II

Basic Function Spaces and Related Inequalities

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VERGILIUS, Bucolica IV, v.60

Introduction

In this chapter we shall introduce some function spaces and enucleate certain properties of fundamental importance for further developments. Particular emphasis will be given to what are called *homogeneous Sobolev spaces*, which will play a fundamental role in the study of flow in exterior domains. We shall not attempt, however, to give an exhaustive treatment of the subject, since this is beyond the scope of the book. Therefore, the reader who wants more details is referred to the specialized literature quoted throughout. As a rule, we give proofs where they are elementary or relevant to the development of the subject, or also when the result is new or does not seem to be widely known.

II.1 Preliminaries

In this section we collect a number of preparatory results. After introducing some basic notation, we shall recall the relevant properties of Banach spaces and of certain classical spaces of smooth functions as well. We shall finally define and analyze the properties of special subsets of the Euclidean space.

G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, Springer Monographs in Mathematics,
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II.1.1 Basic Notation[1]

The symbols $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of all non-negative and of all positive natural numbers, respectively.

For X a set, we denote by X^m, $m \in \mathbb{N}_+$, the Cartesian product of m copies of X. Thus, denoting by $\mathbb{R}$ the real line, $\mathbb{R}^n$ is the n-dimensional Euclidean space. Points in $\mathbb{R}^n$ will be denoted by $x = (x_1, \ldots, x_n) \equiv (x_i)$ and corresponding vectors by $\boldsymbol{u} = (u_1, \ldots, u_n) \equiv (u_i)$. Sometimes, the ith component u_i of the vector $\boldsymbol{u}$ will be denoted by $(\boldsymbol{u})_i$. More generally, for $\boldsymbol{T}$ a tensor of order $m \geq 2$, its generic component $T_{ij\ldots kl}$ will be also denoted by $(\boldsymbol{T})_{ij\ldots kl}$. The components of the *identity tensor* $\boldsymbol{I}$, are denoted by δ_{ij} (*Kronecker delta*).

The *distance between two points* x and y of $\mathbb{R}^n$ is indicated by $|x - y|$, and we have

$$|x - y| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}.$$

More generally, the *distance between two subsets* A *and* B of $\mathbb{R}^n$ is indicated by $\operatorname{dist}(A, B)$, where

$$\operatorname{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|.$$

The *modulus of a vector* $\boldsymbol{u}$ is indicated by $|\boldsymbol{u}|$ (or by u) and it is

$$|\boldsymbol{u}| = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}.$$

Given two vectors $\boldsymbol{u}$, $\boldsymbol{v}$, the second-order tensor having components $u_i v_j$ (*dyadic product of* $\boldsymbol{u}, \boldsymbol{v}$) will be denoted by $\boldsymbol{u} \otimes \boldsymbol{v}$.

The *canonical basis* in $\mathbb{R}^n$ is indicated by

$$\{\boldsymbol{e}_i\} \equiv \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$$

with

$$\boldsymbol{e}_1 = (1, 0, \ldots, 0), \quad \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \boldsymbol{e}_n = (0, \ldots, 0, 1).$$

We also set

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$$

$$\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}.$$

[1] For other notation, we refer the reader to footnotes 8, 9, and 10 of Section I.1

For $r > 0$ and $x \in \mathbb{R}^n$ we denote by $B_r(x)$ the (n-dimensional) *open ball of radius r centered at* x, i.e.,

$$B_r(x) = \{y \in \mathbb{R}^n : \ |x - y| < r\}.$$

For $r = 1$, we shall put

$$B_1(x) \equiv B(x),$$

and for $x = 0$,

$$B_r(0) \equiv B_r.$$

Unless the contrary is explicitly stated, the Greek letter Ω shall always mean a *domain*, i.e., an open connected set of $\mathbb{R}^n$.

Let $\mathcal{A}$ be an arbitrary set of $\mathbb{R}^n$. We denote by $\overline{\mathcal{A}}$ its *closure*, by $\mathcal{A}^c = \mathbb{R}^n - \mathcal{A}$ its *complementary set* (in $\mathbb{R}^n$), by $\overset{\circ}{\mathcal{A}}$ its *interior*, and by $\partial\mathcal{A}$ its *boundary.* For $n \geq 2$, the boundary of the n-dimensional unit ball centered at the origin (i.e., the n-dimensional unit sphere) is denoted by S^{n-1}:

$$S^{n-1} = \partial B_1.$$

Moreover, $\delta(\mathcal{A})$ is the *diameter of* $\mathcal{A}$, that is,

$$\delta(\mathcal{A}) = \sup_{x,y \in \mathcal{A}} |x - y|.$$

If $\Omega^c \subset B_\rho$ for some $\rho \in (0, \infty)$ and with the origin of coordinates in Ω^c, we set

$$\Omega_r = \Omega \cap B_r, \quad r > \rho,$$

$$\Omega^r = \Omega - \overline{\Omega}_r, \quad r > \rho,$$

$$\Omega_{r,R} = \Omega_R - \overline{\Omega}_r, \quad \rho < r < R.$$

If $\mathcal{A}$ is Lebesgue measurable and μ_L *is the (Lebesgue) measure in* $\mathbb{R}^n$, we put

$$|\mathcal{A}| = \mu_L(\mathcal{A}).$$

The measure of the n-dimensional unit ball is denoted by ω_n; therefore,

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$

where Γ is the Euler gamma function

By c, c_i, C, C_i, $i = 1, 2, \ldots$, we denote generic positive constants, whose possible dependence on parameters $\xi_1, \ldots, \xi_m$ will be specified whenever it is needed. In such a case, we write $c = c(\xi_1, \ldots, \xi_m)$, $C = C(\xi_1, \ldots, \xi_m)$, or, especially in formulas, $c_{\xi_1,\dots,\xi_m}$, $C_{\xi_1,\dots,\xi_m}$, *etc.* Sometimes, we shall use the symbol c to denote a positive constant whose numerical value or dependence on parameters is not essential to our aims. In such a case, c may have several different values in a single computation. For example, we may have, in the same line, $2c \le c$.

For a real function u in Ω, we denote by supp (u) the *support of* u, that is,

$$\operatorname{supp}(u) = \overline{\{x \in \Omega : \quad u(x) \neq 0\}}.$$

For a real smooth function u in Ω we set

$$D_j u = \frac{\partial u}{\partial x_j}, \qquad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j};$$

likewise,

$$\nabla u = (D_1 u, \dots, D_n u)$$

denotes the *gradient of* u,

$$D^2 u = \{D_{ij} u\}$$

is the *matrix of the second derivatives.* Occasionally, the gradient of u will be indicated by $D^1 u$ or, more simply, by $D\,u$. We also set[2]

$$\Delta u = D_{ii} u$$

is the *Laplacean of* u.

For a vector function $\boldsymbol{u} = (u_1, \dots, u_n)$, the *divergence of* $\boldsymbol{u}$, $\nabla \cdot \boldsymbol{u}$, is defined by

$$\nabla \cdot \boldsymbol{u} = D_i u_i\,,$$

and, if $n = 3$,

$$\nabla \times \boldsymbol{u} = (D_2 u_3 - D_3 u_2, D_3 u_1 - D_1 u_3, D_1 u_2 - D_2 u_1)$$

denotes the *curl of* $\boldsymbol{u}$. Similarly, if $n = 2$, $\nabla \times \boldsymbol{u}$ has only one component, orthogonal to $\boldsymbol{u}$, given by $(D_1 u_2 - D_2 u_1)$.

If α is an n-tuple of non-negative integers α_i, we set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

[2] According to Einstein's summation convention, unless otherwise explicitly stated, pairs of identical indices imply summation from 1 to n.

The n-tuple α is called a *multi-index*.

If $\mathcal{D}$ is a domain with $|\mathcal{D}| < \infty$, and $\boldsymbol{u} : \mathcal{D} \to \mathbb{R}^n$, $n \geq 1$, we denote by $\overline{\boldsymbol{u}}_{\mathcal{D}}$ the *mean value of the function* $\boldsymbol{u}$ over the domain $\mathcal{D}$, namely,

$$\overline{\boldsymbol{u}}_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \boldsymbol{u} \,,$$

whenever the integral is meaningful.

We shall also use the following standard notation, for functions f and g defined in a neighborhood of infinity:

$$f(x) = O(g(x)) \;\text{ means } |f(x)| \leq M_1 |g(x)| \;\text{ for all } |x| \geq M_2,$$

$$f(x) = o(g(x)) \;\text{ means } \lim_{|x| \to \infty} |f(x)|/|g(x)| = 0$$

where M_1, M_2 denote positive constants.

Finally, the symbols $\square$ and $\blacksquare$ will indicate the end of a proof and of a remark, respectively.

II.1.2 Banach Spaces and their Relevant Properties

For the reader's convenience, in this subsection we shall collect all relevant properties of Banach spaces that will be frequently use throughout this book.

Let X be a *vector* (or *linear*) *space* on the field of real numbers, with corresponding operations of sum of two elements, $x + y$, and multiplication of an element x by a real number α, $\alpha\, x$. Then, X is a *normed space* if there exists a map, called *norm*,

$$\| \cdot \|_X : x \in X \to \|x\|_X \in \mathbb{R}$$

satisfying the following conditions, for all $\alpha \in \mathbb{R}$ and all $x, y \in X$:

(1) $\|x\|_X \geq 0$, and $\|x\|_X = 0$ implies $x = 0$;
(2) $\|\alpha x\|_X = |\alpha| \, \|x\|_X$;
(3) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

In what follows, X denotes a normed space.

Two norms $\|\cdot\|_X$ and $\|\cdot\|_X^*$ on X are *equivalent* if $c_1\|\cdot\|_X \leq \|\cdot\|_X^* \leq c_2\|\cdot\|_X$, for some constants $c_1 \leq c_2$.

A sequence $\{x_k\}$ in X is *convergent* to $x \in X$ if

$$\lim_{k \to \infty} \|x_k - x\|_X = 0 \,, \tag{II.1.1}$$

or, in equivalent notation, $x_k \to x$.

A subset S of X is a *subspace* if $\alpha\, x + \beta\, y$ is in S, for all $x, y \in S$ and all $\alpha, \beta \in \mathbb{R}$.

A subset B of X is *bounded* if there exists a number $M > 0$ such that $\sup_{x\in B}\|x\|_X \leq M$.

A subset C of X is *closed* if for every sequence $\{x_k\} \subset C$ such that $x_k \to x$ for some $x \in X$, implies $x \in C$.

The *closure* of a subset S of X consists of those points of $x \in X$ such that $x_k \to x$ for some $\{x_k\} \subset S$.

A subset K of X is *compact* if from every sequence $\{x_k\} \subset K$ we can find a subsequence $\{x_{k'}\}$ and a point $x \in K$ such that $x_{k'} \to x$.

A subset of X is *precompact* if its closure is compact.

A subset S of X is *dense* in X if for any $x \in X$ there is a sequence $\{x_k\} \subset S$ such that $x_k \to x$.

A subset of X is *separable* if it contains a countable dense set. We have the following result (see, e.g. Smirnov 1964, Theorem in §94).

Theorem II.1.1 *Let X be a separable normed space. Then every subset of X is separable.*

A space X is (continuously) *embedded* in a space Y if X is a linear subspace of Y and the identity map $i : X \to Y$ maps bounded sets into bounded sets, that is, $\|x\|_Y \leq c\,\|x\|_X$, for some constant c and all $x \in X$. In this case, we shall write

$$X \hookrightarrow Y\,.$$

X is *compactly embedded in* Y if $X \hookrightarrow Y$ and, in addition, i maps bounded sets of X into precompact sets of Y. In such a case we write

$$X \hookrightarrow\hookrightarrow Y.$$

Two linear subspaces X_1, Y_1 of normed spaces X and Y, respectively, are *isomorphic* [respectively, *homeomorphic*] if there is a map L from X_1 onto Y_1, called *isomorphism* [respectively, *homeomorphism*], such that (i) L is linear; (ii) L is a bijection, and, moreover, (iii) $\|L(x)\|_X = \|x\|_Y$ [respectively, $c_1\|x\|_X \leq \|L(x)\|_Y \leq c_2\|x\|_X$, for some $c_1 \leq c_2$], for all $x \in X_1$, where $\|\cdot\|_X$, and $\|\cdot\|_Y$ denote the norms in X and Y.

A sequence $\{x_k\} \subset X$ is called *Cauchy* if

$$\text{given } \varepsilon > 0 \text{ there is } \overline{n} = \overline{n}(\varepsilon) \in \mathbb{N}\text{: } \|x_k - x_{k'}\|_X < \varepsilon \text{ for all } k, k' \geq \overline{n}\,.$$

If every Cauchy sequence in X is convergent to an element of X, then X is called *complete*.

A *Banach space* is a normed space where every Cauchy sequence is there convergent or, equivalently, a Banach space is a *complete normed space*.

If X is not complete, namely, there is at least one Cauchy sequence in X that is not convergent to an element of X, we can nevertheless find a uniquely determined[3] Banach space $\widehat{X}$, with the property that X is isomorphic to a

[3] Up to an isomorphism.

dense subset of $\widehat{X}$. The space $\widehat{X}$ is called (Cantor) *completion* of X, and its elements are classes of equivalence of Cauchy sequences, where two such sequences, $\{x_k\}$, $\{x'_m\}$, are called equivalent if $\lim\limits_{l\to\infty} \|x_l - x'_l\|_X = 0$; see, e.g., Smirnov (1964, §85).

Suppose, now, that on the vector space X we can introduce a real-valued function $(\cdot,\cdot)_X$ defined in $X \times X$, satisfying the following properties for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{R}$

(i) $(x, y)_X = (y, x)_X$,
(ii) $(\alpha\, x + \beta\, y, z)_X = \alpha\,(x, z)_X + \beta\,(y, z)_X$,
(iii) $(x, x)_X \geq 0$, and $(x, x)_X = 0$ implies $x = 0$.

Then X becomes a normed space with norm

$$\|x\|_X \equiv \sqrt{(x, x)_X}\,. \tag{II.1.2}$$

The bilinear form $(\cdot,\cdot)_X$ is called *scalar product*, and if X, endowed with the norm (II.1.2), is complete, then X is called *Hilbert space.*

A countable set $\mathfrak{B} \equiv \{x_k\}$ in a Hilbert space X is called a *basis* if (i) $(x_j, x_k) = \delta_{jk}$, for all $x_j, x_k \in \mathfrak{B}$, and $\lim_{N\to\infty} \| \sum_{k=1}^{N} (x, x_k) x_k - x\|_X = 0$, for all $x \in X$.

A linear map $\ell : X \to \mathbb{R}$ on a normed space X, such that

$$s_\ell \equiv \sup_{x\in X;\|x\|_X=1} |\ell(x)| < \infty \tag{II.1.3}$$

is called *bounded linear functional* or, in short, *linear functional* on X. The set, X', of all linear functionals in X can be naturally provided with the structure of vector space, by defining the sum of two functionals ℓ_1 and ℓ_2 as that $\ell \in X'$ such that $\ell(x) = \ell_1(x) + \ell_2(x)$ for all $x \in X$, and the product of a real number α with a functional ℓ as that functional that maps every $x \in X$ into $\alpha\ell(x)$. Moreover, it is readily seen that the map $\ell \in X' \to \|\ell\|_{X'} = s_\ell \in \mathbb{R}$, with s_ℓ defined in (II.1.3), defines a norm in X'. It can be proved that if X is a Banach space, then also X', endowed with the norm $\|\cdot\|_{X'}$, is a Banach space, sometime referred to as *strong dual*; see, e.g. Smirnov (1964, §99).

A Banach space X is naturally embedded into its second dual $(X')' \equiv X''$ via the map $M : x \in X \to J_x \in X''$, where the functional J_x on X' is defined as follows: $J_x(\ell) = \ell(x)$, $\ell \in X'$. One can show that the range, $\mathsf{R}(M)$, of M is closed in X'' and that M is an isomorphism of X onto $\mathsf{R}(M)$; see e.g. Smirnov (1964, Theorem in §99). If $\mathsf{R}(M) = X''$, then X is *reflexive.*

We have the following result (see, e.g. Schechter 1971, Chapter VII, Theorem 1.1, Theorem 3.1 and Corollary 3.2; Chapter VIII, Theorem 1.2).

Theorem II.1.2 *Let X be a Banach space. Then X is reflexive if and only if X' is. Moreover if X' is separable, so is X. Therefore, if X is reflexive and separable, then so is X'. Finally, if X is reflexive, then so is every closed subspace of X.*

A sequence $\{x_k\}$ in a Banach space X is *weakly convergent* to $x \in X$ if

$$\lim_{k\to\infty} \ell(x_k) = x\,, \quad \text{for all } \ell \in X', \tag{II.1.4}$$

or, in equivalent notation, $x_k \overset{w}{\to} x$. In contrast to this latter, convergence in the sense of (II.1.1) will be also referred to as *strong convergence*. It is immediately seen that a strongly convergent sequence is also weakly convergent, while the converse is not generally true, unless X is isomorphic to $\mathbb{R}^n$; see e.g. Schechter (1971, Chapter VIII, Theorem 4.3). The topological definitions given previously (closedness, compactness, etc.) for subsets of X in terms of strong convergence, can be extended to the more general case of weak convergence in an obvious way. We shall then speak of *weakly closed* sets, or *weakly compact* sets, etc. Moreover, we shall say that a sequence $\{x_k\}$ is *weak Cauchy* if the following property holds, for all $\ell \in X'$:

given $\varepsilon > 0$ there is $\overline{n} = \overline{n}(\varepsilon, \ell) \in \mathbb{N}$: $|\ell(x_k - x_{k'})| < \varepsilon$ for all $k, k' \geq \overline{n}$.

A Banach space X is *weakly complete* if every weak Cauchy sequence is weakly convergent to some $x \in X$.

Some significant properties related to weak convergence are collected in the following.

Theorem II.1.3 *Let X be a Banach space. The following properties hold.*

(i) *If $\{x_k\} \subset X$ with $x_k \overset{w}{\to} x$, then there is C independent of k such that $\|x_k\|_X \leq C$. Moreover,*

$$\|x\|_X \leq \liminf_{k\to\infty} \|x_k\|_X\,;$$

see, e.g., Smirnov (1964, §101, Theorem 1 and Theorem 5).

(ii) *The closed unit ball $\{x \in X : \|x\|_X \leq 1\}$, is weakly compact if and only if X is reflexive; see,* e.g., Miranda (1978, §§28, 30).

(iii) *If X is reflexive, then X is also weakly complete; see,* e.g., Smirnov (1964, §101 Theorem 7).

Property (ii) will be sometime referred to as *weak compactness property.*

This property has, in turn, the following interesting consequence.

Theorem II.1.4 *Let X be a reflexive Banach space, and let $\ell \in X'$. Then, there exists $\overline{x} \in X$ such that*

$$\|\ell\|_{X'} = |\ell(\overline{x})|\,, \quad \|\overline{x}\|_X = 1\,. \tag{II.1.5}$$

Proof. If $\ell = 0$, then (II.1.5) is obviously satisfied. So, we assume $\ell \neq 0$. By definition, we have

$$\|\ell\|_{X'} = \sup_{x \in X; \|x\|_X = 1} |\ell(x)| \,.$$

Therefore, there exists a sequence $\{x_k\} \subset X$ such that

$$\|\ell\|_{X'} = \lim_{k\to\infty} |\ell(x_k)| \,, \quad \|x_k\|_X = 1, \text{ for all } k \in \mathbb{N}. \tag{II.1.6}$$

In view of Theorem II.1.3(ii), there exist a subsequence $\{x_{k'}\}$ and $\overline{x} \in X$ such that

$$x_{k'} \xrightarrow{w} \overline{x} \tag{II.1.7}$$

Evaluating (II.1.6) along this subsequence , with the help of Theorem II.1.3(i), we obtain that $\overline{x}$ satisfies the following conditions

$$\|\ell\|_{X'} = |\ell(\overline{x})| \,, \quad \|\overline{x}\|_X \le 1. \tag{II.1.8}$$

If $\overline{x} = 0$, it follows $\|\ell\|_{X'} = 0$ which was excluded, so that $\overline{x} \neq 0$. Thus, since

$$\|\ell\|_{X'} \ge \frac{|\ell(\overline{x})|}{\|\overline{x}\|_X} \,,$$

from this relation and (II.1.8) we prove the result. □

In the sequel, we shall deal with vector functions, namely, with functions with values in $\mathbb{R}^n$, whose components belong to the same Banach space X. We shall, therefore, recall some basic properties of Cartesian products, X^N, of N copies of X. It is readily checked that X^N can be endowed with the structure of vector space by defining the sum of two generic elements $\boldsymbol{x} = (x_1, \ldots, x_N)$ and $\boldsymbol{y} = (y_1, \ldots, y_N)$, and the product of a real number α with $\boldsymbol{x}$ in the following way

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_N + y_N) \,, \quad \alpha\, \boldsymbol{x} = (\alpha\, x_1, \ldots, \alpha x_N) \,.$$

Furthermore, we may introduce in X^N either one of the following (equivalent) norms (or any other norm equivalent to them)

$$\|\boldsymbol{x}\|_{(q)} \equiv \left(\sum_{i=1}^{N} \|x_i\|_X^q \right)^{1/q} , \; q \in [1, \infty) \,, \quad \|\boldsymbol{x}\|_{(\infty)} \equiv \max_{i \in \{1, \ldots N\}} \|x_i\|_X \,, \;\; \boldsymbol{x} \in X^N \,, \tag{II.1.9}$$

in such a way that (as the reader will prove with no pain) X^N becomes a Banach space.

We have the following.

Theorem II.1.5 *If X is separable, so is X^N. Moreover, X^N is reflexive if so is X,*

Proof. The proof of the first property is obvious, while that of the second one is a consequence of Theorem II.1.3(ii). □

The next result establishes the relation between $(X^N)'$ and $(X')^N$.

Theorem II.1.6 *Every $\mathcal{L} \in (X^N)'$ can be written as follows*

$$\mathcal{L} = \sum_{k=1}^{N} \ell_i \,, \tag{II.1.10}$$

where $\ell_i \in X'$, $i = 1, \ldots, N$ are uniquely determined. Moreover, the map

$$T : \mathcal{L} \in (X^N)' \to (\ell_1, \ldots, \ell_N) \in (X')^N$$

is a homeomorphism of $(X^N)'$ onto $(X')^N$. If, in particular, we endow X^N and $(X^N)'$ with the following norms

$$\|\boldsymbol{x}\|_{X^N} \equiv \|\boldsymbol{x}\|_{(1)} \,, \quad \|\mathcal{L}\|_{(X')^N} = \|\mathcal{L}\|_{(\infty)} \,.$$

then T is an isomorphism.

Proof. The generic element $\mathcal{L} \in X^N$ can be represented as in (II.1.10) where $\ell_1(\boldsymbol{x}) \equiv \mathcal{L}(x_1, 0, \ldots, 0)$, $\ell_2(\boldsymbol{x}) \equiv \mathcal{L}(0, x_2, \ldots, 0)$, etc. Obviously, each functional ℓ_i, $i = 1, \ldots N$, can be viewed as an element of X'. We then consider the map T in the way defined above. It is clear that T is surjective and injective and linear. From (II.1.10), it readily follows that

$$\|\mathcal{L}\|_{(X^N)'} \equiv \sup_{\boldsymbol{x} \in X^N; \|\boldsymbol{x}\|_{X^N} = 1} |\mathcal{L}(x)| \leq \|T(\mathcal{L})\|_{(\infty)} \,.$$

Moreover, by definition of supremum, we must have

$$\|\ell_i\|_{X'} \leq \|\mathcal{L}\|_{(X^N)'} \,,$$

so that we conclude $\|\mathcal{L}\|_{(X^N)'} \geq \|T(\mathcal{L})\|_{(\infty)}$, which shows that T is an isomorphism. If, instead, we use any other norm of the type (II.1.9), we can show by a simple calculation that uses (II.3.2) that T is, in general, a homeomorphism. The proof of the lemma is thus completed. □

We next recall the Hahn–Banach theorem and one of its consequences. A proof of these results can be found, e.g., in Schechter (1971, Chapter II Theorem 2.2 and Theorem 3.3).

Theorem II.1.7 *Let M be a subspace of a normed space X. The following properties hold.*

(a) *Let ℓ be a bounded linear functional defined on M, and let*

$$\|\ell\| = \sup_{x \in M; \|x\|_X = 1} |\ell(x)| \,.$$

Then, there exists a bounded linear functional, $\overline{\ell}$, defined on the whole of X, such that (i) $\overline{\ell}(x) = \ell(x)$, for all $x \in M$, and (ii) $\|\overline{\ell}\|_{X'} = \|\ell\|$.

(b) *Let* $x_0 \in X$ *be such that*

$$d \equiv \inf_{x \in M} \|x_0 - x\|_X > 0\,.$$

Then, there is $\ell \in X'$ *such that* $\|\ell\|_{X'} = 1/d$, $\ell(x_0) = 1$, *and* $\ell(x) = 0$, *for all* $x \in M$.

We conclude this section by reporting the classical contraction mapping theorem (see, e.g., Kantorovich & Akilov 1964, p. 625), that we shall often use throughout this book in the following form.

Theorem II.1.8 *Let* M *be a closed subset of the Banach space* X, *and let* T *be a map of* M *into itself. Suppose there exists* $\alpha \in (0, 1)$ *such that*

$$\|T(x) - T(y)\|_X \le \alpha \|x - y\|_X\,, \quad \text{for all } x, y \in M.$$

Then, there is a unique $x_0 \in M$ *such that* $T(x_0) = x_0$.

A map satisfying the assumptions of Theorem II.1.8 is called *contraction*.

II.1.3 Spaces of Smooth Functions

We next define some classical spaces of smooth functions and, for some of them, we recall their completeness properties.

Given a non-negative integer k, we let $C^k(\Omega)$ denote the linear space of all real functions u defined in Ω which together with all their derivatives $D^\alpha u$ of order $|\alpha| \le k$ are continuous in Ω. To shorten notations, we set

$$C^0(\Omega) \equiv C(\Omega).$$

We also set

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega).$$

Moreover, by the symbols $C_0^k(\Omega)$ and $C_0^\infty(\Omega)$ we indicate the (linear) subspaces of $C^k(\Omega)$ and $C^\infty(\Omega)$, respectively, of all those functions having compact support in Ω. Furthermore, $C_0^k(\overline{\Omega})$, $0 \le k \le \infty$, denotes the class of restrictions to Ω of functions in $C_0^k(\mathbb{R}^n)$. As before, we put

$$C_0^0(\Omega) \equiv C_0(\Omega), \quad C_0^0(\overline{\Omega}) \equiv C_0(\overline{\Omega}).$$

We next define $C^k(\overline{\Omega})$ ($C(\overline{\Omega})$ for $k = 0$) as the space of all functions u for which $D^\alpha u$ is bounded and uniformly continuous in Ω, for all $0 \le |\alpha| \le k$. We recall (Miranda 1978, §54) that for $k < \infty$, $C^k(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^k} \equiv \max_{0 \le |\alpha| \le k} \sup_{\Omega} |D^\alpha u|. \tag{II.1.11}$$

Finally, for $\lambda \in (0,1]$ and $k \in \mathbb{N}$, by $C^{k,\lambda}(\overline{\Omega})$ we denote the closed subspace of $C^k(\overline{\Omega})$ consisting of all functions u whose derivatives up to the kth order inclusive are *Hölder continuous* (*Lipschitz continuous* if $\lambda = 1$) in Ω, that is,

$$[u]_{k,\lambda} \equiv \max_{0\le|\alpha|\le k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda} < \infty.$$

$C^{k,\lambda}(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^{k,\lambda}} \equiv \|u\|_{C^k} + [u]_{k,\lambda}, \tag{II.1.12}$$

(Miranda 1978, §54).

Exercise II.1.1 Assuming Ω bounded, use the Ascoli-Arzelà theorem (see, e.g., Rudin 1987, p. 245) to show that from every sequence of functions uniformly bounded in $C^{k+1,\lambda}(\overline{\Omega})$ it is always possible to select a subsequence converging in the space $C^{k,\lambda}(\overline{\Omega})$.

II.1.4 Classes of Domains and their Properties

We begin with a simple but useful result holding for arbitrary domains of $\mathbb{R}^n$.

Lemma II.1.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$. Then there exists an open covering, $\mathfrak{O}$, of Ω satisfying the following properties*

(i) *$\mathfrak{O}$ is constituted by an at most countable number of open balls $\{\mathfrak{B}_k\}$, $k \in I \subseteq \mathbb{N}$, such that*

$$\mathfrak{B}_k \subset \Omega\,, \text{ for all } k \in I\,, \quad \cup_{k\in I}\mathfrak{B}_k = \Omega\,;$$

(ii) *For any family $\mathfrak{F} = \{\mathfrak{B}_l\}$, $l \in I'$ with $I' \subsetneq I$, there is $\mathfrak{B} \in (\mathfrak{O} - \mathfrak{F})$ such that $[\cup_{l\in I'}\mathfrak{B}_l] \cap \mathfrak{B} \neq \emptyset\,;$*

(iii) *For any $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{O}$, there exists a finite number of open balls $\mathfrak{B}_i \in \mathfrak{O}$, $i = 1, \dots, N$, such that*

$$\mathfrak{B} \cap \mathfrak{B}_1 \neq \emptyset\,, \quad \mathfrak{B}_N \cap \mathfrak{B}' \neq \emptyset\,, \quad \mathfrak{B}_j \cap \mathfrak{B}_{j+1} \neq \emptyset\,, \quad j = 1, \dots N-1\,.$$

Proof. Since Ω is open, for each $x \in \Omega$ we may find an open ball $B_{r_x}(x) \subset \Omega$. Clearly, the collection $\mathfrak{C} \equiv \{B_{r_x}(x)\}$, $x \in \Omega$, satisfies $\cup_{x\in\Omega} B_{r_x}(x) = \Omega$. However, since Ω is separable, we may determine an at most countable subcovering, $\mathfrak{O}$, of $\mathfrak{C}$ satisfying condition (i) in the lemma. Next, assume (ii) is not true. Then, there would be at least one family $\mathfrak{F}' = \{\mathfrak{B}_{k'}\}$, $k' \in I'$, with $I' \subsetneq I$ such that

$$\left[\bigcup_{k'\in I'} \mathfrak{B}_{k'}\right] \bigcap \mathfrak{B} = \emptyset\,, \quad \text{for all } \mathfrak{B} \in (\mathfrak{O} - \mathfrak{F}')\,.$$

Consequently, the sets

$$A_1 \equiv \bigcup_{k'\in I'} \mathfrak{B}_{k'}\,, \quad A_2 \equiv \bigcup_{k\in(I-I')} \mathfrak{B}_k$$

are open, disjoint and satisfy $A_1 \cup A_2 = \Omega$, contradicting the assumption that Ω is connected. Finally, let $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{O}$ and denote their centers by x and x', respectively. Since Ω is open and connected, it is, in particular, arc-connected. Therefore, we may find a curve, γ, joining x and x', that is homeomorphic to the interval $[0,1]$. Let $\mathfrak{O}' \subset \mathfrak{O}$ be a covering of γ. Since γ is compact, we can extract from $\mathfrak{O}'$ a finite covering that satisfies the property stated in the lemma. □

We next present certain classes of domains of $\mathbb{R}^n$, along with their relevant properties. We begin with the following.

Definition II.1.1. Let Ω be a domain with a *bounded* boundary, namely, Ω is either a bounded domain or it is a domain complement in $\mathbb{R}^n$ of a compact (not necessarily connected) set, namely, Ω is an *exterior* domain.[4] Assume that for each $x_0 \in \partial\Omega$ there is a ball $B = B_r(x_0)$ and a real function ζ defined on a domain $D \subset \mathbb{R}^{n-1}$ such that in a system of coordinates $\{x_1, \dots, x_n\}$ with the origin at x_0:

(i) The set $\partial\Omega \cap B$ can be represented by an equation of the type $x_n = \zeta(x_1, \dots, x_{n-1})$;
(ii) Each $x \in \Omega \cap B$ satisfies $x_n < \zeta(x_1, \dots, x_{n-1})$.

Then Ω is said to be *of class* C^k (or C^k*-smooth*) [respectively, *of class* $C^{k,\lambda}$ (or $C^{k,\lambda}$*-smooth*), $0 < \lambda \le 1$] if $\zeta \in C^k(\overline{D})$ [respectively, $\zeta \in C^{k,\lambda}(\overline{D})$]. If, in particular, $\zeta \in C^{0,1}(\overline{D})$, we say that Ω is *locally Lipschitz.* Likewise, we shall say that $\sigma \subset \partial\Omega$ is a *boundary portion of class* C^k [respectively, *of class* $C^{k,\lambda}$] if $\sigma = \partial\Omega \cap B_r(x_0)$, for some $r > 0, x_0 \in \partial\Omega$ and σ admits a representation of the form described in (i), (ii) with ζ of class C^k [respectively of class $C^{k,\lambda}$]. If, in particular, $\zeta \in C^{0,1}(\overline{D})$, we say that σ is a *locally Lipschitz boundary portion.*

If Ω is sufficiently smooth, of class C^1, for example, then the unit outer normal, $\boldsymbol{n}$, to $\partial\Omega$ is well defined and continuous. However, in several interesting cases, we need less regularity on Ω, but still would like to have $\boldsymbol{n}$ well-defined. In this regard, we have the following result, for whose proof we refer to Nečas (1967, Chapitre II, Lemme 4.2).

Lemma II.1.2 *Let Ω be locally Lipschitz. Then the unit outer normal $\boldsymbol{n}$ exists almost everywhere on $\partial\Omega$.*

[4] Hereafter, the whole space $\mathbb{R}^n$ will be considered a particular exterior domain.

We shall now consider a special class of bounded domains Ω called *star-shaped* (or *star-like*) with respect to a point. For such domains, there exist $\overline{x} \in \Omega$ (which we may, occasionally, assume to be the origin of coordinates) and a continuous, positive function h on the unit sphere such that

$$\Omega = \left\{ x \in \mathbb{R}^n : |x - \overline{x}| < h \left(\frac{x - \overline{x}}{|x - \overline{x}|} \right) \right\}. \tag{II.1.13}$$

Some elementary properties of star-shaped domains are collected in the following exercises.

Exercise II.1.2 Show that Ω is star-shaped with respect to $\overline{x}$ if and only if every ray starting from $\overline{x}$ intersects $\partial\Omega$ at one and only one point.

Exercise II.1.3 Assume Ω star-shaped with respect to the origin and set

$$\Omega^{(\rho)} = \{x \in \mathbb{R}^n : x = \rho y, \text{ for some } y \in \Omega\}. \tag{II.1.14}$$

Show that $\Omega^{(\rho)} \subset \overline{\Omega}$ if $\rho \in (0,1)$ and $\Omega^{(\rho)} \supset \overline{\Omega}$ if $\rho > 1$.

The following useful result holds.

Lemma II.1.3 *Let Ω be locally Lipschitz. Then, there exist m locally Lipschitz bounded domains $G_1, \ldots, G_m$ such that*

(i) $\partial\Omega \subset \cup_{i=1}^m G_i$;
(ii) *The domains $\Omega_i = \Omega \cap G_i$, $i = 1, \ldots, m$, are (locally Lipschitz and) star-shaped with respect to every point of a ball B_i with $\overline{B}_i \subset \Omega_i$.*

Proof. Let $x_0 \in \partial\Omega$. By assumption, there is $B_r(x_0)$ and a function $\zeta = \zeta(x')$, $x' = (x_1, \ldots, x_{n-1}) \in D \subset \mathbb{R}^{n-1}$ such that

$$|\zeta(\xi') - \zeta(\eta')| < \kappa|\xi' - \eta'|, \quad \xi', \eta' \in D,$$

for some $\kappa > 0$ and, moreover, points $x = (x', x_n) \in \partial\Omega \cap B_r(x_0)$ satisfy

$$x_n = \zeta(x'), \quad x' \in D,$$

while points $x \in \Omega \cap B_r(x_0)$ satisfy

$$x_n < \zeta(x'), \quad x' \in D.$$

We may (and will) take x_0 to be the origin of coordinates. Denote next, by $y_0 \equiv (0, \ldots, 0, y_n)$ the point of Ω intersection of the x_n-axis with $B_r(x_0)$ and consider the cone $\Gamma(y_0, \alpha)$ with vertex at y_0, axis x_n, and semiaperture $\alpha < \pi/2$. It is easy to see that, taking α sufficiently small, every ray ρ starting from y_0 and lying in $\Gamma(y_0, \alpha)$ intersects $\partial\Omega \cap B_r(x_0)$ at (one and) only one point. In fact, assume ρ cuts $\partial\Omega \cap B_r(x_0)$ at two points $z^{(1)}$ and $z^{(2)}$ and denote by $\alpha' < \alpha$ the angle formed by ρ with the x_n-axis. Possibly rotating the coordinate system around the x_n-axis we may assume without loss [5]

$$z^{(1)} = (z_1^{(1)}, 0, \ldots, 0, \zeta(z_1^{(1)}, 0, \ldots, 0)), \quad z_1^{(1)} > 0$$

$$z^{(2)} = (z_1^{(2)}, 0, \ldots, 0, \zeta(z_1^{(2)}, 0, \ldots, 0)), \quad z_1^{(2)} > 0$$

and so, at the same time,

$$\tan\alpha' = \frac{z_1^{(1)}}{\zeta(z_1^{(1)}, 0, \ldots, 0) - y_n}$$

$$\tan\alpha' = \frac{z_1^{(2)}}{\zeta(z_1^{(2)}, 0, \ldots, 0) - y_n}$$

implying

$$\frac{|\zeta(z_1^{(1)}, 0, \ldots, 0) - \zeta(z_1^{(2)}, 0, \ldots, 0)|}{|z_1^{(1)} - z_1^{(2)}|} = \frac{1}{\tan\alpha'} \geq \frac{1}{\tan\alpha}.$$

Thus, if (say)

$$\tan\alpha \leq \frac{1}{2\kappa},$$

ρ will cut $\partial\Omega \cap B_r(x_0)$ at only one point. Next, denote by $\sigma = \sigma(z)$ the intersection of $\Gamma(y_0, \alpha/2)$ with a plane orthogonal to x_n-axis at a point $z = (0, \ldots, z_n)$ with $z_n > y_n$, and set

$$R = R(z) \equiv \text{dist}\,(\partial\sigma, z).$$

Clearly, taking z sufficiently close to y_0 ($z = \overline{z}$, say), $\sigma(\overline{z})$ will be entirely contained in Ω and, further, every ray starting from a point of $\sigma(\overline{z})$ and lying within $\Gamma(y_0, \alpha/2)$ will form with the x_n-axis an angle less than α and so, by what we have shown, it will cut $\partial\Omega \cap B_r(x_0)$ at only one point. Let C be a cylinder with axis coincident with the x_n-axis and such that

$$C \cap \partial\Omega = \Gamma(y_0, \alpha/2) \cap \partial\Omega.$$

Then, setting

$$G = C \cap B_r(x_0),$$

we have that G is locally Lipschitz and that $G \cap \Omega$ is star-shaped with respect to all points of the ball $B_{R(\overline{z})}(\overline{z})$. Since $x_0 \in \partial\Omega$ is arbitrary, we may form an open covering $\mathcal{G}$ of $\partial\Omega$ constituted by domains of the type G. However, $\partial\Omega$ is compact and, therefore, we may select from $\mathcal{G}$ a finite subset $\{G_1, \ldots, G_m\}$ satisfying all conditions in the lemma, which is thus completely proved. □

[5] Clearly, the Lipschitz constant κ is invariant by this transformation.

Other relevant properties related to star-shaped domains are described in the following exercises.

Exercise II.1.4 Assume that the function h in (II.1.13) is Lipschitz continuous, so that, by Lemma II.1.2, the outer unit normal $\boldsymbol{n} = \boldsymbol{n}(x)$ on $\partial\Omega$ exists for a.a. x. Then, setting $F(x) \equiv \boldsymbol{n}(x) \cdot (x - \overline{x})$, show that $\operatorname*{ess\,inf}_{x\in\partial\Omega} F(x) > 0$.

Exercise II.1.5 Assume Ω bounded and locally Lipschitz. Prove that

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \,,$$

where each Ω_i is a locally Lipschitz and star-shaped domain with respect to every point of a ball B_i with $\overline{B}_i \subset \Omega_i$. *Hint:* Use Lemma II.1.3.

We end this section by recalling the following classical result, whose proof can be found, e.g., in Nečas (1967, Chapitre 1, Proposition 2.3).

Lemma II.1.4 *Let K be a compact subset of $\mathbb{R}^n$, and let $\mathcal{O} = \{\mathcal{O}_1, \cdots, \mathcal{O}_N\}$ be an open covering of K. Then, there exist functions ψ_i, $i = 1, \ldots, N$ satisfying the following properties*

(i) $0 \le \psi_i \le 1\,,\ i = 1, \ldots, N\,;$
(ii) $\psi_i \in C_0^\infty(\mathcal{O}_i)\,,\ i = 1, \ldots, N\,;$
(iii) $\sum_{i=1}^{N} \psi_i(x) = 1\,,$ *for all* $x \in K\,.$

The family $\{\psi_i\}$ is referred to as *partition of unity in K subordinate to the covering $\mathcal{O}$.*

II.2 The Lebesgue Spaces L^q

For $q \in [1, \infty)$, let $L^q = L^q(\Omega)$ denote the linear space of all (equivalence classes of) real Lebesgue-measurable functions u defined in Ω such that

$$\|u\|_q \equiv \left(\int_\Omega |u|^q\right)^{1/q} < \infty. \tag{II.2.1}$$

The functional (II.2.1) defines a norm in L^q, with respect to which L^q becomes a Banach space. Likewise, denoting by $L^\infty = L^\infty(\Omega)$ the linear space of all (equivalence classes of) Lebesgue-measurable real-valued functions u defined in Ω with

$$\|u\|_\infty \equiv \operatorname*{ess\,sup}_\Omega |u| < \infty \tag{II.2.2}$$

one shows that (II.2.2) is a norm and that L^∞ endowed with this norm is a Banach space. For a proof of the above properties see, e.g., Miranda (1978, §47). For $q = 2$, L^q is a Hilbert space under the scalar product

$$(u, v) \equiv \int_\Omega uv, \qquad u, v \in L^2.$$

Whenever confusion of domains might occur, we shall use the notation

$$\| \cdot \|_{q,\Omega}, \quad \| \cdot \|_{\infty,\Omega}, \quad \text{and} \;\; (\cdot, \cdot)_\Omega.$$

Given a sequence $\{u_m\} \subset L^q(\Omega)$ and $u \in L^q(\Omega)$, $1 \le q \le \infty$, we thus have that $u_m \to u$, namely, $\{u_m\}$ converges (strongly) to u, if and only if

$$\lim_{k\to\infty} \|u_k - u\|_q = 0\,.$$

The following two basic properties, collected in as many lemmas, will be frequently used throughout. The first one is the classical *Lebesgue dominated convergence theorem* (Jones 2001, Chapter 6 §C), while the other one relates convergence in L^q with pointwise convergence; see Jones (2001, Corollary at p. 234)

Lemma II.2.1 *Let $\{u_m\}$ be a sequence of measurable functions on Ω, and assume that*

$$u(x) \equiv \lim_{m\to\infty} u_m(x) \;\; \textit{exists for a.a. } x \in \Omega\,,$$

and that there is $U \in L^1(\Omega)$ such that

$$|u_m(x)| \le |U(x)| \;\; \textit{for a.a } x \in \Omega\,.$$

Then $u \in L^1(\Omega)$ and

$$\lim_{m\to\infty} \int_\Omega u_m = \int_\Omega u\,.$$

Lemma II.2.2 *Let $\{u_m\} \subset L^q(\Omega)$ and $u \in L^q(\Omega)$, $1 \le q \le \infty$, with $u_m \to u$. Then, there exists $\{u_{m'}\} \subseteq \{u_m\}$ such that*

$$\lim_{m'\to\infty} u_{m'}(x) = u(x)\,, \quad \textit{for a.a. } x \in \Omega\,.$$

We want now to collect some inequalities in L^q spaces that will be frequently used throughout. For $1 \le q \le \infty$, we set

$$q' = q/(q-1);$$

one then shows the *Hölder inequality*

$$\int_\Omega |uv| \le \|u\|_q \|v\|_{q'} \tag{II.2.3}$$

for all $u \in L^q(\Omega)$, $v \in L^{q'}(\Omega)$ (Miranda 1978, Teorema 47.I). The number q' is called the *Hölder conjugate of* q. In particular, (II.2.3) shows that the bilinear form (u,v) is meaningful whenever $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$. In case $q=2$, inequality (II.2.3) is referred to as the *Schwarz inequality.* More generally, one has the *generalized Hölder inequality*

$$\int_\Omega |u_1 u_2 \ldots u_m| \leq \|u_1\|_{q_1} \|u_2\|_{q_2} \cdot \ldots \cdot \|u_m\|_{q_m}, \tag{II.2.4}$$

where

$$u_i \in L^{q_i}(\Omega),\ 1 \leq q_i \leq \infty, \quad i = 1, \ldots, m, \quad \sum_{i=1}^m q_i^{-1} = 1\,.$$

Both inequalities (II.2.3) and (II.2.4) are an easy consequence of the *Young inequality*:

$$ab \leq \frac{\varepsilon a^q}{q} + \varepsilon^{-q'/q} \frac{b^{q'}}{q'} \quad (a, b, \varepsilon > 0) \tag{II.2.5}$$

holding for all $q \in (1,\infty)$. When $q=2$, relation (II.2.5) is known as the *Cauchy inequality.*

Two noteworthy consequences of inequality (II.2.3) are the *Minkowski inequality*:

$$\|u+v\|_q \leq \|u\|_q + \|v\|_q, \quad u, v \in L^q(\Omega), \tag{II.2.6}$$

and the *interpolation* (or *convexity*) *inequality*:

$$\|u\|_q \leq \|u\|_s^\theta \|u\|_r^{1-\theta} \tag{II.2.7}$$

valid for all $u \in L^s(\Omega) \cap L^r(\Omega)$ with $1 \leq s \leq q \leq r \leq \infty$, and

$$q^{-1} = \theta s^{-1} + (1-\theta) r^{-1}, \quad \theta \in [0,1].$$

Another important inequality is the *generalized Minkowski inequality* reported in the following lemma, and for whose proof we refer to Jones (2001, Chapter 11, §E).[6]

Lemma II.2.3 *Let Ω_1, and Ω_2 be domains of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with $m, n \geq 1$. Suppose that $u : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a Lebesgue measurable function such that, for some $q \in [1,\infty]$,*

$$\int_{\Omega_2} \left(\int_{\Omega_1} |u(x,y)|^q dx \right)^{1/q} dy < \infty\,.$$

Then,

$$\left(\int_{\Omega_1} \left| \int_{\Omega_2} u(x,y)\, dy \right|^q dx \right)^{1/q} < \infty\,,$$

and the following inequality holds

[6] Actually, it can be proved that (II.2.6) is just a particular case of (II.2.8), hence the adjective "generalized"; see Jones (2001, p. 272).

$$\left(\int_{\Omega_1}\left|\int_{\Omega_2} u(x,y)\,dy\right|^q dx\right)^{1/q} \le \int_{\Omega_2}\left(\int_{\Omega_1}|u(x,y)|^q dx\right)^{1/q} dy\,. \qquad \text{(II.2.8)}$$

Exercise II.2.1 Assume Ω bounded. Show that if $u \in L^\infty(\Omega)$, then

$$\lim_{q\to\infty}\|u\|_q = \|u\|_\infty.$$

Exercise II.2.2 Prove inequality (II.2.5). *Hint*: Minimize the function

$$t^q/q - t + 1/q'.$$

Exercise II.2.3 Prove inequalities (II.2.6) and (II.2.7).

We shall now list some of the basic properties of the spaces L^q. We begin with the following (see, e.g. Miranda 1978, §51).

Theorem II.2.1 *For $1 \le q < \infty$, L^q is separable, $C_0(\Omega)$ being, in particular, a dense subset*

Note that the above property is not true if $q = \infty$, since $C(\overline{\Omega})$ is a closed subspace of $L^\infty(\Omega)$); see Miranda, *loc. cit.*.

Concerning the density of smooth functions in L^q, one can prove something more than what stated in Theorem II.2.1, namely, that every function in L^q, $1 \le q < \infty$, can be approximated by functions from $C_0^\infty(\Omega)$. This fact follows as a particular case of a general *smoothing procedure* that we are going to describe. To this end, given a real (measurable) function u in Ω, we shall write

$$u \in L^q_{loc}(\Omega)$$

to mean

$$u \in L^q(\Omega'), \quad \textit{for any bounded domain } \Omega' \textit{ with } \overline{\Omega'} \subset \Omega.$$

Likewise, we write

$$u \in L^q_{loc}(\overline{\Omega})$$

to mean

$$u \in L^q(\Omega'), \quad \textit{for any bounded domain } \Omega' \subset \Omega.$$

Clearly, for Ω bounded we have $L^q_{loc}(\overline{\Omega}) = L^q(\Omega)$. Now, let $j \in C_0^\infty(\Omega)$ be a non-negative function such that

(i) $j(x) = 0$, for $|x| \ge 1$,

(ii) $\displaystyle\int_{\mathbb{R}^n} j = 1.$

A typical example is

$$j(x) = \begin{cases} c \exp[-1/(1-|x|^2)] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

with c chosen in such a way that property (ii) is satisfied. The *regularizer* (or *mollifier*) in the sense of Friedrichs u_ε of $u \in L^1_{loc}(\Omega)$ is then defined by the integral

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} j\left(\frac{x-y}{\varepsilon}\right) u(y) dy, \quad \varepsilon < \operatorname{dist}(x, \partial\Omega).$$

This function has several interesting properties, some of which will be recalled now here. First of all, we observe that u_ε is infinitely differentiable at each $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \varepsilon$. Moreover, if $u \in L^q_{loc}(\overline{\Omega})$ we may extend it by zero outside Ω, so that u_ε becomes defined for all $\varepsilon > 0$ and all $x \in \mathbb{R}^n$. Thus, in particular, if $u \in L^q(\Omega)$, $1 \leq q < \infty$, one can show (Miranda 1978, §51; see also Exercise II.2.10 for a generalization)

$$\begin{aligned} &\|u_\varepsilon\|_q \leq \|u\|_q \quad \text{for all } \varepsilon > 0\,, \\ &\lim_{\varepsilon \to 0^+} \|u_\varepsilon - u\|_q = 0. \end{aligned} \tag{II.2.9}$$

Exercise II.2.4 Show that for $u \in C_0(\Omega)$,

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x) = u(x) \quad \text{holds uniformly in } x \in \Omega.$$

Exercise II.2.5 For $u \in L^q(\Omega)$, $1 \leq q < \infty$, show the inequality

$$\sup_{\mathbb{R}^n} |D^\alpha u_\varepsilon(x)| \leq \varepsilon^{-n/q-|\alpha|} \|D^\alpha j\|_{q', \mathbb{R}^n} \|u\|_{q,\Omega}, \quad |\alpha| \geq 0.$$

We next observe that, by writing $u_\varepsilon(x)$ as follows:

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{|\xi|<\varepsilon} j\left(\frac{\xi}{\varepsilon}\right) u(x+\xi) d\xi,$$

it becomes apparent that, if u is of compact support in Ω and ε is chosen less than the distance of the support of u from $\partial\Omega$, then $u_\varepsilon \in C_0^\infty(\Omega)$. The latter, together with $(\text{II.2.9})_2$ and the density of C_0 in L^q, yields that $C_0^\infty(\Omega)$ is a dense subspace of $L^q(\Omega)$, $1 \leq q < \infty$. The proof of this property, along with some of its consequences, is left to the reader in the following exercises.

Exercise II.2.6 Prove that $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)$, $1 \leq q < \infty$. *Hint.* Use the density of $C_0(\Omega)$ in $L^q(\Omega)$ (Miranda 1978, §51) along with the properties of the mollifier.

Exercise II.2.7 Prove the existence of a basis in $L^2(\Omega)$ constituted by functions from $C_0^\infty(\Omega)$. *Hint*: Use the separability of L^2 along with the density of C_0^∞ into L^2.

Exercise II.2.8 Let $u \in L^q(\Omega)$, $1 \le q < \infty$. Extend u to zero in $\mathbb{R}^n - \Omega$ and continue to denote by u the extension. Show the following *continuity in the mean* property: Given $\varepsilon > 0$ there is $\delta > 0$ such that for every $h \in \mathbb{R}^n$ with $|h| < \delta$ the following inequality holds

$$\int_\Omega |u(x+h) - u(x)|^q \, dx < \varepsilon^q .$$

Hint: Show the property for $u \in C_0^\infty(\Omega)$, then use the density of C_0^∞ in L^q.

Exercise II.2.9 Assume $u \in L^1_{loc}(\Omega)$. Prove that

$$\int_\Omega u\psi = 0, \quad \text{for all } \psi \in C_0^\infty(\Omega), \ \text{ implies } u \equiv 0, \text{ a.e. in } \Omega.$$

Hint: Consider the function

$$\operatorname{sign} u = \begin{cases} \ \ 1 \text{ if } u > 0 \\ -1 \text{ if } u \le 0. \end{cases}$$

For a fixed bounded Ω' with $\overline{\Omega'} \subset \Omega$,

$$\operatorname{sign} u \in L^1(\Omega')$$

and so $\operatorname{sign} u$ can be approximated by functions from $C_0^\infty(\Omega')$.

Exercise II.2.10 Let $u \in L^q(\mathbb{R}^n)$, $1 \le q < \infty$, and for $z \in \mathbb{R}^n$ and $k \le n$ set

$$z_{(k)} = (z_1, \dots, z_k), \quad z^{(k)} = (z_{k+1}, \dots, z_n) .$$

Moreover, define

$$u_{(k),\varepsilon}(x) = \varepsilon^{-k} \int_{\mathbb{R}^k} j\left(\frac{x_{(k)} - y_{(k)}}{\varepsilon}\right) u(y_{(k)}, y^{(k)}) \, dy_{(k)} .$$

Show the following properties, for each $y^{(k)} \in \mathbb{R}^{n-k}$:

$$\begin{aligned} &\|u_{(k),\varepsilon}\|_{q,\mathbb{R}^k} \le \|u(\cdot, y^{(k)})\|_{q,\mathbb{R}^k} \quad \text{for all } \varepsilon > 0 , \\ &\lim_{\varepsilon \to 0+} \|u_{(k),\varepsilon} - u(\cdot, y^{(k)})\|_{q,\mathbb{R}^k} = 0. \end{aligned}$$

Hint: Use the generalized Minkowski inequality, the result in Exercise II.2.8 and Lebesgue dominated convergence theorem (Lemma II.2.1).

Let $v \in L^{q'}(\Omega)$, with q' the Hölder conjugate of q. Then, by (II.2.3), the integral

$$\ell(u) = \int_\Omega vu, \quad u \in L^q(\Omega) \tag{II.2.10}$$

defines a linear functional on L^q. However, for $q \in [1, \infty)$, every linear functional must be of the form (II.2.10). Actually, we have the following *Riesz representation theorem* for whose proof we refer to Miranda (1978, §48).

Theorem II.2.2 *Let ℓ be a linear functional on $L^q(\Omega)$, $q \in [1,\infty)$. Then, there exists a uniquely determined $v \in L^{q'}(\Omega)$ such that representation* (II.2.10) *holds. Furthermore*

$$\|\ell(u)\|_{[L^q(\Omega)]'} \equiv \sup_{\|u\|_q=1} |\ell(u)| = \|v\|_{q'} . \tag{II.2.11}$$

From Theorem II.2.2 we thus obtain the following.

Theorem II.2.3 *The (normed) dual of L^q is isomorphic to $L^{q'}$ for $1 < q < \infty$, so that, for these values of q, L^q is a reflexive space.*

Exercise II.2.11 Show the validity of (II.2.11) when $q \in (1,\infty)$. *Hint*: Use the representation (II.2.10).

Exercise II.2.12 Let $u \in L^1_{loc}(\Omega)$, and assume that there exists a constant $C > 0$ such that

$$|(u,\psi)| \le C\|\psi\|_q\,, \quad \text{for some } q \in [1,\infty) \text{ and all } \psi \in C_0^\infty(\Omega).$$

Show that $u \in L^{q'}(\Omega)$ and that $\|u\|_q \le C$. *Hint*: $\psi \to (u,\psi)$ defines a bounded linear functional on a dense set of $L^q(\Omega)$. Then use Hahn–Banach Theorem II.1.7 and the Riesz representation Theorem II.2.2.

Riesz theorem also allows us to give a characterization of weak convergence of a sequence $\{u_k\} \subset L^q(\Omega)$ to $u \in L^q(\Omega)$, $1 < q < \infty$. In fact, we have that $u_k \stackrel{w}{\to} u$ if and only if

$$\lim_{k\to\infty} (v, u_k - u) = 0\,, \quad \text{for all } v \in L^{q'}(\Omega),\ q' = q/(q-1).$$

In view of Theorem II.1.3(iii) and Theorem II.2.3, we find that L^q is *weakly complete*, for $q \in (1,\infty)$. In fact, this property continues to hold in the case $q = 1$; see Miranda (1978, Teorema 48.VII).

We wish now to recall the following results related to weak convergence.

Theorem II.2.4 *Let $\{u_m\} \subset L^q(\Omega)$, $1 \le q \le \infty$. The following properties hold.*

(i) *If $u_m \stackrel{w}{\to} u$, for some , $u \in L^q(\Omega)$, then there is C independent of m such that $\|u_m\|_q \le C$. Moreover,*

$$\|u\|_q \le \liminf_{m\to\infty} \|u_m\|_q.$$

In addition, if $1 < q < \infty$, and

$$\|u\|_q \ge \limsup_{m\to\infty} \|u_m\|_q,$$

then $u_m \to u$.

(ii) *If* $1 < q < \infty$ *and* $\|u_m\|_q \leq C$*, for some* C *independent of* m*, then there exists a subsequence* $\{u_{m'}\}$ *and* $u \in L^q(\Omega)$ *such that* $u_{m'} \xrightarrow{w} u$*.*

Proof. The statement in (ii) follows from Theorem II.1.3(ii), while the first statement in (i) is a consequence of the general result given in Theorem II.1.3(i). A proof of the second statement in (i) can be found, for example, in Brezis (1983, Proposition III.5(iii) and Proposition III.30). However, for $q = 2$ the proof of (i) becomes very simple and it will be reproduced here. By hypothesis and Riesz theorem we have that for all $v \in L^2$ and $\varepsilon > 0$ there exists $m' \in \mathbb{N}$ such that

$$|(u_m - u, v)| < \varepsilon, \quad \text{for all } m \geq m'.$$

If we choose $v = u_m$, with the help of the Schwarz inequality we find

$$\|u_m\|_2^2 \leq \|u\|_2 \|u_m\|_2 + \varepsilon\,.$$

Using Cauchy inequality on the right-hand side of this latter relation we conclude

$$\|u_m\|_2^2 \leq \|u\|_2^2 + 2\varepsilon\,,$$

which proves the boundedness of the sequence. We next choose

$$v = u, \quad \varepsilon = \eta\|u\|_2, \quad \eta > 0,$$

to obtain, again with the aid of Schwarz inequality,

$$\|u\|_2 \leq \|u_m\|_2 + \eta,$$

which completes the proof of the first part of the statement in (i). The second part is a consequence of the assumption and the identity

$$\|u_m - u\|_2^2 = \|u\|_2^2 + \|u_m\|_2^2 - 2(u_m, u).$$

□

We conclude this section with some observations concerning L^q-spaces of *vector-valued* functions. Let $[L^q(\Omega)]^N$ be the direct product of N copies of $L^q(\Omega)$. Then, as we know from Subsection I.1.2, $[L^q(\Omega)]^N$ is a Banach space with respect to any of the following equivalent norms:

$$\|\boldsymbol{u}\|_{q,(r)} \equiv \left(\sum_{i=1}^{N} \|u_i\|_q^r\right)^{1/r}, \quad r \in [1, \infty) \quad \|\boldsymbol{u}\|_{q,(\infty)} \equiv \max_{i \in \{1,\dots,N\}} \|u_i\|_q\,,$$

where $\boldsymbol{u} = (u_1, \dots, u_N)$. Moreover, in view of Theorem II.2.1, Theorem II.2.3, and Theorem II.1.5, we have.

Theorem II.2.5 $[L^q(\Omega)]^N$ *is separable for* $q \in [1, \infty)$*, and reflexive for* $q \in (1, \infty)$ *.*

Also, the Riesz representation theorem can be suitably extended to this more general case. In fact, let

$$(\boldsymbol{v},\boldsymbol{u}) \equiv \sum_{i=1}^{N}(u_i,v_i)\,,\quad \boldsymbol{u}\in[L^q(\Omega)]^N\,,\quad \boldsymbol{v}\in[L^{q'}(\Omega)]^N\,, 1/q+1/q'=1.$$

In view of Theorem II.1.6 and of Theorem II.2.2 , we then have that for every $\mathcal{L}\in\big([L^q(\Omega)]^N\big)'$, there exist uniquely determined $\boldsymbol{v}\in[L^{q'}(\Omega)]^N$, such that

$$\mathcal{L}(\boldsymbol{u}) = (\boldsymbol{v},\boldsymbol{u})\,,$$

and that the map $M:\mathcal{L}\to\boldsymbol{v}$ is a homeomorphism. Actually, if we endow $[L^q(\Omega)]^N$ with the norm $\|\boldsymbol{u}\|_{q,(q)}\equiv\|\boldsymbol{u}\|_q$, the map M is an isomorphism, as stated in the second part of the following lemma, whose proof can be found in Simader (1972, Lemma 4.2).[7]

Theorem II.2.6 *Let Ω be a domain of $\mathbb{R}^n$, and let $q\in(1,\infty)$. Then, for every $\mathcal{L}\in\big([L^q(\Omega)]^N\big)'$, there exists uniquely determined $\boldsymbol{v}\in[L^{q'}(\Omega)]^N$, such that*

$$\mathcal{L}(\boldsymbol{u}) = (\boldsymbol{v},\boldsymbol{u})\,,\quad \boldsymbol{u}\in[L^q(\Omega)]^N\,.$$

Moreover,

$$\|\mathcal{L}\|_{([L^q(\Omega)]^N)'}\equiv\sup_{\boldsymbol{u}\in[L^q(\Omega)]^N,\ \|\boldsymbol{u}\|_q=1}|\mathcal{L}(\boldsymbol{u})| = \|\boldsymbol{v}\|_q\,.$$

II.3 The Sobolev Spaces $W^{m,q}$ and Embedding Inequalities

Let $u\in L^1_{loc}(\Omega)$. Given a multi-index α, we shall say that a function $u^{(\alpha)}\in L^1_{loc}(\Omega)$ is the αth *generalized* (or *weak*) *derivative of* u if and only if the following relation holds:

$$\int_\Omega uD^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega u^{(\alpha)}\varphi,\quad\text{for all }\varphi\in C_0^\infty(\Omega).$$

It is easy to show that $u^{(\alpha)}$ is uniquely determined (use Exercise II.2.9) and that, if $u\in C^{|\alpha|}(\Omega)$, $u^{(\alpha)}$ is the αth derivative of u in the ordinary sense, and the previous integral relation is an obvious consequence of the well-known Gauss formula. Hereafter, the function $u^{(\alpha)}$, whenever it exists, will be indicated by the symbol $D^\alpha u$.

[7] The assumption made in Simader *loc. cit.*, that Ω is bounded, is completely superfluous, since it is never used in the proof, as it was also independently communicated to me by Professor Simader.

Generalized derivatives have several properties in common with ordinary derivatives. For instance, given two functions u, v possessing generalized derivatives $D_j u$, $D_i v$ we have that $\beta u + \gamma v \quad (\beta, \gamma \in \mathbb{R})$ has a generalized derivative and $D_j(\beta u + \gamma v) = \beta D_j u + \gamma D_j v$. In addition, if

$$uv, \quad uD_j v + vD_j u \in L^1_{loc}(\Omega),$$

then uv has a generalized derivative and the familiar *Leibniz rule* holds:

$$D_j(uv) = uD_j v + vD_j u.$$

The proof of these properties is left to the reader as an exercise.

Exercise II.3.1 Generalized differentiation and differentiation almost everywhere are two distinct concepts. Show that a function φ that is continuous in [0,1] but *not* absolutely continuous admits no generalized derivative. *Hint*: Assume, *per absurdum*, that φ has a generalized derivative Φ. Then, it would follow

$$\varphi(x) = \int_0^x \Phi(t)dt + \varphi(0), \quad x \in (0,1),$$

which gives a contradiction. On the other hand, one can give examples of real, continuous functions f on $[0,1]$ that are differentiable *a.e.* in $[0,1]$ and with $f' \in L^1(0,1)$ which are not absolutely continuous (Rudin 1987, pp. 144-145). In this connection, it is worth noticing the following general result (Smirnov 1964, §110): *a function $u \in L^1_{loc}(\Omega)$ $(\Omega \subset \mathbb{R}^n)$ is weakly differentiable if $u = \widetilde{u}$ a.e. in Ω, with $\widetilde{u}$ absolutely continuous on almost all line segments parallel to the coordinate axes and having partial derivatives locally integrable.*

Exercise II.3.2 Let $u \in L^1_{loc}(\Omega)$ and assume that $D^\alpha u$ exists. Show

$$D^\alpha(u_\varepsilon(x)) = (D^\alpha u)_\varepsilon(x)\,, \quad \text{dist}\,(x, \partial\Omega) > \varepsilon.$$

Exercise II.3.3 Let $\Omega \subset \mathbb{R}^n$, and let $\psi \in C^1(\Omega)$ map Ω onto $\Omega_1 \subset \mathbb{R}^n$, with $\psi^{-1} \in C^1(\Omega_1)$. Assume u possesses generalized derivatives $D_j u$, $j = 1, \ldots, n$, and set $v = u \circ \psi^{-1}$. Show that also v possesses generalized derivatives $D_j v$, $j = 1, \ldots, n$, and that the usual change of variable formula applies:

$$D_i u(x) = \frac{\partial \psi_j}{\partial x_i}\, D_j v(y)\,, \quad y = \psi(x)\,,$$

for a.a. $x \in \Omega$ and $y \in \Omega_1$.

For $q \in [1, \infty]$ and $m \in \mathbb{N}$, we let

$$W^{m,q} = W^{m,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega),\ 0 \leq |\alpha| \leq m\}\,.$$

In the linear space $W^{m,q}(\Omega)$ we introduce the following norm:

$$\|u\|_{m,q} = \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|_q^q \right)^{1/q} \quad \text{if } 1 \le q < \infty$$
$$\|u\|_{m,\infty} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty \qquad \text{if } q = \infty. \tag{II.3.1}$$

If confusion of domains arises, we shall write $\|u\|_{m,q,\Omega}$ and $\|u\|_{m,\infty,\Omega}$ in place of $\|u\|_{m,q}$ and $\|u\|_{m,\infty}$. Owing to the completeness of the spaces L^q and taking into account the definition of generalized derivative, it is not hard to show that $W^{m,q}$ endowed with the norm (II.3.1) becomes a Banach space, called *Sobolev space* (of order (m,q)). Along with this space, we shall consider its closed subspace $W_0^{m,q} = W_0^{m,q}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ in the norm (II.3.1). Clearly, we have (see Exercise II.2.6)

$$W^{0,q} = W_0^{0,q} = L^q.$$

In the special case $q = 2$, $W^{m,q}$ (and thus $W_0^{m,q}$) is a Hilbert space with respect to the scalar product

$$(u,v)_m = \sum_{0\le|\alpha|\le m} (D^\alpha u, D^\alpha v)\,.$$

Exercise II.3.4 Prove that, for any Ω, $W_0^{m,q}(\Omega)$ is a closed subspace of $W^{m,q}(\Omega)$. Prove also $W_0^{m,q}(\mathbb{R}^n) = W^{m,q}(\mathbb{R}^n)$, $1 \le q < \infty$. *Hint*: To show the second assertion, take a function $\varphi \in C^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ if $|x| \le 1$, $\varphi(x) = 0$ if $|x| \ge 2$ ("cut-off" function) and set

$$u_m(x) = \varphi(x/m)u(x), \quad u \in W^{m,q}(\mathbb{R}^n), \quad m \in \mathbb{N}.$$

Then, u is approximated in $W^{m,q}(\mathbb{R}^n)$ by $\{(u_m)_\varepsilon\} \subset C_0^\infty(\mathbb{R}^n)$.

Remark II.3.1 Sobolev spaces share several important properties with Lebesgue spaces L^q. Thus, for example, since a closed subspace of a Banach space X is reflexive and separable if X is (see Theorem II.1.1 and Theorem II.1.2), and since $W^{m,q}(\Omega)$ can be naturally embedded in $[L^q(\Omega)]^N$, for a suitable $N = N(m)$, one can readily show, by using Theorem II.2.5 and the fact that $W^{m,q}(\Omega)$ is complete, that $W^{m,q}(\Omega)$ is separable if $1 \le q < \infty$ and reflexive if $1 < q < \infty$; for details, see, e.g., Adams (1975, §3.4). As a consequence, by Theorem II.1.3(ii), we find, in particular, that for $q \in (1,\infty)$, $W^{m,q}$ has the weak compactness property. ■

Exercise II.3.5 Let $u \in L^1_{loc}(\Omega)$ and suppose $\|u_\varepsilon\|_{m,q,B} \le C$, $m \ge 0$, $1 < q < \infty$, where B is an arbitrary open ball with $\overline{B} \subset \Omega$, and C is independent of ε. Show that $u \in W^{m,q}_{loc}(\Omega)$ and that $\|u\|_{m,q,B} \le C$.

Another interesting question is whether elements from $W^{m,q}(\Omega)$ can be approximated by smooth functions. This question is important, for instance, when one wants to establish in $W^{m,q}$ inequalities involving norms (II.3.1). Actually, if such an approximation holds, it then suffices to prove these inequalities for smooth functions only. In the case where Ω is either the whole of $\mathbb{R}^n$ or it is star-shaped with respect to a point, the question is affirmatively answered; cf. Exercise II.3.4 and Exercise II.3.7. In more general cases, we have a fundamental result, given in Theorem II.3.1, which in its second part involves domains having a mild property of regularity, i.e., the *segment property*, which states that, for every $x \in \partial\Omega$ there exists a neighborhood U of x and a vector $\boldsymbol{y}$ such that if $z \in \overline{\Omega} \cap U$, then $z + t\boldsymbol{y} \in \Omega$, for all $t \in (0,1)$.

Exercise II.3.6 Show that a domain having the segment property cannot lie simultaneously on both sides of its boundary.

Theorem II.3.1 *For any domain Ω, every function from $W^{m,q}(\Omega)$, $1 \leq q < \infty$, can be approximated in the norm* (II.3.1)$_1$ *by functions in $C^m(\Omega) \cap W^{m,q}(\Omega)$. Moreover, if Ω has the segment property, it can be approximated in the same norm by elements of $C_0^\infty(\overline{\Omega})$.*

The first part of this theorem is due to Meyers and Serrin (1964), while the second one is given by Adams (1975, Theorem 3.18).

Exercise II.3.7 (Smirnov 1964, §111). Assume Ω star-shaped with respect to the origin. Prove that every function u in $W^{m,q}(\Omega)$, $1 \leq q < \infty$, $m \geq 0$, can be approximated by functions from $C_0^\infty(\overline{\Omega})$. (Compare this result with Theorem II.3.1.) *Hint*: Consider the sequence

$$u_k(x) = \begin{cases} u\left((1-1/k)x\right) & \text{if } x \in \Omega^{(k/(k-1))} \\ 0 & \text{if } x \notin \Omega^{(k/(k-1))} \end{cases} \qquad k = 2, 3, \dots ,$$

with $\Omega^{(\rho)}$ defined in (II.1.14). Then, regularize u_k and use (II.2.9) and Exercise II.3.2.

We wish now to prove some basic inequalities involving the norms (II.3.1). Such results are known as *Sobolev embedding theorems* (see Theorem II.3.2 and Theorem II.3.4). To this end, we propose an elementary inequality due to Nirenberg (1959).

Lemma II.3.1 *For all $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\|u\|_{n/(n-1)} \leq \frac{1}{2\sqrt{n}} \|\nabla u\|_1. \tag{II.3.2}$$

Proof. Just to be specific, we shall prove (II.3.2) for $n = 3$, the general case being treated analogously. We have

$$|u(x)| \le \frac{1}{2}\int_{-\infty}^{\infty} |D_1 u| dx_1 \equiv F_1(x_2, x_3)$$

and similar estimates for x_2 and x_3. With the obvious meaning of the symbols we then deduce

$$|2u(x)|^{3/2} \le [F_1(x_2,x_3)F_2(x_1,x_3)F_3(x_1,x_2)]^{1/2}.$$

Integrating over x_1, and using the Schwarz inequality,

$$\int_{-\infty}^{\infty} |2u(x)|^{3/2} dx_1 \le [F_1(x_2,x_3)]^{1/2}\left(\int_{-\infty}^{\infty} F_2(x_1,x_3)dx_1\right)^{1/2} \times \left(\int_{-\infty}^{\infty} F_3(x_1,x_2)dx_1\right)^{1/2}.$$

Integrating this relation successively over x_2 and x_3 and applying the same procedure, we find

$$2\|u\|_{3/2} \le \left(\int_{\mathbb{R}^3} |D_1 u| \int_{\mathbb{R}^3} |D_2 u| \int_{\mathbb{R}^3} |D_3 u|\right)^{1/3} \le (1/3)\sum_{i=1}^{3}\int_{\mathbb{R}^3} |D_i u|,$$

which, in turn, after employing the inequality [1]

$$(a_1 + a_2 + \ldots + a_m)^q \le m^{q-1}(a_1^q + a_2^q + \ldots + a_m^q), \quad a_i > 0\,, \; q \ge 1 \qquad \text{(II.3.3)}$$

with $m = 3, q = 2$, gives (II.3.2). □

For $q \ge 1$, replacing u with $|u|^q$ in (II.3.2) and using the Hölder inequality, we obtain at once

$$\|u\|_{qn/(n-1)} \le \left(\frac{q}{2\sqrt{n}}\right)^{1/q} \|u\|_q^{1-1/q}\|\nabla u\|_q^{1/q}. \qquad \text{(II.3.4)}$$

Inequalities (II.3.2), (II.3.4), and (II.2.7) allow us to deduce more general relations, which are contained in the following lemma.

Lemma II.3.2 *Let*

$$r \in [q, nq/(n-q)], \quad \textit{if } q \in [1, n),$$

and

$$r \in [q, \infty), \quad \textit{if } q \ge n.$$

Then, for all $u \in C_0^\infty(\mathbb{R}^n)$ *we have*

[1] See Hardy, Littlewood, & Polya 1934, Theorem 16, p. 26.

$$\|u\|_r \le \left(\frac{c_1}{2\sqrt{n}}\right)^{\lambda} \|u\|_q^{1-\lambda}\|\nabla u\|_q^{\lambda}, \tag{II.3.5}$$

where

$$c_1 = \max(q, r(n-1)/n), \quad \lambda = n(r-q)/rq.$$

Proof. We shall distinguish the two cases:

(i) $q \le r \le qn/(n-1)$,
(ii) $r \ge qn/(n-1)$.

In case (i) we have by (II.2.7) and (II.3.4)

$$\|u\|_r \le \|u\|_q^{\theta}\|u\|_{qn/(n-1)}^{1-\theta} \le \left(\frac{q}{2\sqrt{n}}\right)^{(1-\theta)/q} \|u\|_q^{(\theta-1)/q+1}\|\nabla u\|_q^{(1-\theta)/q}$$

with

$$\theta = \frac{r(1-n)+nq}{r}.$$

Substituting the value of θ in the preceding relation furnishes (II.3.5). In case (ii), we replace u in (II.3.2) with $|u|^{r(n-1)/n}$ and apply the Hölder inequality to obtain

$$\|u\|_r^{r(n-1)/n} \le \frac{r(n-1)}{2n\sqrt{n}}\|u\|_{\beta}^{[r(n-1)-n]/n}\|\nabla u\|_q, \qquad \beta = \frac{qr(n-1)-n}{n(q-1)}.$$

Notice that $q \le \beta$. Moreover, it is

$$\beta \le r \ \text{ for } r \le nq/(n-q), \quad \text{if } q < n$$

and

$$\beta \le r \ \text{ for all } r < \infty, \quad \text{if } q \ge n.$$

In either case we may use (II.2.7) to obtain

$$\|u\|_{\beta} \le \|u\|_q^{\theta}\|u\|_r^{1-\theta}, \quad \theta = \frac{r(q-n)+nq}{(r-q)[r(n-1)-n]}.$$

Substituting this inequality in the preceding one gives (II.3.5), and the proof of the lemma is complete. □

Lemma II.3.2 can be extended to include L^q-norms of derivatives of order higher than one. A general multiplicative inequality is given in Nirenberg (1959, p.125). We reproduce here this result, referring the reader to the paper of Nirenberg for a proof. Set

$$|u|_{k,p} \equiv \left(\sum_{|\ell|=k}\int_{\Omega}|D^{\ell}u|^p\right)^{1/p}.$$

We have the following.

Lemma II.3.3 *Let $u \in L^q(\mathbb{R}^n)$, with $D^\alpha u \in L^r(\mathbb{R}^n)$, $|\alpha| = m > 0$, $1 \leq q, r \leq \infty$. Then, $D^\alpha u \in L^s(\mathbb{R}^n)$, $|\alpha| = j$, and the following inequality holds for $0 \leq j < m$ and some $c = c(n, m, j, q, r, a)$:*

$$|u|_{j,s} \leq c\,|u|^a_{m,r}\|u\|_q^{1-a}, \tag{II.3.6}$$

where

$$\frac{1}{s} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1,$$

with the following exceptional cases

1. *If $j = 0$, $rm < n$, $q = \infty$ then we make the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$, or $u \in L^{\overline{q}}(\mathbb{R}^n)$ for some $\overline{q} \in (0, \infty)$.*
2. *if $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then $(*)$ holds only for a satisfying $j/m \leq a < 1$.*

From Lemma II.3.2 we wish to single out some special inequalities that will be used frequently in the theory of the Navier–Stokes equations. First of all, we have the *Sobolev inequality*

$$\|u\|_r \leq \frac{q(n-1)}{2(n-q)\sqrt{n}}\|\nabla u\|_q, \quad 1 \leq q < n, \ r = nq/(n-q), \tag{II.3.7}$$

derived for the first time by Sobolev (1938) by a complete different method and for $q \in (1, n)$.[2] Inequality (II.3.7), holding a priori only for functions $u \in C_0^\infty(\mathbb{R}^n)$, can be clearly extended, by density, to every $u \in W_0^{1,q}(\Omega)$, $1 \leq q < n$. We then deduce, in particular, that every such function is in $L^r(\Omega)$ with r given in (II.3.7).

Exercise II.3.8 Let $\Omega = B_1$ or $\Omega = \mathbb{R}^n$, $n \geq 2$. Show, by means of a counterexample, that the Sobolev inequality does not hold if $q = n$, that is, a (positive, finite) constant γ independent of u such that

$$\|u\|_\infty \leq \gamma\|\nabla u\|_n, \quad u \in C_0^\infty(\Omega), \ n \geq 2,$$

does not exist. (In this respect, see also Section II.9 and Section II.11.)[3]

Remark II.3.2 In connection with (II.3.7) we would like to make some comments. When Ω is an unbounded domain (in particular, exterior to the closure of a bounded domain) the investigation of the asymptotic properties of a solution u to a system of partial differential equations is strictly related to the

[2] In this regard, see Theorem II.11.3 and Exercise II.11.4.

[3] A sharp version of the Sobolev inequality when $q = n$ and Ω is bounded, is due to Trudinger (1967).

Lebesgue space $L^s(\Omega)$ to which u belongs and, roughly speaking, the behavior of u at large distances will be better known when the exponent s is lower.[4] Now, as we shall see in subsequent chapters, the inherent information we derive from the Navier–Stokes equations in such domains is that u (a generic component of the velocity field) has first derivatives $D_i u$ summable with exponents q_i which, however, *may vary with* x_i, $i = 1, \ldots, n$. Therefore, we may wonder if (II.3.7) can be replaced by another inequality which takes into account this different behavior in different directions and leads to an exponent s of summability for u *strictly* less than the exponent r given in (II.3.7). This question finds its answer within the context of *anisotropic Sobolev spaces* (Nikol'skiĭ 1958). Here, we shall limit ourselves to quote, without proof, an inequality due to Troisi (1969, Teorema 1.2) representing the natural generalization of (II.3.7) to the anisotropic case. Let

$$1 \le q_i < \infty, \quad i = 1, \ldots, n.$$

Then, for all $u \in C_0^\infty(\mathbb{R}^n)$ the following *Troisi inequality* holds:

$$\|u\|_s \le c \prod_{i=1}^{n} \|D_i u\|_{q_i}^{1/n}, \quad \sum_{i=1}^{n} q_i^{-1} > 1, \quad s = \frac{n}{\left(\sum_{i=1}^n q_i^{-1} - 1\right)}. \tag{II.3.8}$$

If $q_i = q$, for all $i = 1, \ldots, n$, (II.3.8) reduces to (II.3.7). On the other hand, if for some i (=1, say), $q_1 < q \equiv q_2 = \ldots = q_n$, from (II.3.8) we deduce

$$s = r + \frac{nq(q_1 - q)}{(q - q_1) + q_1(n - q)} < r.$$

■

Other special cases of (II.3.5) are now considered. We choose in Lemma II.3.2 $n = q = 2$ and $r = 4$ to deduce the *Ladyzhenskaya inequality*

$$\|u\|_4 \le 2^{-1/4} \|u\|_2^{1/2} \|\nabla u\|_2^{1/2}, \tag{II.3.9}$$

shown for the first time by Ladyzhenskaya (1958, 1959a, eq. (6)). It should be emphasized that (II.3.9) does *not* hold in three space dimensions with the *same* exponents (see Exercise II.3.9). Rather, for $n = 3$, $q = 2$, and $r = 4$, inequality (II.3.5) delivers

$$\|u\|_4 \le \left(\frac{4}{3\sqrt{3}}\right)^{3/4} \|u\|_2^{1/4} \|\nabla u\|_2^{3/4}. \tag{II.3.10}$$

Furthermore, for $n = 3$, $q = 2$, $r = 6$ the Sobolev inequality (II.3.7) specializes to

[4] It is needless to say that the possibility of lowering the exponent s depends on the particular problem.

$$\|u\|_6 \leq \frac{2}{\sqrt{3}}\|\nabla u\|_2. \tag{II.3.11}$$

In two space dimensions there is no analogue of (II.3.11), and so, in particular, for $n = 2$, a function having all derivatives in $L^2(\mathbb{R}^2)$ need not be in $L^r(\mathbb{R}^2)$, whatever $r \in [1, \infty]$. [5]

Exercise II.3.9 Let φ be the C^∞ "cut-off" function introduced in Exercise II.3.4 and set $u_m(x) = \varphi(x)\exp(-m|x|)$, $m \in \mathbb{N}$. Obviously, $\{u_m\} \subset C_0^\infty(\mathbb{R}^n)$. Show that for $n = 3$ the following inequality holds

$$R(m) \equiv \frac{\|u_m\|_4^4}{\|u_m\|_2^2\|\nabla u_m\|_2^2} \geq c\, m \frac{\int_0^m e^{-y}y^2 dy}{\int_0^m e^{-2y}y^2 dy},$$

with c a positive number independent of m. Since $R(m) \to \infty$ as $m \to \infty$, a constant $\gamma \in (0, \infty)$ such that

$$\|u\|_4 \leq \gamma\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}, \quad u \in C_0^\infty(\mathbb{R}^3),$$

does not exist.

The case $q > n$ of Lemma II.3.2 can be further strengthened, as shown by the following lemma.

Lemma II.3.4 *Let $q > n$. Then, for all $u \in C^1(\overline{B(x)})$ we have*

$$|u(x)| \leq \omega_n^{-1}\|u\|_{1,B(x)} + \omega_n^{-1/q}\left(\frac{q-1}{q-n}\right)^{1-1/q} \|\nabla u\|_{q,B(x)}\,, \tag{II.3.12}$$

and so, in particular, for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq c_2\omega_n^{-1/q}\|u\|_{1,q,\mathbb{R}^n} \tag{II.3.13}$$

with

$$c_2 = \max\left\{1, \left(\frac{q-1}{q-n}\right)^{(q-1)/q}\right\}.$$

Proof. It is enough to prove (II.3.12), since (II.3.13) follows by using the Hölder inequality in the first term of (II.3.12). From the identity

$$u(x) - u(y) = -\int_0^{|x-y|} \frac{\partial u(x + r\boldsymbol{e})}{\partial r} dr, \quad \boldsymbol{e} = \frac{\boldsymbol{y} - \boldsymbol{x}}{|y - x|}, \tag{II.3.14}$$

[5] For example, for $\alpha \in (0, 1/2)$, take $u(x) = \ln^\alpha |x|$, if $|x| > 1$ and $u(x) = 0$ if $|x| \leq 1$. The problem of the behavior at large spatial distances of functions with gradients in $L^q(\Omega)$, Ω an exterior domain, will be fully analyzed in Section II.7 and Section II.9.

we easily show

$$\omega_n|u(x)| \leq \|u\|_{1,B(x)} + \int_{B(x)} |\nabla u(y)|\, |x-y|^{1-n} dy. \tag{II.3.15}$$

Applying the Hölder inequality in the integral in (II.3.15) and dividing the resulting relation by ω_n we prove (II.3.12). □

We want now to draw some consequences from Lemma II.3.2 and Lemma II.3.4. Employing the Young inequality (II.2.5) and the density of $C_0^\infty(\Omega)$ in $W_0^{1,q}(\Omega)$, from (II.3.3), (II.3.5), and (II.3.13) we find, in particular, that a function $u \in W_0^{1,q}(\Omega)$ is also in $L^r(\Omega)$, for all $r \in [q, nq/(n-q)]$, if $1 \leq q < n$, and for all $r \geq q$, if $q = n$. Moreover, if $q > n$, u coincides *a.e.* in Ω with a (uniquely determined) function of $C(\overline{\Omega})$. Finally, u obeys the following inequalities:

$$\begin{aligned} &\|u\|_r \leq C_1\|u\|_{1,q} \quad 1 \leq q < n, \quad q \leq r \leq \frac{nq}{n-q} \\ &\|u\|_r \leq C_2\|u\|_{1,q} \quad q = n, \quad q \leq r < \infty \\ &\|u\|_C \leq C_3\|u\|_{1,q} \quad q > n \end{aligned} \tag{II.3.16}$$

with $C_i = C_i(n,q,r)$, $i = 1,2,3$. Now, using (II.3.16) and an iterative argument we may generalize (II.3.16) to functions from $W_0^{m,q}(\Omega)$, to obtain the following *embedding theorem* whose proof is left to the reader as an exercise.

Theorem II.3.2 *Let $u \in W_0^{m,q}(\Omega)$, $q \geq 1$, $m \geq 0$. If $mq \leq n$ we have*

$$W_0^{m,q}(\Omega) \hookrightarrow L^r(\Omega)$$

for all $r \in [q, \frac{nq}{n-mq}]$ if $mq < n$, and for all $r \in [q, \infty)$ if $mq = n$. In particular, there are constants c_i, $i = 1,2$, depending only on m, q, r and n such that

$$\begin{aligned} &\|u\|_r \leq c_1\|u\|_{m,q} \quad \text{for all } r \in [q, \tfrac{nq}{n-mq}], \quad \text{if } mq < n, \\ &\|u\|_r \leq c_2\|u\|_{m,q} \quad \text{for all } r \in [q, \infty), \quad \text{if } mq = n, \end{aligned} \tag{II.3.17}$$

Finally, if $mq > n$, each $u \in W_0^{m,q}(\Omega)$ is equal a.e. in Ω to a unique function in $C^k(\overline{\Omega})$, $0 \leq k < m - n/q$, and the following inequality holds

$$\|u\|_{C^k} \leq c_3\|u\|_{m,q}, \tag{II.3.18}$$

with $c_3 = c_3(m,q,r,n)$.

We wish now to generalize Theorem II.3.2 to the spaces $W^{m,q}(\Omega)$, $\Omega \neq \mathbb{R}^n$. One of the most usual ways of doing this is to construct an (m,q)-*extension map for Ω*. By this we mean that there exists a linear operator $E : W^{m,q}(\Omega) \to W^{m,q}(\mathbb{R}^n)$ such that

(i) $u(x) = [E(u)](x)$, for all $x \in \Omega$
(ii) $\|E(u)\|_{m,q,\mathbb{R}^n} \le C\|u\|_{m,q,\Omega}$,

for some constant C independent of u. It is then not hard to show that inequalities (II.3.17) and (II.3.18) continue to hold in $W^{m,q}(\Omega)$. For instance, to prove (II.3.17) from (i) and (ii), we notice that

$$\|u\|_{r,\Omega} \le \|E(u)\|_{r,\mathbb{R}^n} \le c\,\|E(u)\|_{m,q,\mathbb{R}^n} \le c\,C\|u\|_{m,q,\Omega}.$$

Results on the existence of an extension map can be proved in a more or less complicated way, depending on the *smoothness* of the domain. In this regard, we shall now state a very general result due to Stein (1970, Chapter VI, Theorem 5; see also Triebel 1978, §§4.2.2, 4.2.3) on the existence of suitable extension maps called *universal* or *total* in that they do not depend on the order of differentiability and summability involved. Specifically, we have the following theorem whose rather deep proof will be omitted.

Theorem II.3.3 *Let Ω be locally Lipschitz.*[6] *Then, there exists an (m,q)-extension map for Ω, for all $q \in [1,\infty]$ and $m \ge 0$.*

On the other hand, results similar to those of Theorem II.3.3 can be proved in an elementary way, provided the domain is of class C^m (see, e.g., Lions 1962, Théorème 4.1, and Friedman 1969, Lemma 5.2). This is because, for such a domain, the boundary can be locally straightened by means of the smooth transformation:

$$y_i = x_i \quad \text{if } 1 \le i \le n-1, \quad y_n = x_n - \zeta(x_1,\dots,x_{n-1}).$$

The extension problem is then reduced to the same problem in $\mathbb{R}^n_+$, for which a simple solution is available, as shown by the following exercise.

Exercise II.3.10 For $x \in \mathbb{R}^n$, we put $x' = (x_1,\dots,x_{n-1})$. Let $u \in C_0^\infty(\overline{\mathbb{R}}^n_+)$ and set

$$\mathcal{E}u(x) = \begin{cases} u(x) & \text{if } x_n \ge 0 \\ \displaystyle\sum_{p=1}^{m+1} \lambda_p u(x', -px_n) & \text{if } x_n < 0 \end{cases}$$

where

$$\sum_{p=1}^{m+1} \lambda_p (-p)^\ell = 1, \quad \ell = 0,1,\dots,m.$$

Show that $\mathcal{E}u \in C_0^m(\mathbb{R}^n)$ and that, moreover, for all $q \in [1,\infty]$ and all $|\beta| \in [0,m]$

$$\|D^\beta \mathcal{E}u\|_{q,\mathbb{R}^n} \le C\|D^\beta u\|_{q,\mathbb{R}^n_+}.$$

Therefore, $\mathcal{E}$ can be extended to an operator $E : W^{m,q}(\mathbb{R}^n_+) \to W^{m,q}(\mathbb{R}^n)$, which is an (m,q)-extension map for $\mathbb{R}^n_+$.

[6] Actually, Stein's theorem applies to much more general domains (with bounded or unbounded boundary) and precisely to those which are "minimally smooth," see Stein (1970, Chapter VI, §3.3).

Exercise II.3.11 Let $u \in W_0^{m,q}(\Omega)$ and set

$$\widetilde{u}(x) = \begin{cases} u(x) \text{ if } x \in \Omega \\ \\ 0 \qquad \text{if } x \in \Omega^c. \end{cases}$$

Show that $\widetilde{u} \in W^{m,q}(\mathbb{R}^n)$.

On the strength of Theorem II.3.3 we thus have

Theorem II.3.4 *Suppose Ω locally Lipschitz. Then all conclusions in* Theorem II.3.2 *remain valid if we replace $W_0^{m,q}(\Omega)$ with $W^{m,q}(\Omega)$ for some constants $c_i = c_i(m, q, r, n, \Omega)$, $i = 1, 2, 3$.*

We wish to remark that, by using alternative methods due to Gagliardo (1958, 1959), one can show the results in Theorem II.3.4 under more general assumptions on Ω (see also Miranda 1978, §58).

Exercise II.3.12 Assume Ω locally Lipschitz. Use Theorem II.3.3 to show that, under the assumptions on r, q, and n stated in Lemma II.3.2 the following inequality holds for $u \in W^{1,q}(\Omega)$:

$$\|u\|_r \leq c\,\|u\|_q^{1-\lambda}\,\|u\|_{1,q}^{\lambda}, \tag{II.3.19}$$

where c is independent of u and $\lambda = n(r-q)/rq$.

Exercise II.3.13 Let $u : \Omega \to \mathbb{R}^n$ and let $\boldsymbol{e}$ be a given unit vector. For $h \neq 0$ the quantity

$$\Delta^h u(x) \equiv \frac{u(x + h\boldsymbol{e}) - u(x)}{h}$$

is called the *difference quotient of u along $\boldsymbol{e}$.* (a) Show that, if Ω' is any domain with $\overline{\Omega'} \subset \Omega$, the following properties hold for all $u \in W^{1,q}(\Omega)$:

(i) $\Delta^h u(x) \in L^q(\Omega')$, for all $h < \operatorname{dist}(\Omega', \Omega)$;
(ii) $\|\Delta^h u(x)\|_{q,\Omega'} \leq \|\nabla u\|_{q,\Omega}$;
(iii) If $\Omega \equiv \mathbb{R}^n_+$ and $\boldsymbol{e}$ is orthogonal to $\boldsymbol{e}_n$:

$$\|\Delta^h u(x)\|_{q,\mathbb{R}^n_+} \leq \|\nabla u\|_{q,\mathbb{R}^n_+}.$$

Hint: For a smooth function u and $\boldsymbol{e}$ parallel to $\boldsymbol{e}_i$ (say) it holds

$$\Delta^h u(x) = \frac{1}{h}\int_0^h D_i u(x_1, \ldots, x_i + \eta, \ldots, x_n)d\eta.$$

(b) Conversely, assume $u \in L^q(\Omega)$ and that for all Ω' with $\overline{\Omega'} \subset \Omega$ and for all $h < \operatorname{dist}(\Omega', \Omega)$ it holds $\|\Delta^h u\|_{q,\Omega'} \leq C$, with a constant C independent of Ω' and h. Then if $\boldsymbol{e}$ is parallel to $\boldsymbol{e}_i$, show that

(iv) $D_i u$ exists;
(v) $\|D_i u\|_{q,\Omega} \leq C$.

We wish to end this section by recalling a useful characterization of the normed dual space $(W_0^{m,q}(\Omega))'$ of the space $W_0^{m,q}(\Omega)$. An analogous result can be given for $W^{m,q}(\Omega)$. A functional ℓ on $W_0^{m,q}(\Omega)$ belongs to $(W_0^{m,q}(\Omega))'$ if and only if

$$\|\ell\|_{(W_0^{m,q}(\Omega))'} \equiv \sup_{\|u\|_{m,q}=1} |\ell(u)| < \infty.$$

Let us consider in $(W_0^{m,q}(\Omega))'$ the subspace constituted by functionals $\mathcal{F}$ of the form

$$\mathcal{F}(u) = (f, u)\,, \quad f \in L^{q'}(\Omega). \tag{II.3.20}$$

Clearly, $\mathcal{F} \in (W_0^{m,q}(\Omega))'$. Setting

$$\|f\|_{-m,q'} = \sup_{u \in W_0^{m,q}(\Omega); \|u\|_{m,q}=1} |\mathcal{F}(u)|, \tag{II.3.21}$$

we easily recognize that (II.3.21) is a norm in $L^{q'}(\Omega)$, and that the following inequalities hold:

$$\begin{aligned} &\|f\|_{-m,q'} \leq \|f\|_{q'} \\ &|\mathcal{F}(u)| \leq \|f\|_{-m,q'} \|u\|_{m,q}. \end{aligned} \tag{II.3.22}$$

Let us denote by $W_0^{-m,q'}(\Omega)$ the *negative Sobolev space* of order $(-m, q')$, obtained by completing $L^{q'}(\Omega)$ in the norm (II.3.21). The following result due to Lax (1955, §2) ensures that for $q \in (1, \infty)$ the two spaces $W_0^{-m,q'}(\Omega)$ and $(W_0^{m,q}(\Omega))'$ can be identified (see also Miranda 1978, §57).

Theorem II.3.5 *The spaces $W_0^{-m,q'}(\Omega)$ and $(W_0^{m,q}(\Omega))'$, $1 < q < \infty$, are isomorphic.*

Throughout this book the value of a functional $\mathcal{F} \in W_0^{-m,q'}(\Omega)$ at $u \in W_0^{m,q}(\Omega)$ will be denoted by

$$\langle \mathcal{F}, u \rangle \quad (\textit{duality pairing}).$$

If, in particular, $\mathcal{F} \in L^{q'}(\Omega)$, we have $\langle \mathcal{F}, u \rangle = (\mathcal{F}, u)$.

Remark II.3.3 A characterization completely similar to that of Theorem II.3.5 can be given also for the space $(W^{m,q}(\Omega))'$. Precisely, denoting by $W^{-m,q'}(\Omega)$ the completion of $L^{q'}(\Omega)$ in the norm

$$\|f\|^*_{-m,q'} = \sup_{u \in W^{m,q}(\Omega); \|u\|_{m,q}=1} |\mathcal{F}(u)|\,,$$

with $\mathcal{F}(u)$ defined in (II.3.20), one shows that $W^{-m,q'}(\Omega)$ and $(W^{m,q}(\Omega))'$, $1 < q < \infty$, are isomorphic; see Miranda *loc. cit.* Notice that, obviously,

$$\|f\|_{-m,q'} \leq \|f\|^*_{-m,q'}\,.$$

■

II.4 Boundary Inequalities and the Trace of Functions of $W^{m,q}$

As a next problem, we wish to investigate if, analogously to what happens for smooth functions, it is possible to ascribe a value at the boundary (the *trace*) to functions in $W^{m,q}(\Omega)$. If Ω is sufficiently regular, the considerations developed in the preceding section assure that this is certainly true if $mq > n$, since, in such a case, every function from $W^{m,q}(\Omega)$ can be redefined on a set of zero measure in such a way that it becomes (at least) continuous up to the boundary. However, if $mq \le n$ we can nevertheless prove some inequalities relating $W^{m,q}$-norms of a smooth function with L^r-norms of the same function at the boundary, which will allow us to define, in a suitable sense, the trace of a function belonging to any Sobolev space of order (m,q), $m \ge 1$. To this end, given a sufficiently smooth domain with a bounded boundary (locally Lipschitz, say) we denote by $L^q(\partial\Omega)$, $1 \le q \le \infty$ the space of (equivalence classes of) real functions u defined on $\partial\Omega$ and such that

$$\|u\|_{q,\partial\Omega} \equiv \left(\int_{\partial\Omega} |u|^q d\sigma\right)^{1/q} < \infty, \quad 1 \le q < \infty,$$

$$\|u\|_{\infty,\partial\Omega} \equiv \operatorname*{ess\,sup}_{\partial\Omega} |u| < \infty, \quad q = \infty,$$

where σ denotes the Lebesgue $(n-1)$-dimensional measure.[1] It can be proved that the space $L^q(\partial\Omega)$ enjoys all the relevant functional properties of the spaces $L^q(\Omega)$. In particular, it is a Banach space with respect to the norm $\|\cdot\|_{q,\partial\Omega}$, $1 \le q \le \infty$, which is separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$ (see Miranda 1978, §60).

In order to accomplish our objective, we need some preliminary considerations and results that we shall next describe.

We shall often use the classical Gauss divergence theorem for smooth vector functions. It is well known that this theorem certainly holds if the domain is (piecewise) of class C^1. However, we need to consider more general situations and, in this respect, we quote the following result of Nečas (1967, Chapitre 2, Lemme 4.2 and Chapitre 3, Théorème 1.1).

Lemma II.4.1 *Let Ω be a bounded, locally Lipschitz domain in $\mathbb{R}^n$. Then the unit outer normal $\boldsymbol{n}$ exists almost everywhere on $\partial\Omega$ (see* Lemma II.1.2*) and the following identity holds*

$$\int_\Omega \nabla \cdot \boldsymbol{u} = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n},$$

for all vector fields $\boldsymbol{u}$ with components in $C^1(\overline{\Omega})$.

---

[1] As usual, if no confusion arises, the infinitesimal surface element $d\sigma$ in the integral will be omitted.

A generalization of this result to functions from $W^{1,q}(\Omega)$ will be considered in Exercise II.4.3.

We are now in a position to perform a study on the traces of functions from $W^{m,q}$. Let Ω' be a locally Lipschitz, star-shaped domain (with respect to the origin) and let u be an arbitrary function from $C_0^\infty(\overline{\Omega'})$. From the identity

$$|u|^r D_j x_j = D_j(x_j|u|^r) - x_j D_j|u|^r, \quad r \in [1,\infty)$$

and Lemma II.4.1 we easily deduce

$$\int_{\partial\Omega'} \boldsymbol{x}\cdot\boldsymbol{n}|u|^r \le n\|u\|_{r,\Omega'}^r + r\delta(\Omega')\int_{\partial\Omega'}|u|^{r-1}|\nabla u|. \tag{II.4.1}$$

Using the Hölder inequality in the last integral in (II.4.1) and noting that

$$\operatorname*{ess\,inf}_{x\in\partial\Omega'}(\boldsymbol{x}\cdot\boldsymbol{n}(x)) \equiv c > 0$$

(see Exercise II.1.4), we obtain

$$\|u\|_{r,\partial\Omega'}^r \le (n/c)\|u\|_{r,\Omega'}^r + (r\delta(\Omega')/c)\|u\|_{q'(r-1),\Omega'}^{r-1}\|\nabla u\|_{q,\Omega'}. \tag{II.4.2}$$

We now choose $r \in [q,(n-1)q/(n-q)]$, if $q<n$, and arbitrary $r \ge q$, if $q \ge n$. Observing that $r \le q'(r-1)$, in the light of Exercise II.3.12 (see (II.3.19)), inequality (II.4.2) then furnishes for all $u \in C_0^\infty(\overline{\Omega'})$

$$\begin{aligned}\|u\|_{r,\partial\Omega'} &\le C\left(\|u\|_{q,\Omega'}^{r(1-\lambda)}\|u\|_{1,q,\Omega'}^{r\lambda} + \|u\|_{q,\Omega'}^{(r-1)(1-\lambda)}\|u\|_{1,q,\Omega'}^{1+\lambda(r-1)}\right)^{1/r}\\ &\le 2^{1/r}C\left(\|u\|_{q,\Omega'}^{1-\lambda}\|u\|_{1,q,\Omega'}^{\lambda} + \|u\|_{q,\Omega'}^{(1-\frac1r)(1-\lambda)}\|u\|_{1,q,\Omega'}^{\frac1r+\lambda(1-\frac1r)}\right)\end{aligned} \tag{II.4.3}$$

where $\lambda = n(r-q)/q(r-1)$, $C = C(n,r,q,\Omega')$, and where we used (II.3.3).

Employing Lemma II.1.3 and Lemma II.1.4, we can now establish (II.4.3) for an arbitrary locally Lipschitz domain Ω. In fact, let $\mathcal{G} = \{G_1,\dots,G_N\}$ be the open covering of $\partial\Omega$ constructed in Lemma II.1.3 and let $\{\psi_i\}$ be a partition of unity in $\partial\Omega$ subordinate to $\mathcal{G}$. Setting $\Omega_i = \Omega\cap G_i$, for $u \in C_0^\infty(\overline{\Omega})$, we have

$$\|u\|_{r,\partial\Omega} = \|\sum_{i=1}^N \psi_i u\|_{r,\partial\Omega} \le \sum_{i=1}^N \|u\|_{r,\partial\Omega\cap G_i} \le \sum_{i=1}^N \|u\|_{r,\partial\Omega_i}\,,$$

and therefore, using in this inequality (II.4.3) with $\Omega' \equiv \Omega_i$, we deduce

$$\|u\|_{r,\partial\Omega} \le 2^{1/r}NC\left(\|u\|_{q,\Omega}^{(1-\lambda)}\|u\|_{1,q,\Omega}^{\lambda} + \|u\|_{q,\Omega}^{(1-\frac1r)(1-\lambda)}\|u\|_{1,q,\Omega}^{\frac1r+\lambda(1-\frac1r)}\right). \tag{II.4.4}$$

Let now Ω be locally Lipschitz, and denote by γ the linear map which to every function $f \in C_0^\infty(\overline{\Omega})$ associates its value at the boundary $\gamma(f) = f|_{\partial\Omega}$, and let $u \in W^{1,q}(\Omega)$. By Theorem II.3.1, there is a sequence $\{f_k\} \subset C_0^\infty(\overline{\Omega})$ converging to u in $W^{1,q}(\Omega)$. On the other hand, by (II.4.4) this sequence will also converge in $L^r(\partial\Omega)$, for suitable r, to a function $\widetilde{u} \in L^r(\partial\Omega)$. Since, as can be easily shown, $\widetilde{u}$ does not depend on the particular sequence, the map γ can be uniquely extended, by continuity, to a map from $W^{1,q}(\Omega)$ into $L^r(\partial\Omega)$ that ascribes, in a well-defined sense, to every function from $W^{1,q}(\Omega)$ a function on the boundary which, for smooth functions u, reduces to the usual trace $u|_{\partial\Omega}$. This result can be fairly generalized to spaces $W^{m,q}$ with $m > 1$. In fact, from Theorem II.3.4 and an iterative argument based on (II.4.4), we obtain the following result whose proof is left to the reader as an exercise.

Theorem II.4.1 *Let Ω be locally Lipschitz. Assume*

$$r \in [q, q(n-1)/(n-mq)]\,, \textit{if } mq < n,$$

$$r \in [q, \infty)\,, \textit{if } mq \geq n.$$

Then there exists a unique, continuous linear map γ from $W^{m,q}(\Omega)$, $1 \leq q < \infty$, $m \geq 1$, into $L^r(\partial\Omega)$ such that for all $u \in C_0^\infty(\overline{\Omega})$ it is $\gamma(u) = u\,|_{\partial\Omega}$. Furthermore, for $m = 1$ the following inequality holds

$$\|\gamma(u)\|_{r,\partial\Omega} \leq C\left(\|u\|_{q,\Omega}^{(1-\lambda)}\|u\|_{1,q,\Omega}^{\lambda} + \|u\|_{q,\Omega}^{(1-\frac{1}{r})(1-\lambda)}\|u\|_{1,q,\Omega}^{\frac{1}{r}+\lambda(1-\frac{1}{r})}\right), \quad \text{(II.4.5)}$$

where $C = C(n, r, q, \Omega)$ and $\lambda = n(r-q)/q(r-1)$.

Exercise II.4.1 Let Ω be locally Lipschitz. Starting from (II.4.5), show that for any $\varepsilon > 0$, there exists $C = C(n, r, q, \Omega, \varepsilon) > 0$ such that

$$\|\gamma(u)\|_{r,\partial\Omega} \leq C\|u\|_{q,\Omega} + \varepsilon\|\nabla u\|_{q,\Omega}\,,$$

with the exponents q and r subject to the restrictions stated in Theorem II.4.1. *Hint*: Use (II.2.5).

Theorem II.4.1 allows us to define, in a natural way, higher-order traces. Actually, since for $u \in W^{m,q}(\Omega)$ we have $D^\alpha u \in W^{m-\ell,q}(\Omega)$ for $0 \leq |\alpha| \leq \ell < m$, the trace of $D^\alpha u$ is well defined and, moreover, it belongs to $L^r(\partial\Omega)$ for suitable exponents $r \geq 1$. In particular, if Ω is sufficiently regular, we can give a precise meaning to the ℓth *normal derivative on* $\partial\Omega$:

$$\frac{\partial^\ell u}{\partial n^\ell} \equiv \sum_{|\alpha|=\ell} n^\alpha D^\alpha u, \quad n^\alpha = n_1^{\alpha_1} n_2^{\alpha_2} \ldots n_n^{\alpha_n},$$

of every function $u \in W^{m,q}(\Omega)$, $m > \ell \geq 0$. Thus, noticing that $n^\alpha \in L^\infty(\partial\Omega)$, we can construct a linear map

$$\Gamma_{(m)} : W^{m,q}(\Omega) \to [L^r(\partial\Omega)]^m \quad \text{(II.4.6)}$$

with

$$\mathit{\Gamma}_{(m)}(u) = \left(u \equiv \gamma_0(u), \frac{\partial u}{\partial n} \equiv \gamma_1(u), \ldots, \frac{\partial^{m-1} u}{\partial n^{m-1}} \equiv \gamma_{m-1}(u) \right). \tag{II.4.7}$$

Obviously, if $u \in W_0^{m,q}(\Omega)$, $\mathit{\Gamma}_m(u) \equiv 0$ *a.e. on* $\partial\Omega$. The converse result also holds and we have (see Nečas 1967, Chapitre 2, Théorème 4.10, 4.12, 4.13).

Theorem II.4.2 *Let Ω be locally Lipschitz if $m = 1, 2$ and of class $C^{m,1}$ if $m \geq 3$. Assume*

$$u \in W^{m,q}(\Omega), \quad 1 \leq q < \infty, \quad m \geq 1,$$

with $\mathit{\Gamma}_m(u) \equiv 0$ a.e on $\partial\Omega$. Then $u \in W_0^{m,q}(\Omega)$.

A more complicated study, which is nonetheless fundamental for solving nonhomogeneous boundary-value problems, is that of determining to which Banach space $\mathcal{B} \subseteq [L^r(\partial\Omega)]^m$ a function $w \equiv (w_0, w_1, \ldots, w_{m-1})$ must belong in order to be considered the trace, via the mapping $\mathit{\Gamma}_{(m)}$, of a function in $W^{m,q}(\Omega)$, i.e., $\gamma_\ell(u) = w_\ell$, for some $u \in W^{m,q}(\Omega)$, for all $\ell = 0, 1, \ldots, m-1$. A counterexample due to J. Hadamard shows that $\mathcal{B}$ is, in general, *strictly* contained in $[L^r(\partial\Omega)]^m$, whatever $r \geq 1$ (Sobolev 1963a, Chapter 2, §5; De Vito 1958). Here we shall only briefly describe the answer to the problem, referring the reader to Gagliardo (1957) and Nečas (1967, Chapitre 2, §§4,5) for a fully detailed description of it. Let us first consider the case $m = 1$. Denote by $W^{1-1/q,q}(\partial\Omega)$ the subspace of $L^q(\partial\Omega)$ constituted by functions u for which the following functional is finite:

$$\|u\|_{1-1/q,q(\partial\Omega)} \equiv \|u\|_{q,\partial\Omega} + \langle\langle u \rangle\rangle_{1-1/q,q}, \tag{II.4.8}$$

where

$$\langle\langle u \rangle\rangle_{1-1/q,q} \equiv \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(y) - u(y')|^q}{|y - y'|^{n-2+q}} d\sigma_y d\sigma_{y'} \right)^{1/q}. \tag{II.4.9}$$

It can be proved (Miranda 1978, §61) that $W^{1-1/q,q}(\partial\Omega)$ is a dense subset of $L^q(\partial\Omega)$ and that it is complete in the norm $\|u\|_{1-1/q,q(\partial\Omega)}$. Furthermore, it is separable for $q \in [1, \infty)$ and reflexive for $q \in (1, \infty)$, and, for Ω smooth enough, the class of smooth functions on $\partial\Omega$ is dense in $W^{1-1/q,q}(\partial\Omega)$. We have the following theorem of Gagliardo (1957), which characterizes the trace operator γ.

Theorem II.4.3 *Let Ω be locally Lipschitz and let $q \in (1, \infty)$. If $u \in W^{1,q}(\Omega)$, then $\gamma(u) \in W^{1-1/q,q}(\partial\Omega)$ and*

$$\|\gamma(u)\|_{1-1/q,q(\partial\Omega)} \leq c_1 \|u\|_{1,q,\Omega}. \tag{II.4.10}$$

Conversely, given $w \in W^{1-1/q,q}(\partial\Omega)$, there exists $u \in W^{1,q}(\Omega)$ with $\gamma(u) = w$ such that

$$\|u\|_{1,q,\Omega} \leq c_2 \|\gamma(u)\|_{1-1/q,q(\partial\Omega)}. \tag{II.4.11}$$

The constants $c_i, i = 1, 2$, depend only on n, q, and Ω.

Since, by Theorem II.4.2, we have, for Ω locally Lipschitz, $u_1, u_2 \in W^{1,q}(\Omega)$ with $\gamma(u_1) = \gamma(u_2)$ then $u_1 - u_2 \in W_0^{1,q}(\Omega)$, Gagliardo's theorem can be equivalently stated by saying: *The trace operator γ is a linear bounded bijective operator from the quotient space $W^{1,q}(\Omega) \Big/ W_0^{1,q}(\Omega)$ onto the space $W^{1-1/q,q}(\partial\Omega)$.*

Remark II.4.1 Gagliardo proved this result by making a clever use of two elementary inequalities due to G. H. Hardy and C. B. Morrey, respectively. Though the proof of Theorem II.4.3 is well beyond the scope of this monograph, we may wish nevertheless to sketch a demonstration of (II.4.10) in the case when Ω is the square

$$S = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\right\}.^2$$

We begin to notice that, in view of Theorem II.4.1, it suffices to show that the double surface integral in (II.4.7) is bounded above by the norm of u in $W^{1,q}(S)$, i.e.,

$$\begin{aligned} &\int_0^1\int_0^1 \left|\frac{u(0,y)-u(0,y')'}{y-y}\right|^q dy\,dy' + \int_0^1\int_0^1 \left|\frac{u(1,y)-u(1,y')}{y-y'}\right|^q dy\,dy' \\ &+\int_0^1\int_0^1 \left|\frac{u(x,0)-u(x',0)}{x-x'}\right|^q dx dx' + \int_0^1\int_0^1 \left|\frac{u(x,1)-u(x',1)}{x-x'}\right|^q dx\,dx' \\ &\qquad\qquad \leq C\,\|u\|_{1,q,S}^q \end{aligned} \tag{II.4.12}$$

with a constant C independent of u. By Theorem II.3.1, we can assume $u \in C_0^\infty(\overline{S})$. Consider the first integral on the left-hand side of (II.4.11) and denote it by $\mathcal{I}$. Making the change of variables

$$\xi = x + y, \quad \eta = y - x,$$

(a rotation of an angle $\pi/4$) we may write

$$\mathcal{I} = \int_0^1\int_0^1 \left|\frac{U(\eta,\eta)-U(\eta,\eta')}{\eta-\eta'}\right|^q d\eta d\eta',$$

where

$$U(\xi,\eta) \equiv u\left(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\right).$$

Setting

$$\phi(\eta) = U(\eta,\eta)$$

[2] In fact, following Gagliardo, it is not difficult to prove that the case of a general locally Lipschitz domain can be reduced to the present one.

for $0 \le \eta' < \eta \le 1$ we have

$$\frac{|\phi(\eta) - \phi(\eta')|}{\eta - \eta'} \le \frac{1}{\eta - \eta'} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \lambda}(\lambda, \eta') \right| d\lambda + \frac{1}{\eta - \eta'} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \mu}(\eta, \mu) \right| d\mu$$

and thus, by (II.3.3),

$$\begin{aligned} f(\eta, \eta') \equiv \left| \frac{\phi(\eta) - \phi(\eta')}{\eta - \eta'} \right|^q \le 2^{q-1} \Bigg\{ \left[\frac{1}{|\eta - \eta'|} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \lambda}(\lambda, \eta') \right| d\lambda \right]^q \\ + \left[\frac{1}{|\eta - \eta'|} \int_{\eta'}^{\eta} \left| \frac{\partial U}{\partial \mu}(\eta, \mu) \right| d\mu \right]^q \Bigg\}. \end{aligned} \tag{II.4.13}$$

We now recall the following inequalities due to G.H. Hardy (Hardy, Littlewood and Polya 1934, p. 240):

$$\begin{aligned} \int_a^b dx \left| \frac{1}{x - a} \int_a^x f(t) dt \right|^q &\le \left(\frac{q}{q-1} \right)^q \int_a^b |f(t)|^q dt, \quad x > a, \ q > 1 \\ \int_a^b dx \left| \frac{1}{b - x} \int_x^b f(t) dt \right|^q &\le \left(\frac{q}{q-1} \right)^q \int_a^b |f(t)|^q dt, \quad x < b, \ q > 1. \end{aligned} \tag{II.4.14}$$

Integrating (II.4.13) first in $\eta \in (\eta', 1]$ and then in $\eta' \in [0, 1]$ and using (II.4.14) we obtain

$$\begin{aligned} \int_0^1 \left(\int_{\eta'}^1 f(\eta, \eta') d\eta \right) d\eta' &\le 2^{q-1} \left(\frac{q}{q-1} \right)^q \left[\int_0^1 d\eta' \int_{\eta'}^1 \left| \frac{\partial U}{\partial \lambda}(\lambda, \eta') \right|^q d\lambda \right. \\ &\qquad \left. + \int_0^1 d\eta \int_0^{\eta} \left| \frac{\partial U}{\partial \mu}(\eta, \mu) \right|^q d\mu \right] \\ &\le c \|\nabla u\|_{q,S}^q, \end{aligned} \tag{II.4.15}$$

with c a suitable constant. Interchanging the roles of η and η' in (II.4.15) and noticing that $f(\eta, \eta') = f(\eta', \eta)$ one also has

$$\int_0^1 \left(\int_{\eta}^1 f(\eta, \eta') d\eta' \right) d\eta \le c \|\nabla u\|_{q,S}^q. \tag{II.4.16}$$

Adding (II.4.15) and (II.4.16) we find

$$\mathcal{I} \le 2c \|\nabla u\|_{q,S}^q.$$

Since the other integrals on the left-hand side of (II.4.12) can be analogously increased, the proof of (II.4.12) is accomplished. ■

Exercise II.4.2 According to the method just described, the case $q = 1$ of Theorem II.4.3 is excluded because Hardy's inequalities (II.4.14) hold if $q > 1$. Show, by means of a counterexample, that (II.4.14) does not hold when $q = 1$. *Hint* (Gagliardo 1957): Take $f(t) = (t-a)^{-1}(\log(t-a))^{-2}$. (For the characterization of the trace when $m = q = 1$, see Gagliardo (1957, Teorema 1.II)).

The extension of Theorem II.4.3 to the space $W^{m,q}(\Omega)$, $m \geq 2$, is formally analogous, provided we introduce a suitable generalization of the space $W^{1-1/q,q}(\partial\Omega)$. To this end, assume Ω of class $C^{m-1,1}$ and let $\{B_k\}$ and $\{\zeta_k\}$, $k = 1, 2, \dots, s$, be a family of open balls centered at $x_k \in \partial\Omega$ with $\partial\Omega \subset B_k$, and of functions of class $C^{m-1,1}(\overline{D}_k)$, respectively, defining the $C^{m-1,1}$ – regularity of $\partial\Omega$ in the sense of Definition II.1.1. Assuming that

$$x_n^{(k)} = \zeta_k(x_1^{(k)}, \dots, x_{n-1}^{(k)}), \quad (x_1^{(k)}, \dots, x_{n-1}^{(k)}) \in D_k$$

is the representation of $\partial\Omega \cap B_k$, for a function u on $\partial\Omega$ we set

$$u_k = u(x_1^{(k)}, \dots, x_{n-1}^{(k)}, \zeta_k(x_1^{(k)}, \dots, x_{n-1}^{(k)}))$$

and define

$$\|u\|_{m-1/q,q(\partial\Omega)} \equiv \sum_{k=1}^{s} \|u_k\|_{m-1/q,q,D_k} \tag{II.4.17}$$

where

$$\|u_k\|_{m-1/q,q,D_k} \equiv \sum_{0\leq|\alpha|\leq m-1} \|D^\alpha u_k\|_{q,D_k} + \langle\langle u_k \rangle\rangle_{m-1/q,q}$$

$$\langle\langle u_k \rangle\rangle_{m-1/q,q} \equiv \sum_{|\alpha|=m-1} \left(\int_{D_k} \int_{D_k} \frac{|D^\alpha u(y) - D^\alpha u(y')|^q}{|y-y'|^{n-2+q}} dy dy' \right)^{1/q}. \tag{II.4.18}$$

We next denote by $W^{m-1/q,q}(\partial\Omega)$ the linear space of functions u for which the functional defined by (II.4.17)–(II.4.18) is finite. It can be shown that the definition of $W^{m-1/q,q}(\partial\Omega)$ does not depend on the particular choice of the local representation $\{B_k\}$, $\{\zeta_k\}$ of the boundary. In fact, if $\{B'_{k'}\}$, $\{\zeta'_{k'}\}$ is another such a representation and $\|u\|'_{m-1/q,q(\partial\Omega)}$ is the corresponding functional associated to u, there exist constants c_1, $c_2 > 0$ such that

$$\|u\|_{m-1/q,q(\partial\Omega)} \leq c_1 \|u\|'_{m-1/q,q(\partial\Omega)} \leq c_2 \|u\|_{m-1/q,q(\partial\Omega)}$$

(Nečas 1967, Chapitre 3, Lemme 1.1). As in the case of $W^{1-1/q,q}(\partial\Omega)$, one shows that the space $W^{m-1/q,q}(\partial\Omega)$ is a dense subset of $L^q(\partial\Omega)$, which is complete in the norm (II.4.17)–(II.4.17), separable for $q \in [1, \infty)$ and reflexive for $q \in (1, \infty)$ (Nečas 1967, Chapitre 2, Proposition 3.1).

Set

$$\mathcal{W}_{m,q}(\partial\Omega) \equiv W^{m-1/q,q}(\partial\Omega) \times W^{m-1-1/q,q}(\partial\Omega) \times \dots \times W^{1-1/q,q}(\partial\Omega).$$

We then have the following characterization of the trace operator $\Gamma_{(m)}$ defined in (II.4.6)–(II.4.7) (Nečas 1967, Chapitre 2, Théorème 5.5, 5.8).

Theorem II.4.4 *Let Ω be of class $C^{m-1,1}$, $m \geq 2$. If*

$$u \in W^{m,q}(\Omega), \quad 1 < q < \infty,$$

then

$$\Gamma_{(m)}(u) \in \mathcal{W}_{m,q}(\partial\Omega)$$

and for all $\ell = 0, 1, \ldots, m-1$ it is

$$\|\gamma_\ell(u)\|_{m-\ell-1/q,q(\partial\Omega)} \leq c_1 \|u\|_{m,q,\Omega}. \tag{II.4.19}$$

Conversely, if Ω is of class $C^{m,1}$, given

$$w \in \mathcal{W}_{m,q}(\partial\Omega)$$

there exists $u \in W^{m,q}(\Omega)$ with

$$\Gamma_{(m)}(u) = w$$

and the following inequality holds

$$\|u\|_{m,q,\Omega} \leq c_2 \sum_{\ell=0}^{m-1} \|\gamma_\ell(u)\|_{m-\ell-1/q,q(\partial\Omega)}. \tag{II.4.20}$$

The constants c_i, $i = 1, 2$, depend only on n, m, q, and Ω.

As in the case of the operator γ, the operator $\Gamma_{(m)}$ can also be characterized, in view of Theorem II.4.2 and Theorem II.4.4, as a bounded linear bijection of $W^{m,q}(\Omega)/W_0^{m,q}(\Omega)$ onto $\mathcal{W}_{m,q}(\partial\Omega)$ (topologized in the obvious way).

Remark II.4.2 If Ω is not globally smooth but has a smooth boundary portion σ, we can still define the trace on σ of functions from $W^{m,q}(\Omega)$ and the space $\mathcal{W}_{m,q}(\sigma)$. In particular, inequality (II.4.19) continues to hold with σ in place of $\partial\Omega$ (see Nečas, *loc. cit.*). ■

Remark II.4.3 Problems of trace on the plane $\{x_n = 0\}$ for functions defined in $\mathbb{R}^{n-1}$ will be considered in Section II.10. ■

Exercise II.4.3 (Nečas 1967, Chapitre 3, Théorème 1.1)**.** Let Ω be bounded and locally Lipschitz. Show the following Gauss identity:

$$\int_\Omega \Phi \nabla \cdot \boldsymbol{u} = \int_{\partial\Omega} \Phi\, \boldsymbol{u} \cdot \boldsymbol{n} - \int_\Omega \boldsymbol{u} \cdot \nabla \Phi \tag{II.4.21}$$

for all vectors $\boldsymbol{u}$ with components in $W^{1,q}(\Omega)$ and scalars Φ from $W^{1,r}(\Omega)$ where q and r satisfy

(i) $q^{-1} + r^{-1} \leq (n+1)/n$ if $1 \leq q < n$, $1 \leq r < n$;
(ii) $r > 1$ if $q \geq n$;
(iii) $q > 1$ if $r \geq n$;

Hint: Use Lemma II.4.1 and Theorem II.3.3 and Theorem II.4.1.

Remark II.4.4 An extension of (II.4.21) to functions $\boldsymbol{u}$ with less regularity than that required in Exercise II.4.3 will be given in Section III.2, see (III.2.14). ■

II.5 Further Inequalities and Compactness Criteria in $W^{m,q}$

We begin to prove some inequalities relating the L^q-norm of a function with that of its first derivatives (Poincaré 1894, §III, and Friedrichs 1933). Throughout this section we shall denote by L_d a layer of width $d > 0$, namely

$$L_d = \{x \in \mathbb{R}^n : -d/2 < x_n < d/2\}.$$

Theorem II.5.1 *Assume $\Omega \subset L_d$, for some $d > 0$. Then, for all $u \in W_0^{1,q}(\Omega)$, $1 \le q \le \infty$,*

$$\|u\|_q \le (d/2)\|\nabla u\|_q. \tag{II.5.1}$$

Proof. It is enough to show the theorem for $u \in C_0^\infty(\Omega)$. For such functions one has

$$|u(x_1, \dots, x_n)| = \left| \int_{-d/2}^{x_n} \frac{\partial u(x_1, \dots, \xi)}{\partial \xi}\, d\xi \right| = \left| \int_{x_n}^{d/2} \frac{\partial u(x_1, \dots, \xi)}{\partial \xi}\, d\xi \right|,$$

which implies

$$|u(x)| \le (1/2) \int_{-d/2}^{d/2} |\nabla u|\, dx_n\,. \tag{II.5.2}$$

From this relation we at once recover (II.5.1) for $q = \infty$. If $q \in [1, \infty)$, employing the Hölder inequality in the right-hand side of (II.5.2) yields

$$|u(x)|^q \le (d^{q-1}/2^q) \int_{-d/2}^{d/2} |\nabla u|^q dx_n$$

which, after integrating over L_d, proves (II.5.1). □

Exercise II.5.1 Inequality (II.5.1) fails, in general, if Ω is not contained in some layer L_d. Suppose, for instance, $\Omega \equiv \mathbb{R}^n$ and consider the sequence

$$u_m = \exp[-|x|/(m+1)],\ m \in \mathbb{N}.$$

Show that

$$\frac{\|u_m\|_q}{\|\nabla u_m\|_q} = \frac{m+1}{q}.$$

Modify this example to prove the invalidity of (II.5.1) for Ω an arbitrary exterior domain or a half-space.

The special case $q = 2$ in (II.5.1) plays an important role in several applications. In particular, it is of great interest in uniqueness and stability questions to determine the smallest constant μ such that

$$\|u\|_2^2 \le \mu \|\nabla u\|_2^2. \tag{II.5.3}$$

The constant μ (sometimes called the *Poincaré constant*) depends on the domain Ω, and when Ω is bounded one easily shows that $\mu = 1/\lambda_1$, where λ_1 is the smallest eigenvalue of the problem

$$-\Delta u = \lambda u \ \text{ in } \Omega\,, \qquad u = 0 \ \text{ at } \partial\Omega\,; \tag{II.5.4}$$

see Sobolev 1963a, Chapter II, §16. An estimate of λ_1 comes from (II.5.1) and one has

$$\lambda_1 \ge 4/[\delta(\Omega)]^2.$$

However, a better estimate can be obtained as a consequence of the following simple argument due to E. Picard (Picone 1946, §160).[1] In fact, assume as before $\Omega \subset L_d$ for some $d > 0$ and consider the function

$$U(x) = \frac{u(x)}{\sin[\pi(x_n + d/2)/d]}, \quad u \in C_0^\infty(\Omega).$$

Since $U(x)$ is bounded in L_d and vanishes at $-d/2$, $d/2$, integrating by parts we find

$$0 \le \int_{-d/2}^{d/2} \left\{ \frac{\partial u}{\partial x_n} - \frac{\pi}{d} u(x) \cot\left[\frac{\pi(x_n + d/2)}{d}\right] \right\}^2 dx_n = \int_{-d/2}^{d/2} \left(\frac{\partial u}{\partial x_n}\right)^2 dx_n$$
$$- \frac{\pi^2}{d^2} \int_{-d/2}^{d/2} u^2 \left\{ \sin^{-2}\left[\frac{\pi(x_n + d/2)}{d}\right] - \cot^2\left[\frac{\pi(x_n + d/2)}{d}\right] \right\} dx_n\,.$$

Hence

$$\int_{-d/2}^{d/2} u^2 dx_n \le (d/\pi)^2 \int_{-d/2}^{d/2} \left(\frac{\partial u}{\partial x_n}\right)^2 dx_n,$$

which implies

$$\|u\|_2 \le (d/\pi)\|\nabla u\|_2.$$

Therefore, one deduces

$$\mu \le d^2/\pi^2$$

and, if Ω is bounded,

$$\mu \le [\delta(\Omega)/\pi]^2.$$

Notice that these estimates are sharp in the sense that when $n = 1$ and $\Omega = L_d$ we have from (II.5.4) $\mu^{-1} = \lambda_1 = [\pi/\delta(\Omega)]^2 = (\pi/d)^2$.

Generalizations of (II.5.1) and (II.5.3) are considered in the following exercises.

[1] This proof was brought to my attention by Professor Luigi Pepe.

Exercise II.5.2 Let $\Omega \subset \{x \in \mathbb{R}^n : -d/2 < x_i < d/2,\ i = 1, \dots, n\}$. Use Picard's argument to show the following estimate for the Poincaré constant μ:

$$\mu \le d^2/n\pi^2.$$

Exercise II.5.3 Let $\Omega \subset L_d$, for some $d > 0$. Show that

$$\|\nabla u\|_2 \le (d/\pi)\|\Delta u\|_2$$

for all $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Thus, in particular,

$$\|u\|_2 \le (d/\pi)^2\|\Delta u\|_2.$$

Hint: Consider the identity: $(u, \Delta u) = -\|\nabla u\|_2$.

Exercise II.5.4 Let Ω be of finite measure and let $u \in W_0^{1,q}(\Omega)$, $1 \le q < \infty$. Show the inequality

$$\|u\|_q \le \beta|\Omega|^{1/n}\|\nabla u\|_q \tag{II.5.5}$$

where

$$\beta = \begin{cases} \dfrac{q(n-1)}{2(n-q)\sqrt{n}} & \text{if } q < n \\[2ex] \dfrac{q}{2\sqrt{n}} & \text{if } q \ge n\,. \end{cases}$$

Hint: Use (II.3.5) and the inequality

$$\|u\|_q \le |\Omega|^{(1/q)-(1/r)}\|u\|_r, \quad r > q.$$

Exercise II.5.5 Let Ω be bounded and let $u \in W_0^{1,q}(\Omega)$, $q > n$. Show that, for all $q_1 \in (n, q)$, the following inequality holds

$$\|u\|_C \le c\,\|u\|_q^{1-q/q_1}\,\|\nabla u\|_q^{q/q_1}\,,$$

with $c = c(n, q, q_1, \Omega)$. *Hint*: From (II.3.18) and (II.5.1) we find $\|u\|_C \le c\,\|\nabla u\|_q$.

Exercise II.5.6 Let Ω be bounded and C^1-smooth, and let $\boldsymbol{u}$ be a vector function with components in $W^{1,q}(\Omega)$, $1 \le q < \infty$, and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ at $\partial\Omega$ ($\boldsymbol{n}$ being the outer normal). Show the inequality

$$\|\boldsymbol{u}\|_q \le C\,\|\nabla \boldsymbol{u}\|_q, \quad C \le \delta(\Omega)(|q-2| + n + 1).$$

Hint (due to L.H. Payne): Integrate the identity:

$$\sum_{i,j=1}^{n} \left(D_i[u_i x_j u_j |u|^{q-2}] - (D_i u_i)x_j u_j |u|^{q-2} - |u|^q - u_i x_j D_i[u_j |u|^{q-2}]\right) = 0.$$

An inequality of the type (II.5.1) continues to hold even though u is not zero at the boundary, provided one replaces u with $u - \overline{u}_\Omega$. We shall begin to prove the following result which traces back to Poincaré (1894).

Lemma II.5.1 *For $a > 0$ let*

$$C = \{x \in \mathbb{R}^n : 0 < x_i < a\}. \tag{II.5.6}$$

Then, for all $u \in W^{1,q}(C)$, $1 \leq q < \infty$,

$$\|u - \overline{u}_C\|_q \leq na\|\nabla u\|_q. \tag{II.5.7}$$

Proof. For simplicity, we shall give the proof in the case $n = 3$. Clearly, in view of Theorem II.3.1, it is enough to show (II.5.6) for $u \in C^1(\overline{\Omega})$. Consider the identity

$$u(x_1, x_2, x_3) - u(y_1, y_2, y_3) = \int_{y_1}^{x_1} \frac{\partial u}{\partial \xi}(\xi, x_2, x_3) d\xi + \int_{y_2}^{x_2} \frac{\partial u}{\partial \eta}(y_1, \eta, x_3) d\eta$$
$$+ \int_{y_3}^{x_3} \frac{\partial u}{\partial \zeta}(y_1, y_2, \zeta) d\zeta.$$

Integrating over the y-variables and raising to the qth power, we deduce

$$|u(x_1, x_2, x_3) - \overline{u}_C|^q \leq |C|^{-q} \left[a^3 \int_0^a |\nabla u(\xi, x_2, x_3)| d\xi\right.$$
$$\left. + a^2 \int_0^a \int_0^a |\nabla u(y_1, \eta, x_3)| dy_1 d\eta + a \int_C |\nabla u| dC\right]^q.$$

Employing in this relation the inequality (II.3.3) along with the Hölder inequality and integrating over the x-variables we obtain

$$\int_C |u - \overline{u}_C|^q \leq 3^q a^q \int_C |\nabla u|^q,$$

which completes the proof. □

Remark II.5.1 An extension of (II.5.7) to arbitrary locally Lipschitz domains will be given in Theorem II.5.4. Here, however, we wish to observe that, unlike Theorem II.5.1, some regularity assumptions on Ω are strictly necessary for inequalities of type (II.5.7) to hold, as shown by means of counterexample in Courant & Hilbert (1937, Kapitel VII, §8.2); see also Fraenkel (1979, and §2 in particular). ■

Let us now analyze some consequences of Lemma II.5.1. Suppose Ω is a cube of side a and subdivide it into N equal cubes C_i, each having sides of length $a/N^{1/n}$. Applying (II.5.7) to each cube C_i and using the Minkowski inequality and (II.3.3) one recovers

$$\|u\|^q_{q,\Omega} \leq \sum_{i=1}^{N} 2^{q-1} \left(\frac{a}{N^{1/n}}\right)^{n(1-q)} \left|\int_{C_i} u dC_i\right|^q + \frac{(2na)^q}{2N^{q/n}} \|\nabla u\|^q_{q,\Omega}.$$

Therefore, introducing the N independent functions

$$\psi_i(x) = 2^{(q-1)/q} \left(\frac{a}{N^{1/n}}\right)^{n(1-q)/q} \chi_i(x),$$

with χ_i characteristic function of the cube C_i, from the previous inequality one has the following result due to Friedrichs (1933).

Lemma II.5.2 *Let C be the cube* (II.5.6) *and let*

$$u \in W^{1,q}(C), \quad 1 \leq q < \infty.$$

Then, given an arbitrary positive integer N, there exist N independent functions $\psi_i \in L^\infty(C)$ depending only on C and N such that

$$\|u\|^q_{q,C} \leq \sum_{i=1}^{N} \left|\int_{C} \psi_i u\right|^q + \frac{(2na)^q}{2N^{q/n}} \|\nabla u\|^q_{q,C}. \tag{II.5.8}$$

Inequality (II.5.8) is very useful in proving compactness results, as we are about to show. In fact, let Ω be bounded and let $\{u_m\} \subset W_0^{1,q}(\Omega)$, $1 \leq q < \infty$, be uniformly bounded in the norm $\|\cdot\|_{1,q}$. Extending u_m by zero outside Ω and denoting again by u_m such an extension, we thus have that $\{u_m\}$ is uniformly bounded in $W^{1,q}(C)$, for some cube C (see Exercise II.3.11), and therefore, by Lemma II.5.2, Theorem II.2.4(ii) and Theorem II.3.2, it is not difficult to show the existence of a subsequence $\{u_{m'}\}$ that is Cauchy in $L^q(C)$ and, as a consequence, converges strongly in $L^q(\Omega)$. On the other hand, by Lemma II.3.2 and by Exercise II.5.5, it follows that $\{u_{m'}\}$ converges also in $L^r(\Omega)$, for all $r \in [1, nq/(n-q))$, if $q < n$, for all $r \in [1, \infty)$ if $q = n$, while it converges in $C(\overline{\Omega})$ if $q > n$. We have proved the following compact embedding result (see Rellich 1930).

Theorem II.5.2 *Assume Ω bounded, and let $q \in [1, \infty)$. Then*

$$W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)\,,$$

with arbitrary $r \in [1, nq/(n-q))$, if $q < n$, and arbitrary $r \in [1, \infty)$, if $q = n$. Finally, if $q > n$, then $W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$

In Theorem II.5.2, when $q < n$, the exponent $q^* = nq/(n-q)$ is excluded. Actually one proves by means of counterexamples that the strong convergence is, in general, ruled out in this case. For, in the ball B_1 consider the sequence of functions

$$u_m(x) = \begin{cases} m^{(n-q)/q}(1 - m|x|) & \text{if } |x| < 1/m \\ 0 & \text{if } |x| \geq 1/m \end{cases} \qquad m = 1, 2, \ldots$$

with $q < n$. One has

$$\|\nabla u_m\|_q = C_1, \quad \|u_m\|_{q*} = C_2,$$

with C_1 and C_2 independent of m. Since

$$\lim_{m\to\infty} u_m(x) = 0 \quad a.e. \text{ in } B_1$$

it follows that no subsequence can converge strongly in $L^{q*}(B_1)$.

Theorem II.5.2 admits the following counterpart in negative Sobolev spaces.

Theorem II.5.3 *Let Ω be bounded. Then $L^q(\Omega) \hookrightarrow\hookrightarrow W_0^{-1,q}(\Omega)$, for any $1 < q < \infty$. Precisely, if $\{u_m\} \subset L^q(\Omega)$ is uniformly bounded, there exists a subsequence $\{u_{m'}\}$ and $u \in L^q(\Omega)$ such that*

$$\lim_{m'\to\infty} \|u - u_{m'}\|_{-1,q} = 0\,.$$

Proof. In view of inequality (II.5.1), we may endow $W_0^{1,q}(\Omega)$ with the equivalent norm $\|\nabla(\cdot)\|_q$. We observe next that, by assumption and by Theorem II.2.4(iii), there are $u \in L^q(\Omega)$ and a subsequence $\{u_{m'}\}$ such that $u_{m'} \xrightarrow{w} u$. Set $U_{m'} = u - u_{m'}$. By Theorem II.3.5 and Theorem II.1.4, for each $m' \in \mathbb{N}$, we can find $w_{m'} \in W_0^{1,q'}(\Omega)$ such that

$$\|U_{m'}\|_{-1,q} = |(U_{m'}, w_{m'})|\,, \quad \|\nabla w_{m'}\|_{q'} = 1\,. \tag{II.5.9}$$

Then, by Theorem II.5.2 and Theorem II.1.3(ii), there exist a subsequence $\{w_{m''}\}$ and $w \in W_0^{1,q'}(\Omega)$ such that $w_{m''} \to w$ in $L^{q'}(\Omega)$, and so (II.5.9) delivers

$$\|U_{m''}\|_{-1,q} \leq |(U_{m''}, w)| + \|U_{m''}\|_q \|w_{m''} - w\|_{q'} \leq |(U_{m''}, w)| + C\,\|w_{m''} - w\|_{q'}\,,$$

which, in turn, gives the desired result since $U_{m''} \xrightarrow{w} 0$ in $L^q(\Omega)$ and $w_{m''} \to w$ in $L^{q'}(\Omega)$. □

Some generalizations of Theorem II.5.2 are proposed to the reader in the following exercises.

Exercise II.5.7 Assume Ω bounded and let $q \in [1, \infty)$, $m \geq 1$. Show that

$$W_0^{m,q}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$$

with arbitrary $r \in [1, nq/(n - mq))$ if $mq < n$ and all $r \in [1, \infty)$ if $mq = n$. Finally, show that if $mq > n$, then $W_0^{m,q}(\Omega) \hookrightarrow\hookrightarrow C^k(\overline{\Omega})$, for all $k \in \mathbb{N}$ such that $0 \leq k < 1 - mq/n$.

Exercise II.5.8 Prove that, when Ω is bounded and locally Lipschitz, Theorem II.5.2 and Exercise II.5.7 continue to hold if $W_0^{m,q}(\Omega)$ is replaced by $W^{m,q}(\Omega)$. *Hint*: Use Theorem II.3.3 and (II.3.19).

We want now to obtain further inequalities as a consequence of the compactness results just derived. The following theorem extends the Poincaré inequality (II.5.7) to more general domains.

Theorem II.5.4 *Let Ω be bounded and locally Lipschitz. Then, for all $u \in W^{1,q}(\Omega)$, $1 \le q < \infty$, we have*

$$\|u - \overline{u}_\Omega\|_q \le c\|\nabla u\|_q, \tag{II.5.10}$$

where $c = c(n, q, \Omega)$.

Proof. To simplify notation, we omit the subscript Ω. If (II.5.10) were not true, a sequence $\{u_m\} \subset W^{1,q}(\Omega)$ would exist such that for all $m \in \mathbb{N}$

$$\overline{u}_m = 0, \quad \|u_m\|_q = 1, \quad \|\nabla u_m\|_q \le 1/m. \tag{II.5.11}$$

Therefore, from $(\text{II.5.11})_{2,3}$ and Exercise II.5.8 there is a subsequence converging in the norm of $W^{1,q}(\Omega)$ to some $u \in W^{1,q}(\Omega)$ which, by (II.5.11), should have $\nabla u = 0$, $\overline{u} = 0$, namely, $u \equiv 0$ *a.e.* in Ω and $\|u\|_q = 1$. This gives a contradiction that proves the theorem. □

Theorem II.5.4 admits several interesting consequences, some of which are left to the reader in the following exercises.

Exercise II.5.9 Let Ω be an arbitrary domain and let $u \in W_{loc}^{1,1}(\Omega)$. Show that, if $Du = 0$, then there is $u_0 \in \mathbb{R}$ such that $u = u_0$ *a.e.* in Ω. Using this result, show that, more generally, if $u \in W_{loc}^{m,1}(\Omega)$ with $D^\alpha u = 0$, $|\alpha| = m$, then $u = P$ *a.e* in Ω, where P is a polynomial of degree $\le m - 1$. *Hint*: Use Lemma II.1.1.

Exercise II.5.10 Assume Ω bounded and locally Lipschitz and let $u \in W^{1,q}(\Omega)$. If $q \in [1, n)$, prove the following *Poincaré-Sobolev inequality*:

$$\|u - \overline{u}_\Omega\|_r \le c\|\nabla u\|_q, \tag{II.5.12}$$

where $r = nq/(n-q)$ and $c = c(n, q, \Omega)$. Moreover, show that, if $q > n$, the following inequality holds

$$\|u - \overline{u}_\Omega\|_C \le c_1\|\nabla u\|_q\,. \tag{II.5.13}$$

Hint: Use Theorem II.5.4 and $(\text{II.3.16})_{1,3}$.

Exercise II.5.11 Let $u \in W^{1,q}(B_r(x_0))$, $q > n$. Show that the following inequality holds

$$\max_{x \in B_r(x_0)} |u(x) - u(x_0)| \le c\, r^{1-n/q}\|\nabla u\|_{q,B_r(x_0)}\,,$$

with $c = c(n, q)$. *Hint*: Use (II.5.13) on the unit ball and then rescale the result for a ball of radius r.

Another consequence of Theorem II.5.4 furnishes an interesting generalization of the *Wirtinger inequality* (Hardy, Littlewood, and Polya 1934, p. 185), which we are going to show. Denote by $\nabla^* u$ the projection of ∇u on the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \geq 2$. We have

$$|\nabla^* u|^2 = r^2 \left[|\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right], \quad r = |x|. \tag{II.5.14}$$

For a function f defined on S^{n-1} we may write

$$\|f - \overline{f}\|^q_{q,S^{n-1}} \leq \frac{2^n n}{2^n - 1} \|f - \overline{f}\|^q_{q,\Omega}, \tag{II.5.15}$$

where

$$\overline{f} = |S^{n-1}|^{-1} \int_{S^{n-1}} f dS^{n-1} \tag{II.5.16}$$

and Ω is the spherical shell of radii 1/2 and 1. Noting that

$$\overline{f} = |\Omega|^{-1} \int_\Omega f,$$

we may employ Theorem II.5.4 to obtain

$$\|f - \overline{f}\|^q_{q,\Omega} \leq c^q \|\nabla f\|^q_{q,\Omega} = c_1 \|\nabla^* f\|^q_{q,S^{n-1}}.$$

Thus, combining (II.5.15) with the latter inequality, we deduce the desired *Wirtinger inequality*:

$$\|f - \overline{f}\|_{q,S^{n-1}} \leq c_2 \|\nabla^* f\|_{q,S^{n-1}}, \quad 1 \leq q < \infty, \tag{II.5.17}$$

with $\overline{f}$ defined in (II.5.16), and $c_2 = c_2(n, q)$.

Exercise II.5.12 (Finn and Gilbarg 1957)**.** Show that, for $q = 2$, the smallest constant c_2 for which (II.5.17) holds is $c_2 = (n-1)^{-1/2}$. *Hint*: Consider the associated eigenvalue problem $\Delta^* u + \lambda u = 0$, where Δ^* denotes the Laplace operator on the unit sphere.

In the exercises that follow, we propose to the reader the proof of some useful inequalities, easily obtainable by using the same compactness argument adopted in the proof of Theorem II.5.4.

Exercise II.5.13 Let Ω be bounded and locally Lipschitz and let Σ be an arbitrary portion of $\partial\Omega$ of positive ($(n-1)$-dimensional) measure. Show that for all $u \in W^{1,q}(\Omega)$, $1 \leq q < \infty$, the following inequality holds

$$\|u\|_q \leq c \left(\|\nabla u\|_q + \left| \int_\Sigma u \right| \right) \tag{II.5.18}$$

with $c = c(n, q, \Omega, \Sigma)$.

Exercise II.5.14 Let Ω be bounded and locally Lipschitz, and let $u \in W^{m,q}(\Omega)$. Then, there exists $c = c(n, q, \Omega, \omega)$ such that

$$\|u\|_{m,q} \le c \left(\sum_{|\alpha|=m} \|D^\alpha u\|_q + \int_\omega |u| \right) \tag{II.5.19}$$

where ω is an arbitrary subdomain of Ω of positive (n-dimensional) measure. *Hint*: Use Exercise II.5.9 .

Exercise II.5.15 Let Ω be bounded and locally Lipschitz and let $\boldsymbol{u}$ be a vector function in Ω with components from $W^{1,q}(\Omega)$, $1 \le q < \infty$. Assuming $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$, show that there exists a constant $c = c(n, q, \Omega)$ such that

$$\|\boldsymbol{u}\|_q \le c \|\nabla \boldsymbol{u}\|_q .$$

Hint: Use Exercise II.5.8.

Exercise II.5.16 (Ehrling inequality) Let Ω be bounded and locally Lipschitz. Show that for any $\varepsilon > 0$ there is $c = c(\varepsilon, n, q, \Omega) > 0$ such that

$$\|\nabla u\|_q \le c\|u\|_q + \varepsilon \|D^2 u\|_q , \tag{II.5.20}$$

for all $u \in W^{2,q}(\Omega)$, $1 \le q < \infty$. The regularity assumption on Ω can be removed if $u \in W_0^{2,q}(\Omega)$. *Hint*: Use Exercise II.5.8 and Theorem II.5.2.

Remark II.5.2 Inequalities of the type given in Exercise II.5.13 and Exercise II.5.14 are relevant in the context of the equivalence of norms in the spaces $W^{m,q}$. A general theorem, that contains these inequalities as a particular case, can be found in Smirnov (1964, §114, Theorem 3). ■

We end this section by giving another significant application of the contradiction-compactness argument used in the proof of Theorem II.5.4, that generalizes the result given in Galdi (2007, Lemma 5.4). To this end, we set

$$\overset{o}{W}{}^{1,q}(\Omega) = \{ u \in W^{1,q}(\Omega) : \quad u|_\Sigma = 0 \} , \tag{II.5.21}$$

where Σ is an arbitrarily fixed locally Lipschitz boundary portion of $\partial\Omega$. It is easily shown that $\overset{o}{W}{}^{1,q}(\Omega)$ is a closed subspace of $W^{1,q}$ (Exercise II.5.17). Moreover, in view of Exercise II.5.13, we find that a norm equivalent to $\|(\cdot)\|_{1,q}$ is given by $\|\nabla(\cdot)\|_q$, and we shall endow $\overset{o}{W}{}^{1,q}(\Omega)$ with this latter.

We recall that a sequence of of linear functionals, $\{\ell_i\}$, on a Banach space X, is called *complete* if

$$\ell_i(u) = 0 , \text{ for all } i \in \mathbb{N}, \text{ implies } u = 0 \text{ in } X .$$

We have the following result.

Lemma II.5.3 *Let Ω be locally Lipschitz, and let $\{l_i\}$ be a complete sequence of linear functionals on $\overset{o}{W}{}^{1,q}(\Omega)$, $1 < q < \infty$. Then, given $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and a positive constant C such that*

$$\|u\| \leq \varepsilon \|\nabla u\|_q + C \sum_{i=1}^{N} |l_i(u)| ,$$

where $\|u\| \equiv \|u\|_r$ with $r \in [1, nq/(n-q))$, if $q < n$, and $r \in [1, \infty)$, if $q = n$, while $\|u\| \equiv \|u\|_C$ if $q > n$. The numbers N and C depend on Ω, ε, q, and also on r if $q \leq n$.

Proof. We give a proof in the case $q < n$, the other two cases being treated in a completely analogous way, with the help of Theorem II.5.2. Thus, assume, by contradiction, that there is $\overline{\varepsilon} > 0$ such that, for all $C > 0$ and all $N \in \mathbb{N}$ we can find at least one $u = u(C, N) \in \overset{o}{W}{}^{1,q}(\Omega)$ such that

$$\|u\|_r \geq \overline{\varepsilon} \|\nabla u\|_q + C \sum_{i=1}^{N} |l_i(u)| .$$

We then fix $N = N_1$ and find a sequence $\{u_m\}$, possibly depending on N_1, such that

$$\|u_m\|_r \geq \overline{\varepsilon} \|\nabla u_m\|_q + m \sum_{i=1}^{N_1} |l_i(u_m)| .$$

Setting $w_m = u_m / \|\nabla u_m\|_q$,[2] from the preceding inequality we find

$$\|w_m\|_r \geq \overline{\varepsilon} + m \sum_{i=1}^{N_1} |l_i(w_m)| , \quad \|\nabla w_m\|_q = 1, \; m \in \mathbb{N}. \tag{II.5.22}$$

From (II.5.22) we then deduce that

$$\|w_m\|_{1,q} \leq C_1 \tag{II.5.23}$$

with $C_1 = C_1(\Omega, \Sigma, q) > 0$. So, by Theorem II.5.2 and by the weak compactness property of the unit closed ball (see Remark II.3.1), there exist a subsequence, again denoted by $\{w_m\}$, and $w^{(1)} \in \overset{o}{W}{}^{1,q}(\Omega)$ such that

$$\begin{aligned} &w_m \to w^{(1)} \ \text{ in } L^r(\Omega) \\ &w_m \overset{w}{\to} w^{(1)} \ \text{ in } \overset{o}{W}{}^{1,q}(\Omega) . \end{aligned} \tag{II.5.24}$$

Using these latter properties along with (II.5.22) we infer, on the one hand,

[2] Of course, we may assume, without loss of generality, that $\|\nabla u_m\|_q \neq 0$, for all $m \in \mathbb{N}$.

$$\sum_{i=1}^{N_1} |l_i(w^{(1)})| = 0\,,$$

and, on the other hand,

$$\|w^{(1)}\|_r \geq \overline{\varepsilon}\,.$$

Moreover, from (II.5.22)$_2$, (II.5.23), and (II.5.24) we obtain

$$\|w^{(1)}\|_r + \|w^{(1)}|_{1,q} \leq C_2$$

with $C_2 = C_2(D, S, r, q)$. We next fix $N = N_2 > N_1$ and, by the same procedure, we can find another $w^{(2)} \in \overset{o}{W}{}^{1,q}(\Omega)$ satisfying the same properties as $w^{(1)}$. By iteration, we can thus construct two sequences, $\{N_k\}$ and $\{w^{(k)}\}$, with $\{N_k\}$ increasing and unbounded, such that

$$\begin{aligned}
&\sum_{i=1}^{N_k} |l_i(w^{(k)})| = 0\,, \\
&\|w^{(k)}\|_r + \|w^{(k)}\|_{1,q} \leq C_2 \\
&\|w^{(k)}\|_r \geq \overline{\varepsilon}\,,
\end{aligned} \tag{II.5.25}$$

for all $k \in \mathbb{N}$. By (II.5.25)$_2$ and again by Theorem II.5.2, it follows that there are a subsequence of $\{w^{(k)}\}$, which we continue to denote by $\{w^{(k)}\}$, and a function $w^{(0)} \in \overset{o}{W}{}^{1,q}(\Omega)$ such that

$$\begin{aligned}
&w^{(k)} \to w^{(0)} \quad \text{in } L^q(\Omega) \\
&w^{(k)} \overset{w}{\to} w^{(0)} \quad \text{in } \overset{o}{W}{}^{1,q}(\Omega)\,.
\end{aligned} \tag{II.5.26}$$

In view of (II.5.25)$_3$ and of (II.5.26)$_1$, we must have

$$\|w^{(0)}\|_q \geq \overline{\varepsilon}\,. \tag{II.5.27}$$

We now claim that $w^{(0)} \equiv 0$, contradicting (II.5.27). In fact, if $w^{(0)} \not\equiv 0$, by the completeness of the family of functionals $\{l_i\}$, we must have, for at least one member of the family, $l_{\overline{i}}$, that

$$l_{\overline{i}}(w^{(0)}) \neq 0\,. \tag{II.5.28}$$

By (II.5.26)$_2$, it is

$$\lim_{k\to\infty} l_{\overline{i}}(w^{(k)}) = l_{\overline{i}}(w^{(0)})\,, \tag{II.5.29}$$

while from (II.5.25)$_1$ evaluated at all $N_k > \overline{i}$, we find

$$l_{\overline{i}}(w^{(k)}) = 0\,, \quad \text{for all sufficiently large } k\,.$$

However, in view of (II.5.29), this condition contradicts (II.5.28). Thus, $w^{(0)} = 0$ and the lemma is proved. □

Exercise II.5.17 Show that the space defined in (II.5.21) is a closed subspace of $W^{1,q}(\Omega)$.

Exercise II.5.18 Prove the following abstract formulation of Lemma II.5.3. Let X, Y be Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Suppose that X is reflexive and compactly embedded in Y. Moreover, let $\{\ell_i\}$ be a complete sequence of functionals in X. Show that, given $\varepsilon > 0$ there exist $N = N(\varepsilon) \in \mathbb{N}$ and a constant $C = C(\varepsilon)$ such that

$$\|u\|_Y \leq \varepsilon \|u\|_X + C \sum_{i=1}^{N} |\ell_i(u)|, \quad \text{for all } u \in X.$$

II.6 The Homogeneous Sobolev Spaces $D^{m,q}$ and Embedding Inequalities

In dealing with boundary-value problems in unbounded domains it can happen that, even for very smooth and rapidly decaying data, the associated solution u does not belong to any space of the type $W^{m,q}$. This is because the behavior at large distances can be different for each derivative of u of a given order and, as a consequence, the corresponding summability properties can be different. As a simple example, consider the Dirichlet problem

$$\begin{aligned}
&\Delta u = 0 \ \text{ in } \Omega \equiv \mathbb{R}^3 - \overline{B}_1\,, \qquad u = 1 \ \text{ at } \partial\Omega,\\
&\lim_{|x|\to\infty} u(x) = 0\,.
\end{aligned}$$

The solution is $u(x) = 1/|x|$ and we have

$$\begin{aligned}
&D^2 u \in L^r(\Omega), \quad 1 < r < \infty,\\
&\nabla u \in L^s(\Omega), \quad 3/2 < s < \infty,\\
&u \in L^t(\Omega), \quad 3 < t < \infty.
\end{aligned}$$

Thus, to formulate boundary-value problems of the above type, one finds it more convenient to introduce spaces more “natural” than the Sobolev spaces $W^{m,q}$, and which, unlike the latter, involve only the derivatives of order m. These classes of functions will be called *homogeneous Sobolev spaces*, and we shall devote this and the next few sections to the study of their relevant properties.

For $m \in \mathbb{N}$ and $1 \leq q < \infty$ we define the following linear space (without topology)

$$D^{m,q} = D^{m,q}(\Omega) = \left\{u \in L^1_{loc}(\Omega) : D^\ell u \in L^q(\Omega),\ |\ell| = m\right\}.$$

In order to investigate some preliminary properties of $D^{m,q}$, we introduce the following notation. If u satisfies

$$D^\ell u \in L^q(\Omega'), \quad 0 \le |\ell| \le m, \quad \textit{for all bounded } \Omega' \textit{ with } \overline{\Omega'} \subset \Omega,$$

we shall write

$$u \in W^{m,q}_{loc}(\Omega).$$

Likewise, if

$$D^\ell u \in L^q(\Omega'), \quad 0 \le |\ell| \le m, \quad \textit{for all bounded } \Omega' \subset \Omega$$

we shall write

$$u \in W^{m,q}_{loc}(\overline{\Omega})\,.$$

We have the following.

Lemma II.6.1 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \ge 2$, and let $u \in D^{m,q}(\Omega)$, $m \ge 0$, $q \in (1,\infty)$. Then $u \in W^{m,q}_{loc}(\Omega)$ and the following inequality holds*

$$\|u\|_{m,q,\omega} \le c \left(\sum_{|\ell|=m} \|D^\ell u\|_{q,\omega} + \|u\|_{1,\omega} \right) \tag{II.6.1}$$

where ω is an arbitrary bounded locally Lipschitz domain with $\overline{\omega} \subset \Omega$. If, in addition, Ω is locally Lipschitz, then $u \in W^{m,q}_{loc}(\overline{\Omega})$, and (II.6.1) *holds for all bounded and locally Lipschitz domains $\omega \subset \Omega$.*

Proof. Clearly, proving that $u \in W^{m,q}(\omega)$, for any ω satisfying the properties stated in the first part of the lemma, implies $u \in W^{m,q}_{loc}(\Omega)$. Let $d = \operatorname{dist}(\partial\omega, \partial\Omega)$ (> 0), and extend u by zero outside Ω. For $d > 1/k > 0$, $k \in \mathbb{N}$, we denote by u_k the regularizer of u corresponding to $\varepsilon = 1/k$. Obviously, $u_k \in W^{m,q}(\omega)$; moreover, by Exercise II.3.2, we have

$$(D^\ell u)_k(x) = (D^\ell u_k)(x), \quad \text{for all } \ell \text{ with } |\ell| = m, \text{ and all } x \in \omega.$$

We may thus use (II.5.19) to find, for any $k, k' \in \mathbb{N}$,

$$\|u_k - u_{k'}\|_{m,q,\omega} \le C \left(\sum_{|\ell|=m} \|(D^\ell u)_k - (D^\ell u)_{k'})\|_{q,\omega} + \|u_k - u_{k'}\|_{1,\omega} \right),$$

for some $C = C(N,q,\omega)$. Observing that, by (II.2.9)$_2$, $(D^\ell u)_k$, $|\ell| = m$, and u_k converge (strongly) in $L^q(\omega)$ and $L^1(\omega)$ to $D^\ell u$ and u, respectively, as $k \to \infty$, from the previous inequality we deduce that $\{u_k\}$ is Cauchy in $W^{m,q}(\omega)$, as well as the validity of (II.6.1). The first part of the lemma is thus proved. In order to show the second part, we begin to observe that, by Exercise II.1.5, we can find a finite number of locally Lipschitz and star-shaped domains Ω_i, $i = 1, \dots, r$, satisfying the following condition

$$\omega \subseteq \bigcup_{i=1}^r \Omega_i \subseteq \Omega\,.$$

If we thus show that $u \in W^{m,q}(\Omega_i)$ for each $i = 1, \dots, r$, the stated property follows with the help of Exercise II.5.14. For a fixed i, we extend $u|_{\Omega_i}$ to zero outside Ω_i, and continue to denote by u this extension. By means of a translation in $\mathbb{R}^n$, we may take the point x_i, with respect to which Ω_i is star-shaped, to be the origin of the coordinates. Then, the domains

$$\Omega_i^{(k)} = \{x \in \mathbb{R}^n : (1 - 1/k)\, x \in \Omega_i\}\,, \quad k \in \mathfrak{N} \equiv \{m \in \mathbb{N} : \ m \geq 2\}\,,$$

satisfy $\Omega_i^{(k)} \supset \overline{\Omega}_i$, for all $k \in \mathfrak{N}$; see Exercise II.1.3. Setting

$$u_k = u_k(x) \equiv u((1 - 1/k)\, x)\,, \quad x \in \Omega_i^{(k)}\,,$$

and $h_0 = \max_{x \in \partial\Omega_i} |x|$, we find that the mollifier, $(u_k)_\varepsilon$, of u_k belongs to $W^{m,q}(\Omega_i)$, if we choose (for example) $\varepsilon = h_0/(2k-2)$. With the aid of (II.2.9)$_1$, we deduce

$$\|u-(u_k)_\varepsilon\|_{1,\Omega_i} \leq \|u-u_\varepsilon\|_{1,\Omega_i} + \|u_\varepsilon - (u_k)_\varepsilon\|_{1,\Omega_i} \leq \|u-u_\varepsilon\|_{1,\Omega_i} + \|u-u_k\|_{1,\Omega_i}\,,$$

which, in turn, by (II.2.9)$_2$ and by Exercise II.2.8, implies

$$\lim_{k\to\infty} \|u - (u_k)_\varepsilon\|_{1,\Omega_i} = 0\,. \tag{II.6.2}$$

We next set $\chi(x) = D^\ell u(x)$, $|\ell| = m$. Observing that, by Exercise II.3.2 and Exercise II.3.3, it is

$$D^\ell (u_k)_\varepsilon = (1 - 1/k)^m [\chi((1 - 1/k)x)]_\varepsilon \quad x \in \Omega_i\,,$$

we may repeat an argument similar to that leading to (II.6.2) to show

$$\lim_{k\to\infty} \|D^\ell u - D^\ell (u_k)_\varepsilon\|_{q,\Omega_i} = 0\,. \tag{II.6.3}$$

Now, with the help of (II.6.2) and (II.6.3), we can use the same procedure used in the proof of the first part of the lemma with $\omega \equiv \Omega_i$, to show the statement contained in the second part. The lemma is thus completely proved. □

Remark II.6.1 From Lemma II.6.1 it follows, in particular, that if Ω is bounded and locally Lipschitz, then $u \in D^{m,q}(\Omega)$ implies $u \in W^{m,q}(\Omega)$, so that $D^{m,q}(\Omega) = W^{m,q}(\Omega)$ algebraically, and, in fact, also topologically, if we endow the space $D^{m,q}(\Omega)$ with the norm $\sum_{|\ell|=m} \|D^\ell u\|_q + \|u\|_1$. On the other hand, if Ω is unbounded in all directions, these latter properties no longer hold, since a priori one loses information on *global* summability of derivatives of order *less* than m, and one can only state *local* properties in the sense specified in Lemma II.6.1. ■

Exercise II.6.1 Let $u \in D^{m,q}(\mathbb{R}^n)$, $n \geq 2$, $m \geq 0$, $q \in (1, \infty)$. Show that $u \in W^{m,q}(B_R)$, for all $R > 0$, and there exists a constant $C = C(R)$ such that

$$\|u\|_{m,q,B_R} \leq C \left(\sum_{\ell=m} \|D^\ell u\|_{q,\mathbb{R}^n} + \|u\|_{1,B_1} \right)\,.$$

Hint: Adapt the arguments used in the proof of the first part of Lemma II.6.1

In $D^{m,q}$ we introduce the seminorm

$$|u|_{m,q} \equiv \left(\sum_{|\ell|=m} \int_\Omega |D^\ell u|^q \right)^{1/q} . \tag{II.6.4}$$

Let $\mathsf{P_m}$ be the class of all polynomials of degree $\leq m-1$ and, for $u \in D^{m,q}$, set

$$[u]_m = \{w \in D^{m,q} : w = u + \mathcal{P}, \text{ for some } \mathcal{P} \in \mathsf{P_m}\}.$$

Denoting by $\dot{D}^{m,q} = \dot{D}^{m,q}(\Omega)$ the space of all (equivalence classes) $[u]_m$, $u \in D^{m,q}$, we see at once that (II.6.4) induces the following norm in $\dot{D}^{m,q}$:

$$|[u]_m|_{m,q} \equiv |u|_{m,q}\,, \quad u \in [u]_m\,. \tag{II.6.5}$$

We shall now show that $\dot{D}^{m,q}$ equipped with the norm (II.6.5) is a Banach space.

Lemma II.6.2 *Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$. Then $\dot{D}^{m,q}(\Omega)$ is a Banach space. In particular, if $q = 2$, it is a Hilbert space with the scalar product*

$$[[u]_m, [v]_m]_m = \sum_{|\ell|=m} \int_\Omega D^\ell u D^\ell v\,, \quad u \in [u]_m\,,\ v \in [v]_m\,.$$

Proof. It is enough to show the first part of the lemma, the second follows easily. We shall consider the case $m = 1$, leaving the more general case as an exercise. We also set $[u]_1 \equiv [u]$. Let $\{[u_s]\}$ be a Cauchy sequence in $\dot{D}^{1,q}(\Omega)$; we have to show the following statements:

(i) For any $\{v_s\}$ with $v_s \in [u_s]$, $s \in \mathbb{N}$, there exists $u \in D^{1,q}(\Omega)$ such that

$$\lim_{s\to\infty} \|D_i v_s - D_i u\|_q = 0\,,\ i = 1, \ldots, n\,;$$

(ii) For any $\{v_s\}$, $\{v'_s\}$, with $v_s, v'_s \in [u_s]$, $s \in \mathbb{N}$, and with u, u' corresponding limits, we have $u' \in [u]$.

It is seen that (ii) easily follows from (i). In fact, since $v_s, v'_s \in [u_s]$, from (i) we have

$$(D_i u, \varphi) = (D_i u', \varphi), \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

which, in view of Exercise II.5.9, implies (ii). Let us show (i). By the completeness of L^q, we find $V_i \in L^q(\Omega)$, $i = 1, \ldots, n$, with

$$D_i v_s \to V_i \ \text{ in } L^q(\Omega). \tag{II.6.6}$$

Let $\mathfrak{O}$ be the open covering of Ω indicated in Lemma II.1.1 and let $\mathfrak{B}_0 \in \mathfrak{O}$. By the Poincaré inequality and (II.6.6) we deduce the existence of $u^{(0)} \in L^q(\mathfrak{B}_0)$ such that

$$v_s - \overline{v_s}_{\mathfrak{B}_0} \to u^{(0)} \quad \text{in } L^q(\mathfrak{B}_0).$$

Since for all $\varphi \in C_0^\infty(\mathfrak{B}_0)$ it is

$$\int_{\mathfrak{B}_0} V_i \varphi = \lim_{s\to\infty} \int_{\mathfrak{B}_0} D_i v_s \varphi = \lim_{s\to\infty} \int_{\mathfrak{B}_0} (v_s - \overline{v_s}_{\mathfrak{B}_0}) D_i \varphi = - \int_{\mathfrak{B}_0} u^{(0)} D_i \varphi,$$

by definition of the weak derivative, it follows

$$V_i = D_i u^{(0)} \quad \text{a.e. in } \mathfrak{B}_0. \tag{II.6.7}$$

By the property (ii) of $\mathfrak{O}$, we can find $\mathfrak{B}_1 \in (\mathfrak{O} - \mathfrak{B}_0)$ with $\mathfrak{B}_1 \cap \mathfrak{B}_0 \equiv \mathfrak{B}_{1,2} \neq \emptyset$. As before, we show the existence of $u^{(1)} \in L^q(\mathfrak{B}_1)$ such that

$$V_i = D_i u^{(1)} \quad \text{a.e. in } \mathfrak{B}_1. \tag{II.6.8}$$

Thus, $u^{(1)} = u^{(0)} + c$ a.e. in $\mathfrak{B}_{1,2}$, for some $c \in \mathbb{R}$. Therefore, we may modify $u^{(1)}$ by the addition of a constant in such a way that $u^{(1)}$ and $u^{(0)}$ agree a.e. in $\mathfrak{B}_{1,2}$. Continue to denote by $u^{(1)}$ the modified function and define a new function $u^{(0,1)}$ that is equal to $u^{(0)}$ in $\mathfrak{B}_0$ and is equal to $u^{(1)}$ in $\mathfrak{B}_1$. By (II.6.6)–(II.6.8) we deduce that $u^{(0,1)}, D_i u^{(0,1)} \in L^q(\mathfrak{B}_0 \cup \mathfrak{B}_1)$, with $V_i = D_i u^{(0,1)}$ a.e. in $\mathfrak{B}_0 \cup \mathfrak{B}_1$. In view of the property (iii) of the covering $\mathfrak{O}$, we can repeat this procedure to show, by a simple inductive argument, the existence of $u \in L^q_{loc}(\Omega)$ satisfying the statement (i) of the lemma, which is thus completely proved. □

Notation. Sometime, and unless confusion arises, the elements of $\dot{D}^{m,q}(\Omega)$ will be denoted simply by u, instead of $[u]_m$, with u a representative of the class $[u]_m$.

The functional (II.6.4) defines a *norm* in the space $C_0^\infty(\Omega)$. We then introduce the Banach space $D_0^{m,q} = D_0^{m,q}(\Omega)$ as the (Cantor) completion of the normed space $\{C_0^\infty(\Omega), |\cdot|_{m,q}\}$.

Remark II.6.2 Since $C_0^\infty(\Omega)$ can be viewed as a subspace of $\dot{D}^{m,q}(\Omega)$ via the natural map

$$i : u \in C_0^\infty(\Omega) \to i(u) = [u]_m \in \dot{D}^{m,q}(\Omega),$$

it follows that, for any domain Ω, $D_0^{m,q}(\Omega)$ is isomorphic to a closed subspace of $\dot{D}^{m,q}(\Omega)$. More specifically, $[u]_m \in \dot{D}^{m,q}(\Omega)$ belongs to $D_0^{m,q}(\Omega)$ if and only if there is $u \in [u]_m$ and corresponding $\{u_k\} \subset C_0^\infty(\Omega)$ such that $\lim_{k\to\infty} |u_k - u|_{m,q} = 0$. Other characterizations of the spaces $D_0^{m,q}$ will be given in Section II.7. We finally observe that (see Exercise II.2.6)

$$D_0^{0,q}(\Omega) = D^{0,q}(\Omega) = L^q(\Omega), \quad q \geq 1.$$

■

Remark II.6.3 If Ω is contained in a layer, then by means of inequality (II.5.1) and Lemma II.6.1 one can easily show that $\|\cdot\|_{m,q}$ is equivalent to $|\cdot|_{m,q}+\|\cdot\|_q$ and to $|\cdot|_{m,q}$. Therefore, if we endow $W_0^{m,q}(\Omega)$ with this latter norm, we find that $D_0^{m,q}(\Omega)$ and $W_0^{m,q}(\Omega)$ are isomorphic. ■

Exercise II.6.2 Show that $\dot{D}^{m,q}$ and $D_0^{m,q}$ are separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$. Thus, for $q \in (1,\infty)$ these spaces are weakly complete and the unit closed ball is weakly compact (see Theorem II.1.3(ii)). *Hint* (for $m = 1$): Let

$$W = \left\{ w \in [L^q]^n : \ w = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right), \text{ for some } u \in \dot{D}^{1,q} \right\}.$$

W is isomorphic to $\dot{D}^{1,q}$, and, since $\dot{D}^{1,q}$ is complete, W is a closed subspace of $[L^q]^n$. Therefore, W is separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$ (see Theorem II.2.5, Theorem II.1.1 and Theorem II.1.2), which, in turn, gives the stated properties for $\dot{D}^{1,q}$. Since $D_0^{1,q}$ is isomorphic to a closed subspace of $\dot{D}^{1,q}$, the same properties are true for $D_0^{1,q}$; see also Simader and Sohr (1997, Theorem I.2.2).

Our next goal will be to investigate global properties of functions from $D^{m,q}(\Omega)$, including their behavior at large distances, when Ω is either *an exterior domain* or *a half-space.*

Remark II.6.4 It will be clear from the context that, in fact, most of the results we shall prove continue to hold for a much larger class of domains. This class certainly includes domains Ω for which any function from $D^{1,q}(\Omega)$ can be extended to one from $D^{1,q}(\mathbb{R}^n)$ with preservation of the seminorm $|\cdot|_{1,q}$. For the existence of such extensions, we refer the reader to the classical paper of Besov (1967); see also Burenkov (1976). ■

Our following objective is to prove some embedding inequalities that ensure that derivatives of u of order less than m belong to suitable Lebesgue or weighted-Lebesgue spaces. Such estimates, unlike the bounded-domain case, where they give information on the "regularity" of u, furnish information on the behavior of u at large distances. We begin to derive these inequalities for the case $m = 1$ (see Theorem II.6.1, Theorem II.6.3), the general case $m \ge 1$ being treated by a simple iterative argument (see Theorem II.6.4).

We recall that, if $q \in [1,n)$ every $u \in C_0^\infty(\Omega)$, satisfies the Sobolev inequality (II.3.7), that we rewrite below for reader's convenience:

$$\|u\|_s \le \frac{q(n-1)}{2(n-q)\sqrt{n}}|u|_{1,q}\,, \quad \text{for all } q \in [1,n)\,,\ s = nq/(n-q). \tag{II.6.9}$$

We shall next consider certain *weighted inequalities* that (in a less general form) were first considered by Leray (1933, p. 47; 1934, §6) and Hardy (Hardy, Littlewood, and Polya 1934, §7.3). Specifically, if $u \in C_0^\infty(\Omega)$, we have

$$\|u|x-x_0|^{-1}\|_q \le \frac{q}{(n-q)}|u|_{1,q}, \quad \text{for all } q \in [1,n). \tag{II.6.10}$$

In fact, consider the identity

$$\nabla \cdot (\boldsymbol{g}|u|^q) = |u|^q \nabla \cdot \boldsymbol{g} + \boldsymbol{g} \cdot \nabla |u|^q \tag{II.6.11}$$

with

$$\boldsymbol{g} = (\boldsymbol{x} - \boldsymbol{x}_0)\, / |x - x_0|^q. \tag{II.6.12}$$

Since

$$\nabla \cdot \boldsymbol{g} = (n-q)/|x-x_0|^q,$$

integrating (II.6.11) and using the Hölder inequality proves (II.6.10). Notice that if $q > n$ and

$$\Omega^c \supset B_a(x_0), \quad \text{some } a > 0,$$

then by the same token one shows the validity of the following inequality:

$$\|u|x-x_0|^{-1}\|_q \leq \frac{q}{(q-n)}|u|_{1,q}, \quad \text{for all } q > n\,; \tag{II.6.13}$$

see also Exercise II.6.7. In case $q = n$ ($\neq 1$) and if

$$\Omega^c \supset B_a(x_0), \quad \text{some } a > 0,$$

we have instead

$$\|u\, [|x-x_0| \ln(|x-x_0|/a)]^{-1}\, \|_n \leq \frac{n}{a\,(n-1)}|u|_{1,n}. \tag{II.6.14}$$

To show this latter, we use again identity (II.6.11) with

$$\boldsymbol{g} = -\frac{(\boldsymbol{x}-\boldsymbol{x}_0)}{|x-x_0|^n[\ln(|x-x_0|/a)]^{n-1}}.$$

Since

$$\nabla \cdot \boldsymbol{g} = \frac{a\,(n-1)}{[|x-x_0| \ln(|x-x_0|/a)]^n}\,,$$

substituting into (II.6.11), integrating over Ω, and applying the Hölder inequality to the last term on the right-hand side of (II.6.11) proves (II.6.14).

We shall next analyze if and to what extent inequalities similar to (II.6.9), (II.6.10), (II.6.13), and (II.6.14) continue to hold for functions from $D^{1,q}(\Omega)$, where the domain Ω can be either an exterior domain or a half-space.[1] In order to perform this study, we need to know more about the behavior at large distances of functions of $D^{1,q}(\Omega)$. In this respect we have

Lemma II.6.3 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an exterior domain and let*

$$u \in D^{1,q}(\Omega), \quad 1 \leq q < n.$$

Then, there exists a unique $u_0 \in \mathbb{R}$ such that, for all $R > \delta(\Omega^c)$,

$$\int_{S^{n-1}} |u(R,\omega) - u_0|^q d\omega \leq \gamma_0 R^{q-n} \int_{\Omega^R} |\nabla u|^q,$$

where $\gamma_0 = [(q-1)/(n-q)]^{q-1}$ if $q > 1$ and $\gamma_0 = 1$ if $q = 1$.

[1] See Remark II.6.4.

Proof. Let $r > R > \delta(\Omega^c)$, and consider first the case $q > 1$. For a smooth u, by the Hölder inequality we have

$$\int_R^r \int_{S^{n-1}} \left|\frac{\partial u}{\partial \rho}\right|^q \rho^{n-1} d\rho dS^{n-1} = \int_{S^{n-1}} \left[\int_R^r \left|\frac{\partial u}{\partial \rho}\right|^q \rho^{n-1} d\rho\right] dS^{n-1}$$
$$\geq \int_{S^{n-1}} \left[\frac{\left|\int_R^r \frac{\partial u}{\partial \rho} d\rho\right|^q}{\left(\int_R^r \rho^{(1-n)/(q-1)} d\rho\right)^{q-1}}\right] = \gamma_0^{-1} R^{n-q} \int_{S^{n-1}} |u(r) - u(R)|^q, \tag{II.6.15}$$

while, by the Wirtinger inequality (II.5.17), it follows that

$$\int_R^r \rho^{n-q-1} \left(\int_{S^{n-1}} |\nabla^* u|^q dS^{n-1}\right) d\rho$$
$$\geq c_1^{-q} \int_R^r \left(\int_{S^{n-1}} |u - \overline{u}|^q dS^{n-1}\right) \rho^{n-q-1} d\rho,$$

where

$$\overline{f} = (n\omega_n)^{-1} \int_{S^{n-1}} f.$$

Therefore, setting

$$D_r(R) = \int_{\Omega_{R,r}} |\nabla u|^q,$$

and taking into account that, by (II.5.14), $|\partial u/\partial r|^q, (|\nabla^* u|/r)^q \leq |\nabla u|^q$, we find

$$\begin{aligned} D_r(R) &\geq \gamma_0^{-1} R^{n-q} \int_{S^{n-1}} |u(r) - u(R)|^q \\ D_r(R) &\geq c_1^{-q} \int_R^r \left(\int_{S^{n-1}} |u - \overline{u}|^q dS^{n-1}\right) \rho^{n-q-1} d\rho. \end{aligned} \tag{II.6.16}$$

In view of Lemma II.6.1, and with the help of Theorem II.3.1, one shows that (II.6.16) continues to hold for all functions merely satisfying the assumption of the lemma. Letting $R, r \to \infty$, into $(\text{II.6.16})_1$, we deduce that u converges (strongly) in $L^q(S^{n-1})$ to some function u^*. Set

$$u_0 = \overline{u^*}, \quad w = u - u_0.$$

Obviously,

$$\lim_{|x|\to\infty} \int_{S^{n-1}} w(x) = 0. \tag{II.6.17}$$

Rewriting (II.6.16) with w instead of u, we recover the existence of a sequence $\{r_m\} \subset \mathbb{R}_+$, with $\lim_{m\to\infty} r_m = \infty$ such that

$$\lim_{m\to\infty}\int_{S^{n-1}}|w(r_m)-\overline{w}(r_m)|^q=0,$$

which, because of (II.6.17), furnishes

$$\lim_{m\to\infty}\int_{S^{n-1}}|w(r_m)|^q=0.$$

Inserting this information into (II.6.16)$_1$ written with w in place of u and letting $r\to\infty$ completes the proof of the lemma when $q>1$. If $q=1$, we easily show that

$$\int_R^r\int_{S^{n-1}}\left|\frac{\partial u}{\partial\rho}\right|\rho^{n-1}d\rho dS^{n-1}\ge R^{n-1}\int_{S^{n-1}}|u(r)-u(R)|\,.$$

Therefore, replacing (II.6.15) with this latter relation and arguing exactly as before, we show the result also when $q=1$ □

Exercise II.6.3 The previous lemma describes the precise way in which a function u, having first derivatives in $L^q(\Omega)$, $1\le q<n$, Ω an exterior domain, must tend to a (finite) limit at large spatial distances. Show by a counterexample that the condition $q<n$ is indeed necessary for the validity of the result. Moreover, prove that if $q\ge n$ the following estimate holds, for all $r\ge r_0>\max\{1,\delta(\Omega^c)\}$:

$$\int_{S^{n-1}}|u(r,\omega)|^q d\omega\le 2^{q-1}\left(\int_{S^{n-1}}|u(r_0,\omega)|^q d\omega+h(r)|u|^q_{1,q,\Omega_{r_0,r}}\right),\qquad\text{(II.6.18)}$$

where

$$h(r)=\begin{cases}(\log r)^{n-1} & \text{if } q=n\\[2ex] [(q-1)/(q-n)]^{q-1}\,r^{q-n} & \text{if } q>n.\end{cases}$$

Finally, using (II.6.18), show

$$\lim_{r\to\infty}(h(r))^{-1}\int_{S^{n-1}}|u(r,\omega)|^q d\omega=0.$$

(For pointwise estimates, see Section II.9.) *Hint*: To show (II.6.18), start with the identity

$$u(r,\omega)=u(r_0,\omega)+\int_{r_0}^r(\partial u/\partial\rho)d\rho,$$

and apply the Hölder inequality.

This preliminary result allows us to prove the following, which answers the question raised previously; see also Finn (1965a), Galdi and Maremonti (1986).

Theorem II.6.1 *Let $\Omega\subseteq\mathbb{R}^n$, $n\ge 2$, be an exterior domain, and let*

$$u\in D^{1,q}(\Omega),\quad 1\le q<\infty\,.$$

The following properties hold.

(i) *If* $q \in [1,n)$, *set*

$$w = u - u_0$$

with u_0 *defined in* Lemma II.6.3. *Then, for any* $x_0 \in \mathbb{R}^n$, *we have*

$$w|x-x_0|^{-1} \in L^q(\Omega^R(x_0)),$$

where

$$\Omega^a(x_0) \equiv \Omega - B_a(x_0), \quad B_a(x_0) \supset \Omega^c,$$

and the following inequality holds:

$$\left(\int_{\Omega^R(x_0)} \left|\frac{w(x)}{x-x_0}\right|^q dx\right)^{1/q} \leq \frac{q}{(n-q)}|w|_{1,q,\Omega^R(x_0)}. \tag{II.6.19}$$

If $|x_0| = \alpha R$, *for some* $\alpha \geq \alpha_0 > 1$ *and some* $R > \delta(\Omega^c)$, *we have*

$$\left(\int_{\Omega^R} \left|\frac{w(x)}{x-x_0}\right|^q dx\right)^{1/q} \leq c|w|_{1,q,\Omega^R}, \tag{II.6.20}$$

where $c = c(n,q,\alpha_0)$. *Furthermore, if* Ω *is locally Lipschitz, then*

$$w \in L^s(\Omega), \quad s = nq/(n-q), \tag{II.6.21}$$

and for some γ_1 *independent of* u

$$\|w\|_s \leq \gamma_1 |w|_{1,q}. \tag{II.6.22}$$

(ii) *If* $q \in [n,\infty)$, *assume* Ω *locally Lipschitz with* $\Omega^c \supset B_a(x_0)$, *for some* $a > 0$, *and set*

$$\mathfrak{w} = \begin{cases} |x-x_0|^{-1} & \text{if } q > n \\ (|x-x_0|\ln(|x-x_0|/a))^{-1} & \text{if } q = n\,. \end{cases} \tag{II.6.23}$$

Then, if u *has zero trace at* $\partial\Omega$, *we have* $\mathfrak{w}\, u \in L^q(\Omega)$, *and the following inequality holds, for all* $R > \delta(\Omega^c)$,

$$\|\mathfrak{w}\, u\|_{q,\Omega_R(x_0)} \leq C_q\, |u|_{1,q,\Omega_R(x_0)}\,, \tag{II.6.24}$$

where $\Omega_R(x_0) \equiv \Omega \cap B_R(x_0)$, *and* $C_q = q/(q-n)$, *if* $q > n$, *while* $C_q = n/[a\,(n-1)]$, *if* $q = n$.

Proof. As in the proof of Lemma II.6.3, it will be enough to consider smooth functions only. We begin to prove part (i). Let us integrate identity (II.6.11), with w in place of u and $\boldsymbol{g}$ given by (II.6.12), over the spherical shell:

$$\Omega^{R,r}(x_0) \equiv \Omega \cap (B_r(x_0) - B_R(x_0))\,, \quad r > R.$$

We have

$$(n-q)\int_{\Omega^{R,r}(x_0)}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \int_{\partial B_R(x_0)} \boldsymbol{g}\cdot\boldsymbol{n}|w|^q + r^{1-q}\int_{\partial B_r(x_0)}|w|^q$$

$$+q\int_{\Omega^{R,r}(x_0)}|\boldsymbol{g}||w|^{q-1}|\nabla w|,$$

where $\boldsymbol{n}$ is the unit normal to $\partial B_R(x_0)$ pointing toward x_0. This yields that the first term on the right-hand side of this latter equation is non-positive. Thus, estimating the integral over $\partial B_r(x_0)$ with the help of Lemma II.6.3, we deduce

$$(n-q)\int_{\Omega^{R,r}(x_0)}\left|\frac{w(x)}{x-x_0}\right|^q dx \le c_1\int_{\Omega^r(x_0)}|\nabla w|^q + q\int_{\Omega^{R,r}(x_0)}|\boldsymbol{g}||w|^{q-1}|\nabla w|,$$

where $c_1 = c_1(n,q)$. Now, if $q=1$ the result follows by letting $r\to\infty$ into this relation; otherwise, employing Young's inequality (II.2.5) with $\varepsilon = [(q-1)/\lambda(n-q)]^{q-1}$, $0<\lambda<1$, in the last integral at the right-hand side we obtain

$$\int_{\Omega^{R,r}(x_0)}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \frac{c_1}{(n-q)(1-\lambda)}\int_{\Omega^r(x_0)}|\nabla w|^q$$

$$+\frac{(q-1)^{q-1}}{(1-\lambda)\lambda^{q-1}(n-q)^q}\int_{\Omega^{R,r}(x_0)}|\nabla w|^q.$$

We now let $r\to\infty$ into this relation and minimize over λ, thus completing the proof of the first part of the lemma. To show the second part, for $r>(\alpha+2)R$ we set

$$\Omega^{R,r}\equiv\Omega\cap(B_r(x_0)-B_R),$$

and so, operating as before, we derive

$$(n-q)\int_{\Omega^{R,r}}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \int_{\partial B_R}\boldsymbol{g}\cdot\boldsymbol{n}|w|^q + r^{1-q}\int_{\partial B_r(x_0)}|w|^q$$

$$+q\int_{\Omega^{R,r}(x_0)}|\boldsymbol{g}||w|^{q-1}|\nabla w|.$$

If $q>1$, we use Young's inequality in the last integral, then Lemma II.6.3 to estimate the surface integral over $\partial B_r(x_0)$. Letting $r\to\infty$ we may then conclude, as in the proof of the first part of the lemma, the validity of the following inequality:

$$\int_{\Omega^R}\left|\frac{w(x)}{x-x_0}\right|^q dx \le \frac{1}{(n-q)(1-\lambda)}\int_{\partial B_R}\boldsymbol{g}\cdot\boldsymbol{n}|w|^q$$

$$+\frac{(q-1)^{q-1}}{(1-\lambda)\lambda^{q-1}(n-q)^q}\int_{\Omega^R}|\nabla w|^q \tag{II.6.25}$$

for all $\lambda \in (0,1)$. Now, if $x \in \partial B_R$ it is

$$|x - x_0| \geq |x_0| - |x| \geq (\alpha_0 - 1)R,$$

and so

$$|\boldsymbol{g}(x)| \leq |x - x_0|^{1-q} \leq [(\alpha_0 - 1)R]^{1-q}, \quad x \in \partial B_R.$$

From this inequality and Lemma II.6.3 we obtain the following:

$$\int_{\partial B_R} \boldsymbol{g} \cdot \boldsymbol{n} |w|^q \leq \frac{R^{n-q}}{(\alpha_0 - 1)^{q-1}} \int_{S^{n-1}} |w|^q \leq \frac{\gamma_0}{(\alpha_0 - 1)^{q-1}} \int_{\Omega^R} |\nabla w|^q,$$

which, once replaced into (II.6.25), proves (II.6.20) for $q > 1$. The proof for $q = 1$ is similar and therefore is left to the reader. To complete the proof of part (i), it remains to show the last statement. To this end, let $\varphi \in C^1(\mathbb{R})$ be a nondecreasing function such that $\varphi(\xi) = 0$ if $|\xi| \leq 1$ and $\varphi(\xi) = 1$ if $|\xi| \geq 2$. We set for $r > 2R > \delta(\Omega^c)$

$$\begin{aligned} \varphi_R(x) &= \varphi(|x|/R), \\ \chi_r(x) &= 1 - \varphi_r(x), \\ w^{\#}(x) &= \varphi_R(x)\chi_r(x)w(x). \end{aligned}$$

Notice that

$$|\nabla \chi_r(x)| \leq c/r, \quad c = c(\varphi).$$

Evidently, $w^{\#} \in W_0^{1,q}(\Omega)$, and we may apply Sobolev inequality (II.3.7) to deduce

$$\|w^{\#}\|_s \leq \gamma |w^{\#}|_{1,q}, \quad s = nq/(n-q),$$

which, by the properties of φ_R and χ_r, in turn implies

$$\|w^{\#}\|_s \leq c_1 \left(|w|_{1,q} + \|w\|_{q,\Omega_{R,2R}} + \|w|x|^{-1}\|_{q,\Omega_{r,2r}} \right),$$

with $c_1 = c_1(R, \varphi, n, q)$. We now let $r \to \infty$ into this relation. By inequality (II.6.19) the last term on the right-hand side must tend to zero. Using this fact along with the monotone convergence theorem, we recover

$$\|w\|_{s,\Omega^{2R}} \leq c_1 \left(|w|_{1,q} + \|w\|_{q,\Omega_{R,2R}} \right). \tag{II.6.26}$$

We next apply the inequality (II.5.18) to the integral over $\Omega_{R,2R}$ to deduce

$$\|w\|_{s,\Omega^{2R}} \leq c_2 \left(|w|_{1,q} + \left(\int_{\partial B_R \cup \partial B_{2R}} |w|^q \right)^{1/q} \right).$$

Using Lemma II.6.3 in this inequality, we finally obtain

$$\|w\|_{s,\Omega^{2R}} \leq c_3 |w|_{1,q}. \tag{II.6.27}$$

We now want to estimate w "near" $\partial\Omega$. We set

$$\zeta_R(x) = 1 - \varphi(|x|/2R)$$

and notice that

$$\zeta_R w \in W^{1,q}(\Omega).$$

Employing the embedding Theorem II.3.4, we obtain

$$\|w\|_{s,\Omega_{2R}} \le c_4 \left(|w|_{1,q} + \|w\|_{q,\Omega_{2R,4R}}\right).$$

We may now bound the last term on the right-hand side of this relation by $|w|_{1,q}$, in the same way as we did for the analogous term in (II.6.26), thus deducing

$$\|w\|_{s,\Omega_{2R}} \le c_5 |w|_{1,q}.$$

The last claim in part (i) of the lemma then follows from this latter inequality and from (II.6.27). We shall prove the claim in part (ii) when $q > n$, the case $q = n$ being treated in exactly the same way. We integrate (II.6.11) over $\Omega_R(x_0)$, with arbitrary $R > \delta(\Omega^c)$. Recalling that u has zero trace at $\partial\Omega$, we find

$$(q-n)\int_{\Omega_R(x_0)} \frac{|u|^q}{|x-x_0|^q} = -\int_{\partial B_R(x_0)} \boldsymbol{g}\cdot\boldsymbol{n}|u|^q - \int_{\Omega_R(x_0)} \boldsymbol{g}\cdot\nabla|u|^q.$$

The surface integral in this relation is non positive, so that, proceeding as in the proof of (II.6.13) we obtain

$$\int_{\Omega_R(x_0)} \frac{|u|^q}{|x-x_0|^q} \le \frac{q}{(q-n)}\int_{\Omega_R(x_0)} |\nabla u|^q, \qquad \text{(II.6.28)}$$

which, in turn, by the arbitrarity of R proves the claim. □

Exercise II.6.4 Let $L^q_{\mathfrak{w}}(\Omega)$, $q \ge n \ge 2$, be the class of (measurable) functions v such that $\mathfrak{w}\, v \in L^q(\Omega)$, with $\mathfrak{w}$ defined in (II.6.24). Show that $L^q_{\mathfrak{w}}(\Omega)$ endowed with the norm $\|\mathfrak{w}(\cdot)\|_q$ is a Banach space.

Exercise II.6.5 Let $u \in D^{1,q}(B^R)$, $q \in [1,n)$. Show that u satisfies (II.6.21), with $\Omega \equiv B^R$, with a constant γ_1 independent of R.

Exercise II.6.6 Let $u \in D^{1,q}(B_R(x_0))$, $n \ge 2$, $q > n$, $R > 0$. Show that the following inequality holds

$$\|(u - u(x_0))/|x-x_0|\|_{q,B_R(x_0)} \le q/(q-n)|u|_{1,q,B_R(x_0)}.$$

Hint: Integrate (II.6.11) over $B_R(x_0) - B_\varepsilon(x_0)$, $\varepsilon < R$. Then, use the results of Exercise II.5.11 and let $\varepsilon \to 0$. (Notice that $u(x_0)$ is well defined, because, for $q > n$, $D^{1,q}(\Omega) \subset W^{1,q}(B_R(x_0)) \subset C(B_R(x_0))$; see Lemma II.6.1 and Theorem II.3.4.)

Exercise II.6.7 Let Ω be an exterior, locally Lipschitz domain, and assume that $u \in D^{1,q}(\Omega)$, $q > n$, with zero trace at $\partial\Omega$. Show that, for all $R > \delta(\Omega^c)$ and all $x_0 \in \Omega_R$,

$$\|\mathfrak{w}(u - u(x_0))\|_{q,\Omega_R} \leq \frac{q}{q-n}|u|_{1,q,\Omega_R}\,,$$

where $\mathfrak{w}$ is defined in (II.6.23)$_1$. *Hint*: Integrate (II.6.11) over $\Omega_R - B_\varepsilon(x_0)$, for sufficiently small ε. Then use the results of Exercise II.5.11 and let $\varepsilon \to 0$.

Exercise II.6.8 Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$, and let $u \in D^{1,q}(\Omega)$, $q \in [1,\infty)$, satisfy the following generalized version of "vanishing of the trace" at $\partial\Omega$:

$$\psi\, u \in W_0^{1,q}(\Omega)\,, \text{ for all } \psi \in C_0^\infty(\mathbb{R}^n)\,. \tag{II.6.29}$$

(a) Assume $q \geq n$ and that $\Omega^c \supset B_a(x_0)$, for some $x_0 \in \mathbb{R}^n$ and $a > 0$. Show that u satisfy (II.6.24)

(b) Assume $q \in [1,n)$, and that the constant u_0 associated to u by Lemma II.6.3 is zero. Show that $u \in L^{nq/(n-q)}(\Omega)$ and that there exists $C = C(n,q,\Omega)$ such that

$$\|u\|_{nq/(n-q)} \leq C\,|u|_{1,q}\,.$$

Theorem II.6.1 ensures, in particular, that, for Ω an exterior locally Lipschitz domain and for $q \in [1,n)$, every function from $D^{1,q}(\Omega)$, possibly modified by the addition of a uniquely determined constant, obeys the Sobolev inequality (II.6.22), even though its trace at the boundary need not be zero. Our next goal is to perform a similar analysis, more generally, for Troisi inequality (II.3.8). Specifically, assuming that the seminorms of u appearing on the right-hand side of (II.3.8) are finite, we wish to investigate if $u \in L^r(\Omega)$ and if (II.3.8) holds. To this end, we will use a special "anisotropic cut-off" function whose existence is proved in the next lemma; see Galdi & Silvestre (2007a) and Galdi (2007). The lemma will also include properties of this function which are not immediately needed, but that will be very useful for future purposes; see, e.g., Chapter VIII.

Lemma II.6.4 *For any $\alpha, R > 0$, there exists a function $\psi_{\alpha,R} \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_{\alpha,R}(x) \leq 1$, for all $x \in \mathbb{R}^n$ and satisfying the following properties*

$$\begin{aligned}
&\lim_{R\to\infty} \psi_{\alpha,R}(x) = 1 \quad \textit{uniformly pointwise, for all } \alpha > 0\,,\\
&\left|\frac{\partial\psi_{\alpha,R}}{\partial x_1}(x)\right| \leq \frac{C_1}{R^\alpha}, \quad \left|\frac{\partial\psi_{\alpha,R}}{\partial x_i}(x)\right| \leq \frac{C_1}{R}\,,\ i = 2,\dots,n\,,\\
&|\Delta\psi_{\alpha,R}(x)| \leq \frac{C_2}{R^2},\\
&(\boldsymbol{e}_1 \times x)\cdot\nabla\psi_{\alpha,R}(x) = 0 \quad \textit{for all } x \in \mathbb{R}^3\,,
\end{aligned} \tag{II.6.30}$$

where C_1, C_2 are independent of x and R. Moreover, the support of $\partial\psi_{\alpha,R}/\partial x_j$, $j = 1,\dots,n$, is contained in the cylindrical shell $\mathcal{S}_R = \mathcal{S}_R^{(1)} \cap \mathcal{S}_R^{(2)}$ where

$$\mathcal{S}_R^{(1)} = \left\{ x \in \mathbb{R}^n : \ \frac{R}{\sqrt{2}} < r < \sqrt{2}R, \ \right\},$$

$$\mathcal{S}_R^{(2)} = \left\{ x \in \mathbb{R}^n : \frac{R^\alpha}{\sqrt{2}} < |x_1| < \sqrt{2}R^\alpha \right\} \cup \left\{ x \in \mathbb{R}^n : \ -\frac{R^\alpha}{\sqrt{2}} \le x_1 \le \frac{R^\alpha}{\sqrt{2}} \right\}, \tag{II.6.31}$$

and where $r = (x_2^2 + \cdots x_n^2)^{1/2}$. *In addition, the following properties hold for all* $\alpha > 0$

$$\frac{\partial \psi_{\alpha,R}}{\partial x_1} \in L^q(\mathbb{R}^3), \quad \textit{for all } q \ge \tfrac{n-1}{\alpha} + 1, \quad \left\| \frac{\partial \psi_{\alpha,R}}{\partial x_1} \right\|_q \le C_3,$$

$$\|(u - u_0)\,|\nabla \psi_{\alpha,R}|\,\|_s \le C_4\, |u|_{1,s,\Omega^{\frac{R^\beta}{\sqrt{2}}}}, \quad \textit{for all } u \in D^{1,s}(\mathbb{R}^n), 1 \le s < n, \tag{II.6.32}$$

where u_0 *is the constant associated to* u *by* Lemma II.6.3, $\beta = \min\{1, \alpha\}$, *and* C_3, C_4 *are independent of* R.

Proof. Let $\psi = \psi(t)$ be a C^∞, non-increasing real function, such that $\psi(t) = 1$, $t \in [0,1]$ and $\psi(t) = 0$, $t \ge 2$. We set

$$\psi_{\alpha,R}(x) = \psi\left(\sqrt{\frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2}} \right), \ x \in \mathbb{R}^n,$$

so that we find

$$\psi_{\alpha,R}(x) = \begin{cases} 1 \ \text{ if } \dfrac{x_1^2}{R^{2\alpha}} + \dfrac{r^2}{R^2} \le 1 \\ 0 \ \text{ if } \dfrac{x_1^2}{R^{2\alpha}} + \dfrac{r^2}{R^2} \ge 4 \,. \end{cases} \tag{II.6.33}$$

The first property in (II.6.30) then follows at once. Moreover, since

$$\frac{\partial \psi_{\alpha,R}}{\partial x_1}(x) = \frac{x_1}{R^\alpha \sqrt{x_1^2 + R^{2\alpha-2} r^2}}\, \psi'\left(\sqrt{\frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2}} \right),$$

$$\frac{\partial \psi_{\alpha,R}}{\partial x_i}(x) = \frac{x_i}{R \sqrt{R^{2-2\alpha} x_1^2 + r^2}}\, \psi'\left(\sqrt{\frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2}} \right), \quad i = 2, \dots n,$$

the uniform bounds for the first derivatives hold with $C := \max_{t \ge 0} |\psi'(t)|$. The estimate for the Laplacean of $\psi_{\alpha,R}$ is easily obtained with C_2 depending on C_1 and $\max_{t\ge0} |\psi''(t)|$. Moreover, the orthogonality relation $(\text{II.6.30})_4$ is immediate if we take account the above components of $\nabla \psi_{\alpha,R}$ and the fact that $e_1 \times x = -x_3 e_2 + x_2 e_3$. Denote next by Σ the support of $\nabla \psi_{\alpha,R}$. From (II.6.33) we deduce that

$$\Sigma \subset \left\{ x \in \mathbb{R}^n : \ 1 < \frac{x_1^2}{R^{2\alpha}} + \frac{r^2}{R^2} < 4 \right\} \equiv \Sigma_1 \,.$$

Consider the following sets

$$\mathcal{S}_1 = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{R^{2\alpha}} < \frac{1}{2} \text{ and } \frac{r^2}{R^2} < \frac{1}{2} \right\},$$

$$\mathcal{S}_2 = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{R^{2\alpha}} > 2 \text{ and } \frac{r^2}{R^2} > 2 \right\}.$$

Clearly, ${\Sigma_1}^c \supset \mathcal{S}_1 \cup \mathcal{S}_2$. Therefore, by de Morgan's law, we get $\Sigma_1 \subset \mathcal{S}_1^c \cap \mathcal{S}_2^c$ and we conclude, from (II.6.31), that $\Sigma_1 \subset \mathcal{S}$, since $\mathcal{S}_1^c \cap \mathcal{S}_2^c = \mathcal{S}$. It remains to prove (II.6.32). The first property follows at once from the estimate for $\partial\psi_{\alpha,R}/\partial x_1$ given in (II.6.30) and the fact that the measure of the support of $\partial\psi_{\alpha,R}/\partial x_1$ is bounded by a constant times $R^{\alpha+n-1}$. Furthermore, we observe that, for all $x \in \mathcal{S}_R$, it is $|x| \le C\sqrt{(R^{2\alpha} + R^2)}$, with C a positive constant independent of R. Thus, from (II.6.30) we find, with $w \equiv u - u_0$,

$$\|w\,|\nabla\psi_{\alpha,R}|\|_{s,\Omega} = \|w\,|\nabla\psi_{\alpha,R}|\|_{s,S_R} \le C_2\,\|w/|x|\|_{s,S_R} \le C_2\|w/|x|\|_{s,B^{\frac{R^\beta}{\sqrt{2}}}},$$

with C_2 a positive constant independent of R and w. The second property in (II.6.32) then follows from this latter inequality and from (II.6.19). The proof of the lemma is complete. □

We are now in a position to prove the following result.

Theorem II.6.2 *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 3$, be an exterior locally Lipschitz domain. Assume $u \in D^{1,2}(\Omega)$ and*

$$\frac{\partial u}{\partial x_1} \in L^{q_1}(\Omega), \quad 1 < q_1 < 2.$$

Then, denoting by u_0 the uniquely determined constant associated to u by Lemma II.6.3, *we have*

$$w = u - u_0 \in L^r(\Omega), \quad r = \frac{2nq_1}{2 + (n-3)q_1},$$

and

$$\|w\|_r^n \le C \left(\left\| \frac{\partial u}{\partial x_1} \right\|_{q_1} \prod_{i=2}^n \|D_i u\|_2 + |u|_{1,2}^n \right), \qquad \text{(II.6.34)}$$

with $C = C(q_1, n, \Omega)$.

Proof. Let $\phi_\rho = \phi_\rho(x)$ be a smooth "cut-off" function that is 1 for $x \in \Omega_\rho$, it is 0 for $x \in \Omega^{2\rho}$, and that satisfies $\max_{x\in\Omega} |\nabla\phi_\rho(x)| \le M$, with M independent of x. We thus have $w = \phi_\rho w + (1 - \phi_\rho)w \equiv w_1 + w_2$. We begin to show the following property: $D_1 w_2$ and $D_i w_2$, $i = 2, \ldots, n$, can be approximated, in $L^{q_1} \cap L^2$ and L^2, respectively, by a sequence of functions from $C_0^\infty(\mathbb{R}^n)$. To this end, we set $\widetilde{w}_{2,k} = \psi_{\alpha,R_k} w_2$, where $\psi_{\alpha,R}$ the function constructed in Lemma

II.6.4 with a choice of α that we specify later in the proof, and where $\{R_k\}$ is an unbounded sequence of positive numbers with R_0 sufficiently large. We thus have that the support of $\widetilde{w}_{2,k}$ is compact in $\mathbb{R}^n$. Therefore, its regularizer, $(\widetilde{w}_{2,k})_\varepsilon$ is in $C_0^\infty(\mathbb{R}^n)$. Observing that $D_j(\widetilde{w}_{2,k})_\varepsilon = (D_j\widetilde{w}_{2,k})_\varepsilon$, $j = 1,\dots,n$ (see Exercise II.3.2), in view of (II.2.9) we may choose a vanishing sequence $\{\varepsilon_k\}$ such that

$$\lim_{k\to\infty} \|D_j\widetilde{w}_{2,k} - D_j w_{2,k}\|_{s_j} = 0\,, \tag{II.6.35}$$

where $w_{2,k} = (\widetilde{w}_{2,k})_{\varepsilon_k}$, $s_1 \in \{q_1, 2\}$, and $s_j = 2$ for $j = 2,\dots,n$. By the Minkowski inequality, we also obtain

$$\|D_j w_2 - D_j w_{2,k}\|_{s_j} \le \|D_j w_2 - D_j\widetilde{w}_{2,k}\|_{s_j} + \|D_j\widetilde{w}_{2,k} - D_j w_{2,k}\|_{s_j}\,, \tag{II.6.36}$$

so that, in view of (II.6.35), to show the stated property we have to show that the first term on the right-hand side of (II.6.36) tends to 0 as $k \to \infty$. We now observe that

$$\|D_j w_2 - D_j\widetilde{w}_{2,k}\|_{s_j} \le \|(1-\psi_{\alpha,R_k})D_j w_2\|_{s_j} + \|D_j\psi_{\alpha,R_k} w_2\|_{s_j}\,, \tag{II.6.37}$$

and so, in view of $(\text{II.6.30})_1$, the property follows if we prove that the second term on the right-hand side of (II.6.37) vanishes as $k \to \infty$. Take $j = 1$ and $s_j = q_1$ first. Since

$$\|D_1\psi_{\alpha,R_k} w_2\|_{q_1} \le \|D_1\psi_{\alpha,R_k}\|_{\frac{2nq_1}{2n-(n-2)q_1}} \|w_2\|_{\frac{2n}{n-2},\Omega^{R_k/\sqrt{2}}}\,,$$

and, by Theorem II.6.1, $w_2 \in L^{2n/(n-2)}(\Omega)$, we take $\alpha \ge (n-1)[2n-(n-2)q_1]/[3nq_1 - 2(n+q_1)]$ to deduce, from the properties of $\psi_{\alpha,R}$,

$$\lim_{k\to\infty} \|D_1 w_2 - D_1\widetilde{w}_{2,k}\|_{q_1} = 0\,. \tag{II.6.38}$$

We next choose $s_j = 2$, $j = 1,\dots,n$, and obtain, with the help of (II.6.32),

$$\|D_j\psi_{\alpha,R_k} w_2\|_2 \le C\,\|w_2/|x|\|_{2,\Omega^{R_k/\sqrt{2}}}\,,$$

which, by (II.6.19) and (II.6.37) implies

$$\lim_{k\to\infty} \|D_j w_2 - D_j\widetilde{w}_{2,k}\|_2 = 0\,. \tag{II.6.39}$$

From (II.6.35), (II.6.36), (II.6.38), and (II.6.39) it then follows

$$\lim_{k\to\infty} \|D_j w_2 - D_j w_{2,k}\|_{s_j} = 0\,, \quad j = 1,\dots,n\,, \tag{II.6.40}$$

which proves the desired property. Notice that, by Theorem II.6.1, (II.6.40) yields

$$\lim_{k\to\infty} \|w_2 - w_{2,k}\|_{2n/(n-2)} = 0\,. \tag{II.6.41}$$

We next observe that each function $w_{2,k}$ obeys, in particular, Troisi inequality (II.3.8) with $s = r$, $q_1 = q_1$ and $q_2 = \cdots = q_n = 2$. In fact, this inequality shows also that $\{w_{2,k}\}$ is Cauchy in $L^r(\Omega)$ and thus it converges there to some $\overline{w}$. In view of (II.6.40) and (II.6.41), it is simple to show that $\overline{w} = w_2$, a.e. in Ω, and that the inequality continues to hold also for the function w_2:

$$\|w_2\|_r^n \leq c \left\| \frac{\partial w_2}{\partial x_1} \right\|_{q_1} \prod_{i=2}^{n} \|D_i w_2\|_2 \,. \tag{II.6.42}$$

Furthermore, by the fact that $w \in L^{2n/(n-2)}(\Omega)$, it follows $w_1 \in L^r(\Omega)$, and since $w = w_1 + w_2$, we deduce $w \in L^r(\Omega)$. It thus remains to prove the validity of (II.6.34) when $\Omega \neq \mathbb{R}^n$. Recalling that $w_2 = \phi_\rho w$, we readily obtain

$$\begin{aligned}\|w_2\|_r^n \leq{}& c_1 \left\| \frac{\partial w}{\partial x_1} \right\|_{q_1} \prod_{i=2}^{n} \|D_i w\|_2 \\ &+ c_2 \left[(\|w\|_{q_1,\sigma} + |w|_{1,2}) \|w\|_{2,\sigma}^{n-1} + \|w\|_{q_1,\sigma} |w|_{1,2}^{n-1} \right] ,\end{aligned} \tag{II.6.43}$$

where σ is the (bounded) support of $\nabla \phi_\rho$. We now suitably apply the Hölder inequality in the σ-terms in square brackets and then use (II.6.22) with $q = 2$. Consequently, (II.6.43) furnishes

$$\|w_2\|_r^n \leq c_1 \left\| \frac{\partial w}{\partial x_1} \right\|_{q_1} \prod_{i=2}^{n} \|D_i w\|_2 + c_3 \, |w|_{1,2}^n \,. \tag{II.6.44}$$

Finally, from Exercise II.3.12, we readily find that

$$\|w_1\|_r \leq c_4 \left(\|w\|_{2,\sigma'} + |w|_{1,2} \right) ,$$

with σ' the (bounded) support of ϕ_ρ. Then, inequality (II.6.34) follows from this latter inequality, from (II.6.22) with $q = 2$ and (II.6.44). □

Exercise II.6.9 Show that if $\Omega = \mathbb{R}^n$, the last term on the right-hand side of (II.6.34) can be omitted.

We would like now to extend the results of Theorem II.6.1 to the case when Ω is a half-space (see Remark II.6.4).[2] We begin to observe that, given $u \in D^{1,q}(\mathbb{R}^n_+)$, $1 \leq q < \infty$, we may extend it to a function $u' \in D^{1,q}(\mathbb{R}^n)$ satisfying (see Exercise II.3.10)

$$\begin{aligned} u(x) &= u'(x), \quad x \in \mathbb{R}^n_+ \,, \\ |u'|_{1,q,\mathbb{R}^n} &\leq c|u|_{1,q,\mathbb{R}^n_+} \leq c|u'|_{1,q,\mathbb{R}^n} \,. \end{aligned} \tag{II.6.45}$$

If $1 \leq q < n$, by Lemma II.6.3, there is a uniquely determined $u_0 \in \mathbb{R}$ such that $(u' - u_0) \in L^s(\mathbb{R}^n)$, $s = nq/(n-q)$, and, moreover,

[2] As a matter of fact, also Theorem II.6.2 can be extended to $\Omega = \mathbb{R}^n_+$. However, for our purposes, this extension would be irrelevant.

$$\|u' - u_0\|_{s,\mathbb{R}^n} \leq \gamma_1 |u'|_{1,q,\mathbb{R}^n}.$$

This relation, together with (II.6.45), then delivers

$$\|u - u_0\|_{s,\mathbb{R}^n_+} \leq \gamma_3 |u|_{1,q,\mathbb{R}^n_+},$$

which is what we wanted to show. It is interesting to observe that if u has zero trace at the boundary $x_n = 0$ then $u_0 = 0$.[3] Actually, denoting by $\widehat{u}$ the function obtained by setting $u \equiv 0$ outside $\mathbb{R}^n_+$, one easily shows

$$\widehat{u} \in D^{1,q}(\mathbb{R}^n)$$

$$|\widehat{u}|_{1,q,\mathbb{R}^n} \leq |u|_{1,q,\mathbb{R}^n_+}$$

(see Exercise II.6.10). Setting $S^{n-1}_- = S^{n-1} \cap \mathbb{R}^n_-$, by Lemma II.6.3 we deduce

$$|u_0|^q |S^{n-1}_-| \leq \int_{S^{n-1}} |\widehat{u}(R,\omega) - u_0|^q d\omega \leq \gamma_0 R^{q-n} |\widehat{u}|_{1,q,\Omega^R},$$

for all $R > 0$, which furnishes $u_0 = 0$. By the same token, we can show weighted inequalities of the type (II.6.19) and (II.6.20). Next, if $q \geq n$, we notice that, if u has zero trace at the plane $x_n = 0$, we may apply the results of part (ii) in Theorem II.6.1 to the extension $\widehat{u}$, to show that the same results continue to hold for $\Omega = \mathbb{R}^n_+$, and with an arbitrary $x_0 \in \mathbb{R}^n_-$. Actually, we can prove a somewhat stronger weighted inequality, holding for any $u \in D^{1,q}(\mathbb{R}^n_+)$, $q \in (1,\infty)$, that vanishes at $x_n = 0$. We start with the identity (valid for smooth u)

$$\frac{\partial}{\partial x_n}\left[\frac{|u|^q}{(1+x_n)^{q-1}}\right] = \frac{1}{(1+x_n)^{q-1}}\frac{\partial |u|^q}{\partial x_n} + (1-q)\frac{|u|^q}{(1+x_n)^q}.$$

Integrating this inequality over the parallelepiped $P_{a,b} = \{x \in \mathbb{R}^n_+ : |x'| < b\,,\ x_n \in (0,a)\}$, $x' \equiv (x_1, \cdots, x_{n-1})$, and using the fact that u vanishes at $x_n = 0$ along with the Hölder inequality, we deduce

$$\|u/(1+x_n)\|_{q,P_{a,b}} \leq \frac{q}{q-1} |u|_{1,q,P_{a,b}}.$$

Since $D^{1,q}(\mathbb{R}^n_+) \subset W^{1,q}(P_{a,b})$, by a density argument we can extend this latter inequality to functions merely belonging to $D^{1,q}(\mathbb{R}^n_+)$ having zero trace at $x_n = 0$. Thus, in particular, letting $b \to \infty$, we find, for all $a > 0$,

$$\|u/(1+x_n)\|_{q,L_a} \leq \frac{q}{q-1} |u|_{1,q,L_a}. \tag{II.6.46}$$

where

$$L_a = \{x \in \mathbb{R}^n_+ :\ x_n \in (0,a)\}. \tag{II.6.47}$$

We may summarize the above considerations in the following.

[3] Notice that since $u \in W^{1,q}(C)$ for every cube C of $\mathbb{R}^n_+$ with a side at $x_n = 0$, the trace of u at $x_n = 0$ is well defined. A more general result for u_0 to be zero is furnished in Exercise II.7.5 and Section II.10.

Theorem II.6.3 *Let $n \geq 2$ and assume*

$$u \in D^{1,q}(\mathbb{R}^n_+), \quad 1 \leq q < \infty.$$

(i) *If $q \in [1, n)$, there exists a uniquely determined $u_0 \in \mathbb{R}$ such that the function*

$$w = u - u_0$$

enjoys the following properties. For any $x_0 \in \mathbb{R}^n$, it is

$$w|x - x_0|^{-1} \in L^q(\Omega^R(x_0)),$$

where

$$\Omega^R(x_0) \equiv \mathbb{R}^n_+ - B_R(x_0)$$

and the following inequality holds:

$$\left(\int_{\Omega^R(x_0)} \left| \frac{w(x)}{x - x_0} \right|^q dx \right)^{1/q} \leq q/(n-q)|w|_{1,q,\Omega^R(x_0)}. \tag{II.6.48}$$

Furthermore, if $x_0 \in \mathbb{R}^n_+$, $|x_0| = \alpha R$, for some $\alpha \geq \alpha_0 > 1$ and some $R > 0$, we have

$$\left(\int_{\Omega^R} \left| \frac{w(x)}{x - x_0} \right|^q dx \right)^{1/q} \leq c|w|_{1,q,\Omega^R}$$

with $\Omega^R = \mathbb{R}^n_+ - B_R$ and $c = c(n, q, \alpha_0)$. In addition,

$$w \in L^s(\mathbb{R}^n_+), \quad s = nq/(n-q) \tag{II.6.49}$$

and for some γ_2 independent of u

$$\|w\|_s \leq \gamma_2 |w|_{1,q}.$$

If the trace of u is zero at $x_n = 0$, then $u_0 = 0$.

(ii) *If $q \geq n$, and u has zero trace at $x_n = 0$ then $\mathfrak{w}\, u \in L^q(\mathbb{R}^n_+)$ and inequality* (II.6.24) *holds with any $x_0 \in \mathbb{R}^n_-$.*[4]

(iii) *If $q \in (1, \infty)$ and u has zero trace at $x_n = 0$, then $u/(1 + x_n) \in L^q(\mathbb{R}^n_+)$ and inequality* (II.6.46) *holds for all $a > 0$.*

By means of a simple procedure based on the iterative use of (II.6.22) and (II.6.49) one can show the following general embedding theorem for functions in $D^{m,q}(\Omega)$, whose proof is left to the reader as an exercise.

Theorem II.6.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be either a locally Lipschitz exterior domain or $\Omega = \mathbb{R}^n_+$, and let $u \in D^{m,q}(\Omega)$, $m \geq 1$, $1 \leq q < \infty$.*

[4] So that (II.6.24) holds with $\Omega_R(x_0) \equiv \mathbb{R}^n_+$.

(a) *If* $q \in [1, n)$, *let* $\ell \in \{1, \dots, m\}$ *be the largest integer such that* $\ell q < n$. *Then there are* ℓ *uniquely determined homogeneous polynomials* $\mathcal{M}_{m-r}$, $r = 1, \dots, \ell$, *of degree* $\leq m - r$ *such that, setting*

$$u_{m-k} = \sum_{r=1}^{k} \mathcal{M}_{m-r}\,, \quad k \in \{1, \dots, \ell\}\,,$$

we have

(i) $(u - u_{m-k}) \in D^{m-k,q_k}(\Omega)\,,$

(ii) $\displaystyle\sum_{k=1}^{\ell} |u - u_{m-k}|_{m-k,q_k} \leq c|u|_{m,q}$,

where $q_k = nq/(n - kq)\,.$

(b) *If* $q \geq n$, $\Omega \neq \mathbb{R}^n$, *and the trace of* $D^\alpha u$, $|\alpha| = m - 1$, *is zero at* $\partial\Omega$, *then* $\mathfrak{w}\, D^\alpha u \in L^q(\Omega_R(x_0))$, *with* $\mathfrak{w}$ *and* $\Omega_R(x_0)$ *given in part* (ii) *of* Theorem II.6.1 *and* Theorem II.6.3, *and* (II.6.24) *holds with* $u \equiv D^\alpha u$.

(c) *If* $u \in D^{m,q}(\mathbb{R}^n_+)$, $q \in (1, \infty)$, *and the trace of* $D^\alpha u$, $|\alpha| = m - 1$, *is zero at* $x_n = 0$, *then* $D^\alpha u/(1 + x_n) \in L^q(\mathbb{R}^n_+)$ *and inequality* (II.6.46), *with* $u \equiv D^\alpha u$, *holds for all* $a > 0$.

Our final objective is to establish embedding inequalities for functions from $D^{m,q}(\Omega)$ that vanish at $\partial\Omega$. We wish to prove these results without assuming any regularity on $\partial\Omega$, and so we introduce the following generalized version of "vanishing of traces at the boundary" for $u \in D^{m,q}(\Omega)$ (see Simader and Sohr 1997, Chapter I)

$$\psi u \in W_0^{m,q}(\Omega)\,, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^n)\,. \tag{II.6.50}$$

Remark II.6.5 In view of Theorem II.4.2, we find at once that, if Ω has the regularity specified in that theorem, condition (II.6.50) is *equivalent* to the condition $\Gamma_m(u) = 0$ at $\partial\Omega$. ■

Theorem II.6.5 *Let* Ω *be an exterior domain of* $\mathbb{R}^n$, $n \geq 2$, *and let* $u \in D^{m,q}(\Omega)$, $m \geq 1$, $q \in [1, \infty)$, *satisfy* (II.6.50).

(i) *Assume* $\Omega^c \supset B_a$, *for some* $a > 0$. *Then, the following inequality holds for all* $R > \delta(\Omega^c)$

$$\|u\|_{m-1,q,\Omega_R} \leq m\, C\, |u|_{m,q,\Omega_R}\,,$$

where $C = n^{-1/q} R^{1+(n-1)/q} a^{(1-n)/q}$.

(ii) *Assume* $q \in [1, n)$ *and let* $\ell \in \{1, \dots, m\}$ *be the largest integer such that* $\ell q < n$. *Then, if the homogeneous polynomials* $\mathcal{M}_{m-r}$, $r = 1, \dots, \ell$, *defined in* Theorem II.6.4(a) *are all zero, the properties* (i) *and* (ii) *of that theorem hold.*

Proof. For any given $R > \delta(\Omega^c)$, let $\psi \in C_0^\infty(\mathbb{R}^n)$ to be 1 in Ω_{2R} and 0 in Ω^{3R}. By (II.6.50), we know that there is $\{u_s\} \subset W_0^{m,q}(\Omega)$ converging to ψu.

Thus, it is enough to show the statement in (i) for $u \in C_0^\infty(\Omega)$ and for $m = 1$. If we extend u to 0 in Ω^c, we find

$$u(x) = \int_a^{|x|} \frac{\partial u}{\partial r}(r\, x/|x|)dr\,.$$

By using the Hölder inequality in this identity, we derive

$$|u(x)|^q \leq R^{q-1}a^{1-n}\int_a^R |\nabla u|^q r^{n-1}dr\,.$$

Therefore, by multiplying both sides of this inequality by r^{n-1}, and by integrating the resulting relations over $r \in [a, R]$ and again over the unit sphere, we obtain the desired inequality. Under the stated assumptions in part (ii), from Theorem II.6.4(a) we find

$$\sum_{k=1}^{\ell} |u|_{m-k,nq/(n-kq),\Omega^R} \leq C\,|u|_{m,q}\,, \tag{II.6.51}$$

while, by a repeated use of (II.6.9), it follows that

$$\sum_{k=1}^{\ell} |u_s|_{m-k,nq/(n-kq),\Omega_R} \leq C\,|u_s|_{m,q}\,.$$

Passing to the limit $s \to \infty$ in this relation, and recalling the properties of ψ, we deduce

$$\sum_{k=1}^{\ell} |u|_{m-k,\frac{nq}{n-kq},\Omega_R} \leq C\left(\sum_{k=1}^{\ell} |u|_{m-k,q,\Omega_{2R,3R}} + \sum_{k=\ell+1}^{m} |u|_{m-k,q,\Omega_{2R,3R}} + |u|_{m,q}\right).$$

Combining this inequality with (II.6.51), we find

$$\sum_{k=1}^{\ell} |u|_{m-k,nq/(n-kq)} \leq C\,\left(\|u\|_{m-\ell-1,q,\Omega_{2R,3R}} + |u|_{m,q}\right)\,, \tag{II.6.52}$$

and the result follows from (II.6.52) and part (i). □

Exercise II.6.10 Let $u \in D^{1,q}(\Omega)$, $1 \leq q < \infty$. Assume $\Omega \cap B_r(x_0)$ locally Lipschitz for every $x_0 \in \partial\Omega$ and some $r > 0$. Show that if u has zero trace at $\partial\Omega$, then its extension $\widehat{u}$ to $\mathbb{R}^n$, obtained by setting $u \equiv 0$ in Ω^c, is in $D^{1,q}(\mathbb{R}^n)$. *Hint:* Take φ arbitrary from $C_0^\infty(\mathbb{R}^n)$. and let B be an open ball with $B \supset \text{supp}\,(\varphi)$. Then $\varphi u \in W_0^{1,q}(\Omega \cap B)$, and one can argue as in Exercise II.3.11.

II.7 Approximation of Functions from $D^{m,q}$ by Smooth Functions and Characterization of the Space $D_0^{m,q}$

In the preceding section, we have defined the space $D_0^{m,q}(\Omega)$ as the (Cantor) completion of the normed space $\{C_0^\infty(\Omega)\,,|\cdot|_{m,q}\}$. As such, the generic element of $D_0^{m,q}(\Omega)$ is an equivalence class of Cauchy sequences. Our main objective in this section is to furnish a "concrete" representation of $D_0^{m,q}(\Omega)$, up to an isomorphism, when *Ω is either an exterior domain or a half-space.*

In order to reach this objective, it is of the utmost importance to investigate the conditions under which an element from $D^{m,q}(\Omega)$ can be approximated by functions from $C_0^\infty(\Omega)$ in the seminorm (II.6.4) (see Galdi and Simader 1990, and Remark II.6.4). As a by-product, we shall also find conditions ensuring the validity of this approximation by functions from $C_0^\infty(\overline{\Omega})$. Like we did previously in analogous circumstances, we shall consider the case $m=1$, leaving the case $m>1$ to the reader (see Theorem II.7.3 through Theorem II.7.8).

Theorem II.7.1 *Let $\Omega\subseteq\mathbb{R}^n$, $n\geq 2$, be an exterior domain, and let $u\in D^{1,q}(\Omega)$, $1\leq q<\infty$. Then, u can be approximated in the seminorm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\Omega)$ under the following assumptions.*

(i) *If $q\in[1,n)$, u satisfies* (II.6.50) *with $m=1$, and $u_0=0$, where u_0 is the constant of* Lemma II.6.3*;*
(ii) *If $q\in[n,\infty)$, u satisfies* (II.6.50) *with $m=1$,.*

Proof. We shall follow the ideas of Sobolev (1963b), combined with the arguments used in the proof of Theorem II.6.2. Let $\psi\in C_0^\infty(\mathbb{R})$ be nonincreasing with $\psi(\xi)=1$ if $|\xi|\leq 1/2$ and $\psi(\xi)=0$ if $|\xi|\geq 1$ and set, for R large enough,

$$\psi_R(x)=\psi\left(\frac{\ln\ln|x|}{\ln\ln R}\right). \tag{II.7.1}$$

Notice that, for a suitable constant $c>0$ independent of R,

$$|D^\alpha\psi_R(x)|\leq\frac{c}{\ln\ln R}\frac{1}{|x|^m\ln|x|},\quad |\alpha|=m\geq 1 \tag{II.7.2}$$

and $D^\alpha\psi_R(x)\not\equiv 0$, $|\alpha|\geq 1$, only if $x\in\widetilde{\Omega}_R$, where

$$\widetilde{\Omega}_R=\left\{x\in\Omega:\exp\sqrt{\ln R}<|x|<R\right\}. \tag{II.7.3}$$

Next, let $u\in D^{1,q}(\Omega)$, $q\in[1,\infty)$, satisfying (II.6.50) with $m=1$, and with $u_0=0$ if $q\in[1,n)$. We write $u=(1-\psi_R)u+\psi_R u$. By (II.6.50) we then have

$$\psi_R u\in W_0^{1,q}(\Omega) \tag{II.7.4}$$

for all $R>\delta(\Omega^c)$. So, given $\varepsilon>0$ we may find a sufficiently large R and a function $u_{R,\varepsilon}\in C_0^\infty(\Omega)$ such that

$$|u_{R,\varepsilon} - \psi_R u|_{1,q} < \varepsilon,$$

and

$$|u - u_{R,\varepsilon}|_{1,q} \leq \|(1-\psi_R)\nabla u\|_q + \|\nabla\psi_R u\|_q + |u_{R,\varepsilon} - \psi_R u|_{1,q} < 2\varepsilon + \|\nabla\psi_R u\|_q.$$

The lemma will then follow from this inequality, provided we show that the last term on its right-hand side tends to zero as $R \to \infty$. Setting

$$\ell(R) \equiv \|\nabla\psi_R u\|_q, \tag{II.7.5}$$

in view of (II.7.2) and (II.7.3) we can find a constant $c_1 > 0$ such that

$$\ell(R)^q \leq \frac{c_1}{(\ln\ln R)^q} \int_{\exp\sqrt{\ln R}}^R \int_{S^{n-1}} \frac{|u(r,\omega)|^q}{(\ln r)^q} r^{n-q-1} d\omega dr.$$

Now, by Lemma II.6.3 and Exercise II.6.3, recalling that $u_0 = 0$ if $q \in [1,n)$, we have

$$\int_{S^{n-1}} |u(r,\omega)|^q \leq c_2 g(r),$$

where, in particular,

$$g(r) = \begin{cases} (\ln r)^{n-1} & \text{if } q = n \\ r^{q-n} & \text{if } q \neq n\,,\ q \neq 1 \\ r^{1-n}|u|_{1,1,\Omega^r} & \text{if } q = 1. \end{cases}$$

Therefore, if $q = n$ we obtain

$$\ell(R)^n \leq \frac{c_2}{(\ln\ln R)^n} \int_{\exp\sqrt{\ln R}}^R (r\ln r)^{-1} dr \leq c_2 (\ln\ln R)^{1-n}; \tag{II.7.6}$$

and if $q \neq n$, $q \neq 1$,

$$\ell(R)^q \leq \frac{c_2}{(\ln\ln R)^q} \int_{\exp\sqrt{\ln R}}^R (\ln r)^{-q} r^{-1} dr \leq \frac{c_2}{(\ln\ln R)^q} \frac{(\ln R)^{(1-q)/2}}{(q-1)}. \tag{II.7.7}$$

Finally, if $q = 1$, we have

$$\ell(R) \leq \frac{c_2}{(\ln\ln R)} \int_{\exp\sqrt{\ln R}}^R (\ln r)^{-1} r^{-1} |u|_{1,1,\Omega^r} dr \leq \frac{c_2}{2} \, |u|_{1,1,\Omega^{\exp\sqrt{\ln R}}}\,. \tag{II.7.8}$$

So, for all $q \in [1,\infty)$, we recover

$$\lim_{R\to\infty} \ell(R) = 0,$$

which completes the proof of the theorem. □

Remark II.7.1 If the trace of u does not vanish at the boundary, that is, if u does not satisfy (II.6.50), Theorem II.7.1 should be suitably modified. In fact, on the one hand, the function $\psi_R u$ does not satisfy the condition (II.7.4) but, rather, it verifies the following one:

$$\psi_R u \in W^{1,q}(\Omega), \quad \text{for all } R > \delta(\Omega^c).$$

So, from Theorem II.3.1 it follows that, if Ω is locally Lipschitz, given $\varepsilon > 0$, we may find a sufficiently large R and a function $u_{R,\varepsilon} \in C_0^\infty(\overline{\Omega})$ such that

$$|u_{R,\varepsilon} - \psi_R u|_{1,q} < \varepsilon$$

and, as in the proof of Theorem II.7.1, we can prove that any $u \in D^{1,q}(\Omega)$ can be approximated in the seminorm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\overline{\Omega})$ for $q \geq n$. However, the same result continues to hold also when $1 \leq q < n$. In fact, it suffices to notice that, for any $u \in D^{1,q}(\Omega)$ with $u_0 \neq 0$, the function $\psi_R(u - u_0)$, with u_0 defined in Lemma II.6.3, is of bounded support in Ω, belongs to $W^{1,q}(\Omega)$ and approaches u in the seminorm $|\cdot|_{1,q}$. We thus have the following.

Theorem II.7.2 *Let Ω be locally Lipschitz, and let $u \in D^{1,q}(\Omega)$. Then, u can be approximated in the norm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\overline{\Omega})$.*

■

Exercise II.7.1 Let Ω be locally Lipschitz. Show that $C_0^\infty(\overline{\Omega})$ is dense in $\dot{D}^{1,q}(\Omega)$.

The technique employed in the proof of Theorem II.7.1 and Theorem II.7.2, along with the results of Theorem II.6.4, allow us to generalize the previous results to the space $D^{m,q}(\Omega)$, $m \geq 1$, in the following theorems, whose proofs we leave to the reader as an exercise.

Theorem II.7.3 *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an exterior domain and let $u \in D^{m,q}(\Omega)$, $1 \leq q < \infty$, $m \geq 1$. Then $u \in D^{m,q}(\Omega)$ can be approximated in the norm $|\cdot|_{m,q}$ by functions from $C_0^\infty(\Omega)$ under the following assumptions.*

(i) *If $q \in [1, n)$, u satisfies* (II.6.50) *and the following conditions hold:*

$$u_{m-\ell} \equiv 0\,, \tag{II.7.9}$$

where $\ell \in \{1, \dots, m\}$ is the largest integers such that $\ell q < n$ and the polynomials $u_{m-\ell}$ are defined in Theorem II.6.4.

(ii) *If $q \in [n, \infty)$, u satisfies* (II.6.50)

Theorem II.7.4 *Let Ω be a locally Lipschitz, exterior domain of $\mathbb{R}^n$, $n \geq 2$. Then, every $u \in D^{m,q}(\Omega)$ can be approximated in the seminorm $|\cdot|_{m,q}$ by functions from $C_0^\infty(\overline{\Omega})$.*

We are now in the position to prove a characterization of the space $D_0^{m,q}(\Omega)$. For the sake of argument, we shall first consider the case $m = 1$.

Set

$$\widetilde{D}_0^{1,q}(\Omega) = \begin{cases} \{u \in D^{1,q}(\Omega) : \|u\|_{\frac{nq}{n-q}} < \infty\,,\ u \text{ satisfies (II.6.50) with } m = 1\}\,, \\ \hfill \text{if } q \in [1, n) \\ \\ \{u \in D^{1,q}(\Omega) :\ u \text{ satisfies (II.6.50) with } m = 1\}\,, \text{ if } q \in [n, \infty) \end{cases} \tag{II.7.10}$$

where, if $q \geq n$, we assume $\Omega^c \supset B_a$, for some $a > 0$.

With the help of Exercise II.6.8, it is not difficult to show that $\widetilde{D}_0^{1,q}(\Omega)$, $1 \leq q < \infty$, endowed with the norm $|\cdot|_{1,q}$ is a Banach space, and that this norm is equivalent to the following one

$$\begin{aligned} &|\cdot|_{1,q} + \|\cdot\|_{nq/(n-q)} \ \text{ if } q \in [1, n) \\ &|\cdot|_{1,q} + \|\mathfrak{w}(\cdot)\|_q \ \text{ if } q \in [n, \infty)\,. \end{aligned} \tag{II.7.11}$$

where $\mathfrak{w}$ is defined in (II.6.23).

Theorem II.7.5 *Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$. Then $D_0^{1,q}(\Omega)$, $q \in [1, \infty)$, is isomorphic to $\widetilde{D}_0^{1,q}(\Omega)$, where $\Omega \neq \mathbb{R}^n$, if $q \geq n$. If $q \geq n$ and $\Omega = \mathbb{R}^n$, then $D_0^{1,q}(\mathbb{R}^n)$ is isomorphic to $\dot{D}^{1,q}(\mathbb{R}^n)$.*

Proof. We first consider the two cases: either (i) $q \in [1, n)$, or (ii) $q \in [n, \infty)$ and $\Omega \neq \mathbb{R}^n$, and begin to construct a suitable map $\mathfrak{T} : D_0^{1,q}(\Omega) \to \widetilde{D}_0^{1,q}(\Omega)$. Let $\widetilde{u}$ be a generic element in $D_0^{1,q}(\Omega)$, that is, an equivalence class of Cauchy sequences, and let $\{u_k\} \in \widetilde{u}$. Then $\{D_j u_k\}$, $j = 1, \ldots, n$, are Cauchy sequences in $L^q(\Omega)$ and, therefore, there exist corresponding $V_j \in L^q(\Omega)$, such that

$$\lim_{k\to\infty} \|D_j u_k - V_j\|_q = 0\,, \quad j = 1, \ldots, n\,. \tag{II.7.12}$$

Moreover, in view of Exercise II.6.4, $\{u_k\}$ is a Cauchy sequence also in $L^{nq/(n-q)}(\Omega)$, if $q \in [1, n)$, and in $L^q_{\mathfrak{w}}(\Omega)$, if $q \geq n$ and $\Omega \neq \mathbb{R}^n$. Thus, there is $u \in L^{nq/(n-q)}(\Omega)$, if $q \in [1, n)$, or $u \in L^q_{\mathfrak{w}}(\Omega)$, if $q \geq n$ and $\Omega \neq \mathbb{R}^n$, such that

$$\begin{aligned} &\lim_{k\to\infty} \|u_k - u\|_{nq/(n-q)} = 0\,, \quad \text{if } q \in [1, n) \\ &\lim_{k\to\infty} \|\mathfrak{w}(u_k - u)\|_q = 0\,, \quad \text{if } q \geq n, \ \ \Omega \neq \mathbb{R}^n\,. \end{aligned} \tag{II.7.13}$$

From the definition of weak derivative and from (II.7.12)–(II.7.13), it immediately follows that $V_j = D_j u$. Next, let $\psi \in C_0^\infty(\mathbb{R}^n)$. We have to show that ψu can be approximated, in $W^{1,q}(\Omega)$–norm, by a sequence $\{v_k\} \subset C_0^\infty(\Omega)$. Take $v_k = \psi u_k$. From (II.7.13) it is clear that $\|\psi u - v_k\|_q \to 0$ as $k \to \infty$. Moreover,

$$|\psi u - v_k|_{1,q} \leq C\,(\,|u - u_k|_{1,q} + \|u - u_k\|_{q,K})$$

with K the support of ψ, so that, from this inequality and (II.7.12), (II.7.13), we find $|\psi u - v_k|_{1,q} \to 0$ as $k \to \infty$, which concludes the proof of the desired property. We may thus infer $u \in \widetilde{D}_0^{1,q}(\Omega)$. Since, as it is readily checked, the function u does not depend on the particular sequence $\{u_k\} \in \widetilde{u}$, we may define a map, $\mathfrak{T}$, that to each $\widetilde{u} \in D_0^{1,q}(\Omega)$ assigns the function $u \in \widetilde{D}_0^{1,q}(\Omega)$ determined in the way described above. Of course, $\mathfrak{T}$ is linear and it is also an isometry, and, in addition,

$$|\widetilde{u}|_{1,q} \equiv \lim_{k\to\infty} |u_k|_{1,q} = |u|_{1,q} \equiv |\mathfrak{T}(\widetilde{u})|_{1,q}.$$

It remains to show that the range of $\mathfrak{T}$ coincides with $\widetilde{D}_0^{1,q}(\Omega)$. This amounts to say that, for each $u \in \widetilde{D}_0^{1,q}(\Omega)$ we can find $\{u_k\} \subset C_0^\infty(\Omega)$ such that $|u_k - u|_{1,q} \to 0$ as $k \to \infty$. However, the validity of this property is assured by Theorem II.7.1. Finally, the case $\Omega = \mathbb{R}^n$ and $q \geq n$. In view of Remark II.6.2, we only have to show that the natural map i is surjective, namely, that for any $[u] \equiv [u]_1 \in \dot{D}_0^{1,q}(\mathbb{R}^n)$, we can find $\{u_k\} \subset C_0^\infty(\Omega)$ such that $|u_k - v|_{1,q} \to 0$, as $k \to \infty$, $v \in [u]$. This property follows from Theorem II.7.1, and the proof of the theorem is complete. □

We may thus summarize the above theorem with the following representation of the spaces $D_0^{1,q}(\Omega)$ (up to an isomorphism).

If $q \in [1, n)$:

$$D_0^{1,q}(\Omega) = \{u \in D^{1,q}(\Omega) : \|u\|_{nq/(n-q)} < \infty\,, u \text{ satisfies (II.6.50) with } m = 1\}, \tag{II.7.14}$$

with equivalent norm given in $(\text{II.7.11})_1$.

If $q \geq n$, and $\Omega^c \supset B_a$, for some $a > 0$:

$$D_0^{1,q}(\Omega) = \{u \in D^{1,q}(\Omega) :\ u \text{ satisfies (II.6.50) with } m = 1\}\,, \tag{II.7.15}$$

with equivalent norm given in $(\text{II.7.11})_2$.

If $q \geq n$ and $\Omega = \mathbb{R}^n$:

$$D_0^{1,q}(\mathbb{R}^n) = \{[u] : u \in D^{1,q}(\mathbb{R}^n)\}\,, \tag{II.7.16}$$

where

$$[u] = \{v \in D^{1,q}(\mathbb{R}^n) \text{ such that } v = u + c\,,\ c \in \mathbb{R}\}\,.$$

By combining Theorem II.7.3 with the arguments used in showing Theorem II.7.5, one is now able to furnish the following representation (up to an isomorphism) of the space $D_0^{m,q}(\Omega)$, for arbitrary $m \geq 1$.

Theorem II.7.6 *Let Ω be an exterior domain of $\mathbb{R}^n$, $n \geq 2$. The following representations hold.*

(i) *If* $q < n$, *let* $\ell \in \{1, \ldots, m\}$ *be the largest integer such that* $\ell q < n$. *If* $\ell < m$, *we assume* $\Omega^c \supset B_a$ *for some* $a > 0$. *Then:*

$$D_0^{m,q}(\Omega) = \left\{ u \in D^{m,q}(\Omega) : \sum_{k=1}^{\ell} |u|_{m-k, \frac{nq}{n-kq}} < \infty \,, \; u \text{ satisfies (II.6.50)} \right\}, \tag{II.7.17}$$

with equivalent norm

$$\|u\|_{m-1,q,\Omega_{R_0}} + \sum_{k=1}^{\ell} |u|_{m-k,nq/(n-kq)} + |u|_{m,q} \,,$$

where R_0 *is a fixed number strictly greater than* $\delta(\Omega^c)$.

(ii) *If* $q \geq n$, *assume* $\Omega^c \supset B_a$ *for some* $a > 0$. *Then:*

$$D_0^{m,q}(\Omega) = \{ u \in D^{m,q}(\Omega) : \; u \text{ satisfies (II.6.50)} \} \,, \tag{II.7.18}$$

with equivalent norm

$$\|u\|_{m-1,q,\Omega_{R_0}} + |u|_{m,q} \,,$$

where R_0 *is a fixed number strictly greater than* $\delta(\Omega^c)$.

(iii) *If* $q < n$, $mq \geq n$, *and* $\Omega = \mathbb{R}^n$*:*

$$D_0^{m,q}(\mathbb{R}^n) = \left\{ [u]_{m-\ell} \,, u \in D^{m,q}(\Omega) : \; \sum_{k=1}^{\ell} |u|_{m-k, \frac{nq}{n-kq}} < \infty \right\} \tag{II.7.19}$$

where ℓ $(< m)$ *is the largest integer such that* $\ell q < n$, *and where, we recall,*

$$[u]_{m-\ell} = \{ v \in D^{m,q}(\mathbb{R}^n) : v = u + \mathcal{P}_{m-\ell-1} \} \,,$$

with $\mathcal{P}_{m-\ell-1}$ *polynomial of degree* $\leq m - \ell - 1$.

(iv) *If* $q \geq n$ *and* $\Omega = \mathbb{R}^n$*:*

$$D_0^{m,q}(\mathbb{R}^n) = \{ [u]_m \,, u \in D^{m,q}(\Omega) \} \tag{II.7.20}$$

The proof of the above theorem is quite straightforward. In fact, it is obtained by combining the procedure used in Theorem II.7.5, with the results of Theorem II.7.3 and Theorem II.6.5. We leave the details to the reader.

Exercise II.7.2 Show that the space defined on the right-hand side of (II.7.19) is a Banach space with respect to the norm $|[u]|_{m,q} \equiv |u|_{m,q}$, $u \in [u]_{m-\ell}$. *Hint.* Follow the arguments of the proof of Theorem II.7.1.

Remark II.7.2 From Theorem II.7.6 we deduce that, unless $mq < n$, the space $D_0^{m,q}(\mathbb{R}^n)$ is a Banach space whose elements are equivalence classes of functions that differ by polynomials of suitable degree. In particular, if $q \geq n$, then $D_0^{m,q}(\mathbb{R}^n) = \dot{D}^{m,q}(\mathbb{R}^n)$. In this respect, see also the following exercise. ■

Exercise II.7.3 Let $\{u_k\}$ be a Cauchy sequence in $D_0^{m,q}(\mathbb{R}^n)$, where $mq \geq n$, and let $[u]_m \in \dot{D}^{m,q}(\mathbb{R}^n)$ be such that $|u_k - u|_{m,q} \to 0$ as $k \to \infty$, $u \in [u]_m$. Show, by means of an example, that even though $u \in L^s(B_R)$, for all $s \in [1,q]$ and all $R > 0$, we may have $\|u_k\|_{1,B_R} \to \infty$ as $k \to \infty$, for all sufficiently large R. *Hint* (Deny & Lions 1954, §4): Take $m = 1$, $q = n = 2$ and choose

$$u_k(x) = -\int_{|x|}^{\infty} (t \ln t)^{-1} a_k(t)\, dt\,,$$

where $a_k = a_k(t)$, $k \in \mathbb{N}$, is a smooth, non-negative function of $C_0^\infty(\mathbb{R})$ which is 0 for $t \leq 2$ and for $t \geq k+4$, and it is 1 for $t \in [5/2, 3+k]$. Then $|u_k - u|_{1,2} \to 0$ as $k \to \infty$, where $u(x) = (\sqrt{|x|} \ln x)^{-1} a(x)$, with $a(x) = 0$ for $|x| \leq 2$ and $= 1$ for $|x| \geq 5/2$, while

$$\lim_{k\to\infty} \int_{B_R} |u_k(x)| = \infty\,, \quad \text{for all } R > 5/2\,.$$

Exercise II.7.4 Let Ω be an exterior domain and let $u \in D_0^{2,2}(\Omega)$. Show that

$$|D^2 u|_{2,2} = \|\Delta u\|_2\,.$$

Hint: It is enough to show the identity for $u \in C_0^\infty(\Omega)$.

Results similar to those of Theorem II.7.3 and Theorem II.7.4 can be proved in the case when $\Omega = \mathbb{R}^n_+$. In fact, as we already noticed, every function $u \in D^{m,q}(\mathbb{R}^n_+)$ can be extended to the whole of $\mathbb{R}^n$ to a function u' satisfying (II.6.45). In particular, if the trace $\Gamma_m(u)$ on every (bounded) portion of the plane $x_n = 0$ is identically zero, we may take u' as the function obtained by setting $u \equiv 0$ outside $\mathbb{R}^n_+$. With this and Theorem II.6.4(c) in mind, one can show the following theorems, whose proofs are left to the reader.

Theorem II.7.7 *The following representation holds, for all $m \geq 0$, $q \in [1,\infty)$.*

$$D_0^{m,q}(\mathbb{R}^n_+) = \{u \in D^{m,q}(\mathbb{R}^n_+) : \ \Gamma_m(u) = 0 \ \text{on } S\}\,,$$

with S arbitrary bounded domain in the plane $x_n = 0$, with equivalent norm

$$|u|_{m,q} + \|u\|_{m-1,q,L_{a_0}}\,,$$

where L_a is defined in (II.6.47) *and a_0 is a fixed positive number.*

Theorem II.7.8 *Let $u \in D^{m,q}(\mathbb{R}^n_+)$, $m \geq 0$, $q \in [1,\infty)$. Then, u can be approximated in the seminorm $|\cdot|_{m,q}$ by functions from $C_0^\infty(\overline{\mathbb{R}}^n_+)$.*

Remark II.7.3 Unlike the case Ω exterior, Theorem II.7.7 does not explicitly impose any restriction at large distances on the behavior of u when $1 \leq q < n$, such as the vanishing condition (II.7.9) on the polynomials $u_{m-\ell}$. Actually by means of an argument completely analogous to that preceding Theorem II.6.3, one can show that the polynomials $u_{m-\ell}$ are identically zero as a consequence of the vanishing of the trace $\Gamma_m(u)$. ■

Exercise II.7.5 (Coscia and Patria 1992, Lemma 5) Let $u \in D^{1,q}(\mathbb{R}^n_+)$, $1 \le q < n$. By Theorem II.6.3 there is $u_0 \in \mathbb{R}$ such that $u - u_0 \in L^s(\mathbb{R}^n_+)$, $s = nq/(n-q)$. Show that if the trace $\gamma(u)$ at $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$ belongs to $L^r(\Sigma)$, for some $r \in [1, \infty)$, then $u_0 = 0$. This fact, together with Theorem II.7.8, implies that every such function can be approximated in the seminorm $|\cdot|_{1,q}$ by functions from $C_0^\infty(\overline{\mathbb{R}}^n_+)$.

II.8 The Normed Dual of $D_0^{m,q}(\Omega)$. The Spaces $D_0^{-m,q}$

We begin to furnish a characterization of the normed dual space $(D_0^{m,q}(\Omega))'$ of $D_0^{m,q}(\Omega)$, when Ω is either an exterior domain or $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, analogous to the one we described at the end of Section II.3 for the space $W_0^{m,q}(\Omega)$. A (bounded) linear functional $\mathcal{F}$ belongs to $(D_0^{m,q}(\Omega))'$ if and only if

$$\|\mathcal{F}\|_{(D_0^{m,q}(\Omega))'} \equiv \sup_{u \in D_0^{m,q}(\Omega)\,,\ |u|_{m,q}=1} |\mathcal{F}(u)| < \infty\,.$$

Let us first take Ω exterior, $\Omega \neq \mathbb{R}^n$ and satisfying the assumptions of Theorem II.7.6, or $\Omega = \mathbb{R}^n_+$. Consider the functional

$$\mathcal{F}(u) = (f, u), \quad f \in C_0^\infty(\Omega), \quad \text{all } u \in D_0^{m,q}(\Omega). \tag{II.8.1}$$

Applying the Hölder inequality in (II.8.1) we obtain

$$|\mathcal{F}(u)| \le \|f\|_{q'} \|u\|_{q,\Omega_0}, \tag{II.8.2}$$

where $\Omega_0 = \operatorname{supp}(f)$. Then, by Theorem II.7.6 and Theorem II.6.5(i), if Ω is exterior, and by Theorem II.7.7, if $\Omega = \mathbb{R}^n_+$, we find that inequality (II.8.2) implies

$$|\mathcal{F}(u)| \le c\,\|f\|_{q'} |u|_{m,q}$$

with $c = c(\Omega_0)$. We now set

$$|f|_{-m,q'} = \sup_{u \in D_0^{m,q}(\Omega)\,,\ |u|_{m,q}=1} |\mathcal{F}(u)|. \tag{II.8.3}$$

Evidently, (II.8.3) is a norm in $C_0^\infty(\Omega)$. Denote by $D_0^{-m,q'}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in this norm. The following result holds.

Lemma II.8.1 *Let Ω be an exterior domain ($\neq \mathbb{R}^n$) satisfying the assumptions of* Theorem II.7.6, *or $\Omega = \mathbb{R}^n_+$. Then, for any $q \in (1, \infty)$, functionals of the form* (II.8.1) *are dense in $(D_0^{m,q}(\Omega))'$, and $(D_0^{m,q}(\Omega))'$ and $D_0^{-m,q'}(\Omega)$ are isomorphic.*

Proof. Let

$$\mathcal{S} = \{\mathcal{F} \in (D_0^{m,q}(\Omega))' :\ \mathcal{F}(u) = (f, u) \text{ for some } f \in C_0^\infty(\Omega)\}\,.$$

Clearly, $\mathcal{S}$ is a subspace of $(D_0^{m,q}(\Omega))'$. Let us prove that $\mathcal{S}$ is dense in $(D_0^{m,q}(\Omega))'$. In fact, assuming by contradiction that $\overline{\mathcal{S}} \neq (D_0^{m,q}(\Omega))'$, by the Hahn–Banach theorem (see Theorem II.1.7(b)) there exists a nonzero element $Z \in (D_0^{m,q}(\Omega))''$ such that

$$Z(\mathcal{F}) = 0, \text{ for all } \mathcal{F} \in \mathcal{S}.$$

Since $D_0^{m,q}(\Omega)$ is reflexive for $q \in (1,\infty)$ (cf. Exercise II.6.2), the preceding condition implies that there exists a nonzero $z \in D_0^{m,q}(\Omega)$ such that

$$\mathcal{F}(z) = 0, \text{ for all } \mathcal{F} \in \mathcal{S},$$

that is

$$(f,z) = 0, \text{ for all } f \in C_0^\infty(\Omega),$$

that is, $z = 0$, which leads to a contradiction. Following Lax (1955, §2), it is now readily seen that $(D_0^{m,q}(\Omega))'$ and $D_0^{-m,q'}(\Omega)$, $1 < q < \infty$, are isomorphic. To this end, let $\mathcal{L} \in (D_0^{m,q}(\Omega))'$ and let $\{f_k\} \subset C_0^\infty(\Omega)$ be such that the sequence $\mathcal{F}_k \equiv (f_k, u)$, $k \in \mathbb{N}$, $u \in D_0^{m,q}(\Omega)$, converges to $\mathcal{L}$ in the norm $|\cdot|_{(D_0^{m,q}(\Omega))'}$ of $(D_0^{m,q}(\Omega))'$. Since

$$|\mathcal{F}_k|_{(D_0^{m,q}(\Omega))'} = |f_k|_{-m,q'}, \tag{II.8.4}$$

$\{f_k\}$ is a Cauchy sequence in $D_0^{-m,q'}(\Omega)$ converging to some $\mathcal{F} \in D_0^{-m,q'}(\Omega)$. Clearly, $\mathcal{F}$ depends only on $\mathcal{L}$ and not on the particular sequence $\{f_k\}$ and, in addition, it is uniquely determined. Likewise, to each $\mathcal{F} \in D_0^{-m,q'}(\Omega)$ we may uniquely associate an $\mathcal{L} \in (D_0^{m,q}(\Omega))'$, thus establishing the existence of a linear bijection, $\mathscr{L}$, between $(D_0^{m,q}(\Omega))'$ and $D_0^{-m,q'}(\Omega)$. However, from (II.8.4), it follows that $\mathscr{L}$ is an isomorphism, and the proof of the lemma is complete. □

Let us now consider the case $\Omega = \mathbb{R}^n$. For $mq < n$, we employ, in (II.8.1), the Hölder inequality and make use m times of the Sobolev inequality (II.3.7) to deduce

$$|\mathcal{F}(u)| \le \|f\|_{nq'/(n+q')}\|u\|_{nq/(n-mq)} \le c\|f\|_{nq'/(n+q')}|u|_{m,q}. \tag{II.8.5}$$

If $mq \ge n$, by Theorem II.7.6 we know that elements from $D_0^{m,q}(\mathbb{R}^n)$ are equivalence classes $[u]_s$ determined by functions that may differ by polynomials $\mathcal{P}_s$ of degree $\le s-1$, where

$$\begin{cases} s = m, & \text{if } q \ge n, \\ s = m - \ell, & \text{if } q < n \text{ and } \ell\ (< m) \text{ is the largest integersuch that } \ell q < n. \end{cases} \tag{II.8.6}$$

Thus, if $mq \ge n$, functionals of the type (II.8.1) must satisfy $\mathcal{F}(u_1) = \mathcal{F}(u_2)$ whenever u_1, u_2 belong to the same class $[u]_s$. This is equivalent to the following condition on f:

$$\int_{\mathbb{R}^n} f\,\mathcal{P}_s = 0, \tag{II.8.7}$$

where $\mathcal{P}_s$ is an *arbitrary* polynomial of degree $\leq s-1$, with s satisfying (II.8.6). As a consequence, from (II.8.7), for $u \in [u]_s$ we have (with $B_R \supset \mathrm{supp}\,(f)$)

$$|\mathcal{F}(u)| = \left|\int_{B_R} fu\right| = \left|\int_{B_R} f(u + \mathcal{P}_s)\right| \leq \|f\|_{q',\mathbb{R}^n}\|u + \mathcal{P}_s\|_{q,B_R}. \tag{II.8.8}$$

We may choose $\mathcal{P}_s$ in such a way that, setting

$$u_s = u - \mathcal{P}_s,$$

it follows

$$\frac{1}{|B_R|}\int_{B_R} D^\alpha u_s = 0, \quad 0 \leq |\alpha| \leq s.$$

In view of these latter conditions, by a repeated use of the Poincaré inequality (II.5.10) in the last term on the right-hand side of (II.8.8), we obtain

$$|\mathcal{F}(u)| \leq c_1\|f\|_{q',\mathbb{R}^n}|u|_{s+1,q,B_R}.$$

Now, if $q \geq n$, from (II.8.6) it is $s = m-1$ and so

$$|u|_{s+1,q,B_R} \leq |u|_{m,q,\mathbb{R}^n}.$$

If $q < n$, again from (II.8.6), the Hölder inequality and (II.7.17) of Remark II.7.2, we deduce

$$|u|_{s+1,q,B_R} = |u|_{m-\ell,q,B_R} \leq |u|_{m-\ell,nq/(n-\ell q),\mathbb{R}^n} \leq c|u|_{m,q,\mathbb{R}^n}.$$

Thus, in all cases, we deduce

$$|\mathcal{F}(u)| \leq c_2\|f\|_{q',\mathbb{R}^n}|u|_{m,q,\mathbb{R}^n}. \tag{II.8.9}$$

Once (II.8.9) has been established, we may again use the arguments of Lemma II.8.1 to show that the spaces $(D_0^{m,q}(\mathbb{R}^n))'$ and $D_0^{-m,q'}(\mathbb{R}^n)$, $1 < q < \infty$, are isomorphic.

Thus, for $q \in (1,\infty)$, let us define $\mathfrak{F}_{q,m}(\Omega)$ as the class of functionals (II.8.1), which, if $\Omega = \mathbb{R}^n$ and $n \leq mq < \infty$, verify, in addition, (II.8.7) for an arbitrary polynomial $\mathcal{P}_s$ of degree $\leq s-1$, with s satisfying (II.8.6). The results just discussed along with those of Lemma II.8.1 can be then summarized in the following.

Theorem II.8.1 *Let $\Omega \subseteq \mathbb{R}^n$ be either an exterior, locally Lipschitz domain, or $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n$. The completion, $D_0^{-m,q'}(\Omega)$, of $\mathfrak{F}_{q,m}(\Omega)$ in the norm* (II.8.3) *is isomorphic to $(D_0^{m,q}(\Omega))'$.*

Remark II.8.1 If $m = 1$, a restriction of the type (II.8.7) occurs if and only if $q \geq n$. In such a case, $\mathcal{P}_s$ reduces to an arbitrary constant so that condition (II.8.7) becomes

$$\int_{\mathbb{R}^n} f = 0. \tag{II.8.10}$$

■

Hereafter, the value of $\mathcal{F} \in D_0^{-1,q'}(\Omega)$ at $u \in D_0^{1,q}(\Omega)$ (duality pairing) will be denoted by

$$[\mathcal{F}, u].$$

Notice that if, in particular, $\mathcal{F} \in C_0^\infty(\Omega)$, we have

$$[\mathcal{F}, u] = (\mathcal{F}, u).$$

By an obvious continuity argument, the same relation holds, more generally, for all $\mathcal{F} \in L^s(\Omega) \cap D_0^{-1,q'}(\Omega)$, $s \in [1, \infty)$.

Our next goal is to provide a useful representation of functionals on $D_0^{1,q}(\Omega)$, valid for an arbitrary domain Ω, as well as another characterization of the space $(D_0^{1,q}(\Omega))'$. Taking into account that $D_0^{1,q}(\Omega)$ is a closed subspace of $\dot{D}^{1,q}(\Omega)$ (see Remark II.6.2), this representation becomes a particular case of the following important general result.

Theorem II.8.2 *Let Ω be a domain in $\mathbb{R}^n$. Then, for any given $\mathcal{F} \in (\dot{D}^{1,q}(\Omega))'$, $q \in (1, \infty)$, there exists $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ such that, for all $u \in \dot{D}^{1,q}(\Omega)$,*

$$\mathcal{F}(u) = (\boldsymbol{f}, \nabla u)\,. \tag{II.8.11}$$

Moreover,

$$\|\mathcal{F}\|_{(\dot{D}^{1,q}(\Omega))'} = \|\boldsymbol{f}\|_{q'}\,. \tag{II.8.12}$$

Proof. We recall that, for any $q \in (1, \infty)$, $\dot{D}^{1,q}(\Omega)$ can be viewed as a subspace of $[L^q(\Omega)]^n$, via the map

$$M : u \in \dot{D}^{1,q}(\Omega) \to \boldsymbol{h} \equiv \nabla u \in [L^q(\Omega)]^n\,. \tag{II.8.13}$$

Therefore, given $\mathcal{F} \in (\dot{D}^{1,q}(\Omega))'$, by the Hahn–Banach theorem (see Theorem II.1.7) there exists a (not necessarily unique) functional $\mathcal{L} \in [[L^q(\Omega)]^n]'$, such that

$$\mathcal{L}(\boldsymbol{h}) = \mathcal{F}(u)\,, \quad u \in \dot{D}^{1,q}(\Omega)\,, \tag{II.8.14}$$

and that, moreover, satisfies

$$\|\mathcal{L}\|_{[[L^q(\Omega)]^n]'} = \|\mathcal{F}\|_{(\dot{D}^{1,q}(\Omega))'}\,. \tag{II.8.15}$$

However, by Theorem II.2.6, we have that, corresponding to the functional $\mathcal{L}$, there exists a uniquely determined $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ such that $\mathcal{L}(\boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w})$ for all $\boldsymbol{w} \in [L^q(\Omega)]^n$, with $\|\boldsymbol{f}\|_{q'} = \|\mathcal{L}\|_{[[L^q(\Omega)]^n]'}$. Therefore, the theorem follows from this latter consideration, and from (II.8.14) and (II.8.15). □

We would like to analyze some significant consequences of this result for the space $D_0^{1,q}(\Omega)$. We begin to observe that, since $D_0^{1,q}(\Omega) \subset \dot{D}^{1,q}(\Omega)$, by Theorem II.8.2 the generic linear functional on $D_0^{1,q}(\Omega)$ can be represented as in (II.8.11), for all $u \in D_0^{1,q}(\Omega)$, where the function $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ is determined up to a function $\boldsymbol{f}_0$ such that

$$(\boldsymbol{f}_0, \nabla u) = 0\,, \quad \text{for all } u \in D_0^{1,q}(\Omega)\,. \tag{II.8.16}$$

Let $\widetilde{L}^{q'}(\Omega)$ be the subspace of $[L^{q'}(\Omega)]^n$ constituted by all those functions satisfying (II.8.16). It is immediately verified that $\widetilde{L}^{q'}(\Omega)$ is closed. Moreover, setting $G_{0,q'}(\Omega) = M(D_0^{1,q'}(\Omega))$, with M defined in (II.8.13), we can readily show that $G_{0,q'}(\Omega)$ is also a closed subspace of $[L^{q'}(\Omega)]^n$; see Exercise II.8.1. Now, let $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$ and consider the problem:

$$\text{Find } w \in D_0^{1,q'}(\Omega) \text{ such that } (\nabla w - \boldsymbol{f}, \nabla u) = 0\,, \text{ for all } u \in D_0^{1,q}(\Omega). \tag{II.8.17}$$

If Ω and $\boldsymbol{f}$ are sufficiently smooth, we can show that this problem is equivalent to the following classical Dirichlet problem

$$\Delta w = \nabla \cdot \boldsymbol{f} \text{ in } \Omega\,, \quad w = 0 \text{ at } \partial\Omega\,, \quad w \in D_0^{1,q'}(\Omega)\,.$$

Lemma II.8.2 *Assume that, for any given* $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$, *problem* (II.8.17) *has one and only one solution* $w \in D_0^{1,q'}(\Omega)$. *Then, the following decomposition holds*

$$[L^{q'}(\Omega)]^n = \widetilde{L}^{q'}(\Omega) \oplus G_{0,q'}(\Omega)\,. \tag{II.8.18}$$

Conversely, if (II.8.18) *holds, then, for any* $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$, *problem* (II.8.17) *is uniquely solvable. Finally, the linear operator* $\Pi_{q'} : \boldsymbol{f} \in [L^{q'}(\Omega)]^n \to \boldsymbol{f}_1 \in G_{0,q'}(\Omega)$ *is a projection (that is,* $\Pi_{q'}^2 = \Pi_{q'}$*) and is continuous.*

Proof. The last statement in the lemma is a consequence of (II.8.18); see Rudin (1973, Theorem 5.16(b)). Since both $L^{q'}(\Omega)$ and $G_{0,q'}(\Omega)$ are closed, in order to prove (II.8.18), under the given assumption, we have to show that (a) $L^{q'}(\Omega) \cap G_{0,q'}(\Omega) = \{0\}$, and that (b) $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{f}_1$, $\boldsymbol{f}_0 \in \widetilde{L}^{q'}(\Omega)$, $\boldsymbol{f}_1 \in G_{0,q'}(\Omega)$. Suppose there are $\boldsymbol{l} \in \widetilde{L}^{q'}(\Omega)$ and $\boldsymbol{g} = \nabla g \in G_{0,q'}(\Omega)$, for some $g \in D_0^{1,q'}(\Omega)$, such that $\boldsymbol{l} = \boldsymbol{g}$. This means, by definition of $\widetilde{L}^{q'}(\Omega)$ that $(\nabla g, \nabla u) = 0$ for all $u \in D_0^{1,q}(\Omega)$, which, in turn, by the uniqueness assumption on problem (II.8.17), implies $\nabla g = \boldsymbol{l} = 0$. Thus, (a) is proved. Next, for the given $\boldsymbol{f}$, let $w \in D_0^{1,q}(\Omega)$ be the corresponding solution to (II.8.17) and set $\boldsymbol{f}_0 = \boldsymbol{f} - \nabla w$ ($\in \widetilde{L}^{q'}(\Omega)$), and $\boldsymbol{f}_1 = \nabla w$ ($\in G_{0,q'}$). Then, $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{f}_1$ which proves (b). The converse claim, namely, that (II.8.18) implies the unique solvability of (II.8.17), is almost obvious and, therefore, it is left to the reader as an exercise $\square$

With the help of Theorem II.8.2 and Lemma II.8.2, we can now show the following result.

Theorem II.8.3 *Assume the hypothesis of* Lemma II.8.2 *is satisfied and $q' \in (1,\infty)$. Then $D_0^{1,q'}(\Omega)$ and $(D_0^{1,q}(\Omega))'$ are homeomorphic. Specifically, the linear map*

$$\mathfrak{M} : w \in D_0^{1,q'}(\Omega) \to \mathfrak{M}(w) \in (D_0^{1,q}(\Omega))' , \tag{II.8.19}$$

where

$$[\mathfrak{M}(w), u] = (\nabla w, \nabla u) , \quad \textit{for all } u \in D_0^{1,q}(\Omega) , \tag{II.8.20}$$

is a bijection and, moreover, for some $c = c(q, n, \Omega) > 0$,

$$c\,|w|_{1,q'} \leq \|\mathfrak{M}(w)\|_{(D_0^{1,q}(\Omega))'} \leq |w|_{1,q'} . \tag{II.8.21}$$

Proof. By assumption, we find that $\mathfrak{M}$ is injective, and, by Theorem II.8.2 ((II.8.11), in particular) and Lemma II.8.2, that $\mathfrak{M}$ is surjective, so that $\mathfrak{M}$ is a bijection. Furthermore, the inequality on the right-hand side of (II.8.21) is an obvious consequence of the Hölder inequality, while the one on the left-hand side follows from the continuity of the projection operator $\Pi_{q'}$ and from (II.8.12). □

In view of the results of Theorem II.8.3, it is of great interest to investigate under what conditions problem (II.8.17) has, for a given $\boldsymbol{f} \in [L^{q'}(\Omega)]^n$, a unique corresponding solution w. As a matter of fact, such unique solvability depends, in general, on the domain Ω and on the exponent q'. In particular, we have the following.

Theorem II.8.4 *Let Ω be either $\mathbb{R}^n$, or $\mathbb{R}^n_+$, or a bounded domain with a boundary of class C^2. Then, for all $q \in (1,\infty)$, the spaces $D_0^{1,q'}(\Omega)$ and $(D_0^{1,q}(\Omega))'$ are homeomorphic, in the sense specified in* Theorem II.8.3*. If Ω is an exterior domain of class C^2 (with $\partial\Omega \neq \emptyset$) the same conclusion holds if and only if $q' \in (n/(n-1), n)$, if $n \geq 3$, and $q' = 2$, if $n = 2$.*

We shall not give a proof of this theorem, mainly, because a completely analogous analysis of unique solvability will be carried out in Chapters IV and V, in the more complicated context of the Stokes problem. Here we shall limit ourselves to observe that the restriction on the exponent q', in the case of the exterior domain, comes from the fact that the Dirichlet problem (II.8.17) for $n \geq 3$ looses existence if $1 < q' \leq n/(n-1)$ ($q' \in (1,2)$ if $n = 2$), while it lacks of uniqueness if $q' \geq n$, $n \geq 3$ ($q' > 2$, if $n = 2$). For further details, we refer the interested reader to the Notes at the end of this chapter.

Exercise II.8.1 Show that $G_{0,q}(\Omega)$, $q \in [1,\infty)$, is a closed subspace of $L^q(\Omega)$.

Exercise II.8.2 Show that the subspace $\mathcal{S}$ of $(\dot{D}^{1,q}(\Omega))'$, $q \in (1,\infty)$, defined as follows

$$\mathcal{S} = \{u \in C_0^\infty(\Omega) : \; u = \nabla \cdot \boldsymbol{\psi}\,, \text{ for some } \boldsymbol{\psi} \in C_0^\infty(\Omega)\}$$

is dense in $(\dot{D}^{1,q}(\Omega))'$. This result generalizes the one proved by Kozono & Sohr (1991, Corollary 2.3). *Hint*: Use Theorem II.8.2.

II.9 Pointwise behavior at Large Distances of Functions from $D^{1,q}$

We begin to give two classical results of potential theory, in a form suitable to our purposes.

Lemma II.9.1 *Let A be a bounded, locally Lipschitz domain of $\mathbb{R}^n$, $n \ge 2$, and let $w \in C^2(\overline{A})$. The following identity holds for all $x \in A$:*

$$w(x) = \frac{1}{n\omega_n}\int_A \frac{\partial w(y)}{\partial y_i}\frac{(x_i - y_i)}{|x-y|^n}dy - \frac{1}{n\omega_n}\int_{\partial A} w(y)\frac{(x_i-y_i)}{|x-y|^n}N_i(y)d\sigma_y$$

where $\boldsymbol{N} \equiv (N_i)$ is the outer unit normal to ∂A.

Proof. Denote by $\mathcal{E}(x-y)$ the *fundamental solution of Laplace's equation*:

$$\mathcal{E}(x-y) = \begin{cases} (2\pi)^{-1}\log|x-y| & \text{if } n=2 \\ [n(2-n)\omega_n]^{-1}\,|x-y|^{2-n} & \text{if } n \ge 3. \end{cases} \tag{II.9.1}$$

Employing the (second) *Green's identity*[1]

$$\int_{A_\varepsilon}(v\Delta u - u\Delta v) = \int_{\partial A_\varepsilon}(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N})$$

with $v(y) \equiv w(y)$, $u(y) = \mathcal{E}(x-y)$, $A_\varepsilon = A - B_\varepsilon(x)$ and integrating by parts we deduce

$$\begin{aligned}\int_{A_\varepsilon}\frac{\partial \mathcal{E}(x-y)}{\partial y_i}\frac{\partial w(y)}{\partial y_i} &= \int_{\partial B_\varepsilon} w(y)\frac{\partial \mathcal{E}(x-y)}{\partial y_i}N_i(y)d\sigma_y \\ &\quad + \int_{\partial A} w(y)\frac{\partial \mathcal{E}(x-y)}{\partial y_i}N_i(y)d\sigma_y\end{aligned}$$

which, in turn, by the properties of $\mathcal{E}$ and a standard procedure, proves the result in the limit $\varepsilon \to 0$. □

Lemma II.9.2 *Let*

[1] As is well known, this identity is obtained by means of the Gauss divergence theorem which, by Lemma II.4.1, holds for locally Lipschitz domains and smooth functions u, v.

$$\mathcal{I}_1(x) = \int_{\mathbb{R}^n} \frac{dy}{|x-y|^\lambda |y|^\mu}, \quad \lambda < n, \quad \mu < n.$$

Then, if $\lambda + \mu > n$*, there exists a constant* $c = c(\lambda, \mu, n)$ *such that*

$$|\mathcal{I}_1(x)| \le c|x|^{-(\lambda+\mu-n)}.$$

Moreover, let

$$\mathcal{I}_2(x) = \int_{\mathcal{A}(x)-B_1(x)} \frac{dy}{|x-y|^n \log|y|},$$

with

$$\mathcal{A}(x) = \{y \in \mathbb{R}^n : \kappa_1|x| < |y| < \kappa_2|x|\}, \quad \kappa_1 \in (0,1), \ \kappa_2 \in (1,\infty)$$

and x *satisfying*

$$|x| > 2/\kappa, \quad \kappa = \min\{1-\kappa_1, \kappa_2 - 1, \kappa_1^2\}.$$

Then, there exist positive constants c_1, c_2 *depending only on* κ_1*,* κ_2*, and* n *such that*

$$\mathcal{I}_2(x) \le c_1 + c_2(\log|x|)^{-1}.$$

Proof. Setting

$$x' = \frac{x}{|x|}, \quad y' = \frac{y}{|x|},$$

it follows that

$$|\mathcal{I}_1(x)| \le c|x|^{-(\lambda+\mu-n)} \int_{\mathbb{R}^n} \frac{dy'}{|x'-y'|^\lambda |y'|^\mu} \equiv c|x|^{-(\lambda+\mu-n)}\mathcal{I}.$$

To estimate $\mathcal{I}$, we rotate the coordinates in such a way that x' goes into $x_0 = (1, 0, \ldots 0)$ so that

$$\mathcal{I} = \int_{\mathbb{R}^n} \frac{dy'}{|x_0 - y'|^\lambda |y'|^\mu}.$$

Thus, $\mathcal{I}$ is convergent, since $\lambda < n$, $\mu < n$ and $\lambda + \mu > n$, and it is independent of x. The first estimate is therefore proved. To show the second one, we put $|x| = R$ and perform into $\mathcal{I}_2$ the same change of coordinates operated before to obtain

$$\mathcal{I}_2(x) = \int_{\mathcal{A}'-B_{1/R}(x_0)} \frac{dy'}{|x_0 - y'|^n \log(R|y'|)},$$

where

$$\mathcal{A}' = \{y' \in \mathbb{R}^n : \kappa_1 < |y'| < \kappa_2\}.$$

Being $R^{1/2}|y'| \ge \kappa_1/\kappa^{1/2} > 1$, we have $\log(R|y'|) \ge (\log R)/2$ and so

$$\mathcal{I}_2(x) \le 2(\log|x|)^{-1}\{I_1 + I_2\},$$

with

$$I_1 = \int_{1/R \le |x_0 - y'| \le 3\kappa/4} |x_0 - y'|^{-n} dy'$$

$$I_2 = \int_{\mathcal{A}' - B_{3\kappa/4}(x_0)} \frac{dy'}{|x_0 - y'|^n}.$$

Clearly,

$$I_2 = b$$

and, since $\kappa < 1$,

$$I_1 \le a \log|x|,$$

where a and b are independent of x. The lemma is thus completely proved. □

The result just shown will be used in the proof of the following one; see also Padula (1990, Lemma 2.6).

Theorem II.9.1 *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be an exterior domain and let*

$$u \in D^{1,r}(\Omega) \cap D^{1,q}(\Omega), \quad \textit{for some } r \in [1,\infty) \textit{ and some } q \in (n,\infty). \tag{II.9.2}$$

Then, if $r < n$, there exists $u_0 \in \mathbb{R}$ such that

$$\lim_{|x|\to\infty} |u(x) - u_0| = 0 \quad \textit{uniformly.} \tag{II.9.3}$$

The same conclusion holds if (II.9.2) *is replaced by the following one: there exists $u_0 \in \mathbb{R}$ such that*

$$(u - u_0) \in L^s(\Omega) \cap D^{1,q}(\Omega), \quad \textit{for some } s \in [1,\infty) \textit{ and some } q \in (n,\infty). \tag{II.9.4}$$

Moreover, under the assumption (II.9.2)*, with $r = n$, we find that*

$$\lim_{|x|\to\infty} |u(x)|/(\log|x|)^{(n-1)/n} = 0\,, \quad \textit{uniformly.} \tag{II.9.5}$$

Finally, if

$$u \in D^{1,q}(\Omega), \quad \textit{for some } q \in (n,\infty)\ ,$$

we have that

$$\lim_{|x|\to\infty} |u(x)|/|x|^{(q-n)/q} = 0\,, \quad \textit{uniformly.} \tag{II.9.6}$$

Proof. We begin to observe that, by density, (II.3.12) continues to hold for all $u \in W^{1,q}(B(x))$, $q > n$, and consequently, by Lemma II.6.1, for all $u \in D^{1,q}(\Omega)$, $q > n$. Thus, we find

$$|v(x)| \leq c\left(\|v\|_{1,B_1(x)} + |v|_{1,q,B_1(x)}\right), \quad \text{for all } v \in D^{1,q}(\Omega),\ q > n\,, \tag{II.9.7}$$

for some c independent of x. Now, under the assumption (II.9.2), by Theorem II.6.1 there exists $u_0 \in \mathbb{R}$ such that

$$\|u - u_0\|_{nr/(n-r)} < \infty. \tag{II.9.8}$$

Relation (II.9.3) then follows with the help of (II.9.8), by setting $v = u - u_0$ in (II.9.7), and then by letting $|x| \to \infty$. Under the assumption (II.9.4), we again use (II.9.7) with $v = u - u_0$, and let $|x| \to \infty$ in the resulting inequality. Let us next prove relation (II.9.6). We take R so large that $\exp\sqrt{\ln R} > 2\delta(\Omega^c)$ and set

$$u^{(1)} = (1 - \psi_R)u,$$

where ψ_R is given in (II.7.1). Putting

$$\Omega^\rho = \Omega - B_\rho, \quad \rho = \exp\sqrt{\ln R},$$

by the properties of the function ψ_R (see (II.7.5), (II.7.7)), it follows for sufficiently large R that

$$|u^{(1)}|_{1,q,\Omega^\rho} \leq |u|_{1,q,\Omega^\rho} + c(\ln\ln R)^{-1}. \tag{II.9.9}$$

Moreover, $u^{(1)} \in D^{1,q}(\Omega^\rho)$ and, since $u^{(1)}$ vanishes at $\partial\Omega^\rho$, by Theorem II.7.1 there exists a sequence $\{u_s\}_{s\in\mathbb{N}} \subset C_0^\infty(\Omega^\rho)$ converging to $u^{(1)}$ in the norm $|\cdot|_{1,q}$. For fixed $s,\ s' \in \mathbb{N}$, we apply Lemma II.9.1 to the function $w(x) \equiv h(x)|x|^{-\gamma}$, where $h(x) = u_s(x) - u_{s'}(x)$ and $A \supset \text{supp}\,(w)$. We thus have

$$\begin{aligned}|h(x)||x|^{-\gamma} \leq & \int_{\Omega^\rho} |\nabla h(y)||y|^{-\gamma}|x-y|^{1-n}dy \\ & + \gamma\int_{\Omega^\rho} |h(y)||y|^{-1-\gamma}|x-y|^{1-n}dy.\end{aligned}$$

Employing the Hölder inequality and (II.6.13) with $x_0 = 0$, there follows

$$|h(x)||x|^{-\gamma} \leq c|h|_{1,q,\Omega^\rho}\left(\int_{\mathbb{R}^n} |y|^{-\gamma q'}|x-y|^{(1-n)q'}dy\right)^{1/q'},$$

where $q' = q/(q-1)$ and $c = c(n,q)$. Taking $\gamma \in (1 - n/q, n - n/q)$ and since $q > n$, we may estimate the integral over $\mathbb{R}^n$ by means of Lemma II.9.2 to deduce

$$|h(x)||x|^{-\gamma} \leq c|h|_{1,q,\Omega^\rho}|x|^{-\gamma+(q-n)/q}.$$

Recalling the definition of the function h and letting $s, s' \to \infty$, from this latter inequality we obtain

$$|u^{(1)}(x)| \leq c|u^{(1)}|_{1,q,\Omega^\rho}|x|^{(q-n)/q}, \quad \text{for all } x \in \Omega^\rho$$

and so, by the properties of ψ_R and by (II.9.9), it follows that

$$|u(x)| \leq c_1 \left(|u|_{1,q,\Omega^\rho} + (\ln\ln R)^{-1}\right)|x|^{(q-n)/q}, \quad \text{for all } x \in \Omega^\rho,$$

which proves (II.9.6). It remains to show (II.9.5). To this end, let $x \in \Omega$ with $|x| = R$, $R > 2\delta(\Omega^c)$ and sufficiently large. Since

$$u \in W^{1,q}(\Omega_{R/2,2R}) \cap W^{1,n}(\Omega_{R/2,2R}),$$

we may use the density Theorem II.3.1 together with Theorem II.3.4 and Theorem II.4.1 to prove the validity of the identity in the statement of Lemma II.9.1 with $A \equiv \Omega_{R/2,2R}$ and $w(y) \equiv u(y)/(\log|y|)^{(n-1)/n}$. We thus obtain for all $x \in \Omega$ with $|x| = R$

$$|u(x)|/(\log|x|)^{(n-1)/n} \leq c(I_1 + I_2 + I_3 + I_4 + I_5 + I_6), \tag{II.9.10}$$

where $c = c(n)$ and

$$I_1 = \int_{\Omega_{R/2,2R} - B_1(x)} |\nabla u(y)|[(\log|y|)^{1/n}|x-y|]^{1-n}dy,$$

$$I_2 = \int_{B_1(x)} |\nabla u(y)|[(\log|y|)^{1/n}|x-y|]^{1-n}dy,$$

$$I_3 = \int_{\Omega_{R/2,2R} - B_1(x)} |u(y)||y|^{-1}(\log|y|)^{1/n-2}|x-y|^{1-n}dy,$$

$$I_4 = \int_{B_1(x)} |u(y)||y|^{-1}(\log|y|)^{1/n-2}|x-y|^{1-n}dy,$$

$$I_5 = \int_{\partial B_{R/2}} |u(y)|(\log|y|)^{(1-n)/n}|x-y|^{-1}d\sigma_y,$$

$$I_6 = \int_{\partial B_{2R}} |u(y)|(\log|y|)^{(1-n)/n}|x-y|^{-1}d\sigma_y.$$

Set

$$\mathcal{I}(x) \equiv \left(\int_{\Omega_{R/2,2R} - B_1(x)} \frac{dy}{|x-y|^n \log|y|}\right)^{(n-1)/n}.$$

The following estimates are a simple consequence of the Hölder inequality:

$$
\begin{aligned}
I_1 &\le \mathcal{I}(x)|u|_{1,n,\Omega_{R/2,2R}},\\
I_2 &\le c_1(\log R)^{(1-n)/n}|u|_{1,q,B_1(x)},\\
I_3 &\le \mathcal{I}(x)\left(\int_{\Omega_{R/2,2R}} \frac{|u(y)|^n}{(|y|\log|y|)^n}dy\right)^{1/n},\\
I_4 &\le c_2(\log R)^{(1-2n)/n}\left(\int_{B_1(x)} \frac{|u(y)|^q}{|y|^q}dy\right)^{1/q}.
\end{aligned}
$$

Moreover, since

$$
|x-y| \ge \begin{cases} R/2 \text{ for } y \in \partial B_{R/2} \\ R \quad\ \text{ for } y \in \partial B_{2R} \end{cases},
$$

it follows that

$$
\begin{aligned}
I_5 + I_6 \le c_3(\log R)^{(1-n)/n}\Bigg\{&\left(\int_{S^{n-1}} |u(R/2,\omega)|^n d\omega\right)^{1/n}\\
&+\left(\int_{S^{n-1}} |u(2R,\omega)|^n d\omega\right)^{1/n}\Bigg\}.
\end{aligned}
$$

By Lemma II.9.2, we have

$$
\mathcal{I}(x) \le c_4 + c_5(\log|x|)^{-1} \tag{II.9.11}
$$

while, by Exercise II.6.3, given $\varepsilon > 0$ there is a sufficiently large $\overline{R}$ such that for all $R > \overline{R}$ it holds that

$$
\int_{S^{n-1}} |u(R/2,\omega)|^n d\omega + \int_{S^{n-1}} |u(2R,\omega)|^n d\omega \le c_6\,\varepsilon\,(\log R)^{n-1}, \tag{II.9.12}
$$

and

$$
\int_{\Omega_{R/2,2R}} \frac{|u(y)|^n}{(|y|\log|y|)^n}dy \le c_7\varepsilon\int_{R/2}^{2R}(r\log r)^{-1}dr \le c_8\,\varepsilon. \tag{II.9.13}
$$

In addition, from (II.9.6), we find

$$
\int_{B_1(x)} \frac{|u(y)|^q}{|y|^q}dy \le c_9 R^{-n}. \tag{II.9.14}
$$

Since, clearly, as $R \to \infty$,

$$
|u|_{1,n,\Omega_{R/2,2R}},\quad |u|_{1,q,B_1} = o(1), \tag{II.9.15}
$$

in view of (II.9.11)–(II.9.15) we deduce in the limit $R \to \infty$

$$\sum_{i=1}^{6} I_i = o(1) \tag{II.9.16}$$

and (II.9.5) follows from (II.9.10) and (II.9.16). The theorem is therefore completely proved. □

Remark II.9.1 The result just shown applies, with no change to domains Ω that possess an extension property of the type specified in Remark II.6.4, such as a half-space. ■

II.10 Boundary Trace of Functions from $D^{m,q}(\mathbb{R}^n_+)$

Our next objective is to investigate the trace space at the boundary of a function $u \in D^{m,q}(\Omega)$, for $\Omega \equiv \mathbb{R}^n_+$. Actually, if Ω is an exterior domain, there is nothing to add to what was said in Section II.4, since, as shown in Lemma II.6.1, if Ω is locally Lipschitz then $u \in W^{m,q}(\Omega_R)$. On the other hand, if $u \in D^{m,q}(\mathbb{R}^n_+)$ then $u \in W^{m,q}(C)$, for any cube $C \subset \mathbb{R}^n_+$, and therefore, by the results of Section II.4, u possesses a well-defined trace $\Gamma_{(m)}(u)$ at the plane $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$ that belongs to the trace space $\mathcal{W}_{m,q}(\Sigma')$, for every *bounded* portion Σ' of Σ. However, from those results we cannot draw any conclusion concerning the finiteness of the norms of $\Gamma_m(u)$ on the *whole* of Σ. Nevertheless, such global information is of primary importance in the resolution of nonhomogeneous boundary-value problems.

A detailed investigation of the properties of the traces on Σ of functions belonging to the spaces $D^{m,q}(\mathbb{R}^n_+)$ has been performed by Kudrjavcev (1966a, 1966b). Here we shall describe some of his results in the case where $m = 1$, since this is the only case we need to consider in the applications. The interested reader is referred to Remark II.10.2 and to the work of Kudrjavcev (1966b, Theorems 2.4′ and 2.7) for generalizations to the case where $m > 1$.

For a function $u \in D^{1,q}(\mathbb{R}^n_+)$, we shall denote throughout by $\overline{u}$ its trace at Σ. From Theorem II.4.1 we derive, in particular, for any bounded (measurable) $\Sigma' \subset \Sigma$,

$$\|\overline{u}\|_{q,\Sigma'} \le c\left(|u|_{1,q,\mathbb{R}^n_+} + \|u\|_{q,B}\right), \tag{II.10.1}$$

where $c = c(\Sigma', n, q, B)$ and B any bounded, locally Lipschitz domain of $\mathbb{R}^n_+$ with $\overline{B} \supset \Sigma'$. Let σ be a non-negative, measurable function in Σ. By the symbol

$$L^q(\Sigma, \sigma), \quad 1 \le q \le \infty,$$

we denote the space of (equivalence classes of) real functions w on Σ that are L^q-summable in with the "weight" σ, namely,

$$\|\sigma w\|_q < \infty.$$

We have

Theorem II.10.1 *Let $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$ and $x' = (x_1, \ldots, x_{n-1})$. Then, for any $u \in D^{1,q}(\mathbb{R}^n_+)$ the trace $\overline{u}$ of u at Σ satisfies*

$$\overline{u} \in L^q(\Sigma, \sigma_1), \quad \sigma_1 = (1 + |x'|)^{(1-n)/q-\varepsilon_1},$$

where ε_1 is an arbitrary positive number, and the following inequality holds:

$$\|\sigma_1 \overline{u}\|_{q,\Sigma} \leq c_1 \left(|u|_{1,q,\mathbb{R}^n_+} + \|u\|_{q,B_+} \right),$$

with $c_1 = c_1(n, q, \varepsilon_1)$ and $B_+ = B_1 \cap \mathbb{R}^n_+$. Moreover, if $1 \leq q < n$, we have

$$\overline{u} - u_0 \in L^q(\Sigma, \sigma_2), \quad \sigma_2 = (1 + |x'|)^{(1-q)/q-\varepsilon_2},$$

where u_0 is the constant associated to u by Theorem II.6.3 *and ε_2 is an arbitrary positive number, and the following inequality holds:*

$$\|\sigma_2(\overline{u} - u_0)\|_{q,\Sigma} \leq c_2 |u|_{1,q,\mathbb{R}^n_+},$$

with $c_2 = c_2(n, q, \varepsilon_2)$.

Proof. The proof of the first part of the theorem is found in Kudrjavcev (1966b, Theorem 2.3′) and it will be omitted here. The second part can be obtained by coupling Kudrjavcev's technique with the results of Theorem II.6.3, as we are going to show. For simplicity, we shall consider the case where $n = 2$, leaving to the reader the simple task of establishing the result for $n \geq 3$. Setting

$$\overline{w} = \overline{u} - u_0,$$

we have to prove the following inequality:

$$\int_{-\infty}^{\infty} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 \leq c_2^q |u|^q_{1,q,\mathbb{R}^2_+}, \quad \sigma_2(x_1) = (1 + |x_1|)^{(1-q)/q-\varepsilon_2}. \tag{II.10.2}$$

Since, by Theorem II.6.3,

$$\begin{gathered} (u - u_0) \in L^{2q/(2-q)}(\mathbb{R}^2_+), \\ \|u - u_0\|_{2q/(2-q)} \leq \gamma_2 |u|_{1,q,\mathbb{R}^2_+}, \end{gathered} \tag{II.10.3}$$

from (II.10.1) we find

$$\int_{-1}^{1} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 \leq c^q |u|^q_{1,q,\mathbb{R}^2_+},$$

and so to show (II.10.2) it suffices to show

$$\int_{1}^{\infty} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1, \quad \int_{-\infty}^{-1} \sigma_2(x_1)^q |\overline{w}(x_1)|^q dx_1 \leq c_3 |u|^q_{1,q,\mathbb{R}^2_+}. \tag{II.10.4}$$

Let us consider the first integral in (II.10.4). In $\mathbb{R}^2_+$ we introduce a polar coordinate system $\rho \in (0,\infty)$, $\theta \in [0,\pi]$ with θ the angle formed by ρ with the positive x_1-axis. Since

$$x_1 = \rho\cos\theta,$$

$$x_2 = \rho\sin\theta,$$

we have

$$\int_1^\infty \sigma_2(x_1)^q|\overline{w}(x_1)|^q dx_1 = \int_1^\infty \sigma_2(\rho)^q|\overline{w}(\rho,0)|^q d\rho. \tag{II.10.5}$$

Setting

$$w = u - u_0,$$

for $x_1 \geq 1$,

$$\overline{w}(x_1) \equiv \overline{w}(\rho,0) = w(\rho,\theta) - \int_0^\theta \frac{\partial u}{\partial\tau}(\rho,\tau)d\tau.$$

Taking the modulus of both sides of this identity, raising them to the qth power, using (II.3.3) and the Hölder inequality, we find

$$|\overline{w}(\rho,0)|^q \leq c_1\left(|w(\rho,\theta)|^q + \int_0^\theta \left|\frac{\partial u}{\partial\tau}(\rho,\tau)\right|^q d\tau\right). \tag{II.10.6}$$

Observing that

$$\left|\frac{\partial u}{\partial\theta}(\rho,\theta)\right| \leq \rho|\nabla u|,$$

from (II.10.6) we derive, for all $\alpha \geq 0$,

$$\begin{aligned}\int_1^\infty \frac{|\overline{w}(\rho,0)|^q}{\rho^{\alpha q}}d\rho \leq c_2\Bigg(&\int_1^\infty\int_0^\pi \frac{|w(\rho,\theta)|^q}{\rho^{\alpha q+1}}\rho\, d\rho\, d\theta \\ &+\int_1^\infty\int_0^\pi \frac{|\nabla u(\rho,\theta)|^q}{\rho^{q(\alpha-1)+1}}\rho\, d\rho\, d\theta\Bigg).\end{aligned} \tag{II.10.7}$$

Taking

$$\alpha > 1 - 1/q, \tag{II.10.8}$$

we have for $\rho \geq 1$

$$\rho^{q(\alpha-1)+1} \geq 1. \tag{II.10.9}$$

Further, from (II.10.3) and (II.10.8)

$$\begin{aligned}\int_1^\infty\int_0^\pi \frac{|w(\rho,\theta)|^q}{\rho^{\alpha q+1}}\rho d\rho d\theta &\leq \left(\pi\int_1^\infty \rho^{1-2(\alpha q+1)/q}d\rho\right)^{q/2} \\ &\quad\times\left(\int_{\mathbb{R}^2_+}|w|^{2q/(2-q)}\right)^{(2-q)/q} \\ &\leq c_5\,|u|^q_{1,q,\mathbb{R}^2_+}.\end{aligned} \tag{II.10.10}$$

Therefore, the first relation in (II.10.4) follows from (II.10.5), (II.10.7), (II.10.9), and (II.10.10). To recover the second one, it is enough to observe that, for $x_1 \leq -1$,

$$\overline{w}(x_1) \equiv \overline{w}(\rho, \pi) = w(\rho, \theta) + \int_\theta^\pi \frac{\partial u(\rho, \tau)}{\partial \tau} d\tau,$$

and to proceed as in the previous case. The theorem is thus completely proved. □

Remark II.10.1 Theorem II.10.1 tells us, in particular, that if $1 \leq q < n$, $\overline{u}$ must tend to the constant u_0 at large distances on Σ, in the sense that for at least a sequence of radii $\{R_m\}$,

$$\lim_{R_m \to \infty} \int_{S^{n-2}} |u(R_m, \omega) - u_0| d\omega = 0,$$

where (R, ω) denotes a system of polar coordinate on Σ. On the other hand, if $q \geq n$, u may even grow at large distance on Σ. ■

Remark II.10.2 We notice, in passing, that Theorem II.10.1 admits of an obvious extension to the case where $m > 1$, in the sense that it selects the weighted L^q-space to which the trace $\overline{u}_\alpha \equiv D^\alpha u$ at Σ, $|\alpha| = m - 1$, of $u \in D^{m,q}(\mathbb{R}^n_+)$ must belong. In particular, if $mq < n$, in the light of Theorem II.6.4, u can be modified by the addition of a suitable polynomial $\mathcal{P}$ in such a way that $\mathsf{u} \equiv u - \mathcal{P}$ and all derivatives of u up to the order $m - 1$ included tend to zero on Σ in the way specified in Remark II.10.1. ■

A weighted space of the type $L^q(\Sigma, \sigma)$, however, does not coincide with the "trace space" of functions from $D^{1,q}(\mathbb{R}^n_+)$. This latter is, in fact, more restricted. To characterize such a space we set, as in the case of a bounded domain,

$$\langle\langle \overline{u} \rangle\rangle_{1-1/q,q} \equiv \left(\int_\Sigma \int_\Sigma \frac{|\overline{u}(x) - \overline{u}(y)|^q}{|x - y|^{n-2+q}} dxdy \right)^{1/q} \tag{II.10.11}$$

and denote by $D^{1-1/q,q}(\Sigma)$ the space of (equivalence classes of) real functions for which the functional (II.10.11) is finite. As in Section II.4, one can show that, provided we identify two functions if they differ by a constant, (II.10.11) defines a norm in $D^{1-1/q,q}(\Sigma)$ and that $D^{1-1/q,q}(\Sigma)$ is complete in this norm.

Exercise II.10.1 (Miranda 1978, Teorema 59.II)**.** Show that

$$u \in W^{1,q}(\Sigma), \quad \text{implies} \quad u \in D^{1-1/q,q}(\Sigma).$$

The following theorem holds, (Kudrjavcev 1966b, Theorems 2.4′ and 2.7 and Corollary 1).

Theorem II.10.2 *Let* Σ *be as in* Theorem II.10.1 *and let* $u \in D^{1,q}(\mathbb{R}^n_+)$, $1 < q < \infty$. *Then the trace* $\overline{u}$ *of* u *at* Σ *belongs to* $D^{1-1/q,q}(\Sigma)$ *and, further,*

$$\langle\langle \overline{u} \rangle\rangle_{1-1/q,q} \leq c_1 |u|_{1,q}$$

with $c_1 = c_1(n,q)$. *Conversely, given* $\overline{u} \in D^{1-1/q,q}(\Sigma)$, $1 < q < \infty$, *there exists* $u \in D^{1,q}(\mathbb{R}^n_+)$ *such that* $\overline{u}$ *is the trace of* u *at* Σ *and, further,*

$$|u|_{1,q} \leq c_2 \, \langle\langle \overline{u} \rangle\rangle_{1-1/q,q},$$

with $c_2 = c_2(n,q)$.

II.11 Some Integral Transforms and Related Inequalities

By *integral transform with kernel* K *of a function* f, we mean the function Ψ defined by

$$\Psi(x) = \int_\Omega K(x,y) f(y) dy. \tag{II.11.1}$$

Our objective in this section is to present some basic inequalities relating Ψ and f, under different assumptions on the kernel. We shall first consider the situation in which

$$K(x,y) = K(x-y),$$

where $K(\xi)$ is defined in the whole of $\mathbb{R}^n$. In this case, the transform (II.11.1) with $\Omega \equiv \mathbb{R}^n$ is called a *convolution*, and it is also denoted by $K * f$. An example of convolution is the regularizer of f, which we already introduced in Section II.2. For these transforms we have the following classical result due to Young (see, e.g., Miranda 1978, Teorema 10.I).

Theorem II.11.1 *Let*

$$K \in L^s(\mathbb{R}^n), \quad 1 \leq s < \infty.$$

If

$$f \in L^q(\mathbb{R}^n), \quad 1 \leq q \leq \infty, \quad 1/q \geq 1 - 1/s,$$

then

$$K * f \in L^r(\mathbb{R}^n), \quad 1/r = 1/s + 1/q - 1,$$

and the following inequality holds:

$$\|K * f\|_r \leq \|K\|_s \|f\|_q. \tag{II.11.2}$$

Exercise II.11.1 Prove inequality (II.11.2) for the case $q = 1$. *Hint*: Use the generalized Minkowski inequality (II.2.8).

Another class of transforms that will be frequently considered is that defined by kernels K of the form

$$K(x,y) = \frac{k(x,y)}{|y|^\lambda}, \quad \lambda > 0, \quad y \in \Omega, \tag{II.11.3}$$

where $k(x,y)$ is a given regular function. If $0 < \lambda < n$ and $k(x,y) \equiv 1$, the kernel (II.11.3) is referred to as *weakly singular* and the corresponding transform (II.11.1) is called the *Riesz potential.* If $\lambda = n$ and $k(x,y)$ is suitable (see (II.11.15)–(II.11.17)), the kernel and the associated transform are called *singular.* The study in Lebesgue spaces L^s of Riesz potentials finds a fundamental contribution in the celebrated paper of Sobolev (1938) (see Theorem II.11.3), while that related to (multidimensional) singular kernels traces back to the work of Calderón and Zygmund (1956) (see Theorem II.11.4).

When Ω is bounded and K is weakly singular one can easily show elementary estimates for $\Psi = K * f$ in terms of f. For example, if

$$\lambda < n(1 - 1/q)$$

one has the inequality

$$\sup_{x\in\Omega} |\Psi(x)| \leq c\|f\|_q \tag{II.11.4}$$

with

$$c = \left(\frac{1}{n - \lambda q'}\right)^{1/q'} \omega_n^{1/q'} \delta(\Omega)^{n/q' - \lambda}. \tag{II.11.5}$$

To show this, it suffices to observe that for all $r > 0$ and $\lambda r < n$,

$$\left(\int_{|x-y|\leq R} |x-y|^{-\lambda r} dy\right)^{1/r} \leq \left(\frac{1}{n-\lambda r}\right)^{1/r} \omega_n^{1/r} R^{n/r - \lambda}. \tag{II.11.6}$$

Thus, (II.11.4) and (II.11.5) follow from (II.11.1), (II.11.3), (II.11.6), and the Hölder inequality. Actually, one can prove an estimate stronger than (II.11.4) under the same assumption on λ, n, and q. In fact, from (II.11.3) with $k(x,y) = 1$, by the mean value theorem it follows that

$$|K(x-y) - K(z-y)| \leq \lambda |x-z| d(y)^{-(\lambda+1)},$$

where $d(y)$ is the distance of y from the segment s with endpoints x and z. Setting $\sigma = |x-z|$ and employing this last inequality, from (II.11.1) we deduce

$$\begin{aligned}|\Psi(x) - \Psi(z)| \leq & \int_{|x-y|<2\sigma} |f(y)||x-y|^{-\lambda} dy + \int_{|z-y|<2\sigma} |f(y)||z-y|^{-\lambda} dy \\ & + \lambda\sigma \int_{\Omega\cap\{|x_0 - y|>\sigma\}} |f(y)| d(y)^{-(\lambda+1)} dy\end{aligned} \tag{II.11.7}$$

with x_0 the midpoint of s. Since $d \geq \sigma/2$, by Carnot's theorem it easily follows that $2d \geq |x - x_0|$. Therefore, assuming $\lambda < n(1 - 1/q)$ and employing the Hölder inequality, the last term in (II.11.7) can be increased by

$$C_1 \left(\sigma + \sigma^{n(1-1/q)-\lambda}\right) \|f\|_q \,, \tag{II.11.8}$$

where $C_1 = C_1(\delta(\Omega), n, q, \lambda)$. On the other hand, by an easy calculation that makes use of (II.11.6) and the Hölder inequality, we show that the first two integrals in (II.11.7) can be dominated by

$$C_2\, \sigma^{n(1-1/q)-\lambda} \|f\|_q,$$

where $C_2 = C_2(n, \lambda)$. Thus, this latter relation along with (II.11.7) and (II.11.8) furnishes

$$|\Psi(x) - \Psi(z)| \leq C \left(\sigma + \sigma^{n(1-1/q)-\lambda}\right) \|f\|_q,$$

where $C = 2\max(C_1, C_2)$. Still retaining the assumption that Ω is bounded, we shall now discuss the case where $\lambda = n(1 - 1/q)$. We set

$$\widetilde{K}(x-y) = \begin{cases} |x-y|^{-\lambda} & \text{if } x, y \in \Omega \\ 0 & \text{if } x, y \notin \Omega \,. \end{cases}$$

Clearly,

$$\Psi(x) = \int_\Omega |x-y|^{-\lambda} f(y) dy = \int_{\mathbb{R}^n} \widetilde{K}(x-y) f(y) dy,$$

and so, by noticing that

$$\widetilde{K} \in L^s(\mathbb{R}^n), \quad \text{for all } s < n/\lambda, \tag{II.11.9}$$

from Young's Theorem II.11.1 it follows that if $f \in L^q(\Omega)$ then

$$\Psi \in L^r(\Omega), \quad 1/r = 1/s + 1/q - 1 \tag{II.11.10}$$

and that the following inequality holds:

$$\|\Psi\|_r \leq c \|f\|_q.$$

Taking into account (II.11.9) and that $\lambda = n(1 - 1/q)$, from (II.11.10) we conclude that

$$\Psi \in L^r(\Omega), \quad \text{for all } r \in [1, \infty).$$

The results established so far are collected in

Theorem II.11.2 *Assume Ω bounded, K weakly singular, and $f \in L^q(\Omega)$, $1 < q < \infty$. Then if $\lambda < n(1 - 1/q)$, the integral transform Ψ defined by*

(II.11.1) *belongs to* $C^{0,\mu}(\overline{\Omega})$ *where* $\mu = \min\{1, n(1-1/q)-\lambda\}$ *and the following estimate holds:*

$$\|\Psi\|_{C^{0,\mu}} \leq C_1 \|f\|_q, \tag{II.11.11}$$

with $C_1 = C_1(\delta(\Omega), n, q, \lambda)$. *Moreover, if* $\lambda = n(1-1/q)$, *then* $\Psi \in L^r(\Omega)$ *for all* $r \in [1, \infty)$, *and the following estimate holds:*

$$\|\Psi\|_r \leq C_2 \|f\|_q, \tag{II.11.12}$$

with $C_2 = C_2(\delta(\Omega), n, q, \lambda)$.

The complementary situation $\lambda > n(1-1/q)$ is considered in Sobolev's theorem which, in addition, does not require the boundedness of Ω. Precisely, we have (Sobolev 1938; for a simpler proof see Stein 1970, Chapter V)

Theorem II.11.3 *Assume* $f \in L^q(\mathbb{R}^n)$, $1 < q < \infty$, *and* K *weakly singular. Then, if* $\lambda > n(1-1/q)$, *the integral transform* Ψ *defined by* (II.11.1) *with* $\Omega \equiv \mathbb{R}^n$ *belongs to* $L^s(\mathbb{R}^n)$, *where* $1/s = \lambda/n + 1/q - 1$. *Moreover, we have*

$$\|\Psi\|_s \leq C \|f\|_q \tag{II.11.13}$$

with $C = C(q, n, \lambda)$.

Remark II.11.1 By means of simple counterexamples one shows that the Sobolev theorem fails either when $q = 1$ or when $s = \infty$ (see Stein 1970, p.119).

Some interesting observations and consequences related to Theorem II.5.1-Theorem II.5.4 are left to the reader in the following exercises. ■

Exercise II.11.2 Show that if (II.11.13) holds, necessarily $1/s = \lambda/n + 1/q - 1$. *Hint*: Use the homogeneity of the Riesz potential.

Exercise II.11.3 For $f \in C_0^\infty(\mathbb{R}^n)$, set $u(x) = (\mathcal{E} * f)(x)$ where $\mathcal{E}$ is the fundamental solution of Laplace's equation (see (II.9.1)). Verify that u is a C^∞ solution of the Poisson equation $\Delta u = f$ in $\mathbb{R}^n$. Moreover, use the Sobolev theorem to show

$$\|\nabla u\|_{nq/(n-q)} \leq c\|f\|_q, \quad 1 < q < n.$$

Exercise II.11.4 Assume $u \in W_0^{1,q}(\mathbb{R}^n)$, $1 < q < \infty$. Starting from the representation given in Lemma II.9.1, prove the following assertions:

(i) If $q < n$, then $u \in L^{nq/(n-q)}(\mathbb{R}^n)$ and $\|u\|_{nq/(n-q)} \leq \gamma \|\nabla u\|_q$;

Hint: Use Theorem II.11.3. Notice that, without using the Sobolev theorem, (i) is obtained directly from Lemma II.3.2 in a much more elementary way (see (2.6)) and with the following advantages: (a) the case $q = 1$ is included; (b) an explicit estimate of the constant γ can be given.

(ii) If $q = n$, then $u \in L^r(\Omega)$, for all $r \in [n, \infty)$ and for any compact domain Ω. *Hint*: Use Theorem II.11.2.

(iii) If $q > n$, then $u \in C^{0,\mu}(\overline{\Omega})$, $\mu = 1 - n/q$, for any compact domain Ω. *Hint*: Use Theorem II.11.2.

Exercise II.11.5 Let Ω be bounded. Show that every function from $W_0^{1,q}(\Omega)$, $q > n$, satisfies the inequality

$$\|u\|_C \le c[\delta(\Omega)]^{1-n/q}\|\nabla u\|_q, \tag{II.11.14}$$

with $c = c(n,q)$. *Hint*: Use the representation formula of Lemma II.9.1 together with relations (II.11.4) and (II.11.5).

We shall now consider the case of singular kernels. We say that a kernel of the form (II.11.3) with $x \in \Omega$, $y \in \mathbb{R}^n - \{0\}$ and $\lambda = n$ is *singular* if and only if

(i) For any admissible x, y and every $\alpha > 0$

$$k(x,y) = k(x,\alpha y); \tag{II.11.15}$$

(ii) For every $x \in \Omega$, $k(x,y)$ is integrable on the sphere $|y| = 1$ and

$$\int_{|y|=1} k(x,y)dy = 0; \tag{II.11.16}$$

(iii) There exists $C > 0$, such that[1]

$$\operatorname*{ess\ sup}_{x\in\Omega;|y|=1} |k(x,y)| \le C\,. \tag{II.11.17}$$

Exercise II.11.6 Show that (II.11.16) is equivalent to the following:

$$\int_{r_1\le|y|\le r_2} K(x,y)dy = 0, \tag{II.11.18}$$

for every x and $r_2 > r_1 > 0$.

Condition (II.11.18) allows us to recognize a noteworthy class of singular kernels. Precisely, we have the following simple but useful result, due to L. Bers and M. Schechter, which we state in the form of a lemma (see Bers, John, & Schechter 1964, p. 223).

Lemma II.11.1 *Let $M(x,y)$ be a function on $\Omega \times (\mathbb{R}^n - \{0\})$, differentiable in y and homogeneous of order $1-n$ with respect to y, that is,*

$$M(x,\alpha y) = \alpha^{1-n}M(x,y), \quad \alpha > 0.$$

[1] This assumption can be weakened; see Calderón & Zygmund (1956, Theorem 2(ii)). However, a weaker assumption would be irrelevant to our purposes.

Assume further that $M_i(x,y) \equiv \partial M(x,y)/\partial y_i$ *satisfies, with some* $C > 0$, *independent of* x,

$$\operatorname*{ess\,sup}_{|y|=1} |M_i(x,y)| \le C\,.$$

Then $M_i(x,y)$ *is a singular kernel.*

Proof. For all $x \in \Omega$ we have

$$\int_{r_1 \le |y| \le r_2} M_i(x,y)dy = \int_{|\eta|=r_2} M(x,\eta)(\eta_i/r_2)d\sigma_\eta - \int_{|\eta|=r_1} M(x,\eta)(\eta_i/r_1)d\sigma_\eta,$$

so that (II.11.18) follows by homogeneity. Therefore, setting

$$k(x,y) = M_i(x,y)|y|^n,$$

by assumption and Exercise II.11.6 we conclude that $M_i(x,y) = k(x,y)|y|^{-n}$ is a singular kernel. □

Exercise II.11.7 Let $\mathcal{E}$ be the fundamental solution to Laplace's equation. Show that $D_{ij}\mathcal{E}(x)$ is a singular kernel.

For integral transforms defined by singular kernels we have the following fundamental result due to Calderón & Zygmund (1956, Theorem 2).

Theorem II.11.4 *Assume* $K(x,y)$ *is a singular kernel and let*

$$N(x,y) \equiv K(x, x-y).$$

Then, if

$$f \in L^q(\mathbb{R}^n), \quad 1 < q < \infty,$$

the P.V. *convolution integral*

$$\Psi(x) = \lim_{\epsilon \to 0} \int_{|x-y| \ge \epsilon} N(x,y)f(y)dy \tag{II.11.19}$$

exists for almost all $x \in \Omega$. *Moreover,*

$$\Psi \in L^q(\mathbb{R}^n)$$

and the following inequality holds:

$$\|\Psi\|_q \le c\|f\|_q. \tag{II.11.20}$$

Exercise II.11.8 Assume K given by (II.11.3), with $k(x,y)$ bounded and $\lambda = n$. Show that, if $f \in C_0^1(\mathbb{R}^n)$, the limit (II.11.19) exists if and only if $k(x,y)$ satisfies condition (II.11.16). *Hint*: Use the identity ($a > \epsilon > 0$)

$$\begin{aligned}\int_{|x-y|\geq\epsilon} N(x,y)f(y)dy = &\int_{|x-y|\geq a} N(x,y)f(y)\\ &+\int_{\epsilon<|x-y|<a}[f(y)-f(x)]N(x,y)dy\\ &+f(x)\int_{\epsilon<|x-y|<a} N(x,y)dy.\end{aligned}$$

Remark II.11.2 Sometimes it is useful to know more about the constant c in (II.11.19) and, particularly, about the way in which it depends on q and k. Here we recall some estimate due to Stein (1970, Chapter II) and to Calderón and Zygmund (1957, §5). Specifically, as far as the dependence on q, one can show:

$$c \leq \begin{cases} c_1/(q-1) \text{ if } 1 < q \leq 2 \\ c_1 q \qquad\quad \text{ if } q \geq 2, \end{cases}$$

with $c_1 = c_1(k)$. Likewise, if $A > 0$ is a constant such that

$$\sup_{x\in\Omega,\ |y|=1} |k(x,y)| \leq A,$$

then one has

$$c \leq c_2\, A, \quad c_2 = c_2(q).$$

■

Two important consequences of the Calderón–Zygmund theorem will be considered. The first one is due to Stein (1957) and is contained in the following.

Theorem II.11.5 *Let the assumptions of* Theorem II.11.4 *be satisfied, and suppose, in addition*

$$f(x)|x|^\beta \in L^q(\mathbb{R}^n), \quad \beta \in (-n/q, n(1-1/q))\,,$$

and that $|k(x,y)| \leq C$, *for some* C *independent of* x *and* y. *Then,*

$$\Psi(x)|x|^\beta \in L^q(\mathbb{R}^n)$$

and the following inequality holds

$$\|\Psi(x)|x|^\beta\|_q \leq c_1 C \|f(x)|x|^\beta\|_q, \tag{II.11.21}$$

where $c_1 = c_1(n,q,\beta)$.

The second consequence is a well-known result of Agmon, Douglis, & Nirenberg (1959, Theorem 3.3), which we are now going to state.

Theorem II.11.6 *Let*

$$K(x', x_n) = \frac{\widetilde{\omega}(x'/|x|, x_n/|x|)}{|x|^{n-1}} \quad x' = (x_1, \ldots, x_{n-1}).$$

Assume that $D_i K$, $i = 1, \ldots, n$, and $D_n^2 K$ are continuous in $\mathbb{R}^n_+$ and bounded in $\mathbb{R}^n_+ \cap S^{n-1}$ by a positive constant κ. Assume further

$$\int_{|x'|=1} \widetilde{\omega}(x', 0) dx' = 0. \tag{II.11.22}$$

Then, setting $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$, given

$$\phi \in L^q(\Sigma), \quad \text{with } \langle\langle \phi \rangle\rangle_{1-1/q,q} \text{ finite,}$$

the integral transform

$$u(x', x_n) = \int_\Sigma K(x' - y', x_n)\phi(y')dy' \tag{II.11.23}$$

belongs to $L^q(\mathbb{R}^n_+)$ and the following inequality holds:

$$|u|_{1,q} \leq c\kappa \langle\langle \phi \rangle\rangle_{1-1/q,q}, \tag{II.11.24}$$

with $c = c(n, q)$.

Theorem II.11.4 and Theorem II.11.6 play a fundamental role in the L^q-theory of elliptic partial differential equations, mainly in deriving a priori estimates for solutions (see, e.g., Agmon, Douglis, & Nirenberg 1959). In the following exercises, we shall propose very simple applications of them to the Poisson equation in $\mathbb{R}^n$ and to the Dirichlet problem for the Poisson equation in $\mathbb{R}^n_+$. Other more relevant applications will be derived, along the same lines as those that follow, in Chapter IV, in the context of steady slow motions of a viscous incompressible fluid (Stokes problem).

Exercise II.11.9 For the problem $\Delta u = f$ in $\mathbb{R}^n$ show that there is a solution u such that

(i) If $f \in W^{m,q}(\mathbb{R}^n)$, $m \geq 0$, $1 < q < \infty$, then $u \in \cap_{k=0}^m D^{k+2}(\mathbb{R}^n)$ and the following inequality holds:

$$|u|_{k+2,q} \leq c_1 \|f\|_{k,q}, \quad k = 0, 1, ..., m, \ c_1(n, q, k);$$

(ii) If $f \in D_0^{-1,q}(\mathbb{R}^n)$, $m \geq 0$, $1 < q < \infty$, then $u \in D_0^{1,q}(\mathbb{R}^n)$ and the following inequality holds:

$$|u|_{1,q} \leq c_2 |f|_{-1,q}, \quad c_2(n, q, k).$$

Hint: Take $f \in C_0^\infty(\mathbb{R}^n)$. Then a solution is given by $u = \mathcal{E} * f$ (see Exercise II.11.3). To show (i), use Theorem II.11.4 and Exercise II.11.7. To show (ii), observe that, for any $\varphi \in L^{q'}(B_r)$ and $i = 1, ..., n$,

$$(D_i u, \varphi) = \int_{\mathbb{R}^n} f(y)\phi(y)dy, \quad \phi = (D_i\mathcal{E}) * \varphi,$$

and that, by Theorem II.11.4,

$$|\phi|_{1,q'} \le c\|\varphi\|_{q',B_r}$$

with c independent of r. Employ, finally, the results of Exercise II.3.4 and Theorem II.8.1.

Exercise II.11.10 It is well known that the function (*Poisson integral*)

$$u(x) = 2\int_\Sigma \phi(y)\frac{\partial \mathcal{E}}{\partial y_n}dy,$$

with $\Sigma = \{x \in \mathbb{R}^n : x_n = 0\}$, $\mathcal{E}$ given in (7.1) and $\phi \in C^m(\overline{\Sigma})$, $m \ge 0$, is a smooth solution to the Dirichlet problem in the half-space:

$$\begin{aligned} \Delta u = 0 \ \text{ in } \mathbb{R}^n_+, \quad n \ge 2 \\ u = \phi \ \text{ at } \Sigma \end{aligned} \tag{II.11.25}$$

(see, e.g., Sobolev 1964, Lecture 13). Use Theorem II.11.6 to show that if

$$\phi \in W^{m,q}(\Sigma) \ \text{ and } \sum_{|k|=m} \langle\langle\, D^k\phi\,\rangle\rangle_{1-1/q,q} < \infty, \quad 1 < q < \infty,$$

then

$$|u|_{s+1,q} \le c\sum_{|k|=s} \langle\langle\, D^k\phi\,\rangle\rangle_{1-1/q,q}, \quad \text{for all } s = 0, 1, ..., m,$$

with $c = c(n, q, s)$.

Uniqueness of solutions determined in the preceding exercises can be easily studied by means of the following result, which the reader is invited to prove.

Exercise II.11.11 Let H be harmonic in the whole of $\mathbb{R}^n$. Assume either

(i) $H = \sum_{i=1}^N H_i$, $N \ge 1$, where $\displaystyle\int_{B^\rho} \frac{|H_i(x)|^{q_i}}{(1+|x|)^{\lambda_i}} < \infty$, for some $q_i \in (1,\infty)$, $\rho > 0$, and $\lambda_i \in [0, n]$;

or

(ii) $\lim_{|x|\to\infty} H(x) = 0$.

Show $H \equiv 0$. *Hint*: By the mean value theorem, we have, for each $x \in \mathbb{R}^n$,

$$|H(x)| \le (n\omega_n)^{-1}\int_{S^{n-1}} |H(R,\omega)|d\omega, \quad R = |x - y|.$$

Remark II.11.3 In virtue of this latter result it follows that solutions determined in Exercise II.11.9(i) are unique in $\dot{D}^{2,q}(\mathbb{R}^n)$, while those in (ii) are unique in $D_0^{1,q}(\mathbb{R}^n)$. As far as solutions considered in Exercise II.11.10, their uniqueness is likewise discussed, since if u solves (II.11.25) with $\phi \equiv 0$, then the function

$$\widetilde{u}(x) = \begin{cases} u(x_1, \dots, x_n) & \text{if } x_n > 0 \\ -u(x_1, \dots, -x_n) & \text{if } x_n \leq 0 \end{cases}$$

is harmonic (and hence smooth; Weyl 1940, Simader 1992) throughout $\mathbb{R}^n$, including $x_n = 0$; see, for instance, Sobolev (1964, Lecture 13). ■

II.12 Notes for the Chapter

Section II.1. A similar (but different in details) proof of Lemma II.1.3 can be found in Erig (1982, Lemma 5.3).

Section II.3. Inequality (II.3.9) was derived by Ladyzhenskaya (1959a) with a larger value of the constant. In this respect, see also Serrin (1963).

Extensions of Lemma II.3.3 to domains with a (sufficiently smooth) bounded boundary can be found in Friedman (1969, Theorem 10.1) for bounded domains, and in Crispo & Maremonti (2004) for exterior domains.

Sequence of functions like that employed in Exercise II.3.9 can also be used to find the best exponents (for fixed dimension) in certain inequalities relating surface and volume integrals, of the type described in Section II.4 (Galdi, Payne, Proctor, & Straughan 1987).

Section II.4. The way of introducing trace inequalities through star-shaped domains is an intrinsic treatment that does not make a *direct* use of the definition of surface integral by means of local representation of the boundary. For this latter approach see, e.g., Nečas (1967, Chapitre 2 Théorème 4.2) and Adams (1975, Chapter 5 Theorem 5.22).

The constant C in Theorem II.4.1 can be simply estimated if the shape of Ω is particular; in this regard see Galdi, Payne, Proctor, & Straughan (1987).

Section II.5. As already remarked, inequality (II.5.1) fails if Ω is not contained in some layer L_d; see Exercise II.5.1. However, in this latter case, (II.5.1) can be replaced by "weighted" inequalities such as (II.6.10), (II.6.13), and (II.6.14). Furthermore, the choice of the "weight" can be suitably related to the "geometry" of Ω at infinity. For instance, if

$$\Omega \subset \{x \in \mathbb{R}^n : |x'| < g(x_n)\},$$

where g satisfies

$$g(t) > g_0, \quad \text{for some } g_0 > 0,$$

then one has

$$\|u/g(x_n)\|_q \le c|u|_{1,q}, \quad u \in C_0^\infty(\Omega).$$

For this and similar inequalities, we refer, among others, to Elcrat and MacLean (1980), Hurri (1990), and Edmunds & Opic (1993).

The Friedrichs inequality (II.5.8) can be a fundamental tool for treating the convergence of approximating solutions of nonlinear partial differential equations. A nontrivial generalization of (II.5.8) is found in Padula (1986, Lemma 3). Extension of the Friedrichs inequality to *unbounded domains* are considered in Birman & Solomjak (1974).

From Theorem II.5.2 and Theorem II.4.1 it is not hard to prove compactness results involving convergence in boundary norms. For example, we have: if $\{u_k\} \subset W^{1,2}(\Omega)$ (Ω bounded and locally Lipschitz) is uniformly bounded, there is a subsequence $\{u_{m'}\}$ such that $u_{m'} \to u$ in $L^q(\partial\Omega)$ with $q = 2(n-1)/(n-2)$ if $n > 2$ and all $q \in [1, \infty)$ if $n = 2$.

The counterexample to compactness after Theorem II.5.2 is due to Benedek & Panzone (see Serrin 1962).

The Poincaré–Sobolev inequality can be proved for a general class of domains, including those with internal cusps. Such a generalization, which is of interest in the context of capillarity theory of fluids, can be found in Pepe (1978). However, in general, embedding inequalities no longer hold if the domain does not possess a certain degree of regularity. For this type of questions we refer to Adams & Fournier (2003, §4.47).

Section II.6. After the pioneering work of Deny & Lions (1954) on the subject ("Beppo Levi Spaces"), a detailed study of homogeneous Sobolev spaces $\dot{D}^{m,q}(\Omega)$ and $D_0^{m,q}(\Omega)$ along with the study of their relevant properties was performed by the Russian school (Uspenskiĭ 1961, Sobolev 1963b, Sedov 1966, Besov 1967). These authors are essentially concerned with the case where $\Omega = \mathbb{R}^n$. For other detailed analysis of the homogeneous Sobolev spaces we refer the reader also to the work of Kozono & Sohr (1991) and Simader & Sohr (1997), and to Chapter I of the book of Maz'ja (1985).

A central role in the study of properties of functions from $D^{m,q}(\Omega)$ is played by the fundamental Lemma II.6.3 which, for $q = 2$ and $n \ge 3$, was first proved by Payne & Weinberger (1957). A slightly weaker version of it was independently provided by Uspenskiĭ (1961, Lemma 1). The proof given in this book is based on a generalization of the ideas of Payne & Weinberger and is due to me. Another proof has been given by Miyakawa & Sohr (1988, Lemma 3.3), which, however, does not furnish the explicit form of the constant u_0. Concerning this issue, from Lemma II.6.3 it follows that

$$u_0 = \lim_{|x|\to\infty} \int_{S^{n-1}} u(|x|, \omega)\, d\omega,$$

or also, as kindly pointed out to me by Professor Christian Simader,

$$u_0 = \lim_{R\to\infty} \frac{1}{|\Omega_R|} \int_{\Omega_R} u.$$

Results contained in Exercise II.6.3 generalize part of those established by Uspenskiĭ (1961, Lemma 1), and for $q = n = 2$ they coincide with those of Gilbarg & Weinberger (1978, Lemma 2.1).

Inequality (II.6.20) with $q = 2$ and $n = 3$ is due to Finn (1965a, Corollary 2.2a); see also Birman & Solomjak (1974, Lemma 2.19) and Padula (1984, Lemma 1), while (II.6.22) for $n = 3$ and $q \in (1, 3)$ is proved by Galdi & Maremonti (1986, Lemma 1.3). Theorem II.6.1, in its generality, is due to me.

The inequality in Theorem II.6.5 is due to Simader and Sohr (1997, Lemma 1.2).

Section II.7. The problem of approximation of functions from $D^{m,q}(\Omega)$ when $\Omega = \mathbb{R}^n$ with functions of bounded support was first considered by Sobolev (1963b). In this section we closely follow Sobolev's ideas to generalize his results to more general domains Ω. In this connection, we refer the reader also to the papers of Besov (1967, 1969) and Burenkov (1976).

The elementary proof of the Hardy-type inequality (II.6.10), (II.6.13) and (II.6.14) presented here and based on the use of the "auxiliary" function $\boldsymbol{g}$ was presented for the first time in Galdi (1994a, §2.5). The same approach was successively rediscovered by Mitidieri (2000).

Section II.8. A slightly weaker version of Theorem II.8.2, with a different proof, can be found in Kozono & Sohr (1991, Lemma 2.2).

The proof of the unique solvability of the Dirichlet problem (II.8.17) in the case $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ is a simple consequence of Exercise II.11.9(ii) and Remark II.11.3. In the case Ω bounded and of class C^∞, a proof was given for the first time by Schechter (1963a, Corollary 5.2). A different proof that requires domains only of class C^2 was later provided by Simader (1972). If Ω is an exterior domain of class C^2, a thorough analysis of the problem can be found in Simader & Sohr (1997, Chapter I). In particular, for $n \geq 3$, the analysis of these authors shows that the problem (II.8.17) has a nonzero one-dimensional null set, if $q' \geq n$. In other words, there exists one and only one non-zero harmonic function $h \in D_0^{1,q'}(\Omega)$, satisfying a normalization condition $\int_{\Omega_R} h^2 = 1$, for some fixed $R > \delta(\Omega)^c$. For instance, if Ω is the exterior of the unit ball in $\mathbb{R}^n$, we have $h(x) = c(|x|^{2-n} - 1)$, for a suitable choice of the constant c depending on R. Consequently, the map $\mathfrak{M}$ defined in (II.8.19)–(II.8.20) is not surjective if $q' \in (1, n/(n-1)]$ and not injective if $q' \in [n, \infty)$.

Section II.9. Results similar to those derived in Theorem II.9.1, in the general context of spaces $D^{m,q}$, $m \geq 1$, have been shown by Mizuta (1989). Estimate (II.9.5) is of a particular interest since, as we shall see in Chapter X, it permits us to derive at once an important asymptotic estimate for solutions to the steady, two-dimensional Navier–Stokes equations in exterior domains having velocity fields with bounded Dirichlet integrals.

Section II.10. The case $1 \leq q < n$ in Theorem II.10.1 is due to me.

Section II.11. If in the Sobolev Theorem II.11.3 one considers the function

$$\psi(x) = \int_{|x-y| \leq R} f(y)|x-y|^{-\lambda} dy,$$

for fixed $R > 0$, the proof of (II.11.13) becomes elementary; however, only for $1/s > \lambda/n + 1/q - 1$ (see Sobolev 1938; 1963a, Chapter 1 §6). For a generalization of the Sobolev theorem in weighted Lebesgue spaces, along the same lines of Theorem II.11.5, we refer to Stein & Weiss (1958).

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