Chapter 2
Principles of Modeling Processes in Moving Media

Abstract It is extremely difficult to model two-phase flows. A scientific model that should adequately reflect the process regularities is, to a considerable extent, "synthetic". It widely uses mathematical apparatus and knowledge from various areas of physics, hydraulics, mineralogy, etc. The scientific analysis is based on quantitative relations between various factors of the process. Mathematical models reflecting even idealized enough relations between the flow parameters allow the generation of principal similarity criteria based on their solution.

Keywords Nature · Model · Process parameters · Linearization of regularities · Simplification · Mathematical models · Model solution · Similarity criteria · Combinations of similarity criteria

2.1 Correlation Between a Full-Scale Process and Its Model

Critical regimes of two-phase flows are characterized by many peculiarities and interrelations. Most often it is impossible to take into account all of them thoroughly and accurately. Therefore, when performing a scientific analysis, the most general ones are singled out. As a result, a simplified idealized approximate enough model of the process is obtained.

Despite this, a serious scientific model is, to a large extent, synthetic. It widely uses mathematical tools and knowledge from various fields of physics, hydraulics, mineralogy, etc. Basing on these, a model is developed which should describe the process exactly enough and adequately reflect its regularities. When developing a mathematical model, we have to put aside a large number of process features. As a result, a model correlates with an object or a process as a caricature – with the reality. Anyway, any caricature should be recognizable, because it contains some features of a real object.

E. Barsky, *Critical Regimes of Two-Phase Flows with a Polydisperse Solid Phase*, Fluid Mechanics and Its Applications 93,
DOI 10.1007/978-90-481-8838-3_2,

Therefore, the results of mathematical investigation of a problem and constructive solutions based on these should not be considered as the only possible ones. Today, when physics of objects consisting of many elements is not sufficiently developed yet, different ways and results of solving the problem should not be ruled out, and it is rather difficult to choose the optimal variant of the solution. Therefore, the choice is made by comparing advantages and drawbacks of the obtained variants.

A scientific analysis is based on the establishment of quantitative relationships between different factors of a process. A strict analysis based on physical laws and mathematical analytical methods gives the most reliable results. Ideally, such an analysis does not need any experimental data. However, since cause-and-effect relations are complicated and diverse, such analysis can be realized very rarely, only in the simplest cases. Meanwhile, in the absence of strict theoretical solutions, practical engineering seeks additional opportunities.

Experimental data obtained on industrial equipment and laboratory experimental equipment are generalized. Based on these generalizations, empirical relations are derived, which are, as a rule, of a particular character. Their application beyond the range of parameters in which they were obtained leads to gross errors, and it is not always possible to extend the range of parameters in the experiments. For example, it is impossible to study a full-scale apparatus under laboratory conditions. Therefore, a transition from a laboratory model to a pilot plant is, most frequently, rather complicated and fraught with numerous mistakes and corrections.

Among the ways of simplification of regularities under study, the linearization of relationships between the phenomena under study and the results of the process is of special importance. The diversity and complexity of these relationships predetermine their nonlinear character, which makes difficult the analysis and mathematical description of the process. A transition to linear relationships significantly simplifies the analysis, but its accuracy is reduced. Therefore, it is important to allow the accuracy decrease only within the limits that do not distort the final result of the analysis to a considerable extent.

The diversity of relationships between the object properties and process parameters aggravated by insufficient understanding of physics of the processes leads to insuperable difficulties in finding quantitative regularities and to cumbersome and confusing calculations. This fact is usually passed over in silence. However, we should clearly point to a gap in the level of turbulent two-phase flow problems, their critical flows and potentialities of analytically derived equations. The arising difficulties call for simplifications both in the derivation of equations and in unambiguity conditions, which leads to accuracy loss. In such cases, numerical methods are frequently used.

Numerical methods of solving differential equations are associated with specific parameters of the process and limited by the accepted range of their variation. The obtained results are not of general character and can be used in a particular case.

Attempts to solve theoretical equations by numerical methods have been made in a sufficiently wide range of variables. Then empirical relationships have been matched to suit the obtained results. Because of a large number of factors, the realization of this method is highly labor-consuming. Besides, it does not guarantee the accuracy of the obtained results.

A transition to generalized variables composed of elementary factors of the process according to certain rules facilitates, to some extent, the overcoming of this situation. These new variables are dimensionless and have a certain physical meaning. They allow establishing connection between generalizing complexes that combine the process factors, and not between the process factors, which are numerous.

The correlations obtained in generalizing complexes possess the following features:

– They are more compact than relationships with dimensional factors.
– They admit analytical solutions in a more laconic form.
– They are useful for the formalization of experimental data.
– They allow calculations in any system of units because they are dimensionless.

It should be emphasized that such combination of factors is not formal. In real processes the influence of individual factors is revealed jointly, and not separately. Therefore, if these factors are combined into a complex, the latter reflects the overall status of the process.

The use of generalized variables involving several factors each, leads to a more general description of a process, since one value of a complex can be realized, strictly speaking, at an infinite number of combinations of numerical values of the factors involved. Hence, these complexes can describe not only single phenomena or processes, but also a group of similar phenomena and processes, for which these complexes have the same numerical value. This is the basis of the notion of physical similarity. Therefore, such complexes are called similarity criteria. They are used in laboratory simulation of complicated processes and apparatuses, in the processing of experimental data and in analytical solving of technological problems. Beyond any doubt, the introduction of similarity criteria is an important stage in the development of science.

It has turned out that cumbersome differential equations derived by analytical methods are valuable *per se*. When initial conditions and unambiguity conditions are used, it is practically impossible to obtain their exact solution. However, on the basis of these equations, similarity criteria can be correctly formulated without solving them. These criteria allow a competent setting of an experiment and processing of the obtained results.

2.2 Mathematical Models Construction

Principal analytical equations of moving flows were obtained proceeding from elementary simple models. Thus, principal equations of hydraulics are based on balance correlations without taking into account such important flow characteristics as the flow structure, turbulence with its developed spectrum of fluctuations or conditions on channel walls, but only the respective balances.

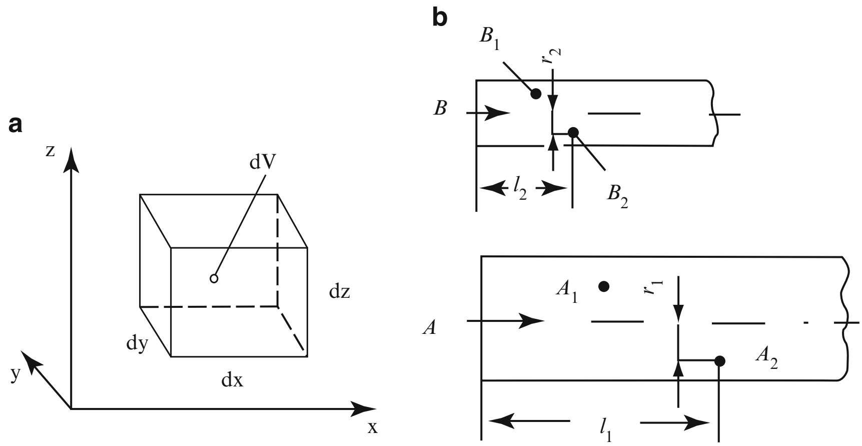

Fig. 2.1 On the modeling of processes in moving media: (**a**) motionless elementary volume; (**b**) on the notion of similarity in flows

However, even such an analytical model allows determining principal regularities and similarity criteria.

Let us examine this procedure, because later on, we will develop a model of critical regimes of two-phase flows using similar methods.

First consider the kinetic aspect of the problem. We place a motionless infinitesimal elementary cuboid with the edges d_x, d_y, d_z into a flow (Fig. 2.1a) and consider the flows through pairs of its parallel faces $dxdy$, $dxdz$ and $dydz$ in an infinitesimal time interval dt, as well as changes inside a certain elementary volume $dV = dxdydz.$

During the time interval dt, an elementary amount of moving medium equal to

$$dG_x^+ = \rho w_x dydzdt$$

enters a motionless contour along the x-axis through the face $dydz$ at the velocity w_x, and the amount

$$dG_x^- = \left[\rho w_x + \frac{\partial(\rho w_x)}{\partial x}\right] dydzdt$$

goes out through a parallel face. Here ρ is the density of the medium in some point of the contour; w_x is the velocity along the x-axis at some point of the contour. After the removal of brackets and cancellations, the difference between dG_x^+ and dG_x^- for the x-axis amounts to

$$-\frac{\partial(\rho w_x)}{\partial x} dxdydzdt = -\frac{\partial(\rho w_x)}{\partial x} dVdt.$$

Similar differences for y- and z-axes can be written as

$$-\frac{\partial(\rho w_y)}{\partial_y}dVdt \quad \text{and} \quad -\frac{\partial(\rho w_z)}{\partial_z}dVdt.$$

The amount of mass in the volume dV at the moment t equals ρdV. Since ρ can vary along the coordinates (x, y, z) and with time (t), the change in mass inside this volume by the moment of time $t + dt$ can be written using partial derivatives:

$$\frac{\partial(\rho dV)}{\partial t}dt = \frac{\partial \rho}{\partial t}dVdt.$$

Now we bring together the obtained elements of balance and, after reducing by dV and dt, we obtain

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho w_x)}{\partial x} + \frac{\partial(\rho w_y)}{\partial y} + \frac{\partial(\rho w_z)}{\partial z}.$$

We can rewrite it in a different form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho w_x)}{\partial x} + \frac{\partial(\rho w_y)}{\partial y} + \frac{\partial(\rho w_z)}{\partial z} = 0.$$

This is an equation of the flow continuity. Integration of this equation using specific unambiguity conditions leads to the mass conservation law in an integral form.

Now consider the dynamic aspect of the problem. A crucial problem in the analysis of the momentum or impulse transfer is to determine pressure (p) and velocity (w) in a certain point of the flow at an arbitrary moment of time t:

$$p = p(x; y; z; t); \quad w = w(x; y; z; t).$$

We consider an elementary cuboid again.

First we carry out the analysis as applied to one coordinate axis (x). The obtained results can be extended to other axes (y, z). We examine, one after another, forces of various nature.

1. Normal forces (pressure) acting along the x-axis on the left face $dydz$ are equal to $pdydz$. On the right, a force $\left(\rho + \frac{\partial \rho}{\partial x}dx\right)dydz$ acts on the parallel face. The difference between these normal forces is

$$pdydz - \left(p + \frac{\partial p}{\partial x}dx\right)dydz = -\frac{\partial p}{\partial x}dxdydz = -\frac{\partial p}{\partial x}dV.$$

2. Tangential forces (of internal friction, shear, viscosity – τ)

The force $\tau_1 dxdy$ acts along the lower face, and for the upper face we can write $\tau_2 dxdy$. We denote the velocity on the lower face by w_x; at the transition to the upper face along the z-axis, the velocity acquires a certain value $w_x + \frac{\partial w_x}{\partial z} dz$.

According to the Newton law, the relation between the tangential force and velocity gradient in a liquid or gas is linear, that is,

$$\tau = -\mu \frac{dw}{dz},$$

where μ is the dynamic viscosity coefficient of the medium. Taking this into account, we can write the following for the upper and lower faces:

$$\tau_1 dxdy = -\mu \frac{\partial w_x}{\partial z} dxdy;$$

$$\tau_2 dxdy = -\mu \frac{\partial}{\partial z}\left(w_x + \frac{\partial w_x}{\partial z} dz\right) dxdy.$$

After the removal of brackets and cancellations, the difference between these forces amounts to

$$(\tau_1 - \tau_2) dxdy = \mu \frac{\partial^2 w_x}{\partial z^2} dxdydz = \mu \frac{\partial^2 w_x}{\partial z^2} dV.$$

Similar tangential forces along the x-axis act on other pairs of faces, and by analogy with the above expressions, we can write

$$\mu \frac{\partial^2 w_x}{\partial_y^2} dV \text{ and } \mu \frac{\partial w_x^2}{\partial_x} dV.$$

Total balance of all tangential forces along the x-axis is expressed by the sum

$$\mu \left(\frac{\partial^2 w_x}{\partial x_2} + \frac{\partial^2 w_x}{\partial y^2} + \frac{\partial w_x}{\partial z^2} \right) dV = \mu \nabla^2 w_x dV$$

where $\nabla^2 -$ is the Laplace operator (Laplacian).

3. External mass forces

External mass forces also act on a mass of liquid or gas with the density ρ in the volume dV. We denote a projection of a resultant unit mass force along the x-axis by P_x. Then the total force acting on the volume under study is $P_x \rho dV$.

4. According to the Newton law, inertia force equal to a product of the volume mass and its acceleration along the x-axis acts on this volume:

$$\rho dV \frac{dw_x}{dt}.$$

Making a total balance of these four components, after reducing by dV with respect to the x-axis, we can write

$$-\frac{\partial p}{\partial x} + \mu \nabla^2 w_x + P_x \rho = \rho \frac{dw_x}{dt}.$$

Dividing each component by ρ and substituting $\frac{\mu}{\rho} = V$ (kinematic viscosity coefficient), we can finally write:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = P_x - \frac{\partial w_x}{d\tau} + v\nabla^2 w_x.$$

If we write, by analogy, similar expressions for y and z axes, the obtained system of equations is called a Navier–Stokes equation or the principal equation of hydrodynamics.

Note that when deriving equations in this chapter, we neglected the signs of differences in all intermediate relationships.

According to mathematical rules, it was assumed that all these increments are positive, and the sign should appear when we determine integration limits or unambiguity conditions (for example, boundary conditions) or specify conditions of the process (for example, gravity force is always directed opposite to an ascending flow).

It is accepted that the system of Navier–Stokes equations together with the continuity equation completely describe the motion of a moving medium. To obtain an unambiguous solution of the system even in the simplest case of a hollow pipe, it is necessary to specify initial velocity field values in space and time and to take into account the fact that the velocity should be zero on the channel walls and on the surface of all solids submerged into the flow. The solution of this system of equations was discussed in many papers.

As early as in 1920s, L. Keller and A. Friedman showed that in order to determine statistical moments of any order for hydrodynamic fields of single-phase turbulent flows, it is necessary to solve an infinite system of equations, that is this system is not closed. It is possible to solve this system using various assumptions idealizing the moving medium. Idealized flows of Newton, Euler, Couette, Poiseille, Haden, etc. are well known. These solutions are of a certain theoretical interest, but of no practical importance in the general case.

Thus, it is impossible to find an exact solution of the obtained system of equations. However, it contains information on the flow motion that can be obtained in the form of dimensionless complex parameters. It becomes possible to determine these parameters using the methods of similarity theory.

2.3 Similarity Criteria Determination

Groups of processes, phenomena or objects that can be mathematically described using similarity criteria are assumed to be similar. The notion of geometrical similarity is studied as early as in secondary school. Physical similarity has a different meaning. One can assume that two physical phenomena are similar, if the respective characteristics of this phenomenon at analogous points of geometrically similar systems differ only by a coefficient that is constant for all the points. Mathematical description of such systems is identical. If we deal with two geometrically similar facilities (Fig. 2.1b), the larger one is usually called an apparatus (A), whereas the smaller one – a model (B). Let moving media with different properties (density, viscosity, velocity, heat capacity, etc.) flow through these facilities.

We choose two pairs of similarly located points A_1 and A_2, B_1 and B_2 in these systems. Geometrical parameters of each pair are characterized by a certain relationship of the following type:

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = \cdots$$

The ratio

$$\frac{x_1}{x_2} = \frac{D_1}{D_2} = \cdots = m_1.$$

By definition of physical similarity, for any pair of analogous points A_1 and B_1, A_2 and B_2 and any other pairs, the following equalities should be satisfied:

$$\frac{\rho_1}{\rho_2} = m_\rho; \quad \frac{\mu_1}{\mu_2} = m_\mu; \quad \frac{w_1}{w_2} = m_w; \quad \frac{c_1}{c_2} = m_c, \text{ etc.}$$

In these expressions, m_i are similarity factors. Naturally, they are different for different parameters ($m_\infty \neq m_w \neq m_c \neq \cdots$), but have a constant value of the same parameter in any pair of analogously located points. Each similarity factor plays the role of a kind of scale of the respective physical magnitude.

The notion of similarity factors is the basis for deriving similarity criteria. By way of example, we derive one of such criteria from the flow continuity equation. For two geometrically similar flows, we can write

$$\frac{\partial \rho_1}{\partial t_1} = \frac{\partial(\rho_1 w_1)}{\partial x_1} + \cdots = 0,$$

$$\frac{\partial \rho_2}{\partial t_2} = \frac{\partial(\rho_2 w_2)}{\partial x_2} + \cdots = 0.$$

Other summands are omitted here, because they structurally coincide with the second summands. We apply scaling transformations to these two equations:

$$\rho_1 = m_\rho \rho_2; \quad t_1 = m_t t_2; \quad w_1 = m_w w_2; \quad x_1 = m_l x_2.$$

Geometrical similarity factor m_{i} is constant for all coordinates (x, y, z), as well as for linear dimensions l_1 and l_2.

Let us substitute the values of all process parameters of the apparatus into the first equation, expressing them through the corresponding similarity factors and process parameters of the model. Then, after the removal of constant multipliers from the derivatives, we obtain a new expression for the apparatus through the parameters of the model:

$$\frac{m_\rho}{m_t}\frac{\partial \rho_2}{\partial t_2} + \frac{m_\rho m_w}{m_l}\frac{\partial(\rho_2 w_2)}{\partial x_2} + \cdots = 0.$$

Compare the obtained expression with the above continuity equation for the model. Their adequacy is possible only if the complexes comprising similarity factors can be factorized and reduced, because the right-hand part is zero. It means that these complexes are equal:

$$\frac{m_\rho}{m_t} = \frac{m_\rho m_w}{m_l}.$$

Here we can reduce by the factor m_ρ and finally obtain:

$$\frac{1}{m_t} = \frac{m_w}{m_l}.$$

However, it is well known that

$$m_t = \frac{t_1}{t_2}; \quad m_w = \frac{w_1}{w_2}; \quad m_l = \frac{l_1}{l_2}.$$

Substitute these values into the final expression:

$$\frac{t_2}{t_1} = \frac{w_1}{w_2}\cdot\frac{l_2}{l_1}$$

and collect magnitudes with the same indices in different parts of the equality. We obtain:

$$\frac{w_1 t_1}{l_1} = \frac{w_2 t_2}{l_2} = \frac{wt}{l} = idem.$$

It means that the obtained result is valid for all similar flows. This complex is dimensionless. The coincidence of the numerical value of a dimensionless complex or group of complexes is a necessary and sufficient similarity condition for several systems, objects, flows, processes, etc. The obtained complex is called a homochronism criterion and denoted by

$$Ho = \frac{wt}{l}.$$

Most often, physical criteria are called after great scientists and denoted by the first two letters of their names.

The physical meaning of the homochronism criterion is clear from the prerequisites of the analysis – two opposite effects are compared, namely, forced medium transfer and accumulation.

Growing Ho value shows that the influence of the factor in the numerator grows, that is mass transfer grows. With decreasing Ho, the role of the factor in the denominator grows, that is mass accumulation is predominant.

In case of stationary processes, where mass accumulation in the working volume does not take place, Ho criterion degenerates, and its numerical values tend to infinity.

Note another important aspect. In many cases, total geometrical similarity is not necessary, an approximate one being sufficient. Geometrical dimensions can affect the course of a process in different ways. For example, it is clear that the shelf thickness in a cascade classifier does not affect the character of the separation process. Therefore, it does not require the fulfillment of geometrical similarity of the type

$$\frac{\delta_1}{l_1} = \frac{\delta_2}{l_2}.$$

A less obvious example concerns the velocity profile of a flow entering the classifier. Its formation is completed at the channel inlet, and then the value of the velocities does not considerably change. Beyond the channel inlet, the influence of the longitudinal coordinate degenerates, and in this sense, geometrical similarity becomes non-obligatory.

Using the same pattern of reasoning, we make an attempt to derive a similarity criterion from the system of Navier–Stokes differential equations. For the process under study, it is most convenient to analyze similar flows with respect to the z-axis. We can write the following for an apparatus and a model at their analogous points:

$$\frac{1}{\rho_1}\frac{\partial p_1}{\partial z_1} = P_1 - \left(w_1\frac{\partial w_1}{\partial z_1} + \cdots\right) + v_1\left(\frac{\partial^2 w_1}{\partial z_1^2} + \cdots\right),$$

$$\frac{1}{\rho_2}\frac{\partial p_2}{\partial z_2} = P_2 - \left(w_2\frac{\partial w_2}{\partial z_2} + \cdots\right) + v_2\left(\frac{\partial^2 w_2}{\partial z_2^2} + \cdots\right).$$

We express all apparatus characteristics through similarity factors and model characteristics:

$$\rho_1 = m_\rho \rho_2; \quad p_1 = m_\rho p_2; \quad P_1 = m_m P_2; \quad z_1 = m_l z_2; \quad w_1 = m_w w_2; \quad v_1 = m_v v_2.$$

Note that m_m is the same for all mass forces. Now we substitute all these ratios into the first equation:

$$\frac{m_p}{m_\rho m_l}\frac{1}{\rho_2}\frac{\partial \rho_2}{\partial z_2} = m_m P_2 - \frac{m_w^2}{m_l}\left(w_2\frac{\partial w_2}{\partial z_2} + \cdots\right) + \frac{m_v m_w}{m_l^2} v_2\left(\frac{\partial w_2}{\partial z_2^2} + \cdots\right).$$

According to the meaning of this mathematical operation, the second derivative is a quotient of the value to be differentiated, divided twice by the argument. Therefore m_l in the last expression is squared. Hence, following the same reasoning as in the previous case,

$$\frac{m_p}{m_\rho m_l} = m_m = \frac{m_w^2}{m_l} = \frac{m_v m_w}{m_l^2}.$$

The physical meaning of these parts of the equality is as follows: $\frac{m_p}{m_\rho m_l}$ – pressure forces; m_m – mass forces; $\frac{m_w^2}{m_l}$ – inertia forces; $\frac{m_v m_w}{m_l}$ – viscosity forces. Usually these parts of the equality are examined pairwise.

1. Comparison of pressure and inertia forces:

$$\frac{m_p}{m_\rho m_l} = \frac{m_w^2}{m_l}.$$

After reducing by m_l and substituting multipliers

$$m_\rho = \frac{\rho_1}{\rho_2}, \quad m_p = \frac{p_1}{p_2}, \quad m_w = \frac{w_1}{w_2},$$

we obtain

$$\frac{p_1 \rho_2}{p_2 \rho_1} = \frac{w_1^2}{w_2^2},$$

and hence

$$\frac{p_1}{\rho_1 w_1^2} = \frac{p_2}{\rho_2 w_2^2} = \frac{p}{\rho w^2} = Eu = idem.$$

Here Eu is Euler's criterion. It is used to determine the relationship of inertia and pressure forces. In practical problems, most often the pressure drop in a certain interval is of interest, and not the absolute value of pressure. Therefore, a somewhat different expression is used:

$$Eu = \frac{\Delta p}{\rho w^2}.$$

2. Comparison of mass and inertia forces

   In this case,

$$m_m = \frac{m_w^2}{m_l}.$$

Along the vertical z-axis, a value numerically equal to acceleration g corresponds to a unit mass gravity force. Taking this into account, substitution of $m_{w_1} m_l$ and $m_m = \frac{g_1}{g_2}$ values leads to

$$\frac{g_1}{g_2} = \frac{w_1^2 l_2}{w_2^2 l_1}.$$

Hence

$$\frac{g_1 l_1}{w_1^2} = \frac{g_2 l_2}{w_2^2} = \frac{gl}{w^2} = Fr = idem$$

where Fr is Froude's criterion, which is a measure of the ratio of mass forces and inertia forces. The comparison of these forces predetermines the character of critical flow processes under study, and therefore, this criterion plays a crucial role, which is confirmed experimentally.

3. Comparison of inertia and viscous forces

$$\frac{m_w^2}{m_l} = \frac{m_v m_w}{m_l^2}.$$

After reducing by m_w/m_l and substituting all similarity factors, we can obtain

$$\frac{w_1}{w_2} = \frac{v_1 l_2}{v_2 l_1}.$$

Hence

$$\frac{w_1 l_1}{v_1} = \frac{w_2 l_2}{v_1} = \frac{wl}{v} = Re = idem.$$

Thus, we obtain the Reynolds criterion, which is widely used for describing liquid and gas flows, as well as particles displacements with respect to moving media.

Usually the linear dimension in the Reynolds criterion is either channel diameter or particle diameter:

$$Re = \frac{wd}{\nu}.$$

4. To derive other similarity complexes, these three principal ones can be used, for example, a relationship between the gravity force and viscosity force. In this case, we can write $Fr \cdot Re$. Other combinations of similarity criteria having a clear physical meaning can be also obtained.

Usually it is impossible to establish quantitative relationships between similarity criteria purely theoretically. In each specific case they are established by means of a specially set experiment. These relationships are called criterial equations of $Eu = f(Re)$ type. Such relationships are valid only within experimentally checked ranges of similarity criteria variation.

Similarity criteria should not be considered as parameters determining the ratio between respective forces, because the value of this ratio is different at different points of the flow. They should be considered as a measure characterizing correctly the relationship between respective forces. For example, the higher the Reynolds number, the larger are inertia forces with respect to friction forces in a specific flow.

http://www.springer.com/978-90-481-8837-6

Critical Regimes of Two-Phase Flows with a Polydisperse
Solid Phase
Barsky, E.
2010, XVI, 348 p., Hardcover
ISBN: 978-90-481-8837-6