2

The Ancients

Man propounds negotiations, Man accepts the compromise.
Very rarely will he squarely push the logic of a fact
To its ultimate conclusion in unmitigated act.

—Rudyard Kipling

For a charm of powerful trouble,
Like a hell-broth boil and bubble.
Double, double toil and trouble;
Fire burn and cauldron bubble.

—William Shakespeare, *Macbeth*

... If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did ...

—August De Morgan

One normally thinks that everything that is true is true for a reason. I've found mathematical truths that are true for no reason at all. These mathematical truths are beyond the power of mathematical reasoning because they are accidental and random.

—G. J. Chaitin

I should like to ask the same question that Descartes asked. You are proposing to give a precise definition of logical correctness which is to be the same as my vague intuitive feeling for logical correctness. How do you intend to show that they are the same? ... The average mathematician should not forget that intuition is the final authority.

—J. Barkley Rosser

2.1 Eudoxus and the Concept of Theorem

Perhaps the first mathematical proof in recorded history is due to the Babylonians. They seem (along with the Chinese) to have been aware of the Pythagorean theorem (discussed in

S.G. Krantz, *The Proof is in the Pudding: The Changing Nature of Mathematical Proof*,
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detail below) well before Pythagoras.[29] The Babylonians had certain diagrams that indicate why the Pythagorean theorem is true, and tablets have been found to validate this fact.[30] They also had methods for calculating Pythagorean triples—that is, triples of integers (or whole numbers) a, b, c that satisfy

$$a^2 + b^2 = c^2$$

as in the Pythagorean theorem.

The Babylonians were remarkably sophisticated in a number of ways. As early as 1200 BCE they had calculated $\sqrt{2}$ to (what we would call) six decimal places.[31] They did not "prove" theorems as we conceive of the activity today, but they had well-developed ideas about mathematics (not just arithmetic).

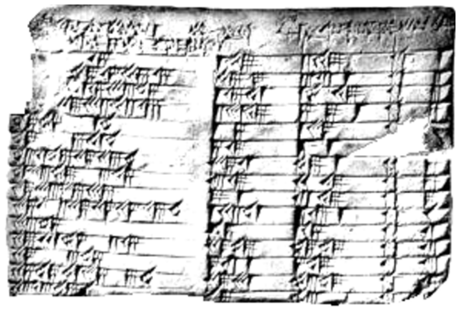


Figure 2.1. The Plimpton 322 tablet.

It was the Greek Eudoxus (408–355 BCE) who began the grand tradition of organizing mathematics into theorems.[32] Eudoxus was one of the first to use this word in the context of mathematics.

What Eudoxus gained in the rigor and precision of his mathematical formulations, he lost because he did not prove anything. Formal proof was not yet the tradition in mathematics. As we have noted elsewhere, mathematics in its early days was a largely heuristic and empirical subject. It had never occurred to anyone that there was any need to prove anything. When you asked yourself whether a certain table would fit in your dining room, you did not prove a theorem; you just checked it out.[33] When you wondered whether a certain amount of fence would surround your pasture, you did not seek a rigorous argument; you simply unrolled the fence and determined whether it did the job. In its earliest days, mathematics was intimately bound up with questions precisely like these. Thus mathematical thinking was almost inextricable from practical thinking. And that is how its adherents viewed mathematical facts. They were just practical information, and their assimilation and verification was a strictly pragmatic affair.

[29]Although it must be stressed that they did not have Pythagoras's sense of the structure of mathematics, of the importance of rigor, or of the nature of formal proof.

[30]This is the famous Plimpton 322 tablet. See Figure 2.1.

[31]This can be found in the YBC 7289 tablet. Of course the Babylonians did *not* have decimal notation.

[32]The word "theorem" comes from the Greek root *theorema*, meaning "speculation."

[33]William (Willy) Feller (1906–1970) was a prominent mathematician at Princeton University. He was one of the fathers of modern probability theory. Feller and his wife were once trying to move a large circular table from their living room into the dining room. They pushed and pulled and rotated and maneuvered, but try as they might they could not get the table through the door. It seemed to be inextricably stuck. Frustrated and tired, Feller sat down with a pencil and paper and devised a mathematical model of the situation. After several minutes he was able to *prove* that what they were trying to do was impossible. While Willy was engaged in these machinations, his wife had continued struggling with the table, and she managed to get it into the dining room.

2.2 Euclid the Geometer

Euclid (325–265 BCE) is hailed as the first scholar to systematically organize mathematics (i.e., a substantial portion of the mathematics that had come before him), formulate definitions and axioms, and prove theorems. This was a monumental achievement, and a highly original one.

Euclid is not known as much (as were Archimedes and Pythagoras) for his original and profound insights—although there are some important theorems and ideas named after him—but he has had overall an incisive effect on human thought. After all, Euclid wrote a treatise (consisting of 13 Books)—now known as Euclid's *Elements*—which has been continuously available for over 2000 years and has been through a large number of editions. It is still studied in detail today, and continues to have a substantial influence over the way we think about mathematics.

Not a great deal is known about Euclid's life, although it is fairly certain that he had a school in Alexandria. In fact "Euclid" was a common name in his day, and various accounts of Euclid the mathematician's life confuse him with other Euclids (one a prominent philosopher). One appreciation of Euclid comes from Proclus, one of the last of the ancient Greek philosophers:

> Not much younger than these [pupils of Plato] is Euclid, who put together the *Elements*, arranging in order many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefutable demonstration the things which had been only loosely proved by his predecessors. This man lived in the time of the first Ptolemy; for Archimedes, who followed closely upon the first Ptolemy makes mention of Euclid, and further they say that Ptolemy once asked him if there were a shorter way to study geometry than the *Elements*, to which he replied that "there is no royal road to geometry." He is therefore younger than Plato's circle, but older than Eratosthenes and Archimedes; for these were contemporaries, as Eratosthenes somewhere says. In his aim he was a Platonist, being in sympathy with this philosophy, whence he made the end of the whole *Elements* the construction of the so-called Platonic figures.

As often happens with scientists and artists and scholars of immense accomplishment, there is disagreement, and some debate, over exactly who or what Euclid actually was. The three schools of thought are these:

- Euclid was a historical character—a single individual—who in fact wrote the *Elements* and the other scholarly works that are commonly attributed to him.
- Euclid was the leader of a team of mathematicians working in Alexandria. They all contributed to the creation of the complete works that we now attribute to Euclid. They even continued to write and disseminate books under Euclid's name after his death.
- Euclid was not a historical character at all. In fact "Euclid" was a *nom de plume* adopted by a group of mathematicians working in Alexandria. They took their inspiration from Euclid of Megara (who *was* in fact an historical figure), a prominent philosopher who lived about one hundred years before Euclid the mathematician is thought to have lived.

Most scholars today subscribe to the first theory—that Euclid was a unique person who created the *Elements*. But we acknowledge that there is evidence for the other two scenarios. Almost surely Euclid had a vigorous school of mathematics in Alexandria, and there is little doubt that his students participated in his projects.

It is thought that Euclid must have studied in Plato's (430–349 BCE) academy in Athens, for it is unlikely that there would have been another place where he could have learned the geometry of Eudoxus and Theaetetus on which the *Elements* is based.

Another famous story[34] and quotation about Euclid is this: A certain pupil of Euclid, at his school in Alexandria, came to Euclid after learning just the first proposition in the geometry of the *Elements*. He wanted to know what he would gain by arduous study, doing all the necessary work, and learning the theorems of geometry. At this, Euclid called over his slave and said, "Give him three drachmas since he must needs make gain by what he learns."

What is important about Euclid's *Elements* is the paradigm it provides for the way that mathematics should be studied and recorded. He begins with several definitions of terminology and ideas for geometry, and then he records five important postulates (or axioms) of geometry. A version of these postulates is as follows:

P1 Through any pair of distinct points there passes a line.

P2 For each segment $\overline{AB}$ and each segment $\overline{CD}$ there is a unique point E (on the line determined by A and B) such that B is between A and E and the segment $\overline{CD}$ is congruent to $\overline{BE}$ (Figure 2.2).

P3 For each point C and each point A distinct from C there exists a circle with center C and radius CA (Figure 2.3a).

P4 All right angles are congruent (Figure 2.3b).

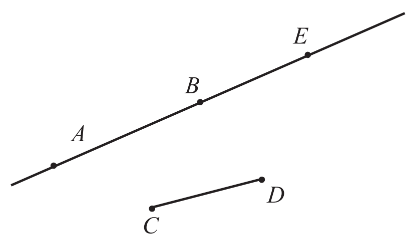
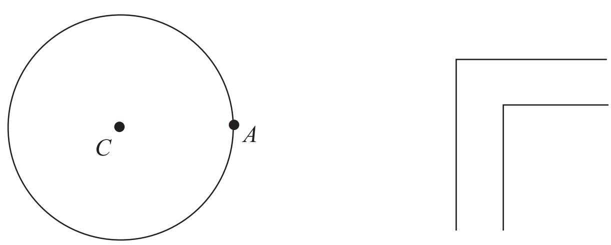


Figure 2.2. Euclid's second postulate.



(a) (b)

Figure 2.3. The circle and the right angle.

[34]A similar story is told of Plato.

These are the standard four axioms that give our Euclidean conception of geometry. The fifth axiom, a topic of intense study for 2000 years, is the so-called parallel postulate (in *Playfair's* formulation):

P5 For each line ℓ and each point P that does not lie on ℓ there is a unique line ℓ' through P such that ℓ' is parallel to ℓ (Figure 2.4).

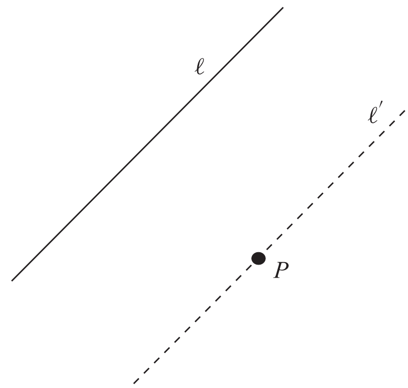


Figure 2.4. The parallel postulate.

Prior to this enunciation of his celebrated five axioms, Euclid had defined "point," "line," "circle," and the other terms that he uses. Although Euclid borrowed freely from mathematicians both earlier and contemporaneous with himself, it is generally believed that the famous "parallel postulate," that is, Postulate P5, is Euclid's own creation.

We should note, however, that the work *Elements* is not simply about plane geometry. In fact, Books VII–IX deal with number theory, in which Euclid proves his famous result that there are infinitely many primes (treated elsewhere in this text) and also his celebrated "Euclidean algorithm" for division with remainder. Book X deals with irrational numbers, and Books XI–XIII treat 3-dimensional geometry. In short, Euclid's *Elements* is an exhaustive treatment of a good deal of the mathematics that was known at the time. It is presented in a strictly rigorous and axiomatic manner that has set the tone for the way that mathematics is recorded and studied today. Euclid's *Elements* is perhaps most notable for the clarity with which theorems are formulated and proved. The standard of rigor that Euclid set was to be a model for the inventors of calculus nearly 2000 years later.

Noted algebraist B. L. van der Waerden (1903–1996) assesses the impact of Euclid's *Elements* in this way:

> Almost from the time of its writing and lasting almost to the present, the *Elements* has exerted a continuous and major influence on human affairs. It was the primary source of geometric inference, theorems, and methods at least until the advent of non-Euclidean geometry in the nineteenth century. It is sometimes said that, next to the Bible, the *Elements* may be the most translated, published, and studied of all the books produced in the Western world.

Indeed, there have been more than 1000 editions of Euclid's *Elements*. It is arguable that Euclid was and still is the most important and most influential mathematics teacher of all time. It may be added that a number of other books by Euclid survive. These include *Data* (which studies geometric properties of figures), *On Divisions* (which studies the division of geometric regions into subregions having areas of a given ratio), *Optics* (which is the first Greek work on perspective), and *Phaenomena* (which is an elementary introduction to

Number Theory

Number theory is concerned with properties of the whole numbers $1, 2, 3, \ldots$. Are there infinitely many prime numbers? How are they distributed? Can every even number greater than 4 be written as the sum of two odd primes? If N is a large, positive number, then how many primes are less than or equal to N?

These are great, classical questions—some of them answered today and some not. Today number theory plays a vital role in cryptography. The National Security Agency in Washington, D.C. employs over 2,000 Ph.D. mathematicians to study this subject area.

mathematical astronomy). Several other books of Euclid—including *Surface Loci*, *Porisms*, *Conics*, *Book of Fallacies*, and *Elements of Music*—have been lost.

2.2.1 Euclid the Number Theorist

Most of us remember Euclid's *Elements* as a work on geometry. But Books VII–IX of the *Elements* deal with number theory. One of the particular results presented there has stood the test of time, and the proof is taught today to every mathematics student. We shall discuss it now.

Recall that a *prime number* is a positive whole number that has no divisors except for 1 and itself. By tradition we do not consider 1 to be a prime. So the prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots.$$

A number greater than 1 that is not prime is called *composite*. For example, 126 is composite. Notice that

$$126 = 2 \cdot 3^2 \cdot 7.$$

It is *not* prime. Any composite number can be factored in a unique fashion into prime factors—that is the fundamental theorem of arithmetic.

The question that Euclid considered (and, unlike many of the other results in the *Elements*, this result seems to have originated with Euclid himself) is whether there are infinitely many prime numbers. And Euclid's dramatic answer is yes.

Theorem: *There are infinitely many prime integers.*

For the proof, assume the contrary. So there are only finitely many primes. Call them $p_1, p_2, \ldots, p_N$. Now consider the number $P = (p_1 \cdot p_2 \cdots p_N) + 1$. What kind of number is P? Notice that if we divide P by p_1, then we get a remainder of 1 (since p_1 goes evenly into $p_1 \cdot p_2 \cdots p_N$). Also if we divide p_2 into P, then we get a remainder of 1. And it is the same if we divide any of p_3 through p_N into P.

Now, if P were a composite number, then it would have to be evenly divisible by some prime. But we have just shown that it is not: We have divided every known prime number into P and obtained a nonzero remainder in each instance. The only possible conclusion is that P is another prime, obviously greater than any of the primes on the original list. That is a contradiction. So there cannot be only finitely many primes. There must be infinitely many.

Euclid's argument is one of the first known instances of proof by contradiction.[35] This important method of formal proof has actually been quite controversial over the years. We shall discuss it in considerable detail as the book develops.

2.3 Pythagoras

Pythagoras (ca. 569–500 BCE) was both a person and a society (i.e., the *Pythagoreans*). He was also a political figure and a mystic. He was special in his time, among other reasons, because he involved women as equals in his activities. One critic characterized the man as "one tenth of him genius, nine-tenths sheer fudge." Pythagoras died, according to legend, in the flames of his own schools fired by political and religious bigots who stirred up the masses to protest against the enlightenment that Pythagoras sought to bring them.

The Pythagorean society was intensely mathematical in nature, but it was also quasi-religious. Among its tenets (according to [RUS]) were:

- To abstain from beans.
- Not to pick up what has fallen.
- Not to touch a white cock.
- Not to break bread.
- Not to step over a crossbar.
- Not to stir the fire with iron.
- Not to eat from a whole loaf.
- Not to pluck a garland.
- Not to sit on a quart measure.
- Not to eat the heart.
- Not to walk on highways.
- Not to let swallows share one's roof.
- When the pot is taken off the fire, not to leave the mark of it in the ashes, but to stir them together.
- Not to look in a mirror beside a light.
- When you rise from the bedclothes, roll them together and smooth out the impress of the body.

The Pythagoreans embodied a passionate spirit that is remarkable to our eyes:

> Bless us, divine Number, thou who generatest gods and men.

and

> Number rules the universe.

[35] There are many other ways to prove Euclid's result, including direct proofs and proofs by induction. In other words, it is not *necessary* to use proof by contradiction.

The Pythagoreans are remembered for two monumental contributions to mathematics. The first was establishing the importance of, and the necessity for, *proofs* in mathematics: that mathematical statements, especially geometric statements, must be verified by way of rigorous proof. Prior to Pythagoras, the ideas of geometry were generally rules of thumb that were derived empirically, merely from observation and (occasionally) measurement. Pythagoras also introduced the idea that a great body of mathematics (such as geometry) could be derived from a small number of postulates. Clearly Euclid was infuenced by Pythagoras.

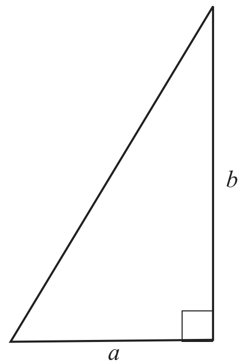


Figure 2.5. The fraction $\frac{b}{a}$.

The second great contribution was the discovery of, and proof of, the fact that not all numbers are commensurate. More precisely, the Greeks prior to Pythagoras believed with a profound and deeply held passion that everything was built on the whole numbers. Fractions arise in a concrete manner: as ratios of the sides of triangles with integer length (and are thus *commensurable*—this antiquated terminology has today been replaced by the word “rational”)—see Figure 2.5.

Pythagoras proved the result now called the *Pythagorean theorem*. It says that the legs a, b and hypotenuse c of a right triangle (Figure 2.6) are related by the formula

$$a^2 + b^2 = c^2. \qquad (*)$$

This theorem has perhaps more proofs than any other result in mathematics—well over 50 altogether. And in fact it is one of the most ancient mathematical results. There is evidence that the Babylonians and the Chinese knew this theorem at least five hundred years before Pythagoras.

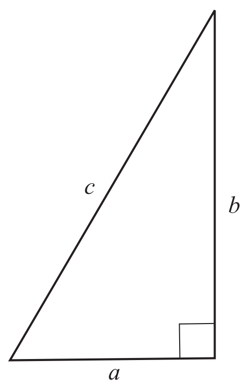


Figure 2.6. The Pythagorean theorem.

Remarkably, one proof of the Pythagorean theorem was devised by U.S. President James Garfield (1831–1881). We now provide one of the simplest and most classical arguments.

Proof of the Pythagorean Theorem. Refer to Figure 2.7. Observe that we have four right triangles and a square packed into a large square. Each of the triangles has legs a and b and hypotenuse c, just as in the Pythagorean theorem. Of course, on the one hand, the area of the large square is c^2. On the other hand, the area of the large square is the sum of the areas of its component pieces.

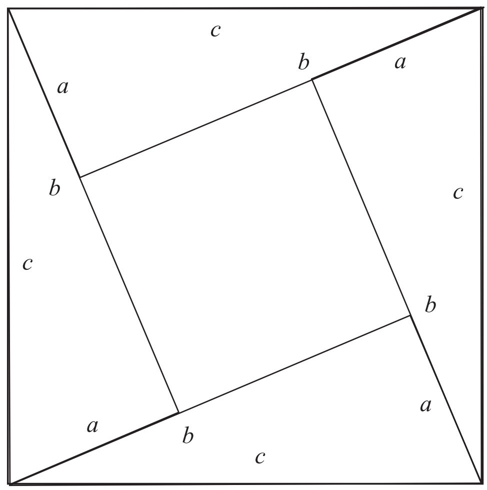


Figure 2.7. Proof of the Pythagorean theorem.

Thus we calculate that

$$\begin{aligned}
c^2 &= (\text{area of large square}) \\
&= (\text{area of triangle}) + (\text{area of triangle}) \\
&\quad + (\text{area of triangle}) + (\text{area of triangle}) \\
&\quad + (\text{area of small square}) \\
&= \frac{1}{2} \cdot ab + \frac{1}{2} \cdot ab + \frac{1}{2} \cdot ab + \frac{1}{2} \cdot ab + (b-a)^2 \\
&= 2ab + [a^2 - 2ab + b^2] \\
&= a^2 + b^2.
\end{aligned}$$

This proves the Pythagorean theorem. □

It is amusing to note that (according to legend) the Egyptians had, as one of their standard tools, a rope with twelve equally spaced knots. They used this rope to form a triangle with sides 3, 4, 5—see Figure 2.8. In this way they took advantage of the Pythagorean theorem to construct right angles.

Pythagoras noticed that, if $a = 1$ and $b = 1$, then $c^2 = 2$. He wondered whether there was a rational number c that satisfied this last identity. His stunning conclusion was this:

Theorem. *There is no rational number c such that $c^2 = 2$.*

Pythagoras is telling us that a right triangle with each leg having length 1 will have a hypotenuse with irrational length. This is a profound and disturbing assertion!

Proof of the Theorem. Suppose that the conclusion is false. Then there *is* a rational number $c = \alpha/\beta$, expressed in lowest terms (i.e., α and β are integers with no factors in common) such that $c^2 = 2$. This translates to

$$\frac{\alpha^2}{\beta^2} = 2$$

or

$$\alpha^2 = 2\beta^2.$$

We conclude that the right-hand side is even; hence so is the left-hand side. Therefore α is even, so $\alpha = 2m$ for some integer m (see Propositions 12.1.3 and 12.1.4 below).

But then

$$(2m)^2 = 2\beta^2$$

or

$$2m^2 = \beta^2.$$

We see that the left-hand side is even, so β^2 is even. Hence β is even.

But now both α and β are even—the two numbers have a common factor of 2. This statement contradicts the hypothesis that α and β have no common factors, so it cannot be that c is a rational number; c must be irrational. □

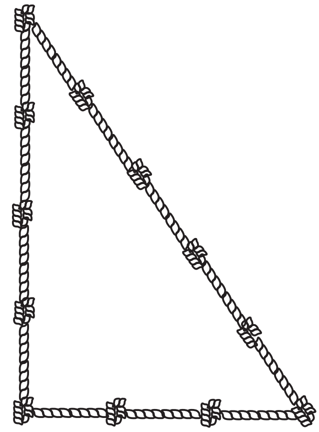


Figure 2.8. The Egyptian Pythagorean rope.

It is notable that T. Apostol has found a strictly graphical proof of the irrationality of $\sqrt{2}$—see [BAB1, p. 73].

The Pythagoreans realized the profundity and potential social importance of this discovery. It was ingrained in the ancient Greek consciousness that all numbers were rational. To claim the contrary would have been virtually heretical. For a time the Pythagoreans kept this new fact a secret. Ultimately, so legend has it, the Pythagoreans were destroyed by (ignorant) marauding hordes. But their universal ideas live on.

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