1.2 Einstein Velocity Addition

Let c be any positive constant and let $(\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n = 1, 2, 3, \ldots$, equipped with the common vector addition, $+$, and inner product, $\cdot$. Furthermore, let

$$\mathbb{R}^n_c = \left\{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\right\} \tag{1.1}$$

be the c-ball of all relativistically admissible velocities of material particles. It is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ of all relativistically admissible velocities, given by the equation [58], [49, (2.9.2)], [40, p. 55], [18],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{1.2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_\mathbf{u}$ is the gamma factor given by the equation

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \tag{1.3}$$

Here $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = \mathbf{v}^2$. A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid*.

In the Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its space $\mathbb{R}^n$, as we see from (1.1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (1.2) and (1.3).

In physical applications, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. Furthermore, the constant c represents in physical applications the vacuum speed of light. Since we are interested in applications to geometry, we allow n to be any positive integer.

Einstein addition (1.2) of relativistically admissible velocities, with $n = 3$, was introduced by Einstein in his 1905 paper [12], [13, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (1.2) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [12] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (1.2) of Einstein addition.

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$, $\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}$ and, in particular,

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{1.4}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{1.5}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (1.4) is known as the *automorphic inverse property*, and Identity (1.5) is known as the *left cancellation law* of Einstein addition [63]. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (1.5) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u} \tag{1.6}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired in Sect. 1.9, p. 21.

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{1.7}$$

which can be equivalently written as

$$\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{1.8}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. Here, (1.8) is obtained from (1.7) by replacing $\mathbf{u}$ by $\ominus\mathbf{u} = -\mathbf{u}$ in (1.7).

A frequently used identity that follows immediately from (1.3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \tag{1.9}$$

and, similarly, a useful identity that follows immediately from (1.8) is

$$\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} = 1 - \frac{\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{1.10}$$

It is the gamma identity (1.7) that signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld [51] and Varičak [66, 67] in terms of *rapidities*, a term coined by Robb [47]. In fact, the gamma identity plays a role in hyperbolic geometry, analogous to the law of cosines in Euclidean geometry, as we will see in Sect. 6.3, p. 132. Historically, it formed the first link between special relativity and the hyperbolic geometry of Bolyai and Lobachevsky, recently leading to the novel trigonometry in hyperbolic geometry that became known as *gyrotrigonometry*, developed in [63, Chap. 12], [64, Chap. 4], [57, 62] and in Part II of this book.

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

$$\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\| \tag{1.11}$$

we have, in general,

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u} \tag{1.12}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Moreover, Einstein addition is also nonassociative since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) \tag{1.13}$$

for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$.

It seems that following the breakdown of commutativity and associativity in Einstein addition some mathematical regularity has been lost in the transition from Newton's velocity vector addition in $\mathbb{R}^n$ to Einstein's velocity addition (1.2) in $\mathbb{R}_c^n$. This is, however, not the case since Thomas gyration comes to the rescue, as we will see in Sect. 1.4. Owing to the presence of Thomas gyration, the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ has a grouplike structure [56] that we naturally call the *Einstein gyrogroup* [58]. The formal definition of the resulting abstract gyrogroup will be presented in Definition 1.5, p. 12.

1.3 Einstein Addition With Respect to Cartesian Coordinates

Like any physical law, Einstein velocity addition law (1.2) is coordinate independent. Indeed, it is presented in (1.2) in terms of vectors, noting that one of the great advantages of vectors is their ability to express results independent of any coordinate system.

However, in order to generate numerical and graphical demonstrations of physical laws, we need coordinates. Accordingly, we introduce Cartesian coordinates into the Euclidean n-space $\mathbb{R}^n$ and its ball $\mathbb{R}_c^n$, with respect to which we generate the graphs of this book. Introducing the Cartesian coordinate system Σ into $\mathbb{R}^n$ and $\mathbb{R}_c^n$, each point $P \in \mathbb{R}^n$ is given by an n-tuple

$$P = (x_1, x_2, \ldots, x_n), \qquad x_1^2 + x_2^2 + \cdots + x_n^2 < \infty \tag{1.14}$$

of real numbers, which are the coordinates, or components, of P with respect to Σ. Similarly, each point $P \in \mathbb{R}_c^n$ is given by an n-tuple

$$P = (x_1, x_2, \ldots, x_n), \qquad x_1^2 + x_2^2 + \cdots + x_n^2 < c^2 \tag{1.15}$$

of real numbers, which are the coordinates, or components of P with respect to Σ.

Equipped with a Cartesian coordinate system Σ and its standard vector addition given by component addition, along with its resulting scalar multiplication, $\mathbb{R}^n$ forms the standard Cartesian model of n-dimensional Euclidean geometry. In full analogy, equipped with a Cartesian coordinate system Σ and its Einstein addition, along with its resulting scalar multiplication (to be studied in Sect. 2.1), the ball $\mathbb{R}_c^n$ forms in this book the Cartesian–Beltrami–Klein ball model of n-dimensional hyperbolic geometry.

As an illustrative example, we present below the Einstein velocity addition law (1.2) in $\mathbb{R}^3_c$ with respect to a Cartesian coordinate system.

Let $\mathbb{R}^3_c$ be the c-ball of the Euclidean 3-space, equipped with a Cartesian coordinate system Σ. Accordingly, each point of the ball is represented by its coordinates $(x_1, x_2, x_3)^t$ (exponent t denotes transposition) with respect to Σ, satisfying the condition $x_1^2 + x_2^2 + x_3^2 < c^2$.

Furthermore, let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3_c$ be three points in $\mathbb{R}^3_c \subset \mathbb{R}^3$ given by their coordinates with respect to Σ,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \tag{1.16}$$

where

$$\mathbf{w} = \mathbf{u} \oplus \mathbf{v} \tag{1.17}$$

The dot product of $\mathbf{u}$ and $\mathbf{v}$ is given in Σ by the equation

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1.18}$$

and the squared norm $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$ of $\mathbf{v}$ is given by the equation

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2 \tag{1.19}$$

Hence, it follows from the coordinate independent vector representation (1.2) of Einstein addition that the coordinate dependent Einstein addition (1.17) with respect to the Cartesian coordinate system Σ takes the form

$$\begin{aligned} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= \frac{1}{1 + \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{c^2}} \\ &\quad \times \left\{ \left[1 + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (u_1 v_1 + u_2 v_2 + u_3 v_3) \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \frac{1}{\gamma_{\mathbf{u}}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} \end{aligned} \tag{1.20}$$

where

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{u_1^2 + u_2^2 + u_3^2}{c^2}}} \tag{1.21}$$

The three components of Einstein addition (1.17) are w_1, w_2 and w_3 in (1.20). For a two-dimensional illustration of Einstein addition (1.20) one may impose the condition $u_3 = v_3 = 0$, implying $w_3 = 0$.

In the Newtonian–Euclidean limit, $c \to \infty$, the ball $\mathbb{R}^3_c$ expands to the Euclidean 3-space $\mathbb{R}^3$, and Einstein addition (1.20) reduces to the common vector addition in $\mathbb{R}^3$,

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{1.22}$$

1.4 Einstein Addition vs. Vector Addition

Vector addition, $+$, in $\mathbb{R}^n$ is both commutative and associative, satisfying

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u}, && \text{(Commutative Law)} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{(Associative Law)} \end{aligned} \tag{1.23}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ is neither commutative nor associative.

In order to measure the extent to which Einstein addition deviates from associativity we introduce *gyrations*, which are maps that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ onto itself. Gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}^3_c, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\bigl\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\bigr\} \tag{1.24}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3_c$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}^3_c, \oplus)$.

We recall that an automorphism of a groupoid $(S, \oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, denoted $\mathrm{Aut}(S, \oplus)$. To emphasize that the gyrations of an Einstein gyrogroup $(\mathbb{R}^3_c, \oplus)$ are automorphisms of the gyrogroup, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, is *trivial* if $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^3_c$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0}, \mathbf{v}]$, $\mathrm{gyr}[\mathbf{v}, \mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v}, \ominus\mathbf{v}]$ are trivial for all $\mathbf{v} \in \mathbb{R}^3_c$, as we see from (1.24).

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the gyrocommutative and the gyroassociative laws of Einstein addition in the following identities [58, 60, 63]:

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}), && \text{(Gyrocommutative Law)}\\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}, && \text{(Left Gyroassociative Law)}\\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus\big(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}\big), && \text{(Right Gyroassociative Law)}\\
\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}], && \text{(Gyration Left Loop Property)}\\
\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}], && \text{(Gyration Right Loop Property)}\\
\mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}], && \text{(Gyration Even Property)}\\
\big(\mathrm{gyr}[\mathbf{u},\mathbf{v}]\big)^{-1} &= \mathrm{gyr}[\mathbf{v},\mathbf{u}], && \text{(Gyration Inversion Law)}
\end{aligned} \tag{1.25}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$.

Einstein addition is thus regulated by gyrations to which it gives rise owing to its nonassociativity, so that Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [55]. Interestingly, (Thomas) gyrations are the mathematical abstraction of the relativistic effect known as *Thomas precession* [63, Sect. 10.3].

The loop properties in (1.25) present important gyration identities. These two gyration identities are, however, just the tip of a giant iceberg. Many other useful gyration identities are studied in [58, 60, 63] and will be studied in the sequel.

1.5 Gyrations

Owing to its nonassociativity, Einstein addition gives rise in (1.24) to gyrations

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]:\mathbb{R}^n_c\to\mathbb{R}^n_c \tag{1.26}$$

for any $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$ in an Einstein groupoid $(\mathbb{R}^n_c,\oplus)$. Gyrations, in turn, regulate Einstein addition, endowing it with the rich structure of a gyrocommutative gyrogroup, as we will see in Sect. 1.6, and a gyrovector space, as we will see in Sect. 2.1. Clearly, gyrations measure the extent to which Einstein addition is nonassociative, where associativity corresponds to trivial gyrations.

An explicit presentation of the gyrations of Einstein groupoids $(\mathbb{R}^n_c,\oplus)$ is, therefore, desirable. Indeed, the gyration equation (1.24) can be manipulated into the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}=\mathbf{w}+\frac{A\mathbf{u}+B\mathbf{v}}{D} \tag{1.27}$$

Chapter 2
Einstein Gyrovector Spaces

Abstract Einstein addition admits scalar multiplication between any real number and any relativistically admissible velocity vector, giving rise to the Einstein gyrovector spaces. As an example, Einstein scalar multiplication enables hyperbolic lines to be calculated with respect to Cartesian coordinates just as Euclidean lines are calculated with respect to Cartesian coordinates. Along with remarkable analogies that Einstein scalar multiplication shares with the common scalar multiplication in vector spaces there is a striking disanalogy. Einstein scalar multiplication does not distribute over Einstein addition. However, a weaker law, called the *monodistributive law*, remains valid. It is shown in this chapter that Einstein gyrovector spaces form the setting for the Cartesian–Beltrami–Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry.

2.1 Einstein Scalar Multiplication

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the setting for the Cartesian–Beltrami–Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry, as we will see in this book.

Let $k\otimes\mathbf{v}$ be the Einstein addition of k copies of $\mathbf{v}\in\mathbb{R}_c^n$, that is $k\otimes\mathbf{v}=\mathbf{v}\oplus\mathbf{v}\cdots\oplus\mathbf{v}$ (k terms). Then,

$$k\otimes\mathbf{v}=c\frac{(1+\frac{\|\mathbf{v}\|}{c})^k-(1-\frac{\|\mathbf{v}\|}{c})^k}{(1+\frac{\|\mathbf{v}\|}{c})^k+(1-\frac{\|\mathbf{v}\|}{c})^k}\frac{\mathbf{v}}{\|\mathbf{v}\|}. \tag{2.1}$$

The definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing k off the positive integers, thus obtaining the following definition:

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Definition 2.1 (Einstein Scalar Multiplication; Einstein Gyrovector Spaces) An Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is an Einstein gyrogroup $(\mathbb{R}_s^n, \oplus)$ with scalar multiplication $\otimes$ given by

$$r\otimes\mathbf{v} = s\frac{(1+\frac{\|\mathbf{v}\|}{s})^r - (1-\frac{\|\mathbf{v}\|}{s})^r}{(1+\frac{\|\mathbf{v}\|}{s})^r + (1-\frac{\|\mathbf{v}\|}{s})^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} = s\tanh\left(r\tanh^{-1}\frac{\|\mathbf{v}\|}{s}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \tag{2.2}$$

where r is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

Example 2.2 (The Einstein Half) In the special case when $r = 1/2$, (2.2) reduces to

$$\frac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \tag{2.3}$$

so that

$$\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} = \mathbf{v}. \tag{2.4}$$

Einstein gyrovector spaces are studied in [63, Sect. 6.18]. Einstein scalar multiplication does not distribute with Einstein addition, but it possesses other properties of vector spaces. For any positive integer k, and for all real numbers $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}_s^n$, we have

$$\begin{aligned} k\otimes\mathbf{v} &= \mathbf{v}\oplus\cdots\oplus\mathbf{v}, && (k \text{ terms}) \\ (r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}, && \text{(Scalar Distributive Law)} \\ (r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{(Scalar Associative Law)} \end{aligned} \tag{2.5}$$

in any Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$.

Additionally, Einstein gyrovector spaces possess the *scaling property*

$$\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{2.6}$$

for $\mathbf{a} \in \mathbb{R}_s^n$, $\mathbf{a} \neq \mathbf{0}$, $r \in \mathbb{R}$, $r \neq 0$, the *gyroautomorphism property*

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \tag{2.7}$$

for $\mathbf{a}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, $r \in \mathbb{R}$, and the identity gyroautomorphism

$$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I \tag{2.8}$$

for $r_1, r_2 \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$.

Any Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ inherits an inner product and a norm from its vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations, that is,

$$\begin{aligned} \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{u} \cdot \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v} &= \mathbf{u} \cdot \mathbf{v}, \\ \big\|\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v}\big\| &= \|\mathbf{v}\| \end{aligned} \tag{2.9}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$.

Unlike vector spaces, Einstein gyrovector spaces $(\mathbb{R}^n_s, \oplus, \otimes)$ do not possess the distributive law since, in general,

$$r \otimes (\mathbf{u} \oplus \mathbf{v}) \neq r \otimes \mathbf{u} \oplus r \otimes \mathbf{v} \tag{2.10}$$

for $r \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$. However, a weak form of the distributive law does exist, as we see from the following theorem:

Theorem 2.3 (The Monodistributive Law) *Let* $(\mathbb{R}^n_s, \oplus, \otimes)$ *be an Einstein gyrovector space. Then,*

$$r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) = r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}) \tag{2.11}$$

for all $r, r_1, r_2 \in \mathbb{R}$ *and* $\mathbf{v} \in \mathbb{R}^n_s$.

Proof By the scalar distributive and associative laws, (2.5), we have

$$\begin{aligned} r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) &= r \otimes \big\{(r_1 + r_2) \otimes \mathbf{v}\big\} \\ &= \big(r(r_1 + r_2)\big) \otimes \mathbf{v} \\ &= (rr_1 + rr_2) \otimes \mathbf{v} \\ &= (rr_1) \otimes \mathbf{v} \oplus (rr_2) \otimes \mathbf{v} \\ &= r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}), \end{aligned} \tag{2.12}$$

as desired. □

Since scalar multiplication in Einstein gyrovector spaces does not distribute with Einstein addition, the following theorem is interesting.

Theorem 2.4 (The Two-Sum Identity) *Let* $\mathbf{u}, \mathbf{v}$ *be any two points of an Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$. *Then*

$$2 \otimes (\mathbf{u} \oplus \mathbf{v}) = \mathbf{u} \oplus (2 \otimes \mathbf{v} \oplus \mathbf{u}). \tag{2.13}$$

Proof Employing the right gyroassociative law in (1.25), the identity $\mathrm{gyr}[\mathbf{v}, \mathbf{v}] = I$, Theorem 1.8(4), the left gyroassociative law, and the gyrocommutative law in (1.25) we have the following chain of equations that gives (2.13),

$$
\begin{aligned}
\mathbf{u}\oplus(2\otimes\mathbf{v}\oplus\mathbf{u}) &= \mathbf{u}\oplus\big((\mathbf{v}\oplus\mathbf{v})\oplus\mathbf{u}\big) \\
&= \mathbf{u}\oplus\big(\mathbf{v}\oplus\big(\mathbf{v}\oplus \mathrm{gyr}[\mathbf{v},\mathbf{v}]\mathbf{u}\big)\big) \\
&= \mathbf{u}\oplus\big(\mathbf{v}\oplus(\mathbf{v}\oplus\mathbf{u})\big) \\
&= (\mathbf{u}\oplus\mathbf{v})\oplus \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) \\
&= (\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus\mathbf{v}) \\
&= 2\otimes(\mathbf{u}\oplus\mathbf{v}).
\end{aligned}
\tag{2.14}
$$

□

As an application of Theorem 2.4, we prove the following theorem:

Theorem 2.5 *Let* $\mathbf{u}, \mathbf{v}$ *be any two points of an Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$. *Then*

$$
\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes\frac{1}{2} = \frac{1}{2}\otimes(\mathbf{u}\boxplus\mathbf{v}). \tag{2.15}
$$

Proof The proof is given by the following chain of equations, which are numbered for subsequent derivation:

$$
\begin{aligned}
2\otimes\left\{\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes\frac{1}{2}\right\} &\overbrace{=\!=\!=}^{(1)} \mathbf{u}\oplus\big\{(\ominus\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{u}\big\} \\
&\overbrace{=\!=\!=}^{(2)} \big\{\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\big\}\oplus \mathrm{gyr}[\mathbf{u},\ominus\mathbf{u}\oplus\mathbf{v}]\mathbf{u} \\
&\overbrace{=\!=\!=}^{(3)} \mathbf{v}\oplus \mathrm{gyr}[\mathbf{u},\ominus\mathbf{u}\oplus\mathbf{v}]\mathbf{u} \\
&\overbrace{=\!=\!=}^{(4)} \mathbf{v}\oplus \mathrm{gyr}[\mathbf{v},\ominus\mathbf{u}]\mathbf{u} \\
&\overbrace{=\!=\!=}^{(5)} \mathbf{v}\boxplus\mathbf{u} \\
&\overbrace{=\!=\!=}^{(6)} \mathbf{u}\boxplus\mathbf{v},
\end{aligned}
\tag{2.16}
$$

implying (2.15) by the scalar associative law in (2.5). Derivation of the numbered equalities in (2.16) follows.

1. Follows from the Two-Sum Identity in Theorem 2.4 and the scalar associative law in (2.5).
2. Follows from Item 1 by the left gyroassociative law.
3. Follows from Item 2 by a left cancellation.
4. Follows from Item 3 by applying successively the left loop property and the right loop property.

5. Follows from Item 4 by Definition 1.7, p. 13, of the gyrogroup cooperation $\boxplus$.
6. Follows from Item 5 by the commutativity of Einstein coaddition $\boxplus$ according to (1.37), p. 13. □

2.2 Linking Einstein Addition to Hyperbolic Geometry

The Einstein distance function, $d(\mathbf{u}, \mathbf{v})$, in an Einstein gyrovector space $(\mathbb{R}_c^n, \oplus, \otimes)$ is given by the equation

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \ominus \mathbf{v}\| \tag{2.17}$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. We call it a *gyrodistance function* in order to emphasize the analogies it shares with its Euclidean counterpart, the distance function $\|\mathbf{u} - \mathbf{v}\|$ in $\mathbb{R}^n$. Among these analogies is the gyrotriangle inequality according to which

$$\|\mathbf{u} \oplus \mathbf{v}\| \le \|\mathbf{u}\| \oplus \|\mathbf{v}\| \tag{2.18}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. For this and other analogies that distance and gyrodistance functions share, see [60, 63].

In a two dimensional Einstein gyrovector space $(\mathbb{R}_c^2, \oplus, \otimes)$, the squared gyrodistance between a point $\mathbf{x} \in \mathbb{R}_c^2$ and an infinitesimally nearby point $\mathbf{x} + d\mathbf{x} \in \mathbb{R}_c^2$, $d\mathbf{x} = (dx_1, dx_2)$, is defined by the equation [63, Sect. 7.5], [60, Sect. 7.5]

$$\begin{aligned} ds^2 &= \left\|(\mathbf{x} + d\mathbf{x}) \ominus \mathbf{x}\right\|^2 \\ &= E\,dx_1^2 + 2F\,dx_1\,dx_2 + G\,dx_2^2 + \cdots, \end{aligned} \tag{2.19}$$

where, if we use the notation $r^2 = x_1^2 + x_2^2$, we have

$$\begin{aligned} E &= c^2 \frac{c^2 - x_2^2}{(c^2 - r^2)^2}, \\ F &= c^2 \frac{x_1 x_2}{(c^2 - r^2)^2}, \\ G &= c^2 \frac{c^2 - x_1^2}{(c^2 - r^2)^2}. \end{aligned} \tag{2.20}$$

The triple $(g_{11}, g_{12}, g_{22}) = (E, F, G)$ along with $g_{21} = g_{12}$ is known in differential geometry as the metric tensor g_{ij} [31]. It turns out to be the metric tensor of the Beltrami–Klein disc model of hyperbolic geometry [37, p. 220]. Hence, ds^2 in (2.19)–(2.20) is the Riemannian line element of the Beltrami–Klein disc model of hyperbolic geometry, linked to Einstein velocity addition (1.2), p. 4, and to Einstein gyrodistance function (2.17) [61].

The link between Einstein gyrovector spaces and the Beltrami–Klein ball model of hyperbolic geometry, already noted by Fock [18, p. 39], has thus been established

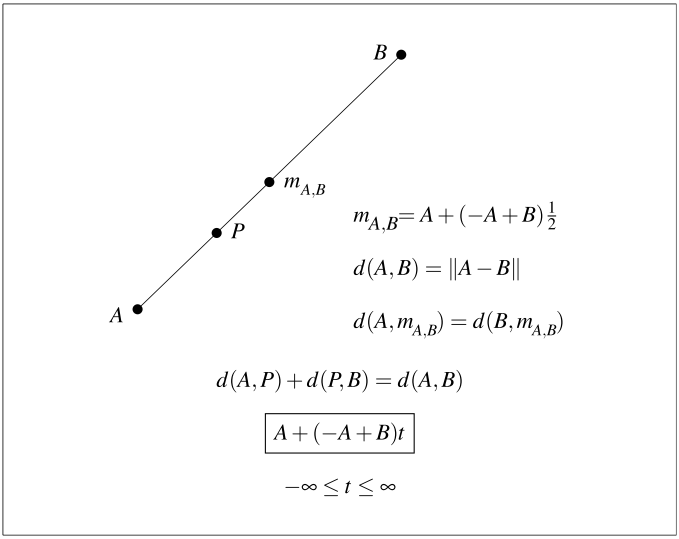

Fig. 2.1 The Euclidean line. The line $A + (-A + B)t$, $t \in \mathbb{R}$, in a Euclidean plane is shown. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the line through the points A and B lying between these points. The Einstein sum, $+$, of the distance from A to P and from P to B equals the distance from A to B. The point $m_{A,B}$ is the midpoint of the points A and B, corresponding to $t = 1/2$

in (2.17)–(2.20) in two dimensions. The extension of the link to higher dimensions is presented in [58, Sect. 9, Chap. 3], [63, Sect. 7.5], [60, Sect. 7.5], and [61]. For a brief account of the history of linking Einstein's velocity addition law with hyperbolic geometry, see [44, p. 943].

2.3 The Euclidean Line

We introduce Cartesian coordinates into $\mathbb{R}^n$ in the usual way in order to specify uniquely each point P of the Euclidean n-space $\mathbb{R}^n$ by an n-tuple of real numbers, called the coordinates, or components, of P. Cartesian coordinates provide a method of indicating the position of points and rendering graphs on a two-dimensional Euclidean plane $\mathbb{R}^2$ and in a three-dimensional Euclidean space $\mathbb{R}^3$.

As an example, Fig. 2.1 presents a Euclidean plane $\mathbb{R}^2$ equipped with an unseen Cartesian coordinate system Σ. The position of points A and B and their midpoint m_{AB} with respect to Σ are shown. The missing Cartesian coordinates in Fig. 2.1 are shown in Fig. 2.3.

The set of all points

$$A + (-A + B)t, \tag{2.21}$$

$t \in \mathbb{R}$, forms a Euclidean line. The segment of this line, corresponding to $1 \le t \le 1$, and a generic point P on the segment, are shown in Fig. 2.1. Being collinear, the

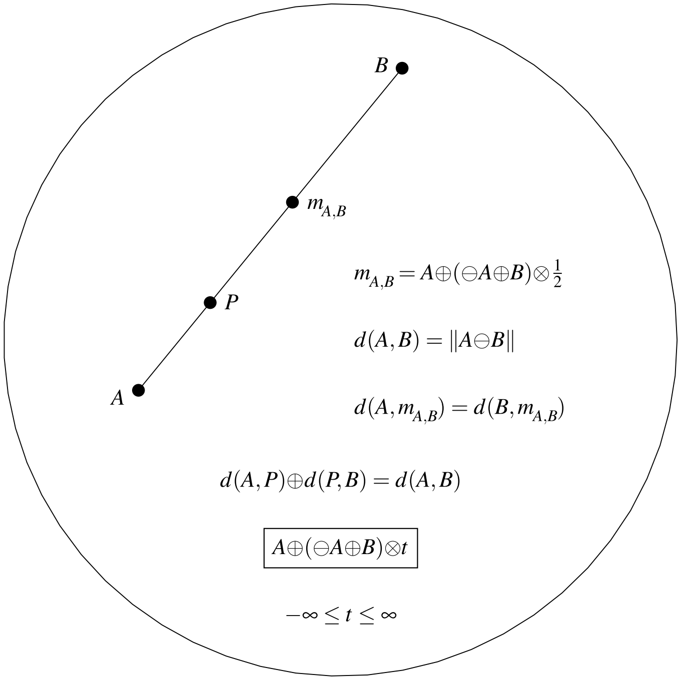

Fig. 2.2 Gyroline, the hyperbolic line. The gyroline $A\oplus(\ominus A\oplus B)\otimes t$, $t \in \mathbb{R}$, in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a geodesic line in the Beltrami–Klein ball model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in the Euclidean geometry of $\mathbb{R}^n$. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the gyroline through the points A and B lying between these points. The Einstein sum, $\oplus$, of the gyrodistance from A to P and from P to B equals the gyrodistance from A to B. The point $m_{A,B}$ is the gyromidpoint of the points A and B, corresponding to $t = 1/2$. The analogies between lines and gyrolines, as illustrated in Figs. 2.1 and 2.2, are obvious

points A, P and B obey the triangle equality $d(A, P) + d(P, B) = d(A, B)$, where $d(A, B) = \|-A + B\|$ is the Euclidean distance function in $\mathbb{R}^n$.

Figure 2.1 demonstrates the use of the standard Cartesian model of Euclidean geometry for graphical presentations. In a fully analogous way, Fig. 2.2 demonstrates the use of the Cartesian–Beltrami–Klein model of hyperbolic geometry, as we will see in Sects. 2.4 and 2.5.

2.4 Gyrolines—the Hyperbolic Lines

In the study of triangles and gyrotriangles, we use extensively the letters a, b, c to denote triangle side-lengths and gyrotriangle side-gyrolengths. Hence, it is convenient in applications to geometry to replace the notation $\mathbb{R}^n_c$ for the c-ball of an Einstein gyrovector space by the s-ball, $\mathbb{R}^n_s$. Moreover, it is understood in this book that $n \ge 2$ is any integer greater than 2, unless specified otherwise.

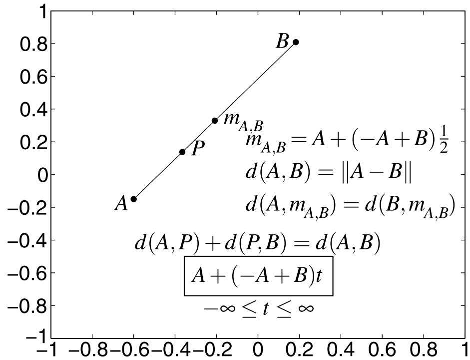

Fig. 2.3 The Cartesian coordinates for the Euclidean plane $\mathbb{R}^2$, (x_1, x_2), $x_1^2 + x_2^2 < \infty$, unseen in Fig. 2.1, are shown here. The points A and B are given, with respect to these Cartesian coordinates by $A = (-0.60, -0.15)$ and $B = (0.18, 0.80)$

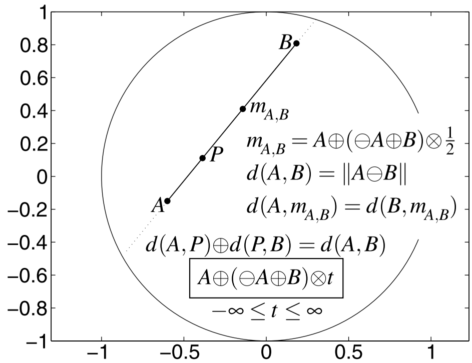

Fig. 2.4 The Cartesian coordinates for the unit disc in the Euclidean plane $\mathbb{R}^2$, (x_1, x_2), $x_1^2 + x_2^2 < 1$, unseen in Fig. 2.2, are shown here. The points A and B are given, with respect to these Cartesian coordinates by $A = (-0.60, -0.15)$ and $B = (0.18, 0.80)$

Let $A, B \in \mathbb{R}_s^n$ be two distinct points of the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, in full analogy with the Euclidean line (2.21), the graph of the set of all points, Fig. 2.2,

$$A\oplus(\ominus A\oplus B)\otimes t \tag{2.22}$$

for $t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}_s^n$. As such, it is a geodesic line of the Cartesian–Beltrami–Klein ball model of hyperbolic geometry, shown in Fig. 2.2 for $n = 2$. The geodesic line (2.22) is the unique geodesic passing through the points A and B. It passes through the point A when $t = 0$ and, owing to the left cancellation law, (1.38), it passes through the point B when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of A and B when $t = 1/2$. Accordingly, the *gyrosegment* that joins the points A and B in Fig. 2.2 is obtained from gyroline (2.22) with $0 \le t \le 1$.

Each point of (2.22) with $0 < t < 1$ is said to lie *between* A and B. Thus, for instance, the point P in Fig. 2.2 lies between the points A and B. As such, the points A, P and B obey the *gyrotriangle equality* according to which $d(A, P)\oplus d(P, B) = d(A, B)$, in full analogy with Euclidean geometry. The points in Fig. 2.2 are drawn with respect to an unseen Cartesian coordinate system. The missing Cartesian coordinates for the hyperbolic disc in Fig. 2.2 are shown in Fig. 2.4.

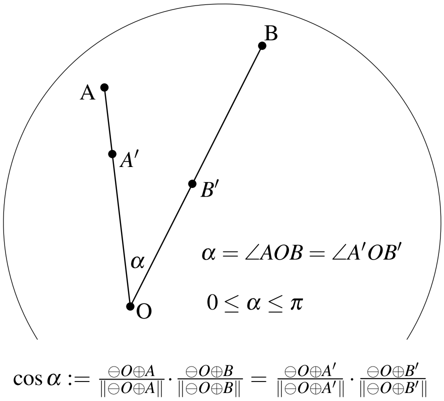

Fig. 2.5 Gyroangles share remarkable analogies with angles, allowing the use of the elementary trigonometric functions cos, sin, etc., in gyrotrigonometry as well. Let A' and B' be points different from O, lying arbitrarily on the gyrosegments OA and OB, respectively, that emanate from a common point O in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ as shown here for $n = 2$. The measure of the gyroangle α formed by the two gyrosegments OA and OB or, equivalently, formed by the two gyrosegments OA' and OB', is given by $\cos\alpha$, as shown here. In full analogy with angles, the measure of gyroangle α is independent of the choice of A' and B'

2.5 The Cartesian Model of Euclidean and Hyperbolic Geometry

The introduction of Cartesian coordinates $(x_1, x_2, \ldots, x_n)$, $x_1^2 + x_2^2 + \cdots + x_n^2 < \infty$, (1.14), p. 6, into the Euclidean n-space $\mathbb{R}^n$, along with the common vector addition in Cartesian coordinates, results in the Cartesian model of Euclidean geometry. The latter, in turn, enables Euclidean geometry to be studied analytically.

In full analogy, the introduction of Cartesian coordinates $(x_1, x_2, \ldots, x_n)$, $x_1^2 + x_2^2 + \cdots + x_n^2 < s^2$, (1.15), p. 6, into the s-ball $\mathbb{R}^n_s$ of the Euclidean n-space $\mathbb{R}^n$, along with the common Einstein addition in Cartesian coordinates, presented in Sect. 1.3, p. 6, results in the Cartesian model of hyperbolic geometry. The latter, in turn, enables hyperbolic geometry to be studied analytically. Indeed, Figs. 2.3 and 2.4 of Sect. 2.4 and Figs. 2.5 and 2.6 of Sect. 2.6 indicate the way we study analytic hyperbolic geometry, guided by analogies with analytic Euclidean geometry.

2.6 Gyroangles—the Hyperbolic Angles

The analogies between lines and gyrolines suggest corresponding analogies between angles and gyroangles. Let O, A and B be any three distinct points in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. The resulting gyrosegments OA and OB that emanate from the point O form a gyroangle $\alpha = \angle AOB$ with vertex O, as shown in Fig. 2.5

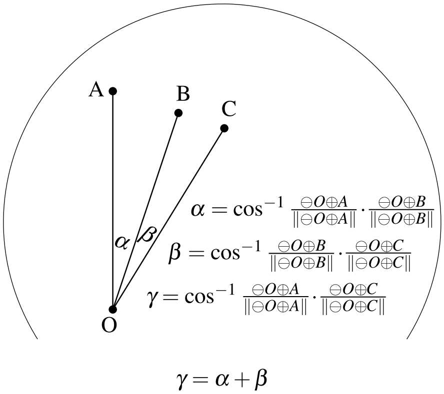

Fig. 2.6 Let A and C be two distinct points, let O be a point not on gyroline AC, and let B be a point between A and C in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. Furthermore, let $\alpha = \angle AOB$ and $\beta = \angle BOC$ be the two adjacent gyroangles that the three gyrosegments OA, OB and OC form, and let γ be their composite gyroangle, formed by gyrosegments OA and OC. Then, $\gamma = \alpha + \beta$, demonstrating that, like angles, gyroangles are additive. We call $(\ominus O \oplus A)/\|\ominus O \oplus A\|$ a unit gyrovector. When applied to an inner product of unit gyrovectors, the common cosine function of trigonometry becomes the gyrocosine function of gyrotrigonometry

for $n = 2$. Following the analogies between gyrolines and lines, the radian measure of gyroangle α is, suggestively, given by the equation

$$\cos\alpha = \frac{\ominus O \oplus A}{\|\ominus O \oplus A\|} \cdot \frac{\ominus O \oplus B}{\|\ominus O \oplus B\|}. \tag{2.23}$$

Here, $(\ominus O \oplus A)/\|\ominus O \oplus A\|$ and $(\ominus O \oplus B)/\|\ominus O \oplus B\|$ are unit *gyrovectors*, and cos is the common cosine function of trigonometry, which we apply to the inner product between unit gyrovectors rather than unit vectors. Accordingly, in the context of gyrovector spaces rather than vector spaces, we refer the function "cosine" of trigonometry to as the function "gyrocosine" of gyrotrigonometry. Similarly, all the other elementary trigonometric functions and their interrelationships survive unimpaired in their transition from the common trigonometry in Euclidean spaces $\mathbb{R}^n$ to a corresponding gyrotrigonometry in Einstein gyrovector space $\mathbb{R}_s^n$, as we will see in Chap. 6.

The center $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}_s^n$ of the ball $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is conformal (to Euclidean geometry) in the sense that the measure of any gyroangle with vertex $\mathbf{0}$ is equal to the measure of its Euclidean counterpart. Indeed, if $O = \mathbf{0}$ then (2.23) reduces to

$$\cos\alpha = \frac{A}{\|A\|} \cdot \frac{B}{\|B\|}, \tag{2.24}$$

which is indistinguishable from its Euclidean counterpart.

2.7 The Euclidean Group of Motions

The Euclidean group of motions of $\mathbb{R}^n$ consists of the commutative group of all translations of $\mathbb{R}^n$ and the group of all rotations of $\mathbb{R}^n$ about its origin.

For any $\mathbf{x} \in \mathbb{R}^n$, a translation of $\mathbb{R}^n$ by $\mathbf{x} \in \mathbb{R}^n$ is the map $L_\mathbf{x} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$L_\mathbf{x}\mathbf{v} = \mathbf{x} + \mathbf{v} \tag{2.25a}$$

for all $\mathbf{v} \in \mathbb{R}^n$.

A rotation R of $\mathbb{R}^n$ about its origin is an element of the group $SO(n)$ of all $n \times n$ orthogonal matrices with determinant 1. The rotation of $\mathbf{v} \in \mathbb{R}^n$ by $R \in SO(n)$ is given by $R\mathbf{v}$. The map $R \in SO(n)$ is a linear map of $\mathbb{R}^n$ that keeps the inner product invariant, that is,

$$\begin{aligned} R(\mathbf{u} + \mathbf{v}) &= R\mathbf{u} + R\mathbf{v}, \\ R\mathbf{u}\cdot R\mathbf{v} &= \mathbf{u}\cdot\mathbf{v} \end{aligned} \tag{2.25b}$$

for all $R \in \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

The *Euclidean group of motions* is the *semidirect product group*

$$\mathbb{R}^n \times SO(n) \tag{2.26}$$

of the Euclidean commutative group $\mathbb{R}^n = (\mathbb{R}^n, +)$ and the rotation group $SO(n)$. It is a group of pairs $(\mathbf{x}, R)$, $\mathbf{x} \in (\mathbb{R}^n, +)$, $R \in SO(n)$, acting on elements $\mathbf{v} \in \mathbb{R}^n$ according to the equation

$$(\mathbf{x}, R)\mathbf{v} = \mathbf{x} + R\mathbf{v}. \tag{2.27}$$

The group operation of the semidirect product group (2.26) is given by action composition. The latter, in turn, is determined by the following chain of equations, in which we employ the associative law:

$$\begin{aligned} (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)\mathbf{v} &= (\mathbf{x}_1, R_1)(\mathbf{x}_2 + R_2\mathbf{v}) \\ &= \mathbf{x}_1 + R_1(\mathbf{x}_2 + R_2\mathbf{v}) \\ &= \mathbf{x}_1 + (R_1\mathbf{x}_2 + R_1R_2\mathbf{v}) \\ &= (\mathbf{x}_1 + R_1\mathbf{x}_2) + R_1R_2\mathbf{v} \\ &= (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1R_2)\mathbf{v} \end{aligned} \tag{2.28}$$

for all $\mathbf{v} \in \mathbb{R}^n$.

Hence, by (2.28), the group operation of the semidirect product group (2.26) is given by the *semidirect product*

$$(\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2) = (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1R_2) \tag{2.29}$$

for any $(\mathbf{x}_1, R_1), (\mathbf{x}_2, R_2) \in \mathbb{R}^n \times SO(n)$.

Definition 2.6 (Covariance) An identity in $\mathbb{R}^n$ that remains invariant in form under the action of the Euclidean group of motions of $\mathbb{R}^n$ is said to be covariant.

We will see in Chap. 4 that Euclidean barycentric coordinate representations of points of $\mathbb{R}^n$ are covariant, by Theorem 4.3, p. 87.

2.8 The Hyperbolic Group of Motions

The hyperbolic group of motions of $\mathbb{R}^n_s$ consists of the gyrocommutative gyrogroup of all left gyrotranslations of $\mathbb{R}^n_s$ and the group of all rotations of $\mathbb{R}^n_s$ about its center.

For any $\mathbf{x} \in \mathbb{R}^n_s$, a left gyrotranslation of $\mathbb{R}^n_s$ by $\mathbf{x} \in \mathbb{R}^n_s$ is the map $L_{\mathbf{x}} : \mathbb{R}^n_s \to \mathbb{R}^n_s$ given by

$$L_{\mathbf{x}}\mathbf{v} = \mathbf{x} \oplus \mathbf{v} \tag{2.30a}$$

for all $\mathbf{v} \in \mathbb{R}^n_s$.

The group of all rotations of the ball $\mathbb{R}^n_s$ about its center is $SO(n)$. Following (2.25b), we have

$$\begin{aligned} R(\mathbf{u} \oplus \mathbf{v}) &= R\mathbf{u} \oplus R\mathbf{v}, \\ R\mathbf{u} \cdot R\mathbf{v} &= \mathbf{u} \cdot \mathbf{v} \end{aligned} \tag{2.30b}$$

for all $R \in SO(n)$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

The *hyperbolic group of motions* is the *gyrosemidirect product group*

$$\mathbb{R}^n_s \times SO(n) \tag{2.31}$$

of the Einsteinian gyrocommutative gyrogroup $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus)$ and the rotation group $SO(n)$. It is a group of pairs $(\mathbf{x}, R)$, $\mathbf{x} \in (\mathbb{R}^n_s, \oplus)$, $R \in SO(n)$, acting on elements $\mathbf{v} \in \mathbb{R}^n_s$ according to the equation

$$(\mathbf{x}, R)\mathbf{v} = \mathbf{x} \oplus R\mathbf{v}. \tag{2.32}$$

The group operation of the gyrosemidirect product group (2.31) is given by action composition. The latter, in turn, is determined by the following chain of equations, in which we employ the left gyroassociative law:

$$\begin{aligned} (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)\mathbf{v} &= (\mathbf{x}_1, R_1)(\mathbf{x}_2 \oplus R_2\mathbf{v}) \\ &= \mathbf{x}_1 \oplus R_1(\mathbf{x}_2 \oplus R_2\mathbf{v}) \\ &= \mathbf{x}_1 \oplus (R_1\mathbf{x}_2 \oplus R_1R_2\mathbf{v}) \\ &= (\mathbf{x}_1 \oplus R_1\mathbf{x}_2) \oplus \mathrm{gyr}[\mathbf{x}, R_1\mathbf{x}_2]R_1R_2\mathbf{v} \\ &= \big(\mathbf{x}_1 \oplus R_1\mathbf{x}_2, \mathrm{gyr}[\mathbf{x}, R_1\mathbf{x}_2]R_1R_2\big)\mathbf{v} \end{aligned} \tag{2.33}$$

for all $\mathbf{v} \in \mathbb{R}^n_s$.

Hence, by (2.33), the group operation of the gyrosemidirect product group (2.31) is given by the *gyrosemidirect product*

$$(\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2) = \big(\mathbf{x}_1 \oplus R_1\mathbf{x}_2, \operatorname{gyr}[\mathbf{x}, R_1\mathbf{x}_2]R_1 R_2\big) \tag{2.34}$$

for any $(\mathbf{x}_1, R_1), (\mathbf{x}_2, R_2) \in \mathbb{R}_s^n \times SO(n)$. Indeed, the gyrosemidirect product is a group operation, as demonstrated in Sect. 1.11, p. 23.

Definition 2.7 (Gyrocovariance) An identity in $\mathbb{R}_s^n$ that remains invariant in form under the action of the hyperbolic group of motions of $\mathbb{R}_s^n$ is said to be gyrocovariant.

We will see in Chap. 4 that hyperbolic barycentric (that is, gyrobarycentric) coordinate representations of points of $\mathbb{R}_s^n$ are gyrocovariant, by the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 4.6, p. 90.

2.9 Problems

Problem 2.1 Einstein Scalar Multiplication:
Show that $k\otimes\mathbf{v} := \mathbf{v}\oplus\cdots\oplus\mathbf{v}$ (k terms) is given by (2.1), p. 45.

Problem 2.2 Einstein Scalar Multiplication:
Prove the second equation in (2.2), p. 46.

Problem 2.3 The Einstein Half:
Prove the Einstein-half identities (2.3)–(2.4), p. 46.

Problem 2.4 Einstein Scalar Distributive Law:
Prove the scalar distributive law in (2.5), p. 46.

Problem 2.5 Einstein Scalar Associative Law:
Prove the scalar associative law in (2.5), p. 46.

Problem 2.6 Scaling Property:
Prove the scaling property (2.6), p. 46.

Problem 2.7 A Gyroautomorphism Property:
Prove the gyroautomorphism property (2.7), p. 46.

Problem 2.8 Inner Product Invariance Under Gyrations:
Prove the identities in (2.9), p. 47.

Problem 2.9 Rotations Respect Einstein Addition:
Show that the first identity in (2.30b), p. 56, follows from (2.25b), p. 55.

Sect. 4.2 of Chap. 4 that the relativistically invariant mass m_0 in (3.62) is what we need for the introduction of barycentric coordinates into hyperbolic geometry. The latter, in turn, is what we need for the determination of hyperbolic triangle centers.

3.9 Remarkable Analogies

In this section, we emphasize the analogies in Theorems 3.2, p. 65, and 3.3, p. 71, that the classical mass and center of momentum velocity of a particle system in (3.78a)–(3.78d) below share with their relativistic counterparts in (3.79a)–(3.79d) below.

Seeking a way to place the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of a particle system S under the umbrella of the Minkowskian four-vector formalism of special relativity, we have uncovered the novel, relativistically invariant, or rest, mass m_0 of a particle system, presented in (3.79d) below. Furthermore, following the discovery of m_0 in (3.62), we have uncovered remarkable analogies that Newtonian and Einsteinian mechanics share.

To see the analogies clearly, let us consider the following well known classical results, (3.78a)–(3.78d) below, which are involved in the determination of the Newtonian resultant mass m_0 and the classical center of momentum velocity of a Newtonian system of particles, and to which we will subsequently present our Einsteinian analogs that have been discovered in Theorem 3.2. Let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \dots, N), \qquad \mathbf{v}_k \in \mathbb{R}^n \tag{3.78a}$$

be an isolated Newtonian system of N noninteracting material particles the kth particle of which has mass m_k and Newtonian uniform velocity $\mathbf{v}_k$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \dots, N$. Furthermore, let m_0 be the resultant mass of S, considered as the mass of a virtual particle located at the center of momentum of S, and let $\mathbf{v}_0$ be the Newtonian velocity relative to $\Sigma_{\mathbf{0}}$ of the Newtonian center of momentum frame of S. Then we have the following well-known identities:

$$1 = \frac{1}{m_0}\sum_{k=1}^{N} m_k \tag{3.78b}$$

and

$$\begin{aligned} \mathbf{v}_0 &= \frac{1}{m_0}\sum_{k=1}^{N} m_k \mathbf{v}_k, \\ \mathbf{w} + \mathbf{v}_0 &= \frac{1}{m_0}\sum_{k=1}^{N} m_k (\mathbf{w} + \mathbf{v}_k), \end{aligned} \tag{3.78c}$$

where the binary operation $+$ is the common vector addition in $\mathbb{R}^n$, and where

$$m_0 = \sum_{k=1}^{N} m_k \tag{3.78d}$$

for $\mathbf{v}, \mathbf{w}_k \in \mathbb{R}^3$, $m_k > 0$, $k = 0, 1, \ldots, N$.

In full analogy with (3.78a), let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N), \qquad \mathbf{v}_k \in \mathbb{R}_c^n \tag{3.79a}$$

be an isolated Einsteinian system of N noninteracting material particles the kth particle of which has invariant mass m_k and Einsteinian uniform velocity $\mathbf{v}_k$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$. Furthermore, let m_0 be the resultant mass of S, considered as the mass of a virtual particle located at the center of mass of S (calculated in (3.29)), and let $\mathbf{v}_0$ be the Einsteinian velocity relative to $\Sigma_{\mathbf{0}}$ of the Einsteinian center of momentum of the Einsteinian system S. Then, as shown in Theorem 3.2, the relativistic analogs of the Newtonian expressions in (3.78b)–(3.78d) are, respectively, the following Einsteinian expressions in (3.79b)–(3.79d),

$$\begin{aligned} \gamma_{\mathbf{v}_0} &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}, \\ \gamma_{\mathbf{u}\oplus\mathbf{v}_0} &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{u}\oplus\mathbf{v}_k}, \end{aligned} \tag{3.79b}$$

and

$$\begin{aligned} \gamma_{\mathbf{v}_0} \mathbf{v}_0 &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k, \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_0} (\mathbf{w}\oplus\mathbf{v}_0) &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w}\oplus\mathbf{v}_k), \end{aligned} \tag{3.79c}$$

where the binary operation $\oplus$ is the Einstein velocity addition in $\mathbb{R}_c^n$, given by (1.2), p. 4, and where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \tag{3.79d}$$

for $\mathbf{w}, \mathbf{v}_k \in \mathbb{R}_c^3$, $m_k > 0$, $k = 0, 1, \ldots, N$. Here m_0 is the relativistic invariant mass of the Einsteinian system S, supposed concentrated at the relativistic center of mass of S, and $\mathbf{v}_0$ is the Einsteinian velocity relative to $\Sigma_{\mathbf{0}}$ of the Einsteinian center of momentum frame of the Einsteinian system S.

To conform with the Minkowskian four-vector formalism of special relativity, both m_0 and $\mathbf{v}_0$ are determined in Theorem 3.2 as the unique solution of the Minkowskian four-vector equation (3.19).

We finally wrote (3.62) as (3.65), i.e.,

$$m_0 = \sqrt{m_{newton}^2 + m_{dark}^2}, \tag{3.80}$$

viewing the relativistically invariant, or rest, mass m_0 of the system S as a Pythagorean composition of the Newtonian rest mass, m_{newton} and the dark mass, m_{dark} of S. The mass m_{dark} is *dark* in the sense that it is the mass of virtual matter that does not collide and does not emit radiation. Following observations in cosmology, one may postulate that our dark mass reveals its presence only gravitationally. We have shown qualitatively that (3.80) explains observations in both astrophysics and particle physics.

We should remark that the presence of our dark mass is predicted by theoretic special relativistic techniques. Hence, it need not account for the whole mass of dark matter observed by astrophysicists in the cosmos because there could be contributions from general relativistic considerations and, perhaps, other unknown sources.

3.10 Problems

Problem 3.1 Matrix Representation of the Lorentz Boost:
Show that the Lorentz boost $L(\mathbf{u})$, given vectorially by (3.5), p. 61, is a linear map that possesses the matrix representation (3.1), p. 60.

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