

known from (S) with abbreviations

$$A_k := \left[I_{\mathbb{C}^N} + h_k \tilde{\Delta}_h\right]^{-1}, \qquad f_k(x') := h_k \left[I_{\mathbb{C}^N} + h_k \tilde{\Delta}_h\right]^{-1} F(t_{k+1}, x')$$

for all $k \in \mathbb{I}'$. On basis and terminology of Lemma 3.7.11 we obtain

Lemma 4.9.20. *The matrices $A_k \in \mathbb{C}^{N\times N}$, $k \in \mathbb{I}'$, are invertible and with complementary orthogonal projections $P_1^n, Q_1^n \in \mathbb{C}^{N\times N}$,*

$$Q_1^n x := \sum_{j=n+1}^{N-(n+1)} \langle x, \phi_j \rangle \phi_j, \qquad\qquad P_1^n x := x - P_n x,$$

one has the following properties for all integers $k \in \mathbb{I}'$, $n = 1, \dots, \left\lfloor \frac{N-2}{2} \right\rfloor$:

(a) $A_k P_1^n = P_1^n A_k$,
(b) $\|A_k Q_1^n\|_{L(H_N^1)} \le |1 + (1 + i\nu)h_k(1 + \nu_{n+1})|^{-1}$,
(c) $\|A_k P_1^n\|_{L(H_N^1)} \ge |1 + (1 + i\nu)h_k(1 + \nu_n)|^{-1}$,

where $\phi_1, \dots, \phi_N \in \mathbb{C}^N$ are the orthonormal eigenvectors from Lemma 3.7.11.

Proof. Let $k \in \mathbb{I}'$ and n be an integer with $1 \le n \le \left\lfloor \frac{N-2}{2} \right\rfloor$. Referring to the spectral mapping theorem (cf., e.g., [96, p. 204]) one derives the explicit relation $\sigma(A_k) = \{\upsilon_j(k) \in \mathbb{C} : j = 1, \dots, N\}$ with eigenvalues

$$\upsilon_j(k) := [1 + h_k(1 + i\nu)(1 + \nu_j)]^{-1}$$

and consequently $0 \notin \sigma(A_k)$. In conclusion, $A_k \in \mathbb{C}^{N\times N}$ is an invertible matrix. For later use we introduce the discrete intervals $\mathbb{I}_n^+ := \{n+1, \dots, N-(n+1)\}$, $\mathbb{I}_n^- := \{1, \dots, N\} \setminus \mathbb{I}_n^+$ and choose $x \in \mathbb{C}^N$ with $x = \sum_{j=1}^N x_j \phi_j$ and $x_j = \langle x, \phi_j \rangle$. We get claim (a) from

$$\begin{aligned} A_k P_1^n x &= \sum_{j\in\mathbb{I}_n^-} x_j A_k \phi_j = \sum_{j\in\mathbb{I}_n^-} \upsilon_j(k) x_j \phi_j = P_1^n \sum_{j=1}^N x_j \upsilon_j(k) \phi_j \\ &= P_1^n A_k \sum_{j=1}^N x_j \phi_j = P_1^n A_k x. \end{aligned}$$

In addition, due to $|\upsilon_j(k)| \le |\upsilon_{n+1}(k)|$ for all $j \in \mathbb{I}_n^+$ one has the forward estimate

$$\|A_k Q_1^n x\|_{H_N^1}^2 = \sum_{j\in\mathbb{I}_n^+} (1 + \nu_j) |\upsilon_j(k)|^2 |x_j|^2 \le |\upsilon_{n+1}(k)|^2 \|x\|_{H_N^1}^2,$$

which implies (b), and claim (c) follows by the corresponding backward estimate

$$\|A_k P_1^n x\|_{H_N^1}^2 = \sum_{j\in\mathbb{I}_n^-} (1 + \nu_j) |\upsilon_j(k)|^2 |x_j|^2 \ge |\upsilon_n(k)|^2 \|x\|_{H_N^1}^2,$$

since we have $|\upsilon_j(k)| \ge |\upsilon_n(k)|$ for all $j \in \mathbb{I}_n^-$. $\square$

Lemma 4.9.21. *For $r>0$ and $u,v\in\bar{B}_r(0,H_N^1)$ the nonlinearity $f_k:\mathbb{C}^N\to\mathbb{C}^N$ satisfies $\|f_k(u)-f_k(v)\|_{H_N^1}\le L(r)T\,\|u-v\|_{H_N^1}$ for all $k\in\mathbb{I}'$ with*

$$L(r):=\sqrt{2(1+R_1^2+\nu^2)+360(1+R_2^2)r^4}.$$

Proof. Let $r>0$, $u,v\in\bar{B}_r(0,H_N^1)$ and $n\in\left\{1,\dots,\left\lfloor\frac{N-2}{2}\right\rfloor\right\}$.

(I) We derive a Lipschitz condition for the function $F_N:\mathbb{C}^N\to\mathbb{C}^N$. Here, our approach is based on relation (3.7j). The mean value inequality (see [295, p. 342, Corollary 4.3]) leads to

$$\left|\left|u_j\right|^2u_j-\left|v_j\right|^2v_j\right|\le 2\sup_{t\in[0,1]}\left|u_j+t(v_j-u_j)\right|^2\left|u_j-v_j\right|\quad\text{for all } j=1,\dots,N.$$

From Lemma 3.7.13(b) we borrow the relation $|u_j|\le\sqrt{3}\,\|u\|_{H_N^1}$, $j=1,\dots,N$, (cf. (3.7n)) and arrive at

$$\left|\left|u_j\right|^2u_j-\left|v_j\right|^2v_j\right|\le 6r^2\left|u_j-v_j\right|\quad\text{for all } j=1,\dots,N \tag{4.9r}$$

and $u,v\in\bar{B}_r(0,H_N^1)$, which, in turn, equips us with the first L_N^2-estimate

$$\|F_N(u)-F_N(v)\|_{L_N^2}^2=\frac{1}{N}\sum_{j=1}^N\left|\left|u_j\right|^2u_j-\left|v_j\right|^2v_j\right|^2\le 36r^4\,\|u-v\|_{L_N^2}^2 \tag{4.9s}$$

for all $u,v\in\bar{B}_r(0,H_N^1)$. Moreover, for notational convenience we identify u_{N+1} with u_1 (and v_{N+1} with v_1) to obtain

$$\begin{aligned}
&\left|\delta_h^+(\left|u_j\right|^2u_j)-\delta_h^+(\left|v_j\right|^2v_j)\right|^2\\
\le\ &\left(\left|\left|u_{j+1}\right|^2u_{j+1}-\left|v_{j+1}\right|^2v_{j+1}\right|+\left|\left|u_j\right|^2u_j-\left|v_j\right|^2v_j\right|\right)^2\\
\overset{(4.9r)}{\le}\ &\left(6r^2\left|u_{j+1}-v_{j+1}\right|+6r^2\left|u_j-v_j\right|\right)^2\le 72r^4\left(\left|u_{j+1}-v_{j+1}\right|^2+\left|u_j-v_j\right|^2\right)
\end{aligned}$$

for all $j=1,\dots,N$ from the elementary inequality

$$(x+y)^2\le 2x^2+2y^2\quad\text{for all } x,y\in\mathbb{R}. \tag{4.9t}$$

Therefore, with relation (3.7k) for the seminorm $|\cdot|_{\Delta_h}$ we get

$$\begin{aligned}
|F_N(u)-F_N(v)|_{\Delta_h}^2&=\frac{1}{N}\sum_{j=1}^N\delta_h^+(F_N(u)-F_N(v))_j\overline{\delta_h^+(F_N(u)-F_N(v))_j}^{\,2}\\
&=\frac{1}{N}\sum_{j=1}^N\left|\delta_h^+(F_N(u)-F_N(v))_j\right|^2\le 144r^4\,\|u-v\|_{L_N^2}^2
\end{aligned}$$

and combining this with (4.9s) we obtain from (3.7j) for all $u, v \in \bar{B}_r(0, H_N^1)$ that

$$\|F_N(u) - F_N(v)\|_{H_N^1}^2 \le 180r^4 \|u - v\|_{L_N^2}^2 \overset{(3.7l)}{\le} 180r^4 \|u - v\|_{H_N^1}^2 \tag{4.9u}$$

(II) Now we aim at a Lipschitz estimate for the full nonlinearity F resp. f_k. By definition, adopting the notation from Lemma 4.9.20 and its proof one has

$$\|f_k(u) - f_k(v)\|_{H_N^1}^2 \le T^2 \sum_{j=1}^{N} (1 + \nu_j) |v_j(k)|^2 |\langle F(t_{k+1}, u) - F(t_{k+1}, v), \phi_j\rangle|^2$$

and referring again to the basic inequality (4.9t) we proceed to

$$\begin{aligned} \|f_k(u) - f_k(v)\|_{H_N^1}^2 &\le 2T^2 \left(1 + R_1^2 + \nu^2\right) \sum_{j=1}^{N} (1 + \nu_j) |\langle u - v, \phi_j\rangle|^2 \\ &\quad + 2T^2 \left(1 + R_2^2\right) \sum_{j=1}^{N} (1 + \nu_j) |\langle F_N(u) - F_N(v), \phi_j\rangle|^2 \\ &\overset{(4.9u)}{\le} 2T^2 \left[(1 + R_1^2 + \nu^2) + 180(1 + R_2^2) r^4\right] \|u - v\|_{H_N^1}^2 . \end{aligned}$$

Taking the square root of this estimate yields the assertion. □

After all these preparations we eventually arrive at

Theorem 4.9.22 (fully discretized Ginzburg–Landau equation). *Let* $\mathbb{I} = \mathbb{Z}$. *Choose reals* $\omega \in (0, 1)$ *and* $\rho > \rho_1$*, where* ρ_1 *is the radius of the absorbing ball from* (2.6p)*. If* $N \ge 5$ *fulfills*

$$N^2 \sin\left(\tfrac{\pi}{N}\right) > \frac{L(\rho)}{\sqrt{3}\varpi\omega^{3/2}} \tag{4.9v}$$

and if the stepsize bound $T \in (0, 1]$ *is so small that*

$$L(\rho)T < 1, \qquad \left[(3 + \nu^2)(1 + \nu_{n+1})^2\right] T < \omega^{-1} - 1 \tag{4.9w}$$

with $n := \left\lceil \frac{N-3}{6} \right\rceil$*, then the following holds:*

(a) The full finite difference discretization (ΔGL) *of* (GL) *has a* $(2n + 1)$*-dimensional inertial fiber bundle* $\mathcal{W} \subseteq \mathbb{I} \times \bar{B}_\rho(0, H_N^1)$ *as in Theorem 4.7.3.*
(b) Under the stepsize restrictions (2.6m), (2.6q) *and* (2.6s) *there exists a unique global attractor* $\mathcal{A}^*$ *for* (ΔGL) *with* $\mathcal{A}^* \subseteq \mathcal{W}$,

where the constant $L(\rho) > 0$ *is defined in Lemma 4.9.21.*

Remark 4.9.23. The formulation of Theorem 4.9.22 (as well as of Theorem 4.9.18) is quantitative in the sense that the dimension of the inertial fiber bundle $\mathcal{W}$ can actually be computed for given values of ν and bounds $R_1, R_2 > 0$ (Fig. 4.8). Related estimates for the continuous problem (GL) (and constant μ_1, μ_2) are given in [122].

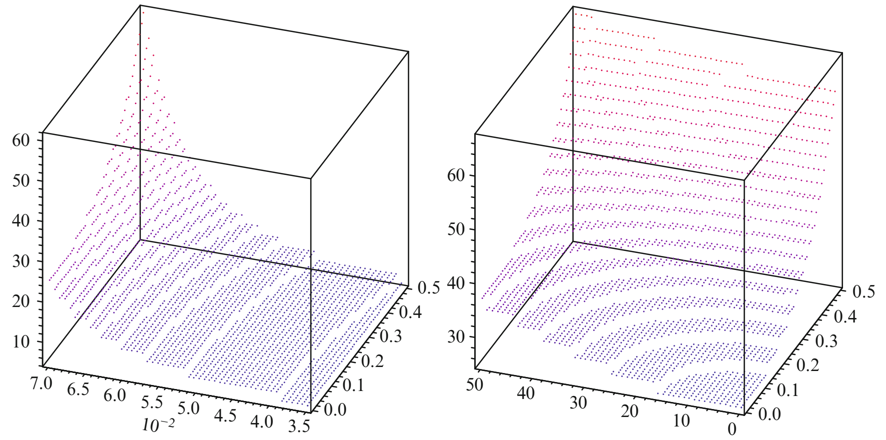

Fig. 4.8 Dimension of the inertial fiber bundle $\mathcal{W}$ from Theorem 4.9.22. *Left*: $\dim \mathcal{W}$ over (R_1, R_2)-plane for $\nu = 1$, $R_1 \in [0.035, 0.07]$, $R_2 \in [0, 0.5]$. *Right*: $\dim \mathcal{W}$ over (R_1, ν)-plane for parameters $\nu \in [0, 0.5]$, $R_1 = 0.07$ and $R_2 = [0, 50]$

Proof. Unfortunately, we cannot apply Theorem 4.7.3 directly, since the forward solutions of (ΔGL) need not to be unique (cf. Lemma 2.6.18). However, this problem can be circumvented as follows:

Choose $\rho > \rho_1$ as required above. We modify the nonlinearity of (ΔGL) as in the proof of Theorem 4.7.3 and directly employ Theorems 4.2.9 and 4.3.7 to equation

$$x' = A_k x + f_k^\rho(x'). \tag{4.9x}$$

For this, we verify the corresponding assumptions with $Y_k = H_N^1$ and the extended state space $\mathcal{X} = \mathbb{Z} \times H_N^1$ for an appropriate spatial discretization with $N \geq 5$. Since H_N^1 is a Hilbert space, we are in the scope of Remark 4.7.4(1) and radial retractions on H_N^1 have Lipschitz constant 1.

ad Hypothesis 4.2.1: Thanks to Lemma 4.9.20 the linear part of (4.9x) possesses a strongly regular exponential $\lfloor \frac{N+1}{2} \rfloor$-splitting. More precisely, for each positive integer $n \leq \frac{N+1}{2}$ we obtain an exponential dichotomy with constant projectors P_1^n, constants $K_n^\pm = 1$ and the growth rates $a_n(k) := |1 + (1 + i\nu)h_k(1 + \lambda_{n+1})|^{-1}$, $b_n(k) := |1 + (1 + i\nu)h_k(1 + \lambda_n)|^{-1}$; from an elementary calculation we obtain $a_n \ll b_n$. Indeed, Lemma 4.9.20 implies the desired dichotomy estimates

$$\|\Phi(k,l)Q_1^n\|_{L(H_N^1)} \leq \prod_{j=l}^{k-1} \|A_j Q_1^n\|_{L(H_N^1)} \leq e_{a_n}(k,l) \quad \text{for all } l \leq k$$

and analogously $\|\Phi(k,l)P_1^n\|_{L(H_N^1)} \leq e_{b_n}(k,l)$ for all $k \leq l$.

ad Hypothesis 4.3.1: Referring to Lemma 4.9.21 and Proposition C.2.5 one has the Lipschitz condition $\operatorname{lip} f_k^\rho \leq L(\rho)T$. Hence, Proposition 4.1.3 ensures that the general forward solution of (4.9x) exists as a continuous mapping. Moreover, also the global Lipschitz conditions (4.2a) hold with $L_1 = 0$ and $L_2 = L(\rho)T$.

ad $(\Gamma_\kappa^-(n))$: This growth condition trivially holds due to $f_k(0) \equiv 0$ on $\mathbb{Z}$.

ad $(\hat{G}_n)$: In the present setting the spectral gap condition reads as

$$\exists \varsigma \in \left(0, \tfrac{\lfloor b_n - a_n \rfloor}{2}\right) : \frac{2\lceil b_n \rceil TL(\rho)}{1 + 2TL(\rho)} < \varsigma$$

and it is slightly more involved to verify it. For this, we abbreviate

$$\begin{aligned}\alpha &:= 2h_k(1+\nu_{n+1}) + h_k^2(1+\nu_{n+1})^2 + h_k^2\nu^2(1+\nu_{n+1})^2,\\ \beta &:= 2h_k(1+\nu_n) + h_k^2(1+\nu_n)^2 + h_k^2\nu^2(1+\nu_n)^2\end{aligned}$$

and observe $a_n(k) = (1+\alpha)^{-1/2}$, $b_n(k) = (1+\beta)^{-1/2}$. Our conditions (4.9w) ensure

$$\begin{aligned}\beta < \alpha &\le \left[2(1+\nu_{n+1}) + (1+\nu_{n+1})^2 + \nu^2(1+\nu_{n+1})^2\right] T\\ &\le (3+\nu^2)(1+\nu_{n+1})^2 T < \omega^{-1} - 1,\end{aligned}$$

one has $\sqrt{1+\beta} \le \sqrt{1+\alpha} \le \omega^{-1/2}$, thus $\sqrt{1+\alpha}(\sqrt{1+\alpha} + \sqrt{1+\beta}) \le \frac{2}{\omega}$ and

$$\tfrac{\omega}{2\sqrt{1+\beta}}(\alpha - \beta) \le \tfrac{\alpha-\beta}{\sqrt{1+\alpha}\sqrt{1+\beta}(\sqrt{1+\alpha}+\sqrt{1+\beta})} = \tfrac{1}{\sqrt{1+\beta}} - \frac{1}{\sqrt{1+\alpha}}.$$

Since we also have $\beta < \omega^{-1} - 1$, one deduces $\frac{1}{\sqrt{1+\beta}} \ge \sqrt{\omega}$ and we assemble

$$b_n(k) - a_n(k) = \frac{1}{\sqrt{1+\beta}} - \frac{1}{\sqrt{1+\alpha}} \ge \frac{\omega}{2\sqrt{1+\beta}}(\alpha-\beta) \ge \frac{\omega^{3/2}}{2}(\alpha - \beta).$$

Directly from the definition one has $\alpha - \beta \ge 2\varpi T(\lambda_{n+1} - \lambda_n)$ and for $n = \left\lceil \frac{N-2}{6} \right\rceil$ using Lemma 3.7.11(b) it is $b_n(k) - a_n(k) \ge 2\sqrt{3}\varpi\omega^{3/2}N^2 \sin\left(\frac{\pi}{N}\right)$. Thanks to the estimate $b_n(k) \le 1$ this ensures $\frac{\lfloor b_n - a_n \rfloor}{\lceil b_n \rceil} \ge 2\sqrt{3}\varpi\omega^{3/2}N^2 \sin\left(\frac{\pi}{N}\right)$, which implies

$$\frac{2TL(\rho)}{1+2TL(\rho)} \le 2TL(\rho) \overset{(4.9v)}{<} 2\sqrt{3}\varpi\omega^{3/2}N^2 \sin\left(\tfrac{\pi}{N}\right) T \le \frac{\lfloor b_n - a_n \rfloor}{\lceil b_n \rceil}$$

for $n = \left\lceil \frac{N-2}{6} \right\rceil$. Note that by Remark 3.7.12 we can satisfy (4.9v).

(a) Having verified all assumptions of Theorem 4.3.7 we know that the implicit difference equation (4.9x) in $\mathbb{C}^N$ has a global inertial fiber bundle $\tilde{\mathcal{W}}$, if $N \ge 5$ satisfies (4.9v). Each fiber $\tilde{\mathcal{W}}(k)$, $k \in \mathbb{Z}$, is a graph over $P_1^n\mathbb{C}^N$ and by definition of the projector P_1^n in Lemma 4.9.20 we have $\dim \tilde{\mathcal{W}}(k) = \dim P_1^n\mathbb{C}^N = 2n+1$, $k \in \mathbb{Z}$. Since the nonautonomous set $\mathbb{Z} \times \bar{B}_\rho(0, H_N^1)$ is uniformly absorbing for the original equation (ΔGL) (see Lemma 2.6.20), all solutions of (ΔGL) eventually enter $\bar{B}_\rho(0, H_N^1)$. We define $\mathcal{W} := \tilde{\mathcal{W}} \cap \mathcal{B}_\rho \subseteq \mathbb{Z} \times H_N^1$ and the positive invariance of $\mathbb{Z} \times \bar{B}_\rho(0, H_N^1)$ implies that $\mathcal{W}$ is an inertial fiber bundle for (ΔGL).

(b) By Proposition 2.6.21 there exists a of unique global attractor $\mathcal{A}^* \subseteq \mathbb{Z} \times B_\rho(0, H_N^1)$. Clearly, $b_n \ll 1$ and $f_k(0) \equiv 0$ on $\mathbb{Z}$ implies that Corollary 4.7.5 can be applied, which yields $\mathcal{A}^* \subseteq \mathcal{W}$. Therewith, Theorem 4.9.22 is established. $\square$

4.10 Remarks

Semilinear difference equations: Semilinear (also called *quasilinear*) difference equations inherit their dynamical properties from the linear part and consequently, features like stability might be global in nature. In fact, the dominant linear part invites to apply various perturbation techniques, and we strongly benefit from this observation when it comes to the construction of invariant fiber bundles and foliations in Sects. 4.2 resp. 4.3. Nevertheless, as illustrated by Theorem 4.1.8, semilinear equations can also possess properties typical for nonlinear problems, like nontrivial attractors.

Boundedness criteria for quasilinear problems are given in [175, p. 174ff]. Concerning stability questions, there is an enormous literature and for example, issues like asymptotic equivalence of solutions have been studied in [330,331,363,364]. In discretization theory, approximations to dissipative linear parts like elliptic differential operators, yield uniformly stable linear difference equations (see Sects. 3.7.3–3.7.4).

Existence of invariant fiber bundles: As indicated by the quotation in the beginning of Sect. 4.2, the literature on the existence of invariant manifolds for various kinds of evolutionary equations is vast. Hence, in the following (understandably incomplete) survey on methods for the construction of invariant manifolds, we skip important contributions like [91, 247] and essentially restrict to discrete systems:

- *Graph transform*: This historically first method due to [186] is of geometric nature. It characterizes the graph of an invariant manifold using a functional equation in an appropriate space of bounded Lipschitz functions, whose graph is the desired manifold – a contraction mapping argument enables a constructive proof for the existence of a solution. The references [138, 141, 210, 211, 227, 230, 236, 254, 319, 343, 434, 447, 462] use this approach. A graph transform technique for nonautonomous problems can be found in [245, 277].
- In this book, we excessively use of the *Lyapunov–Perron method* (see [316, 361, 362]) with a rather functional analytical flavor. One relies on a dynamical characterization of stable manifolds as set of initial points for solutions decaying exponentially to zero. Such solutions can be characterized as fixed points of so-called Lyapunov–Perron operators in sequence or function spaces (cf. [77, 83, 90, 110, 114, 152, 232, 233, 456]). This abstract method lifts to nonautonomous problems (see [20, 26, 82, 205, 355, 374, 385, 458]). In fact, the occurring Lyapunov–Perron operators are omnipresent throughout the nonautonomous theory.

- In the *deformation method* from [320] one formulates the problem of finding an invariant manifold as solution of a family of vector fields. These differential equations can be integrated and their solutions define the invariant manifold.
- The *parametrization method* (cf. [71–73]) allows to establish the existence of smooth invariant manifolds associated to linear subspaces, invariant by linearization, which satisfy non-resonance conditions; the basic idea is to construct a conjugacy between a map and its linearization.
- The scheme of [214] is much more geometrical than the approaches relying on abstract fixed point theorems. It is based on the convergence of a canonical sequence of "finite time local stable manifolds", which are related to the dynamics of a finite number of iterations.

For differential equations there are further methods to construct invariant manifolds. They can be traced back to the work of [411, 412] (based on PDE techniques like elliptic regularization in order to solve the invariance equation), or [270] (see also [306], where the manifolds are constructed via appropriate boundary value problems). However, to the authors knowledge, these approaches have not been applied to difference equations yet.

A fairly flexible approach to construct attractive invariant manifolds of autonomous difference equations based on the graph transform method is due to [254, p. 207ff] and [343], which also has a variety of applications. It is remarkable that already the early contribution [191] constructed invariant manifolds for nonautonomous equations. More contemporary results of this kind have been successively developed in [110, 355], [458, p. 86, Satz 2.3.1], [20, Theorem 4.1], [23, 233] and finally also in the monograph [245, pp. 242–243, Theorem 6.2.8]. Invariant fiber bundles for difference equations with an almost periodic or recurrent time-dependence are investigated in [82, Theorems 6.10–6.11]. Invariant manifolds under the assumption of a nonuniformly exponentially dichotomic linear part have been investigated in [41] (dealing with ODEs). For the origins of the pseudo-stable and -unstable hierarchies of invariant fiber bundles we refer to [19, 430]; in [12, p. 339, Corollary 7.3.12] these hierarchies are denoted as *flag* of (un)stable manifolds. Further references will be given below.

Finally, conditions for smooth finite-dimensional mappings precluding the existence of invariant Lipschitz compact submanifolds, are given in [158].

Our goal in Sect. 4.2 was to provide a flexible existence result for invariant fiber bundles, which carries the main technical load in global (inertial manifolds) as well as local situations (pseudo-stable and -unstable manifolds). Due to our nonautonomous and implicit setting we followed the Lyapunov–Perron method of [374]. Nonetheless, our Lyapunov–Perron construction differs from the one in [20, 114] or the continuous counterpart in [432, p. 569ff, Chap. 8]. As a benefit, we require a weaker spectral gap condition and our smoothness proofs (see Sect. 4.4) are less involved. Moreover, differing from the previous approaches [20, 458], our invariant fiber bundles are not associated to the trivial solution (the growth condition $(\Gamma_i^{\pm})$ circumvents this) and our method works for equations without a decoupled linear part.

An example that even in the classical situation of an exponentially trichotomic autonomous equations, not all invariant subspaces persist under nonlinear perturbations as Lipschitzian manifolds, is given in [18, Sect. 8] in the ODE case.

Invariant foliations and asymptotic phase: In the autonomous situation, a construction of invariant foliations has been given in [83, 157, 253] and smoothness issues have been addressed in [153]. We remark that the construction of invariant fiber bundles, as well as of invariant foliations, can be put in a common framework of general "Lyapunov–Perron equations" (cf. [83]). The nonautonomous case is investigated in [458, p. 99ff, Abschnitt 2.5], as well as in [33]. Invariant foliations near normally hyperbolic invariant manifolds of diffeomorphisms have been constructed in [211] and generalizations to maps on Banach spaces are due to [44]. Finally, a basic application of invariant foliations is the decoupling and linearization theory studied in Chap. 5.

Using geometrical arguments, invariant foliations yield a crucial asymptotic phase property of an invariant manifold resp. fiber bundle. Instead of an asymptotic phase one also speaks of an *exponential tracking*.

An invariant fiber bundle $\mathcal{W}$ is called *hyperbolic*, if for all pairs $(\kappa, \xi) \in \mathcal{W}$, the variational equation along the solution $\varphi(\cdot; \kappa, \xi)$ admits an appropriate exponential trichotomy. For discrete dynamical systems, the asymptotic phase property of such invariant manifolds consisting of equilibria has been investigated in [16, p. 59ff, Chap. 2] and was extended to infinite-dimensional problems in [197, Theorem 2.8], [69]. The related nonautonomous situation is addressed in [25, 307, 308].

Smoothness of fiber bundles and foliations: At least the graph transform and the Lyapunov–Perron method work for equations, whose right-hand side is merely Lipschitz in the state space variable. The corresponding proofs are based on the contraction mapping or the Lipschitz inverse function theorem (cf. Theorem B.3.1). For smooth mappings with hyperbolic linear part, the classical stable and unstable manifolds inherit their differentiability properties from the equation – including being C^∞ or analytic. Here, the implicit function theorem (see [228]) or the uniform contraction principle is used (see [227, p. 132ff]).

For more general splittings of the linear part, i.e., particularly for pseudo-stable and -unstable manifolds, C^1-smooth right-hand sides yield continuously differentiable invariant manifolds. The question for a higher-order smoothness is more subtle, since substitution operators on spaces of exponentially bounded sequences need not to be as smooth as the mappings inducing them (see Example 4.6.6). Whence, more delicate techniques and tools come into play, which fall into the following categories (see [21, Remark 8] for another survey):

- Uniform contraction principles on scales of continuously embedded Banach spaces (see [205, 406, 409, 457]).
- The fiber contraction principle is due to [209] and applications to derive smoothness assertions of invariant manifolds of difference equations can be found in [138, 141, 456].
- A lemma of Henry (cf. [87, p. 324, Lemma 2.1] or [21, Lemma 1]) yielding a condition when locally Lipschitzian mappings are of class C^1.

Our argument to prove smoothness assertions for invariant fiber bundles does not rely on the sophisticated tools mentioned above. Rather, it is based on a formal differentiation of the Lyapunov–Perron equation and has its origins in [385]. As advantage we point out that the only nonelementary tools involved in the proof are the Neumann series and Lebesgue's convergence theorem.

Smoothness results for invariant foliations can are due to [83, 153], and the non-autonomous differential equations case is treated in [89, 435].

Normal hyperbolicity: The property of normal hyperbolicity is a key issue in the general theory of invariant and inertial manifolds, since it guarantees their robustness, which in turn, is essential for discretizations matters. Indeed, a normally hyperbolic invariant manifold is stable under small perturbations of the right-hand side in the problem (see [160, 370] or [432, p. 494ff] for the differential equations situation); in case of inertial manifolds this is guaranteed by the spectral gap condition (see, e.g., [404]).

Pseudo-stable and pseudo-unstable fiber bundles: Stable and unstable invariant manifolds for maps with hyperbolic linear part are considered in [228], or in the infinite-dimensional case in [227, p. 132ff] and [462]. Stable and unstable manifolds for time discretizations of PDEs were constructed in [8].

A generalization to arbitrary spectral splittings of the linear part, i.e., to pseudo-stable and -unstable manifolds, can be traced back to monographs like, e.g., [200, pp. 234–236, Lemma 5.1 and Example 5.2] for C^1-mappings on $\mathbb{R}^n$, [211, pp. 53–54, Theorem 5.1], [434] for C^m-diffeomorphisms, $m \geq 1$, and to [430] for the almost periodic ODE case. In a pseudo-hyperbolic context, we also refer to the research papers [229, 293, 460]. Finally, in a globally Lipschitzian setting, the nonautonomous pseudo-hyperbolic case has been addressed in [20]. On the basis of these results, [392, p. 116ff, Sect. 5.1] constructed forward resp. backward invariant fiber bundles meeting the requirements of pullback attraction. Applications of invariant fiber bundles have been given in [218, 220].

Rigorous smoothness proofs for invariant manifolds are due to the contributions [90, 138, 139, 205]. In the nonautonomous situation, we refer to [26, 385] for (sharp) differentiability assertions. Our Example 4.6.6 illustrating sharpness of the gap condition (4.6c) implying smoothness of the pseudo-stable fiber bundle is taken from [385, Example 4.1]; a similar example for the pseudo-unstable fiber bundle may be found in [26, Example 4.13].

Pointing at their relevance in stability and bifurcation theory, dynamical properties of center manifolds for maps are discussed at various sources like [319, p. 28, Theorem (2.1)], [227, p. 145ff], [77, p. 33ff] or [275]. In particular, see [232] for center manifolds of mappings with an unbounded linear part.

The celebrated reduction principle of Pliss (see [366]) found its discrete counterpart in [77, p. 35, Theorem 8]. A reduction principle for nonautonomous difference equations is due to [458, p. 104, Satz 2.6.1], [34, Theorem 2.1] and [234, 399] under global assumptions on the nonlinearities. Using an ad hoc approach, these restrictive conditions have been removed in [373] yielding an applicable form. Note that our approach to deduce Theorem 4.6.15 differs from [373] and does not require a finite-dimensional center-unstable bundle.

Taylor approximations of invariant manifolds are discussed in [59] with a focus on the homological equation as an algebraic problem in spaces of multilinear mappings. A nonautonomous generalization has been given in [382, 383]. Here, the homological equation (4.6q) is a linear difference equation and finding Taylor coefficients becomes a dynamical, rather than an algebraic problem. Such nonautonomous Taylor approximations make the reduction principle from Theorem 4.6.15 applicable. We illustrated this via critical nonautonomous stability problems in [383].

We have tackled the lacking uniqueness of locally invariant manifolds in form of Proposition 4.6.17. A different approach has been given in [68, Theorem 4.1], where a single manifold is picked by requiring it to contain the graph of a particular function defined on a sphere in the range of the pseudo-stable spectral projector.

An interesting topic is the connection between center manifolds of (semi-) flows and their time-h-map. In general, a center manifold of a time-h-map is not flow-invariant (see [285, Sect. 3] for an explicit example) and it is demonstrated in [285] that invariance depends of the correct choice of an appropriate cut-off function. On the other hand, once a center manifold of a time-h-map is unique, then it is also flow-invariant (cf. [275]).

The lack of differentiable cut-off functions on $C[-r, 0]$ (see Example C.2.8(4)) makes it difficult to apply our results on locally invariant fiber bundles, e.g., to temporal discretizations of DDEs or FDEs. Here, one has to employ more subtle techniques elaborated in [286, p. 173ff] and [152, Theorem 5.1].

Inertial fiber bundles: The theory of inertial manifolds for evolutionary differential equations is wide and we refer to [453, p. 498ff, Chap. 8], [432, pp. 569, Chap. 8] for a survey and to [164] for an early contribution.

In general, attractive invariant manifolds are a theoretically useful tool in numerical analysis. For instance, [252, 444] show that multistep methods (or general linear methods) to solve ODE initial value problems are asymptotically equivalent to one-step methods. An extension to methods involving varying stepsizes, as well as applications to delay difference equations, is due to [376].

Discrete inertial manifolds in discretization theory have been investigated in [114, 216, 236, 237, 272–274, 309, 433] for autonomous equations. In addition, a linearly implicit temporal Euler discretization of an autonomous RDE was considered in [464]. The nonautonomous case as well as issues like normal hyperbolicity is due to [375].

Inertial manifolds and also fiber bundles from Theorem 4.7.3 satisfy a weaker form of an asymptotic forward phase. In fact, they are *asymptotically complete*, which means that forward solutions have to enter the absorbing set first, before they are grasped by a solution on the inertial bundle via its asymptotic phase.

The main obstacle for the existence of inertial fiber bundles is the spectral gap condition, which is hardly satisfied in spatial discretizations of partial differential equations in more than one dimension. To avoid this problem, the concept of an *exponential attractor* (cf. Definition 1.6.3) has been introduced in [124, pp. 9–24, Chap. 2]. In a way, exponential attractors are "realistic" objects intermediate between the two "ideal" ones which are global attractors and inertial manifolds.

Approximation of invariant fiber bundles: As summarized above, most methods for the construction of invariant manifolds are iterative and consequently in some sense algorithmic. Hence, there is a tremendous literature on the approximation of invariant manifolds for autonomous dynamical systems – for a survey we refer to [283]. As usual, the situation is different in the nonautonomous case. Unstable fiber bundles have been characterized as pullback attractors and approximated using set-valued numerics in [27].

The method presented here, generalizes our approach from [384] to implicit equations; this paper illustrates that both (4.8i) and (4.8j) can be solved efficiently using Newton-methods. Numerical tests showed that the best performance is obtained for inexact Quasi-Newton techniques. Moreover, for 1-dimensional fibers, pseudo-arc length continuation methods have been applied in order to compute relatively long arcs of global fiber bundles for nonlinear problems.

Applications: First of all, the construction of invariant manifolds for autonomous evolutionary differential equations by applying discrete results to their time-h-map has been exemplified in [83, Sects. 5–6]. Invariant foliations over inertial manifolds have been investigated in [157].

Concerning flexible C^1-perturbation results, which apply to general normally hyperbolic invariant manifolds, we quote [368] – this forms the basics for a corresponding discretization theory.

For numerical methods to solve DDEs we refer to [47, 467] and [172] gives a survey on the corresponding numerical dynamics. Unstable manifolds of discretized retarded FDEs are the topic of [155] and see [156] for small delay inertial manifolds and numerical structural stability.

The behavior of invariant manifolds for ODEs under numerical discretization has a long history. Pioneering papers are [53] (stable and unstable manifolds), [60] (center manifolds), [158], and [171] (pseudo-stable and -unstable case). The results of [53] were generalized to parabolic PDEs in [8]. For periodic ODEs we refer to [468] and the general nonautonomous situation has been investigated in [246].

When discretizing PDEs one is confronted with the problem of nonsmooth initial data, which have an effect on the convergence rates under approximation. The corresponding error estimates to show convergence results can be found, for instance, in [55, 314, 315].

Evolutionary equations having an inertial manifold and corresponding discretizations are the topic of various papers. Early contributions on temporal discretizations are [114, 272], general perturbation results applicable to a variety of (full) discretizations have been considered in [236], and we furthermore refer to [216], [121] (temporal and full Runge–Kutta schemes), [273], [237, 238, 274] (C^1-convergence). Abstract convergence results of inertial manifolds are due to [400]. Moreover, [464] analyzes the inertial manifold of a linearly-implicit Euler discretization of a sectorial evolution equation. While many papers restrict to one-step methods, see [433] for generalizations to multistep schemes. Finally, time-discrete inertial manifolds for nonautonomous equations have been discussed in [374, 375]. Inertial manifold for the fully discretized Ginzburg–Landau equations are constructed in [309], which we generalized to the time-variant case. A similar analysis to ours from Sect. 4.9.5, but for the Kuramoto–Sivashinsky equation, can be found in [162, 163, 165, 273, 274]. A numerical computation scheme for inertial manifolds based on approximations of the continuous Lyapunov–Perron method has been suggested in [235], while we have preferred an initial time discretization in Sect. 4.9.4. Two further approximation schemes are due to [401].

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