Preface

The seeds of this book were planted in 1988 [55], when the author discovered that the seemingly structureless Einstein addition of relativistically admissible velocities possesses rich nonassociative algebraic structures that became known as a *gyrocommutative gyrogroup* and a *gyrovector space*. Einstein gyrovector spaces turn out to form the algebraic setting for the Cartesian–Beltrami–Klein ball model of the hyperbolic geometry of János Bolyai and Nikolai Ivanovich Lobachevsky, just as vector spaces form the algebraic setting for the standard Cartesian model of Euclidean geometry.

This book presents the novel approach to triangle centers in hyperbolic geometry that Einstein's special theory of relativity offers. Writing the book became possible following the adaption of Cartesian coordinates and vector algebra for use in hyperbolic geometry in its forerunners, the author's four earlier books [58, 60, 63, 64]. This adaption enables in this book the Möbius barycentric coordinates in Euclidean geometry to be embedded into hyperbolic geometry. The resulting hyperbolic barycentric coordinates form a tool for determining various hyperbolic triangle centers and relations between them, just as Euclidean barycentric coordinates form a tool for determining various Euclidean triangle centers and relations between them.

Following Möbius idea, let $\mathbb{R}^3$ be the Euclidean 3-space whose points represent Newtonian velocities in classical mechanics. The barycentric coordinates of a point $P \in \mathbb{R}^3$ with respect to a triangle in $\mathbb{R}^3$ are viewed as masses suspended at the triangle vertices in such a way that P is their center of momentum. Clearly, the masses are determined by P up to a multiplicative common factor, so that barycentric coordinates of a point with respect to a triangle are homogeneous.

Incorporating Einstein's ideas, let $\mathbb{R}^3_c$ be the c-ball of $\mathbb{R}^3$, that is, the ball of $\mathbb{R}^3$ that contains all vectors of $\mathbb{R}^3$ with magnitude smaller than the vacuum speed of light, c. Viewing the points of the ball as Einsteinian velocities in relativistic mechanics, we obtain hyperbolic barycentric coordinates that are fully analogous to their Euclidean counterpart. Thus, the hyperbolic barycentric coordinates of a point $P \in \mathbb{R}^3_c$ with respect to a hyperbolic triangle in $\mathbb{R}^3_c$ are viewed as Einsteinian, relativistic masses suspended at the triangle vertices in such a way that P is their center of momentum. Clearly, the relativistic masses are determined by P up to a multiplicative common factor, so that hyperbolic barycentric coordinates of a point with respect to a hyperbolic triangle are homogeneous.

Seemingly "unfortunately", (i) the Einstein relativistic mass is velocity dependent so that, as a result, (ii) it seemingly does not sit under the umbrella of the Minkowskian four-vector formalism of Einstein's special theory of relativity. However, it is demonstrated in this book that (i) the velocity dependence of the Einstein relativistic mass is precisely what is needed to successfully determine various triangle centers in hyperbolic geometry and that, contrary to general belief, (ii) the Einstein relativistic mass meshes up extraordinarily well with the Minkowskian four-vector formalism of Einstein's special relativity.

Accordingly, this book takes an in-depth look at one of the aspects of gyrocommutative gyrogroups and gyrovector spaces where Einstein's special relativity and the hyperbolic geometry of Bolyai and Lobachevsky meet.

A most convincing way to describe the success of the author's adaption of Cartesian coordinates and vector algebra for use in hyperbolic geometry is presented by the renowned historian of relativity physics Scott Walter in his review of the author's 2001 book [58], which is the first forerunner of this book. Therefore, part of Scott Walter's review is quoted below.

> Over the years, there have been a handful of attempts to promote the non-Euclidean style for use in problem solving in relativity and electrodynamics, the failure of which to attract any substantial following, compounded by the absence of any positive results must give pause to anyone considering a similar undertaking. Until recently, no one was in a position to offer an improvement on the tools available since 1912. In his [2001] book, Ungar furnishes the crucial missing element from the panoply of the non-Euclidean style: an elegant nonassociative algebraic formalism that fully exploits the structure of Einstein's law of velocity composition. The formalism relies on what the author calls the "missing link" between Einstein's velocity addition formula and ordinary vector addition: Thomas precession …
>
> Scott Walter, 2002 [71]

Indeed, the special relativistic effect known as *Thomas precession* is mathematically abstracted into a version called *Thomas gyration*. The latter, in turn, justifies the prefix "gyro" that is extensively used in this book. Thus, as a matter of a few examples, Einstein's velocity addition is neither commutative nor associative, but it turns out to be both *gyrocommutative* and *gyroassociative*, thus giving rise to the algebraic structures known as gyrogroups and gyrovector spaces. Remarkably, the mere introduction of *gyrations* turns Euclidean geometry, the geometry of classical mechanics, into hyperbolic geometry, the geometry of relativistic mechanics. This remarkable result gives rise to the gyro-language of this book, in which one prefixes a gyro to a classical term to mean the analogous modern term.

As a mathematical prerequisite for a fruitful reading of this book, it is assumed familiarity with Euclidean geometry from the point of view of vectors and with basic elements of linear algebra and classical mechanics. In particular, there is no demand upon readers of this book as to a prior acquaintance with either special relativity, nonassociative algebra or hyperbolic geometry.

The immediate purpose of this book is to build a momentum toward the hunt of more hyperbolic triangle centers and the study of relationships between them. It is hoped that this book will be in the forefront of shaping and popularizing the future of the study under one umbrella of both (i) the special theory of relativity of Einstein and (ii) the hyperbolic geometry of Bolyai and Lobachevsky.

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