Chapter 2
The Great River Civilizations

2.1 Civilizations Long Dead: And Yet Alive

In the first part of the fourth millennium B.C. a group of cultural centers developed in southwestern Asia. Probably emerging through coalescence from a web of small and, it would seem, insignificant Neolithic villages, impressive cities formed in the river valleys of the Indus, the Euphrates–Tigris and the Nile. Spreading out to form nets with other, in part more peripheral, urban centers, the classical civilizations bearing their names were born in these river valleys (Fig. 2.1).

Today their once flourishing life is attested to only by the mounds or *tells* covering their ruins. And by no means have we managed to uncover them all. Great finds are awaiting future archaeologists. The fascinating story of how these tells in some cases were persuaded to reveal their secrets, is not the subject of the present narrative. Nor is it our concern here to elaborate on the heroic endeavor and the almost superhuman tenacity, fortitude and resilience shown by those pioneers who managed to decipher the writings left behind by the vanished civilizations, thousands of years dead.

Suffice to recall that the pyramids and ruins in the Nile valley had been the objects of legends for millennia. Tales of treasures and curses.

Bedouins were roaming the deserts and fever-infested marshlands which now covered the once flourishing river basins of the Euphrates and the Tigris. They still worshiped with awe the mysterious *tells*, calling them by mystic names, the origins of which were long forgotten.

But in present day Iraq, not too far from the present city of Baghdad, were the crumbling ruins of the once so marvellous city of *Babylon*, the Gate of The Gods, the *Bab-Ili*. Once a center of science and mathematics, literature and history, astronomy and medicine, astrology and worship. As well as the bearer and center of a complex business structure which extended throughout the known world. With exquisite restaurants and a most sinful and hedonistic nightlife. In the rubble of what once were, there remained a ruin of what evidently had been a gigantic structure. Locally, it still was called reverently by the name of the ancient *Tower of Babel*.

Napoleon's invasion of Egypt led to the discovery of the Rosetta Stone and subsequently the decipherment of its inscriptions by *Jean-Francçois Champollion*.

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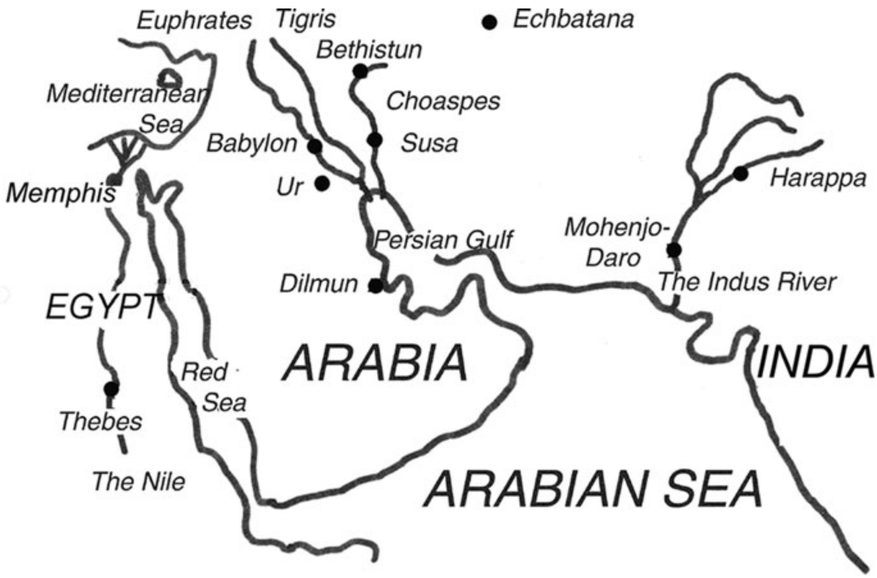

Fig. 2.1 This map shows the areas of the three Great River Civilizations

The main rivers shown are the Nile, the Euphrates and the Tigris. Euphrates flowed through the city of Babylon at this time, and the two rivers joined before they met the Persian Gulf. Also shown is the river of Choaspes, as well as the Indus River. Trade routes over land and sea connected them, and the presence of trade as well as other kinds of contact and interaction are very much in evidence. The ancient cities of Babylon and Ur of Mesopotamia are shown, Memphis and Thebes of Egypt, and Mohenjo-Daro and Harappa of the Indus Valley Civilization. Susa was the capital city of Persia before Darius founded the magnificent capital of Persepolis (in 518 B.C.). Dilmun was, according to some legends, the location of *the Garden of Eden*, central to three great religions. In ancient times this whole area was much more fertile than today, the climate was better and wildlife was abundant. Lion-hunting was a favorite activity for the Kings of Mesopotamia, as a result of which lions became extinct relatively early there.

Thus were opened up insights into events extending a thousand years beyond those recorded by the Bible and those accounted by the classical Greek historians and travellers. The decipherment of the hieroglyphic script of Egypt provided insights into all aspects of life in this ancient land.

The original script of ancient Egypt was, of course, *the hieroglyphs*. Eventually the need for a more practical and faster way of writing led to the development of the so called *hieratic* (sacred) script, a kind of simplified hieroglyphs. From the eight century B.C. a third type of script, which is even easier, is used, the so called *demotic* script, *the script of the people*. The Rosetta Stone is a fragment dating from 196 B.C., found in 1799 by one of Napoleon's officers. On it a decree had been written in Egyptian with hieroglyphs and with hieratic characters, and in Greek in demotic script. The breakthrough in deciphering the hieroglyphs and hieratic script came when Champollion could locate the names *Ptolemy* and *Cleopatra* in both

Egyptian scripts. It was then possible to compare with the Greek text, and piece together the alphabets.

The insights provided by the finds in the pyramids are subject to an important restriction. Almost exclusively all we have is material intended as support for the deceased in after-life. This is not to say that we are lacking texts pertaining to daily life or practical matters. But the criterion for preservation have been that of inclusion into the burial chamber.

Mesopotamia was now part of the once powerful Turk Ottoman Empire which was entering its final decades. The mounds and the areas around them had long been a source of building material of exquisite quality: Bricks of clay, baked at such high temperature that it surpassed by far what could be accomplished with contemporary technology. A number of these bricks were decorated with strange patterns of wedge-formed marks. These ubiquitous *decorated bricks* were excellent as building material, and served useful purposes as landfill as well as for building dykes along the banks of the Euphrates. In fact, as one of the first expeditions arrived at the old cite of Babylon to commence excavations there, workers were busily occupied with constructing huge embankments along the river banks. Using as building material not earth, but the books from the Imperial Library, housed at the Imperial Museum which had been located at the northernmost corner of ancient Babylon.

Indeed, the decorated bricks were books. The key to the decipherment of the *cuneiform* script in – or rather, on – these books, was provided by inscriptions found at Behistun near Persepolis, in the highlands of present day Persia. As in the case of the Rosetta Stone the inscriptions had been made in more than one language, one of which was Old Persian, which was known. An other version of the inscription was in the cuneiform script found on the *"decorated bricks."* Ordered by the Persian King Darius to commemorate his victories, the inscription had been carved on a great limestone cliff near the present village of Behistun, about 300 ft above the ground. Just getting an exact copy of the ancient letters was a strenuous and quite dangerous task. The story about the decipherment of the ancient script of the Sumerians and the Babylonians is suspenseful and fascinating. Building on work by the German philologist George Friedrich Grotefeld (1775–1853), the British diplomat and scholar, later to be called *the Father of Assyriology*, Henry Creswick Rawlinson (1810–1895), finally unravelled the script in 1846.

The assignment undertaken and carried out successfully by the archaeologists and linguists is humbling: A totally unknown script, writing texts in a completely unknown language, several thousand years dead. Nevertheless, the script was deciphered and the language was slowly reconstructed and pieced together. At first the results were viewed with skepticism by the scholarly world, suspecting a conspiracy of swindles. In the end a curious but convincing test was undertaken: A guaranteed new tablet was copied and given to a number of experts. Secluded they were then each required to make their individual versions of translation of the text! They passed. Even though there were discrepancies, there were enough common elements in all the translations to clearly demonstrate that they had indeed red and understood a common text which they had been given, à priori unknown to all of them.

The most mysterious of the Great River Civilizations is undoubtedly the *Indus Civilization*, contemporary to the early stages of the Mesopotamian and Egyptian ones. Excavations have uncovered what could be intriguing relations to the *Sumerian* civilization of ancient Mesopotamia.

In the time span between 2700 B.C. and 1500 B.C. the cities in the Indus valley developed into remarkable urban centers, carriers of an advanced civilization second to none from this epoch. Then decline set in, around 1,500 it is over. We do not know the cause of the demise of this great human achievement. Over the next 1,000 years a different way of life is in evidence, of a totally rural nature.

The three best known cities are *Harappa*, *Mohenjo-Daro* and *Chanhu-Daro*. The layout of the cities remarkably resemble that of ancient Mesopotamian ones: In Mohenjo-Daro the center is dominated by the *Citadel*, an elevated area surrounded by a wall about 50 ft high. Here we find *the Great Bath*, a watertight pool which may have had a sacred function. Below the Citadel lies the city, with broad avenues and more narrow side streets, arranged in a regular grid. Houses are build with baked brick, are usually two stories high around a central courtyard. All this closely resemble the layout and architecture found in Mesopotamian cities like ancient Ur and Babylon. Running water is supplied, and we find a covered system for drainage and sewage.

An intriguing feature is the lack of imposing structures immediately identifiable as *palaces* of kings or rulers. This has led some scholars to speculate that perhaps no ruling class existed at all, or possibly the ruling class harbored values which made them shun outwardly trappings of their elevated position. Also in evidence are a larger number of female sculptures, leading to hypothesizing of a matriarchal society.

More than 2,000 seals and seal impressions have been found. Again we find a close parallel to seals uncovered in Mesopotamia. As in Mesopotamia, they were carved from stone, and probably were used as the signature of the owner on various documents, letters and packets. The script, the *Harappan script,* found on them has not been deciphered as of this writing.

As of this writing no evidence of the mathematics or the geometry of this civilization has been uncovered.

This notwithstanding that evidence of their architecture and technology is everywhere. Also, they had a standardized system for weights and measure. One may, therefore, speculate that spectacular breakthroughs in revealing their science and mathematics may well lie in the future. This possibility is also borne out by the ubiquity of sophisticated geometric patterns and ornaments in decorations found throughout the Indus Valley area. These finds are of a clear *protogeometric nature*, which constitutes strong evidence of the sophistication required to support geometric ideas. Another intriguing piece of information is the following: In the first Indian mathematical text, presumably of Hindu origin, there are specific geometric rules for constructing *altars*. The tool for doing so is a set of *ropes* or *string*. The title of the work is the *Sulva-Sutra*, which means *"string-rules"*. The methods employed document knowledge of the so-called *Pythagorean Theorem* as well as *similar right triangles*. Now, the altars are supposed to be made of *burned bricks*, a technology the Hindus of that time did not possess, according to knowledgeable sources. But in the cities of the Indus civilization this technique is to be found everywhere. This has led some historians of mathematics to speculate that the Sulva-Sutra may have originated here. It certainly should be admitted that this is a speculative hypothesis, but it should be worth some serious digging at the cites in question.

2.2 Birth of Geometry as We Know It

Some historians have tended to dismiss the early science as *"merely magic and sorcery"*. But others have forcefully espoused the diametrically opposite view: *The ancients employed precisely the same method as modern scientists!* Indeed, the model of explanation they had for events in nature, for disease, for astronomical phenomena, and so on, was tried out. Corrections were attempted for the shortcomings. Eventually, through trial and error, with failures and mistakes, humanity arrived at our state of today. It may have taken a long time. Or did it really? The invention of the wheel, the first written records, may date from around the fourth millennium B.C. That makes 6,000 years up to our time. But compare that to the cave paintings of 30,000 years ago!

Amazingly, however, the earliest mathematics we encounter is qualitatively of the same nature as the mathematics of today. For no other science can one assert the same.

An important precondition for humans to be able to live in a well organized society, based on agriculture, is the existence of a reliable calendar. Indeed, without secure knowledge on the changes of the seasons, it is not possible to sow the grain or other seeds at the right time. Sowing too early may destroy the crops by nightly frost early on, and sowing too late may not leave enough time for it to ripen.

These needs were of the outmost importance, literally a question of life and death. And knowledge of a calendar is not possible without insights in astronomy, which again requires knowledge of geometry. Geometry and mathematics did also play an important role in measuring land, constructing irrigation channels or dykes along major rivers and in other engineering tasks. Some historians of mathematics speculate that the capricious and often unpredictably violent behavior of the Euphrates and the Tigris accounts for the fact that mathematics seems to have been better developed in ancient Mesopotamia than it was in ancient Egypt, where the more benign Nile behaved with exemplary regularity. But this comparison is not uncontroversial: Other historians argue that we know more about Mesopotamian mathematics than we do of the Egyptian, simply because the former was written on baked clay tablets, a practically imperishable medium, while the Egyptians wrote on papyrus which has a much shorter life under normal circumstances.

2.3 Geometry in the Land of the Pharaoh

The Egyptian civilization erected itself a proliferation of monuments in the form of huge geometric objects: The Great Pyramids. Such is the immenseness of these artifacts that some writers have speculated that they were left behind by extraterrestrials visitors to Earth. How could people without a sophisticated technology make plans for these structures, let alone carry out the actual constructions?

And the pyramids themselves have been surrounded by mysticism and speculations by puzzled observers. But the greatest pyramid of them all was not, according to some historians of mathematics, one found in the Egyptian desert. Instead, it is found on an ancient piece of papyrus, named the *Moscow Papyrus* (Fig. 2.2).

The so-called Moscow Papyrus dates from approximately 1850 B.C. The papyrus contains 25 problems or *examples*, already old when the papyrus was written. It was bought in Egypt in 1893 by the Russian collector *Golenischev*, and now resides in the Moscow Museum. The text was translated and published in 1930 by *W.W. Struve*, in [56]. This papyrus may show that the mathematical knowledge of the Egyptians went considerably further than the so-called *Rhind Papyrus* (see below) demonstrates.

In one of the problems treated there, a formula for the volume of a frustum of a square pyramid is given. If a and b are the sides of the base and the top, respectively, and h is the height, then the formula for its volume is

$$V = \frac{h}{3}(a^2 + ab + b^2)$$

This is exactly right, and its beauty and simplicity has led some historians of mathematics to reverently refer to it as *the greatest of all the Egyptian Pyramids.*

It is often asserted that this formula was unknown to the Babylonians, thus documenting a rare instance where Egyptian mathematics surpassed the Babylonian. But whereas there does exist tablets from Babylonia where the (obviously false) formula

$$V = \frac{h}{2}(a^2 + b^2)$$

is used, there also exists at least one tablet where a formula equivalent to the Egyptian one may have been employed, for a frustum of a cone.

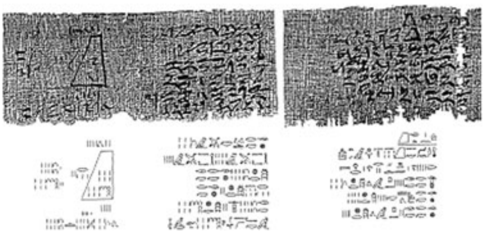

Fig. 2.2 The Moscow Papyrus with the geometry described in the text

This is according to a controversial interpretation by Neugebauer, see [58, pp. 75–76]. Much of the interpretation hinges on whether there is an error in the calculation on the tablet. By the way, some of the tablets we find from ancient Babylonia are the "papers" prepared by the students of the *Temple Schools* or the *Business Schools* which could be found in the larger cities, certainly in Babylon itself. So some sources must be treated with caution. On the other hand, there are some 22,000 tablets from the Royal Library of the last of the great Assyrian Kings *Ashurbanipal* at Nineveh.

Another explanation for the mistake might be that the correct formula can be written as

$$V = h\frac{(A + \sqrt{AB} + B)}{3},$$

where A and B denote the surface area of the base and the top (Fig. 2.3). Now $\frac{(A+\sqrt{AB}+B)}{3}$ is known as the *Heronian Mean* (named after Heron of Alexandria) of A and B. So one might speculate if two different kinds of "mean" have been confused here.

But be this as it may, the geometric insights documented by the *Greatest of the Egyptian Pyramids* is surely prodigious. It is instructive to attempt deducing this formula by our present day High School Math. We proceed as illustrated in Fig. 2.4.

We let the side of the base be a and the side of the top be b. The height of the big, uncut pyramid is $OC = T$, and the height of the small one, which has been removed, is $OC' = t$. Thus the height of the frustum is $h = T - t$, in other words the distance between the base and the top. Further $AB = a$ and $A'B' = b$ so that the similar triangles given by O, A', B', and O, A, B yield

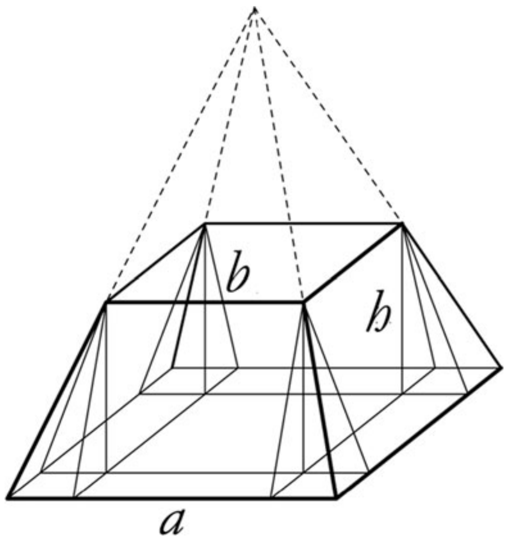

Fig. 2.3 Proof by cutting and reassembling

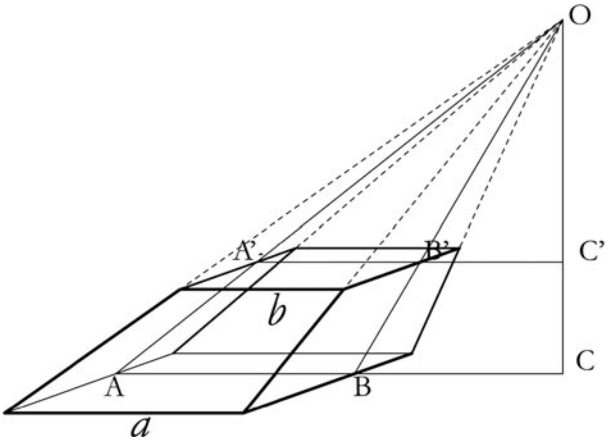

Fig. 2.4 Deducing the formula by our present day High School Math

$$\frac{T}{a} = \frac{t}{b}$$

We are now ready to compute the volume V of the frustum.

$$V = \frac{1}{3}Ta^2 - \frac{1}{3}tb^2 = \frac{1}{3}\left(\frac{T}{a}a^3 - \frac{t}{b}b^3\right) = \frac{1}{3}\frac{T}{a}(a^3 - b^3) =$$
$$\frac{1}{3}\frac{T}{a}(a-b)(a^2+ab+b^2) = \frac{1}{3}(T - \frac{T}{a}b)(a^2+ab+b^2) =$$
$$\frac{1}{3}(T-t)(a^2+ab+b^2) = \frac{1}{3}h(a^2+ab+b^2)$$

The Moscow-papyrus also contains another problem of great interest. Struve, in [56], claims that in it, Egyptian mathematicians document that they know how to compute the surface area of the sphere (actually, the hemisphere). He interprets the text as computing this area with a value for π implicitly given by the formula

$$\frac{\pi}{4} = \left(1 - \frac{1}{9}\right)^2,$$

which gives $\pi = 3\frac{13}{81} \approx 3\frac{1}{6}$. Other researchers disagree sharply with Struve's interpretation. Van der Waerden writes as follows in [58]:

> The genius of the Egyptians would have been wonderful and indeed incomprehensible, if they had succeeded in obtaining the correct formula for the area of the hemisphere.

The situation is not improved by the presence of an unfortunate hole at a decisive spot in the papyrus. Thus it must be regarded as an open question whether the

Egyptians knew the formula for the surface area of the sphere. But the claim is supported by the fact that Papyrus Rhind also does give this value for the number π.

The so-called *Papyrus Rhind* is in fact the most important papyrus for our understanding of Egyptian mathematics. It has been given this name because it was bought in Luxor by the Scottish Egyptologist *A. Henry Rhind* in 1858. Rhind, who was in poor health, had to spend some winters in Egypt. He died on his way home from his last visit there in 1863, and the papyrus was purchased from his executor by British Museum, together with another Egyptian mathematical document known as *the Leather Scroll.*

A more appropriate name for this important papyrus would be *the Ahmes Papyrus*, after the *Egyptian Scribe*[1] who copied it from a considerably older papyrus. This name is now being used more frequently. Ahmes relates on it that the original stems from the Middle Kingdom, which dates to about 2000 to 1800 B.C.

The copy by Ahmes is from around 1650 B.C. Together with the Moscow Papyrus and the Leather Scroll, the Ahmes Papyrus forms our main source for Egyptian mathematics. The Egyptians used the value given above for π, and with this value a computation which appears on the papyrus, uses the correct formula for the area of a circular disc. Altogether the papyrus has the appearance of a practical handbook of math, explaining basic methods by doing a total of 85 examples.

A very beautifully booklet has been published recently with photographs in color of the entire papyrus, transcription of the hieroglyphs and figures on it and explanation of the mathematics in a modern language. Highly recommended reading [48].

2.4 Babylonian Geometry

When we use the term *Babylonians* we actually mean the civilization residing in the whole of Mesopotamia, not just the citizens of that marvellous city Babylon. This culture was already highly developed at the time from which we find the earliest records, the ancient culture of the *Sumerians*. The main city was not Babylon, until comparatively recent times. The ancient city of *Ur* in southern Mesopotamia was the spiritual and political center for a long time. The Sumerians arrived in this region with their culture already well developed, we do not know from where. The political hegemonies shifted over time, most notably with the arrival of the Akkadians, of which the Babylonians eventually were part. But new rulers carefully preserved the old culture of the Sumerians, and the Kings carefully collected ancient books, baked clay tablets, in Libraries, and made translations into the Akkadian from the Sumerian. In fact we have preserved elaborate dictionaries for the two languages, as well as parallel translations.

The Babylonians had a sophisticated way of representing numbers and computing. They represented numbers to the base 60, in the same way as we represent

[1] *Ahmes* is the earliest individual name associated with mathematics which we know.

Fig. 2.5 Some sexagesimal digits. Above (10), (20), (30), (40) and (50). Below (1), (2), (21), (19) and (59)

numbers to the base 10. Thus for instance they would represent the number 61 as (1)(1), while the number 6,359 would be represented as (1)(45)(59).

The name *sexagesimal* comes from the Latin term *sexagesimus*, which means "sixtieth". The word *sex* is Latin for *six*. In Greek "six" is *hex*, hence the terms *hexadecimal*, meaning the number system with base 16, used extensively in Computer Science. Further, the term *hexagon* means *6-gon*. In Fig. 2.5 we have written the *sexagesimal* digits in parenthesis. Those possible digits are of course (0), (1), ..., (59). Using a stylus usually cut from reed, the Babylonians impressed wedges on clay tablets, which were subsequently baked if the writing was to be preserved. Wedges of different shapes were used, thus making it possible to codify a large set of characters. The digits from 1 to 59 were build up of two types of wedges, in the simplest script in use (others were also present at different epochs). In Fig. 2.5 we see some digits, ending with (59). Note the mixture of base, as the individual digits in the base 60-system were represented with symbols for 1's and 10's.

The Babylonians did not directly use the digit (0) in the beginning, but did so indirectly by leaving an open space: Nothing there! But as *scribes*, writers and copiers, copied old tablets to new clay to be baked, mistakes were easily made. So to clarify matters, they started to write a symbol which meant *None* or *Not*. But trailing zeroes were not used. Thus context would have to determine whether (1) meant 3,600, 60, 1, $\frac{1}{60}$, ... Even though we would find this clumsy, it represented a *numerology*, a representation and understanding of numbers, far superior to that of the Egyptians, Greek or the Romans.

We know a great deal about the mathematics of the Babylonians. This research was to a large degree initiated by *Otto Neugebauer* and his collaborators and associates. Like many others Neugebauer had to flee Germany during the Nazi era, and came to the United States. He uncovered and interpreted many tablets from Babylonia, and made the striking discovery of the meaning of the most famous of all tablets which have been found until now, and which we shall return to below.

While realizing that the Babylonians had admirable mathematical insights, historians of mathematics had no clear understanding of the motivation behind it. In fact, it was a widespread view that all mathematics prior to the Greek period only consisted of simple practical computations for everyday applications in trade, agriculture and simple engineering tasks. Mathematics as the science we know it, they maintained, did not exist until the advent of the Greek. This view would be espoused since it was the Greek who introduced the concept of a mathematical proof.

But it is a fundamental misunderstanding that there can be no mathematics as a science without our modern notion of *proof*. Indeed, the creative process which every research mathematician engages in when mathematics is discovered is almost the complete opposite of a formal proof. Only *à posteriori* do we mathematicians cloak our work in the formal style of *Satz–Beweis*, so beloved by some professors but equally hated by the majority of their students. Of course proofs are necessary so as to ensure correctness of results. And actually *finding* a proof of a conjecture everyone believes to be true is also very much central to mathematics, as in the case of Andrew Wiles' proof of the famous Fermat Conjecture in the last decade of the twentieth century, or *Grigori Perelman's* recent proof of the *Poincaré Conjecture.* But it really is not necessary to have produced a formal proof of a mathematical theorem in order to document complete knowledge of why the theorem is indeed true.

As it happens, a careful analysis of a baked clay tablet from ancient Babylon elucidates this point very well.

The tablet which is perhaps the most famous one, has been given the name *Plimpton 322.* It signifies that it is the tablet numbered 322 in the *A.G. Plimpton* collection at Columbia University in New York. The tablet is written in old Babylonian characters, dating from the period 1900–1600 B.C. We follow some of the description of the tablet in E. Robson [51]: The tablet is about 13 by 9 by 2 cm. Its second and third column list the smallest and largest member of Pythagorean triples, one may think of the shortest side and the hypothenuse of a right angled triangle. The final column contains the line count from 1 to 15. Unfortunately the tablet is damaged, in that a piece along the entire left edge is missing. Moreover, there is a deep indentation at the middle of the right hand side. Finally, it is also somewhat damaged at the upper left corner. So the first column is partly broken away. It may have contained either the square of the hypothenuse divided by the square of the longest side, or the square of the shortest side divided by the square of the longest side.

Whatever interpretation of these incomplete data, however, the tablet documents that the Babylonians had firm knowledge of so called Pythagorean triples.

Some claim that it has been found traces of modern glue along the rupture-edge at the left, thus indicating that it was complete at excavation, but broke thereafter in the possession of individuals with access to such amenities as glue, who attempted repairing it.

If so, it would be interesting if the missing piece could somehow be traced. It could reside in one of the many bins of unclassified and unintelligible fragments of Babylonian tablets. As it happens, this was the gravest danger facing the ancient tablets: Destruction at the time of their excavation, which was often – at least in the beginning – done quite crudely.

The tablet was acquired by an interesting character named *Edgar James Banks*, (1866–1945).[2] He was an American college professor, antiquities collector and dealer, and adventurer. He was active in the Ottoman Empire, at the end of its

[2] We follow [61] among other sources.

existence, and is probably an original for the figure of *Indiana Jones*. He started out as American consul in Baghdad in 1898, and bought many cuneiform tablets on the markets of the decaying Ottoman Empire. These he resold in carefully planned small installments, so as not to flood the international market and thus deflate prices. The tablets went to museums, libraries, universities, and theological seminaries. One of the tablets which Banks sold, was, according to the information he gave, from Senkereh in southern Irak, near the ruins of the ancient city of Larsa. He sold this tablet to Professor Eugene Smith of Columbia University in New York. Smith willed his books to the university, and the tablet is today number 322 in the A.G. Plimpton library's collection of rare books.

The contents of Plimpton 322 demonstrates that the Babylonians had firm knowledge Pythagorean triples. They probably also knew the so called Pythagorean Theorem. In what sense did they know this? In the absence of firm knowledge we may ask questions and speculate. Before proceeding with Plimpton 322, I shall present the simplest and most beautifully proof I know of this theorem.

Is it the *Babylonian proof*, the proof they knew? But they would not call it a proof, but regard it as an example of using the *rule by which certain areas may be added.* And, of course, we give the proof here in modern language and symbolism. But first we give a more conventional proof, the principle behind it might also have been known to the Babylonians, in Fig. 2.6. See Howard Eves, [14].

In Fig. 2.6 the three sides in the right triangle are labelled as above: The hypothenuse as d, the two others as a and b, where $a \geq b$. We then set

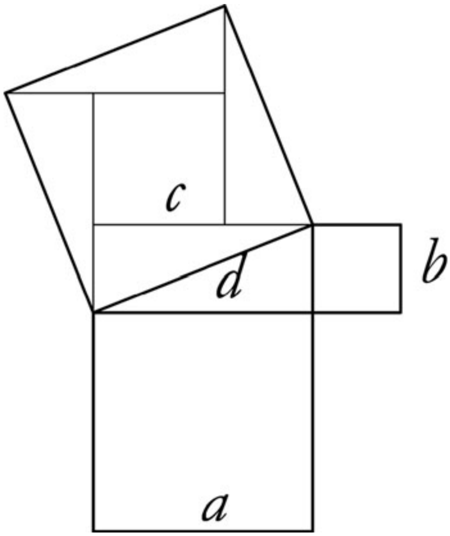

Fig. 2.6 A (very hypothetical) Babylonian proof of "Pythagoras' Theorem". The essential part of this figure, namely the subdivision of the largest square, appears in the oldest Chinese mathematical text we know, the *Chóu-pü*, from the second millennium B.C. Thus evidence suggests that this insight formed part of a common wisdom in the ancient world

$$c = a - b$$

From the figure we now see that the area of the square on the hypothenuse, d^2, is equal to c^2 plus the areas of the four right triangles congruent with the given one. As the area of a triangle is equal to *half the base times the height*, a fact well known to the Babylonians, we get

$$d^2 = c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$$

But as the Babylonians also knew,

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

which, using the formula in the case of the minus-sign, finally yields

$$d^2 = a^2 + b^2,$$

as desired.

Figure 2.6 and the corresponding proof is one possibility. A variation of the same theme, less familiar to us in our usual thinking concerning "Pythagoras' Theorem", but even more in line with the way the *Babylonians* thought, is a proof derived from Fig. 2.7.

Indeed, the Fig. 2.7 yields

$$(a + b)^2 = d^2 + 2ab,$$

from which follows $d^2 = a^2 + b^2$.

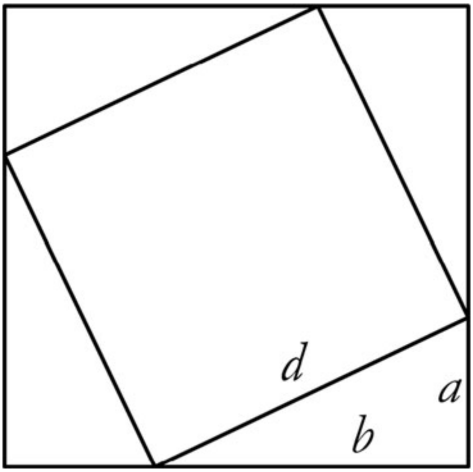

Fig. 2.7 The Putative Babylonian Proof of "Pythagoras' Theorem"

Such methods for dealing with sums of squares is well documented from Babylonian tablets. From [55, pp. 27–28], we reproduce the following example, to be found on a tablet in Strasbourg's *Bibliothèque National et Universitàire*. Phrased in modern language:

> An area A, consisting of the sum of two squares, is 1,000. The side of one square is $\frac{2}{3}$ of the side of the other square, diminished by 10. What are the sides of the square?

The Babylonians would solve this as follows, again presented in modern language: The sides of the respective squares are denoted by x and y. We then have $x^2 + y^2 = 1{,}000$, as well as the relation $y = \frac{2}{3}x - 10$. Squaring the latter yields

$$y^2 = \frac{4}{9}x^2 - 2 \cdot \frac{2}{3}x \cdot 10 + 10 \cdot 10 = \frac{4}{9}x^2 - \frac{40}{3}x + 100.$$

Substitution into the first equation yields

$$\frac{13}{9}x^2 - \frac{40}{3}x - 900 = 0.$$

Having thus transformed the geometric problem into an algebraic one, the Babylonian scholars and scribes – rather, in the present case presumably students doing their homework – could find the solution utilizing their knowledge about equations and systems of equations. The answer to the present problem is 30, the one positive solution of the equation.

The presentation of the solution starts like this: *"Square 10, this gives (1)(40) (i.e., 100). Subtract (1)(40) from (16)(40) (i.e., 1,000), this gives (15)(0) (i.e., 900)..."*

We return to Plimpton 322 (Fig. 2.8). The tablet contains a table of numbers, arranged in four columns of 15 numbers each. The rightmost column just consists of the numbers $1, 2, \ldots, 15$. The column to the left is partly destroyed by the missing part.

Today Neugebauer and Sachs' explanation is no longer generally accepted. An alternative explanation is by so called *reciprocal pairs*. The explanation is due to a number of authors. Accounts of this work, with references, may be found in Eleanor Robson [51] and [50], as well as in *Jöran Friberg's* book [15]. Friberg ties Plimpton 322 to an Old Babylonian generating rule, which has been ascribed to Pythagoras and Plato, and also appear in Euclid's Elements, Book X. Below we shall start with the explanation which was given by Neugebauer and his collaborators, then give a briefly summary of Friberg's and Robson's explanation of the method of regular reciprocal pairs.

However, the competing explanations are mathematically related, and they have similar far reaching consequences. They demonstrate first of all that the Babylonians knew ways to generate such triples (a, b, d). It is also fairly certain that they knew the so-called Pythagorean Theorem. But exactly how did they work out their list of the Pythagorean triples?

As already stated, the rightmost column only serves to *number* the entries in the other columns. But the two next columns look at first rather haphazard and arbitrary.

Fig. 2.8 The tablet Plimpton 322

At first this led some to assume that the tablet merely constituted a fragment of some business-files, which are actually very much present in quantity among the ancient tablets from Babylon. But the column to the left bears the heading "*diagonal*", while the next has the heading "*breadth*". As with most of the numbers in the first row, to the left, also the heading here is illegible. But the consensus of opinion among the experts is that the numbers constitute in some way a list of Pythagorean triples. How they are presented, however, cannot be ascertained with certainty.

Clearly, given any Pythagorean triple (a, b, d), we get another by multiplying each number by the same natural number r, obtaining (ra, rb, rd). Thus we need only to generate the so-called *primitive* Pythagorean triples, that is to say the triples where the numbers do not have a common factor >1. Now there is an elegant way of generating all possible primitive Pythagorean triples. Usually the method and its proof is attributed to *Diophantus*, as it is explained in his *Arithmetica*. But recent detective work might indicate that *Hypatia of Alexandria* deserves some of the credit for this work, see Chap. 4, Sect. 4.20, as well as [11].

In Sect. 2.7 we shall give a completely modern account of the method for finding the primitive Pythagorean triples, according to Diophantus as reconstructed by *Fermat and Newton* much later, still. The reader who is only interested in Plimpton 322, may skip that section.

Only a small fragment of this theory is really needed in the explanation of the Pythagorean triples on Plimpton 322. In particular the full statement of Theorem 1 in Sect. 2.7 is not needed, it suffices to carry out the obvious verification

$$(v^2 - u^2)^2 + (2uv)^2 = (v^2 + u^2)^2,$$

thus

$$a = v^2 - u^2,\ b = 2uv \text{ and } d = v^2 + u^2$$

form a Pythagorean triple. But when the significance of Plimpton 322 was first discovered by Neugebauer and Sachs, there was a tendency to interpret it as evidence of a much more far reaching mathematical knowledge on the *parametrization of primitive Pythagorean triples*.

2.5 The (u,v) Explanation of Plimpton 322

It is generally accepted that the tablet contains four errors. Three of them are easy to explain as a simple mistake with the stylus, whereas the fourth is more mysterious. Several explanations have been offered, but as long as we only have this one table of this type, and in view of the missing part, it is difficult to decide what the correct explanation is.

At any rate, except for these presumed errors the second and third column from the right consists of the numbers b and d described above, for the choices of u and v shown in the table presented as Fig. 2.9.

We have written the corrected numbers with the presumed erroneous ones in parenthesis.[3] We start with entry number 11: The values 2 and 1 should give $(b, a, d) = (4, 3, 5)$ which is not shown. Instead this triple is multiplied with 15, to give more palatable digits in the Babylonian number system. Next we note that the last entry, in line number 15, is not a primitive triple.

b	a "Breadth"	d "Diagonal"	No.	v	u
120	119	169	1	12	5
3456	3367	4825 (11521)	2	64	27
4800	4601	6649	3	75	32
13500	12709	18541	4	125	54
72	65	97	5	9	4
360	319	481	6	20	9
2700	2291	3541	7	54	25
960	799	1249	8	32	15
600	481 (541)	769	9	25	12
6480	4961	8161	10	81	40
60	45	75	11	2	1
2400	1679	2929	12	48	25
240	161 (25921)	289	13	15	8
2700	1771	3229	14	50	27
90	56	106 (53)	15	9	5

Fig. 2.9 The reconstructed Plimpton 322

[3] Also note that these are all *primitive* triples corresponding to the given values of v, u, with the exception of two entries. This observation is only interesting for the readers who will study Sect. 2.7.

If this explanation of the numbers on Plimpton 322 is correct, the numbers u and v would be carefully chosen. First, they would all be regular sexagesimal numbers: Their inverses are finite sexagesimal fractions. That such choices are possible at all for the entire table is due to the choice of base 60, which has the prime factors $2, 3, 5$, whereas base 10 only has $2, 5$. Thus for instance, in Babylonia they would have $\frac{1}{3} = (0) \cdot (20)$ and $\frac{1}{15} = (0) \cdot (4)$. Then the tricky long division in the sexagesimal system could be avoided in many cases, and replaced by multiplication, which they easily performed using multiplication tables on baked clay tablets.

With our base of 10, *we* have a special relationship to the numbers 3, 7 and 13, as being, respectively, *lucky, sacred and unlucky*. The Babylonians do not seem to have offered 3 much thought, but 7 was sacred and 13 was very unlucky, *The Number of the Raven*.

As stated above the leftmost column may have contained either the square of the hypothenuse divided by the square of the longest side of the triangle, or the square of the shortest side divided by the square of the longest side. Therefore it has been speculated that this tablet might have been used in computations as equivalent to a table over $\cot an(\varphi)$ or $\cos(\varphi)$ for angles φ between $44°46'$ and $31°53'$. According to Robson [51], page 112 the concept of angle is anachronistic, in that the Babylonians did not have this concept.[4] The decrement in the values of φ are not constant, and $\sec(\varphi)$ decreases by very roughly $\frac{1}{60}$ from one line to the next. Is Plimpton 322 part of a set of "trigonometric tables" for use in astronomy and engineering? Some might still like to believe that, but there is no evidence for such a usage. On the contrary, Babylonian astronomy and astrology flourished much later than the Old Babylonian Epoch, which the tablet comes from.

2.6 Regular Reciprocal Pairs, Babylonian Number-Work and Plimpton 322

The Babylonians did most of their number-work relying on tables. For example, multiplication could be carried out using the tables of squares by the formula

$$xy = \frac{1}{4}((x+y)^2 - (x-y)^2).$$

Moreover, one should note that

$$\frac{1}{4} = (0) \cdot (15),$$

and multiplication with this number is especially simple in base 60, much like multiplying by 0.2 or 0.5 in base 10.

[4] But this would not preclude that there might exit tables which have served a similar purpose to trigonometric tables.

In addition to tables of squares, the students of the ancient scribal schools had to learn sexagesimal multiplication tables by heart, and also had to learn tables of *regular sexagesimal reciprocal pairs*.[5] These tables were important for a handy conversion of a problem of division into a problem of multiplication.

As an illustration of a division using this, we look at

$$123 : 12 = 10.25,$$

with our decimal system, in modern sexagesimal notation $(10) \cdot (15)$, while the Babylonians would write the answer as $(10)(15)$.

The Babylonians would very probably *not* handle such an easy division by their Method of Reciprocal Pairs, but nevertheless, here is how it works: First observe that $12 \times 5 = 60$, thus in modern sexagesimal notation $\frac{1}{12} = (0) \cdot (5)$, and in Babylonian notation the reciprocal of (12) is (5). Since, as we would write

$$123 : 12 = 123 \times \frac{1}{12} = (2)(3) \times (0) \cdot (5),$$

the Babylonians would proceed to multiply $(2)(3)$ with (5), obtaining the answer $(10)(15)$, immediately and without having to consult tables of squares. Finally this answer has to be interpreted right, going back to the context. The correct answer is $10 + \frac{15}{60} = 10\frac{1}{4}$ rather than, for instance, $10 \times 60 + 15 = 615$.

Now we return to Pythagorean triples. We have worked above with a particular reciprocal pair, namely $(12, \frac{1}{12})$ in our notation. Now it turns out that every such pair of reciprocals x and $x' = \frac{1}{x}$ yields two rational numbers $b' = \frac{x-x'}{2}$ and $d' = \frac{x+x'}{2}$ such that with $a' = 1$ we get $a'^2 + b'^2 = d'^2$, in other words (a', b', d') is a rational Pythagorean triple. In fact, since $xx' = 1$ we get

$$\begin{aligned} a'^2 + b'^2 &= \left(\frac{x - x'}{2}\right)^2 + 1 = \frac{x^2 - 2xx' + x'^2 + 4}{4} \\ &= \frac{x^2 + 2xx' + x'^2}{4} = \left(\frac{x + x'}{2}\right)^2 = d'^2. \end{aligned}$$

With $x = 12$ we obtain $b' = \frac{143}{24}$ and $d' = \frac{145}{24}$. Scaling this rational triple we get a Pythagorean triple of integers $(24, 143, 145)$, which by the way does not appear on Plimpton 322.

Now, going back to the (u, v)-explanation, we have that

$$a = v^2 - u^2,\ b = 2uv \text{ and } d = v^2 + u^2,$$

thus

[5] See Robson [51, p. 113].

$$a' = \frac{a}{b} = \frac{v^2 - u^2}{2uv} = \frac{1}{2}(x - x'),\ b' = \frac{b}{b} = 1 \text{ and } d' = \frac{v^2 + u^2}{2uv} = \frac{1}{2}(x + x'),$$

where $x = \frac{v}{u}$ and $x' = \frac{u}{v} = \frac{1}{x}$. Hence from a mathematical point of view the two explanations are equivalent. However, the point is that regular reciprocal pairs are ubiquitous in Babylonian mathematics, whereas primitiveness and parametrization appears nowhere else. This argument alone would lead one to discard the (u, v)-version of the explanation in favor of the regular reciprocal pairs.

Friberg, in [15, p. 92], refers to the rule

$$d, b, a = \frac{x + x'}{2}, 1, \frac{x - x'}{2}$$

as the *Old Babylonian generating rule*, and he argues on page 88 for the following tentative translation of the headings of Plimpton 322, although as he states "The meaning [...] is far from obvious":

> The square of the holder for the diagonal (from) which 1 is subtracted, then [the square of the holder for] the front comes up. The square side of [the square of the holder for] the front. The square side of [the square of the holder for] the diagonal. Its line number.

2.7 Parametrization of Pythagorean Triples

We now explain the complete theory of parameterizing primitive Pythagorean triples. Let (a, b, d) be a Pythagorean triple. We then have

$$\left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 = 1,$$

i.e., the point $(x, y) = (\frac{a}{d}, \frac{b}{d})$ lies on the unit circle which has the equation

$$x^2 + y^2 = 1.$$

So the problem is equivalent to finding all points with rational coefficients on this circle. We now pull one of today's standard tricks, taught in every class of first year calculus: We wish to find *a rational parametrization of the circle*, that is to say, to find rational expressions in some variable t, $x = \varphi(t)$, $y = \psi(t)$, such that when t varies, then $(\varphi(t), \psi(t))$ runs through all points on the circle. The trick is to let t be the slope of the line through the point $(-1, 0)$, see Fig. 2.10.

The equation of this line is

$$y = t(x + 1),$$

which we substitute into the equation for the circle, thus obtaining

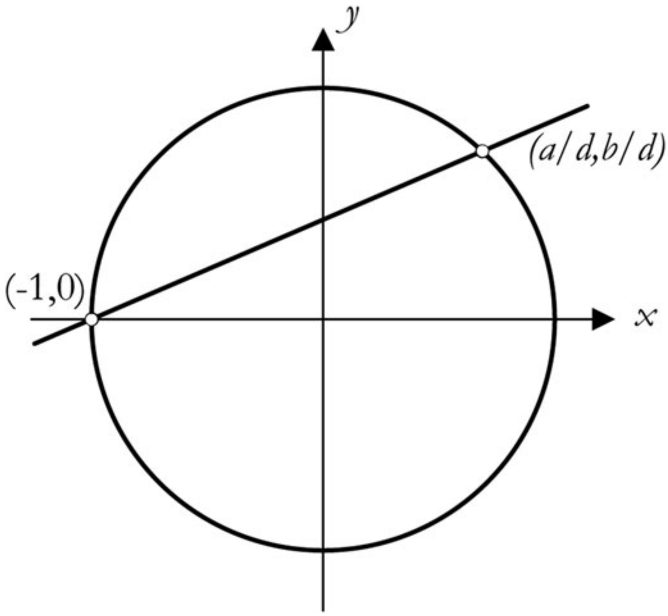

Fig. 2.10 Finding all rational points on the *circle*

$$x^2 + t^2(x+1)^2 = 1,$$

and hence

$$(1+t^2)x^2 + 2t^2x + t^2 - 1 = 0,$$

which, as $1 + t^2$ is never zero, may be written as

$$x^2 + \frac{2t^2}{1+t^2}x + \frac{t^2-1}{1+t^2} = 0.$$

Now the formula for the roots of the general second degree equation,

$$x^2 + px + q = 0,$$

is

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q},$$

which when applied to the equation in question here yields

$$x = -\frac{t^2}{1+t^2} \pm \sqrt{\left(\frac{t^2}{1+t^2}\right)^2 - \frac{t^2-1}{1+t^2}} = -\frac{t^2}{1+t^2} \pm \frac{1}{1+t^2},$$

after a short computation. We thus obtain

$$x = -1 \text{ or } x = \frac{1-t^2}{1+t^2}$$

Substituting the last solution into the equation for the line, we get

$$y = t\left(\frac{1-t^2}{1+t^2} + 1\right) = \frac{2t}{1+t^2}$$

Since the points (x, y) have rational coordinates, we may write $t = \frac{u}{v}$ for natural numbers u and v. Here we must have $v > u$ since the slope t of our line lies in the interval $<0, 1>$. Substituting this into the expressions for x and y, we obtain the following formulas:

$$x = \frac{a}{d} = \frac{v^2 - u^2}{v^2 + u^2}, \quad \text{and} \quad y = \frac{b}{d} = \frac{2vu}{v^2 + u^2}.$$

We have essentially completed all ingredients needed to prove the following:

Theorem 1. *All primitive Pythagorean triples (a, b, d) are given by*

$$a = v^2 - u^2, b = 2uv \quad \textit{and} \quad d = v^2 + u^2,$$

where u and v are positive integers, $v > u$, without a common factor > 1. Moreover, u and v are not both odd numbers.

Proof. First of all, numbers of the form $a = v^2 - u^2, b = 2uv, d = v^2 + u^2$ where u and v are natural numbers do form a Pythagorean triple, as is seen by computing $a^2 + b^2$. If we assume that u and v have no common factor > 1, then the triple is also primitive, *except for the possibility that* $a = 2\overline{a}, b = 2\overline{b}$ and $d = 2\overline{d}$. Indeed, a, b, d can have no other common factor than 2, and it is easily seen that this happens if and only if u and v are both odd numbers. Then the overlined numbers do form a primitive Pythagorean triple. In this case we introduce new versions of u and v by putting

$$\overline{v} = \frac{v+u}{2}, \quad \overline{u} = \frac{v-u}{2},$$

from which we find

$$2\overline{uv} = \frac{v^2 - u^2}{2} = \overline{a}, \overline{v}^2 - \overline{u}^2 = uv = \overline{b} \text{ and } \overline{v}^2 + \overline{u}^2 = \frac{v^2 + u^2}{2} = \overline{d}$$

It is not difficult to verify that $\overline{v}, \overline{u}$ have no common factor > 1, and are not both odd numbers.

Now, given a primitive Pythagorean triple (a, b, d). From the considerations preceding the formulation of the theorem, we can always find natural numbers v and u, such that

$$\frac{a}{d} = \frac{v^2 - u^2}{v^2 + u^2}, \text{ and } \frac{b}{d} = \frac{2vu}{v^2 + u^2}.$$

Unless u and v are both odd numbers, we therefore have that $a, b,$ and d must be as claimed in the theorem. If u and v are both odd, then we proceed as above, obtaining new u and v's, $\overline{v} = \frac{v+u}{2}, \overline{u} = \frac{v-u}{2}$, also without common factors, but now not both odd numbers, such that

$$2\overline{v}\overline{u} = \frac{v^2 - u^2}{2}, \overline{v}^2 - \overline{u}^2 = uv, \text{ and } \overline{v}^2 + \overline{u}^2 = \frac{v^2 + u^2}{2}.$$

Thus the primitive Pythagorean triple a, b, d is described as in the theorem, but with the roles of a and b interchanged. □

Exercises

Exercise 2.1 An ancient method for computing the area of a circle is to take the average of the areas of the inscribed and the circumscribed squares. What value for π does this method correspond to?

The following exercises are modern generalizations of problems which come from Babylonian clay tablets. You are free to use all your modern algebra and calculus. See [14, pp. 58–59], for the original formulation and more information on these problems.

Exercise 2.2

(a) An Old Babylonian tablet, that is to say a tablet from the period 1900–1600 B.C., the same time as Plimpton 322, poses a problem about a *ladder* standing upright against a wall.
This problem deals with a ladder of known length b stands upright against a wall. The ladder is then allowed to slide down a known distance a. The question is how far out from the wall the lower end of the ladder will be.
(b) A similar problem comes from a much later period, namely the Seleucian epoch, about 300 B.C.–300 A.D. This problem states the following: A reed stands up against a wall, and then slides down a known distance a, which results in the lower end moving out a known distance b from the wall. The question is how long the reed is.

Exercise 2.3 Find the radius of the circumscribed circle of an isosceles triangle with sides b, b and $a > b$. On the tablet $a = 60, b = 50$. For these values, write the answer in the sexagesimal system.

Exercise 2.4 Find the sides x and y of a rectangle, when it is given that $xy = A$ and that $x^3d = B$, where d is the diagonal. Find the answer when $A = 12$ and $B = 320(= (5)(20))$. Then compute the answer for the values on the tablet,

$A = (20)(0)$, $B = (14)(48)(53)(20)$. For these two sets of values, write the answer in the sexagesimal system.

Exercise 2.5 Find the area A of an isosceles trapezoid with bases a and b and sides s. On the tablet $a = (50)$, $b = (14)$ and $s = (30)$. For these values, write the answer in the sexagesimal system.

Exercise 2.6 One leg of a right triangle is a. A line parallel to the other leg at a distance h from it cuts off a right trapezoid of area A. Find the lengths of the bases of the trapezoid. On the tablet $a = (50)$, $h = (20)$ and $A = (5)(20)$. For these values, write the answer in the sexagesimal system.

Exercise 2.7 An area consisting of the sum of two squares is A. The side of one square is 10 less than $\frac{2}{3}$ of the side of the other square. What are the sides of the square? On the tablet $A = (16)(40)(= 1{,}000)$. For this value, write the answer in the sexagesimal system.

Exercise 2.8 A rectangle has area A and perimeter B. Find the lengths of the sides x and y. Take $A = (1)(40)$, $B = (1)(44)$. For these values, write the answer in the sexagesimal system.

Exercise 2.9 An Old Babylonian tablet, found at Susa, gives the ratio of perimeter and circumference of the circumscribed circle as $(0)\cdot(57)(36)$ for a regular hexagon. Use this to find an approximate value for π, written sexagesimally.

The following two problems are inspired by the Moscow Papyrus.

Exercise 2.10 The area of a rectangle is A, and the width is the fraction $\frac{p}{q}$ of its length. Find the dimensions of the rectangle. Compute the answer when $A = 12$, $p = 3$ and $q = 4$.

Exercise 2.11 The area of a right triangle is A, and one leg is m times the other. Find the dimensions of the rectangle. Compute the answer when $A = 20$ and $m = 2.5$.

The two following exercises are based on information from [58].

Exercise 2.12 The ancient Egyptians computed the area of a triangle and a trapezoid correctly. But the quadrangles were some times treated as follows: Half the sum of two opposite sides was multiplied by half the sum of the other two sides. Is this method correct? If no, when does the method yield a correct answer?

Exercise 2.13 To find the area of a circle, the Egyptians squared the diameter and multiplied by $\frac{8}{9}$. What value for π does this give?

Exercise 2.14 As stated in the text, it has been speculated that the tablet Plimpton 322 might have been used in computations as equivalent to a table over $\cot(\varphi)$ or $\cos(\varphi)$ for angles φ between $44°46'$ and $31°53'$, or at least perhaps served a similar purpose to such a table. The decrement in the values of $\sec^2(\varphi)$ is very close to $\frac{1}{60}$ from one line to the next. Assuming that Plimpton 322 were part of such a collection, try to compute the 15 numbers the preceding tablet would have contained.

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