Chapter 2
On the Number of Conjugacy Classes of Symmetries of Riemann Surfaces

As said in the Introduction, under the correspondence between compact Riemann surfaces and smooth irreducible complex projective algebraic curves, the fact that a Riemann surface S is symmetric means that the corresponding complex curve $\mathcal{C}$ can be defined over the field $\mathbb{R}$ of real numbers. This is why such a symmetry is often called a *real form* of $\mathcal{C}$. Symmetries which are non-conjugate in the automorphism group $\mathrm{Aut}(S)$ of S correspond to non-isomorphic real forms of $\mathcal{C}$. In this chapter we shall pay attention to quantitative results concerning the number of conjugacy classes of symmetries. We will distinguish cases according to whether the sets of fixed points of the symmetries are empty or not.

We start with a study of conjugacy classes of involutions in 2-groups at large.

2.1 Conjugacy Classes of Involutions in 2-Groups

Given an abstract group G, it makes sense to say that an involution $x \in G$ is a "symmetry" provided that a concept of orientation in such a group is defined. This is done in the following definitions.

Definitions 2.1.1. Let G be an abstract group.

(1) G is said to be *abstractly orientable* if there exists an epimorphism $\alpha : G \to \mathrm{Z}_2 = \{\pm 1\}$. In such a case, α is an *orientation* of the group G. If an orientation α is chosen then we say that G is *abstractly oriented.*
(2) Let α be an orientation of G. An element $x \in G$ is *orientation preserving* (respectively *orientation reversing*) with respect to the orientation α if $\alpha(x) = +1$ (respectively $\alpha(x) = -1$).

Examples of orientable groups are provided by proper NEC groups and groups of automorphisms of symmetric Riemann surfaces.

Lemma 2.1.2. *Let G be a 2-group containing a cyclic group $\mathrm{Z}_N = \langle x \rangle$ as a subgroup of index 2^r. Then G has at most $2^{r+1} - 1$ conjugacy classes of involutions. Furthermore, if G is abstractly oriented and x preserves the orientation then G has at most 2^r conjugacy classes of orientation reversing involutions.*

E. Bujalance et al., *Symmetries of Compact Riemann Surfaces*, Lecture Notes in Mathematics 2007, DOI 10.1007/978-3-642-14828-6_2,

Proof. Let

$$\mathrm{Z}_N = H_0 \overset{2}{\leq} H_1 \overset{2}{\leq} H_2 \overset{2}{\leq} \cdots \overset{2}{\leq} H_{r-1} \overset{2}{\leq} H_r = G$$

be a subnormal series for G and let $x_i \in H_i \setminus H_{i-1}$ for $i = 1, \dots, r$. Then each element $g \in G$ can be uniquely represented as $g = x^{\varepsilon} x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ for some integers $\varepsilon \in \{0, \dots, N-1\}$ and $\varepsilon_i \in \{0, 1\}$. Let us denote $w = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ and observe that there are $2^r - 1$ non-trivial elements of this form. We shall show that for any such element $w \neq 1$ there are at most 2 conjugacy classes of involutions among the elements of the set $\{w, xw, x^2w, \dots, x^{N-1}w\}$. This will complete the proof of the first part of the lemma since for $w = 1$ this set has one involution. Observe that w may not be an element of order 2 and, furthermore, among these elements there may not even exist elements of order 2.

Assume then that there are at least two elements $x^k w$ and $x^{\ell} w$ of order 2 and assume also that k and ℓ are chosen so that $k > \ell$ and $m = k - \ell$ is minimal. We shall show that each involution $x^n w$ is conjugate either to $x^k w$ or to $x^{\ell} w$. We have

$$1 = (x^k w)^2 = x^m (x^{\ell} w)^2 w^{-1} x^m w = x^m w^{-1} x^m w.$$

So $wx^{-m} = x^m w$ and therefore $x^{\ell + sm} w$ has order 2 for each integer s. Moreover,

$$x^{sm}(x^{\ell} w)x^{-sm} = x^{\ell + 2sm} w, \tag{2.1}$$

$$x^{sm}(x^k w)x^{-sm} = x^{k+2sm} w = x^{\ell + (2s+1)m} w. \tag{2.2}$$

Now let $x^n w$ be an arbitrary element of order 2. Then $n = \ell + tm + j$ for some integers t, j, where $0 \leq j < m$, and since both $x^n w$ and $x^{\ell + tm} w$ have order 2, it follows by the minimality of m that $j = 0$. Thus $x^n w = x^{\ell + tm} w$ which, by (2.1) and (2.2), is conjugate to $x^{\ell} w$ if t is even, and it is conjugate to $x^k w$ if t is odd. This shows the first part of the lemma.

Assume now that G is abstractly oriented and that x preserves the orientation. Then half of the 2^r elements $w = x_1^{\varepsilon_1} \cdots x_r^{\varepsilon_r}$ reverses orientation and the other half preserves it. For each of the 2^{r-1} orientation reversing ones, the (at most two) conjugacy classes of involutions in the set $\{w, xw, x^2w, \dots, x^{N-1}w\}$ are the only ones that reverse the orientation. This shows the second part of the lemma. □

Let D_N be a dihedral group and let $x, y \in \mathrm{D}_N$ be two generating involutions. If D_N is abstractly oriented and x and y reverse the orientation then $\mathrm{Z}_N = \langle xy \rangle$ is a subgroup of D_N of index 2 generated by an orientation preserving element. So, as a consequence of the above Lemma 2.1.2, we get the following result.

Corollary 2.1.3. *Let G be a 2-group containing a dihedral group D_N as a subgroup of index 2^r. Then G has at most $2^{r+2} - 1$ conjugacy classes of involutions. Furthermore if G is abstractly oriented and D_N is generated by two involutions which reverse the orientation then G has at most 2^{r+1} conjugacy classes of orientation reversing involutions.*

The next technical lemma deals with 2-groups of automorphisms of a Riemann surface. Together with Lemma 2.1.2 and Corollary 2.1.3, it will play a key role in the sequel.

Lemma 2.1.4. *Let S be a Riemann surface of genus $g \geq 2$, and let 2^{r-1} be the largest power of 2 dividing $g-1$. Let G be a 2-group of automorphisms of S of order 2^t and assume that $t \geq r+1$. Then G contains a cyclic or a dihedral subgroup of index 2^r.*

Proof. Let us write $S = \mathcal{H}/\Gamma$ for some surface Fuchsian group Γ and $G = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup. Assume that Λ has signature

$$\mathtt{s}(\Lambda) = (h; \pm; [m_1, \ldots, m_\nu]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \tag{2.3}$$

We claim that $\mathtt{s}(\Lambda)$ has either a proper period or a link period. In fact, by the Hurwitz–Riemann formula we have

$$\frac{g-1}{2^{r-1}} = 2^{t-r}\left(\eta h - 2 + k + \sum_{i=1}^{\nu} \frac{m_i - 1}{m_i} + \sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{n_{ij}-1}{2n_{ij}}\right)$$

where either $\eta = 1$ or $\eta = 2$ depending on the sign of $\mathtt{s}(\Lambda)$. Since $(g-1)/2^{r-1}$ is odd and $t - r \geq 1$, the expression in brackets cannot be an integer. So there must be a non-trivial m_i or n_{ij}, as claimed. Moreover, since G is a 2-group, all periods of Λ are powers of 2, since otherwise Γ would have elements of finite order. It follows that $m_i \geq 2^{t-r}$ for some i or $n_{ij} \geq 2^{t-r-1}$ for some i, j.

Assume first that Λ has a proper period $m \geq 2^{t-r}$; in this case the image x in G of an elliptic generator of Λ of order m is still an element of order m and so for $m' = m/2^{t-r}$, the element $x^{m'}$ generates a cyclic subgroup of G of index 2^r. Assume now that Λ has a link period $n \geq 2^{t-r-1}$; in this case the images c and c' in G of two consecutive reflections of Λ whose product has order n are involutions, since otherwise Γ would be a proper NEC group. Moreover, for $n' = n/2^{t-r-1}$, the element $(cc')^{n'}$ has order 2^{t-r-1} and so c and $(cc')^{n'}$ generate a dihedral subgroup of G of index 2^r. This completes the proof. □

Remark 2.1.5. The proof shows that in fact G contains a cyclic subgroup generated by an orientation preserving element or a dihedral subgroup generated by two orientation reversing elements, of index 2^r in both cases.

Remark 2.1.6. Let G^+ denote the subgroup of G consisting of its orientation preserving elements. With the notations in the above proof of Lemma 2.1.4, the existence of a proper period or a link period in the signature of Λ shows that G^+ acts on S with fixed points.

2.2 Symmetries with Non-Empty Set of Fixed Points

The quantitative study of conjugacy classes of symmetries started with a seminal result of Natanzon [95] who proved, using topological methods, that a complex algebraic curve of genus $g \geq 2$ has at most $2(\sqrt{g}+1)$ non-isomorphic real forms with real points. He also showed that this bound is attained for infinitely many values of g, those being of the form $(2^n - 1)^2$. Here we go further, namely, we determine the maximal number of conjugacy classes of symmetries with fixed points that a compact Riemann surface S of genus $g \geq 2$ can admit.

Assume that $\sigma_1, \ldots, \sigma_k$ are representatives of the conjugacy classes of symmetries of S. Since each σ_i belongs to a Sylow 2-subgroup of $\mathrm{Aut}(S)$ and all Sylow 2-subgroups are conjugate, we may assume that all these symmetries generate a 2-group G. We now establish a fundamental result on this topic, whose first proof appeared in [23].

Theorem 2.2.1. *Let S be a Riemann surface of genus $g \geq 2$ and let us write $g = 2^{r-1}u + 1$ with u odd. Then the maximum number of non-conjugate symmetries with fixed points that S admits is 2^{r+1}. Furthermore, this bound is attained if and only if $u \geq 2^{r+1} - 3$.*

Proof. Let k be the number of conjugacy classes of symmetries with fixed points of S. As we observed above, we can choose representatives of these classes such that they generate a 2-group, say of order 2^t. If $t \leq r$ then $k < 2^t \leq 2^r$ and so the first part of the statement is proved in this case. If $t \geq r + 1$ then the first part is a direct consequence of Lemma 2.1.4, Corollary 2.1.3 and Lemma 2.1.2.

Let now $S = \mathcal{H}/\Gamma$ be a Riemann surface with the maximum number 2^{r+1} of conjugacy classes of symmetries with fixed points and let G be a 2-group generated by 2^{r+1} representatives of these classes. Let us write $G = \Lambda/\Gamma$ for some NEC group Λ with signature (2.3). Let $C_1, \ldots, C_n$ be the different period cycles of Λ involving these symmetries, and assume that $C_1, \ldots, C_m$ are non-empty and $C_{m+1}, \ldots, C_n$ are empty. Observe that $n > 0$ because they are symmetries with fixed points. As each empty period cycle involves at most one symmetry we see that $C_1, \ldots, C_m$ involve at least $2^{r+1} - (n - m)$ symmetries. As each non-empty period cycle C_i of length s_i involves at most s_i non-conjugate symmetries, see Remark 1.1.6, we get

$$s_1 + \cdots + s_m \geq 2^{r+1} - n + m.$$

We shall show that

$$\mathrm{Area}(\Lambda) \geq 2\pi \left(\frac{2^{r+1} - 3}{4} - \frac{2^r}{|G|} \right).$$

Observe that each term $(1/2)(1 - 1/n_{ij})$ occurring in the formula of $\mathrm{Area}(\Lambda)$ is not smaller than $1/4$ because $n_{ij} \geq 2$.

Since G is generated by 2^{r+1} orientation reversing involutions, we see that $|G| \geq 2^{r+2}$. In particular, we may repeat the proof of Lemma 2.1.4 to show that Λ has a proper period $\geq |G|/2^r$ or a link period $\geq |G|/2^{r+1}$. In the first case

$$\begin{aligned}\mathrm{Area}(\Lambda) &\geq 2\pi \left(n - 2 + \left(1 - \frac{2^r}{|G|}\right) + \frac{s_1 + \cdots + s_m}{4} \right) \\ &\geq 2\pi \left(\frac{2^{r+1} + 3n + m - 4}{4} - \frac{2^r}{|G|} \right) > 2\pi \left(\frac{2^{r+1} - 3}{4} - \frac{2^r}{|G|} \right).\end{aligned}$$

If Λ has a link period $\geq |G|/2^{r+1}$ then $m > 0$ and

$$\begin{aligned}\text{Area}(\Lambda) &\geq 2\pi\left(n - 2 + \frac{s_1 + \cdots + s_m - 1}{4} + \frac{1}{2} - \frac{2^r}{|G|}\right)\\ &\geq 2\pi\left(\frac{2^{r+1} + 3n + m - 7}{4} - \frac{2^r}{|G|}\right) \geq 2\pi\left(\frac{2^{r+1} - 3}{4} - \frac{2^r}{|G|}\right).\end{aligned}$$

So, in both cases,

$$4\pi(g-1) = \text{Area}(\Gamma) = |G|\text{Area}(\Lambda) \geq 2\pi\left(\frac{2^{r+1} - 3}{4} - \frac{2^r}{|G|}\right)|G|.$$

Since $|G| \geq 2^{r+2}$ we get

$$g - 1 \geq (2^{r+1} - 3)\frac{|G|}{8} - 2^{r-1} \geq (2^{r+1} - 3)2^{r-1} - 2^{r-1} = 2^{r-1}(2^{r+1} - 4).$$

Therefore $u = (g-1)/2^{r-1} \geq 2^{r+1} - 4$. However, u is odd by assumption and consequently $u \geq 2^{r+1} - 3$.

Conversely, let $g = 2^{r-1}u + 1$ with $u \geq 2^{r+1} - 3$ and let $s = u + 3$. Consider a maximal NEC group Λ with signature $(0; +; [-]; \{(2, \overset{s+1}{\ldots}, 2)\})$, and let $\{c_0, \ldots, c_{s+1}\}$ be a canonical set of generators of Λ. Let us consider the group $G = \mathrm{Z}_2^{r+2} = \langle x_1 \rangle \oplus \cdots \oplus \langle x_{r+2} \rangle$ and let $a_1, \ldots, a_{2^{r+1}}$ be the involutions in G whose length in $x_1, \ldots, x_{r+2}$ is odd. We define a homomorphism $\theta : \Lambda \to G$ by choosing $\theta(c_i) \in \{a_1, \ldots, a_{2^{r+1}}\}$ for $0 \leq i \leq s+1$ so that $\theta(c_i) \neq \theta(c_{i+1})$ for $0 \leq i \leq s$, and such that θ is in fact an epimorphism. Observe that this is indeed possible because $s \geq r + 1$.

Clearly, $\ker\theta$ is a surface Fuchsian group. The orbit space $S = \mathcal{H}/\ker\theta$ is a Riemann surface of genus $2^{r-1}u + 1$ (by the Hurwitz–Riemann formula) having G as its full group $\text{Aut}(S)$ of automorphisms (by the maximality of Λ). Since the image under θ of each canonical reflection c_i is a symmetry with fixed points we see that S has 2^{r+1} non-conjugate symmetries with fixed points. $\square$

Every even value of g can be written as $2^{r-1}u + 1$ with $r = 1$ and u odd. In this way we obtain the main result in [53] as a corollary of Theorem 2.2.1.

Corollary 2.2.2. *A Riemann surface of even genus g has at most 4 non-conjugate symmetries with fixed points. Furthermore this bound is attained for every even genus $g \geq 2$.*

Remark 2.2.3. Given an arbitrary integer $g \geq 2$, there is an integer $r \geq 1$ and an odd integer $u \geq 1$ such that $g = 2^{r-1}u + 1$. Fix $r \geq 1$ and consider all values of g of this form. Observe that the numbers $g - 1$ are just the solutions of the congruence $x \equiv 2^{r-1} \pmod{2^r}$. Suppose that $u \geq 2^{r+1} - 3$. Then

$$g \geq 2^{r-1}(2^{r+1} - 3) + 1 = 2^{2r} - 3 \cdot 2^{r-1} + 1 > 2^{2r} - 4 \cdot 2^{r-1} + 1 = (2^r - 1)^2.$$

Henceforth $2(\sqrt{g} + 1) > 2^{r+1}$ and thus, for the values of g corresponding to $u \geq 2^{r+1} - 3$, the bound $2(\sqrt{g} + 1)$ obtained by Natanzon in [95] for the number of non-conjugate symmetries with fixed points is not sharp. On the other hand, if $u \leq 2^{r+1} - 5$ then

$$g \leq 2^{r-1}(2^{r+1} - 5) + 1 = 2^{2r} - 5 \cdot 2^{r-1} + 1 < 2^{2r} - 4 \cdot 2^{r-1} + 1 = (2^r - 1)^2.$$

Hence $2(\sqrt{g} + 1) < 2^{r+1}$ in this case and so, for the values of g corresponding to $u \leq 2^{r+1} - 5$, Natanzon's bound is better than the one in Theorem 2.2.1. We now calculate sharp bounds for the remaining values of g.

Notation 2.2.4. For each integer $g \geq 2$ we denote by $\mu_f(g)$ the maximal number of conjugacy classes of symmetries with fixed points that a genus g Riemann surface can admit.

With this notation, Theorem 2.2.1 can be stated as

$$\mu_f(g) = 2^{r+1} \quad \text{for } g = 2^{r-1}u + 1 \text{ with } u \text{ odd and } u \geq 2^{r+1} - 3.$$

Next we shall calculate the remaining values of this function, as stated in [23]. To that end we fix $r \geq 2$ (since the case $r = 1$ is solved in Corollary 2.2.2) and consider the function

$$f(s) = \frac{2^{s-4} - 1}{2^{r-s}},$$

which is strictly increasing for $s > 3$. Since $f(4) = 0$ and $f(r + 3) = 2^{r+2} - 8$ we see that for each odd positive integer $u < 2^{r+2} - 7$ there exists a unique integer $s \in \{4, \ldots, r + 2\}$ such that $f(s) < u \leq f(s + 1)$, that is,

$$\frac{2^{s-4} - 1}{2^{r-s}} < u \leq \frac{2^{s-3} - 1}{2^{r-s-1}}.$$

The next theorem shows that $\mu_f(g)$ depends on this value of s; in fact, it shows that $\mu_f(g) = \min\{2^{r-s+2}u + 4,\ 2^{s-1}\}$.

Theorem 2.2.5. *Let* $g = 2^{r-1}u + 1$*, where* $r \geq 2$ *and* $u < 2^{r+2} - 7$ *is odd. Let* s *be defined as above. Then*

$$\mu_f(g) = \begin{cases} 2^{r-s+2}u + 4 & \textit{if} \quad \dfrac{2^{s-4}-1}{2^{r-s}} < u \leq \dfrac{2^{s-3}-1}{2^{r-s}}; \\[2ex] 2^{s-1} & \textit{if} \quad \dfrac{2^{s-3}-1}{2^{r-s}} < u \leq \dfrac{2^{s-2}-2}{2^{r-s}}. \end{cases}$$

Proof. Let S be a Riemann surface of genus $g = 2^{r-1}u + 1$ where $u < 2^{r+2} - 7$ such that S has k non-conjugate symmetries with fixed points. First we shall show that k is not greater than the proposed value for $\mu_f(g)$. As before, by Sylow theory, we may assume that the k symmetries generate a 2-group G, say of order 2^t. Observe that the product of two of these symmetries is an orientation preserving element of G which generates a cyclic subgroup of index $\leq 2^{t-1}$. So Lemma 2.1.2 yields

$$k \leq 2^{t-1}. \tag{2.4}$$

Let us write $S = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and a proper NEC group Λ containing Γ as a normal subgroup. It is easy to see, using arguments similar to those in the proof of Theorem 2.2.1, that $\mathrm{Area}(\Lambda) \geq \pi(k-4)/2$. So, by the Hurwitz–Riemann formula, $4\pi(g-1) = |G|\mathrm{Area}(\Lambda) \geq 2^{t-1}\pi(k-4)$, which gives

$$k \leq 2^{r-t+2}u + 4. \tag{2.5}$$

Let us suppose first that

$$(2^{s-4}-1)/2^{r-s} < u \leq (2^{s-3}-1)/2^{r-s}. \tag{2.6}$$

If $t \geq s$ then $k \leq 2^{r-s+2}u + 4$ by (2.5). If $t < s$ then $k \leq 2^{t-1} \leq 2^{s-2}$ by (2.4), and so $k < 2^{r-s+2}u + 4$, because $2^{s-2} < 2^{r-s+2}u + 4$ by (2.6).

We now suppose that

$$(2^{s-3}-1)/2^{r-s} < u \leq (2^{s-2}-2)/2^{r-s}. \tag{2.7}$$

If $t \geq s+1$ then $k \leq 2^{r-s+1}u + 4 \leq 2^{s-1}$, where we have used (2.5) for the first inequality and (2.7) for the second. If $t \leq s$ then $k \leq 2^{t-1} \leq 2^{s-1}$ by (2.4).

To finish the proof we consider an arbitrary integer $s \in \{4, \ldots, r+2\}$ and an arbitrary odd integer u in the range

$$(2^{s-4}-1)/2^{r-s} < u \leq (2^{s-2}-2)/2^{r-s}.$$

Let $G = \mathrm{Z}_2^s = \langle x_1 \rangle \oplus \cdots \oplus \langle x_s \rangle$. Let A be the set consisting of the 2^{s-1} involutions of G which can be written as words of odd length in $x_1, \ldots, x_s$. Let us write $k = 2^{r-s+2}u + 4$ ($k \geq 5$) and let Λ be a maximal NEC group with signature $(0; +; [-]; \{(2, \overset{k}{\ldots}, 2)\})$. Observe that $k > s$ because $u > (2^{s-4}-1)/2^{r-s}$. Hence there exists an epimorphism $\theta : \Lambda \to G$ such that the image $\theta(c_i)$ of each canonical reflection belongs to A and $\theta(c_i) \neq \theta(c_{i+1})$. In addition, if $k \leq 2^{s-1}$ then θ can be defined so that the k canonical reflections $c_1, \ldots, c_k$ are mapped onto distinct elements of A.

Then $\ker\theta$ is a surface Fuchsian group and the orbit space $S = \mathcal{H}/\ker\theta$ is a Riemann surface of genus $2^{r-1}u+1$ having G as its full group of automorphisms. Now, if $k \le 2^{s-1}$, which happens if and only if $u \le (2^{s-3}-1)/2^{r-s}$, then $\{\theta(c_i) : i = 1,\ldots,k\}$ are representatives of the conjugacy classes of symmetries with fixed points in S. On the other hand, if $k > 2^{s-1}$, which happens if and only if $u > (2^{s-3}-1)/2^{r-s}$, then the 2^{s-1} elements in A are representatives of the conjugacy classes of symmetries with fixed points in S. □

Remark 2.2.6. The values of u in the range $2^{r+1}-3 \le u < 2^{r+2}-7$ are covered by both Theorems 2.2.1 and 2.2.5. Let us check that the formulae of $\mu_f(g)$ given by these theorems coincide for these values of u. First, Theorem 2.2.1 gives directly $\mu_f(g) = 2^{r+1}$. To apply the formula of Theorem 2.2.5 we observe that the value of the parameter s corresponding to those u in the range $2^{r+1}-3 \le u < 2^{r+2}-7$ is $s = r+2$. So,

$$\mu_f(g) = \min\{2^{r-s+2}u+4,\ 2^{s-1}\} = \min\{u+4,\ 2^{r+1}\} = 2^{r+1},$$

because $u+4 \ge 2^{r+1}+1$.

Example 2.2.7. The function $g \mapsto \mu_f(g)$ is not increasing because $\mu_f(g) = 4$ for all even values of g (see Corollary 2.2.2). However, if we write $g = 2^{r-1}u+1$ and fix a value of r then the function $u \mapsto \mu_f(2^{r-1}u+1)$ is increasing (but not strictly) as a function of u. It attains the maximal value 2^{r+1} for $u = 2^{r+1}-3$ and remains constant from that moment onwards. We illustrate this in Table 2.1, where the pairs $(g, \mu_f(g))$ are computed for small values of r.

Table 2.1 Values of the pair $(g, \mu_f(g))$ where $g = 2^{r-1}u+1$ with u odd for small values of r

$r=1$	$r=2$	$r=3$	$r=4$
$(2,4)$	$(3,5)$	$(5,6)$	$(9,8)$
$(4,4)$	$(7,7)$	$(13,8)$	$(25,10)$
$(6,4)$	$(11,8)$	$(21,9)$	$(41,14)$
$(8,4)$	$(15,8)$	$(29,11)$	$(57,16)$
$(10,4)$	$(19,8)$	$(37,13)$	$(73,16)$
$(12,4)$	$(23,8)$	$(45,15)$	$(89,16)$
$(14,4)$	$(27,8)$	$(53,16)$	$(105,17)$
$(18,4)$	$(31,8)$	$(61,16)$	$(121,19)$
$(20,4)$	$(35,8)$	$(69,16)$	$(137,21)$
$(22,4)$	$(39,8)$	$(77,16)$	$(153,23)$
$(24,4)$	$(43,8)$	$(85,16)$	$(169,25)$
$(26,4)$	$(47,8)$	$(93,16)$	$(185,27)$
$(28,4)$	$(51,8)$	$(101,16)$	$(201,29)$
$(30,4)$	$(55,8)$	$(109,16)$	$(217,31)$
$(32,4)$	$(59,8)$	$(117,16)$	$(233,32)$
$(34,4)$	$(63,8)$	$(125,16)$	$(249,32)$

2.3 Symmetries with Empty Set of Fixed Points

In this section we shall deal with symmetries without fixed points. These symmetries correspond to the so called *purely imaginary curves*, that is, complex algebraic curves which can be defined over the reals but have no $\mathbb{R}$-rational points.

For an arbitrary value of $g \geq 2$, let $\mu_i(g)$ denote the maximal number of conjugacy classes of fixed point free symmetries that can be admitted by a Riemann surface S of genus g which has no symmetry with fixed points.

Theorem 2.3.1. *Let us write $g = 2^{r-1}u + 1$ with u odd. Then $\mu_i(g) \leq 2^r$. Furthermore, this bound is attained whenever $u \geq 2r + 1$.*

Proof. Let S be a compact Riemann surface of genus g having no symmetry with fixed points and let G be a 2-group of automorphisms of S generated by representatives of all the conjugacy classes of fixed point free symmetries. Let us write $S = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$, where Γ has signature $(g; -)$ and Λ is a proper NEC group. Since S has no symmetry with fixed points, Λ contains no reflection, and so its signature is $(h; -; [m_1, \ldots, m_v]; \{-\})$ for some $h \geq 1$. Let 2^s be the largest proper period in $\mathtt{s}(\Lambda)$, if any (observe that each proper period m_i is a power of 2). Then, by the Hurwitz–Riemann formula, $4\pi(g-1) = |G|2\pi(h - 2 + m/2^s)$ for some non-negative integer m. Since $g - 1 = 2^{r-1}u$ we get

$$u = \frac{|G|}{2^r}\left(h - 2 + \frac{m}{2^s}\right) = \frac{|G|}{2^{r+s}}\left(2^s(h-2) + m\right).$$

This yields that the order of G divides 2^{r+s} because u is odd. The image in G of the elliptic element of order 2^s is an orientation preserving element which generates a cyclic subgroup of index 2^r. Hence $\mu_i(g) \leq 2^r$ by Lemma 2.1.2.

To prove the second part, let $u \geq 2r + 1$ and let Λ be a maximal NEC group with signature $(h; -; [2, 2, 2]; \{-\})$, where $h = (u+1)/2 \geq r + 1$. Take $G = \mathrm{Z}_2^{r+1}$ with generating basis $\{z_1, \ldots, z_{r+1}\}$, and let $\theta : \Lambda \to G$ be the epimorphism given by $\theta(d_i) = z_i$ for $1 \leq i \leq r+1$, $\theta(d_i) = z_1$ for $r + 2 \leq i \leq h$ and $\theta(x_1) = z_1 z_2,\ \theta(x_2) = z_2 z_3,\ \theta(x_3) = z_1 z_3$. Then $\Gamma = \ker\theta$ is a surface Fuchsian group and $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus $g = 2^{r-1}u + 1$, without symmetries with fixed points and admitting 2^r conjugacy classes of fixed point free symmetries. $\square$

The results are more precise if we restrict our considerations to Riemann surfaces whose full group $\mathrm{Aut}(S)$ acts without fixed points, that is, no automorphism of S (either analytic or antianalytic) fixes points in S. These are the surfaces S for which the normal covering $S \to S/\,\mathrm{Aut}(S)$ is unramified.

For each $g \geq 3$, let $\mu_i^w(g)$ denote the maximal number of conjugacy classes of symmetries that a genus g Riemann surface whose full group $\mathrm{Aut}(S)$ acts without fixed points may admit. Observe that $\mu_i^w(g)$ does not make sense for $g = 2$ since all surfaces of genus 2 are hyperelliptic and the hyperelliptic involution fixes points.

Theorem 2.3.2. *Let us write $g = 2^{r-1}u + 1$ with u odd. Then $\mu_i^w(g) \leq 2^{r-1}$. Assume that $g \notin \{3, 5\}$. Then the bound is attained if and only if $u \geq r - 2$. For $g \in \{3, 5\}$ we have $\mu_i^w(3) = 1$ and $\mu_i^w(5) = 2$.*

Proof. Let S be a genus g Riemann surface such that $\mathrm{Aut}(S)$ acts fixed point freely and let G be a 2-group of automorphisms of S generated by representatives of the conjugacy classes of its symmetries. Let us write $S = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$, where Γ has signature $(g; -)$ and Λ is a proper NEC group. Since the automorphisms in G act fixed point freely, the group Λ contains no reflection and no elliptic element; hence $\mathtt{s}(\Lambda) = (h; -; [-]; \{-\})$ for some integer $h > 2$.

By the Hurwitz–Riemann formula, $2\pi|G|(h-2) = 4\pi(g-1) = 2^{r+1}\pi u$, which implies that $|G|$ divides 2^r because G is a 2-group and u is odd. The product of two symmetries generates a dihedral group of index $\leq 2^{r-2}$ and so Corollary 2.1.3 yields $\mu_i^w(g) \leq 2^{r-1}$.

Assume that $g \notin \{3, 5\}$. If this bound is attained then $|G| = 2^r$, $h - 2 = u$ and, by Lemma 2.1.2, no element of G has order greater than two. So $G = \mathrm{Z}_2^r$. Moreover, since G is generated by the cosets Γd_i, where $d_1, \ldots, d_h$ form a set of canonical generators of Λ, it follows that $h \geq r$ and so $u \geq r - 2$.

Conversely, if $u \geq r - 2$ then $u + 2 \geq 4$ since otherwise $g = 3$ or 5. So we may take a maximal NEC group Λ with signature $(u + 2; -; [-]; \{-\})$. Let $\{d_1, \ldots, d_{u+2}\}$ be a set of canonical generators of Λ. Take $G = \mathrm{Z}_2^r$ with generating basis $\{z_1, \ldots, z_r\}$, and let $\theta : \Lambda \to G$ be the epimorphism induced by the assignment $\theta(d_i) = z_i$ for $1 \leq i \leq r$ and $\theta(d_j) = z_1$ for $r + 1 \leq j \leq u + 2$. Then $\ker\theta$ is a surface Fuchsian group and $S = \mathcal{H}/\ker\theta$ is a Riemann surface of genus $g = 2^{r-1}u+1$ with exactly 2^{r-1} conjugacy classes of symmetries whose full group $\mathrm{Aut}(S)$ acts fixed point freely.

If the bound were attained for $g = 3$ or $g = 5$ then, with the above notations, $\mathtt{s}(\Lambda) = (3; -; [-]; \{-\})$, which is not a maximal signature. Indeed, according to the list of normal pairs of NEC signatures given in [12], for each Λ with the above signature, there exists an NEC group Λ' with signature $\mathtt{s}(\Lambda') = (0; +; [2, 2, 2]; \{(-)\})$ containing Λ as a normal subgroup of index 2. Up to automorphisms in Λ and Λ', there is a unique embedding of Λ in Λ', given by $d_1 = x_1c$, $d_2 = cx_2$ and $d_3 = x_2cx_3x_2$, see [12, Proposition 4.8], where $\{x_1, x_2, x_3, c\}$ is a set of canonical generators of Λ'. Using this embedding it is easy to see that any smooth epimorphism $\theta : \Lambda \to G$, where $G = \mathrm{Z}_2^2$ if $g = 3$ and $G = \mathrm{Z}_2^3$ if $g = 5$, can be extended to a smooth epimorphism $\theta' : \Lambda' \to G'$ where $G' = \mathrm{Z}_2^3$ if $g = 3$ and $G' = \mathrm{Z}_2^4$ if $g = 5$. Hence $\ker\theta = \ker\theta'$ and so the Riemann surface $\mathcal{H}/\ker\theta = \mathcal{H}/\ker\theta'$ admits automorphisms with fixed points, namely, the images under θ' of the elliptic elements of Λ'. This is a contradiction and so $\mu_i^w(3) < 2$ and $\mu_i^w(5) < 4$.

Let us consider now a maximal NEC group Λ with signature $(4; -; [-]; \{-\})$ and define the epimorphisms $\theta_1 : \Lambda \to \mathrm{Z}_2 = \langle\sigma\rangle$ by $\theta_1(d_i) = \sigma$ for $1 \leq i \leq 4$ and $\theta_2 : \Lambda \to \mathrm{Z}_2^2 = \langle\sigma_1, \sigma_2\rangle$ by $\theta_2(d_1) = \theta_2(d_2) = \sigma_1$ and $\theta_2(d_3) = \theta_2(d_4) = \sigma_2$. The group $\mathrm{Aut}(S_j)$ of the Riemann surface $S_j = \mathcal{H}/\ker\theta_j$ acts fixed point freely and has one conjugacy class of symmetries if $j = 1$ and two if $j = 2$. This yields $\mu_i^w(3) = 1$ because S_1 has genus 3, and $\mu_i^w(5) \geq 2$ because S_2 has genus 5.

We finally show that $\mu_i^w(5) < 3$. Suppose, to get a contradiction, that there exists a Riemann surface $S = \mathcal{H}/\Gamma$ of genus 5 with three conjugacy classes of symmetries and let G be a 2-group generated by representatives of them. Observe that $|G| \geq 2^3$. Writing $G = \Lambda/\Gamma$ and using that G acts fixed point freely, we get, by the Hurwitz–Riemann formula, that $\mathtt{s}(\Lambda) = (3; -; [-]; \{-\})$ and $|G| = 8$. Then $G = \mathbb{Z}_2^3$ and we may repeat the above arguments to show that the action of G extends to a group G' which does not act fixed point freely. □

2.4 Symmetries of Surfaces Admitting a Fixed Point Free Symmetry

In the previous sections we have studied either collections of symmetries having fixed points or collections of fixed point free symmetries of surfaces that do not admit symmetries with fixed points. Now we shall study hybrid configurations. In this section we calculate the maximal number of conjugacy classes of symmetries that can be admitted by a Riemann surface S of genus g which has a fixed point free symmetry. The computation of this bound was also carried out in [18]. It turns out that this bound is the same as in Theorem 2.2.1 (for symmetries with fixed points) but now it is attained for a wider range of genera.

Theorem 2.4.1. *Let S be a compact Riemann surface of genus g admitting a fixed point free symmetry. Let us write $g = 2^{r-1}u + 1$ with u odd. Then the number of conjugacy classes of symmetries of S is at most 2^{r+1}. Furthermore this bound is attained whenever $u \geq r - 2$.*

Proof. Let G be a 2-group of automorphisms of S generated by representatives of the conjugacy classes of symmetries of S. Let us write $|G| = 2^t$. Clearly, G has at most 2^{t-1} conjugacy classes of symmetries. If $t \geq r + 1$ (otherwise there is nothing to prove) then Lemma 2.1.4 yields that G contains either a cyclic or a dihedral subgroup of index 2^r. In the first case the number of conjugacy classes of symmetries in G is $\leq 2^r$ by Lemma 2.1.2 (see also Remark 2.1.5) and $\leq 2^{r+1}$ in the second one by Corollary 2.1.3. Therefore S has at most 2^{r+1} conjugacy classes of symmetries. In fact, the above shows that this bound is attained only if G contains a dihedral subgroup of index 2^r and the subgroup G^+ of orientation preserving elements acts with fixed points on S, see Remark 2.1.6.

Suppose now that $u \geq r - 2$. Let $s = u + 3 \geq 4$ and take a maximal NEC group Λ with signature $(0; +; [-]; \{(2, \overset{s+1}{\ldots}, 2)\})$. Let $\{c_0, \ldots, c_{s+1}\}$ be a canonical set of generators for Λ and let $G = \mathbb{Z}_2^{r+2} = \langle x_1 \rangle \oplus \cdots \oplus \langle x_{r+2} \rangle$. Since $s \geq r + 1$, the assignment $c_i \mapsto x_i$ for $1 \leq i \leq s + 1$, where the indices of x_i are modulo $r + 1$, induces an epimorphism $\theta : \Lambda \to G$. In fact, $\ker \theta$ is a surface Fuchsian group and so $S = \mathcal{H}/\Gamma$ is a Riemann surface; its genus equals $2^{r-1}u + 1$ and its full automorphism group $\mathrm{Aut}(S) = \mathbb{Z}_2^{r+2}$ has 2^{r+1} symmetries which are pairwise non-conjugate. In addition, S admits fixed point free symmetries since, for instance, the image under θ of the glide reflection $c_1c_2c_3$ is one of them. Hence the bound 2^{r+1} is achievable when $u \geq r-2$. □

Recall from Theorem 2.2.1 that 2^{r+1} is also the upper bound for the number of conjugacy classes of symmetries with fixed points. As a consequence of Theorems 2.2.1 and 2.4.1 we get the following.

Corollary 2.4.2. *The maximum number of non-conjugate symmetries (of any type) that a Riemann surface of genus g may admit is 2^{r+1}, where 2^{r-1} is the largest power of 2 dividing $g-1$.*

Corollary 2.4.3. *A Riemann surface with the maximum number of non-conjugate symmetries with fixed points admits no symmetry with empty set of fixed points.*

Springer

http://www.springer.com/978-3-642-14827-9

Symmetries of Compact Riemann Surfaces
Bujalance, E.; Cirre, F.J.; Gamboa, J.M.; Gromadzki, G.
2010, XX, 164 p., Softcover
ISBN: 978-3-642-14827-9