Preface

The title of this book is not entirely frivolous. There are many who will claim that the correct aphorism is "The proof of the pudding is in the eating."—that it makes no sense to say, "The proof is in the pudding." Yet people say it all the time, and the intended meaning is usually clear. So it is with mathematical proof. A *proof* in mathematics is a psychological device for convincing some person, or some audience, that a certain mathematical assertion is true. The structure, and the language used, in formulating such a proof will be a product of the person creating it; but it also must be tailored to the audience that will be receiving it and evaluating it. Thus there is no "unique" or "right" or "best" proof of any given result. A proof is part of a situational ethic: situations change, mathematical values and standards develop and evolve, and thus the very *way* that mathematics is done will alter and grow.

This is a book about the changing and growing nature of mathematical proof. In the earliest days of mathematics, "truths" were established heuristically and/or empirically. There was a heavy emphasis on calculation. There was almost no theory, no formalism, and there was little in the way of mathematical notation as we know it today. Those who wanted to consider mathematical questions were thereby hindered: they had difficulty expressing their thoughts. They had particular trouble formulating general statements about mathematical ideas. Thus it was virtually impossible for them to state theorems and prove them.

Although there are some indications of proofs even on ancient Babylonian tablets (such as Plimpton 322) from 1800 BCE, it seems that it is in ancient Greece that we find the identifiable provenance of the concept of proof. The earliest mathematical tablets contained numbers and elementary calculations. Because of the paucity of texts that have survived, we do not know how it came about that someone decided that some of these mathematical procedures required *logical justification*. And we really do not know how the formal concept of proof evolved. The *Republic* of Plato contains a clear articulation of the proof concept. The *Physics* of Aristotle not only discusses proofs, but treats minute distinctions of proof methodology. Many of the ancient Greeks, including Eudoxus, Theaetetus, Thales, Euclid, and Pythagoras, either used proofs or referred to proofs. Protagoras was a sophist, whose work was recognized by Plato. His *Antilogies* were tightly knit, rigorous arguments that could be thought of as the germs of proofs.

It is acknowledged that Euclid was the first to systematically use precise definitions, axioms, and strict rules of logic, and to carefully enunciate and *prove* every statement (i.e., every theorem). Euclid's formalism, as well as his methodology, has become the model—even to the present day—for establishing mathematical facts.

What is interesting is that a mathematical statement of fact is a freestanding entity with intrinsic merit and value. But a proof is a device of communication. The creator or discoverer of this new mathematical result wants others to believe it and accept it. In the physical sciences—chemistry, biology, or physics for example—the method for achieving this end is the *reproducible experiment*.[1] For the mathematician, the reproducible experiment is a proof that others can read and understand and validate.

The idea of "proof" appears in many aspects of life other than mathematics. In the courtroom, a lawyer (either for the prosecution or the defense) must establish a case by means of an accepted version of proof. For a criminal case this is "beyond a reasonable doubt," while for a civil case it is "the preponderance of evidence shows." Neither of these is mathematical proof, or anything like it. The real world has no formal definitions and no axioms; there is no sense of establishing facts by strict exegesis. The lawyer uses logic—such as "the defendant is blind so he could not have driven to Topanga Canyon on the night of March 23," or "the defendant has no education and therefore could not have built the atomic bomb that was used to ... "—but the principal tools are *facts*. The lawyer proves the case beyond a reasonable doubt by amassing an overwhelming preponderance of evidence in favor of that case.

At the same time, in ordinary, family-style parlance there is a notion of proof that is different from mathematical proof. A husband might say, "I believe that my wife is pregnant," while the wife may *know* that she is pregnant. Her pregnancy is not a permanent and immutable fact (like the Pythagorean theorem), but instead is a "temporary fact" that will be false after several months. So, in this context, the concept of truth has a different meaning from the one used in mathematics, and the means of verification of a truth are also rather different. What we are really seeing here is the difference between knowledge and belief—something that never plays a formal role in mathematics.

In modern society it takes far less "proof" to convict someone of speeding than to get a murder conviction. But, ironically, it seems to take even less evidence to justify waging war.[2] A great panorama of opinions about the modern concept of proof—in many different contexts—may be found in [NCBI]. It has been said—see [MCI]—that in some areas of mathematics (such as low-dimensional topology) a proof can be conveyed by a combination of gestures.

The French mathematician Jean Leray (1906–1998) perhaps sums up the value system of the modern mathematician:

> ... all the different fields of mathematics are as inseparable as the different parts of a living organism; as a living organism mathematics has to be permanently recreated; each generation must reconstruct it wider, larger and more beautiful. The death of mathematical research would be the death of mathematical thinking which constitutes the structure of scientific language itself and by consequence the death of our scientific civilization. Therefore we must transmit to our children strength of character, moral values and drive towards an endeavouring life.

---

[1] More precisely, it is the reproducible experiment *with control*. For the careful scientist compares the results of his/her experiment with some standard or norm. That is the means of evaluating the result.

[2] See [SWD] for the provenance of these ideas. Fascinating related articles are [ASC], [MCI]. An entire issue of the *Philosophical Transactions of the Royal Society* in the fall of 2005 is devoted to articles of this type, all deriving from a meeting in Britain to discuss issues such as those treated in this book. See [PTRS].

What Leray is telling us is that mathematical ideas travel well and stand up under the test of time, because we have such a rigorous and well-tested standard for formulating and recording the ideas. It is a grand tradition, and one well worth preserving.

There is a human aspect to proof that cannot be ignored. The acceptance of a new mathematical truth is a sociological process. It is something that takes place in the mathematical community. It involves understanding, internalization, rethinking, and discussion. And even the most eminent mathematicians make mistakes, and announce new results that in fact they *do not* know how to prove. In 1879, A. Kempe published a proof of the four-color theorem that stood for eleven years before P. Heawood found a fatal error in the work. The first joint work of G. H. Hardy and J. E. Littlewood was announced at the June, 1911 meeting of the London Mathematical Society. The result was never published because they later discovered that their proof was incorrect. A. L. Cauchy, G. Lamé, and E. E. Kummer all thought at one point or another in their careers that they had proved Fermat's last theorem; they were all mistaken. H. Rademacher thought in 1945 that he had disproved the Riemann hypothesis; his achievement was even written up in *Time Magazine*. He later had to withdraw the claim because C. L. Siegel found an error. Considerable time is spent here in this book exploring the social workings of the mathematical discipline, and how the interactions of different mathematicians and different mathematical cultures serve to shape the subject. Mathematicians' errors are corrected, not by formal mathematical logic, but by other mathematicians. This is a seminal point about this discipline.[3]

The early twentieth century saw L. E. J. Brouwer's dramatic proof of his fixed-point theorem followed by his wholesale rejection of proof by contradiction (at least in the context of existence proofs—which is precisely what the proof of his fixed-point theorem is an instance of) and his creation of the intuitionist movement. This program was later taken up by Errett Bishop, and his *Foundations of Constructive Analysis* (written in 1967) has made quite a mark (see also the revised version, written jointly with Douglas Bridges, published in 1985). These ideas are of particular interest to the theoretical computer scientist, for proof by contradiction has questionable meaning in computer science (this despite the fact that Alan Turing cracked the Enigma code by applying ideas of proof by contradiction in the context of computing machines).

In the past thirty years or so it has come about that we have rethought, and reinvented, and decisively amplified our concept of proof. Computers have played a strong and dynamic role in this reorientation of the discipline. A computer can make hundreds of millions of calculations in a second. This opens up possibilities for trying things, calculating, and visualizing things that were unthinkable fifty years ago. It should be borne in mind that mathematical thinking involves concepts and reasoning, while a computer is a device for manipulating data, two quite different activities. It appears unlikely (see Roger Penrose's remarkable book *The Emperor's New Mind*) that a computer will ever be able to think and prove mathematical theorems in the way a human being performs these activities. Nonetheless, the computer can provide valuable information and insights. It can enable the user to see things that would not be envisioned otherwise. It is a valuable tool. We shall

[3]It is common for mathematicians to make errors. Probably every published mathematical paper has errors in it. The book [LEC] documents many important errors in the literature up to the year 1935.

definitely spend a good deal of time in this book pondering the role of the computer in modern human thought.

In trying to understand the role of the computer in mathematical life, it is perhaps worth drawing an analogy with history. Tycho Brahe (1546–1601) was one of the great astronomers of the Renaissance. Through painstaking scientific procedure, he recorded reams and reams of data about the motions of the planets. His gifted student Johannes Kepler (1571–1630) was anxious to get his hands on Brahe's data because he had ideas about formulating mathematical laws about the motions of the planets. But Brahe and Kepler were both strong-willed men. They did not see eye-to-eye on many things. And Brahe feared that Kepler would use his data to confirm the Copernican theory about the solar system (namely that the *sun*, not the earth, was the center of the system—a notion that ran counter to Christian religious dogma). As a result, during Brahe's lifetime Kepler did not have access to Brahe's numbers.

But providence intervened in a strange way. Tycho Brahe had been given an island by his sponsor on which to build and run his observatory. As a result, he was obliged to attend certain social functions—just to show his appreciation and to report on his progress. At one such function, Brahe drank an excessive amount of beer, his bladder burst, and he died. Kepler was then able to negotiate with Brahe's family to get the data that he so desperately needed. And thus the course of scientific history was forever altered.

Kepler did *not* use deductive thinking or reasoning, or the axiomatic method, or the strategy of mathematical proof to derive his three laws of planetary motion. Instead he simply stared at the hundreds of pages of planetary data that Brahe had provided, and he performed numerous calculations.

At around this same time John Napier (1550–1617) was developing his theory of logarithms. These are terrific calculational tools, which would have simplified Kepler's task immensely. But Kepler could not understand the derivation of logarithms and refused to use them. He did everything the hard way. Imagine what Kepler could have done with a computer!—but he probably would have refused to use one just because he would not have understood how the central processing unit (CPU) worked.

In any event, we tell here of Kepler and Napier because the situation is perhaps a harbinger of modern agonizing over the use of computers in mathematics. There are those who argue that the computer can enable us to see things—both calculationally and visually—that we could not see before. And there are those who say that all those calculations are well and good, but they do not constitute a mathematical proof. Nonetheless it seems that the first can inform the second, and a productive symbiosis can be created. We shall discuss these matters in detail as the book develops.

Now let us return to our consideration of changes that have come about in mathematics in the past thirty years, in part because of the advent of high-speed digital computers. Here is a litany of some of the components of this process:

(a) In 1974, Appel and Haken [APH1] announced a proof of the four-color conjecture. This is the question of how many colors are needed to color any map, so that adjacent countries are colored differently. Their proof used 1200 hours of computer time on a supercomputer at the University of Illinois. Mathematicians found this event puzzling because this "proof" was not something that anyone could study or check. Or understand. To this day there does not exist a proof of the four-color theorem that can be read and checked by humans.

(b) Over time, people have become more and more comfortable with the use of computers in proofs. In its early days, the theory of wavelets (for example) depended on the estimation of a certain constant—something that could be done only with a computer. De Branges's original proof of the Bieberbach conjecture [DEB2] seemed to depend on a result from special function theory that could be verified only with the aid of a computer (it was later discovered to be a result of Askey and Gasper that was proved in the traditional manner).

(c) The evolution of new teaching tools such as the software `The Geometer's Sketchpad` has suggested to many—including Fields Medalist William Thurston—that traditional proofs may be set aside in favor of experimentation, that is, testing of thousands or millions of examples, on the computer.

Thus the use of the computer has truly reoriented our view of what a proof might comprise. Again, the point is to convince someone else that something is true. There are evidently many different means of doing this.

Perhaps more interesting are some of the new social trends in mathematics and the resulting construction of nonstandard proofs (we shall discuss these in detail in the text that follows):

(a) One of the great efforts of twentieth-century mathematics has been the classification of the finite simple groups. Daniel Gorenstein, of Rutgers University, was in some sense the lightning rod who orchestrated the effort. It is now considered to be complete. What is remarkable is that this is a single theorem that is the aggregate effort of many hundreds of mathematicians. The "proof" is in fact the union of hundreds of papers and tracts spanning more than 150 years. At the moment this proof comprises over 10,000 pages. It is still being organized and distilled today. The final "proof for the record" will consist of several volumes, and it is not clear that the living experts will survive long enough to see the fruition of this work.

(b) Thomas Hales's resolution of the Kepler sphere-packing problem uses a great deal of computer calculation, much as with the four-color theorem. It is particularly interesting that his proof supplants the earlier proof of Wu-Yi Hsiang that relied on spherical trigonometry and *no computer calculation*. Hales allows that his "proof" cannot be checked in the usual fashion. He has organized a worldwide group of volunteers called `FlySpeck` to engage in a checking procedure for his computer-based arguments.

(c) Grisha Perelman's "proof" of the Poincaré conjecture and the geometrization program of Thurston are currently in everyone's focus. In 2003, Perelman wrote three papers that describe how to use Richard Hamilton's theory of Ricci flows to carry out Thurston's idea (called the "geometrization program") of breaking up a 3-manifold into fundamental geometric pieces. One very important consequence of this result would be a proof of the important Poincaré conjecture. Although Perelman's papers are vague and incomplete, they are full of imaginative and deep geometric ideas. This work set off a storm of activity and speculation about how the program might be assessed and validated. There have been huge efforts by John Lott and Bruce Kleiner (at the University of Michigan) and Gang Tian (Princeton) and John Morgan (Columbia) to complete the Hamilton/Perelman program and produce a bona fide, recorded proof that others can study and verify.

(d) In fact, Thurston's geometrization program is a tale in itself. He announced in the early 1980s that he had a result on the structure of 3-manifolds, at least for certain important subclasses of the manifolds, and he knew how to prove it. The classical Poincaré conjecture would be an easy corollary of Thurston's geometrization program. He wrote an extensive set of notes [THU3]—of book length—and these were made available to the world by the Princeton mathematics department. For a nominal fee, the department would send a copy to anyone who requested it. These notes, entitled *The Geometry and Topology of Three-Manifolds* [THU3], were extremely exciting and enticing. But the notes, for all the wealth of good mathematics that they contained, were written in a rather informal style. They were difficult to assess and evaluate.

The purpose of this book is to explore all the ideas and developments outlined above. Along the way, we are able to acquaint the reader with the culture of mathematics: who mathematicians are, what they care about, and what they do. We also give indications of why mathematics is important, and why it is having such a powerful influence in the world today. We hope that by reading this book the reader will become acquainted with, and perhaps charmed by, the glory of this ancient subject, and will realize that there is so much more to learn.

December, 2010

Steven G. Krantz
St. Louis, Missouri

http://www.springer.com/978-0-387-48908-7

The Proof is in the Pudding
The Changing Nature of Mathematical Proof
Krantz, S.G.
2011, XVII, 264 p. 88 illus., 5 illus. in color., Hardcover
ISBN: 978-0-387-48908-7