A Review on Regression-based Monte Carlo Methods for Pricing American Options

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Abstract In this article we give a review of regression-based Monte Carlo methods for pricing American options. The methods require in a first step that the generally in continuous time formulated pricing problem is approximated by a problem in discrete time, i.e., the number of exercising times of the considered option is assumed to be finite. Then the problem can be formulated as an optimal stopping problem in discrete time, where the optimal stopping time can be expressed by the aid of so-called continuation values. These continuation values represent the price of the option given that the option is exercised after time t conditioned on the value of the price process at time t. The continuation values can be expressed as regression functions, and regression-based Monte Carlo methods apply regression estimates to data generated by the aid of artificial generated paths of the price process in order to approximate these conditional expectations. In this article we describe various methods and corresponding results for estimation of these regression functions.

1 Pricing of American Options as Optimal Stopping Problem

In many financial contracts it is allowed to exercise the contract early before expiry. E.g., many exchange traded options are of American type and allow the holder any exercise date before expiry, mortgages have often embedded prepayment options such that the mortgage can be amortized or repayed, or life insurance contracts allow often for early surrender. In this article we are interested in pricing of options with early exercise features.

It is well-known that in complete and arbitrage free markets the price of a derivative security can be represented as an expected value with respect to the so called martingale measure, see for instance Karatzas and Shreve (1998). Furthermore, the

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price of an American option with maturity T is given by the value of the optimal stopping problem

$$V_0 = \sup_{\tau \in \mathcal{T}([0,T])} \mathbf{E}\left\{d_{0,\tau} g_\tau(X_\tau)\right\}, \tag{1}$$

where g_t is a nonnegative payoff function, $(X_t)_{0 \le t \le T}$ is a stochastic process, which models the relevant risk factors, $\mathcal{T}([0,T])$ is the class of all stopping times with values in $[0,T]$, and $d_{s,t}$ are nonnegative $\mathcal{F}((X_u)_{s \le u \le t})$-measurable discount factors satisfying $d_{0,t} = d_{0,s} \cdot d_{s,t}$ for $s < t$. Here, a stopping time $\tau \in \mathcal{T}([0,T])$ is a measurable function of $(X_t)_{0 \le t \le T}$ with values in $[0,T]$ with the property that for any $r \in [0,T]$ the event $\{\tau \le r\}$ is contained in the sigma algebra $\mathcal{F}_r = \mathcal{F}((X_s)_{0 \le s \le r})$ generated by $(X_s)_{0 \le s \le r}$.

There are various possibilities for the choice of the process $(X_t)_{0 \le t \le T}$. The most simple examples are geometric Brownian motions, as for instance in the celebrated Black-Scholes setting. More general models include stochastic volatility models, jump-diffusion processes or general Levy processes. The model parameters are usually calibrated to observed time series data.

The first step in addressing the numerical solution of (1) is to pass from continuous time to discrete time, which means in financial terms to approximate the American option by a so-called Bermudan option. The convergence of the discrete time approximations to the continuous time optimal stopping problem is considered in Lamberton and Pagès (1990) for the Markovian case but also in the abstract setting of general stochastic processes.

For simplicity we restrict ourselves directly to a discrete time scale and consider exclusively Bermudan options. In analogy to (1), the price of a Bermudan option is the value of the discrete time optimal stopping problem

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E}\{f_\tau(X_\tau)\}, \tag{2}$$

where $X_0, X_1, \ldots, X_T$ is now a discrete time stochastic process, f_t is the discounted payoff function, i.e., $f_t(x) = d_{0,t} g_t(x)$, and $\mathcal{T}(0,\ldots,T)$ is the class of all $\{0,\ldots,T\}$-valued stopping times. Here a stopping time $\tau \in \mathcal{T}(0,\ldots,T)$ is a measurable function of $X_0, \ldots, X_T$ with the property that for any $k \in \{0,\ldots,T\}$ the event $\{\tau = k\}$ is contained in the sigma algebra $\mathcal{F}(X_0,\ldots,X_k)$ generated by $X_0, \ldots, X_k$.

2 The Optimal Stopping Time

In the sequel we assume that X_0, X_1, …, X_T is a $\mathbb{R}^d$-valued Markov process recording all necessary information about financial variables including prices of the underlying assets as well as additional risk factors driving stochastic volatility or stochastic interest rates. Neither the Markov property nor the form of the payoff as a function of the state X_t are very restrictive and can often be achieved by including supplementary variables.

The computation of (2) can be done by determination of an optimal stopping time $\tau^* \in \mathcal{T}(0, \ldots, T)$ satisfying

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E}\{f_\tau(X_\tau)\} = \mathbf{E}\{f_{\tau^*}(X_{\tau^*})\}. \tag{3}$$

For $0 \leq t < T$ let

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1,\ldots,T)} \mathbf{E}\{f_\tau(X_\tau)|X_t = x\} \tag{4}$$

be the so-called continuation value describing the value of the option at time t given $X_t = x$ and subject to the constraint of holding the option at time t rather than exercising it. For $t = T$ we define the corresponding continuation value by

$$q_T(x) = 0 \quad (x \in \mathbb{R}^d), \tag{5}$$

because the option expires at time T and hence we do not get any money if we sell it after time T.

In the sequel we will use techniques from the general theory of optimal stopping (cf., e.g., Chow et al. 1971 or Shiryayev 1978) in order to show that the optimal stopping time τ^* is given by

$$\tau^* = \inf\{s \in \{0, 1, \ldots, T\} : q_s(X_s) \leq f_s(X_s)\}. \tag{6}$$

Since $q_T(x) = 0$ and $f_T(x) \geq 0$ there exists always some index where $q_s(X_s) \leq f_s(X_s)$, so the right-hand side above is indeed well defined. The above form of τ^* allows a very nice interpretation: in order to sell the option in an optimal way, we have to sell it as soon as the value we get if we sell it immediately is at least as large as the value we get in the mean in the future, if we sell it in the future in an optimal way.

In order to prove (6) we need the following notations: Let $\mathcal{T}(t, t+1, \ldots, T)$ be the subset of $\mathcal{T}(0, \ldots, T)$ consisting of all stopping times which take on values only in $\{t, t+1, \ldots, T\}$ and let

$$V_t(x) = \sup_{\tau \in \mathcal{T}(t,t+1,\ldots,T)} \mathbf{E}\left\{f_\tau(X_\tau)\middle|X_t = x\right\} \tag{7}$$

be the so-called value function which describes the value we get in the mean if we sell the option in an optimal way after time $t-1$ given $X_t = x$. For $t \in \{-1, 0, \ldots, T-1\}$ set

$$\tau_t^* = \inf\{s \geq t+1 : q_s(X_s) \leq f_s(X_s)\}, \tag{8}$$

hence $\tau^* = \tau_{-1}^*$. Then the following result holds:

Theorem 1. *Under the above assumptions we have for any* $t \in \{-1, 0, \ldots, T\}$ *and* $\mathbf{P}_{X_t}$*-almost all* $x \in \mathbb{R}^d$*:*

$$V_t(x) = \mathbf{E}\left\{ f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}}) \middle| X_t = x \right\}. \tag{9}$$

Furthermore we have

$$V_0 = \mathbf{E}\left\{ f_{\tau^*}(X_{\tau^*}) \right\}. \tag{10}$$

The above theorem is well-known in literature (cf., e.g., Chap. 8 in Glasserman 2004), but usually not proven completely. For the sake of completeness we present a complete proof next.

Proof. We prove (9) by induction. For $t = T$ we have

$$\tau^*_{T-1} = T$$

and any $\tau \in \mathcal{T}(T)$ satisfies

$$\tau = T.$$

So in this case we have

$$\begin{aligned} V_T(x) &= \sup_{\tau \in \mathcal{T}(T)} \mathbf{E}\left\{ f_\tau(X_\tau) \middle| X_T = x \right\} = \mathbf{E}\left\{ f_T(X_T) \middle| X_T = x \right\} \\ &= \mathbf{E}\left\{ f_{\tau^*_{T-1}}(X_{\tau^*_{T-1}}) \middle| X_T = x \right\}. \end{aligned}$$

Let $t \in \{0, \ldots, T-1\}$ and assume that

$$V_s(x) = \mathbf{E}\left\{ f_{\tau^*_{s-1}}(X_{\tau^*_{s-1}}) \middle| X_s = x \right\}$$

holds for all $t < s \leq T$. In the sequel we prove (9). To do this, let $\tau \in \mathcal{T}(t, \ldots, T)$ be arbitrary. Then

$$\begin{aligned} f_\tau(X_\tau) &= f_\tau(X_\tau) \cdot 1_{\{\tau = t\}} + f_\tau(X_\tau) \cdot 1_{\{\tau > t\}} \\ &= f_t(X_t) \cdot 1_{\{\tau = t\}} + f_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}}) \cdot 1_{\{\tau > t\}}. \end{aligned}$$

Since $1_{\{\tau = t\}}$ and $1_{\{\tau > t\}} = 1 - 1_{\{\tau \leq t\}}$ are measurable with respect to $X_0, \ldots, X_t$ and since $(X_t)_{0 \leq t \leq T}$ is a Markov process we have

$$\begin{aligned} &\mathbf{E}\{f_\tau(X_\tau) | X_t\} \\ &\quad = \mathbf{E}\{f_t(X_t) \cdot 1_{\{\tau = t\}} | X_0, \ldots, X_t\} \\ &\qquad + \mathbf{E}\{f_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}}) \cdot 1_{\{\tau > t\}} | X_0, \ldots, X_t\} \\ &\quad = f_t(X_t) \cdot 1_{\{\tau = t\}} + 1_{\{\tau > t\}} \cdot \mathbf{E}\{f_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}}) | X_0, \ldots, X_t\} \\ &\quad = f_t(X_t) \cdot 1_{\{\tau = t\}} + 1_{\{\tau > t\}} \cdot \mathbf{E}\{f_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}}) | X_t\}. \end{aligned}$$

Using the definition of V_{t+1} together with $\max\{\tau, t+1\} \in \mathcal{T}(t+1, \ldots, T)$ and the Markov property we get

$$\begin{aligned} \mathbf{E}\{f_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}}) | X_t\} &= \mathbf{E}\{\mathbf{E}\{f_{\max\{\tau, t+1\}}(X_{\max\{\tau, t+1\}}) | X_{t+1}\} | X_t\} \\ &\leq \mathbf{E}\{V_{t+1}(X_{t+1}) | X_t\}, \end{aligned}$$

from which we can conclude

$$\begin{aligned}\mathbf{E}\{f_\tau(X_\tau)|X_t\} &\le f_t(X_t)\cdot 1_{\{\tau=t\}} + 1_{\{\tau>t\}}\cdot \mathbf{E}\{V_{t+1}(X_{t+1})|X_t\}\\ &\le \max\{f_t(X_t), \mathbf{E}\{V_{t+1}(X_{t+1})|X_t\}\}.\end{aligned}$$

Now we make the same calculations using $\tau = \tau^*_{t-1}$. We get

$$\begin{aligned}&\mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|X_t\}\\ &\quad = f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{f_{\max\{\tau^*_{t-1},t+1\}}(X_{\max\{\tau^*_{t-1},t+1\}})|X_t\}.\end{aligned}$$

By definition of τ^*_t we have on $\{\tau^*_{t-1} > t\}$

$$\max\{\tau^*_{t-1}, t+1\} = \tau^*_t.$$

Using this, the Markov property and the induction hypothesis we can conclude

$$\begin{aligned}\mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|X_t\} &= f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{\mathbf{E}\{f_{\tau^*_t}(X_{\tau^*_t})|X_{t+1}\}|X_t\}\\ &= f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{V_{t+1}(X_{t+1})|X_t\}.\end{aligned}$$

Next we show

$$\mathbf{E}\{V_{t+1}(X_{t+1})|X_t\} = q_t(X_t). \tag{11}$$

To see this, we observe that by induction hypothesis, Markov property and because of $\tau^*_t \in \mathcal{T}(t+1,\dots,T)$ we have

$$\begin{aligned}\mathbf{E}\{V_{t+1}(X_{t+1})|X_t\} &= \mathbf{E}\{\mathbf{E}\{f_{\tau^*_t}(X_{\tau^*_t})|X_{t+1}\}|X_t\} = \mathbf{E}\{f_{\tau^*_t}(X_{\tau^*_t})|X_t\}\\ &\le \sup_{\tau\in\mathcal{T}(t+1,\dots,T)} \mathbf{E}\{f_\tau(X_\tau)|X_t\} = q_t(X_t).\end{aligned}$$

Furthermore the definition of V_{t+1} implies

$$\begin{aligned}\mathbf{E}\{V_{t+1}(X_{t+1})|X_t\} &= \mathbf{E}\left\{\sup_{\tau\in\mathcal{T}(t+1,\dots,T)} \mathbf{E}\{f_\tau(X_\tau)|X_{t+1}\}\,|X_t\right\}\\ &\ge \sup_{\tau\in\mathcal{T}(t+1,\dots,T)} \mathbf{E}\{\mathbf{E}\{f_\tau(X_\tau)|X_{t+1}\}\,|X_t\} = q_t(X_t).\end{aligned}$$

Using the definition of τ^*_{t-1} we conclude

$$\begin{aligned}&f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{V_{t+1}(X_{t+1})|X_t\}\\ &\quad = f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}}\cdot q_t(X_t)\\ &\quad = \max\{f_t(X_t), q_t(X_t)\}.\end{aligned}$$

Summarizing the above results we have

$$V_t(x) = \sup_{\tau \in \mathcal{T}(t,t+1,\ldots,T)} \mathbf{E}\left\{ f_\tau(X_\tau) \middle| X_t = x \right\} \leq \max\{f_t(x), \mathbf{E}\{V_{t+1}(X_{t+1})|X_t = x\}\}$$
$$= \max\{f_t(x), q_t(x)\} = \mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|X_t = x\},$$

which proves

$$V_t(x) = \max\{f_t(x), q_t(x)\} = \mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|X_t = x\}. \tag{12}$$

In order to prove (10) we observe that by arguing as above we get

$$\begin{aligned}
V_0 &= \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E}\{f_\tau(X_\tau)\} \\
&= \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E}\left\{ f_0(X_0) \cdot 1_{\{\tau=0\}} + f_{\max\{\tau,1\}}(X_{\max\{\tau,1\}}) \cdot 1_{\{\tau>0\}} \right\} \\
&= \mathbf{E}\left\{ f_0(X_0) \cdot 1_{\{f_0(X_0) \geq q_0(X_0)\}} + f_{\tau_0^*}(X_{\tau_0^*}) \cdot 1_{\{f_0(X_0) < q_0(X_0)\}} \right\} \\
&= \mathbf{E}\left\{ f_0(X_0) \cdot 1_{\{f_0(X_0) \geq q_0(X_0)\}} + \mathbf{E}\{V_1(X_1)|X_0\} \cdot 1_{\{f_0(X_0) < q_0(X_0)\}} \right\} \\
&= \mathbf{E}\left\{ f_0(X_0) \cdot 1_{\{f_0(X_0) \geq q_0(X_0)\}} + q_0(X_0) \cdot 1_{\{f_0(X_0) < q_0(X_0)\}} \right\} \\
&= \mathbf{E}\{\max\{f_0(X_0), q_0(X_0)\}\} \\
&= \mathbf{E}\{f_{\tau^*}(X_{\tau^*})\}.
\end{aligned}$$

□

Remark 1. The continuation values and the value function are closely related. As we have seen already in the proof of Theorem 1 (cf., (11) and (12)) we have

$$q_t(x) = \mathbf{E}\{V_{t+1}(X_{t+1})|X_t = x\}$$

and

$$V_t(x) = \max\{f_t(x), q_t(x)\}.$$

Remark 2. Remark 1 shows that $q_s(X_s) \leq f_s(X_s)$ is equivalent to $V_s(X_s) \leq f_s(X_s)$. Hence the optimal stopping time can be also expressed via

$$\tau^* = \inf\{s \in \{0, \ldots, T\} : V_s(X_s) \leq f_s(X_s)\}. \tag{13}$$

3 Regression Representations for Continuation Values

The previous section shows that it suffices to determine the continuation values $q_0, \ldots, q_{T-1}$ in order to determine the optimal stopping time. We show in our next theorem three different regression representations for q_t, which have been introduced in Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999) and Egloff (2005), resp. In principle they allow a direct (and sometimes recursive) computation of the continuation values by computing conditional expectations.

Theorem 2. *Under the above assumptions for any $t \in \{0, \ldots, T-1\}$ and $\mathbf{P}_{X_t}$-almost all $x \in \mathbb{R}^d$ the following relations hold:*

(a)

$$q_t(x) = \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_t = x \right\}, \tag{14}$$

(b)

$$q_t(x) = \mathbf{E}\left\{ \max\left\{ f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1}) \right\} \middle| X_t = x \right\} \tag{15}$$

(c)

$$q_t(x) = \mathbf{E}\left\{ \Theta_{t+1,t+w+1}^{(w)} \middle| X_t = x \right\} \tag{16}$$

for any $w \in \{0, 1, \ldots, T-t-1\}$, *where*

$$\begin{aligned}
&\Theta_{t+1,t+w+1}^{(w)} \\
&\quad = \sum_{s=t+1}^{t+w+1} f_s(X_s) \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{s-1}(X_{s-1}) < q_{s-1}(X_{s-1}), f_s(X_s) \geq q_s(X_s)\}} \\
&\qquad + q_{t+w+1}(X_{t+w+1}) \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+w+1}(X_{t+w+1}) < q_{t+w+1}(X_{t+w+1})\}}.
\end{aligned}$$

Proof. (a) By (11), Theorem 1 and Markov property we get

$$\begin{aligned}
q_t(X_t) &= \mathbf{E}\left\{ V_{t+1}(X_{t+1}) \middle| X_t \right\} \\
&= \mathbf{E}\left\{ \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_{t+1} \right\} \middle| X_t \right\} \\
&= \mathbf{E}\left\{ \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_0, \ldots, X_{t+1} \right\} \middle| X_0, \ldots, X_t \right\} \\
&= \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_0, \ldots, X_t \right\} \\
&= \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_t \right\}.
\end{aligned}$$

(b) Because of

$$\begin{aligned}
f_{\tau_t^*}(X_{\tau_t^*}) &= f_{t+1}(X_{t+1}) \cdot 1_{\{\tau_t^* = t+1\}} + f_{\tau_t^*}(X_{\tau_t^*}) \cdot 1_{\{\tau_t^* > t+1\}} \\
&= f_{t+1}(X_{t+1}) \cdot 1_{\{f_{t+1}(X_{t+1}) \geq q_{t+1}(X_{t+1})\}} \\
&\quad + f_{\tau_{t+1}^*}(X_{\tau_{t+1}^*}) \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1})\}}
\end{aligned}$$

we can conclude from (a) and Markov property

$$\begin{aligned}
q_t(X_t) &= \mathbf{E}\Big\{ f_{t+1}(X_{t+1}) \cdot 1_{\{f_{t+1}(X_{t+1}) \geq q_{t+1}(X_{t+1})\}} \\
&\qquad + f_{\tau_{t+1}^*}(X_{\tau_{t+1}^*}) \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1})\}} \Big| X_t \Big\} \\
&= \mathbf{E}\left\{ \mathbf{E}\left\{ \ldots \middle| X_0, \ldots, X_{t+1} \right\} \middle| X_0, \ldots, X_t \right\} \\
&= \mathbf{E}\Big\{ f_{t+1}(X_{t+1}) \cdot 1_{\{f_{t+1}(X_{t+1}) \geq q_{t+1}(X_{t+1})\}} \\
&\qquad + \mathbf{E}\Big\{ f_{\tau_{t+1}^*}(X_{\tau_{t+1}^*}) \Big| X_{t+1} \Big\} \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1})\}} \Big| X_t \Big\} \\
&= \mathbf{E}\Big\{ f_{t+1}(X_{t+1}) \cdot 1_{\{f_{t+1}(X_{t+1}) \geq q_{t+1}(X_{t+1})\}} \\
&\qquad + q_{t+1}(X_{t+1}) \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1})\}} \Big| X_t \Big\} \\
&= \mathbf{E}\Big\{ \max\Big\{ f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1}) \Big\} \Big| X_t \Big\}.
\end{aligned}$$

(c) For any $w \in \{0, 1, \ldots, T-t-1\}$ we have

$$
\begin{aligned}
&f_{\tau_t^*}(X_{\tau_t^*}) \\
&\quad = \sum_{s=0}^{w} f_{t+s+1}(X_{t+s+1}) \cdot 1_{\{\tau_t^* = t+s+1\}} + f_{\tau_t^*}(X_{\tau_t^*}) \cdot 1_{\{\tau_t^* > t+w+1\}} \\
&\quad = \sum_{s=0}^{w} f_{t+s+1}(X_{t+s+1}) \\
&\qquad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+s}(X_{t+s}) < q_{t+s}(X_{t+s}), f_{t+s+1}(X_{t+s+1}) \geq q_{t+s+1}(X_{t+s+1})\}} \\
&\qquad + f_{\tau_{t+w}^*}(X_{\tau_{t+w}^*}) \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+w+1}(X_{t+w+1}) < q_{t+w+1}(X_{t+w+1})\}}.
\end{aligned}
$$

Using a) and Markov property we conclude

$$
\begin{aligned}
q_t(X_t) &= \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_t \right\} \\
&\quad \times \mathbf{E}\Bigg\{ \sum_{s=0}^{w} f_{t+s+1}(X_{t+s+1}) \\
&\quad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+s}(X_{t+s}) < q_{t+s}(X_{t+s}), f_{t+s+1}(X_{t+s+1}) \geq q_{t+s+1}(X_{t+s+1})\}} \\
&\quad + f_{\tau_{t+w}^*}(X_{\tau_{t+w}^*}) \\
&\quad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+w+1}(X_{t+w+1}) < q_{t+w+1}(X_{t+w+1})\}} \Big| X_t \Bigg\} \\
&= \mathbf{E}\{\mathbf{E}\{\ldots | X_0, \ldots, X_{t+w+1}\} | X_0, \ldots, X_t\} \\
&= \mathbf{E}\Bigg\{ \sum_{s=0}^{w} f_{t+s+1}(X_{t+s+1}) \\
&\quad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+s}(X_{t+s}) < q_{t+s}(X_{t+s}), f_{t+s+1}(X_{t+s+1}) \geq q_{t+s+1}(X_{t+s+1})\}} \\
&\quad + \mathbf{E}\{ f_{\tau_{t+w}^*}(X_{\tau_{t+w}^*}) | X_{t+w+1} \} \\
&\quad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+w+1}(X_{t+w+1}) < q_{t+w+1}(X_{t+w+1})\}} \Big| X_t \Bigg\} \\
&= \mathbf{E}\Bigg\{ \sum_{s=0}^{w} f_{t+s+1}(X_{t+s+1}) \\
&\quad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+s}(X_{t+s}) < q_{t+s}(X_{t+s}), f_{t+s+1}(X_{t+s+1}) \geq q_{t+s+1}(X_{t+s+1})\}} \\
&\quad + q_{t+w+1}(X_{t+w+1}) \\
&\quad \cdot 1_{\{f_{t+1}(X_{t+1}) < q_{t+1}(X_{t+1}), \ldots, f_{t+w+1}(X_{t+w+1}) < q_{t+w+1}(X_{t+w+1})\}} \Big| X_t \Bigg\},
\end{aligned}
$$

which implies the assertion. □

Remark 3. Because of

$$
\Theta_{t+1,t+1}^{(0)} = \max\{ f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1}) \}
$$

and

$$
\Theta_{t+1,T}^{(T-t-1)} = f_{\tau_t^*}(X_{\tau_t^*})
$$

the regression representation (16) includes (14) (for $t = T - t - 1$) and (15) (for $t = 0$) as special cases.

Remark 4. There exists also regression representations for the value functions. E.g., as we have seen already in Theorem 1 and its proof we have

$$V_t(x) = \mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|X_t = x\}$$

and

$$V_t(x) = \max\{f_t(x), \mathbf{E}\{V_{t+1}(X_{t+1})|X_t = x\}\}.$$

Furthermore, similarly to Theorem 2 it can be shown

$$V_t(x) = \mathbf{E}\{\Theta^{(w+1)}_{t,t+w+1}|X_t = x\}.$$

Using Theorem 2 or Remark 4 we can compute the continuation values and the value functions by (recursive) evaluation of conditional expectations. However, in applications the underlying distributions will be rather complicated and therefore it is not clear how to compute these conditional expectations in practice.

4 Outline of Regression-based Monte Carlo Methods

The basic idea of regression-based Monte Carlo methods is to use regression estimates as numerical procedures to compute the above conditional estimations approximately. To do this artificial samples of the price process are generated which are used to construct data for the regression estimates. The algorithms either construct estimates $\hat{q}_{n,t}$ of the continuation values q_t or estimates $\hat{V}_{n,t}$ of the value functions. Comparing the regression representations for the continuation values like

$$q_t(x) = \mathbf{E}\left\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\}\,\middle|\,X_t = x\right\}$$

with the regression representation for the value function like

$$V_t(x) = \max\{f_t(x), \mathbf{E}\{V_{t+1}(X_{t+1})|X_t = x\}\},$$

we see that in the later relation the maximum occurs outside of the expectation and as a consequence the value function will be in generally not differentiable. In contrast in the first relation the maximum will be smoothed by taking its conditional expectation. Since it is always easier to estimate smooth regression functions there is some reason to focus on continuation values, which we will do in the sequel.

Let $X_0, X_1, \ldots, X_T$ be a $\mathbb{R}^d$-valued Markov process and let f_t be the discounted payoff function. We assume that the data generating process is completely known, i.e., that all parameters of this process are already estimated from historical data. In order to estimate the continuation values q_t recursively, we generate in a first step artificial independent Markov processes $\{X_{i,t}\}_{t=0,\ldots,T}$ $(i = 1, 2, \ldots, n)$ which

are identically distributed as $\{X_t\}_{t=0,\dots,T}$. Then we use these so-called Monte Carlo samples in a second step to generate recursively data to estimate q_t by using one of the regression representation given in Theorem 2.

We start with

$$\hat{q}_{n,T}(x) = 0 \quad (x \in \mathbb{R}^d).$$

Given an estimate $\hat{q}_{n,t+1}$ of q_{t+1}, we estimate

$$\begin{aligned} q_t(x) &= \mathbf{E}\left\{ f_{\tau_t^*}(X_{\tau_t^*}) \middle| X_t = x \right\}, \\ &= \mathbf{E}\left\{ \max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} \middle| X_t = x \right\} \\ &= \mathbf{E}\left\{ \Theta_{t+1,t+w+1}^{(w)} \middle| X_t = x \right\} \end{aligned}$$

by applying a regression estimate to an "approximative" sample of (X_t, Y_t) where

$$Y_t = Y_t(X_{t+1}, \dots, X_T, q_{t+1}, \dots, q_T)$$

is either given by

$$\begin{aligned} Y_t = Y_t(X_{t+1}, \dots, X_t, q_{t+1}, \dots, q_T) = f_{\tau_t^*}(X_{\tau_t^*}), \\ Y_t = Y_t(X_{t+1}, q_{t+1}) = \max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} \end{aligned}$$

or

$$Y_t = Y_t(X_{t+1}, \dots, X_{t+w+1}, q_{t+1}, \dots, q_{t+w+1}) = \Theta_{t+1,t+w+1}^{(w)}.$$

With the notation

$$\hat{Y}_{i,t} = Y_t(X_{i,t+1}, \dots, X_{i,T}, \hat{q}_{n,t+1}, \dots, \hat{q}_{n,T})$$

(where we have suppressed the dependency of $\hat{Y}_{i,t}$ on n) this "approximative" sample is given by

$$\left\{ \left(X_{i,t}, \hat{Y}_{i,t} \right) : i = 1, \dots, n \right\}. \tag{17}$$

After having computed the estimates $\hat{q}_{0,n}, \dots, \hat{q}_{n,T}$ we can use them in two different ways to produce estimates of V_0. Firstly we can estimate

$$V_0 = \mathbf{E}\{\max\{f_0(X_0), q_0(X_0)\}\}$$

(cf. proof of Theorem 1) by just replacing q_0 by its estimate, i.e., by a Monte Carlo estimate of

$$\mathbf{E}\left\{ \max\{f_0(X_0), \hat{q}_{0,n}(X_0)\} \right\}. \tag{18}$$

Secondly, we can use our estimates to construct a plug-in estimate

$$\hat{\tau} = \inf\{s \in \{0, 1, \dots, T\} \geq 0 : \hat{q}_{n,s}(X_s) \leq f_s(X_s)\} \tag{19}$$

of the optimal stopping rule τ^* and estimate V_0 by a Monte Carlo estimate of

$$\mathbf{E}\{f_{\hat{\tau}}(X_{\hat{\tau}})\}. \tag{20}$$

Here in (19) and in (20) the expectation is taken only with respect to $X_0, \dots, X_T$ and not with respect to the random variables used in the definition of the estimates $\hat{q}_{n,s}$.

This kind of recursive estimation scheme was firstly proposed by Carriér (1996) for the estimation of value functions. In Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001) it was used to construct estimates of continuation values.

In view of a theoretical analysis of the estimates it usually helps if new variables of the price process are used for each recursive estimation step. In this way the error propagation (i.e., the influence of the error of $\hat{q}_{n,t+1}, \dots, \hat{q}_{n,T}$) can be analyzed much easier, cf. Kohler et al. (2010) or Kohler (2008a).

5 Algorithms Based on Linear Regression

In most applications the algorithm of the previous section is applied in connection with linear regression. Here basis functions

$$B_1, \dots, B_K : \mathbb{R}^d \to \mathbb{R}$$

are chosen and the estimate $\hat{q}_{n,t}$ is defined by

$$\hat{q}_{n,t} = \sum_{k=1}^{K} \hat{a}_k \cdot B_k, \tag{21}$$

where $\hat{a}_1, \dots, \hat{a}_K \in \mathbb{R}$ are chosen such that

$$\frac{1}{n}\sum_{i=1}^{n}\left|\hat{Y}_{i,t} - \sum_{k=1}^{K} \hat{a}_k \cdot B_k(X_{i,t})\right|^2 = \min_{a_1,\dots,a_K \in \mathbb{R}} \frac{1}{n}\sum_{i=1}^{n}\left|\hat{Y}_{i,t} - \sum_{k=1}^{K} a_k \cdot B_k(X_{i,t})\right|^2. \tag{22}$$

Here $\hat{Y}_{i,t}$ are defined either by

$$\hat{Y}_{i,t} = \max\{f_{t+1}(X_{i,t+1}), \hat{q}_{n,t+1}(X_{i,t+1})\}$$

in case of the Tsitsiklis-Van-Roy algorithm, or by

$$\hat{Y}_{i,t} = f_{\hat{\tau}_{i,t}}(X_{i,\hat{\tau}_{i,t}})$$

where

$$\hat{\tau}_{i,t} = \inf\{s \in \{t+1, \dots, T\} : f_s(X_{i,s}) \geq \hat{q}_{n,s}(X_{i,s})\}$$

in case of the Longstaff-Schwartz algorithm.

The estimate can be computed easily by solving a linear equation system. Indeed, it is well-known from numerical analysis (cf., e.g., Stoer 1993, Chap. 4.8.1) that (22) is equivalent to

$$\mathbf{B}^T \mathbf{B}\hat{\mathbf{a}} = \mathbf{B}^T \mathbf{Y} \tag{23}$$

where

$$\mathbf{Y} = (\hat{Y}_{1,t}, \ldots, \hat{Y}_{n,t})^T, \qquad \mathbf{B} = (B_k(X_{i,t}))_{i=1,\ldots,n,k=1,\ldots,K}$$

and

$$\hat{\mathbf{a}} = (\hat{a}_1, \ldots, \hat{a}_K)^T.$$

It was observed e.g. in Longstaff and Schwartz (2001) that the above estimate combined with the corresponding plug-in estimate (19) of the optimal stopping rule is rather robust with respect to the choice of the basis functions. The most simplest possibility are monomials, i.e.,

$$B_k(u_1, \ldots, u_d) = u_1^{s_{1,k}} \cdot u_2^{s_{2,k}} \cdots u_d^{s_{d,k}}$$

for some nonnegative integers $s_{1,k}, \ldots s_{d,k}$. For $d = 1$ this reduce to fitting a polynomial of a fixed degree (e.g., $K - 1$) to the data. For d large the degree of the multinomial polynomial (e.g. defined by $s_{1,k} + \cdots + s_{d,k}$ or by $\max_{j=1,\ldots,d,k=1,\ldots,K} s_{j,k}$ has chosen to be small in order to avoid that there are too many basis functions. It is well-known in practice that the estimate gets much better if the payoff function is chosen as one of the basis functions.

The Longstaff-Schwartz algorithm was proposed in Longstaff and Schwartz (2001). It was further theoretical examined in Clément et al. (2002). The Tsitsiklis-Van-Roy algorithm was introduced and theoretical examined in Tsitsiklis and Van Roy (1999, 2001).

6 Algorithms Based on Nonparametric Regression

Already in Carriér (1996) it was proposed to use nonparametric regression to estimate value functions. In the sequel we describe various nonparametric regression estimates of continuation values.

According to Györfi et al. (2002) there are four (related) paradigms for defining nonparametric regression estimates. The first is local averaging, where the estimate is defined by

$$\hat{q}_{n,t}(x) = \sum_{i=1}^{n} W_{n,i}(x, X_{1,t}, \ldots, X_{n,t}) \cdot \hat{Y}_{i,t} \tag{24}$$

with weights $W_{n,i}(x, X_{1,t}, \ldots, X_{n,t}) \in \mathbb{R}$ depending on the x-values of the sample. The most popular example of local averaging estimates is the Nadaraya-Watson kernel estimate, where a kernel function

$$K : \mathbb{R}^d \to \mathbb{R}$$

(e.g., the so-called naive kernel $K(u) = 1_{\{\|u\| \leq 1\}}$ or the Gaussian kernel $K(u) = \exp(-\|u\|^2/2)$) and a so-called bandwidth $h_n > 0$ are chosen and the weights are

defined by

$$W_{n,i}(x, X_{1,t}, \ldots, X_{n,t}) = \frac{K(\frac{x-X_{i,t}}{h_n})}{\sum_{j=1}^{n} K(\frac{x-X_{j,t}}{h_n})}.$$

Here the estimate is given by

$$\hat{q}_{n,t}(x) = \frac{\sum_{i=1}^{n} K(\frac{x-X_{i,t}}{h_n}) \cdot \hat{Y}_{i,t}}{\sum_{j=1}^{n} K(\frac{x-X_{j,t}}{h_n}).}$$

The second paradigm is global modeling (or least squares estimation), where a function space $\mathcal{F}_n$ consisting of functions $f : \mathbb{R}^d \to \mathbb{R}$ is chosen and the estimate is defined by

$$\hat{q}_{n,t} \in \mathcal{F}_n \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} |\hat{Y}_{i,t} - \hat{q}_{n,t}(X_{i,t})|^2 = \min_{f \in \mathcal{F}_n} \frac{1}{n}\sum_{i=1}^{n} |\hat{Y}_{i,t} - f(X_{i,t})|^2. \quad (25)$$

In case that $\mathcal{F}_n$ is a linear vector space (with dimension depending on the sample size) this estimate can be computed by solving a linear equation system corresponding to (23). Such linear function spaces occur e.g. in the definition of least squares spline estimates with fixed knot sequences, where the set $\mathcal{F}_n$ is chosen as a set of piecewise polynomials satisfying some global smoothness conditions (like differentiability).

Especially for large d it is also useful to consider nonlinear function spaces. The most popular example are neural networks, where for the most simple model $\mathcal{F}_n$ is defined by

$$\mathcal{F}_n = \left\{ \sum_{i=1}^{k_n} c_i \cdot \sigma(a_i^T x + b_i) + c_0 : a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R} \right\} \quad (26)$$

for some sigmoid function $\sigma : \mathbb{R} \to [0, 1]$. Here it is assumed that the sigmoid function σ is monotonically increasing and satisfies

$$\sigma(x) \to 0 \quad (x \to -\infty) \quad \text{and} \quad \sigma(x) \to 1 \quad (x \to \infty).$$

An example of such a sigmoid function is the logistic squasher defined by

$$\sigma(x) = \frac{1}{1 + e^{-x}} \quad (x \in \mathbb{R}).$$

There exists a deepest decent algorithm (so-called backfitting) which computes the corresponding least squares estimate approximately (cf., e.g., Rumelhart and McClelland 1986).

The third paradigm is penalized modeling. Instead of restricting the set of functions over which the so called empirical L_2 risk

$$\frac{1}{n}\sum_{i=1}^{n} |\hat{Y}_{i,t} - f(X_{i,t})|^2$$

is minimized, in this case a penalty term penalizing the roughness of the function is added to the empirical L_2 risk and this penalized empirical L_2 risk is basically minimized with respect to all functions.The most popular example of this kind of estimates are smoothing spline estimates. Here the estimate is defined by

$$\hat{q}_{n,t}(\cdot) = \arg\min_{f \in W^k(\mathbb{R}^d)} \left(\frac{1}{n}\sum_{i=1}^{n} |f(X_{i,t}) - \hat{Y}_{i,t}|^2 + \lambda_n \cdot J_k^2(f) \right), \tag{27}$$

where $k \in \mathbb{N}$ with $2k > d$, $W^k(\mathbb{R}^d)$ denotes the Sobolev space

$$\left\{ f : \frac{\partial^k f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \in L_2(\mathbb{R}^d) \text{ for all } \alpha_1, \ldots, \alpha_d \in \mathbb{N} \text{ with } \alpha_1 + \cdots + \alpha_d = k \right\},$$

and

$$J_k^2(f) = \sum_{\alpha_1,\ldots,\alpha_d \in \mathbb{N}, \alpha_1+\cdots+\alpha_d=k} \frac{k!}{\alpha_1! \cdot \ldots \cdot \alpha_d!} \int_{\mathbb{R}^d} \left| \frac{\partial^k f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x) \right|^2 dx.$$

Here $\lambda_n > 0$ is the smoothing parameter of the estimate.

The fourth (and last) paradigm is local modeling. It is similar to global modeling, but this time the function is fitted only locally to the data and a new function is used for each point in $\mathbb{R}^d$. The most popular example of this kind of estimate are local polynomial kernel estimates. Here the estimate, which depends on a nonnegative integer M and a kernel function $K : \mathbb{R}^d \to \mathbb{R}$, is given by

$$\hat{q}_{n,t}(x) = \hat{p}_x(x) \tag{28}$$

where

$$\hat{p}_x(\cdot) \in \mathcal{F}_M = \left\{ \sum_{0 \le j_1,\ldots,j_d \le M} a_{j_1,\ldots,j_d} \cdot \ldots \cdot (x^{(1)})^{j_1} \cdots (x^{(d)})^{j_d} : a_{j_1,\ldots,j_d} \in \mathbb{R} \right\} \tag{29}$$

satisfies

$$\frac{1}{n}\sum_{i=1}^{n} |\hat{p}_x(X_{i,t}) - \hat{Y}_{i,t}|^2 K\left(\frac{x - X_i}{h_n}\right) = \min_{p \in \mathcal{F}_M} \frac{1}{n}\sum_{i=1}^{n} |p(X_{i,t}) - \hat{Y}_{i,t}|^2 K\left(\frac{x - X_i}{h_n}\right). \tag{30}$$

The estimate can be computed again by solving a linear equation system, but this time of size n times n (instead K_n times K_n as for least squares estimates).

Each estimate above contains a smoothing parameter which determines how smooth the estimate should be. E.g., for the Nadaraya-Watson kernel estimate it is

the bandwidth $h_n > 0$, where a small bandwidth leads to a very rough estimate. For the smoothing spline estimate it is the parameter $\lambda_n > 0$, and for the least squares neural network estimate the smoothing parameter is the number k_n of neurons. For a successful application of the estimates these parameters need to be chosen data-dependent. The most simple way of doing this is splitting of the sample (cf., e.g., Chap. 7 in Györfi et al. 2002): Here the sample is divided into two parts, the first part is used to compute the estimate for different values of the parameter, and the second part is used to compute the empirical error of each of these estimates and that estimate is chosen where this empirical error is minimal. Splitting of the sample is in case of regression-based Monte Carlo methods the best method to choose the smoothing parameter, because there the data is chosen artificially with arbitrary sample size so it does not hurt at all if the estimate depends primary on the first part of the sample (since this first part can be as large as possible in view of computation of the estimate).

The first article where the use of nonparametric regression for the estimation of continuation values was examined theoretically was Egloff (2005). There nonparametric least squares estimates have been used, where the parameters where chosen by complexity regularization (cf., e.g., Chap. 12 in Györfi et al. 2002) and the consistency for general continuation values and the rate of convergence of the estimate in case of smooth continuation values has been investigated. For smooth continuation values Egloff (2005) showed the usual optimal rate of convergence for estimation of smooth regression functions. However, due to problems with the error propagation the estimate was defined such that it was very hard to compute it in practice, and it was not possible to check with simulated data whether nonparametric regression is not only useful asymptotically (i.e., for sample size tending to infinity, as was shown in the theoretical results), but also for finite sample size.

In Egloff et al. (2007) the error propagation was simplified by generating new data for each time point which was (conditioned on the data corresponding to time t) independent of all previously used data. In addition, a truncation of the estimate was introduced which allowed to choose linear vector spaces as function spaces for the least squares spline estimates, so that they can be computed by solving a linear equation system. The parameter (here the vector space dimension of the function space) of the least squares estimates were chosen by splitting of the sample. As regression representation the general formula of Egloff (2005) (cf. Theorem 2(c)) has been used. Consistency and rate of convergence results for these estimates have been derived, where as a consequence of the truncation of the estimate the rates contained an additional logarithmic factor. But the main advantage of these estimates is that they are easy to compute, so it was possible to analyze the finite sample size behavior of the estimates.

In Kohler et al. (2010), Kohler (2008a) and Kohler and Krzyżak (2009) the error propagation was further simplified by generation of new paths of the price process for each recursive estimation step and by using only the simple regression representation of Tsitsiklis and Van Roy (1999) (cf. Theorem 2(b)). As a consequence it was possible to analyze the estimates by using results derived in Kohler (2006) for regression estimation in case of additional measurement errors in the dependent vari-

able. Kohler et al. (2010) investigated least squares neural network estimates, which are very promising in case of large d, and Kohler (2008a) considered smoothing spline estimates. In both papers results concerning consistency and rate of convergence of the estimates have been derived. Kohler and Krzyżak (2009) presents a unifying theory which contains the results of the previous papers as well as results concerning new estimates (e.g., orthogonal series estimates).

The above papers focus on properties of the estimates of the continuation values, i.e., they consider the error between the continuation values and its estimates. As was pointed out by Belomestny (2009), sometimes much better rate of convergence results can be derived for the Monte Carlo estimate of (20) considered as estimate of the price V_0 of the option. Because in view of a good performance of the stopping time it is not important that the estimate of the continuation values are close to the continuation values, instead it is important that they lead to the same decision as the optimal stopping rule. And for this it is only important that

$$f_t(X_t) \geq \hat{q}_{n,t}(X_t)$$

is equivalent to

$$f_t(X_t) \geq q_t(X_t)$$

and not that $\hat{q}_{n,t}(X_t)$ and $q_t(X_t)$ are close. Belomestny (2009) introduces a kind of margin condition (similar to margin conditions in pattern recognition) measuring how quickly $q_t(X_t)$ approaches $f_t(X_t)$, and shows under this margin condition much better rate of convergence for the estimate (20) than previous results on the rates of convergence of the continuation values imply for the estimate (19).

7 Dual Methods

The above estimates yield estimates

$$\hat{\tau} = \inf\left\{s \in \{0, \ldots, T\} : \hat{q}_s(X_{n,s}) \leq f_s(X_s)\right\}$$

of the optimal stopping time τ^*. By Monte Carlo these estimates yields estimates of V_0, such that expectation

$$\mathbf{E}\left\{f_{\hat{\tau}}(X_{\hat{\tau}})\right\}$$

of the estimate is less than or equal to the true price V_0. It was proposed independently by Rogers (2001) and Haugh and Kogan (2004) that by using a dual method Monte Carlo estimates can be constructed such that the expectation of the estimate is greater than or equal to V_0. The key idea is the next theorem, which is already well-known in literature (cf., e.g., Sect. 8.7 in Glasserman 2004).

Theorem 3. *Let $\mathcal{M}$ be the set of all martingales $M_0, \ldots, M_T$ with $M_0 = 0$. Then*

$$V_0 = \inf_{M \in \mathcal{M}} \mathbf{E}\left\{\max_{t=0,\ldots,T}\left(f_t(X_t) - M_t\right)\right\} = \mathbf{E}\left\{\max_{t=0,\ldots,T}\left(f_t(X_t) - M_t^*\right)\right\}, \quad (31)$$

where

$$M_t^* = \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - \mathbf{E}\{\max\{f_s(X_s), q_s(X_s)\}|X_{s-1}\}). \tag{32}$$

For the sake of completeness we present next a complete proof of Theorem 3.

Proof. We first prove

$$\max_{t=0,\ldots,T} \left(f_t(X_t) - \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - \mathbf{E}\{\max\{f_s(X_s), q_s(X_s)\}|X_{s-1}\}) \right) = \max\{f_0(X_0), q_0(X_0)\}. \tag{33}$$

To do this, we observe that we have by Theorem 2(b)

$$\max_{t=0,\ldots,T} \left(f_t(X_t) - \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - \mathbf{E}\{\max\{f_s(X_s), q_s(X_s)\}|X_{s-1}\}) \right) = \max_{t=0,\ldots,T} \left(f_t(X_t) - \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})) \right).$$

For any $t \in \{1, \ldots, T\}$ we have

$$\begin{aligned} & f_t(X_t) - \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})) \\ & \leq f_t(X_t) - \sum_{s=1}^{t-1} (q_s(X_s) - q_{s-1}(X_{s-1})) - (f_t(X_t) - q_{t-1}(X_{t-1})) \\ & = q_0(X_0), \end{aligned}$$

furthermore in case $t = 0$ we get

$$f_t(X_t) - \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})) = f_0(X_0),$$

which shows

$$\max_{t=0,\ldots,T} \left(f_t(X_t) - \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})) \right) \leq \max\{f_0(X_0), q_0(X_0)\}.$$

But for $t = \tau^*$ we get in case of $q_0(X_0) > f_0(X_0)$ by definition of τ^*

$$
\begin{aligned}
&f_{\tau^*}(X_{\tau^*}) - \sum_{s=1}^{\tau^*} (\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})) \\
&\quad = f_{\tau^*}(X_{\tau^*}) - \sum_{s=1}^{\tau^*-1} (q_s(X_s) - q_{s-1}(X_{s-1})) - (f_{\tau^*}(X_{\tau^*}) - q_{\tau^*-1}(X_{\tau^*-1})) \\
&\quad = q_0(X_0),
\end{aligned}
$$

and in case of $q_0(X_0) \leq f_0(X_0)$ (which implies $\tau^* = 0$) we have

$$
f_{\tau^*}(X_{\tau^*}) - \sum_{s=1}^{\tau^*} (\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})) = f_0(X_0).
$$

This completes the proof of (33).

As shown at the end of the proof of Theorem 1 we have

$$
V_0 = \mathbf{E}\{\max\{f_0(X_0), q_0(X_0)\}\}.
$$

Using this together with (33) we get

$$
\mathbf{E}\left\{\max_{t=0,\ldots,T} \left(f_t(X_t) - M_t^*\right)\right\} = \mathbf{E}\{\max\{f_0(X_0), q_0(X_0)\}\} = V_0.
$$

Thus it suffices to show: For any martingale $M_0, \ldots, M_T$ with $M_0 = 0$ we have

$$
\mathbf{E}\left\{\max_{t=0,\ldots,T} (f_t(X_t) - M_t)\right\} \geq \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E}\{f_\tau(X_\tau)\} = V_0.
$$

But this follows from the optional sampling theorem, because if $M_0, \ldots, M_T$ is a martingale with $M_0 = 0$ and τ is a stopping time we know

$$
\mathbf{E}M_\tau = \mathbf{E}M_0 = 0
$$

and hence

$$
\mathbf{E}\{f_\tau(X_\tau)\} = \mathbf{E}\{f_\tau(X_\tau) - M_\tau\} \leq \mathbf{E}\left\{\max_{t=0,\ldots,T} (f_t(X_t) - M_t)\right\}.
$$

This completes the proof. □

Given estimates $\hat{q}_{n,s}$ ($s \in \{0, 1, \ldots, T\}$) of the continuation values, we can estimate the martingale (32) by

$$
\hat{M}_t = \sum_{s=1}^{t} \left(\max\{f_s(X_s), \hat{q}_{n,s}(X_s)\} - \mathbf{E}^*\left\{\max\{f_s(X_s), \hat{q}_{n,s}(X_s)\} | X_{s-1}\right\}\right). \quad (34)
$$

Provided we use unbiased and $\mathcal{F}(X_0, \ldots, X_t)$-measurable estimates $\mathbf{E}^*$ of the inner expectation in (32) (which can be constructed, e.g., by nested Monte Carlo) this leads to a martingale, too. This in turn can be used to construct Monte Carlo estimates of V_0, for which the expectation

$$\mathbf{E}\left\{\max_{t=0,\ldots,T}\left(f_t(X_t) - \hat{M}_t\right)\right\}$$

is greater than or equal to V_0. As a consequence we get two kind of estimates with expectation lower and higher than V_0, resp., so we have available an interval in which our true price should be contained.

In connection with linear regression these kind of estimates have been studied in Rogers (2001) and Haugh and Kogan (2004). Jamshidian (2007) studies multiplicative versions of this method. A comparative study of multiplicative and additive duals is contained in Chen and Glasserman (2007). Andersen and Broadie (2004) derive upper and lower bounds for American options based on duality. Belomestny et al. (2009) propose in a Brownian motion setting estimates with expectation greater than or equal to the true price, which can be computed without nested Monte Carlo (and hence are quite easy to compute).

In Kohler (2008b) dual methods have been combined with nonparametric smoothing spline estimates of the continuation values and consistency of the resulting estimates was shown for all bounded Markov processes. In Kohler et al. (2008) it is shown how these estimates can be modified such that less nested Monte Carlo steps are needed in an application.

8 Application to Simulated Data

The PhD thesis Todorovic (2007) contains various comparisons of regression-based Monte Carlo methods on simulated data. Using the standard monomial basis for linear regression (without including the payoff function) it turns out that for linear regression the regression representation of Longstaff and Schwartz (2001) produces often better results than the regression representation of Tsitsiklis and Van Roy (1999) in view of the performance of the estimated stopping rule on new data. But for nonparametric regression it does not seem to make a difference whether the regression representation of Longstaff and Schwartz (2001), of Tsitsiklis and Van Roy (1999) or the more general form of Egloff (2005) is used. Furthermore Todorovic (2007) shows that nonparametric regression estimate lead sometimes to much better performance than the linear regression estimates (and in his simulations never really worse performance) as long as the payoff function is not included in the basis function.

It turns out that this is less obvious if the payoff function is used as one of the basis functions for linear regression. But as we show below, in this case a very high sample size for the Monte Carlo estimates leads again to better results for

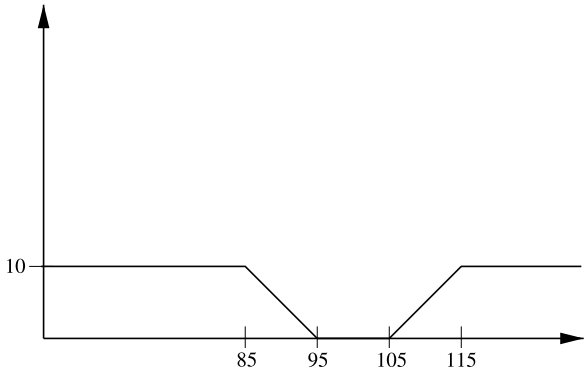

Fig. 1 Strangle spread payoff with strike prices 85, 95, 105 and 115

the nonparametric regression estimate. The reason for this is that the bias of the nonparametric regression estimates can be decreased by increasing the sample size, which is not true for linear regression.

In the sequel we consider an American option based on the average of three correlated stock prices. The stocks are ADECCO R, BALOISE R and CIBA. The stock prices were observed from Nov. 10, 2000 until Oct. 3, 2003 on weekdays when the stock market was open for the total of 756 days. We estimate the volatility from data observed in the past by the historical volatility

$$\sigma = (\sigma_{i,j})_{1 \le i,j \le 3} = \begin{pmatrix} 0.3024 & 0.1354 & 0.0722 \\ 0.1354 & 0.2270 & 0.0613 \\ 0.0722 & 0.0613 & 0.0717 \end{pmatrix}.$$

We simulate the paths of the underlying stocks with a Black-Scholes model by

$$X_{i,t} = x_0 \cdot e^{r \cdot t} \cdot e^{\sum_{j=1}^{3} (\sigma_{i,j} \cdot W_j(t) - \frac{1}{2} \cdot \sigma_{i,j}^2 t)} \quad (i = 1, \ldots, 3),$$

where $\{W_j(t) : t \in \mathbb{R}_+\}$ $(j = 1, \ldots, 3)$ are three independent Wiener processes and where the parameters are chosen as follows: $x_0 = 100$, $r = 0.05$ and components $\sigma_{i,j}$ of the volatility matrix as above. The time to maturity is assumed to be one year. To compute the payoff of the option we use a strangle spread function (cf. Fig. 1) with strikes 85, 95, 105 and 115 applied to the average of the three correlated stock prices.

We discretize the time interval [0, 1] by dividing it into $m = 48$ equidistant time steps with $t_0 = 0 < t_1 < \cdots < t_m = 1$ and consider a Bermudan option with payoff function as above and exercise dates restricted to $\{t_0, t_1, \ldots, t_m\}$. We choose discount factors $e^{-r \cdot t_j}$ for $j = 0, \ldots, m$. For all three algorithms we use sample size $n = 40000$ for the regression estimates of the continuation values.

For the nonparametric regression estimate we use smoothing splines as implemented in the routine Tps from the library "fields" in the statistics package *R*, where the smoothing parameter is chosen by generalized cross-validation. For the

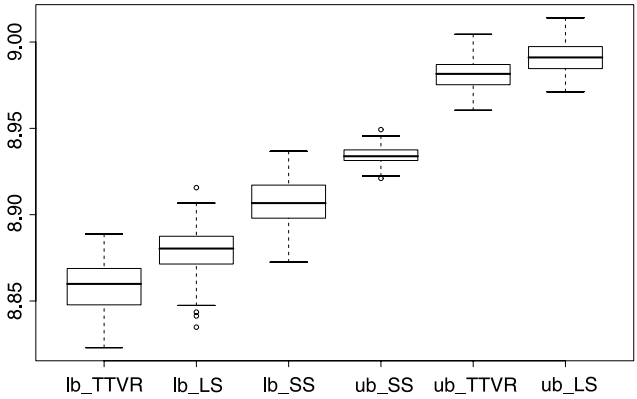

Fig. 2 Boxplots for 100 Monte Carlo estimates of lower bounds (lb) and upper bounds (ub) based on the estimates of the continuation values generated by the algorithm of Tsitsiklis and Van Roy (TTVR), Longstaff and Schwartz (LS) and nonparametric smoothing splines (SS)

Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms we use linear regression as implemented in *R* with degree 1 and payoff function included in the basis.

For each of these algorithms we compute Monte Carlo estimates of lower bounds on the option price defined using the corresponding estimated stopping rule, and Monte Carlo estimates of upper bounds on the option price using the corresponding estimated optimal martingale. Here we use 100 nested Monte Carlo steps to approximate the conditional expectation occurring in the optimal martingale. The sample size of the Monte Carlo estimates is 10000 in case of estimation of upper bounds and 40000 in case of estimation of lower bounds.

We apply all six algorithms for computing lower or upper bounds to 100 independently generated sets of paths and we compare the algorithms using boxplots for the 100 lower or upper bounds computed for each algorithm. We would like to stress that for all three algorithms computing upper bounds the expectation of the values are upper bounds to the true option price, hence lower values indicates a better performance of the algorithms, and that for all three algorithms computing lower bounds the expectation of the values are lower bounds to the true option price, hence higher values indicates a better performance of the algorithms.

As we can see in Fig. 2, the algorithms based on nonparametric regression are superior to Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms, since the lower boxplot of the upper bounds for this algorithm and the higher boxplot of the lower bounds for this algorithm indicate better performance.

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