Preface

This is a revised edition of the first printing which appeared in 2002. The book is based on lectures at the University of Bergen, Norway. Over the years these lectures have covered many different aspects and facets of the wonderful field of geometry. Consequently it has never been possible to give a full and final account of geometry as such, at an undergraduate level: A carefully considered selection has always been necessary. The present book constitutes the main central themes of these selections.

One of the groups I am aiming at, is future teachers of mathematics. All too often the texts dealing with geometry which go into the syllabus for teacher-students present the material in ways which appear pedantic and formalistic, suppressing the very powerful and dynamic character of this old field, which at the same time so young. Geometry is a field of mathematical insight, research, history and source of artistic inspiration. And not least important, an integral part of our common cultural heritage.

Another motivation is to provide an invitation to *mathematics in general*. It is an unfortunate fact that today, at a time when mathematics and knowledge of mathematics is more important than ever, the phenomena of *math avoidance* and *math anxiety* are very much present under different names all over the world. It is an important task to attempt seriously to heal these ills. Perhaps they are inflicted on students at an early age, through deficient or even harmful teaching practices. Thus the book also aims at an informed public, interested in making a new beginning in mathematics. And in doing so, learning more about this part of our cultural heritage.

The book is divided in two parts. Part I is called *A Cultural Heritage.* This section contains material which is normally not found in a mathematical text. For example, we relate some of the stories told in [28] by the Greek historian *Herodotus*. We also include some excursions into the history of geometry. These excursions do not represent an attempt at writing the history of geometry. To write an introduction to the history of geometry would be a quite different and very challenging undertaking.

To write the History of Geometry is therefore definitely not my aim with Part I of the present book. Instead, I wish to seek out the roots of the themes to be treated in Part II, *Introduction to Geometry.* These roots include not only the geometric ideas and their development, but also the historical context. Also relevant are the legends and tales – really *fairy-tales* – told about, for example, *Pythagoras*. Even if some of the more or less fantastic events in *Iamblichus'* writings are unsubstantiated, these stories very much became our *perception* of the geometry of Pythagoras, and thus became part of the *heritage of geometry*, if not of its *history*.

In Chaps. 1 and 2, we go back to the beginnings of science. As geometry represents one of the two oldest fields of mathematics, we find it in evidence from the early beginnings. The other field being *Number Theory*, they go back as far as written records exist. Moreover, in the first written accounts from ancient civilizations they present themselves as already well developed and sophisticated disciplines.

Thus we find that problems which ancient mathematicians thought about several thousand years ago, in many cases are the same problems which are difficult to handle for the students of today. As we move on in Chaps. 3 and 4, we find that great minds like Archimedes, Pythagoras, Euclid and many others should be allowed to speak to the people of today, young and old. They are unsurpassable tutors.

The mathematical insight which Archimedes regarded as his most profound theorem, was a theorem on geometry which was inscribed on Archimedes' tombstone. All of us, from college student to established mathematician, must feel humbled by it. What does it say? Simply that if a sphere is inscribed in a cylinder, then the proportion of the volume of the cylinder to that of the sphere, is equal to the proportion of the corresponding surface areas, counting of course top and bottom of the cylinder. The common proportion is $3 : 2$. This is a truly remarkable achievement for someone who did not know about integration, not know about limits, not know about... Its beauty and simplicity beckons us. How did Archimedes arrive at this result? Archimedes deserves to be remembered for this, rather for the silly affair that he ran out into the street as God had created him, shouting – *Eureka, Eureka!* But the story may well be true, his absentmindedness under pressure cost him dearly in the end.

A new addition to the present revised edition is a more extensive treatment of the Archimedean polyhedra, starting in Sect. 4.4. The Archimedean polyhedra are also treated in Sect. 6.9, where the process of finding these polyhedra is also tied to finding all semi regular tessellations of the plane. It becomes evident that the tessellations and the polyhedra are intimately connected. This relies on the pioneering work of *Johannes Kepler* (1571–1630), long after Archimedes. Our study of the Archimedean polyhedra is concluded in Sect. 20.6, where the mathematics of symmetries in space is outlined.

Pythagoras and his followers certainly did not discover the so-called *Pythagorean Theorem*. The Babylonians, and before them the Sumerians, not only knew this fact very well, they also knew how to construct all *Pythagorean triples*, that is to say, all natural numbers a, b and d such that $a^2+b^2 = d^2$. This is documented by a famous babylonian clay tablet, now in the library of Columbia University in New York, it is known today as the tablet *Plimpton 322*. The tablet was originally found or otherwise obtained by an American collector and adventurer who is a probable model for the character of *Indiana Jones*.

Some have speculated whether the Babylonians used these insights to construct the equivalent of trigonometric tables. Such tables would be simple, accurate and powerful thanks to the *sexagesimal* system they used for representing numbers. But this theory has few if any followers today, for one thing the very concept of *an angle* lay far into the future at the time of the creation of Plimpton 322.

In any case it would be a challenging project for interested college students to understand the mathematics of the Plimpton 322 tablet at Columbia University, and to correct and explain the four mistakes in it. Or perhaps even to construct the successor or the predecessor of this tablet if we allow ourselves to imagine that it were indeed one of several in a series of tablets, constituting “trigonometric tables”.

So what did Pythagoras discover himself? We know nothing with certainty of Pythagoras’ life before he appeared on the Greek scene in midlife. Some say that he travelled to Egypt, where he was taken prisoner by the legendary, in part infamous, Persian King *Cambyses II*, who also ruled Babylon, which had been captured by his father *Cyrus II*. Pythagoras was subsequently brought to Babylon as a prisoner, but soon befriended the priests, the *Magi*, and was initiated into the priesthood in the temple of Marduk. We tell this story as related by Herodotus and Iamblichus in [28] and [37]. However, the accounts given in these classical books are not always historically correct, the reader should consult the footnotes in [28] to get a flavor of the present state of Herodotus, *The Father of History*, who by some of his critics is called *The Father of Lies.* But Herodotus is a fascinating story-teller, and the place occupied by Pythagoras today has considerably more to do with the legends told about him than with what actually happened. So with this warning, do enjoy the story.

Euclid’s Elements represents a truly towering masterpiece in the development of mathematics. Its influence runs strong and clear throughout, leading to non-Euclidian geometry, Hilbert’s axioms and a deeper understanding of the foundations of mathematics. The era which Euclid was such an eminent representative of, ended with the murder of another geometer: Hypatia of Alexandria.

Chapter 5 is a new addition to this revised edition. It deals with the Arabic contribution to geometry and mathematics in the epoch following the decline of European mathematics and civilization. The central themes include the House of Wisdom in Baghdad, the life and work of Al Khwarizmi, who invented and gave rise to the name of the the field of Algebra in our sense of the word. Ibn Qurra, who was one of the first real reformers of the Ptolemaic astronomy, and also did important work in mechanics. Al Battani also did important astronomical work, and used a new method in geometry which we today call *trigonometry*. Al Wafa al Buzjani did work on mensuration, and as a method for craftsmen devised constructions with a so called *rusty compass*. Al Quhi solved a problem on spheres posed by Archimedes, by intricate procedures using methods developed by Euclid, Apollonius and Archimedes. We give a modern treatment of this material, we see methods as powerful as our own advanced college calculus! Ibn Hud who was King of Saragossa and the real discoverer of the so called Ceva’s Theorem. Omar al Khayyam was both mathematician an an eminent poet and philosopher. His work on algebraic equations, as continued by Sharaf al Din, significantly paved the road for the Italian geometers who respectively found or rediscovered and found methods to solve a general equations of degree three and four by radicals. He is some times referred to as the *Voltaire of*

Persia. We finally give some glimpses of the work and biography of the fascinating mathematician ant thinker *Nasir al Din al Tusi.*

The cultural and scientific contribution of Arabic civilization is a vast and important field, and in this book it has only been possible to scratch the surface.

In Chap. 6, we describe how the foundation of present day geometry was created. Elementary Geometry is tied to straight lines and *circles*. The theorems are closely tied to constructions with straightedge and compass, reflecting the postulates of Euclid. In *higher geometry* one moves on to the more general class of conic sections, as well as curves of higher degrees.

Descartes introduced – or reintroduced, depending on your point of view – algebra into the geometry. At any rate, he is credited with the invention of the *Cartesian coordinate system*, which is named after him.

In the last section of this chapter we return to the theme of Archimedean polyhedra as well as tessellations, as already mentioned.

In Chap. 7, the last chapter of Part I, we discuss the relations between geometry and the real world. The qualitative study of *catastrophes* is of a geometric nature. We explain the simplest one among *Thom's Elementary Catastrophes*, the so-called *Cusp catastrophe.* It yields an amazing insight into occurrences of *abrupt events* in the real world.

Also tied to the real world are the fractal structures in nature. Fractals are geometric objects whose *dimensions are not integers,* but which instead have a *real number* as dimension. Strange as this sounds, it is a natural outgrowth of *Felix Hausdorff's* theory of dimensions. Hausdorff was one of the pioneers of the modern transformation of geometry, referred to in his time as the *High Priest of point-set topology.* In the end, this all did not help him. He knew, being a Jew, what to expect when he was ordered to report the next morning for deportation. This was in 1942 in his home town of Bonn, Germany. Instead of doing so, Hausdorff and his wife committed suicide.

The *Geometry of fractals* shows totally new and unexpected geometric phenomena. Amazingly, what was thought of as *pathology*, as useless curiosities, may turn out to give the most precise description of the world we live in.

In Part II, *Introduction to Geometry*, we take as our starting point the axiomatic treatment of geometry flowing from Euclid.

Considering that Euclid's original system of axioms and postulates is well over 2,000 years old, we must say that it has passed the test of modern demands to rigor remarkably well. To say the least it is the precursor to modern axiomatic theories. But the original system was set on a shaky foundation by our current mathematical standards. A clarifying explanation of the foundations was provided by Hilbert.

The search for a proof of Euclid's Fifth Postulate had gone on ever since the Elements were written, but met with no success. One version of this postulate asserts that there is one and only one line parallel to a given line through a point outside it.

A plausible approach to the problem of proving the Fifth Postulate was to assume the converse, and then derive a contradiction. This approach is usually referred to as an *indirect proof.* But instead of producing a contradiction, this relentless toil ended up producing collections of theorems belonging to *alternative geometries,* to non-Euclidian geometry. This was a highly troubling development for an age in which non-Euclidian Geometry would appear as controversial as "Darwinism" appears in some circles today.

We explain non-Euclidian Geometry in Chap. 10. But first we need to do some work on foundations. We start with *Logic and Set Theory.* In fact, the Intuitive Set Theory, even as put on a firmer foundation by *Cantor*, turned out to contain contradictions. The best known is the so-called *Russell's Paradox*, which we explain in Sect. 8.3.

Thus arouse the need for *Axiomatic set theory*, to which we give an introduction. The aim is to give a flavor of the field without going into the technical details at all.

We then explain the interplay between *axiomatic theories* and their *models* in Sects. 8.3 and 8.4. The troubling result of *Gödel* is explained, in simplified terms, showing that a *mathematical Tower of Babel* as perhaps dreamt of by Hilbert, is not possible: Any axiomatic system without contradictions among its possible consequences, will have to live with some *undecidable* statements. This means that it may happen that statements which are perfectly legal constructions within the system, are inherently undecidable: Their truth or falsehood cannot be ascertained from the system itself.

In Chap. 9, we apply these insights to axiomatic projective geometry. This is an extensive field in itself, and a complete treatment does of course, fall outside the scope of this book. But we give a basic set of axioms, to which others may be added, thus in the end culminating with a set which determines uniquely the *real, projective plane*. This is not on our agenda here. But we do give, in some detail, two important models for the basic system of axioms. The *Seven Point Plane* and the *real projective plane* $\mathbb{P}^2(\mathbb{R})$. In Sect. 9.2, we see that the simple axioms still leads to intriguing open problems. Use of powerful computers and dexterous programming have led to new insights in axiomatic projective geometry, and there are good possibilities for further research. The question the following: *Given a projective plane* $\mathcal{P}$, as defined in the first section of Chap. 9. How many points can there be on each line? It is not difficult to see that this number m is the same for every line in the geometry $\mathcal{P}$. And we also see easily by the standard theory that m can be any power of any prime number p. But no other possible value is known. This question is related to the existence of a sufficient supply of *mutually orthogonal Latin squares*, and goes back to *Leonhard Euler*.

In Chap. 10, we are ready to explain models for non-Euclidian Geometry. In the *hyperbolic plane* there are infinitely many lines parallel to a given line through a point outside it. In the *elliptic plane* there are no parallel lines: Two lines always intersect. A model for this version is provided by $\mathbb{P}^2(\mathbb{R})$.

Plane non-Euclidian geometries have, of course, their spatial versions. This is best understood by turning to some of the basic facts from *Riemannian Geometry*, which we do in Sect. 10.5.

Chapter 11 contains some much needed mathematical tools, simple but essential. We need them for constructions to be carried out in the next chapters. The reader is advised to take the moments needed to ingest this material, which may well appear somewhat dry and barren at the first encounter.

In Chap. 12, we are then able to give coordinates in the projective plane, introduce projective n-space and discuss affine and projective coordinate systems. Again, the material may appear dry, but the reader will be rewarded in Chap. 13. There we use these techniques to give the remarkably simple proof of the theorem of Desargues. We introduce *duality for* $\mathbb{P}^2(\mathbb{R})$ and start the theory of conic sections in $\mathbb{R}^2$ and $\mathbb{P}^2(\mathbb{R})$ discussing tangency, degeneracy and the familiar classification of the conic sections. Pole and Polar belong to this picture, as well as a very simple proof of a famous theorem of Pascal. Using it, we then prove the theorem of Pappus by a classical technique known as *degeneration*, or some times as the *principle of continuity*. Here we give the first, naive, definition of an algebraic curve.

In Chap. 14, we move on to study curves of degrees greater than 2. This forms the fundament for Algebraic Geometry, and gives a glimpse into an important and very rich, active and expanding mathematical field. Here we encounter the *cubic parabola*, merely a fancy name for a familiar curve, but also the enigmatic *semi-cubic parabola*, so important in modern *Catastrophe Theory*. However, as we shall see in the following Chap. 15, from a projective point of view these two kinds of affine curves are the same. This is shown at the end of Sect. 15.5. We also learn about the *Folium of Descartes*, the *Trisectrix of Maclaurin*, of *Elliptic Curves* – which are by no means ellipses – and much more. Chapter 15 concludes with Pascal's *Mysterium Hexagrammicum*, which may be obtained as a beautiful application of Pascal's Theorem: Dualizing it the Mystery of the Hexagram is revealed.

In Chap. 16, as the title says, we sharpen the Sword of Algebra. The aim is to show how one finally disposes of the three so called *Classical Problems*. They have haunted mathematicians and amateurs for two millennia. And unfortunately, still does haunt the latter. The algebra derives in large part from the heritage of Euclid, relying as it does on *Euclid's algorithm*. This mathematics also constitutes the foundation for the important field of *Galois theory* and the theory of equations and their solvability by radicals. That theme is, however, not treated in the present book.

In Chap. 17, we use this algebra for proving that the three classical problems are insoluble: Trisecting an angle with legal use of straightedge and compass, doubling the cube using straightedge and compass, and finally we see how the transcendency of the number π precludes the squaring of the circle using straightedge and compass. Gauss' towering achievement on constructibility of regular polygons conclude the chapter. The solution of this problem by Gauss transformed the answer to a geometric question into a number theoretic problem on the existence of certain primes, namely primes of the form $F_r = 2^{2^r} + 1$, the so-called *Fermat primes*. For $r = 0, 1, 2, 3, 4$ the numbers P_r are $3, 5, 17, 257$ and 65,537. They are all primes, but then no case of an r yielding a prime is known. Gauss proved that if q is a product of such primes p_r, all of them distinct, then the regular $n = 2^m q$-gon may be constructed with straightedge and compass, and that this are precisely all the constructible cases. Thus for example the regular 3-gon, the regular 5-gon and the regular 15-gon are all constructible with straightedge and compass, as is the regular 30-gon and the regular 60-gon. The first impossible case is the regular 7-con. Now Archimedes constructed the regular 7-gon, but he used means beyond legal use of straightedge and compass. In Sect. 4.4 we have given Archimedes' construction of the regular 7-gon, the regular *heptagon*, by a so-called *verging construction*. It is not possible by the *legal use* of compass and straightedge, but may be carried out by conic sections or by a curve of degree 3. In fact, such constructions were part of the motivation for passing from *elementary geometry* to *higher geometry*.

In Chap. 18 we take a closer look at the theory of fractals. We explain the computation of fractal dimensions.

Chapter 19 contains a mathematical treatment of introductory Catastrophe Theory. We explain the Cusp Catastrophe as an application of geometry on a cubic surface. For this we also explain some rudiments of Control Theory.

The final chapter is Chap. 20. Here we return to polyhedra and tessellations, and study them in light of their groups of symmetry. This also applies to the more general situation of patterns and their groups of symmetry. We start out with the important groups of symmetries in the Euclidian plane and the Euclidian 3-space. This chapter presupposes more knowledge of linear algebra and group theory than the earlier parts of the book. Good sources and references for some of the material in this section are the books [9] and [57].

Some of the historical material giving historical context has been extended, and and a large number of illustrations have been added. In revising the historical part, I have tried to follow the guideline that when an interpretation is controversial, this should be noted, and in some cases I provide the alternatives interpretations. In particular this applies to the explanation of the numbers on the Babylonian tablet Plimpton 322, where the first edition only treats the original theory of Otto Neugebauer and his collaborators. Today this is not a justified exposition, pathbreaking as this work was at the time. It is certainly true that a controversial theory should not be presented as a fact.

The main difference from the first edition, however, is the inclusion of a large number of exercises with some suggestions for solutions. Some of the exercises are simple, others more challenging.

Several historical topics, which were not treated at all in the first edition, now have been included in the form of exercises with hints or complete solutions. In Part II the exercises are more or less of the standard type which might appear on a college test. In some cases I include complete solutions, in other cases just more or less extensive hints of just the answer, while some exercises are left open without answers.

Some of the material in this book has been published in the author's [31] and [33]. The material is included here with the permission of *Fagbokforlaget*, the publisher of [31] and [33]. A large number of the illustrations are created with the marvellous system Cinderella, [47], some of them were made by Ulrich H. Kortenkamp, one of the authors of the system. Others were made by the author, who would like to take this opportunity to thank Professor Kortenkamp for his efforts in making these illustrations, as well as for his valuable advice and assistance during this work. Some illustrations are made with the aid of the Computer Algebra system MAPLE, and finally some were made by Springer's illustrator, based on sketches by the author. In addition there are a number of images from various sources which are listed after the bibliography.

Another nice treatment of the material in Chapters 15 and 16 may be found in [16]. For children and young people [35] may be suitable as introduction to our history. An interesting historical source is [41]. My interest was much inspired by [45].

It is also a great pleasure to thank Springer Verlag, in particular the Mathematics Editor Dr. Martin Peters as well as Mrs. Ruth Allewelt, for their enthusiasm and support. Mrs. Jayalakshmi Gurupatham of SPI Publisher Services provided great support and assistance in getting the manuscript of the second edition in shape for the printer.

Some illustrations were made by Professor Ulrich H. Kortenkamp, one of the authors of the Cinderella system, some were made by the author as drawings or computer aided, and some were made by Springer's illustrator, based on sketches by the author. The maps were drawn by the illustrator of Fagbokforlaget, Bergen, Norway, publisher of two volumes on the History of mathematic by the author, namely [33] and [34]. These maps are based on sketches by the author. The remaining illustrations are acknowledged as follows:

Figures 2.2, 2.8, 3.1, 3.6, 4.45, 4.52, 5.2, 6.1, 6.2, 10.11. From Wikimedia, Wikimedia Commons is *"a media file repository making available public domain and freely-licensed educational media content (images, sound and video clips) to all."* The description page in [62] has the following entry: *This image (or other media file) is in the public domain because its copyright has expired.* The name of the creator, if known, as well as other relevant information concerning the image is linked to the image in [61].

Figure 1.1. The illustration was provided by the authors of the article cited.

Figure 8.2: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

Figure 4.54: Courtesy of the Mittag-Leffler Institute, Djursholm, Sweden.

Figure 4.22. Drawing by Nils Henrik Gran, Moss, Norway.

Figures 4.42, 4.43: Downloaded from From Wikimedia. Courtesy of The General Libraries, The University of Texas at Austin. According to the collection's title page, the image is in the public domain and no permission is needed to use it.

Bergen *Audun Holme*

In the spring of 2010

http://www.springer.com/978-3-642-14440-0

Geometry
Our Cultural Heritage
Holme, A.
2010, XVII, 519 p., Hardcover
ISBN: 978-3-642-14440-0