Chapter 2
MASH Digital Delta–Sigma Modulator with Multi-Moduli

Tao Xu and Marissa Condon

Abstract This chapter proposes a novel design methodology for a Multi-stAge noise SHaping (MASH) digital delta–sigma modulator (DDSM) which employs multi-moduli. The structure is termed the MM-MASH. The adequacy and benefit of using such a structure is demonstrated. In particular, the sequence length is lengthened if the quantizer modulus of the first-order error feedback modulator (EFM1) of each stage are co-prime numbers. An expression for the sequence length of an MM-MASH is derived.

Keywords Delta–sigma modulation · sequence length · multi-moduli · co-prime numbers

2.1 Introduction

The digital delta–sigma modulator (DDSM) sometimes acts as the controller of the multi-modulus frequency divider in the feedback loop of the Fractional-N Frequency Synthesizers [4]. Since the DDSM is a finite state machine, when the input is a DC rational constant, the output is always a repeating pattern (limit cycle) [2, 8]. The period of the cycle is termed the sequence length. For this type of input, the quantization noise is periodic. When a sequence length is short, the power is distributed among spurious spurs that appear in the DDSM output spectrum. Hence, there is a desire to break short sequences. Dithering [10, 11] is one of the most commonly employed methods to break the short sequence length. However, it requires extra hardware and inherently introduces additional inband noise. Recently,

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some design methodologies have been proposed to maximise the sequence length. Borkowski [2] summarises the maximum period obtained by setting the initial condition of the registers in an EFM. Hosseini [7] introduced a digital delta–sigma modulator structure termed the HK-MASH with a very long sequence. The period of the HK-MASH is proven by mathematical analysis [6, 7]. This paper proposes a novel architecture to further increase the modulator sequence length.

This chapter proposes that the modulus of each quantizer in a DDSM is set as a different value from each other. Note that each quantizer has only one modulus. Furthermore, all of the moduli are co-prime numbers. Co-prime numbers [3] are two real numbers whose greatest common divisor is 1. They do NOT have to be prime numbers. A novel design methodology based on this concept is proposed. It will be shown to increase the modulator period and consequently reduce the effect of quantization noise on the useful output frequency spectrum of the DDSM.

In Section 2.2, the architectures of the classic MASH DDSM and the HK-MASH are reviewed. In Section 2.3, a novel structure is proposed that results in the maximum sequence length. The expression for the sequence length is derived as well. The simulation results are shown in Section 2.5.

2.2 Previous MASH Architectures

The architecture of an lth order MASH digital delta–sigma modulator (DDSM) is illustrated in Fig. 2.1. It contains l first-order error-feedback modulators (EFM1). $x[n]$ and $y[n]$ are an n_0-bit input digital word and an m-bit output, respectively. The relationship between them is

$$mean(y) = \frac{X}{M} \tag{2.1}$$

where X is the decimal number corresponding to the digital sequence $x[n]$ [1], i.e., $x[n] = X \in \{1, 2, \ldots, M\}$, and M is the quantizer modulus which is set as 2^{n_0} in the conventional DDSM.

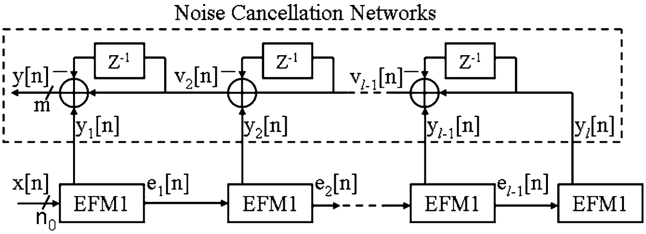

Fig. 2.1 MASH DDSM architecture

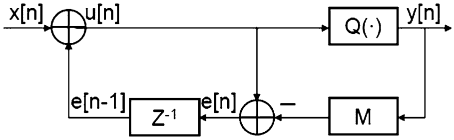

Fig. 2.2 EFM1: first-order error-feedback modulator

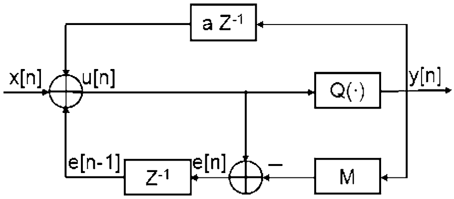

Fig. 2.3 The modified EFM1 used in HK-MASH

The model of the EFM1 is shown in Fig. 2.2. This is a core component in the make-up of the MASH digital delta–sigma modulator (DDSM). The rectangle Z^{-1} represents the register which stores the error $e[n]$ and delays it for one time sample. $Q(\cdot)$ is the quantization function:

$$y[n] = Q(u[n]) = \begin{cases} 1, & u[n] \geq M \\ 0, & u[n] < M \end{cases} \tag{2.2}$$

where

$$u[n] = x[n] + e[n-1]. \tag{2.3}$$

The maximum sequence lengths for this structure are 2^{n_0+1} or 2^{n_0+2} when the modulator order is below 5, which is found from simulations [2]. To achieve both of these sequence lengths, the first stage EFM1 must have an odd initial condition. This is implemented by setting the register.

The architecture of the modified EFM1 used in the HK-MASH is illustrated in Fig. 2.3. The only difference between it and the conventional EFM1 in Fig. 2.2 is the presence of the feedback block aZ^{-1}. a is a specifically-chosen small integer to make $(M - a)$ the maximum prime number below 2^{n_0} [7]. The sequence length of it is $(2^{n_0} - a)^l \approx (2^{n_0})^l$. This value will be compared with that of the proposed novel MM-MASH in Section 2.4.

2.3 The Adequacy and Effect of the Multi-Modulus MASH-DDSM

We assume that the quantizer modulus of the ith stage EFM1 in an lth order MASH-DDSM is represented by M_i, where $i \in \{1, 2, \ldots, l\}$. It shall first be shown that the MM-MASH is an accurate modulator. Then the effect of the multi-moduli on the sequence length shall be investigated mathematically.

2.3.1 The Suitability of the Multi-Modulus MASH-DDSM

In a fractional-N frequency synthesizer, the static frequency divider is controlled by the average value of the delta–sigma modulator output, $mean(y)$. The goal is to show that in an MM-MASH, $mean(y)$ is affected only by the quantizer modulus of the first stage EFM1, M_1, and is independent of the moduli in other stages. With this being true, having a multi-modulus architecture does not affect the accuracy of the digital delta–sigma modulator. Hence, it is a suitable digital delta–sigma modulator. Required to prove:

$$mean(y) = \frac{X}{M_1}. \tag{2.4}$$

Proof. As seen in Fig. 2.1, at the output of the last adder,

$$v_{l-1}[1] = y_{l-1}[1] + y_l[1] - y_l[0] \tag{2.5}$$

$$v_{l-1}[2] = y_{l-1}[2] + y_l[2] - y_l[1] \tag{2.6}$$

$$\vdots$$

$$v_{l-1}[N] = y_{l-1}[N] + y_l[N] - y_l[N-1] \tag{2.7}$$

where N is assumed as the sequence length of the MASH delta–sigma modulator. Adding all of the above equations yields:

$$\sum_{k=1}^{N} v_{l-1}[k] = \sum_{k=1}^{N} y_{l-1}[k] + \sum_{k=1}^{N} y_l[k] - \sum_{k=0}^{N-1} y_l[k] \tag{2.8}$$

where the period of y_l is assumed as N_l. As seen in Fig. 2.1, the output of the MASH modulator is obtained by simply summing and/or subtracting the output of each EFM1. Hence, the period of the MASH DDSM is the least common multiple of the sequence length of each stage. In other words, N is a multiple of N_i, where N_i is the period of the ith stage EFM1 and $i \in \{1, 2, \ldots, l\}$. It follows that

$$\sum_{k=1}^{N} y_l[k] = \sum_{k=0}^{N-1} y_l[k]. \tag{2.9}$$

Thus (2.8) becomes:

$$\sum_{k=1}^{N} v_{l-1}[k] = \sum_{k=1}^{N} y_{l-1}[k]. \tag{2.10}$$

Similarly, each of the other adders' output is obtained as

$$\sum_{k=1}^{N} v_{l-2}[k] = \sum_{k=1}^{N} y_{l-2}[k] \tag{2.11}$$

$$\vdots$$

$$\sum_{k=1}^{N} v_{2}[k] = \sum_{k=1}^{N} y_{2}[k] \tag{2.12}$$

$$\sum_{k=1}^{N} y[k] = \sum_{k=1}^{N} y_{1}[k]. \tag{2.13}$$

Each side of (2.13) may be expressed as

$$\sum_{k=1}^{N} y[k] = N \cdot mean(y) \tag{2.14}$$

$$\sum_{k=1}^{N} y_1[k] = K \sum_{k=1}^{N_1} y_1[k] \tag{2.15}$$

where N_1 is the sequence length of y_1, K is an integer and $N = K \cdot N_1$. Since y_1 is the output of a first-order delta–sigma modulator EFM1,

$$\sum_{k=1}^{N_1} y_1[k] = N_1 \cdot mean(y_1) = N_1 \cdot \frac{X}{M_1}. \tag{2.16}$$

On substitution of (2.16) into (2.15), the right-hand side of (2.13) becomes

$$\sum_{k=1}^{N} y_1[k] = K \cdot N_1 \cdot \frac{X}{M_1} = N \cdot \frac{X}{M_1}. \tag{2.17}$$

By substituting (2.14) and (2.17) into (2.13), the average value of the MASH DDSM output y is determined as

$$mean(y) = \frac{X}{M_1}. \tag{2.18}$$

■

2.3.2 The Effect of the Multi-Moduli on the Modulator Sequence Length

It is required to prove that the sequence length of the MASH modulator depends on the product of all the quantizer moduli. The expression for the lth order MASH DDSM sequence length is

$$N = \frac{M_1 \cdot M_2 \cdot \ldots \cdot M_l}{\lambda} \tag{2.19}$$

where λ is a parameter to make N the least common multiple of the sequence length of each stage N_i.

In addition, if the following two conditions, C1 and C2, are satisfied:

1. X and M_1 are co-prime numbers
2. $\{M_1, M_2, \ldots, M_l\}$ are co-prime numbers

then the sequence length of the MASH DDSM attains the maximum value:

$$N_{max} = M_1 \cdot M_2 \cdot \ldots \cdot M_l. \tag{2.20}$$

Proof: In the first-stage EFM1 shown in Fig. 2.2,

$$\begin{aligned} e_1[1] &= u[1] - y_1[1]M_1 \\ &= X + e_1[0] - y_1[1]M_1 \end{aligned} \tag{2.21}$$

$$e_1[2] = X + e_1[1] - y_1[2]M_1 \tag{2.22}$$

$$\vdots$$

$$e_1[N_1] = X + e_1[N_1 - 1] - y_1[N_1]M_1 \tag{2.23}$$

where $e_1[0]$ is the initial condition of the register. The sum of all of the above equations is

$$\sum_{k=1}^{N_1} e_1[k] = N_1 X + \sum_{k=0}^{N_1-1} e_1[k] - \sum_{k=1}^{N_1} y_1[k]M_1. \tag{2.24}$$

Since in the steady state, the first EFM1 is periodic with a period N_1 [5],

$$\sum_{k=1}^{N_1} e_1[k] = \sum_{k=0}^{N_1-1} e_1[k]. \tag{2.25}$$

Hence, (2.24) may be modified to

$$\sum_{k=1}^{N_1} y_1[k] = \frac{N_1 X}{M_1}. \tag{2.26}$$

In practice, the input DC X is set as $0 < X < M_1$. So in order to make the right side of (2.26) an integer, the minimum nonzero solution of N_1 has to be

$$N_1 = \frac{M_1}{\lambda_1} \tag{2.27}$$

where λ_1 is the greatest common divisor of M_1 and X. If M_1 and X are co-prime numbers, λ_1 equals to 1.

If the process of (2.21)–(2.26) is repeated with the second EFM1, the sum of its output which has a period N_2 is obtained as

$$\sum_{k=1}^{N_2} y_2[k] = \frac{\sum\limits_{k=1}^{N_2} e_1[k]}{M_2}. \tag{2.28}$$

If the relationship between the sequence lengths of the first and second stages is

$$N_2 = K_1 N_1 \tag{2.29}$$

(2.28) becomes

$$\sum_{k=1}^{N_2} y_2[k] = \frac{\sum\limits_{k=1}^{K_1 N_1} e_1[k]}{M_2}. \tag{2.30}$$

Since e_1 is periodic with the sequence length N_1 [7],

$$\sum_{k=1}^{N_2} y_2[k] = \frac{K_1 \sum\limits_{k=1}^{N_1} e_1[k]}{M_2} \tag{2.31}$$

where

$$\sum_{k=1}^{N_1} e_1[k] = N_1 \cdot mean(e_1). \tag{2.32}$$

Recalling (2.27),

$$\sum_{k=1}^{N_1} e_1[k] = \frac{M_1 \cdot mean(e_1)}{\lambda_1}. \tag{2.33}$$

On substitution of (2.33) into (2.31), the following expression is obtained

$$\sum_{k=1}^{N_2} y_2[k] = \frac{K_1 \cdot M_1 \cdot mean(e_1)}{\lambda_1 \cdot M_2} \tag{2.34}$$

Normally $mean(e_1)$ is a decimal fraction. However, if both sides of (2.33) are multiplied by λ_1, the result is

$$\lambda_1 \sum_{k=1}^{N_1} e_1[k] = M_1 \cdot mean(e_1). \tag{2.35}$$

Thus, $M_1 \cdot mean(e_1)$ is always an integer.

Then the minimum solution of K_1 so that the right-hand-side of (2.34) is an integer is obtained as

$$K_1 = \frac{\lambda_1 M_2}{\lambda_2} \tag{2.36}$$

where λ_2 is the greatest common divisor of $\lambda_1 M_2$ and $M_1 mean(e_1)$. Substituting (2.27) and (2.36) into (2.29), the sequence length of the second stage is

$$N_2 = \frac{M_1 M_2}{\lambda_2}. \tag{2.37}$$

If M_1 and M_2 are co-prime numbers, the greatest common divisor of $\lambda_1 M_2$ and $M_1 mean(e_1)$ is λ_1, i.e., $\lambda_2 = \lambda_1$. Hence,

$$N_2 = \frac{M_1 M_2}{\lambda_1}. \tag{2.38}$$

When X and M_1 are also co-prime numbers, λ_1 equals to 1. Thus the maximum sequence length for y_2 is obtained as:

$$N_{2\_max} = M_1 M_2. \tag{2.39}$$

Continuing in this manner, the sequence length of the ith effective stage EFM1 in an lth order MASH modulator is

$$N_i = \frac{M_1 M_2, \ldots, M_i}{\lambda_i} \tag{2.40}$$

where $i \in \{1, 2, 3, \ldots, l\}$ and λ_i is the maximum common divisor of $\lambda_{i-1} M_i$ and $M_1 M_2, \ldots, M_{i-1} mean(e_{i-1})$. Note that when $i = 1$, $mean(e_0) = X$ and $\lambda_0 = M_0 = 1$.

If $\{M_1, M_2, \ldots, M_i\}$ are co-prime numbers, M_i and $M_1 M_2, \ldots, M_{i-1}$ have to be co-prime numbers as well. Thus $\lambda_i = \lambda_{i-1}$. Since M_{i-1} and $M_1 M_2, \ldots, M_{i-2}$ are also co-prime numbers, $\lambda_{i-1} = \lambda_{i-2}$. Repeating this manner, it is follows that

$$\lambda_i = \lambda_{i-1} = \ldots = \lambda_1. \tag{2.41}$$

Then the sequence length of the ith EFM1 becomes

$$N_i = \frac{M_1 M_2, \ldots, M_i}{\lambda_1} \tag{2.42}$$

where λ_1 is the greatest common divisor of X and M_1. In practice, if the input X and M_1 are set as co-prime numbers, the maximum sequence length of the ith stage EFM1 is

$$N_{i\_max} = M_1 M_2, \ldots, M_i. \tag{2.43}$$

Since N is the least common multiple of $\{N_1, N_2, \ldots, N_l\}$, as is proven in Section 2.3.1, the sequence length of the MASH DDSM is obtained as

$$N = \frac{M_1 \cdot M_2 \cdot \ldots \cdot M_l}{\lambda} \tag{2.44}$$

where λ is the least common multiple of $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$.

When $\{M_1, M_2, \ldots, M_l\}$ are co-prime numbers, (2.41) is true. Then

$$N = \frac{M_1 \cdot M_2 \cdot \ldots \cdot M_l}{\lambda_1}. \tag{2.45}$$

In addition, if X and M_1 are co-prime numbers as well, λ_1 becomes 1. Thus the maximum sequence length is

$$N_{max} = M_1 \cdot M_2 \cdot \ldots \cdot M_l. \tag{2.46}$$

■

2.4 The Proposed Structure and Simulation Results

A novel structure for the MASH digital delta–sigma modulator (DDSM) employing multi-moduli is proposed in this section. It is termed the Multi-Modulus MASH – MM-MASH. As illustrated in Fig. 2.4, M_i represents the quantizer modulus in ith stage of MM-EFM1.

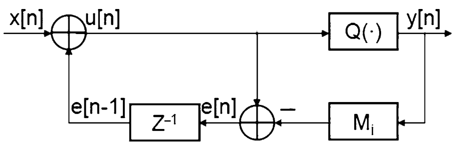

Fig. 2.4 MM-EFM1: The modified first-order error-feedback modulator used in MM-MASH

Table 2.1 Some sample moduli of the third order MM-MASH

Word length	M_1	M_2	M_3
5 bit	31	32	29
8 bit	251	256	255
9 bit	509	512	511
10 bit	1,021	1,024	1,023
11 bit	2,039	2,048	2,047

M_1 is set as a prime number below 2^{n_0}. This is to make X and M_1 always co-prime numbers and therefore satisfy the first condition, C1, stated in the previous section. This condition must be satisfied to maximise the sequence length of the MASH DDSM and to make the sequence length independent of the value of input. In order to maintain the modulator output accuracy, the value of the input DC X is adjusted to

$$X = M_1 \cdot mean(y) \tag{2.47}$$

where $mean(y)$ is the required output to control the static frequency divider in a fractional-N frequency synthesizer.

In an lth order MM-MASH, there are l co-prime numbers around 2^{n_0} that need to be found in order to satisfy the second condition, C2. The higher the modulator order, the greater difficulty in finding suitable values for these moduli. Fortunately, the most popular MASH DDSM in modern communication systems is the third order one [2]. Note that all of the quantizer moduli should be chosen no bigger than 2^{n_0} to avoid necessitating extra hardware. Some quantizer moduli chosen by the authors for a third order MM-MASH are given in Table 2.1.

2.5 The Simulation Results

All of the models of the EFM1 and MASH DDSM are built and simulated in **Simulink**. The simulation confirms that the average value of the output $mean(y)$ equals to $\frac{X}{M_1}$. The sequence length of the MASH DDSM is determined using the autocorrelation function [2]. Figure 2.5 shows that the sequence length of a third-order 5-bit MM-MASH is 28,768 and this equals $M_1 \cdot M_2 \cdot M_3$ as given in Table 2.1. The sequence length is only 64 from a third-order 5-bit conventional MASH. The sequence lengths of the HK-MASH and the MM-MASH are compared in Table 2.2. The MM-MASH achieves a longer sequence when the word length is 8, 9, 10 and 11, and hence is deemed superior.

The power spectral density [9] of the third-order 9-bit MM-MASH and a dithered [10] third-order 9-bit conventional MASH DDSM is compared in Fig. 2.6. It is evident from the figure that the MM-MASH is significantly more effective than the conventional MASH DDSM at the important useful lower frequencies.

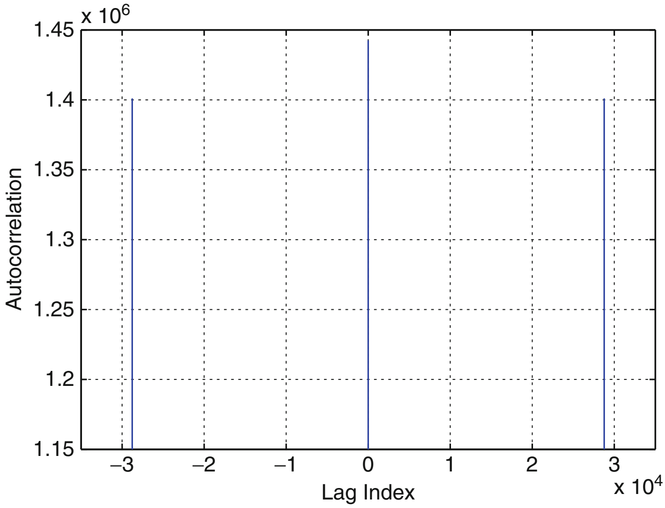

Fig. 2.5 The autocorrelation result for the third-order 5-bit MM-MASH

Table 2.2 A comparison of the sequence lengths for the HK-MASH and MM-MASH

Word length	HK-MASH	MM-MASH	Difference
8 bit	15.81×10^6	16.39×10^6	$+0.58 \times 10^6$
9 bit	131.87×10^6	133.17×10^6	$+1.3 \times 10^6$
10 bit	1.06×10^9	1.07×10^9	$+10 \times 10^6$
11 bit	8.48×10^9	8.55×10^9	$+70 \times 10^6$

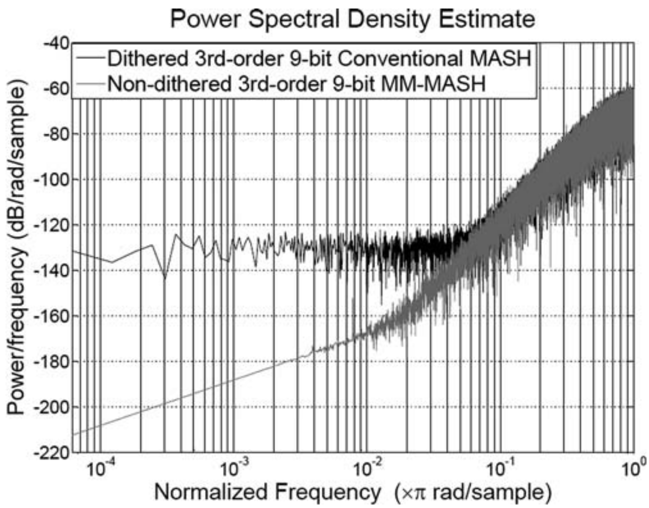

Fig. 2.6 The power spectral density of the dithered conventional MASH DDSM and non-dithered MM-MASH

2.6 Conclusions

A novel structure for the MASH digital delta–sigma modulator employing the multi-moduli (MM-MASH). This method employs different moduli in each stage of the EFM1. It is proven that the multi-modulus architecture is suitable because the output of the MASH modulator is only dependent on the quantizer modulus of the first stage EFM1 and independent of the others. The expressions for the sequence length of the EFM1 of each stage and for the complete MASH DDSM are derived. There are two conditions given that must be satisfied to yield the maximum modulator period. The outcome is a structure with an increased sequence length and hence an improved noise performance.

References

1. Borkowski, M.J., Kostamovaara, J.: Spurious tone free digital delta-sigma modulator design for dc inputs. In: Proceedings of the IEEE International Symposium on Circuits and Systems, pp. 5601–5604. Kobe, Japan (2005)
2. Borkowski, M.J., Riley, T.A.D., Häkkinen, J., Kostamovaara, J.: A practical Δ-Σ modulator design method based on periodical behavior analysis. IEEE Trans. Circ. Syst.-II: Express Briefs **52**(10), 626–630 (2005)
3. Crandall, R.E., Pomerance, C.: Prime Numbers: A Computational Perspective. Springer, Germany (2001)
4. Gu, R., Ramaswamy, S.: Fractional-*N* phase locked loop design and applications. In: Proceedings of the 7th International Conference on ASIC, pp. 327–332. Guilin, China (2007)
5. Hosseini, K., Kennedy, M.P.: Mathematical analysis of digital mash delta-sigma modulator for fractional-*N* frequency synthesizers. In: Proceedings of the IEEE Ph.D. Research in Microelectronics and Electronics, pp. 309–312. Otranto (Lecce), Italy (2006)
6. Hosseini, K., Kennedy, M.P.: Mathematical analysis of a prime modulus quantizer mash digital delta-sigma modulator. IEEE Trans. Circ. Syst.-II: Express Brief **54**(12), 1105–1109 (2007)
7. Hosseini, K., Kennedy, M.P.: Maximum sequence length mash digital delta-sigma modulators. IEEE Trans. Circ. Syst.-I: Regular Papers **54**(12), 2628–2638 (2007)
8. Kozak, M., Kale, İ.: Rigorous analysis of delta-sigma modulators for fractional-*n* pll frequency synthesis. IEEE Trans. Circ. Syst.-I: Regular Papers **51**(6), 1148–1162 (2004)
9. Oppenheim, A.V., Schafer, R.W.: Discrete-Time Signal Processing. Prentice-Hall, Upper Saddle River, NJ (1999)
10. Pamarti, S., Galton, I.: Lsb dithering in mash delta-sigma d/a converters. IEEE Trans. Circ. Syst.-I: Regular Papers **54**(4), 779–790 (2007)
11. Pamarti, S., Welz, J., Galton, I.: Statistics of the quantization noise in 1-bit dithered single-quantizer digital delta-sigma modulators. IEEE Trans. Circ. Syst.-I: Regular Papers **54**(3), 492–503 (2007)

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