

## Chapter VI

# POISSON RANDOM MEASURES

The aim is to give a reasonably detailed account of Poisson random measures and their uses. Such random measures are often the primitive elements from which more evolved processes are constructed. We shall illustrate several such constructions which yield Lévy processes and pure-jump Markov processes.

Throughout,  $(\Omega, \mathcal{H}, \mathbb{P})$  is the probability space in the background. We shall have many occasions to use the notation  $\exp_- x$  for  $e^{-x}$ .

## 1 RANDOM MEASURES

Let  $(E, \mathcal{E})$  be a measurable space. A *random measure* on  $(E, \mathcal{E})$  is a transition kernel from  $(\Omega, \mathcal{H})$  into  $(E, \mathcal{E})$ .

More explicitly, a mapping  $M : \Omega \times \mathcal{E} \mapsto \bar{\mathbb{R}}_+$  is called a random measure if  $\omega \mapsto M(\omega, A)$  is a random variable for each  $A$  in  $\mathcal{E}$  and if  $A \mapsto M(\omega, A)$  is a measure on  $(E, \mathcal{E})$  for each  $\omega$  in  $\Omega$ . We shall denote by  $M(A)$  the former random variable; then, we may regard  $M$  as the collection of random variables  $M(A)$ ,  $A \in \mathcal{E}$ . We shall denote by  $M_\omega$  the latter measure  $A \mapsto M(\omega, A)$ ; the term “random measure” signifies that  $M$  is a random variable that assigns a measure  $M_\omega$  to every outcome  $\omega$  in  $\Omega$ .

The terms such as finite,  $\sigma$ -finite, diffuse, etc. are used for a random measure  $M$  if the measure  $M_\omega$  has the corresponding properties for almost every  $\omega$  in  $\Omega$ . In particular, it is said to be integer-valued if it takes values in  $\bar{\mathbb{N}} = \{0, 1, \dots, +\infty\}$ . It is said to be a *random counting measure* if, for almost every  $\omega$ , the measure  $M_\omega$  is purely atomic and its every atom has weight one.

## Mean measures and integrals

Let  $M$  be a random measure on  $(E, \mathcal{E})$ . Recall that  $\mathcal{E}_+$  denotes the set of all positive  $\mathcal{E}$ -measurable functions. We recall also the results and notations of the measure-kernel-function setup: According to Fubini's theorem, I.6.3,

$$1.1 \quad Mf(\omega) = \int_E M(\omega, dx) f(x), \quad \omega \in \Omega,$$

defines a positive random variable  $Mf$  for each  $f$  in  $\mathcal{E}_+$ , and

$$1.2 \quad \mu(A) = \mathbb{E} M(A) = \int_{\Omega} \mathbb{P}(d\omega) M(\omega, A), \quad A \in \mathcal{E},$$

defines a measure  $\mu$  on  $(E, \mathcal{E})$ , called the mean of  $M$ , and

$$1.3 \quad \mathbb{E} Mf = \mu f, \quad f \in \mathcal{E}_+.$$

## Laplace functionals

Let  $M$  be a random measure on  $(E, \mathcal{E})$ . It may be regarded as a collection  $\{M(A) : A \in \mathcal{E}\}$  indexed by  $\mathcal{E}$  or as a collection  $\{Mf : f \in \mathcal{E}_+\}$  indexed by  $\mathcal{E}_+$ . Thus, to specify the probability law of  $M$ , it is sufficient to specify the joint distribution of  $Mf_1, \dots, Mf_n$  for every choice of integer  $n \geq 1$  and functions  $f_1, \dots, f_n$  in  $\mathcal{E}_+$ . Such a joint distribution can be specified implicitly via the joint Laplace transform (recall the notation  $\exp_- x = e^{-x}$ )

$$\mathbb{E} \exp_-(r_1 Mf_1 + \dots + r_n Mf_n), \quad r_1, \dots, r_n \in \mathbb{R}_+.$$

Noting that this is the same as  $\mathbb{E} \exp_- Mf$  with  $f = r_1 f_1 + \dots + r_n f_n$ , we obtain the proof of the following.

1.4 PROPOSITION. *The probability law of a random measure  $M$  on  $(E, \mathcal{E})$  is determined uniquely by*

$$\mathbb{E} e^{-Mf}, \quad f \in \mathcal{E}_+. \quad \square$$

The mapping  $f \mapsto \mathbb{E} e^{-Mf}$  from  $\mathcal{E}_+$  into  $[0, 1]$  is called the *Laplace functional* of  $M$ . It is continuous under increasing limits:

1.5 PROPOSITION. *If  $(f_n) \subset \mathcal{E}_+$  is increasing to  $f$ , then*

$$\lim_n \mathbb{E} \exp_- Mf_n = \mathbb{E} \exp_- Mf.$$

*Proof.* If  $(f_n)$  is increasing to  $f$ , then  $(Mf_n)$  is increasing to  $Mf$  by the monotone convergence theorem for the measure  $M_\omega$  applied for each outcome  $\omega$ , and thus,  $(\exp_- Mf_n)$  is decreasing to  $\exp_- Mf$ . The desired conclusion is now immediate from the bounded convergence theorem for expectations.  $\square$

Laplace functionals have uses similar to those of Laplace transforms:

**1.6 PROPOSITION.** *Let  $M$  and  $N$  be random measures on  $(E, \mathcal{E})$ . They are independent if and only if*

$$\mathbb{E} e^{-(Mf+Ng)} = \mathbb{E} e^{-Mf} \mathbb{E} e^{-Ng}, \quad f, g \in \mathcal{E}_+.$$

*Proof.* To show that  $M$  and  $N$  are independent, it is sufficient to show that the vectors  $(Mf_1, \dots, Mf_m)$  and  $(Ng_1, \dots, Ng_n)$  are independent for all choices of  $m \geq 1$  and  $n \geq 1$  and  $f_1, \dots, f_m, g_1, \dots, g_n$  in  $\mathcal{E}_+$ . For this, it is enough to show the following equality of joint Laplace transforms:

$$\mathbb{E} \exp_- \left( \sum_1^m p_i Mf_i + \sum_1^n q_j Ng_j \right) = \left( \mathbb{E} \exp_- \sum_1^m p_i Mf_i \right) \left( \mathbb{E} \exp_- \sum_1^n q_j Ng_j \right).$$

But this is immediate from the condition of the proposition upon taking  $f = \sum p_i f_i$  and  $g = \sum q_j g_j$ . This proves the sufficiency of the condition. The necessity is obvious.  $\square$

The preceding has a useful corollary concerning the Laplace functional of the sum  $M + N$ ; we leave it to Exercise 1.15.

**1.7 EXAMPLE.** Let  $\lambda$  be a probability measure on  $(E, \mathcal{E})$ . Let  $X = \{X_i : i \in \mathbb{N}^*\}$  be an independency of random variables taking values in  $(E, \mathcal{E})$  according to the common distribution  $\lambda$ . Let  $K$  be independent of  $X$  and have the Poisson distribution with mean  $c$ , the last being a constant in  $(0, \infty)$ . We define  $M$  by (recall that  $I(x, A) = \delta_x(A) = 1_A(x)$  defines the identity kernel  $I$ )

$$1.8 \quad M(\omega, A) = \sum_{i=1}^{K(\omega)} I(X_i(\omega), A), \quad \omega \in \Omega, \quad A \in \mathcal{E},$$

where, by the usual conventions, the sum is zero if  $K(\omega) = 0$ . This defines a measure  $M_\omega$  for each  $\omega$ , and re-writing 1.8 as

$$1.9 \quad Mf = \sum_{i=1}^K f \circ X_i = \sum_{i=1}^{\infty} f \circ X_i 1_{\{K \geq i\}}, \quad f \in \mathcal{E}_+,$$

we see that  $Mf$  is a random variable for each  $f$  in  $\mathcal{E}_+$ . Hence  $M$  is a random measure. It is integer-valued. Its mean measure is equal to  $c\lambda$ :

$$\mu f = \mathbb{E} Mf = \sum_1^{\infty} \mathbb{E} f \circ X_i \mathbb{E} 1_{\{K \geq i\}} = (\lambda f) \mathbb{E} K = c\lambda f.$$

To compute its Laplace functional, we use 1.9 and the assumptions on independence and the distributions of  $K$  and the  $X_i$ . With the notation  $e^{-f}$  for the function  $x \mapsto e^{-f(x)}$ , we have

$$\mathbb{E} e^{-Mf} = \mathbb{E} \prod_{i=1}^K e^{-f \circ X_i} = \sum_{k=0}^{\infty} \frac{e^{-c} c^k}{k!} (\lambda e^{-f})^k = \exp_- c\lambda(1 - e^{-f}).$$

## Atoms, point processes

The preceding example is the prototype of the random measures of primary interest, namely, purely atomic random measures.

Let  $(\bar{E}, \bar{\mathcal{E}})$  be a measurable space containing  $(E, \mathcal{E})$ , that is,  $\bar{E} \supset E$  and  $\bar{\mathcal{E}} \supset \mathcal{E}$ . Let  $X = (X_i)$  be a countable collection of random variables taking values in  $(\bar{E}, \bar{\mathcal{E}})$ . Define

$$1.10 \quad Mf(\omega) = \sum_i f \circ X_i(\omega), \quad \omega \in \Omega, \quad f \in \mathcal{E}_+,$$

with  $f$  extended automatically onto  $\bar{E}$  by putting  $f(x) = 0$  for all  $x$  in  $\bar{E} \setminus E$ . This defines a random measure  $M$  on  $(E, \mathcal{E})$ . To indicate it, we say that  $X$  forms  $M$  on  $(E, \mathcal{E})$  and, conversely, call the  $X_i$  atoms of  $M$ . See Remarks 1.14a and 1.14b below for the need to define  $X$  on a larger space  $(\bar{E}, \bar{\mathcal{E}})$  in general; often, it is possible to take  $E = \bar{E}$ .

Assuming that the singleton  $\{x\}$  belongs to  $\mathcal{E}$  for every  $x$  in  $E$ , the preceding use of the term “atom” is well-justified: for every  $\omega$ , the measure  $M_\omega$  defined by 1.10 is purely atomic, and each  $X_i(\omega)$  in  $E$  is an atom of  $M_\omega$ . In particular, if all the  $X_i(\omega)$  in  $E$  are distinct then  $M_\omega$  is a counting measure. And if  $M_\omega$  is such, the simplest way to visualize it is by marking its atoms in  $E$ ; this yields a set  $S_\omega = \{x \in E : x = X_i(\omega) \text{ for some } i\}$ . The resulting random set  $S$  is called the *point process* associated with  $M$ .

In practice, often, the random variables  $X_i$  are defined first and  $M$  is introduced by 1.10. We now treat the converse situation where  $M$  is already defined and we need to specify the  $X_i$ . The following does this for the case  $E = \mathbb{R}_+$  and  $\mathcal{E} = \mathcal{B}(\mathbb{R}_+)$ ; we think of  $\mathbb{R}_+$  as time and of atoms as times of occurrence of some physical event. See Exercise 1.19 for the general case.

1.11 PROPOSITION. *Let  $M$  be an integer-valued random measure on  $\mathbb{R}_+$ . Suppose that it is finite over bounded intervals. Then, there exists an increasing sequence  $(T_n)$  of random variables taking values in  $\mathbb{R}_+ = [0, \infty]$  such that, for almost every  $\omega$ ,*

$$1.12 \quad M(\omega, A) = \sum_{n=1}^{\infty} I(T_n(\omega), A), \quad A \in \mathcal{B}(\mathbb{R}_+).$$

*Proof.* Define  $L_t(\omega) = M(\omega, [0, t])$  for  $t$  in  $\mathbb{R}_+$  and  $\omega$  in  $\Omega$ . Since  $M$  is a random measure,  $L_t$  is a random variable for each  $t$ , and by assumption,  $L_t < \infty$  almost surely. For each integer  $n \geq 1$ , let

$$1.13 \quad T_n(\omega) = \inf\{t \in \mathbb{R}_+ : L_t(\omega) \geq n\}, \quad \omega \in \Omega,$$

with the usual convention that  $\inf \emptyset = +\infty$ . Since  $\{T_n \leq t\} = \{L_t \geq n\}$  and the latter is an event,  $T_n$  is a random variable for each  $n$ . Note that  $T_n$  takes values in  $\bar{\mathbb{R}}_+$  and the sequence  $(T_n)$  is increasing. Let  $\Omega_0$  be the intersection of  $\{L_n < \infty\}$  over all finite integers  $n$ ; then,  $\Omega_0$  is almost sure

since  $L_t < \infty$  almost surely for every  $t < \infty$ . For every  $\omega$  in  $\Omega_0$ , the definition 1.13 implies 1.12 for  $A = [0, t]$  for every  $t$  and, therefore, through a monotone class argument, for every Borel subset  $A$  of  $\mathbb{R}_+$ .  $\square$

1.14 REMARKS. a) In the preceding proposition, if one assumes in addition that  $M(\mathbb{R}_+) = +\infty$  almost surely, then  $T_n < \infty$  almost surely for every  $n$  and can be defined to take values in  $\mathbb{R}_+$ . Otherwise, in general, it is impossible to avoid  $+\infty$  as a value for the  $T_n$ . For instance, in Example 1.7 with  $E = \mathbb{R}_+$ , the event  $\{K = 3\}$  has a strictly positive probability, and for every  $\omega$  with  $K(\omega) = 3$  we have  $0 \leq T_1(\omega) \leq T_2(\omega) \leq T_3(\omega) < +\infty$  and  $T_4(\omega) = T_5(\omega) = \dots = +\infty$ . Incidentally, if  $K(\omega) = 3$  then  $T_1(\omega)$  is the smallest number in  $\{X_1(\omega), X_2(\omega), X_3(\omega)\}$ , and  $T_2(\omega)$  is the second smallest, and  $T_3(\omega)$  is the largest.

b) Going back to Example 1.7 on an arbitrary space  $E$ , we now re-define the  $X_n$  in order that the re-defined sequence  $\bar{X}$  form the random measure in the sense of 1.10. Take a point  $\partial$  that is not in  $E$ , let  $\bar{E} = E \cup \{\partial\}$ , and let  $\bar{\mathcal{E}}$  be the  $\sigma$ -algebra on  $\bar{E}$  generated by  $\mathcal{E}$ . For each  $n \geq 1$ , let

$$\bar{X}_n(\omega) = \begin{cases} X_n(\omega) & \text{if } n \leq K(\omega), \\ \partial & \text{otherwise.} \end{cases}$$

Then,  $\bar{X} = \{\bar{X}_n : n \geq 1\}$  is a sequence of random variables taking values in  $(\bar{E}, \bar{\mathcal{E}})$ , and  $\bar{X}$  forms  $M$  by 1.10.

## Exercises and complements

1.15 *Sums of independent measures.* Let  $M$  and  $N$  be independent random measures on  $(E, \mathcal{E})$ . Show that, then,

$$\mathbb{E} e^{-(M+N)f} = \mathbb{E} e^{-Mf} \mathbb{E} e^{-Nf}, \quad f \in \mathcal{E}_+.$$

In words, the Laplace functional of the sum is the product of the Laplace functionals.

1.16 *Mean measure.* Let  $M$  be a random measure on  $(E, \mathcal{E})$  with mean  $\mu$ . Suppose that the singleton  $\{x\}$  is in  $\mathcal{E}$  for every  $x$  in  $E$ .

- a) For  $f$  in  $\mathcal{E}_+$ , if  $\mu f < \infty$  then  $Mf < \infty$  almost surely.
- b) If  $\mu$  is  $\sigma$ -finite, then  $M$  is  $\sigma$ -finite.
- c) If  $x$  is an atom of  $\mu$ , then it is an atom of  $M$  with a strictly positive probability. If  $\mu$  is purely atomic, then  $M$  is purely atomic.

Show these. The converses of these statements are generally false.

1.17 *Uniform points on  $E = [0, 1]$ .* Fix an integer  $n \geq 1$ . Let  $X_1, \dots, X_n$  be independent and uniformly distributed over  $E = [0, 1]$ . Define  $M$  by 1.10 but for Borel functions  $f$  on  $E$ . Compute the Laplace functional of  $M$ .

1.18 *Increasing processes.* Let  $L = (L_t)_{t \in \mathbb{R}_+}$  be an increasing right-continuous process with state space  $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ . Show that there is a random measure  $M$  on  $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$  such that  $M(A) = L_t$  for  $A = [0, t]$ . Hint: Define  $M_\omega$  to be the measure corresponding to the “distribution function”  $t \mapsto L_t(\omega)$ .

1.19 *Atoms.* Let  $(E, \mathcal{E})$  be a standard measurable space. Let  $M$  be a random counting measure with a  $\sigma$ -finite mean measure  $\mu$ . Let  $\partial$  be a point outside  $E$ ; put  $\bar{E} = E \cup \{\partial\}$  and let  $\bar{\mathcal{E}}$  be the  $\sigma$ -algebra on  $\bar{E}$  generated by  $\mathcal{E}$ . Show that there exists a sequence  $(X_i)$  of random variables taking values in  $(\bar{E}, \bar{\mathcal{E}})$  such that 1.10 holds for  $M$ . Hint: Follow the steps of Exercises I.5.15 and I.5.16 to carry  $M$  from  $E$  to a random measure  $\hat{M}$  on  $\mathbb{R}_+$ , use 1.11 above to pick the atoms  $\hat{T}_i$  for  $\hat{M}$ , and transport the  $\hat{T}_i$  back into  $\bar{E}$  as the  $X_i$ .

1.20 *Lexicographic ordering of atoms.* Let  $M$  be a random counting measure on  $\mathbb{R}_+ \times F$ , where  $F$  is a Borel subset of  $\mathbb{R}$  (can be replaced with  $\mathbb{R}^d$ ). Suppose that, for almost every  $\omega$ , the set  $D_\omega$  of atoms of  $M_\omega$  has the following pleasant properties:

- i) if  $(s, y)$  and  $(t, z)$  are atoms then  $s \neq t$ ,
- ii) the number of atoms in  $[0, t] \times F$  is finite for every  $t < \infty$ , but the total number of atoms in  $\mathbb{R}_+ \times F$  is infinite.

For such a good  $\omega$ , label the atoms  $(T_1(\omega), Z_1(\omega)), (T_2(\omega), Z_2(\omega)), \dots$  going from left to right, so that  $0 \leq T_1(\omega) < T_2(\omega) < \dots$ . For the remaining negligible event of bad  $\omega$ , put  $T_k(\omega) = 0$  for all  $k \geq 1$  and do something similar for the  $Z_k(\omega)$ . Show that  $T_1, T_2, \dots$  and  $Z_1, Z_2, \dots$  are random variables and that the pairs  $(T_i, Z_i)$  form  $M$ .

1.21 *Continuation.* Let  $M$  be a random counting measure on  $\mathbb{R}_+ \times \mathbb{R}$  with mean measure  $\mu = \text{Leb} \times \lambda$ , where  $\lambda$  is a  $\sigma$ -finite measure on  $\mathbb{R}$ . Choose a partition  $(F_n)$  of  $\mathbb{R}$  such that  $\lambda(F_n) < \infty$  for every  $n$ . Assume that the condition (i) of 1.20 holds, and note that the condition (ii) follows for the restriction of  $M$  to  $\mathbb{R}_+ \times F_n$  for each  $n$ . For  $n$  in  $\mathbb{N}^*$ , let  $(T_{n,i}, Z_{n,i})$  with  $i = 1, 2, \dots$  be the lexicographic ordering of the atoms in  $\mathbb{R}_+ \times F_n$ . Then, the collection  $(T, Z) = \{(T_{n,i}, Z_{n,i}) : n \in \mathbb{N}^*, i \in \mathbb{N}^*\}$  forms  $M$ .

## 2 POISSON RANDOM MEASURES

In this section we introduce Poisson random measures, give some examples, and illustrate their elementary uses.

Recall that a random variable  $X$  taking values in  $\bar{\mathbb{N}} = \{0, 1, \dots, \infty\}$  is said to have the Poisson distribution with mean  $c$  in  $(0, \infty)$  if

$$2.1 \quad \mathbb{P}\{X = k\} = \frac{e^{-c} c^k}{k!}, \quad k \in \mathbb{N},$$

and then  $X < \infty$  almost surely and  $\mathbb{E} X = \text{Var} X = c$ . We extend this definition to allow 0 and  $+\infty$  as values for  $c$ :

$$2.2 \quad c = 0 \iff X = 0 \text{ almost surely, } c = +\infty \iff X = +\infty \text{ almost surely.}$$

Indeed, the case where  $c = 0$  is covered by 2.1, and the case where  $c = +\infty$  is consistent with 2.1 in a limiting sense.

Recall, also, that if  $X$  and  $Y$  are independent and have the Poisson distributions with means  $a$  and  $b$ , then  $X + Y$  has the Poisson distribution with mean  $c = a + b$ . This property extends to countable sums even when the means sum to  $c = +\infty$ ; see Exercise 2.18. These remarks should make it clear that the following definition is without internal contradictions.

**2.3 DEFINITION.** Let  $(E, \mathcal{E})$  be a measurable space and let  $\nu$  be a measure on it. A random measure  $N$  on  $(E, \mathcal{E})$  is said to be Poisson with mean  $\nu$  if

a) for every  $A$  in  $\mathcal{E}$ , the random variable  $N(A)$  has the Poisson distribution with mean  $\nu(A)$ , and

b) whenever  $A_1, \dots, A_n$  are in  $\mathcal{E}$  and disjoint, the random variables  $N(A_1), \dots, N(A_n)$  are independent, this being true for every  $n \geq 2$ .

**2.4 REMARKS.** a) In some cases, the condition (a) in the definition implies the condition (b). In fact, the necessary condition that

$$\mathbb{P}\{N(A) = 0\} = e^{-\nu(A)}, \quad A \in \mathcal{E},$$

is also sufficient for  $N$  to be Poisson with mean  $\nu$  at least in the case  $E = \mathbb{R}^d$  and  $\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$  and  $\nu$  is diffuse, and then it is known that  $N$  is also a random counting measure. See Theorem 5.12 for this deep result.

b) *Deterministic transformations.* Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ . Let  $h$  be a measurable transformation from  $(E, \mathcal{E})$  into another measurable space  $(F, \mathcal{F})$ . Let  $M$  be the image of  $N$  under  $h$ , that is,

$$M(\omega, B) = N(\omega, h^{-1}B), \quad \omega \in \Omega, \quad B \in \mathcal{F};$$

notation:  $M = N \circ h^{-1}$ . Then,  $M$  is a Poisson random measure on  $(F, \mathcal{F})$  with mean  $\mu = \nu \circ h^{-1}$ . This can be shown by checking the conditions in the definition above. In Section 3, this result will be generalized by replacing  $h$  with a random transformation.

## Examples

**2.5 Particles in boxes.** Let  $E$  be countable and  $\mathcal{E} = 2^E$ , and let  $\nu$  be a measure on it. For each  $x$  in  $E$ , let  $W_x$  be a Poisson distributed random variable with mean  $\nu(\{x\})$ . Assume that the countable collection  $\{W_x : x \in E\}$  is an independency. Then,

$$N(\omega, A) = \sum_{x \in E} W_x(\omega) I(x, A) \quad \omega \in \Omega, \quad A \in \mathcal{E},$$

defines a Poisson random measure  $N$  on  $(E, \mathcal{E})$  with mean  $\nu$ . This is the form of the most general Poisson random measure on a space like this. We may think of  $E$  as a countable collection of boxes and of  $W_x$  as the number of particles in the box  $x$ .

**2.6 Stones in a field.** Take a Poisson distributed number of stones. Throw each into a space  $E$ , using the same mechanism every time, without regard to the total number of stones or where the previous ones have landed. The final configuration of stones in  $E$  describe a Poisson random measure.

The precise version is the construction of Example 1.7. To re-capitulate: let  $X = (X_1, X_2, \dots)$  be an independency of random variables taking values in the measurable space  $(E, \mathcal{E})$  and having some probability measure  $\lambda$  as their common distribution. Let  $K$  be independent of  $X$  and have the Poisson distribution with some number  $c$  in  $(0, \infty)$  as its mean. For each outcome  $\omega$ , we think of  $K(\omega)$  as the total number of stones and  $X_1(\omega), \dots, X_{K(\omega)}(\omega)$  as the landing points of those stones. The configuration formed by the stones (without regard to their identities) can be described by the measure  $N_\omega$  where

$$N_\omega(A) = N(\omega, A) = \sum_{n=1}^{K(\omega)} I(X_n(\omega), A), \quad A \in \mathcal{E},$$

that is, the number of stones in the set  $A$ . The claim is that the random measure  $N$  is Poisson with mean  $\nu = c\lambda$ . Every Poisson random measure with a finite mean measure can be assumed to have this construction.

To prove the claim, we check the conditions of Definition 2.3. To that end, letting  $\{A, \dots, B\}$  be a finite measurable partition of  $E$ , it is enough to show that  $N(A), \dots, N(B)$  are independent and Poisson distributed with respective means  $\nu(A), \dots, \nu(B)$ . In view of the assumptions on  $K$  and the  $X_n$ , we have, for  $i, \dots, j$  in  $\mathbb{N}$  with  $i + \dots + j = k$ ,

$$\begin{aligned} \mathbb{P}\{N(A) = i, \dots, N(B) = j\} \\ &= \mathbb{P}\{K = k\} \mathbb{P}\{N(A) = i, \dots, N(B) = j | K = k\} \\ &= \frac{e^{-c} c^k}{k!} \cdot \frac{k!}{i! \dots j!} \lambda(A)^i \dots \lambda(B)^j = \frac{e^{-\nu(A)} \nu(A)^i}{i!} \dots \frac{e^{-\nu(B)} \nu(B)^j}{j!}, \end{aligned}$$

which is as needed to be shown.

**2.7 Homogeneous counting measures on the plane.** Let  $N$  be a Poisson random measure on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$  with mean  $\nu = c \text{ Leb}$ , where  $c$  is a constant in  $(0, \infty)$  and  $\text{Leb}$  stands for the Lebesgue measure on  $\mathbb{R}^2$ . If  $h$  is a rigid transformation of the plane, since Lebesgue measure is invariant under such transformations, we have  $\nu \circ h^{-1} = \nu$ . Thus, in the notation of Remark 2.4b, the image  $N \circ h^{-1}$  is again a Poisson random measure with mean  $\nu$ . In other words, the probability law of  $N$  is invariant under rigid transformations; this is the meaning of *homogeneity* for  $N$ . Moreover, as will be shown in Theorem 2.17 below,  $N$  is a random counting measure, that is, for almost every outcome  $\omega$ , the measure  $N_\omega$  is purely atomic with weight one on each atom. In Figure 5 below, the atoms are shown for a typical  $\omega$ . Poisson random measures are best visualized by means of such pictures. We now give two computations occasioned by such thoughts.

Consider the distance  $R$  from the origin to the nearest atom. A glance at the figure shows that  $R(\omega) > r$  if and only if  $N(\omega, B_r) = 0$ , where  $B_r$  is



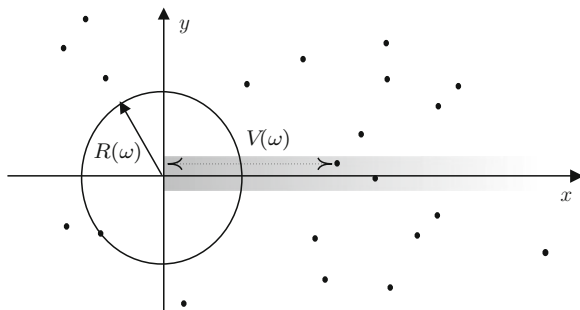


Figure 5: The dots mark the atoms of  $N(\omega, \cdot)$ , the distance from the origin to the nearest atom is  $R(\omega)$ , the visibility in the positive  $x$ -direction is  $V(\omega)$  (the shaded area is  $D = [0, \infty) \times [-a, a]$  where  $a$  is the “size” of the discs).

the closed disk of radius  $r$  centered at the origin. Since  $N(B_r)$  is a random variable,  $\{R > r\} = \{N(B_r) = 0\}$  is an event; this is true for every  $r$  in  $\mathbb{R}_+$  and, hence,  $R$  is a random variable. Since the area of  $B_r$  is  $\pi r^2$ ,

$$\mathbb{P}\{R > r\} = \mathbb{P}\{N(B_r) = 0\} = e^{-\nu(B_r)} = e^{-\pi c r^2}, \quad r \in \mathbb{R}_+.$$

For another small computation, we now think of the atoms as centers of small disks of radius  $a$ . The disks represent trees in a thin forest with intensity  $c$  of trees; assuming that  $a$  and  $c$  are small, the difficulty of explaining overlapping trees can be ignored. Since  $N$  is homogeneous, we may take the positive  $x$ -axis as an arbitrary direction. We are interested in the distance  $V$  from the origin to the nearest tree intersecting the positive  $x$ -axis; this is the visibility in the forest in that direction. Note that a disk intersects the  $x$ -axis if and only if its center is at a distance at most  $a$  from the  $x$ -axis. Thus  $V(\omega) > x$  if and only if  $N(\omega, D_x) = 0$  where  $D_x = [0, x] \times [-a, a]$ . It follows that  $V$  is a random variable and

$$\mathbb{P}\{V > x\} = \mathbb{P}\{N(D_x) = 0\} = \exp_{-} \nu(D_x) = e^{-2acx}, \quad x \in \mathbb{R}_+.$$

## Mean, variance, Laplace functional

Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ . For  $A$  in  $\mathcal{E}$ , if  $\nu(A) < \infty$  then  $N(A)$  has a proper Poisson distribution with mean  $\nu(A)$  and variance  $\nu(A)$ . If  $\nu(A) = +\infty$  then  $N(A) = +\infty$  almost surely, the mean of  $N(A)$  is still  $\nu(A)$ , but the variance is undefined. These carry over to functions as follows: for  $f$  in  $\mathcal{E}_+$ ,

$$2.8 \quad \mathbb{E} Nf = \nu f, \quad \text{Var} Nf = \nu(f^2) \quad \text{if } \nu f < \infty.$$

The claim for the mean is immediate from 1.3. The one for the variance can be shown by computing  $\mathbb{E} (Nf)^2$  first for simple  $f$  and then for arbitrary  $f$

by approximating  $f$  via simple functions. The same steps are used to prove the following important characterization via Laplace functionals.

**2.9 THEOREM.** *Let  $N$  be a random measure on a measurable space  $(E, \mathcal{E})$ . It is Poisson with mean  $\nu$  if and only if*

$$2.10 \quad \mathbb{E} e^{-Nf} = e^{-\nu(1-e^{-f})}, \quad f \in \mathcal{E}_+.$$

**2.11 REMARKS.** a) The compact notation on the right side stands for  $e^{-\nu g}$  where  $\nu g$  is the integral of the function  $g = 1 - e^{-f}$  with respect to the measure  $\nu$ . In other words, 2.10 can be re-written as

$$\mathbb{E} \exp_- \int_E N(dx) f(x) = \exp_- \int_E \nu(dx) (1 - e^{-f(x)}), \quad f \in \mathcal{E}_+.$$

b) Since the Laplace functional of  $N$  determines its probability law, we conclude from the theorem above that the probability law of a Poisson random measure is determined by its mean measure.

c) The proof below will show that  $N$  is Poisson with mean  $\nu$  if and only if 2.10 holds for every simple  $f$  in  $\mathcal{E}_+$ .

### Proof of Theorem 2.9

*Necessity.* Suppose that  $N$  is Poisson with mean  $\nu$ . For  $a$  in  $\mathbb{R}_+$  and  $A$  in  $\mathcal{E}$  with  $\nu(A) < \infty$ , since  $N(A)$  has the Poisson distribution with mean  $\nu(A)$ ,

$$\mathbb{E} \exp_- aN(A) = \sum_0^\infty \frac{e^{-\nu(A)} \nu(A)^k}{k!} e^{-ak} = \exp_- \nu(A)(1 - e^{-a});$$

the result remains true even when  $\nu(A) = +\infty$ . Next, let  $f$  in  $\mathcal{E}_+$  be simple, say  $f = \sum_1^n a_i 1_{A_i}$  with the  $A_i$  disjoint. Then,  $Nf = \sum a_i N(A_i)$  and the variables  $N(A_i)$  are independent by the definition of Poisson random measures. So,

$$\mathbb{E} e^{-Nf} = \prod_i \mathbb{E} \exp_- a_i N(A_i) = \exp_- \sum_i \nu(A_i)(1 - e^{-a_i}),$$

which shows that 2.10 holds when  $f$  is simple. Finally, let  $f$  in  $\mathcal{E}_+$  be arbitrary. Choose  $(f_n) \subset \mathcal{E}_+$  increasing to  $f$  such that each  $f_n$  is simple. By the continuity (Proposition 1.5) of Laplace functionals, using 2.10 for each  $f_n$ , we get

$$\mathbb{E} e^{-Nf} = \lim_n \mathbb{E} \exp_- Nf_n = \lim_n \exp_- \nu(1 - e^{-f_n}).$$

The last limit is equal to the right side of 2.10: as  $n \rightarrow \infty$ , the functions  $g_n = 1 - e^{-f_n}$  increase to  $g = 1 - e^{-f}$ , and the integrals  $\nu g_n$  increase to  $\nu g$  by the monotone convergence theorem.

*Sufficiency.* This is immediate from the necessity part coupled with the one-to-one relationship between Laplace functionals and probability laws of random measures (Proposition 1.4).  $\square$

**2.12 EXAMPLE.** *Shot noise.* Arrivals of electrons at an anode form a Poisson random measure  $N$  on  $\mathbb{R}$  with mean  $\nu = c \text{ Leb}$ , where  $c$  is a constant in  $(0, \infty)$ . We view  $\mathbb{R}$  as the time axis. Since the Lebesgue measure is diffuse, it follows from Theorem 2.17 below that  $N$  is a random counting measure, that is, for almost every  $\omega$ , no two electrons arrive at the same time. Since the number of electrons arriving during a bounded interval is finite (because the mean is finite), we may assume that the arrival times  $T_n$  are ordered so that  $\dots < T_{-1} < T_0 < 0 \leq T_1 < T_2 < \dots$  almost surely.

Each arriving electron produces a current whose intensity is  $g(u)$  after a lapse of  $u$  time units, and the currents produced by the different electrons are additive. Thus, the resulting current's intensity at time  $t$  is

$$X_t = \sum_{n=-\infty}^{\infty} g(t - T_n) 1_{(-\infty, t]} \circ T_n = \int_{(-\infty, t]} N(ds) g(t - s).$$

The function  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is generally continuous and decreases rapidly to 0; all we need here is that  $g$  be Borel and integrable (over  $\mathbb{R}_+$  with respect to the Lebesgue measure). Note that  $X_t = Nf$ , where for  $t$  in  $\mathbb{R}$  fixed,

$$f(s) = g(t - s) 1_{(-\infty, t]}(s), \quad s \in \mathbb{R}.$$

Thus, according to 2.8,

$$\begin{aligned} \mathbb{E} X_t &= \nu f = c \int_{(-\infty, t]} ds g(t - s) = c \int_0^\infty du g(u), \\ \text{Var} X_t &= \nu(f^2) = c \int_{(-\infty, t]} ds g(t - s)^2 = c \int_0^\infty du g(u)^2, \end{aligned}$$

since the mean is finite by the assumed integrability of  $g$ . Finally, we obtain the Laplace transform of  $X_t$  by using the preceding theorem on the Laplace functional of  $N$ :

$$\begin{aligned} \mathbb{E} e^{-rX_t} &= \mathbb{E} e^{-N(rf)} \\ &= \exp_- c \int_{(-\infty, t]} ds (1 - e^{-rg(t-s)}) = \exp_- c \int_{\mathbb{R}_+} du (1 - e^{-rg(u)}). \end{aligned}$$

The formulas for the expected value and variance are well-known as *Campbell's theorem*. Variants and generalizations occur frequently. See Exercise 2.25 also.

## Finiteness of $Nf$

The following provides a criterion for the almost sure finiteness of the random variable  $Nf$ . Recall that  $f \wedge g$  is the function whose value at  $x$  is the minimum of  $f(x)$  and  $g(x)$ .

**2.13 PROPOSITION.** *Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ . Let  $f$  in  $\mathcal{E}_+$  be real-valued.*

- a) If  $\nu(f \wedge 1) < \infty$  then  $Nf < \infty$  almost surely.  
 b) If  $\nu(f \wedge 1) = +\infty$  then  $Nf = +\infty$  almost surely.

*Proof.* We start by recalling that, in view of 2.10 and II.2.31,

$$2.14 \quad \mathbb{P}\{Nf < \infty\} = \lim_{r \rightarrow 0} \mathbb{E} e^{-rNf} = \lim_{r \rightarrow 0} e^{-\nu(1-e^{-rf})}.$$

Moreover, for every  $t = f(x)$ , the mapping  $r \mapsto 1 - e^{-rt}$  is bounded by  $t \wedge 1$  on  $(0, 1)$  and has the limit 0 at  $r = 0$  since  $t < \infty$  by the hypothesis that  $f$  is real-valued. Thus, as  $r \rightarrow 0$ , the function  $1 - e^{-rf}$  is dominated by  $f \wedge 1$  and goes to 0. Hence, by the dominated convergence theorem,

$$\nu(f \wedge 1) < \infty \implies \lim_{r \rightarrow 0} \nu(1 - e^{-rf}) = 0,$$

and this proves the claim (a) via 2.14.

Note that  $1 - e^{-t} \geq (1 - e^{-1})(t \wedge 1)$  for  $t \geq 0$ . This shows, together with the form 2.10 of the Laplace functional, that

$$\nu(f \wedge 1) = +\infty \implies \nu(1 - e^{-f}) = +\infty \implies \mathbb{E} e^{-Nf} = 0,$$

which means that  $Nf = \infty$  almost surely, proving the claim (b).  $\square$

## Existence of Poisson random measures

This is to show that, given a  $\Sigma$ -finite measure  $\nu$  on a measurable space  $(E, \mathcal{E})$ , there exists a probability space and a random measure defined over it such that the latter is Poisson with mean  $\nu$ . The proof is constructive; it is the formal version of the stone throwing we did earlier, repeated a few times.

2.15 THEOREM. *Let  $\nu$  be a  $\Sigma$ -finite measure on  $(E, \mathcal{E})$ . Then, there exists a probability space  $(W, \mathcal{G}, P)$  and a measure  $N(w, \cdot)$  on  $(E, \mathcal{E})$  for each  $w$  in  $W$  such that  $N$  is Poisson with mean  $\nu$ .*

*Proof.* a) First, suppose that  $\nu$  is finite. Let  $c = \nu(E) < \infty$  and define the probability measure  $\mu$  on  $(E, \mathcal{E})$  so that  $\nu = c\mu$ . Let  $\pi$  be the Poisson distribution on  $(\mathbb{N}, 2^{\mathbb{N}})$  with mean  $c$ . Define,

$$(W, \mathcal{G}, P) = (\mathbb{N}, 2^{\mathbb{N}}, \pi) \times (E, \mathcal{E}, \mu)^{\mathbb{N}^*};$$

the existence and construction of this follows from the theorem of Ionescu-Tulcea; see Sections 4 and 5 of Chapter IV. Each point  $w$  in  $W$  is a sequence  $w = (x_0, x_1, x_2, \dots)$ ; for it, define

$$K(w) = x_0; \quad X_i(w) = x_i, \quad i \in \mathbb{N}^*.$$

Then,  $K, X_1, X_2, \dots$  are totally independent,  $K$  has the Poisson distribution  $\pi$  with mean  $c$ , and the  $X_i$  take values in  $(E, \mathcal{E})$  with the distribution  $\mu$ .

Define  $N$  as in Example 2.6 from these variables. As was shown there, then,  $N$  is a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu = c\mu$ . This completes the proof if  $\nu$  is finite.

b) Suppose that  $\nu$  is  $\Sigma$ -finite but not finite. Then, there are finite measures  $\nu_1, \nu_2, \dots$  such that  $\nu = \sum \nu_n$ . For each  $n$ , construct  $(W_n, \mathcal{G}_n, P_n)$  and  $N_n$  as in the part (a) above, but for the measure  $\nu_n$ . Now, put

$$(W, \mathcal{G}, P) = \otimes_{n=1}^{\infty} (W_n, \mathcal{G}_n, P_n);$$

see Section 5 of Chapter IV again. For  $w = (w_1, w_2, \dots)$  in  $W$ , each  $w_n$  is in  $W_n$  and we put  $\hat{N}_n(w, A) = N_n(w_n, A)$ , and finally, define

$$N(w, A) = \sum_{n=1}^{\infty} \hat{N}_n(w, A).$$

Then,  $\hat{N}_1, \hat{N}_2, \dots$  are independent Poisson random measures on  $(E, \mathcal{E})$  with mean measures  $\nu_1, \nu_2, \dots$ , all defined over the probability space  $(W, \mathcal{G}, P)$ . Thus, for  $f$  in  $\mathcal{E}_+$ , writing  $E$  for the expectation operator corresponding to  $P$ ,

$$E \exp_- \sum_1^n \hat{N}_i f = \prod_1^n \exp_- \nu_i (1 - e^{-f}) = \exp_- \sum_1^n \nu_i (1 - e^{-f})$$

according to Proposition 1.6 and Theorem 2.9. Letting  $n \rightarrow \infty$  on both sides we obtain, since  $N = \hat{N}_1 + \hat{N}_2 + \dots$  and  $\nu = \nu_1 + \nu_2 + \dots$ ,

$$E e^{-Nf} = e^{-\nu(1-e^{-f})}, \quad f \in \mathcal{E}_+,$$

which means by Theorem 2.9 that  $N$  is Poisson with mean  $\nu$  as claimed.  $\square$

**2.16 REMARK. Monte-Carlo.** In Monte-Carlo studies using random measures, the starting point is often the construction of the realization  $N_\omega$ , for a typical outcome  $\omega$ , of a Poisson random measure  $N$  on a standard measure space. We illustrate the technique for  $N$  on  $E = \mathbb{R}_+ \times \mathbb{R}_+$  with mean measure  $\nu = \text{Leb}$ ; it is easy to extend the technique to  $E = \mathbb{R}^d$ . The problem is to get a typical realization of  $N$  from a sequence of “random numbers,” the latter being the realizations of independent uniform variables over  $(0, 1)$ . This is a description of the preceding construction in the simulation language, utilizing the  $\sigma$ -finiteness of  $\nu$ : Pick an appropriately large number  $a$ , consider the square  $E_0 = (0, a] \times (0, a]$ . Generate a Poisson distributed random variable with mean  $a^2$  using the initial random number  $u_0$ ; if it turns up to be  $k$ , use the random numbers  $u_1, \dots, u_{2k}$  to form pairs  $(au_1, au_2), (au_3, au_4), \dots, (au_{2k-1}, au_{2k})$ ; these  $k$  pairs are the atoms, each with unit weight, of a realization of Poisson  $N_0$  on  $E_0$  with  $\nu_0 = \text{Leb}$ . Repeat this procedure, using fresh random numbers, with the obviously required shifts, to obtain realizations on the squares  $E_{ij} = (ia, ja) + E_0$  with  $i$  and

$j$  in  $\mathbb{N}$ ; of course,  $E_{0,0} = E_0$  and  $E_{ij}$  is the translation of  $E_0$  by putting its lower left corner at the point  $(ia, ja)$ . The resulting collection of atoms in  $E$  are the atoms of a typical realization of the Poisson random measure  $N$  on  $E = \mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\nu$  equal to the Lebesgue measure.

## Poisson counting measures

Let  $(E, \mathcal{E})$  be a measurable space and assume that the singleton  $\{x\}$  belongs to  $\mathcal{E}$  for every  $x$  in  $E$ . This is the case if  $(E, \mathcal{E})$  is a standard measurable space, and, in particular, if  $E$  is a Borel subset of some Euclidean space  $\mathbb{R}^d$  and  $\mathcal{E}$  is the  $\sigma$ -algebra of the Borel subsets of  $E$ . The following exploits the preceding construction.

**2.17 THEOREM.** *Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$ . Suppose that its mean  $\nu$  is  $\Sigma$ -finite. Then,  $N$  is a random counting measure if and only if  $\nu$  is diffuse.*

*Proof. Necessity.* Fix an arbitrary point  $x$  in  $E$  and let  $c = \nu(\{x\})$ . Assuming that  $N$  is a random counting measure, we need to show that  $c = 0$ . Indeed, the assumption implies that the event  $\{N(\{x\}) \geq 2\}$  is negligible. Whereas, the hypothesis that  $N$  is Poisson implies that the same event has probability  $1 - e^{-c} - ce^{-c}$ . Hence the last probability must vanish, which means that  $c = 0$  as needed.

*Sufficiency.* Assume that  $\nu$  is diffuse and  $\Sigma$ -finite. Since the probability law of a Poisson random measure is determined by its mean measure, we may assume that  $N$  is constructed as in Theorem 2.15; the construction is applicable since  $\nu$  is  $\Sigma$ -finite. Thus,  $N$  has the form

$$N = \sum_1^\infty N_n, \quad N_n = \sum_{i \leq K_n} I(X_{n,i}, \cdot),$$

where the collection  $X = \{X_{n,i} : n \geq 1, i \geq 1\}$  is an independency whose every member has a diffuse distribution (since  $\nu$  is diffuse).

For a pair of distinct indices  $(n, i)$  and  $(m, j)$ , the event  $\{X_{n,i} = X_{m,j}\}$  is negligible in view of what was said about  $X$ . Thus, the countable union  $\Omega_0$  of all such events is negligible. This shows that  $N$  is a counting measure, since the form of  $N_\omega$  is that of a counting measure for every  $\omega$  outside  $\Omega_0$ .  $\square$

## Atomic structure

This is to extend the preceding theorem in a special case of some importance. Consider the case where  $E = \mathbb{R}_+ \times \mathbb{R}_+$  and  $\mathcal{E} = \mathcal{B}(E)$ . For each possible outcome  $\omega$ , we visualize the atoms of the counting measure  $N_\omega$  as solid objects, and, if  $(t, z)$  in  $\mathbb{R}_+ \times \mathbb{R}_+$  is an atom, we create a sense of dynamics by calling  $t$  the arrival time of that atom and  $z$  its size.

**2.18 PROPOSITION.** *Let  $N$  be a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\nu = \text{Leb} \times \lambda$ , where  $\lambda\{0\} = 0$  and  $\lambda(\varepsilon, \infty) < \infty$  for every  $\varepsilon > 0$ . Then, for almost every  $\omega$ , the measure  $N_\omega$  is a counting measure whose atoms are such that*

- a) no atom arrives at time 0, no atom has size 0, no two atoms arrive simultaneously;*
- b) for every  $t < \infty$  and  $\varepsilon > 0$ , only finitely many atoms arrive before  $t$  that have sizes exceeding  $\varepsilon$ ;*
- c) the preceding statement is true for  $\varepsilon = 0$  as well provided that the measure  $\lambda$  be finite; otherwise, during every interval of non-zero length there are infinitely many arrivals of atoms of size at most  $\varepsilon$ , however small  $\varepsilon > 0$  may be.*

*Proof.* a) Almost surely,  $N$  puts no mass on  $\{0\} \times \mathbb{R}_+$  because  $\text{Leb}\{0\} = 0$ , and no mass on  $\mathbb{R}_+ \times \{0\}$  because  $\lambda\{0\} = 0$ ; let  $\Omega_0$  be the almost sure set involved. Fix  $\varepsilon > 0$  and consider the random measure  $M$  on  $\mathbb{R}_+$  defined by  $M(A) = N(A \times (\varepsilon, \infty))$ ,  $A \in \mathcal{B}(\mathbb{R}_+)$ . Then,  $M$  is Poisson with mean  $\mu = \lambda(\varepsilon, \infty)\text{Leb}$ , and it is a counting random measure by the last theorem. Hence, there is an almost sure set  $\Omega_\varepsilon$  such that, for every  $\omega$  in it, if  $(t, z)$  and  $(t', z')$  are atoms of  $N_\omega$  with  $z > \varepsilon$  and  $z' > \varepsilon$ , then  $t \neq t'$ . Let  $\Omega_a$  be the intersection of  $\Omega_0$  and all the  $\Omega_\varepsilon$  with  $\varepsilon = \frac{1}{2}, \frac{1}{3}, \dots$ ; it is almost sure and the statement (a) is true for every  $\omega$  in it.

b) Since  $\nu$  puts the mass  $t \cdot \lambda(\varepsilon, \infty) < \infty$  on the set  $[0, t] \times (\varepsilon, \infty)$ , there is an almost sure event  $\Omega_{t, \varepsilon}$  on which  $N$  has only finitely many atoms in  $[0, t] \times (\varepsilon, \infty)$ . Let  $\Omega_b$  be the intersection of  $\Omega_{t, \varepsilon}$  over  $t = 1, 2, \dots$  and  $\varepsilon = \frac{1}{2}, \frac{1}{3}, \dots$ ; it is almost sure, and the statement (b) is true for every  $\omega$  in it.

c) If  $\lambda$  is finite, let  $\Omega_c$  be the intersection of  $\Omega_{t, 0}$  over  $t = 1, 2, \dots$ . Otherwise, if  $\lambda$  is an infinite measure, then

$$\nu((t, t + \delta) \times (0, \varepsilon]) = \delta \lambda(0, \varepsilon] = +\infty$$

since  $\lambda(\varepsilon, \infty) < \infty$  by assumption; this means that there is an almost sure event  $\Omega_{t, \delta, \varepsilon}$  on which  $N$  has infinitely many atoms in  $(t, t + \delta) \times (0, \varepsilon]$ ; let  $\Omega_c$  be the intersection of all those almost sure events over  $t = 1, 2, \dots$  and  $\varepsilon, \delta = \frac{1}{2}, \frac{1}{3}, \dots$ . The statement (c) is true for every  $\omega$  in  $\Omega_c$ .

It is now obvious that the statements (a), (b), (c) hold simultaneously for every  $\omega$  in the almost sure event  $\Omega_a \cap \Omega_b \cap \Omega_c$ .  $\square$

The preceding proposition will be used in clarifying the jump structure of certain processes constructed from Poisson random measures; see Proposition 4.6 for instance.

## Exercises and complements

**2.19 Sums of Poisson variables.** Let  $X_1, X_2, \dots$  be independent random variables having the Poisson distributions with respective means  $c_1, c_2, \dots$ . Show

that, then,  $X = \sum_1^\infty X_n$  has the Poisson distribution with mean  $c = \sum_1^\infty c_n$ . Discuss the particular case where all the  $c_n$  are finite and  $c = +\infty$ .

**2.20 Covariances.** Let  $N$  be a Poisson random measure with mean  $\nu$  on some measurable space  $(E, \mathcal{E})$ . Show that  $\mathbb{E} N(A)N(B) = \nu(A \cap B) + \nu(A)\nu(B)$  for arbitrary  $A$  and  $B$  in  $\mathcal{E}$ ; thus, when it exists, covariance of  $N(A)$  and  $N(B)$  is  $\nu(A \cap B)$ . Extend these to functions  $f$  and  $g$  in  $\mathcal{E}_+$ :

$$\mathbb{E} Nf Ng = \nu(fg) + \nu f \nu g.$$

**2.21 Higher moments.** These can be obtained either directly or via Laplace functionals. For instance, formally, show that

$$\mathbb{E} (Nf \cdot Ng)^2 = \lim_{q,r \rightarrow 0} \frac{\partial^2}{\partial q^2} \frac{\partial^2}{\partial r^2} \mathbb{E} e^{-N(qf+rg)}.$$

**2.22 Product random measure.** Let  $N$  be Poisson on  $(E, \mathcal{E})$  with mean  $\nu$ . For each  $\omega$ , let  $M(\omega, \cdot)$  be the product measure  $N(\omega, \cdot) \times N(\omega, \cdot)$  on the product space  $(E \times E, \mathcal{E} \otimes \mathcal{E})$ . Show that  $M$  is a random measure whose mean is  $\nu \times I + \nu \times \nu$ , that is, for every positive  $h$  in  $\mathcal{E} \otimes \mathcal{E}$ ,

$$\mathbb{E} Mh = \mathbb{E} \int_{E \times E} N(dx)N(dy)h(x, y) = \int_E \nu(dx)h(x, x) + \int_{E \times E} \nu(dx)\nu(dy)h(x, y)$$

Hint: Use 2.20 with  $h = 1_{A \times B}$  and a monotone class argument.

**2.23 Arrival processes.** Let  $N$  be Poisson on  $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$  with mean  $\nu = c \text{ Leb}$ . Think of  $N$  as an arrival process (see Example 2.12) with  $\mathbb{R}$  as the time axis, that is, the atoms of  $N$  represent the times of arrivals into a store. Let  $V_t$  be the length of the interval from  $t$  to the first arrival time after  $t$ , and let  $U_t$  be the amount of time passed since the last arrival before  $t$ . Define these carefully. Show that they are random variables; compute

$$\mathbb{P}\{U_t > x, V_t > y\}, \quad x, y \in \mathbb{R}_+,$$

to conclude that  $U_t$  and  $V_t$  are independent exponential variables.

**2.24 Continuation.** Let  $N$  be as in the preceding exercise except that the space now is  $\mathbb{R}_+$  instead of  $\mathbb{R}$ . Define  $V_t$  as before, but the definition of  $U_t$  needs modification: if  $N(\omega, [0, t]) = 0$  let  $U_t(\omega) = t$ . Re-do the computations of 2.23 for this case. Show that distribution of  $U_t$  converges weakly to the exponential distribution as  $t \rightarrow \infty$ .

**2.25 Shot noise.** Let  $N$  be as in Example 2.12 but on the space  $\mathbb{R}_+$ . Let  $T_1, T_2, \dots$  be the successive arrival times after time 0, as before. With  $b > 0$  fixed, redefine  $X_t$  by

$$X_t = X_0 e^{-bt} + \sum_{n=1}^{\infty} a e^{-b(t-T_n)} 1_{[0, t] \circ T_n}.$$



- a) Assuming that  $X_0$  is independent of  $(T_n)$ , compute the mean, variance, and Laplace transform of  $X_t$ .
- b) Show that the distribution of  $X_t$  converges weakly, as  $t \rightarrow \infty$ , to the distribution of  $X_0$  in Example 2.12 with  $g(t) = a e^{-bt}$ .
- c) Suppose that  $X_0$  here has the limiting distribution found in the preceding part. Show that, then,  $X_t$  has the same distribution as  $X_0$  for every  $t$  in  $\mathbb{R}_+$ .
- d) Show that  $(X_t)$  satisfies the differential equation

$$dX_t = -b X_t dt + a N(dt), \quad t > 0,$$

in other words, for almost every  $\omega$ ,

$$X_t(\omega) = X_0(\omega) - b \int_0^t X_s(\omega) ds + aN(\omega, [0, t]).$$

The differential equation shows that this process  $(X_t)$  is an *Ornstein-Uhlenbeck* process driven by a Poisson process.

**2.26 Configurations on  $\mathbb{R}^3$ .** This is similar to Example 2.7. Let  $N$  be Poisson on  $\mathbb{R}^3$  with mean  $\nu = c$  Leb. Again,  $N$  is homogeneous and is a random counting measure. Think of its atoms as stars in  $\mathbb{R}^3$ .

a) Let  $R$  be the distance from the origin to the nearest star. Show that it is a random variable. Find its distribution.

b) Let  $X$  be the location of the nearest star expressed in spherical coordinates. Find its distribution.

c) Suppose now that each star is a ball of radius  $a$ . Let  $V$  be the visibility in the  $x$ -direction. Find its distribution.

d) Suppose that a camera located at the origin has an angle of vision of  $2\alpha$  radians and is directed in the  $x$ -direction (so that the  $x$ -axis is the axis of revolution that defines the cone of vision). Let  $R_\alpha$  be the distance from the origin to the nearest star within the cone of vision. Find its distribution; note that  $R_\pi = R$  in part (a).

**2.27 Conditional structure.** Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ . Let  $D$  in  $\mathcal{E}$  have  $\nu(D) < \infty$ . Let  $A, \dots, B$  form a finite measurable partition of  $D$ .

- a) Show that, for integers  $i, \dots, j$  in  $\mathbb{N}$  with  $i + \dots + j = k$ ,

$$\mathbb{P}\{N(A) = i, \dots, N(B) = j | N(D) = k\} = \frac{k!}{i! \dots j!} p^i \dots q^j,$$

(multinomial distribution) where  $p = \nu(A)/\nu(D), \dots, q = \nu(B)/\nu(D)$ . This is another way of saying that, as in Example 2.6, given that  $N(D) = k$ , the locations of those  $k$  stones in  $D$  are as if the stones have been thrown into  $D$  independently and according to the distribution  $\mu(C) = \nu(C)/\nu(D), C \in \mathcal{E} \cap D$ .

b) For  $f$  in  $\mathcal{E}_+$  that vanishes outside  $D$ , show that

$$\mathbb{E}(e^{-Nf} | N(D) = k) = \left[ \int_D \mu(dx) e^{-f(x)} \right]^k, \quad k \in \mathbb{N}.$$

Compute the same conditional expectation for arbitrary  $f$  in  $\mathcal{E}_+$ .

2.28 *Sums.* Let  $L$  and  $M$  be independent Poisson random measures on  $(E, \mathcal{E})$  with means  $\lambda$  and  $\mu$ . Then, show that  $L + M$  is a Poisson random measure with mean  $\lambda + \mu$ .

2.29 *Continuation.* Let  $N_1, N_2, \dots$  be independent Poisson random measures on  $(E, \mathcal{E})$  with means  $\nu_1, \nu_2, \dots$ . Show that, then,  $N = N_1 + N_2 + \dots$  is Poisson with mean  $\nu = \nu_1 + \nu_2 + \dots$ .

2.30 *Superpositions.* Let  $F$  be a countable set and put  $\mathcal{F} = 2^F$ . Suppose that  $\{N_m : m \in F\}$  is an independency of Poisson random measures on  $(E, \mathcal{E})$  with mean  $\nu_m$  for  $N_m$ . Define a random measure  $M$  on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$  by letting

$$M(\omega, A \times \{m\}) = N_m(\omega, A), \quad A \in \mathcal{E}, \quad m \in F.$$

Show that  $M$  is a Poisson random measure and compute its mean  $\mu$ . We call  $M$  the *superposition* of the  $N_m, m \in F$ , because the atoms of  $M$  are pictured by superposing the atoms of the  $N_m$ . Make a picture for the case  $E = \mathbb{R}_+$ ,  $F = \{1, 2, 3\}$ ,  $\nu_1 = \nu_2 = \nu_3 = \text{Leb}$ .

2.31 *Traces.* Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ . Let  $D \in \mathcal{E}$ . Define

$$\nu_D(B) = \nu(B \cap D), \quad N_D(\omega, B) = N(\omega, B \cap D), \quad B \in \mathcal{E}, \quad \omega \in \Omega.$$

Then,  $N_D$  is called the *trace* of  $N$  on  $D$ , and  $\nu_D$  the trace of  $\nu$  on  $D$ . Show that  $N_D$  is Poisson on  $(E, \mathcal{E})$  with mean  $\nu_D$ . Show that, if  $C$  and  $D$  are disjoint sets in  $\mathcal{E}$ , then  $N_C$  and  $N_D$  are independent.

2.32 *Singular mean measures.* Let  $M$  and  $N$  be Poisson random measures on  $(E, \mathcal{E})$  with means  $\mu$  and  $\nu$ . Suppose that  $\mu$  and  $\nu$  are singular with respect to each other (see 5.21 in Chapter I). Show that, then,  $M$  and  $N$  are independent if  $M + N$  is a Poisson random measure.

2.33 *Decomposition into fixed and moving atoms.* Let  $(E, \mathcal{E})$  be a standard measurable space; all we need is that  $\{x\} \in \mathcal{E}$  for every  $x$  in  $E$ . Let  $N$  be a Poisson random measure on  $(E, \mathcal{E})$  with a  $\Sigma$ -finite mean  $\nu$ . Recall that such  $\nu$  have at most countably many atoms. Let  $A$  be the set of all those atoms and let  $D = E \setminus A$ . Define  $\nu_A, \nu_D, N_A, N_D$  as the traces as in 2.31. Then,

$$N = N_A + N_D$$

where  $N_A$  and  $N_D$  are independent Poisson random measures with respective means  $\nu_A$  and  $\nu_D$ . Note that  $N_D$  is a random counting measure; explain the structure of  $N_A$ . Each  $x$  in  $A$  is an atom of  $N(\omega, \cdot)$  for a set of  $\omega$  with strictly

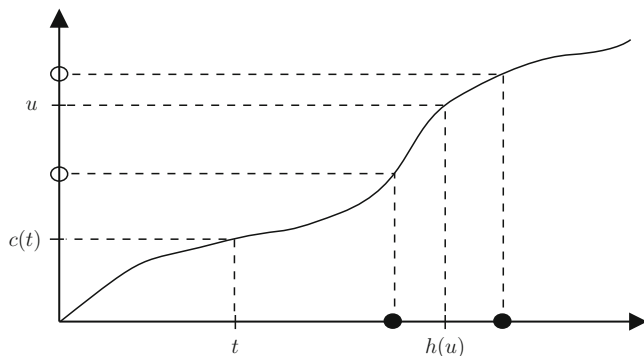


Figure 6: Dots mark the atoms of  $N$ , circles mark the atoms of  $L$ .

positive probability; such  $x$  are called the *fixed* atoms of  $N$ . By contrast, the atoms of the counting measure  $N_D(\omega, \cdot)$  vary with  $\omega$  and are called the moving atoms of  $N$ .

**2.34 Arrival processes.** Let  $N$  be a Poisson random measure on  $\mathbb{R}_+$  (with its Borel  $\sigma$ -algebra) with mean  $\nu$  such that  $c(t) = \nu(0, t] < \infty$  for every  $t$  in  $\mathbb{R}_+$ . It is called an arrival process if, in addition,  $\nu$  is diffuse. Here is a way of constructing such  $N$  by means of time changes. Let  $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be the functional inverse of the increasing continuous function  $t \mapsto c(t)$ . Recall that, then,  $\nu = \lambda \circ h^{-1}$  where  $\lambda$  is the Lebesgue measure on  $\mathbb{R}_+$ . Now, let  $L$  be a Poisson random measure on  $\mathbb{R}_+$  with mean  $\lambda$ , and define  $N = L \circ h^{-1}$  in the notation of Remark 2.4b. See Figure 6 above.

**2.35 Intensities.** In the setup of the preceding exercise, suppose that  $\nu$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_+$ , say,

$$\nu(B) = \int_B dt \, r(t), \quad B \text{ Borel},$$

for some positive Borel function  $r(t)$ ,  $t \in \mathbb{R}_+$ . Then,  $N$  is said to have the intensity function  $r$ , or  $r(t)$  is called the expected arrival rate at time  $t$ . Especially when  $r$  is bounded, the following technique is effective for constructing  $N$  in Monte-Carlo studies.

Let  $M$  be a Poisson random measure on  $E = \mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\mu = \text{Leb}$ . Then, for almost every  $\omega$ , the measure  $M_\omega$  is a counting measure; let  $N_\omega$  be the counting measure on  $\mathbb{R}_+$  whose atoms are those points  $t$  such that  $(t, z)$  is an atom of  $M_\omega$  and  $z \leq r(t)$ . See Figure 7 below.

a) Let  $M_D$  be the trace of  $M$  on  $D = \{(t, z) \in E : z \leq r(t)\}$ , the last being the region under  $r$ . Note that  $N = M_D \circ h^{-1}$  where  $h : E \mapsto \mathbb{R}_+$  is defined as the projection mapping  $(t, z) \mapsto t$ . Show that  $N$  is Poisson with intensity function  $r$ .

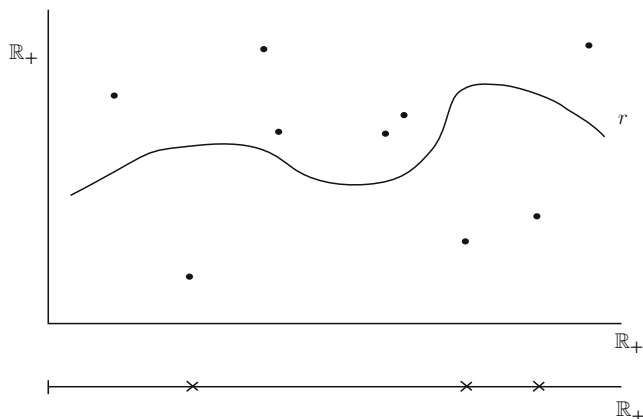


Figure 7: Dots mark the atoms of  $M$  on the positive plane. Crosses mark the atoms of  $N$ .

b) Show that, for Borel subsets  $B$  of  $\mathbb{R}_+$ , and outcomes  $\omega$ ,

$$N(\omega, B) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} M(\omega; dt, dz) 1_B(t) 1_{[0, r(t)]}(z).$$

Use this to obtain the Laplace functional of  $N$ .

**2.36 Random intensities.** Let  $R = (R_t)_{t \in \mathbb{R}_+}$  be a bounded positive left-continuous stochastic process. Let  $M$  be as in the preceding exercise, a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  with unit intensity. Define  $N$  from  $M$  as before, but with  $R_t(\omega)$  replacing  $r(t)$  in 2.35b.

Then,  $N$  is called an arrival process with the random intensity process  $R$ . Generally,  $N$  is not Poisson. If  $R$  is independent of  $M$ , show that

$$\mathbb{E} e^{-Nf} = \mathbb{E} \exp_{-} \int_{\mathbb{R}_+} dt R_t (1 - e^{-f(t)}), \quad f \geq 0 \text{ Borel.}$$

If  $M$  and  $R$  are independent, then the conditional law of  $N$  given  $R$  is that of a Poisson random measure on  $\mathbb{R}_+$  with intensity function  $R$ ; such  $N$  are said to be conditionally Poisson, or doubly stochastic Poisson, or Cox processes.

**2.37 Conditionally Poisson random measures.** Let  $L$  and  $N$  be random measures on  $(E, \mathcal{E})$ . Suppose that the conditional expectation of  $e^{-Nf}$  given the  $\sigma$ -algebra  $\mathcal{G}$  generated by  $L$  has the form

$$\mathbb{E}_{\mathcal{G}} e^{-Nf} = \exp_{-} \int_E L(dx) (1 - e^{-f(x)}), \quad f \in \mathcal{E}_+.$$

Then,  $N$  is said to be conditionally Poisson given  $L$ . Preceding exercise is a special case where  $E = \mathbb{R}_+$  and  $L(dx) = R_x dx$ .

**2.38 Weak convergence to Poisson laws.** Take  $n$  stones. Throw each into the interval  $[0, n]$  uniformly at random. For large  $n$ , the configuration formed on  $\mathbb{R}_+$  is approximately Poisson with unit intensity. Here is the precise version.

Let  $U_1, U_2, \dots$  be independent uniformly distributed random variables on  $(0, 1)$ . For each integer  $n \geq 1$ , let  $M_n$  be the random measure on  $\mathbb{R}_+$  whose atoms are  $nU_1, nU_2, \dots, nU_n$ .

a) Show that, for positive Borel  $f$  on  $\mathbb{R}_+$ ,

$$\mathbb{E} e^{-M_n f} = \left[ \frac{1}{n} \int_0^n du e^{-f(u)} \right]^n = \left[ 1 - \frac{1}{n} \int_0^n du (1 - e^{-f(u)}) \right]^n.$$

b) Assuming that  $f$  is continuous, positive, with compact support, note that

$$\lim_{n \rightarrow \infty} \mathbb{E} e^{-M_n f} = \mathbb{E} e^{-M f},$$

where  $M$  is some Poisson random measure on  $\mathbb{R}_+$  with unit intensity.

**2.39 Continuation.** In the preceding exercise, the result is that  $(M_n)$  “converges in distribution” to a Poisson random measure  $M$  with unit intensity. We now explain the meaning of “convergence in distribution” in this context.

Let  $\mathcal{M}$  be the collection of all measures on  $\mathbb{R}$ . Let  $C_K = C_K(\mathbb{R} \mapsto \mathbb{R}_+)$  be the collection of all positive continuous functions on  $\mathbb{R}$  with compact support. A sequence  $(\mu_n)$  in  $\mathcal{M}$  is said to *converge vaguely* to the measure  $\mu$  if  $\mu_n f \rightarrow \mu f$  for every  $f$  in  $C_K$ . With the topology induced by this mode of convergence,  $\mathcal{M}$  becomes a topological space, in fact, a Polish space.

The Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{M})$  is also the  $\sigma$ -algebra generated by the coordinate functions  $\mu \mapsto \mu(A)$  from  $\mathcal{M}$  into  $\bar{\mathbb{R}}_+$ . Thus, we may regard a random measure  $M$  as the random variable  $\omega \mapsto M_\omega$  taking values in  $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ .

Given a sequence  $(M_n)$  of random measures on  $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ , we say that  $(M_n)$  *converges in distribution* to the random measure  $M$  if

$$2.40 \quad \mathbb{E} \varphi \circ M_n \rightarrow \mathbb{E} \varphi \circ M$$

for every bounded continuous function  $\varphi$  from  $\mathcal{M}$  into  $\mathbb{R}$ . This is the natural generalization of the concept discussed in Chapter III to the space  $\mathcal{M}$ . In fact, it is sufficient to check 2.40 for functions  $\varphi$  of the form  $\varphi(\mu) = e^{-\mu f}$  with  $f$  in  $C_K$ . We state this here without proof. The following statements are equivalent:

- a)  $(M_n)$  converges in distribution to  $M$ .
- b)  $(M_n f)$  converges in distribution to  $M f$  for every  $f$  in  $C_K$ .
- c)  $(\mathbb{E} \exp_- M_n f)$  converges to  $\mathbb{E} \exp_- M f$  for every  $f$  in  $C_K$ .

### 3 TRANSFORMATIONS

Let  $(E, \mathcal{E})$  and  $(F, \mathcal{F})$  be measurable spaces. Let  $X = \{X_i : i \in I\}$  and  $Y = \{Y_i : i \in I\}$  be collections, indexed by the same countable set  $I$ , of random variables taking values in  $(E, \mathcal{E})$  and  $(F, \mathcal{F})$  respectively.

Suppose that  $X$  forms a Poisson random measure on  $(E, \mathcal{E})$ , that is, the random measure  $N$  on  $(E, \mathcal{E})$  defined by

$$3.1 \quad Nf = \sum_{i \in I} f \circ X_i, \quad f \in \mathcal{E}_+,$$

is Poisson with some mean measure  $\nu$ . Suppose also that, for some measurable transformation  $h : E \mapsto F$ , we have  $Y_i = h \circ X_i$  for each  $i$ . Then, the random measure formed by  $Y$  on  $(F, \mathcal{F})$  is the image  $N \circ h^{-1}$  of  $N$  under  $h$  and is again Poisson; see Remark 2.4b. In this section we consider two generalizations of this: first, we let each  $Y_i$  be a random transform of the corresponding  $X_i$  according to the heuristic that  $Y_i$  falls in  $B$  with probability  $Q(x, B)$  if  $X_i = x$ . Second, we regard each  $Y_i$  as an indicator of some property associated with the atom  $X_i$ , which leads us to the random measure  $M$  formed by  $(X, Y) = \{(X_i, Y_i) : i \in I\}$  as a marked version of  $N$ .

We present the main ideas in terms of the setup above, which is convenient in most applications. Afterward, in Theorem 3.19, we handle the more general case where the atoms  $X_i$  and  $Y_i$  take values in spaces larger than  $E$  and  $F$ . Finally, in Theorem 3.26, we give a modern re-formulation of the main results directly in terms of counting measures.

## Main theorem

This is important; used with some art, it simplifies many a complex problem to mere computations.

3.2 THEOREM. *Let  $\nu$  be a measure on  $(E, \mathcal{E})$ , and  $Q$  a transition probability kernel from  $(E, \mathcal{E})$  into  $(F, \mathcal{F})$ . Assume that (i) the collection  $X$  forms a Poisson random measure with mean  $\nu$ , and (ii), given  $X$ , the variables  $Y_i$  are conditionally independent and have the respective distributions  $Q(X_i, \cdot)$ . Then,*

- a)  *$Y$  forms a Poisson random measure on  $(F, \mathcal{F})$  with mean  $\nu Q$ , and*
- b)  *$(X, Y)$  forms a Poisson random measure on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$  with mean  $\nu \times Q$ .*

3.3 REMARK. Recall from I.6.23 that  $\mu = \nu \times Q$  means that

$$\mu(dx, dy) = \nu(dx)Q(x, dy),$$

and  $\nu Q$  is the marginal of  $\mu$  on  $F$ .

*Proof.* Let  $N$  be the random measure formed by  $X$  on  $(E, \mathcal{E})$ , and  $M$  the one formed by  $(X, Y)$  on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$ . Note that the random measure formed by  $Y$  is the image of  $M$  under the projection mapping  $h(x, y) = y$ . Thus, (a) is immediate from (b), and we shall prove (b) by showing that the Laplace functional of  $M$  has the form required by Theorem 2.9. Note that, for positive real-valued  $f$  in  $\mathcal{E} \otimes \mathcal{F}$ ,

$$e^{-Mf} = \prod_{i \in I} e^{-f \circ (X_i, Y_i)}.$$

In view of the assumption (ii), the conditional expectation of  $e^{-Mf}$  given  $X$  is equal to

$$\prod_i \int_F Q(X_i, dy) e^{-f \circ (X_i, y)} = \prod_i e^{-g \circ X_i} = e^{-Ng},$$

where  $g$  is defined by

$$e^{-g(x)} = \int_F Q(x, dy) e^{-f(x, y)}.$$

It follows that

$$3.4 \quad \mathbb{E} e^{-Mf} = \mathbb{E} e^{-Ng} = \exp_- \nu(1 - e^{-g})$$

where we used Theorem 2.9 on the Laplace functional of  $N$  after noting that, by the assumption (i),  $N$  is Poisson with mean  $\nu$ . Since  $Q(x, F) = 1$ ,

$$\nu(1 - e^{-g}) = \int_E \nu(dx) \int_F Q(x, dy) (1 - e^{-f(x, y)}) = (\nu \times Q)(1 - e^{-f}),$$

and, putting this into 3.4, we conclude from Theorem 2.9 that  $M$  is Poisson with mean  $\nu \times Q$  as claimed.  $\square$

The following is an immediate corollary where  $Q(x, B)$  is specialized to become  $\pi(B)$  for some probability measure  $\pi$ . No proof is needed.

3.5 COROLLARY. *Suppose that  $X$  forms a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ , and that  $Y$  is independent of  $X$  and is an independency of variables with distribution  $\pi$  on  $(F, \mathcal{F})$ . Then,  $(X, Y)$  forms a Poisson random measure on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$  with mean  $\nu \times \pi$ .*

In certain applications, it is convenient to think of  $Y_i$  as some mark (weight, velocity, etc.) associated with the atom  $X_i$ . Then,  $(X, Y)$  is sometimes called a *marked point process* on  $E$  with mark space  $F$ , and we may think of the Poisson random measure on  $E \times F$  as a magnification of that on  $E$ . The next four sub-sections provide examples on the uses of the theorem and corollary above.

## Compound Poisson processes

The arrival times  $T_i$  of customers at a store form a Poisson random measure  $N$  on  $\mathbb{R}_+$  with intensity  $c$ , that is, the mean measure is  $\nu = c\text{Leb}$ . The customers spend, independently of each other, random amounts of money at the store, the mean being  $a$ , variance  $b^2$ , and distribution  $\pi$ . We are interested in  $Z_t$ , the cumulative amount spent by all who arrived at or before  $t$ .

More precisely, we are assuming that the customer who arrives at  $T_i$  spends an amount  $Y_i$ , where  $Y = (Y_i)$  is independent of  $T = (T_i)$  and the

variables  $Y_i$  are independent and have the distribution  $\pi$  on  $\mathbb{R}_+$  in common (with mean  $a$  and variance  $b^2$ ). It follows from the preceding corollary that  $(T, Y)$  forms a Poisson random measure  $M$  on  $\mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\mu = \nu \times \pi$ . For fixed  $t$ , the variable  $Z_t$  is defined by  $M$  through

$$3.6 \quad Z_t = \sum_{i=1}^{\infty} Y_i 1_{(0,t]} \circ T_i = \int_{[0,t] \times \mathbb{R}_+} M(ds, dy) y = Mf,$$

where  $f(s, y) = y 1_{[0,t]}(s)$ . It follows from 2.8 and Theorem 2.9 applied to  $M$  with this  $f$  that

$$3.7 \quad \begin{aligned} \mathbb{E} Z_t &= \mu f = act, & \text{Var } Z_t &= \mu(f^2) = (a^2 + b^2)ct \\ \mathbb{E} \exp_- rZ_t &= \mathbb{E} e^{-M(rf)} = \exp_- ct \int_{\mathbb{R}_+} \pi(dy)(1 - e^{-ry}). \end{aligned}$$

The process  $Z = (Z_t)_{t \in \mathbb{R}_+}$  is an example of compound Poisson processes. The most general cases are obtained by allowing the  $Y_i$  to be  $\mathbb{R}^d$ -valued, without restrictions on the distribution  $\pi$  on  $\mathbb{R}^d$ , with the same assumptions of independence.

## Money in the bank

This is the same as the shot noise process of Example 2.12, but the deterministic function  $g$  is replaced by a randomized one. Let  $T_1, T_2, \dots$  form a Poisson random measure  $N$  on  $\mathbb{R}_+$  with mean  $\nu = a$  Leb. We think of  $T_i$  as the arrival time of the  $i^{\text{th}}$  person to a bank in order to open an account; let  $Y_i(u)$  be the balance at time  $T_i + u$  for that account. Then, the sum of all balances at time  $t$  is (assuming  $X_0 = 0$ )

$$3.8 \quad X_t = \sum_{i=1}^{\infty} Y_i(t - T_i) 1_{[0,t]} \circ T_i.$$

We suppose that the processes  $Y_1, Y_2, \dots$  are independent of each other and of the collection  $(T_i)$ , and let

$$P_u(B) = \mathbb{P}\{Y_i(u) \in B\}, \quad u \in \mathbb{R}_+, B \in \mathcal{B}(\mathbb{R}_+).$$

We are interested in the mean, variance, and Laplace transform of  $X_t$  for fixed  $t$ . Exercise 2.25 is the special case where  $Y_i(u) = g(u)$  for all  $i$  and  $u$ .

We assume that the  $Y_i$  are right-continuous and bounded. Thus, each  $Y_i$  takes values in the space  $F$  of all bounded right-continuous functions  $y : \mathbb{R}_+ \mapsto \mathbb{R}_+$  with  $\mathcal{F}$  the Borel  $\sigma$ -algebra corresponding to the supremum norm. Since  $(T_i)$  forms the Poisson random measure  $N$  on  $\mathbb{R}_+$ , it follows from Corollary 3.5 that the pairs  $(T_i, Y_i)$  form a Poisson random measure  $M$  on  $(\mathbb{R}_+ \times F, \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F})$  with mean  $\mu = \nu \times \pi$ , where  $\nu = a$  Leb and  $\pi$  is the probability law of  $Y_i$ . The law  $\pi$  is not specified, but we are given

$$\pi\{y \in F : y(u) \in B\} = P_u(B).$$



Finally, we note that 3.8 is the same as

$$3.9 \quad X_t = Mf, \quad \text{where } f(s, y) = y(t-s) 1_{[0,t]}(s), \quad s \in \mathbb{R}_+, y \in F.$$

We leave the mean and variance as an exercise to compute and do the Laplace transform of  $X_t$  for fixed  $t$  by using Theorem 2.9 on  $M$  and  $f$  here: for  $r$  in  $\mathbb{R}_+$ ,

$$\begin{aligned} 3.10 \quad \mathbb{E} e^{-rX_t} &= \mathbb{E} e^{-M(rf)} = e^{-\mu(1-e^{-rf})} \\ &= \exp_- a \int_{[0,t]} ds \int_F \pi(dy) (1 - e^{-ry(t-s)}) \\ &= \exp_- a \int_{[0,t]} ds \mathbb{E}(1 - e^{-rY_i(t-s)}) \\ &= \exp_- a \int_{[0,t]} du \int_{\mathbb{R}_+} P_u(dx)(1 - e^{-rx}). \end{aligned}$$

## Closed particle systems

Imagine some particles moving about in space  $E$ . At time 0, the configuration of particles form a Poisson random measure  $N_0$  on  $(E, \mathcal{E})$  with some mean  $\nu_0$ . Each particle moves in  $E$  according to a probability law  $P^x$  if its initial position is  $x$ . Other than this dependence on the initial positions, the particle motions are independent. We are interested, for fixed time  $t$ , in the random measure  $N_t$  formed by the positions of the particles at time  $t$ .

To make the picture more precise, we label the particles with the integers  $i \geq 1$ , let  $X_i$  be the initial position of the particle  $i$ , and let  $Y_i = (Y_i(t))_{t \in \mathbb{R}_+}$  be the corresponding motion with  $Y_i(0) = X_i$ . Each  $Y_i$  is a stochastic process with state space  $(E, \mathcal{E})$ ; we view it as a random variable taking values in the function space  $(F, \mathcal{F}) = (E, \mathcal{E})^{\mathbb{R}_+}$ . We are given  $P^x$  as the conditional distribution of  $Y_i$  given that  $X_i = x$ , and the  $X_i$  form a Poisson random measure  $N_0$  on  $(E, \mathcal{E})$  with mean  $\nu_0$ , and we may assume that  $Q(x, B) = P^x(B)$  defines a transition kernel. Then, Theorem 3.2 applies, and the  $Y_i$  form a Poisson random measure  $M$  on  $(F, \mathcal{F})$  with mean  $\mu = \nu_0 Q$ , that is, with

$$\mu(B) = \int_E \nu_0(dx) P^x(B), \quad B \in \mathcal{F}.$$

Most everything about this particle system can be posed in terms of  $M$ . In particular,  $N_t = M \circ h^{-1}$  where  $h(w) = w(t)$  for  $w$  in  $F$  and  $t$  fixed. Thus,  $N_t$  is a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu_t = \mu \circ h^{-1}$ , that is,

$$3.11 \quad \nu_t(A) = \int_E \nu_0(dx) \int_F P^x(dw) 1_A(w(t)) = \int_E \nu_0(dx) P_t(x, A), \quad A \in \mathcal{E},$$

where

$$3.12 \quad P_t(x, A) = P^x\{w \in F : w(t) \in A\}$$

is the probability that a particle started at  $x$  is in  $A$  at time  $t$ . Indeed, this could have been obtained more directly by noting that  $N_t$  is formed by the atoms  $Y_i(t)$ , each of which has the conditional distribution  $P_t(x, \cdot)$  given that  $Y_i(0) = X_i = x$ .

Here is a particular case of interest. Take  $E = \mathbb{R}$ ,  $\mathcal{E} = \mathcal{B}_{\mathbb{R}}$ ,  $\nu_0 = \text{Leb}$ , and assume that every particle motion is Brownian (independent of all others). Then,

$$3.13 \quad P_t(x, A) = \int_A dz \frac{1}{\sqrt{2\pi t}} e^{-(z-x)^2/2t}, \quad A \in \mathcal{E},$$

and we observe that  $\nu_t = \nu_0 P_t = \nu_0 = \text{Leb}$ . Thus, in this case, the particle configuration at time  $t$  is Poisson with mean  $\nu_t = \text{Leb}$  for all times  $t$ . In this sense, the particle system is in equilibrium, even though individual particles never experience equilibrium.

In the theory of Markov processes on general state spaces  $(E, \mathcal{E})$ , the family  $(P_t)_{t \in \mathbb{R}_+}$  of transition kernels is called the *transition semigroup*, and measures  $\nu$  satisfying  $\nu = \nu P_t$  are said to be *invariant*. When  $\nu$  is an infinite invariant measure, the particle system above with  $\nu_0 = \nu$  provides a dynamic meaning for  $\nu$ .

## Particle systems with birth and death

Particles arrive over time according to a Poisson process on  $\mathbb{R}$  with intensity  $a$ . Each arriving particle lands somewhere in  $E$  and starts moving in  $E$  in some random fashion until it dies. Thus, at each time, a snapshot of  $E$  will show the locations of the particles that are alive (born but not dead) at that time. We are interested in the evolution of this picture over time.

We label the particles with integers  $i$  in  $\mathbb{Z}$ , let  $T_i$  be the arrival time for  $i$ , and  $Y_i$  its “motion”. Each  $Y_i$  is a stochastic process  $(Y_i(t))_{t \in \mathbb{R}_+}$  with state space  $(\bar{E}, \bar{\mathcal{E}})$ , where  $\bar{E} = E \cup \{\partial\}$  and  $\bar{\mathcal{E}}$  is the  $\sigma$ -algebra on  $\bar{E}$  generated by  $\mathcal{E}$ . We regard  $\partial$  as the cemetery attached to  $E$ ; it is a trap; once a particle enters  $\partial$  it must stay there forever. We interpret  $Y_i(t)$  as the location of  $i$  at time  $T_i + t$ , being in state  $\partial$  means being dead. We regard  $Y_i$ ,  $i \in \mathbb{Z}$ , as random variables taking values in the function space  $(F, \mathcal{F}) = (\bar{E}, \bar{\mathcal{E}})^{\mathbb{R}_+}$ ; they are assumed to be independent of each other and of the  $T_i$ , and we let  $P$  their common probability law. It follows from Corollary 3.5 that the pairs  $(T_i, Y_i)$  form a Poisson random measure  $M$  on  $(\mathbb{R} \times F, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{F})$  with mean  $\mu = a \text{Leb} \times P$ .

Consider the snapshot at time  $t$ ; it shows the locations  $Y_i(t - T_i)$  of those particles  $i$  that are born but not dead, that is,  $T_i \leq t$  and the location

not  $\partial$ . The snapshot can be represented by the random measure  $M_t$  on  $(E, \mathcal{E})$  defined by

$$3.14 \quad M_t(A) = \sum_{i=-\infty}^{\infty} 1_A \circ Y_i(t - T_i) 1_{(-\infty, t]} \circ T_i, \quad A \in \mathcal{E}.$$

Essentially,  $M_t$  is the trace on  $E$  of  $\bar{M}_t = M \circ h^{-1}$ , where  $h : \mathbb{R} \times F \mapsto E$  is given by

$$h(s, w) = \begin{cases} w(t - s) & \text{if } s \leq t, \\ \partial & \text{if } s > t. \end{cases}$$

Since  $M$  is Poisson,  $\bar{M}_t$  is Poisson on  $\bar{E}$  by Remark 2.4b, and  $M_t$  is Poisson on  $E$  by Exercise 2.31 on traces. As to the mean measure  $\mu_t$  of  $M_t$ , we have, for  $A \in \mathcal{E}$ ,

$$\begin{aligned} 3.15 \quad \mu_t(A) &= a \int_{(-\infty, t]} ds \int_F P(dw) 1_A \circ w(t - s) \\ &= a \int_{\mathbb{R}_+} du \mathbb{E} 1_A \circ Y_0(u) = a \mathbb{E} \int_{\mathbb{R}_+} du 1_A \circ Y_0(u) \end{aligned}$$

In summary,  $M_t$  is a Poisson random measure on  $(E, \mathcal{E})$ , and the mean number of particles in  $A$  at anytime  $t$  is equal to the arrival rate  $a$  times the expected amount of time spent in  $A$  by one particle during its lifetime.

The computations can be made more specific by making the law  $P$  of the  $Y_i$  more precise. We illustrate this by assuming that the  $Y_i$  are independent replicas of the process  $X = (X_t)_{t \in \mathbb{R}_+}$  with state space  $E = \mathbb{R}$ ,  $\mathcal{E} = \mathcal{B}_{\mathbb{R}}$ , defined by

$$3.16 \quad X_t(\omega) = \begin{cases} X_0(\omega) + W_t(\omega) & \text{if } t < \zeta(\omega), \\ +\infty & \text{if } t \geq \zeta(\omega), \end{cases}$$

where  $X_0, W, \zeta$  are independent,  $X_0$  has the Gaussian distribution with mean 0 and variance  $b$ , and  $W$  is a Wiener process, and  $\zeta$  has the exponential distribution with parameter  $c$ . Note that  $X$  describes the motion of a particle that starts at  $X_0$ , moves as a standard Brownian motion, and dies at age  $\zeta$  and is carried to the point  $\partial = +\infty$ . In this case, 3.15 becomes, for  $A \in \mathcal{B}_{\mathbb{R}}$ ,

$$\begin{aligned} 3.17 \quad \mu_t(A) &= a \int_{\mathbb{R}_+} du \mathbb{P}\{X_u \in A\} \\ &= \frac{a}{c} \int_{\mathbb{R}_+} du c e^{-cu} \int_{\mathbb{R}} \pi(dx) P_u(x, A) \end{aligned}$$

where  $\pi$  is the distribution of  $X_0$ , and  $P_u(x, A)$  is as in 3.13. The integral over  $\mathbb{R}$  is the probability that  $X_0 + W_u$  belongs to  $A$ . Thus, the double integral yields the probability that  $X_0 + W_{\zeta}$  is in  $A$ . And, we know by calculating the characteristic function of  $W_{\zeta}$  that  $W_{\zeta}$  has the same distribution as  $Z_1 - Z_2$ , where  $Z_1$  and  $Z_2$  are independent and exponentially distributed random

variables with parameter  $\sqrt{2c}$ . Letting  $\nu$  be the distribution of  $X_0 + Z_1 - Z_2$ , we see that 3.17 is the same as

$$3.18 \quad \mu_t(A) = \frac{a}{c} \nu(A).$$

In particular, we note that the total number of particles in  $E$  has the Poisson distribution with mean  $\mu_t(E) = \frac{a}{c}$ , the expected number of new arrivals during one lifetime.

## Generalization of the main theorem

The setup and assumptions of the main theorem, 3.2, include two points: the index set  $I$  is countable, and  $X$  forms a Poisson random measure with mean  $\nu$ . Note that, being deterministic,  $I$  has to be infinite, and thus,  $\nu$  must be an infinite measure. These conditions are caused by letting the  $X_i$  take values in  $E$ . To remedy the situation, we let them take values in a larger space  $\bar{E}$  as in the discussion on atoms; see 1.10 *et seq.* and Remark 1.14b.

Let  $(\bar{E}, \bar{\mathcal{E}})$  be a measurable space that contains  $(E, \mathcal{E})$ , let  $(\bar{F}, \bar{\mathcal{F}})$  similarly contain  $(F, \mathcal{F})$  by setting  $\bar{F} = F \cup \{\Delta\}$  with an extra point  $\Delta$  outside  $F$ . Let  $I$  be a countably infinite index set as before,  $X = \{X_i : i \in I\}$  a collection of random variables taking values in  $(\bar{E}, \bar{\mathcal{E}})$ , and  $Y = \{Y_i : i \in I\}$  in  $(\bar{F}, \bar{\mathcal{F}})$ . Every function on  $E$  is extended onto  $\bar{E}$  by letting it vanish on  $\bar{E} \setminus E$ , and similarly for extension from  $F$  onto  $\bar{F}$  and from  $E \times F$  onto  $\bar{E} \times \bar{F}$ . Given a transition probability kernel  $Q$  from  $(E, \mathcal{E})$  into  $(F, \mathcal{F})$ , we extend it to a kernel  $\bar{Q}$  from  $(\bar{E}, \bar{\mathcal{E}})$  into  $(\bar{F}, \bar{\mathcal{F}})$  by the requirement that  $\bar{Q}(x, F) = 0$  and  $\bar{Q}(x, \bar{F}) = 1$  for  $x$  in  $\bar{E} \setminus E$ . Finally, recall the meaning of “ $X$  forms a random measure  $N$  on  $(E, \mathcal{E})$ ”, namely, that  $Nf = \sum_i f \circ X_i$  for  $f$  in  $\mathcal{E}_+$  extended onto a function on  $\bar{\mathcal{E}}_+$  as prescribed. With this setup, the following is the generalization of Theorem 3.2.

3.19 THEOREM. *Suppose that  $X$  forms a Poisson random measure on  $(E, \mathcal{E})$  with mean  $\nu$ . Assume that, given  $X$ , the  $Y_i$  are conditionally independent with corresponding distributions  $\bar{Q}(X_i, \cdot)$ . Then,  $Y$  forms a Poisson random measure on  $(F, \mathcal{F})$  with mean  $\nu Q$ , and  $(X, Y)$  forms a Poisson random measure on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$  with mean  $\nu \times Q$ .*

*Proof.* This follows the proof of 3.2 word for word except for the substitution of  $\bar{Q}$  for  $Q$  in some places.  $\square$

The preceding is the most general result on transformations of Poisson random measures. Unfortunately, its formulation is in terms of the atoms rather than being directly in terms of the random measures. The following is aimed at this direct formulation.

## Random transformations of Poisson

Let  $(E, \mathcal{E})$  and  $(F, \mathcal{F})$  be measurable spaces. By a *random transformation* from  $E$  into  $F$  we mean a mapping

$$3.20 \quad \varphi : (\omega, x) \mapsto \varphi(\omega, x)$$

that is measurable relative to  $\mathcal{H} \otimes \mathcal{E}$  and  $\mathcal{F}$ . We write  $\varphi x$  for the random variable  $\omega \mapsto \varphi(\omega, x)$  and  $\varphi_\omega$  for the transformation  $x \mapsto \varphi_\omega x = \varphi(\omega, x)$ . Of course,  $\varphi$  can be regarded as a collection  $\varphi = \{\varphi x : x \in E\}$ , and it is said to be an independency if the collection is such in the usual sense. In that case, the probability law of  $\varphi$  is specified by the marginal distributions

$$3.21 \quad Q(x, B) = \mathbb{P}\{\varphi x \in B\}, \quad x \in E, \quad B \in \mathcal{F}.$$

The joint measurability assumed for the mapping 3.20 assures that the preceding defines a transition probability kernel  $Q$  from  $(E, \mathcal{E})$  into  $(F, \mathcal{F})$ .

Given a random measure  $N$  on  $(E, \mathcal{E})$  and a random transformation  $\varphi$  from  $(E, \mathcal{E})$  into  $(F, \mathcal{F})$ , we define the image of  $N$  under  $\varphi$  as the random measure  $\hat{N}$  on  $(F, \mathcal{F})$ , and write  $N \circ \varphi^{-1}$  for  $\hat{N}$ , defined by

$$3.22 \quad \hat{N}_\omega f = (N_\omega \circ \varphi_\omega^{-1})f = \int_E N(\omega, dx) f(\varphi_\omega x), \quad \omega \in \Omega, \quad f \in \mathcal{F}_+.$$

Magnification  $M$  of  $N$  is defined similarly

$$3.23 \quad M_\omega f = \int_E N(\omega, dx) f(x, \varphi_\omega x), \quad \omega \in \Omega, \quad f \in (\mathcal{E} \otimes \mathcal{F})_+.$$

Assuming that  $N$  and  $\varphi$  are independent, with  $\nu$  as the mean of  $N$  and  $Q$  as in 3.21, we observe, by conditioning on  $\varphi$  first and using Fubini's theorem repeatedly, that

$$3.24 \quad \mathbb{E} \hat{N} f = \mathbb{E} \int_E \nu(dx) f(\varphi x) = \int_E \nu(dx) \int_F Q(x, dy) f(y) = \nu Q f, \quad f \in \hat{\mathcal{F}},$$

$$3.25 \quad \begin{aligned} \mathbb{E} M f &= \mathbb{E} \int_E \nu(dx) f(x, \varphi x) \\ &= \int_E \nu(dx) \int_F Q(x, dy) f(x, y) = (\nu \times Q) f, \quad f \in \mathcal{E} \otimes \mathcal{F}, \quad f \geq 0. \end{aligned}$$

The following is the promised direct formulation.

3.26 THEOREM. *Let  $(E, \mathcal{E})$  and  $(F, \mathcal{F})$  be standard measurable spaces. Let  $\nu$  be a  $\sigma$ -finite diffuse measure on  $(E, \mathcal{E})$ , and  $Q$  a transition probability kernel from  $(E, \mathcal{E})$  into  $(F, \mathcal{F})$ . Suppose that  $N$  is Poisson on  $(E, \mathcal{E})$  with mean  $\nu$ , and that  $\varphi$  is independent of  $N$  and is an independency of variables  $\varphi x$  with distributions 3.21. Then,  $\hat{N}$  is Poisson on  $(F, \mathcal{F})$  with mean  $\nu Q$ , and  $M$  is Poisson on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$  with mean  $\nu \times Q$ .*

3.27 **REMARKS.** a) The conditions of the theorem imply some side conclusions:  $N$  exists and is a random counting measure by Theorems 2.15 and 2.16; and there exist random variables  $X_1, X_2, \dots$  taking values in  $(\bar{E}, \bar{\mathcal{E}})$  with  $\bar{E} = E \cup \{\partial\}$  such that  $N$  is formed by them, this is by Exercise 1.18. Moreover, since  $(F, \mathcal{F})$  is standard, Kolmogorov's extension theorem IV.4.18 applies to show that  $\varphi$  exists as an independency indexed by  $E$ .

b) The condition that  $\nu$  be diffuse is to ensure that  $N$  be a random counting measure. Otherwise, the claims of the theorem are false. The reason is that, if  $\nu$  has an atom  $x$ , then  $N$  will have a (Poisson distributed) number of stones at  $x$  and all those stones will be transferred to the same random point  $\varphi x$  in  $F$ . This makes  $\hat{N}$  not Poisson.

*Proof.* Let  $\bar{E} = E \cup \{\partial\}$  and  $\bar{F} = F \cup \{\Delta\}$ , and let  $\bar{\mathcal{E}}$  and  $\bar{\mathcal{F}}$  be the  $\sigma$ -algebras on  $\bar{E}$  and  $\bar{F}$  respectively, generated by  $\mathcal{E}$  and  $\mathcal{F}$  respectively. As remarked in 3.27a, there exist random variables  $X_1, X_2, \dots$  taking values in  $(\bar{E}, \bar{\mathcal{E}})$  that form  $N$  on  $(E, \mathcal{E})$ . Define  $Y_n$  by setting  $Y_n(\omega) = \varphi(\omega, X_n(\omega))$  after extending  $\varphi$  onto  $\bar{E}$  by letting  $\varphi(\omega, \partial) = \Delta$  for all  $\omega$ . The joint measurability of  $(\omega, x) \mapsto \varphi(\omega, x)$  ensures that  $Y_1, Y_2, \dots$  are random variables taking values in  $(\bar{F}, \bar{\mathcal{F}})$ . Now,  $(X, Y)$  satisfy all the conditions of Theorem 3.19, and the proof is immediate.  $\square$

## Exercises and complements

3.28 *Heuristics.* This is to give an informal “proof” of Theorem 3.19 at least for the case where  $\nu$  is finite. Take a Poisson distributed number of stones with mean  $c$ . Throw each into  $E$  as in Example 2.6; the resulting configuration is Poisson with mean measure  $\nu = c\lambda$ . Next, take each stone in  $E$  and throw into  $F$ , independently of all others, so that the stone at the point  $x$  of  $E$  lands in the subset  $B$  of  $F$  with probability  $Q(x, B)$ . The resulting configuration of stones in  $F$  must form a Poisson random measure with mean  $\nu Q$ , because the net effect of all the stone throwing is that a Poisson distributed number of stones with mean  $c$  got thrown into  $F$  according to the distribution  $\lambda Q$ . Replacing  $F$  by  $E \times F$  we also get the marking result.

3.29 *Marked point processes.* Let  $X$  and  $Y$  be as in the setup leading to Theorem 3.2. Consider the atoms  $X_i$  as the points of a point process on  $E$ , and regard each  $Y_i$  as a *mark* associated with the corresponding atom  $X_i$ . Then, some authors refer to  $(X, Y)$  as a *marked point process* on  $E$  with mark space  $F$ .

3.30 *Random decompositions.* This is the converse to the superposition described in Exercise 2.29. Let  $X = \{X_i : i \in I\}$  form a Poisson random measure  $N$  on  $(E, \mathcal{E})$  with mean  $\nu$ . Suppose that each atom  $X_i$  has a mark  $Y_i$ , the latter being  $m$  with probability  $p_m(x)$  if the atom is at  $x$ , where  $x \in E$  and  $m \in F = \{1, 2, \dots\}$ . Let  $N_m$  be the random measure on  $(E, \mathcal{E})$  formed by the atoms of  $N$  marked  $m$ . Formulate this story in precise terms

and supply the missing assumptions. Show that  $N_1, N_2, \dots$  are independent Poisson random measures with means  $\nu_1, \nu_2, \dots$  respectively, where

$$\nu_m(dx) = \nu(dx)p_m(x), \quad x \in E, \quad m \in F.$$

**3.31 Continuation.** Arrivals of male and female customers at a store form independent Poisson random measures  $N_1$  and  $N_2$  on  $\mathbb{R}_+$  with intensities  $a$  and  $b$  respectively, that is, the mean of  $N_1$  is  $\nu_1 = a$  Leb, and of  $N_2$  is  $\nu_2 = b$  Leb. What is the probability that exactly 5 males arrive during the interval from 0 to the time of first female arrival. Answer:  $p^5(1-p)$ , where  $p = a/(a+b)$ , obviously!

**3.32 Translations.** Let  $X$  and  $Y$  be as in the setup preceding 3.2, but with  $E = F = \mathbb{R}^d$ . Suppose that the conditions of Corollary 3.5 are satisfied. Show that, then,  $\{X_i + Y_i : i \in I\}$  forms a Poisson random measure  $\hat{N}$  on  $\mathbb{R}^d$  with mean  $\hat{\nu} = \nu * \pi$ , the last being the convolution of  $\nu$  and  $\pi$ , that is,

$$\hat{\nu}f = \int_E \nu(dx) \int_F \pi(dy) f(x+y).$$

Show that, if  $\nu = c$  Leb for some constant  $c$ , then  $\hat{\nu} = \nu$ . Much of the sub-section on closed particle systems can be reduced to this case.

**3.33 Particle systems with birth and death.** In the setup of the corresponding subsection, suppose that the law  $P$  of the  $Y_i$  gives

$$\pi_t(A) = P\{w : w(t) \in A\} = \mathbb{P}\{Y_0(t) \in A\}, \quad A \in \mathcal{E},$$

for  $t \in \mathbb{R}_+$ . Here  $\pi_t$  is a defective probability measure, the defect  $1 - \pi_t(E)$  being the probability that the particle died before  $t$ . Let  $f \in \mathcal{E}_+$ , and interpret  $f(x)$  as the rate, per unit time, at which a particle pays “rent” when its position is  $x$ ; of course,  $f(\partial) = 0$  extends  $f$  onto  $\bar{E}$ . Then,

$$W_t = \sum_i \int_0^t ds \, e^{-rs} f(Y_i(s - T_i)) \, 1_{(-\infty, s]} \circ T_i$$

is the total rent paid by all the particles during  $[0, t]$ , discounted at rate  $r$ . Compute the expected value and variance of  $W_t$  and of  $W_\infty$ .

**3.34 Compound Poisson random measures.** Let  $X$  and  $Y$  satisfy the setup and conditions of Corollary 3.5. Then,  $(X, Y)$  forms a Poisson random measure  $M$  on  $(E \times F, \mathcal{E} \otimes \mathcal{F})$  with mean  $\mu = \nu \times \pi$ . Now, take  $F = \mathbb{R}_+$  and define

$$L(A) = \int_{A \times F} M(dx, dy) \, y, \quad A \in \mathcal{E}.$$

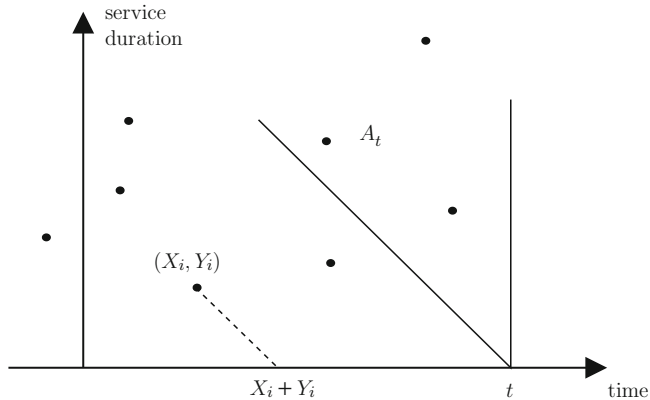


Figure 8: The pairs  $(X_i, Y_i)$  form a Poisson random measure,  $Q_t$  is the number of them in the wedge  $A_t$ .

This defines a random measure  $L$  on  $(E, \mathcal{E})$ ; it is said to be a compound Poisson random measure. This is a slight generalization of the processes with the similar name.

a) Show that  $L(A_1), \dots, L(A_n)$  are independent whenever  $A_1, \dots, A_n$  are disjoint sets in  $\mathcal{E}$ .

b) Compute the mean and the Laplace functional of  $L$ .

**3.35 Infinite server queues.** Consider a service facility with an infinite number of servers, a Poisson process of customer arrivals with intensity  $a$ , and independent service times with a common distribution  $\pi$ . Let  $X_i$  be the arrival time of the customer labeled  $i$ , let  $Y_i$  be the corresponding service time. It is assumed that  $X$  and  $Y$  satisfy the conditions of Corollary 3.5 with  $E = \mathbb{R}$ ,  $F = \mathbb{R}_+$ ,  $\nu = a$  Leb, and  $\pi$  as the service distribution here.

a) Show that the departure times  $X_i + Y_i$  form a Poisson random measure on  $\mathbb{R}$  with mean  $\nu = a$  Leb.

b) Let  $Q_t$  be the number of customers in the system at time  $t$ ; this is the number of pairs  $(X_i, Y_i)$  in the wedge  $A_t = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : x \leq t < x + y\}$ . What is the distribution of  $Q_t$ ? Describe the joint distribution of  $Q_s$  and  $Q_t$  for times  $s < t$ ; what is the covariance. See Figure 8 above.

c) Suppose that the customer  $i$  pays an amount  $f(Y_i)$  at his departure time  $X_i + Y_i$ . Consider the present worth  $W$  of all payments after time 0 if the discount rate is  $r$ , that is,

$$W = \sum_i f(Y_i) e^{-r(X_i + Y_i)} 1_{\mathbb{R}_+}(X_i + Y_i).$$

Compute the mean, variance, and Laplace transform of  $W$ .

d) Re-do the answers assuming that the arrival process has intensity  $a(x)$  at time  $x$  and the service distribution is  $\pi(x, \cdot)$  depending on the arrival time  $x$ .



3.36 *Continuation.* Think of the service facility as a large national park. Customer  $i$  arrives at time  $X_i$  and stays in the park for  $Y_i$  units of time and, during the interval  $[X_i, X_i + Y_i)$  the customer moves in the forest  $F$  according to some process  $Z_i = \{Z_i(u) : 0 \leq u \leq Y_i\}$ . Let  $Q_t(B)$  be the number of customers in the subset  $B$  of  $F$  at time  $t$ . Characterize  $Q_t$  under reasonable assumptions regarding the motions  $Z_i$ .

3.37 *Traffic flow.* This is a reasonable account of low density traffic flows. It applies well in situations where vehicles can pass each other freely, and, therefore, the speed of a vehicle remains nearly constant over time and does not depend on the speeds or positions of other vehicles. Under these conditions, it can be shown that the configuration of vehicles on the road approximates a Poisson configuration.

We take the road to be  $\mathbb{R}$ . Vehicles are labeled with the integers,  $X_i$  is the position at time 0 of the vehicle  $i$ , and  $V_i$  is its velocity (random, but constant over time). We suppose that  $X = \{X_i : i \in \mathbb{Z}\}$  forms a Poisson random measure on  $\mathbb{R}$  with mean  $\nu = c \text{ Leb}$ , and that  $V = \{V_i : i \in \mathbb{Z}\}$  is independent of  $X$  and is an independency of variables with the same distribution  $\pi$  on  $\mathbb{R}$ .

a) Describe the random measure  $N_t$  representing the configuration of vehicles at time  $t$ , that is,  $N_t$  is formed by  $X_i + tV_i$ ,  $i \in \mathbb{Z}$ .

b) Fix a point  $x$  on the highway. Let  $M_x$  be formed by the times  $T_i$  at which vehicles pass by the point  $x$ . Show that it is Poisson on  $\mathbb{R}$  with intensity  $a = cb$  where  $b = \mathbb{E} |V_i|$ .

c) Consider the relative motions of the vehicles with respect to a marked (observer's) vehicle. Suppose that the marked vehicle is at  $x_0$  at time 0 and travels with the constant deterministic velocity  $v_0$  in  $(0, \infty)$ . Assume that all the  $V_i$  are positive (that is, we are considering the traffic moving in the positive direction). Let  $M_a$  be formed by the times at which vehicles with speeds above  $v_0$  pass the marked one, and  $M_b$  by the times at which the marked vehicle passes the vehicles with speeds below  $v_0$ . Show that  $M_a$  and  $M_b$  are independent Poisson random measures on  $\mathbb{R}_+$ . Find their mean measures. Safety hint: What should  $v_0$  be in order to minimize the expected rate of passings of both kinds?

3.38 *Continuation.* We now put in entrances and exits on the highway. The vehicle  $i$  enters the highway at time  $T_i$  at the point  $X_i$ , moves with speed  $V_i$ , and exits at point  $Y_i$ . It is reasonable to assume that the  $T_i$  form a Poisson random measure with mean  $\mu(dt)$ , the pairs  $(X_i, Y_i)$  have a joint distribution  $Q(t, dx, dy)$  depending on time  $t = T_i$ , and that the  $V_i$  has the distribution  $R(t, x, y, dv)$  depending on  $T_i = t$ ,  $X_i = x$ ,  $Y_i = y$ . Make these precise. Do the same problems as in 3.37.

3.39 *Stereology.* A piece of metal contains chunks of some foreign substance distributed in it at random. A plane section of the metal is inspected and the disks of foreign substance seen are counted and their radii are noted. The problem is to infer, from such data, the volume occupied by the foreign

substance. A similar problem arises in studies of growth of cancerous tissue. Mice are injected with carcinogens when they are a few days old, and their livers are taken out for inspection a few weeks after. Each liver is sliced, and sizes and locations of cancerous parts (seen on that plane) are measured. The problem is to infer the volume of liver occupied by the cancerous tissue. This exercise is about the essential issues.

Let  $X$  and  $Y$  be as in Corollary 3.5, but with  $E = \mathbb{R}^3$ ,  $F = \mathbb{R}_+$ ,  $\nu = c \text{ Leb}$ , and  $\pi$  a distribution on  $\mathbb{R}_+$ . We replace  $Y$  with  $R$  for our present purposes. According to the corollary, then,  $(X, R)$  forms a Poisson random measure  $M$  on  $\mathbb{R}^3 \times \mathbb{R}_+$  with mean  $\mu = c \text{ Leb} \times \pi$ . We think of an atom  $(X_i, R_i)$  as a ball of radius  $R_i$  centered at  $X_i$ .

Consider the intersections of these balls with the plane  $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ . A ball with center  $(x, y, z)$  and radius  $r$  intersects this plane if and only if  $r > |z|$ , and if it does, then the intersection is a disc of center  $(x, y)$  and radius  $q = \sqrt{r^2 - z^2}$ . Let  $L$  be the random measure on  $\mathbb{R}^2 \times \mathbb{R}_+$  formed by such “disks”. Show that  $L$  is Poisson with mean measure  $bc \text{ Leb} \times \lambda$ , where  $b$  is the mean radius of a ball and  $\lambda$  is the probability measure (for the disk radius)

$$\lambda(dq) = \frac{1}{b} \text{Leb}(dq) \int_q^\infty \pi(dr) \frac{q}{\sqrt{r^2 - q^2}}.$$

Let  $V_t$  be the total volume of all the balls whose centers are within a distance  $t$  from the origin, and let  $A_t$  be the total area of all the disks on the plane  $z = 0$  whose centers are within a distance  $t$  from the origin. Compute the ratio  $(\mathbb{E} V_t)/(\mathbb{E} A_t)$ .

**3.40 Poisson fields of lines.** Consider a system of random lines in  $\mathbb{R}^2$ . We are interested in the sizes and shapes of polygons formed by those lines. In Exercise 3.37 on traffic flow, if we draw the paths of all the vehicles on a time  $\times$  space coordinate system, the resulting collection of lines would be an example. Since each line corresponds to an atom  $(X_i, V_i)$  of a Poisson random measure, it is appropriate to call the system a Poisson field of lines. In metallurgy, in discussions of sizes and shapes of grains that make up the granular structure of the metal, the grains are approximated by polygons formed by such lines (the model is poor, but it seems to be the only tractable one).

Let  $g$  be an (infinite straight) line in  $\mathbb{R}^2$ . By its *distance* from the origin we mean the length of the perpendicular drawn from the origin to the line. By its *orientation* is meant the angle that the perpendicular makes with the  $x$ -axis. Orientation is an angle between 0 and  $\pi$ , distance is positive or negative depending on whether the perpendicular is above or below the  $x$ -axis. Note that, if  $g$  has distance  $d$  and orientation  $\alpha$  then

$$(x, y) \in g \iff d = x \cos \alpha + y \sin \alpha.$$

Let  $(D_i)$  form a Poisson random measure  $N$  on  $\mathbb{R}$  with mean measure  $\nu = \pi c \text{ Leb}$  where  $c > 0$  is fixed; let  $(A_i)$  be independent of  $(D_i)$ , and the  $A_i$  be independent of each other with uniform distribution on  $[0, \pi]$ .

a) Describe the random measure  $M$  formed by the pairs  $(D_i, A_i)$  on  $\mathbb{R} \times [0, \pi]$ . The lines  $G_i$  corresponding to the pairs  $(D_i, A_i)$  are said to form a Poisson field of lines with intensity  $c$ .

b) Let  $X_i$  be the intersection of the line  $G_i$  with the  $x$ -axis and  $B_i$  the angle (between 0 and  $\pi$ ) between  $G_i$  and the  $x$ -axis. Show that  $(X_i, B_i)$  form a Poisson random measure on  $\mathbb{R} \times [0, \pi]$  with mean measure  $2c \, dx \, \beta(db)$ , where  $\beta$  is the distribution  $\beta(db) = \frac{1}{2}(1 - \cos b)$ ,  $0 \leq b \leq \pi$ .

c) The Poisson field of lines  $G_i$  is invariant in law under translations and rotations of the plane. Show this by considering the translation  $(x, y) \rightarrow (x_0 + x, y_0 + y)$  for some fixed  $(x_0, y_0) \in \mathbb{R}^2$ , and then by considering the rotation of the plane by an angle  $\alpha_0$ .

d) The number  $K$  of random lines intersecting a fixed convex region  $C \subset \mathbb{R}^2$  with perimeter  $p$  has the Poisson distribution with mean  $cp$ . Show this. Hints: Whether a line  $G_i$  intersects  $C$  is merely a function of  $(D_i, A_i)$ . If  $G_i$  intersects  $C$ , it intersects  $C$  twice. So  $2K$  is the total number of intersections. Suppose  $C$  is a polygon with  $n$  sides of lengths  $p_1, \dots, p_n$  (then the perimeter is  $p = p_1 + \dots + p_n$ ). Use the results of (b) and (c) above to compute the expected numbers of intersections with each side. The sum of these numbers is  $2\mathbb{E} K$ . Finally, approximate  $C$  by polygons.

e) This requires much knowledge of geometry. The lines  $G_i$  partition the plane into random polygons (some of these are triangles, some have four sides, some 17, etc.). Take an arbitrary one of these (can you make this precise so that the following are random variables?). Let  $S$  be the number of sides,  $P$  the perimeter,  $A$  the area,  $D$  the diameter of the in-circle. Then,  $D$  has the exponential distribution with parameter  $\pi c$ , and  $\mathbb{E} S = 4$ ,  $\mathbb{E} P = 2/c$ ,  $\mathbb{E} A = 1/\pi c^2$ .

## 4 ADDITIVE RANDOM MEASURES AND LÉVY PROCESSES

Our aim is to illustrate the uses of Poisson random measures to construct more evolved random measures. These are related intimately to Lévy processes to be studied in the next chapter; we give a few examples here.

Throughout,  $(E, \mathcal{E})$  will be a fixed measurable space. If  $\mu$  is a measure on a measurable space  $(F, \mathcal{F})$ , we shall omit mentioning the  $\sigma$ -algebra and merely say that  $\mu$  is a measure on  $F$ . This convention extends to random measures naturally. Indeed, on spaces such as  $\mathbb{R}_+$  or  $\mathbb{R}^d$ , we shall omit specifying the  $\sigma$ -algebra; they will always be the Borel  $\sigma$ -algebras.

**4.1 DEFINITION.** Let  $M$  be a random measure on  $E$ . It is said to be additive if  $M(A), \dots, M(B)$  are independent for all choices of the finitely many disjoint sets  $A, \dots, B$  in  $\mathcal{E}$ .

Every deterministic measure is additive. Every Poisson random measure is additive. We shall see shortly that archetypical additive random measures on  $E$  are constructed from Poisson random measures on  $E \times \mathbb{R}_+$ .

The probability law of an additive random measure  $M$  is specified once the distribution of  $M(A)$  is specified for each  $A$  in  $\mathcal{E}$ . To see this, note that the joint distribution of  $M(B)$  and  $M(C)$  can be computed from the marginal distributions of  $M(B \setminus C)$ ,  $M(B \cap C)$ ,  $M(C \setminus B)$  using the independence of the last three variables, and that this argument extends to finite-dimensional distributions of  $M$ .

## Construction of additive measures

The following seems to require no proof. It shows the construction of a purely atomic additive random measure whose atoms are fixed; only the weights on the atoms are random.

**4.2 LEMMA.** *Let  $D$  be a countable subset of  $E$ , and let  $\{W_x : x \in D\}$  be an independency of positive random variables. Define*

$$K(\omega, A) = \sum_{x \in D} W_x(\omega) I(x, A), \quad \omega \in \Omega, \quad A \in \mathcal{E}.$$

*Then,  $K$  is an additive random measure.*

At the other extreme, a Poisson counting measure has weight one on each of its atoms, but the atoms themselves are generally random. The following is the construction of additive measures of general interest.

**4.3 LEMMA.** *Let  $N$  be a Poisson random measure on  $E \times \mathbb{R}_+$  with mean measure  $\nu$ . Define*

$$L(\omega, A) = \int_{A \times \mathbb{R}_+} N(\omega; dx, dz) z, \quad \omega \in \Omega, \quad A \in \mathcal{E}.$$

*Then,  $L$  is an additive random measure on  $E$ . The Laplace transform for  $L(A)$  is, for  $A$  in  $\mathcal{E}$ ,*

$$\mathbb{E} e^{-rL(A)} = \exp_- \int_{A \times \mathbb{R}_+} \nu(dx, dz) (1 - e^{-rz}), \quad r \in \mathbb{R}_+.$$

*Proof.* By Fubini's theorem,  $L$  is a random measure. Note that  $L(A)$  is determined by the trace of  $N$  on  $A \times \mathbb{R}_+$ . If  $A, \dots, B$  are finitely many disjoint sets in  $\mathcal{E}$ , then the traces of  $N$  over  $A \times \mathbb{R}_+, \dots, B \times \mathbb{R}_+$  are independent by Exercise 2.31, and, hence,  $L(A), \dots, L(B)$  are independent. So,  $L$  is additive. The formula for the Laplace transform follows from Theorem 2.9 on the Laplace functional of  $N$  after noting that  $rL(A) = Nf$  with  $f(x, z) = rz1_A(x)$ .  $\square$

**4.4 THEOREM.** *Let  $\alpha$  be a deterministic measure on  $E$ . Let  $K$  be as in Lemma 4.2 and  $L$  as in Lemma 4.3, and suppose that  $K$  and  $L$  are independent. Then,*

$$M = \alpha + K + L$$

*is an additive random measure on  $E$ .*

*Proof.* It is immediate from the lemmas above and the observation that the sum of independent additive random measures is again additive.  $\square$

Conversely, it can be shown that the preceding is, basically, the general form of an additive random measure. More precisely, if  $M$  is an additive random measure on a standard measurable space  $(E, \mathcal{E})$  and it is  $\Sigma$ -bounded as a kernel, then it has the decomposition  $M = \alpha + K + L$  with the components as described in the preceding theorem; moreover, this decomposition is unique provided that both  $\alpha$  and  $\nu(\cdot \times \mathbb{R}_+)$  be diffuse measures on  $E$ , that is, all the fixed atoms (if any) belong to  $K$ .

## Increasing Lévy processes

This is to establish a connection between additive random measures and the processes to be studied in the next chapter in greater generality. We start by introducing an important class of such processes.

**4.5 DEFINITION.** Let  $S = (S_t)_{t \in \mathbb{R}_+}$  be an increasing right-continuous stochastic process with state space  $\mathbb{R}_+$  and  $S_0 = 0$ . It is said to be an increasing Lévy process (or subordinator) if

- a) the increments  $S_{t_1} - S_{t_0}, S_{t_2} - S_{t_1}, \dots, S_{t_n} - S_{t_{n-1}}$  are independent for  $n \geq 2$  and  $0 \leq t_0 < t_1 < \dots < t_n$ , and
- b) the distribution of the increment  $S_{t+u} - S_t$  is the same as that of  $S_u$  for every  $t$  and  $u$  in  $\mathbb{R}_+$ .

The property (a) is called the independence of increments, and (b) the stationarity of the increments. More general Lévy processes are defined by the properties (a) and (b), but for right-continuous left-limited processes with state spaces  $\mathbb{R}^d$ ; see Chapter VII.

Given an additive random measure  $M$  on  $\mathbb{R}_+$ , putting  $S_t(\omega) = M(\omega, [0, t])$  yields an increasing right-continuous process. Once we assure that  $S_t < \infty$  almost surely for all  $t$ , independence of increments follows from the additivity of  $M$ . Stationarity of increments is achieved by making sure that the mean measure is chosen appropriately and there be no fixed atoms and the deterministic measure  $\alpha$  be a constant multiple of the Lebesgue measure. In other words, the following proposition is in fact a complete characterization of increasing Lévy processes; here we state and prove the sufficiency part, the necessity will be shown in the next chapter; see also 4.13 below.

**4.6 PROPOSITION.** Let  $b$  be a constant in  $\mathbb{R}_+$ . Let  $N$  be a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  whose mean has the form  $\nu = \text{Leb} \times \lambda$ , where the measure  $\lambda$  satisfies

$$4.7 \quad \int_{\mathbb{R}_+} \lambda(dz) (z \wedge 1) < \infty.$$

Define

$$S_t(\omega) = bt + \int_{[0,t] \times \mathbb{R}_+} N(\omega; dx, dz) z, \quad t \in \mathbb{R}_+, \quad \omega \in \Omega.$$

Then,  $S = (S_t)_{t \in \mathbb{R}_+}$  is an increasing Lévy process, and

$$4.8 \quad \mathbb{E} e^{-rS_t} = \exp_{-} t \left[ br + \int_{\mathbb{R}_+} \lambda(dz)(1 - e^{-rz}) \right], \quad r \in \mathbb{R}_+.$$

REMARK. Let  $L$  be defined on  $E = \mathbb{R}_+$  as in Lemma 4.3 from the Poisson random measure  $N$  here. Let  $\alpha = b \text{ Leb}$  on  $E = \mathbb{R}_+$ . Then,  $M = \alpha + L$  is an additive random measure on  $\mathbb{R}_+$ , and we have  $S_t(\omega) = M(\omega, [0, t])$  for all  $t$  and  $\omega$ .

*Proof.* It is obvious that  $S$  is increasing, right-continuous, and  $S_0 = 0$ . Note that  $S_t = bt + Nf$ , where  $f(x, z) = z1_{[0,t]}(x)$ . In view of 4.7,

$$\nu(f \wedge 1) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \nu(dx, dz)(f(x, z) \wedge 1) = t \int_{\mathbb{R}_+} \lambda(dz)(z \wedge 1)$$

is finite, which shows that  $Nf$  and, thus,  $S_t$  are almost surely finite. Now, the increments are well-defined, and their independence follows from the Poisson nature of  $N$ . The formula for the Laplace transform is immediate from Theorem 2.9 with the present  $\nu$  and  $f$ . Finally, a similar computation shows that the Laplace transforms of  $S_{t+u} - S_t$  and  $S_u$  are the same, and hence the stationarity of increments.  $\square$

The constant  $b$  is called the drift coefficient, and the measure  $\lambda$  the *Lévy measure*, of the process  $S$ . Obviously, together, they determine the probability law of  $S$ . We shall give two particular examples of such  $\lambda$  below. For the present, we add that every finite measure  $\lambda$  on  $\mathbb{R}_+$  satisfies the condition 4.7 for the Lévy measure.

## Examples

4.9 *Gamma processes.* Let  $S$  be as in the last proposition with  $b = 0$  and

$$\lambda(dz) = dz \cdot a \frac{e^{-cz}}{z}, \quad z > 0$$

for some constants  $a$  and  $c$  in  $(0, \infty)$ ; note that 4.7 is satisfied. Thus,  $S$  is an increasing Lévy process. Now, 4.8 becomes

$$\begin{aligned} \mathbb{E} e^{-rS_t} &= \exp_{-} t \int_0^\infty dz a \frac{e^{-cz}}{z} (1 - e^{-rz}) \\ &= \exp_{-} t a \log \frac{c+r}{c} = \left( \frac{c}{c+r} \right)^{at}; \end{aligned}$$

thus,  $S_t$  has the gamma distribution with shape index  $at$  and scale parameter  $c$ . For this reason,  $S$  is said to be a *gamma process with shape rate  $a$*  and scale parameter  $c$ .

**4.10 Increasing stable processes.** Let  $S$  be as in Proposition 4.6 again, but with  $b = 0$  and

$$\lambda(dz) = dz \frac{ac}{\Gamma(1-a)} z^{-1-a}, \quad z > 0,$$

where  $a$  is a constant in  $(0, 1)$ , and  $c$  is a constant in  $(0, \infty)$ , and  $\Gamma$  denotes the gamma function. Again,  $\lambda$  satisfies the condition 4.7, and  $S$  is an increasing Lévy process. Even though  $S_t < \infty$  almost surely for every  $t$  in  $\mathbb{R}_+$ ,

$$\mathbb{E} S_t = t \int_{\mathbb{R}_+} \lambda(dz)z = t \cdot (+\infty) = +\infty$$

for every  $t > 0$ , however small  $t$  may be. This process  $S$  is said to be *stable with index  $a$*  in  $(0, 1)$ . *Stability* refers to the fact that  $(S_{ut})_{t \in \mathbb{R}_+}$  has the same probability law as  $(u^{1/a} S_t)_{t \in \mathbb{R}_+}$  for every  $u > 0$ ; this can be seen by recalling that the probability law of an additive measure is specified by its one-dimensional distributions and that

$$\mathbb{E} e^{-rS_t} = e^{-tcr^a}, \quad t \in \mathbb{R}_+, \quad r \in \mathbb{R}_+,$$

in view of the formula 4.8 and the form of  $\lambda$  here.

The distribution of  $S_t$  does not have an explicit form in general. However, for  $a = 1/2$ , we have

$$\mathbb{P}\{S_t \in dz\} = dz \cdot \frac{ct}{\sqrt{4\pi z^3}} e^{-c^2 t^2 / 4z}, \quad z > 0.$$

The further special case where  $a = 1/2$  and  $c = \sqrt{2}$  plays an important role in the theory of Brownian motion: as can be seen in Proposition V.5.20, there,  $S_x$  becomes the time it takes for the Wiener process to go from 0 to  $x$ .

**4.11 Stable random measures.** These are generalizations of the preceding example. Let  $L$  be as in Lemma 4.3, but with  $E = \mathbb{R}^d$  for some dimension  $d \geq 1$ . Assume that the mean of the Poisson random measure there has the form  $\nu = \text{Leb} \times \lambda$ , where the measure  $\lambda$  on  $\mathbb{R}_+$  is as in Example 4.10 above. The formula for the Laplace transform in 4.3 becomes

$$\mathbb{E} e^{-rL(A)} = \exp_{-}(\text{Leb } A)cr^a, \quad r \in \mathbb{R}_+.$$

If  $\text{Leb } A = \infty$ , then  $L(A) = +\infty$  almost surely. If  $\text{Leb } A < \infty$ , then  $L(A) < \infty$  almost surely. But  $\mathbb{E} L(A) = +\infty$  for every  $A$  with  $\text{Leb } A > 0$ , however small its measure may be. This additive random measure on  $\mathbb{R}^d$  is *stable* in the following sense: Let  $u$  be a constant in  $(0, \infty)$  and note that the Lebesgue measure of  $uA = \{ux : x \in A\}$  is equal to  $u^d \text{Leb } A$ . Thus, the Laplace transform of  $\hat{L}(A) = u^{-d/a} L(uA)$  is the same as that of  $L(A)$ . This means

that the probability laws of the additive random measures  $L$  and  $\hat{L}$  are the same. In other words, the probability law of  $L$  remains *stable* under the transformation  $(x, z) \mapsto (ux, u^{-d/a}z)$  of  $\mathbb{R}^d \times \mathbb{R}_+$  into itself, this being true for every scalar  $u > 0$ .

**4.12 Gamma random measures.** These are counterparts, on arbitrary spaces, of the processes of Example 4.9. Let  $L$  and  $N$  be as in Lemma 4.3 with  $(E, \mathcal{E})$  arbitrary, but the mean  $\nu$  of  $N$  having the form  $\nu = \mu \times \lambda$ , with  $\mu$  an arbitrary measure on  $E$ , and  $\lambda$  the measure on  $\mathbb{R}_+$  given in Example 4.9. Then,  $L$  is an additive random measure on  $E$ , and

$$\mathbb{E} e^{-rL(A)} = \exp_{-} \mu(A) \int_{\mathbb{R}_+} \lambda(dz)(1 - e^{-rz}) = \left( \frac{c}{c+r} \right)^{a\mu(A)}.$$

If  $\mu(A) = +\infty$  then  $L(A) = \infty$  almost surely. If  $\mu(A) < \infty$ , then  $L(A)$  is almost surely finite and has the gamma distribution with shape index  $a\mu(A)$  and scale  $c$ . Example 4.9 is, basically, the special case where  $E = \mathbb{R}_+$  and  $\mu = \text{Leb}$ .

## Homogeneity and stationarity

Suppose that  $E$  is  $\mathbb{R}_+$  or  $\mathbb{R}^d$ , and  $\mathcal{E}$  is the corresponding Borel  $\sigma$ -algebra. An additive random measure on  $E$  is said to be *homogeneous* if its probability law remains invariant under shifts of the origin in  $E$ . If  $E$  is  $\mathbb{R}_+$  or  $\mathbb{R}$  and is regarded as time, the term *stationary* is preferred instead.

Let  $M$  be an additive random measure on  $E$ . Then, its probability law is determined by specifying the distribution of  $M(A)$  for each  $A$  in  $\mathcal{E}$ . Thus, homogeneity of  $M$  is equivalent to requiring that, for every  $A$  in  $\mathcal{E}$ , the distribution of  $M(x+A)$  remains the same while  $x$  ranges over  $E$ ; here and below,  $x+A = \{x+y : y \in A\}$ . This observation is the primary ingredient in the proof of the following proposition, which is a slightly more general version of random measures associated with increasing Lévy processes of Proposition 4.6 above. Recall that the kernel  $M$  is  $\sigma$ -bounded if there is a countable partition  $(A_n)$  of  $E$  such that  $M(A_n) < \infty$  almost surely for each  $n$ .

**4.13 PROPOSITION.** *Suppose that  $E$  is  $\mathbb{R}_+$  or  $\mathbb{R}^d$ . Let  $M$  be a  $\sigma$ -bounded additive random measure on  $E$  as in Theorem 4.4. Then,  $M$  is homogeneous if and only if it has the form*

$$M(\omega, A) = b \text{ Leb } A + \int_{A \times \mathbb{R}_+} N(\omega; dx, dz)z, \quad \omega \in \Omega, \quad A \in \mathcal{E},$$

for some constant  $b$  in  $\mathbb{R}_+$  and some Poisson random measure  $N$  on  $E \times \mathbb{R}_+$  with mean  $\nu = \text{Leb} \times \lambda$ , where  $\lambda$  satisfies 4.7.

*Proof. Sufficiency.* Let  $M$  be as described. It is obviously additive. To show that  $M$  is  $\sigma$ -bounded, it is enough to show that  $M(A) < \infty$  almost surely for every  $A$  in  $\mathcal{E}$  with  $\text{Leb } A < \infty$ ; then, the  $\sigma$ -finiteness of the Lebesgue



measure does the rest. So, let  $A$  be such that  $c = \text{Leb } A$  is finite. Note that  $M(A) = bc + Nf$  where  $f(x, z) = z1_A(x)$ , and that

$$\nu(f \wedge 1) = c \int_{\mathbb{R}_+} \lambda(dz)(z \wedge 1) < \infty$$

by condition 4.7 on  $\lambda$ . Thus,  $Nf < \infty$  almost surely by Proposition 2.13, and thus  $M(A) < \infty$  almost surely. Finally,  $M$  is homogeneous since  $M(A)$  and  $M(x + A)$  have the same distribution: we have

$$\mathbb{E} e^{-rM(A)} = \exp_{-}(\text{Leb } A) \left( br + \int_{\mathbb{R}_+} \lambda(dz)(1 - e^{-rz}) \right),$$

and  $\text{Leb } (x + A) = \text{Leb } A$ .

*Necessity.* Suppose that  $M = \alpha + K + L$  as in Theorem 4.4 and, in addition, is homogeneous and  $\sigma$ -bounded. The homogeneity has two consequences:

First,  $M$  cannot have fixed atoms. If there were a fixed atom  $x_0$  then  $x + x_0$  would be a fixed atom for every  $x$  in  $E$ , which contradicts the  $\sigma$ -boundedness of  $M$  (recall that a  $\sigma$ -finite measure can have at most countably many atoms). Thus,  $K = 0$ , and  $\alpha$  is diffuse, and the Poisson random measure  $N$  defining  $L$  has no fixed atoms.

Second, homogeneity implies that  $M(A)$  and  $M(x + A)$  have the same distribution for all  $x$  and  $A$ . Thus,  $\alpha(A) = \alpha(x + A)$  and  $\nu(A \times B) = \nu((x + A) \times B)$  for all  $x, A, B$  appropriate. These imply that  $\alpha = b \text{Leb}$  for some constant  $b$  in  $\mathbb{R}_+$  and that  $\nu = \text{Leb} \times \lambda$  for some measure  $\lambda$  on  $\mathbb{R}_+$ .

Finally,  $\sigma$ -boundedness of  $M$  implies that there is  $A$  in  $\mathcal{E}$  with  $c = \text{Leb } A$  belonging to  $(0, \infty)$  such that  $M(A) < \infty$  almost surely. Since  $M(A) = bc + Nf$  with  $f(x, z) = z1_A(x)$ , we must have  $Nf < \infty$  almost surely, which in turn implies via Proposition 2.13 that

$$\nu(f \wedge 1) = c \int_{\mathbb{R}_+} \lambda(dz)(z \wedge 1)$$

must be finite. Hence, the condition 4.7 holds on  $\lambda$ . □

## Alternative constructions

Consider an increasing Lévy process  $S$  constructed as in Proposition 4.6. The condition 4.7 on  $\lambda$  implies that  $\lambda(\varepsilon, \infty) < \infty$  for every  $\varepsilon > 0$ , and it follows from Proposition 2.18 that no two atoms of  $N$  arrive at the same time, that is, for almost every  $\omega$ , if  $(t, z)$  and  $(t', z')$  are distinct atoms of the counting measure  $N(\omega, \cdot)$ , then  $t$  and  $t'$  are distinct as well. It follows that, for almost every  $\omega$ , the path  $t \mapsto S_t(\omega)$  has a jump of size  $z$  at time  $t$  if and only if  $(t, z)$  is an atom of  $N(\omega, \cdot)$ .

Fix  $z > 0$  and consider the times at which  $S$  has a jump of size exceeding  $z$ . Those times form a Poisson random measure on  $\mathbb{R}_+$  whose mean measure

is  $\lambda(z, \infty) \cdot \text{Leb}$ ; in other words,  $\lambda(z, \infty)$  is the time rate of jumps of size exceeding  $z$ . More generally,  $\lambda(B)$  is the rate of jumps of size belonging to  $B$ .

In the construction of Proposition 4.6, then, the burden of describing the probabilistic jump structure is on the measure  $\lambda$ , which in turn specifies the probability law of  $N$ . Next, we describe a construction where the law of  $N$  is parameter free and the jump mechanism is more explicit: to every atom  $(t, u)$  of  $N(\omega, \cdot)$  there corresponds a jump of size  $j(u)$  at time  $t$ . We ignore the drift term.

**4.14 PROPOSITION.** *Let  $N$  be a standard Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  (with mean  $\text{Leb} \times \text{Leb}$ ). Let  $j : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a Borel function satisfying*

$$4.15 \quad \int_{\mathbb{R}_+} du \, (j(u) \wedge 1) < \infty.$$

*Define*

$$S_t(\omega) = \int_{[0,t] \times \mathbb{R}_+} N(\omega; dx, du) \, j(u), \quad \omega \in \Omega, \quad t \in \mathbb{R}_+.$$

*Then,  $S = (S_t)$  is an increasing Lévy process with*

$$4.16 \quad \mathbb{E} e^{-rS_t} = \exp -t \int_0^\infty du \, (1 - e^{-rj(u)}), \quad r \in \mathbb{R}_+.$$

*Proof.* Observe that  $S_t = Nf$  where  $f(x, u) = j(u) 1_{[0,t]}(x)$ . The condition 4.15 ensures that  $Nf < \infty$  almost surely. That  $S$  is an increasing Lévy process follows from the Poisson character of  $N$  and the particular mean measure. Finally, the formula 4.16 is immediate from Theorem 2.9 on the Laplace functionals.  $\square$

Let  $j$  be as in the preceding proposition and define

$$4.17 \quad \lambda = \text{Leb} \circ j^{-1}.$$

Then, the condition 4.15 on the jump function  $j$  is equivalent to the condition 4.7 on  $\lambda$ , the Laplace transforms 4.16 and 4.8 are the same, and, hence, the processes  $S$  of Propositions 4.14 and 4.6 have the same probability law. These conclusions remain true when, in the converse direction, we start with the measure  $\lambda$  and define  $j$  so that 4.17 holds. This is easy to do at least in principle: since  $\lambda$  satisfies 4.7, the function  $z \mapsto \lambda(z, \infty)$  is real-valued and decreasing over  $(0, \infty)$ , and defining

$$4.18 \quad j(u) = \inf\{z > 0 : \lambda(z, \infty) \leq u\}, \quad u \in \mathbb{R}_+,$$

we see that 4.17 holds.

4.19 EXAMPLE. *Stable processes.* Let  $\lambda$  be as in Example 4.10 and recall that  $a \in (0, 1)$ . Then,  $\lambda(z, \infty) = c/z^a \Gamma(1-a)$ , and 4.18 yields  $j(u) = \hat{c}u^{-1/a}$  for all  $u > 0$  with  $\hat{c} = (c/\Gamma(1-a))^{1/a}$ . With this  $j$ , the process  $S$  of the last proposition has the same probability law as that of Example 4.10. Incidentally, in the particular case where  $a = 1/2$  and  $c = \sqrt{2}$ , we have

$$\lambda(dz) = dz \frac{1}{\sqrt{2\pi z^3}}, \quad j(u) = \frac{2}{\pi u^2}. \quad \square$$

In Monte Carlo studies, the construction of Proposition 4.14 is preferred over that of Proposition 4.6, because constructing a standard Poisson random measure is easier; see Remark 2.16. More importantly, 4.14 enables us to construct two (or any number of) processes using the same standard Poisson random measure, but with different jump functions for different processes; this is advantageous in making stochastic comparisons.

The practical limitations to the method of the last proposition come from the difficulties in implementing 4.18. For instance, this is the case for the gamma process of Example 4.9 because the tail  $\lambda(z, \infty)$  does not have an explicit expression in that case. Every such instance requires special numerical methods. The following is an analytical solution in the case of gamma processes.

4.20 EXAMPLE. *Gamma processes.* Let  $\lambda(dz) = dz \, ae^{-cz}/z$ ,  $z > 0$ , the same as in Example 4.9. The formula 4.18 for  $j$  is difficult to use because  $\lambda(z, \infty)$  does not have an explicit expression. However, it is possible to write

$$4.21 \quad \lambda = \mu \circ j^{-1}$$

for a pleasant measure  $\mu$  and explicit  $j$ , but  $\mu$  and  $j$  must be defined on  $\mathbb{R}_+ \times \mathbb{R}_+$ : for  $(u, v)$  in  $\mathbb{R}_+ \times \mathbb{R}_+$ , let

$$4.22 \quad \mu(du, dv) = du \, dv \, ace^{-cv}, \quad j(u, v) = e^{-u}v.$$

It is easy to check that 4.21 holds. The measure  $\mu$  can be thought as  $\mu = (a \text{ Leb}) \times \eta$  where  $\eta$  is the exponential distribution on  $\mathbb{R}_+$  with parameter  $c$ . Thus, constructing a Poisson random measure  $N$  on  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\text{Leb} \times \mu$  is easy, and the following construction should be preferred over that in Example 4.9. For motivation and comments on this construction we refer to Exercises 4.29 and 4.30 below.

Let  $N$  be a Poisson random measure on  $\mathbb{R}_+ \times (\mathbb{R}_+ \times \mathbb{R}_+)$  with mean  $\nu = \text{Leb} \times \mu$ . Recalling  $j(u, v) = e^{-u}v$ , define

$$S_t(\omega) = \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}_+} N(\omega; dx, du, dv) \, j(u, v), \quad t \in \mathbb{R}_+, \quad \omega \in \Omega.$$

Then,  $S = (S_t)$  is a gamma process with shape rate  $a$  and scale parameter  $c$ , just as the process  $S$  in Example 4.9.

To see the truth of the last assertion, it is enough to note that  $S$  is an increasing Lévy process obviously, and then note that the Laplace transform of  $S_t$  is

$$\mathbb{E} e^{-rS_t} = \exp_- \int_{[0,t]} dx \int_{\mathbb{R}_+} du \int_{\mathbb{R}_+} dv ace^{-cv} (1 - e^{-re^{-u}v}) = \left( \frac{c}{c+r} \right)^{at}$$

as needed. Another, simpler, way of seeing the result is by noting that

$$S_t = \int_{[0,t] \times \mathbb{R}_+} \hat{N}(dx, dz) z,$$

where  $\hat{N} = N \circ h^{-1}$  with  $h(x, u, v) = (x, e^{-u}v) = (x, j(u, v))$ . Then,  $\hat{N}$  is Poisson with mean  $\nu \circ h^{-1} = (\text{Leb} \times \mu) \circ h^{-1} = \text{Leb} \times \lambda$  since  $\mu \circ j^{-1} = \lambda$ , which shows that  $\hat{N}$  has the same law as the Poisson random measure in Example 4.9.  $\square$

## Exercises

**4.23 Additive measures with fixed atoms.** Let  $K$  be as in Lemma 4.2, but with  $E = [0, 1]$  and  $D$  the set of all strictly positive rational numbers in  $E$ . Suppose that the independent variables  $W_x$ ,  $x \in D$ , are exponentially distributed with means  $m_x$ ,  $x \in D$ , chosen such that  $\sum_{x \in D} m_x = 1$ . Then  $K$  is an additive random measure whose atoms are fixed. Show that  $K$  is almost surely finite. Compute the Laplace functional  $\mathbb{E} e^{-Kf}$  for positive Borel functions  $f$  on  $E$ .

**4.24 Continuation.** Choose the numbers  $m_x$  for rational numbers  $x$  in  $(0, 1]$  such that their sum is equal to 1. For rational  $x$  in  $(n, n+1]$ , define  $m_x = m_{x-n}$ . Let  $W_x$  have the exponential distribution with mean  $m_x$  for every  $x$  in the set  $D$  of all strictly positive rationals, and assume that the  $W_x$  are independent. Let  $K$  be defined as in Lemma 4.2 with  $E = \mathbb{R}_+$  and this  $D$ . Then,  $K$  is an additive random measure on  $\mathbb{R}_+$ . Show that it is  $\sigma$ -bounded. Define  $S_t(\omega) = K(\omega, [0, t])$  for  $\omega$  in  $\Omega$  and  $t$  in  $\mathbb{R}_+$ . Show that  $S = (S_t)$  is a strictly increasing right-continuous process with state space  $\mathbb{R}_+$ . Show that it has independent increments, that is, the condition (a) of Definition 4.5 holds, but the increments are not stationary, that is, the condition (b) does not hold.

**4.25 Continuation.** Let  $S$  be as in the preceding exercise. Let  $f$  be an arbitrary Borel function on  $\mathbb{R}_+$ . Define  $\hat{S}_t(\omega) = f(t) + S_t(\omega)$ . Show that  $\hat{S}$  has independent increments. If  $f$  fails to be right-continuous,  $\hat{S}$  will fail to be right-continuous.

**4.26 Laplace functionals.** Let  $L$  be as in Lemma 4.3. Show that, for every  $f$  in  $\mathcal{E}_+$ ,

$$\mathbb{E} e^{-Lf} = \exp_- \int_{E \times \mathbb{R}_+} \nu(dx, dz) (1 - e^{-zf(x)}).$$

4.27 *Compound Poisson processes.* Show that the condition 4.7 is satisfied by every finite measure  $\lambda$  on  $\mathbb{R}_+$ . Let  $S$  be defined as in Proposition 4.6 but with  $b = 0$  and  $\lambda$  finite with  $\lambda\{0\} = 0$ .

a) Show that the atoms of  $N$  can be labeled as  $(T_1, Z_1), (T_2, Z_2), \dots$  so that  $0 < T_1 < T_2 < \dots$  almost surely. Show that the  $T_i$  form a Poisson random measure on  $\mathbb{R}_+$  with mean equal to  $c$  Leb, where  $c = \lambda(0, \infty)$ ; see 1.10 *et seq.*

b) Show that the paths  $t \mapsto S_t(\omega)$  are increasing step functions; the jumps occur at  $T_1(\omega), T_2(\omega), \dots$ ; the size of the jump at  $T_i(\omega)$  is equal to  $Z_i(\omega)$ ; between the jumps the paths remain constant.

4.28 *Compound Poisson continued.* Let  $N$  be a standard Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  (with mean  $\text{Leb} \times \text{Leb}$ ). Let  $j : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a Borel function with a compact support, that is, there is  $b$  in  $(0, \infty)$  such that  $j$  vanishes outside  $[0, b]$ . Define

$$S_t = \int_{[0, t] \times \mathbb{R}_+} N(dx, du) j(u), \quad t \in \mathbb{R}_+.$$

Show that  $S$  is a compound Poisson process.

4.29 *Gamma distribution.* Let  $(U_i)$  form a Poisson random measure on  $\mathbb{R}_+$  with mean  $a$  Leb; see 1.10 *et seq.* for the terms. Let  $(V_i)$  be independent of  $(U_i)$  and be an independency of exponential variables with parameter  $c$ . Show that

$$X = \sum_i e^{-U_i} V_i$$

has the gamma distribution with shape index  $a$  and scale parameter  $c$ . Hint: Let  $N$  be the Poisson random measure formed by the pairs  $(U_i, V_i)$ , and note that  $X = Nf$  with  $f(u, v) = e^{-u}v$  to compute the Laplace transform of  $X$ .

4.30 *Gamma processes.* Let the pairs  $(T_i, U_i)$  form a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  with mean equal to  $a$  Leb  $\times$  Leb. Let the family  $(V_i)$  be an independency of exponential variables with parameter  $c$ . Assume that the collections  $(V_i)$  and  $(T_i, U_i)$  are independent. Define

$$S_t = \sum_i e^{-U_i} V_i 1_{[0, t]} \circ T_i, \quad t \in \mathbb{R}_+.$$

Show that  $(S_t)$  is a gamma process with shape rate  $a$  and scale parameter  $c$ . This representation is equivalent to that of Example 4.20.

4.31 *Lévy measure as an image.* Let  $\lambda, \mu$ , and  $j$  be as in Example 4.20. Show the claim that  $\lambda = \mu \circ j^{-1}$ . It is possible to change  $\mu$  and  $j$  while keeping intact the relationship  $\lambda = \mu \circ j^{-1}$ . Show this for

- a)  $\mu(du, dv) = du dv e^{-v}$  and  $j(u, v) = e^{-u/a}v/c$ ,  $(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+$ ,
- b)  $\mu(du, dv) = du dv$  and  $j(u, v) = e^{-u/a}(-\log v)/c$ ,  $(u, v) \in \mathbb{R}_+ \times (0, 1)$ .

4.32 *Another construction.* This is the generalization of the foregoing ideas to arbitrary additive random measures. Let  $N$  be a Poisson random measure on some measurable space  $(D, \mathcal{D})$ . Let  $j : D \mapsto \mathbb{R}_+$  be  $\mathcal{D}$ -measurable, and let  $h : D \mapsto E$  be measurable with respect to  $\mathcal{D}$  and  $\mathcal{E}$ . Define

$$L(\omega, A) = \int_D N(\omega, dx) j(x) 1_{A \circ h}(x), \quad \omega \in \Omega, \quad A \in \mathcal{E}.$$

Show that  $L$  is an additive random measure on  $(E, \mathcal{E})$ .

4.33 *Dirichlet random measures.* In Lemma 4.3, suppose that  $\mu$  is a finite measure on  $E$  and  $\lambda$  is the measure on  $\mathbb{R}_+$  given by  $\lambda(dz) = dz(e^{-z}/z)$ . Then,  $L$  is a gamma random measure on  $E$  with  $L(E) < \infty$  almost surely; see Example 4.12. Define

$$P(\omega, A) = \frac{L(\omega, A)}{L(\omega, E)}, \quad \omega \in \Omega, A \in \mathcal{E}.$$

Then,  $P$  is a random probability measure on  $(E, \mathcal{E})$ . We call it a Dirichlet random measure with shape measure  $\mu$ .

The name comes from the Dirichlet distribution, which is a generalization of the beta distribution. For every finite measurable partition  $\{A, \dots, B\}$  of  $E$ , the vector  $(P(A), \dots, P(B))$  has the Dirichlet distribution with shape vector  $(a, \dots, b)$  where  $a = \mu(A), \dots, b = \mu(B)$ . This distribution has the density function

$$\frac{\Gamma(a + \dots + b)}{\Gamma(a) \dots \Gamma(b)} x^{a-1} \dots y^{b-1}, \quad (x, \dots, y) \in \Delta,$$

where  $\Delta$  is the simplex (of appropriate dimension) of positive vectors  $(x, \dots, y)$  with  $x + \dots + y = 1$ .

In the particular case  $a = \dots = b = 1$ , the distribution becomes the uniform distribution on  $\Delta$ .

4.34 *Poisson-Dirichlet process.* Let  $L$  and  $P$  be as in the preceding exercise. Note that  $c = \mu(E) < \infty$  and that  $\lambda(z, \infty) < \infty$  for every  $z > 0$ . Thus the atoms of  $L$  can be labeled as  $(Y_n, Z_n), n = 1, 2, \dots$ , so that  $Z_1 > Z_2 > \dots$ . The sequence  $(Z_n)$  is called the Poisson-Dirichlet process in the statistical literature.

a) Show that  $(Z_n)$  forms a Poisson random measure on  $\mathbb{R}_+$  with mean measure  $c\lambda$ .

b) Show that  $(Y_n)$  is independent of  $(Z_n)$  and is an independency of  $E$ -valued random variables with distribution  $\frac{1}{c}\mu$ .

c) Show that the Dirichlet random measure  $P$  has the form, with  $S = \sum_1^\infty Z_n$ ,

$$P(A) = \frac{1}{S} \sum_{n=1}^\infty Z_n I(Y_n, A), \quad A \in \mathcal{E}.$$

4.35 *Sampling from Dirichlet.* Let  $P$  be a Dirichlet random measure on  $(E, \mathcal{E})$  with shape measure  $\mu$ . A collection  $\{X_1, \dots, X_n\}$  of  $E$ -valued random

variables is called a sample from  $P$  if, given  $P$ , the conditional law of  $\{X_1, \dots, X_n\}$  is that of an independency with the common distribution  $P$ , that is,

$$\mathbb{P}\{X_1 \in A_1, \dots, X_n \in A_n | P\} = P(A_1) \dots P(A_n), \quad A_1, \dots, A_n \in \mathcal{E}.$$

Let  $\{X_1, \dots, X_n\}$  be such a sample. Show that, given  $\{X_1, \dots, X_n\}$ , the conditional law of  $P$  is that of a Dirichlet random measure with shape measure

$$\mu_n(A) = \mu(A) + \sum_{i=1}^n I(X_i, A), \quad A \in \mathcal{E}.$$

Show that, assuming  $\{X_1, \dots, X_n, X_{n+1}\}$  is also a sample from  $P$ ,

$$\mathbb{P}\{X_{n+1} \in A | X_1, \dots, X_n\} = \frac{\mu_n(A)}{\mu_n(E)}, \quad A \in \mathcal{E}.$$

**4.36 Random fields.** By a positive random field on  $(E, \mathcal{E})$  we mean a collection  $F = \{F(x) : x \in E\}$  of positive random variables  $F(x)$  such that the mapping  $(\omega, x) \mapsto F(\omega, x)$  is measurable relative to  $\mathcal{H} \otimes \mathcal{E}$  and  $\mathcal{B}(\mathbb{R}_+)$ . The probability law of  $F$  is specified by giving the finite-dimensional distributions, that is, the distribution of  $(F(x_1), \dots, F(x_n))$  with  $n \geq 1$  and  $x_1, \dots, x_n$  in  $E$ . Equivalently, the probability law is specified by the Laplace transforms

$$\mathbb{E} e^{-\alpha F} = \mathbb{E} \exp_- \int_E \alpha(dx) F(x)$$

as  $\alpha$  varies over all finite measures on  $(E, \mathcal{E})$ .

An expedient method of defining a positive random field is as follows: Let  $N$  be a Poisson random measure on some measurable space  $(D, \mathcal{D})$ , and let  $k : D \times E \mapsto \mathbb{R}_+$  be  $\mathcal{D} \otimes \mathcal{E}$ -measurable. Define

$$F(\omega, y) = \int_D N(\omega, dx) k(x, y), \quad \omega \in \Omega, \quad y \in E.$$

Show that  $F$  is a positive random field on  $(E, \mathcal{E})$ . Show that, for every finite measure  $\alpha$  on  $E$ , with  $\nu$  the mean of  $N$ ,

$$\mathbb{E} e^{-\alpha F} = \exp_- \int_D \nu(dx) \left[ 1 - \exp_- \int_E \alpha(dy) k(x, y) \right].$$

**4.37 Continuation.** Suppose that  $E = \mathbb{R}^d$ ,  $D = \mathbb{R}^d \times \mathbb{R}_+$ ,  $\nu = \text{Leb} \times \lambda$  for some measure  $\lambda$  on  $\mathbb{R}_+$ , and take

$$k(x, r, y) = r \, 1_{\mathbb{B}}(y - x), \quad (x, r) \in D, \quad y \in E,$$

where  $\mathbb{B}$  is the unit ball in  $\mathbb{R}^d$  centered at the origin. Give a condition on  $\lambda$  to make  $F$  real-valued. For  $d = 1$  and  $\lambda(dz) = dz e^{-z}/z$ , compute the marginal distribution of  $F(y)$ .

**4.38 Random vector fields.** By a *velocity field* on  $\mathbb{R}^d$  is meant a mapping  $u : \mathbb{R}^d \mapsto \mathbb{R}^d$ , where one interprets  $u(x)$  as the velocity of the particle at  $x$ . We now describe a random version, which is useful modeling the medium scale eddy motions in  $\mathbb{R}^2$ .

Let  $v$  be a smooth velocity field on the unit disk  $D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ . Let  $N$  be a Poisson random measure on  $E = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$  with mean  $\nu = \text{Leb} \times \alpha \times \beta$  where  $\alpha$  is a probability measure on  $\mathbb{R}$ , and  $\beta$  on  $\mathbb{R}_+$ . Define

$$u(\omega, x) = \int_E N(\omega; dz, da, db) a v\left(\frac{x-z}{b}\right), \quad \omega \in \Omega, \quad x \in \mathbb{R}^2.$$

This defines a two-dimensional random velocity field  $u$  on  $\mathbb{R}^2$ .

Think of  $v$  as the velocity field corresponding to an eddy motion over the unit disk  $D$ . Then,  $x \mapsto v(\frac{x-z}{b})$  corresponds to a similar eddy motion over a disk of radius  $b$  centered at  $z$ . Thus,  $u(\omega, \cdot)$  is the superposition of velocity fields  $x \mapsto av(\frac{x-z}{b})$ , each one corresponding to an atom  $(z, a, b)$  of the counting measure  $N(\omega, \cdot)$ .

Show that  $\mu$  is homogeneous, that is, its probability law is invariant under shifts of the origin. If  $v$  is isotropic, show that the probability law of  $u$  is invariant under rotations as well.

## 5 POISSON PROCESSES

This section is devoted to a closer examination of simple Poisson processes and, by extension, of Poisson random measures on  $\mathbb{R}_+$ . The presence of time and the order properties of the real-line allow for a deeper understanding of such processes. This, in turn, illustrates the unique position occupied by things Poisson in the theories of martingales, point processes, and Markov processes.

### Counting processes

We introduce the setup to be kept throughout this section. In keeping with traditional notation, we let  $N = (N_t)_{t \in \mathbb{R}_+}$  be a counting process: for almost every  $\omega$ , the path  $t \mapsto N_t(\omega)$  is an increasing right-continuous step function with  $N_0(\omega) = 0$  and whose every jump is of size one. Such a process is defined by its jump times: there is an increasing sequence of random variables  $T_k$  taking values in  $\bar{\mathbb{R}}_+$  such that

$$5.1 \quad N_t(\omega) = \sum_{k=1}^{\infty} 1_{[0, t]} \circ T_k(\omega), \quad t \in \mathbb{R}_+, \quad \omega \in \Omega.$$

It is usual to call  $(T_k)$  the point process associated with  $N$ . Since  $N_0 = 0$  and each jump of  $N$  is of size one,

$$5.2 \quad 0 < T_1(\omega) < T_2(\omega) < \cdots < T_k(\omega) \quad \text{if } T_k(\omega) < \infty,$$



this being true for almost every  $\omega$ . Of course, if  $T_k(\omega) = \infty$  then  $T_n(\omega) = \infty$  for all  $n > k$ . The sequence  $(T_k)$  forms a random counting measure  $M$  on  $\mathbb{R}_+$ ; for positive Borel functions  $f$  on  $\mathbb{R}_+$ , extending  $f$  onto  $\mathbb{R}_+$  by setting  $f(\infty) = 0$ ,

$$5.3 \quad Mf = \sum_{k=1}^{\infty} f \circ T_k.$$

Obviously,  $N_t(\omega) = M(\omega, [0, t])$ . Finally, we let  $\mathcal{F}$  be the filtration generated by  $N$ , that is,

$$5.4 \quad \mathcal{F}_t = \sigma\{N_s : s \leq t\} = \sigma\{M(A) : A \in \mathcal{B}_{[0, t]}\}.$$

## Poisson processes

We recall Definition V.2.20 in the present context: The counting process  $N$  is called a *Poisson process* with rate  $c$  if, for every  $s$  and  $t$  in  $\mathbb{R}_+$ , the increment  $N_{s+t} - N_s$  is independent of  $\mathcal{F}_s$  and has the Poisson distribution with mean  $ct$ . Equivalently,  $N$  is Poisson with rate  $c$  if and only if it has stationary and independent increments (see Definition 4.5a,b) and each  $N_t$  has the Poisson distribution with mean  $ct$ .

The following theorem characterizes Poisson processes in terms of random measures, Markov processes, martingales, and point processes. Much of the proof is easy and is of a review nature. The exceptions are the proofs that (c)  $\Rightarrow$  (d)  $\Rightarrow$  (a).

5.5 THEOREM. *For fixed  $c$  in  $(0, \infty)$ , the following are equivalent:*

- a)  $M$  is a Poisson random measure with mean  $\mu = c \text{ Leb}$ .
- b)  $N$  is a Poisson counting process with rate  $c$ .
- c)  $N$  is a counting process and  $\tilde{N} = (N_t - ct)_{t \in \mathbb{R}_+}$  is an  $\mathcal{F}$ -martingale.
- d)  $(T_k)$  is an increasing sequence of  $\mathcal{F}$ -stopping times, and the differences  $T_1, T_2 - T_1, T_3 - T_2, \dots$  are independent and exponentially distributed with parameter  $c$ .

REMARK. The limitation on  $c$  is natural: Since  $N$  is a counting process,  $N_t$  is finite almost surely and can have the Poisson distribution with mean  $ct$  only if  $c$  is finite. The other possibility,  $c = 0$ , is without interest, because it implies that  $N = 0$  almost surely.

*Proof.* We shall show that (a)  $\Rightarrow$  (b)  $\iff$  (c)  $\Rightarrow$  (d)  $\Rightarrow$  (a).

i) Assume (a). Since the Lebesgue measure is diffuse,  $M$  is a random counting measure by Theorem 2.14, and  $\mu[0, t] = ct < \infty$ . Thus,  $N$  is a counting process. The independence of  $N_{s+t} - N_s$  from  $\mathcal{F}_s$  and the associated distribution being Poisson with mean  $ct$  follow from the definition of Poisson random measures. So, (a)  $\Rightarrow$  (b).

ii) The equivalence (b)  $\iff$  (c) was shown in Chapter V on martingales; see Theorem V.2.23 and Proposition V.6.13.

iii) Assume (c) and, therefore, (b). It follows from 5.1 that, for each integer  $k \geq 1$ , we have  $\{T_k \leq t\} = \{N_t \geq k\} \in \mathcal{F}_t$  for every  $t$ . Thus, each  $T_k$  is a stopping time of  $\mathcal{F}$ . Moreover, since  $N$  is a counting process and  $\lim_{t \rightarrow \infty} N_t = M(\mathbb{R}_+) = +\infty$  almost surely, for almost every  $\omega$ , we have 5.2, and  $T_k(\omega) < \infty$  for all  $k$ , and  $\lim_k T_k(\omega) = \infty$ .

On the other hand, Corollary V.6.7 applies to the martingale  $\tilde{N}$ : we have, with  $\mathbb{E}_S = \mathbb{E}(\cdot | \mathcal{F}_S)$  as usual,

$$5.6 \quad \mathbb{E}_S \int_{(S,T]} F_t dN_t = \mathbb{E}_S \int_{(S,T]} F_t c dt$$

for bounded predictable processes  $F$  and stopping times  $S$  and  $T$  with  $S \leq T$ . Put  $T_0 = 0$  for convenience, take  $S = T_k$  and  $T = T_{k+1}$  for fixed  $k$  in  $\mathbb{N}$ , and let  $F_t = re^{-rt}$  with  $r > 0$  fixed. On the left side of 5.6, the integral becomes equal to  $re^{-rT}$  since  $N$  remains constant over the interval  $(S, T)$  and jumps by the amount one at  $T$ . So, 5.6 becomes

$$r \mathbb{E}_S e^{-rT} = c \mathbb{E}_S (e^{-rS} - e^{-rT}).$$

Multiplying both sides by  $e^{rS}$ , which can be passed inside the conditional expectations since  $S$  is in  $\mathcal{F}_S$ , and re-arranging, we obtain

$$\mathbb{E}_S e^{-r(T-S)} = \frac{c}{c+r}, \quad r > 0.$$

This means that  $T - S = T_{k+1} - T_k$  is independent of  $\mathcal{F}_S = \mathcal{F}_{T_k}$  and has the exponential distribution with parameter  $c$ . Thus (c)  $\Rightarrow$  (d).

iv) Assume (d). Let  $\hat{M}$  be a Poisson random measure on  $\mathbb{R}_+$  with mean  $\hat{\mu} = \mu = c$  Leb, constructed over some auxiliary probability space  $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{P}})$ ; see Theorem 2.13 for its existence. Using the already proved implications (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d) on the Poisson random measure  $\hat{M}$ , we conclude that, for positive Borel  $f$ ,

$$5.7 \quad \hat{M}f = \sum_1^\infty f \circ \hat{T}_k,$$

where  $\hat{T}_1, \hat{T}_2 - \hat{T}_1, \hat{T}_3 - \hat{T}_2, \dots$  are independent and have the exponential distribution with parameter  $c$ . Observe: 5.3 and 5.7 hold, and  $(T_k)$  and  $(\hat{T}_k)$  have the same probability law. It follows that the random variables  $Mf$  and  $\hat{M}f$  have the same distribution. Hence, writing  $\hat{\mathbb{E}}$  for expectation on  $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{P}})$ ,

$$\mathbb{E} e^{-Mf} = \hat{\mathbb{E}} e^{-\hat{M}f} = e^{-\mu(1-e^{-f})},$$

where the last equality follows from Theorem 2.9 on Laplace functionals applied to the Poisson random measure  $\hat{M}$  with mean  $\hat{\mu} = \mu$ . This shows, via Theorem 2.9, that  $M$  is Poisson with mean  $\mu$ . Thus, (d)  $\Rightarrow$  (a).  $\square$

5.8 REMARK. The unorthodox proof that (d) $\Rightarrow$ (a) is an application of the following principle. Let  $X$  and  $Y$  be random variables taking values on some spaces, and suppose that they define each other, that is,  $Y = h \circ X$  for some isomorphism  $h$  and  $X = g \circ Y$  with  $g$  being the functional inverse of  $h$ . Let  $\lambda$  be the distribution of  $X$  and let  $\mu = \lambda \circ h^{-1}$ . The principle, rather trivial, is that  $Y$  has the distribution  $\mu$  if and only if  $X$  has the distribution  $\lambda$ . The proof of (d) $\Rightarrow$ (a) uses this principle with  $X = M$ ,  $Y = (T_k)$ , and  $\lambda$  the probability law of  $M$ . Unfortunately, a direct application of the principle requires showing that  $h$  is a bimeasurable bijection, which is technically difficult: in this application,  $h$  is a mapping from the space of all counting measures on  $\mathbb{R}_+$  onto the space of all increasing sequences on  $\mathbb{R}_+$ . The auxiliary process  $\tilde{M}$  was introduced to circumvent the technical difficulties.

## Characterization as a Lévy process

The next theorem's characterization is often used as a definition: A Poisson process is a counting process with stationary and independent increments. We refer to Definition 4.5 for increasing Lévy processes and the allied terminology.

5.9 THEOREM. *The counting process  $N$  is a Lévy process if and only if it is a Poisson process.*

*Proof.* Sufficiency part is trivial. As to the necessity part, assuming that  $N$  is a counting Lévy process, the only thing to show is that  $N_t$  has the Poisson distribution with mean  $ct$  for some constant  $c$  in  $\mathbb{R}_+$  and for all times  $t$ .

a) We start by showing that, for some fixed  $c$  in  $\mathbb{R}_+$ ,

$$5.10 \quad q(t) = \mathbb{P}\{N_t = 0\} = e^{-ct}, \quad t \in \mathbb{R}_+.$$

For  $s$  and  $t$  in  $\mathbb{R}_+$ , because  $N$  is Lévy, the increments  $N_s$  and  $N_{s+t} - N_s$  are independent and the latter has the same distribution as  $N_t$ . Thus,

$$\mathbb{P}\{N_{s+t} = 0\} = \mathbb{P}\{N_s = 0, N_{s+t} - N_s = 0\} = \mathbb{P}\{N_s = 0\}\mathbb{P}\{N_t = 0\},$$

that is,

$$5.11 \quad q(s+t) = q(s)q(t), \quad s, t \in \mathbb{R}_+.$$

Moreover,  $q(0) = 1$ , and  $q(t) = \mathbb{E} 1_{\{0\}} \circ N_t$  is right-continuous by the almost sure right-continuity of  $t \mapsto 1_{\{0\}} \circ N_t$  and the bounded convergence theorem for expectations. The only solution of 5.11 with these properties is as claimed in 5.10 with  $c$  in  $\mathbb{R}_+$ . If  $c = 0$ , then  $N_t = 0$  almost surely for all  $t$ , which makes  $N$  a trivial Poisson process, and the proof is complete. For the remainder of the proof we assume that  $0 < c < \infty$ .

b) It is convenient to recall  $M$ , the random counting measure associated with  $N = (N_t)$ . Fix  $t > 0$ . Let  $A_1, \dots, A_n$  be equi-length intervals of the form  $(\cdot, \cdot]$  that constitute a partition of  $A = (0, t]$ . Let  $X_i$  be the indicator of the event  $\{M(A_i) \geq 1\}$  and put  $S_n = X_1 + \dots + X_n$ . Since  $N$  is Lévy, the increments  $M(A_1), \dots, M(A_n)$  are independent and have the same distribution as  $M(A_1) = N_{t/n}$ . Thus,  $X_1, \dots, X_n$  are independent Bernoulli variables with the same success probability  $p = 1 - q(t/n) = 1 - e^{-ct/n}$ . Hence, for each  $k$  in  $\mathbb{N}$ ,

$$\begin{aligned} \mathbb{P}\{S_n = k\} &= \frac{n!}{k!(n-k)!} (1 - e^{-ct/n})^k (e^{-ct/n})^{n-k} \\ &= \frac{e^{-ct}}{k!} n(n-1) \cdots (n-k+1) (e^{ct/n} - 1)^k. \end{aligned}$$

c) For almost every  $\omega$ , the counting measure  $M_\omega$  has only finitely many atoms in  $A$ ; let  $\delta(\omega)$  be the minimum distance between the atoms; if  $n > t/\delta(\omega)$  then we have  $S_n(\omega) = M(\omega, A) = N_t(\omega)$ . In other words,  $S_n \rightarrow N_t$  almost surely as  $n \rightarrow \infty$ . Hence,

$$\mathbb{P}\{N_t = k\} = \lim_{n \rightarrow \infty} \mathbb{P}\{S_n = k\} = \frac{e^{-ct}(ct)^k}{k!}, \quad k \in \mathbb{N}.$$

This completes the proof that  $N$  is a Poisson process.  $\square$

The preceding proof is the most elementary of all possible ones. In the next chapter we shall give another proof. Assuming that  $N$  is a counting Lévy, we shall show that  $\mathbb{E} N_t = ct$  necessarily for some finite constant  $c$ ; this uses the strong Markov property (to be shown). Then,  $(N_t - ct)$  is a martingale, and  $N$  is Poisson by Theorem 5.5 or, more correctly, by Proposition V.6.13.

## A minimalist characterization

A careful examination of the preceding proof would show that the Lévy property of  $N$  is used only to conclude that the Bernoulli variables  $X_i$  are independent and have the same success probability. This observation leads to the following characterization theorem; compare the minimal nature of the condition here with the extensive conditions of Definition 2.3. See 5.16 below for a generalization.

5.12 THEOREM. *Let  $M$  be a random counting measure on  $\mathbb{R}_+$ . Let  $\mu = c$  Leb on  $\mathbb{R}_+$  for some constant  $c$  in  $\mathbb{R}_+$ . Then  $M$  is Poisson with mean  $\mu$  if and only if*

$$5.13 \quad \mathbb{P}\{M(A) = 0\} = e^{-\mu(A)}$$

*for every bounded set  $A$  that is a finite union of disjoint intervals.*

*Proof.* The necessity of the condition follows trivially from Definition 2.3. To prove the sufficiency, assume that 5.13 holds for every  $A$  as described, and let  $\mathcal{A}$  be the collection of all such  $A$ .

Fix a finite number of disjoint intervals  $A, \dots, B$ ; let  $a, \dots, b$  be the corresponding lengths; and let  $i, \dots, j$  be positive integers. We shall show that

$$5.14 \quad \mathbb{P}\{M(A) = i, \dots, M(B) = j\} = \frac{e^{-ca}(ca)^i}{i!} \dots \frac{e^{-cb}(cb)^j}{j!}.$$

This will prove that  $N$  is a Poisson process with rate  $c$  and, hence, that  $M$  is Poisson with mean  $\mu$  as claimed.

a) Let  $A_1, \dots, A_n$  form a partition of  $A$  into  $n$  equi-length sub-intervals,  $\dots$ , and  $B_1, \dots, B_n$  form a partition of  $B$  similarly. Let

$$\mathcal{D} = \{A_1, \dots, A_n; \dots; B_1, \dots, B_n\}.$$

For each  $D$  in  $\mathcal{D}$ , let  $X_D$  be the indicator of  $\{N(D) \geq 1\}$ , and define

$$5.15 \quad S_n(A) = \sum_1^n X_{A_m}, \dots, S_n(B) = \sum_1^n X_{B_m}.$$

Arguments of the proof of Theorem 5.9, part (c), show that

$$M(A) = \lim_n S_n(A), \dots, M(B) = \lim_n S_n(B)$$

almost surely. Thus, to prove 5.14, it is enough to show that  $S_n(A), \dots, S_n(B)$  are independent and have the binomial distributions for  $n$  trials with respective success probabilities  $1 - e^{-ca/n}, \dots, 1 - e^{-cb/n}$ .

b) Consider the collection  $\{X_D : D \in \mathcal{D}\}$  of Bernoulli variables. Let  $\mathcal{C}$  be a subset of  $\mathcal{D}$ , and let  $C$  be the union of the elements of  $\mathcal{C}$ . Observe that  $1 - X_D$  is the indicator of  $\{M(D) = 0\}$ , and the product  $\prod (1 - X_D)$  over  $D$  in  $\mathcal{C}$  is the indicator of  $\{M(C) = 0\}$ . Since  $C \in \mathcal{A}$  and  $\mathcal{D} \subset \mathcal{A}$ , the condition 5.13 holds for the sets  $D$  and  $C$ ; thus,

$$\mathbb{E} \prod_{D \in \mathcal{C}} (1 - X_D) = e^{-\mu(C)} = \prod_{D \in \mathcal{C}} e^{-\mu(D)} = \prod_{D \in \mathcal{C}} \mathbb{E} (1 - X_D).$$

This implies, via II.5.33, that the collection  $\{1 - X_D : D \in \mathcal{D}\}$  is an independency. Hence,  $\{X_D : D \in \mathcal{D}\}$  is an independency, and the success probabilities are

$$\mathbb{P}\{X_D = 1\} = 1 - e^{-\mu(D)},$$

where  $\mu(D)$  is equal to  $ca/n$  for the first  $n$  elements of  $\mathcal{D}$ , and  $\dots$ , and to  $cb/n$  for the last  $n$  elements. Thus,  $S_n(A), \dots, S_n(B)$  are independent and have the binomial distribution claimed in part (a). This completes the proof.  $\square$

The preceding can be generalized to abstract spaces. The basic idea is the following: Let  $(E, \mathcal{E})$  be a standard measurable space and let  $\mu$  be a diffuse  $\Sigma$ -finite measure on it. Put  $b = \mu(E) \leq +\infty$ . As was explained in Exercise I.5.16, it is possible to find measurable bijections  $g : E \mapsto [0, b)$  and  $h : [0, b) \mapsto E$  such that  $\hat{\mu} = \mu \circ g^{-1}$  is the Lebesgue measure on  $[0, b)$ , and  $\mu = \hat{\mu} \circ h^{-1}$  in return.

5.16 THEOREM. *Let  $\mu$  be a diffuse  $\Sigma$ -finite measure on a standard measurable space  $(E, \mathcal{E})$ . Let  $M$  be a random counting measure on  $E$ . Suppose that*

$$5.17 \quad \mathbb{P}\{M(A) = 0\} = e^{-\mu(A)}, \quad A \in \mathcal{E}.$$

*Then,  $M$  is a Poisson random measure with mean  $\mu$ .*

*Proof.* Let  $g$  and  $h$  be as described preceding the theorem. Since  $g$  is a bijection,  $\hat{M} = M \circ g^{-1}$  is still a random counting measure, and, with  $\hat{\mu} = \mu \circ g^{-1} = \text{Lebesgue on } [0, b)$ , we have

$$\mathbb{P}\{\hat{M}(B) = 0\} = \mathbb{P}\{M(g^{-1}B) = 0\} = e^{-\mu(g^{-1}B)} = e^{-\hat{\mu}(B)}$$

for every Borel subset of  $[0, b)$ . Since  $\hat{\mu}$  is the Lebesgue measure on  $[0, b)$ , Theorem 5.12 applies to conclude that  $\hat{M}$  is Poisson on  $[0, b)$  with mean  $\hat{\mu}$ . It follows that  $M = \hat{M} \circ h^{-1}$  is Poisson on  $E$  with mean  $\hat{\mu} \circ h^{-1} = \mu$ .  $\square$

## Strong Markov property

As was remarked in V.6.16, if  $N$  is a Poisson process with rate  $c$ , then the independence of increments can be extended to increments over random intervals  $(S, S+t]$ : For every finite stopping time  $S$  of the filtration  $\mathcal{F}$ , the increment  $N_{S+t} - N_S$  is independent of  $\mathcal{F}_S$  and has the Poisson distribution with mean  $ct$ . We may remove the finiteness condition on  $S$  and express the same statement as

$$5.18 \quad \mathbb{E}_S f(N_{S+t} - N_S) 1_{\{S < \infty\}} = \sum_{k=0}^{\infty} \frac{e^{-ct}(ct)^k}{k!} f(k) 1_{\{S < \infty\}}.$$

In fact, we could have used this result to shorten the proof of (c) $\Rightarrow$ (d) in Theorem 5.5: take  $f = 1_{\{0\}}$  and  $S = T_k$  to conclude that the event  $\{N_{T_k+t} - N_{T_k} = 0\} = \{T_{k+1} - T_k > t\}$  is independent of  $\mathcal{F}_{T_k}$  and has probability  $e^{-ct}$ .

The property 5.18 is called the strong Markov property for the Poisson process  $N$ . It expresses the independence of future from the past when the present is a stopping time.

5.19 EXAMPLE. Let  $S$  be the first time an interval of length  $a$  passes without a jump, that is,

$$S = \inf\{t \geq a : N_t = N_{t-a}\}.$$

Clearly,  $S = T_k + a$  if and only if the first  $k$  interjump intervals are at most  $a$  in length and the  $(k+1)^{\text{th}}$  interval is greater than  $a$  in length. Thus,  $S < \infty$  almost surely. Let  $T$  be the time of first jump after  $S$ . Note that the interval that includes  $S$  has length  $a + (T - S)$ , and for  $a$  large, the raw intuition expects  $T - S$  to be small. Instead, noting that  $\{T - S > t\} = \{N_{S+t} - N_S = 0\}$ , we see that  $T - S$  is independent of  $\mathcal{F}_S$  and has the same exponential distribution as if  $S$  is a jump time.

## Total unpredictability of jumps

Let  $T = T_1$  be the time of first jump for the Poisson process  $N$  with rate  $c$ . Is it possible to predict  $T$ ? Is there a sequence of stopping times  $S_n$  increasing to  $T$  and having  $S_n < T$  almost surely? The following shows that the answer is no.

**5.20 PROPOSITION.** *Let  $S$  be a stopping time of  $\mathcal{F}$ . Suppose that  $0 \leq S < T$  almost surely. Then,  $S = 0$  almost surely.*

*Proof.* Since  $S < T$ , the event  $\{T - S > t\}$  is the same as the event  $\{N_{S+t} - N_S = 0\}$ , and the latter is independent of  $\mathcal{F}_S$  and has probability  $e^{-ct}$  by the strong Markov property 5.18. Thus, in particular,

$$\mathbb{E}_S(T - S) = 1/c.$$

Taking expectations, and recalling that  $\mathbb{E} T = 1/c$ , we conclude that  $\mathbb{E} S = 0$ . Thus,  $S = 0$  almost surely.  $\square$

A similar proof will show that if  $T_k \leq S < T_{k+1}$  almost surely then  $S = T_k$  almost surely, this is for each  $k$ . Thus, it seems impossible to predict the jump time  $T_k$  for fixed  $k$ . We list the following stronger result without proof. See V.7.30 and V.7.31 for the terms. Recall the definition 5.4 of  $\mathcal{F}$ .

**5.21 THEOREM.** *Let  $S$  be a stopping time of  $\mathcal{F}$ . Then,  $S$  is predictable if and only if  $N_{S-} = N_S$  almost surely on  $\{S > 0\}$ ; and  $S$  is totally unpredictable if and only if  $N_{S-} \neq N_S$  almost surely.*

## Exercises

**5.22 Crossing times.** Let  $S$  be as in Example 5.19. Find its expected value.

**5.23 Logarithmic Poisson random measures.** Let  $U_1, U_2, \dots$  be independent and uniformly distributed over  $E = (0, 1)$ . Let  $X_n = U_1 U_2 \cdots U_n$ ,  $n \geq 1$ , and define  $M$  to be the random counting measure on  $E$  whose atoms are those  $X_n$ . Show that  $M$  is a Poisson random measure on  $E$ .

**5.24 Continuation.** Let  $M$  be a Poisson random measure on  $E = (0, \infty)$  with mean  $\mu$  given by

$$\mu(dx) = dx \frac{1}{x}, \quad x \in E.$$

Note that  $M(A) < \infty$  almost surely for every closed interval  $A = [a, b]$  in  $E$ , but that  $M(A) = +\infty$  almost surely for  $A = (0, a)$  and for  $A = (b, \infty)$ . Label the atoms  $X_i$  such that  $\cdots < X_{-1} < X_0 < 1 \leq X_1 < X_2 < \cdots$ . Describe the probability law of  $(X_i)$ .

**5.25 Atom counts.** Let  $L$  be a purely atomic additive random measure on  $\mathbb{R}_+$ . Suppose that

$$\mathbb{P}\{L(A) = 0\} = e^{-\text{Leb } A}, \quad A \in \mathcal{B}_{\mathbb{R}_+}.$$

Describe the atomic structure of  $L$ .

**5.26 Importance of additivity.** Let  $(T_k)$  be the increasing sequence of jump times for a Poisson process  $(N_t)$  with unit rate. Let  $w_1, w_2, \dots$  be arbitrarily chosen from  $(0, \infty)$ . Define

$$L(A) = \sum_{i=1}^{\infty} w_i I(T_i, A), \quad A \in \mathcal{B}_{\mathbb{R}_+}.$$

Show that  $\mathbb{P}\{L(A) = 0\} = e^{-\text{Leb } A}$ . Is the random measure  $L$  additive?

**5.27 Another warning.** Let  $(N_t)_{t \in \mathbb{R}_+}$  be a counting process. Suppose that  $N_t$  has the Poisson distribution with mean  $t$  for every  $t$  in  $\mathbb{R}_+$ . Give an example where  $N$  is *not* a Poisson process.

**5.28 Importance of diffusivity.** In Theorem 5.16, the diffusivity of  $\mu$  is essential. On  $\mathbb{R}$ , define  $\mu = c\delta_0$  and  $M = K\delta_0$ , where  $c > 0$  is a constant, and  $K$  is equal to 0 or 2 with respective probabilities  $e^{-c}$  and  $1 - e^{-c}$ . Show that  $M$  is not Poisson.

## 6 POISSON INTEGRALS AND SELF-EXCITING PROCESSES

Our aim is the to introduce some martingale-theoretic tools in dealing with random measures. Some such tools were introduced in Chapter V on martingales and were used in the martingale characterization for Poisson processes. Here, we introduce similar ideas for applications to stochastic processes with jumps, because the jumps are often regulated by Poisson random measures. As an example, we introduce counting processes with random intensities and a particular self-exciting two point process. Further applications will appear in the chapters on Lévy and Markov processes.

Throughout,  $(E, \mathcal{E})$  will be a measurable space, and  $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$  an augmented filtration on the probability space  $(\Omega, \mathcal{H}, \mathbb{P})$ .

### Poisson on time and space

This is about Poisson random measures on spaces of the form  $\mathbb{R}_+ \times E$ , where  $\mathbb{R}_+$  is interpreted as time and  $E$  as some physical space, and the filtration  $\mathcal{F}$  represents the flow of information over time. As usual  $\mathbb{R}_+ \times E$  is



furnished with the product  $\sigma$ -algebra  $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{E}$ , and we shall omit the mention of the  $\sigma$ -algebra in phrases like “random measure on  $\mathbb{R}_+ \times E$ .”

**6.1 DEFINITION.** *Let  $M$  be a Poisson random measure on  $\mathbb{R}_+ \times E$ . It is said to be so relative to the filtration  $\mathcal{F}$  if, for every  $t$  in  $\mathbb{R}_+$ ,*

- a)  $M(A)$  is in  $\mathcal{F}_t$  for every  $A$  in  $\mathcal{B}_{[0,t]} \otimes \mathcal{E}$ , and*
- b) the trace of  $M$  over  $(t, \infty) \times E$  is independent of  $\mathcal{F}_t$ .*

The condition (a) is called adaptedness, and (b) the independence of the future of  $M$  from the past. It is obvious that, if  $M$  is Poisson relative to  $\mathcal{F}$ , then it is Poisson relative to the filtration  $\mathcal{G}$  generated by itself (that is,  $\mathcal{G}_t$  is the  $\sigma$ -algebra generated by  $M(A)$ ,  $A \in \mathcal{B}_{[0,t]} \otimes \mathcal{E}$ ).

Given a Poisson random measure  $M$  on  $\mathbb{R}_+ \times E$ , it is convenient to think of its atoms as solid objects. For a fixed outcome  $\omega$ , if  $(t, z)$  is an atom of the measure  $M_\omega$ , then we think of  $t$  as the arrival time of that atom and of  $z$  as the landing point in space; see also Proposition 2.18 where  $E = \mathbb{R}_+$  and is interpreted as “size” space.

## Poisson integrals of predictable processes

This is to extend Theorem V.6.5 to the current case of processes on  $\mathbb{R}_+ \times E$ . Recall Definition 6.1 and Proposition 6.2 from Chapter V: The  $\mathcal{F}$ -predictable  $\sigma$ -algebra is the  $\sigma$ -algebra  $\mathcal{F}^p$  on  $\Omega \times \mathbb{R}_+$  generated by the collection of sets having the form  $H \times A$ , where  $A$  is an interval  $(a, b]$  and  $H$  is an event in  $\mathcal{F}_a$ , or  $A$  is the singleton  $\{0\}$  and  $H$  is in  $\mathcal{F}_0$ . A process  $F = (F_t)_{t \in \mathbb{R}_+}$  is said to be  $\mathcal{F}$ -predictable, or is said to be in  $\mathcal{F}^p$ , if the mapping  $(\omega, t) \mapsto F_t(\omega)$  from  $\Omega \times \mathbb{R}_+$  into  $\mathbb{R}$  is  $\mathcal{F}^p$ -measurable. In particular, if  $F$  is adapted to  $\mathcal{F}$  and is left-continuous, then it is  $\mathcal{F}$ -predictable. The next theorem is important.

**6.2 THEOREM.** *Let  $M$  be a Poisson random measure on  $\mathbb{R}_+ \times E$  with mean measure  $\mu$  satisfying  $\mu(\{0\} \times E) = 0$ . Let  $G = \{G(t, z) : t \in \mathbb{R}_+, z \in E\}$  be a positive process in  $\mathcal{F}^p \otimes \mathcal{E}$ . Suppose that  $M$  is Poisson relative to  $\mathcal{F}$ . Then,*

$$6.3 \quad \mathbb{E} \int_{\mathbb{R}_+ \times E} M(dt, dz) G(t, z) = \mathbb{E} \int_{\mathbb{R}_+ \times E} \mu(dt, dz) G(t, z).$$

*Proof.* The hypothesis on  $G$  is that the mapping  $(\omega, t, z) \mapsto G(\omega, t, z)$  from  $\Omega \times \mathbb{R}_+ \times E$  into  $\mathbb{R}_+$  is  $(\mathcal{F}^p \otimes \mathcal{E})$ -measurable. Consider the collection of all such  $G$  for which 6.3 holds. That collection includes constants, is a linear space, and is closed under increasing limits, the last being by the monotone convergence theorem applied to 6.3 with  $G_n \nearrow G$  on both sides. Thus, by the monotone class theorem, the proof is reduced to showing 6.3 for  $G$  that are indicators of sets  $H \times A \times B$ , where  $B \in \mathcal{E}$  and  $A = (a, b]$  and  $H \in \mathcal{F}_a$ ,

and also when  $B \in \mathcal{E}$  and  $A = \{0\}$  and  $H \in \mathcal{F}_0$ . In the former case, the left side of 6.3 is equal to

$$\begin{aligned}\mathbb{E} 1_H M(A \times B) &= \mathbb{E} \mathbb{E}_a 1_H M(A \times B) \\ &= \mathbb{E} 1_H \mathbb{E}_a M(A \times B) \\ &= \mathbb{E} 1_H \mathbb{E} M(A \times B) = \mathbb{E} 1_H \mu(A \times B),\end{aligned}$$

and the last member is exactly the right side of 6.3; here we used the notation  $\mathbb{E}_a$  for  $\mathbb{E}(\cdot | \mathcal{F}_a)$  as in earlier chapters and noted that  $\mathbb{E} = \mathbb{E} \mathbb{E}_a$ , and  $H \in \mathcal{F}_a$ , and  $A \times B \subset (a, \infty] \times E$ , and the trace of  $M$  over  $(a, \infty) \times E$  is independent of  $\mathcal{F}_a$ . The case where  $A = \{0\}$  is trivial: then,  $\mu(A \times E) = 0$  by hypothesis, which implies that  $M(A \times E) = 0$  almost surely, which together imply that the integrals vanish on both sides of 6.3.  $\square$

6.4 REMARKS. a) In most applications the mean  $\mu$  will have the form  $\mu = \text{Leb} \times \lambda$  for some measure  $\lambda$  on  $E$ . Then,  $\mu(\{0\} \times E) = 0$  automatically.

b) Let  $S$  and  $T$  be stopping times of  $\mathcal{F}$  with  $S \leq T$ , and let  $V$  be a positive random variable in  $\mathcal{F}_S$ . Then, putting  $F_t = V 1_{(S, T]}(t)$  for  $t$  in  $\mathbb{R}_+$ , we obtain a positive predictable (in fact, left-continuous) process  $F$ . Letting  $M$  and  $G$  be as in the preceding theorem, we note that  $\tilde{G}(t, z) = F_t G(t, z)$  defines a process  $\tilde{G}$  that is again positive and in  $\mathcal{F}^p \otimes \mathcal{E}$ . Thus, replacing  $G$  by  $\tilde{G}$  in 6.3, we see that

$$\mathbb{E}_S \int_{\llbracket S, T \rrbracket \times E} M(dt, dz) G(t, z) = \mathbb{E}_S \int_{\llbracket S, T \rrbracket \times E} \mu(dt, dz) G(t, z),$$

where  $\llbracket S, T \rrbracket$  should be interpreted as  $(S, T] \cap \mathbb{R}_+$  in order to accommodate possibly infinite values for  $S$  and  $T$ .

c) Let  $M$  and  $G$  be as in the theorem. Suppose that

$$\mathbb{E} \int_{[0, t] \times E} \mu(ds, dz) G(s, z) < \infty, \quad t \in \mathbb{R}_+.$$

Then,

$$X_t = \int_{[0, t] \times E} M(ds, dz) G(s, z) - \int_{[0, t] \times E} \mu(ds, dz) G(s, z)$$

is integrable for each  $t$ , and is in  $\mathcal{F}_t$  for each  $t$ . The preceding remark applied with deterministic  $s$  and  $t$  show that  $X = (X_t)_{t \in \mathbb{R}_+}$  is a martingale with  $X_0 = 0$ .

6.5 REMARK. In fact, Theorem 6.2 has a partial converse that can be used to characterize Poisson random measures: Let  $M$  be a random counting measure on  $\mathbb{R}_+ \times E$  whose mean  $\mu$  is equal to  $\text{Leb} \times \lambda$ , where  $\lambda$  is a  $\sigma$ -finite measure on  $E$ . Suppose that  $M$  is adapted to  $\mathcal{F}$ , that is, 6.1a holds, and that 6.3 holds for every positive  $G$  in  $\mathcal{F}^p \otimes \mathcal{E}$ . Finally, suppose that no two atoms

arrive simultaneously, that is, for almost every  $\omega$ , we have  $M(\omega, \{t\} \times E) \leq 1$  for all  $t$ . Then, it can be shown that  $M$  is a Poisson random measure relative to  $\mathcal{F}$ . Here is an outline of the proof.

For  $B$  in  $\mathcal{E}$ , put  $N_t = M([0, t] \times B)$ , and assume that  $\lambda(B) < \infty$ . Then,  $\mathbb{E} N_t = \lambda(B)t$ , and by the assumption that no two atoms of  $M$  arrive simultaneously,  $(N_t)$  is a counting process. In the formula 6.3 being assumed, taking  $G(t, z) = F_t 1_B(z)$  for some positive predictable process  $F$ , we see that

$$\mathbb{E} \int_{\mathbb{R}_+} F_t dN_t = \lambda(B) \mathbb{E} \int_{\mathbb{R}_+} F_t dt.$$

Now, it follows from Proposition V.6.13 (see also V.6.3 - V.6.12) that  $(N_t)$  is a Poisson process with rate  $\lambda(B)$ . Further, it can be shown that the Poisson processes corresponding to disjoint sets  $B_1, \dots, B_n$  are independent, because no two such processes can jump at the same time; this will be proved in the next chapter. It follows that  $M$  is a Poisson random measure on  $\mathbb{R}_+ \times E$ .

## Self-exciting processes

For a Poisson arrival process with constant rate  $c$ , the expected number of arrivals during an interval of length  $t$  is equal to  $ct$ , and, hence,  $c$  is the arrival intensity in this sense. In Exercise 2.36, the rate  $c$  was replaced by a random rate  $R_t$  varying with time, but the randomness of  $R$  stemmed from a source exogeneous to the arrival process  $N$ . Here, we extend the concept to cases where the rate  $R_t$  at time  $t$  is allowed to depend on the history of arrivals during  $(0, t)$ . Recall that  $\mathcal{F} = (\mathcal{F}_t)$  is an arbitrary filtration over time. We use the shorthand notation  $\mathbb{E}_t$  for the conditional expectation  $\mathbb{E}_{\mathcal{F}_t} = \mathbb{E}(\cdot | \mathcal{F}_t)$ .

**6.6 DEFINITION.** Let  $N = (N_t)_{t \in \mathbb{R}_+}$  be a counting process adapted to  $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ . Let  $R = (R_t)_{t \in \mathbb{R}_+}$  be a positive  $\mathcal{F}$ -predictable process. Then,  $R$  is called the intensity process for  $N$  relative to  $\mathcal{F}$  if

$$6.7 \quad \mathbb{E} \int_{\mathbb{R}_+} F_t dN_t = \mathbb{E} \int_{\mathbb{R}_+} F_t R_t dt$$

for every positive  $\mathcal{F}$ -predictable process  $F$ .

The processes  $F$  satisfying 6.7 form a positive monotone class. Noting that  $N_0 = 0$  almost surely, we conclude that 6.7 holds for every positive predictable  $F$  if and only if it holds for those  $F$  that are the indicators of sets of the form  $H \times (t, u]$  with  $0 \leq t < u$  and  $H$  in  $\mathcal{F}_t$ . Hence, the condition 6.7 is equivalent to the following:

$$6.8 \quad \mathbb{E}_t (N_u - N_t) = \mathbb{E}_t \int_{(t, u]} ds R_s, \quad 0 \leq t < u.$$

Heuristically, putting  $u = t + \Delta t$  for small  $\Delta t$ , we may say that  $R_t \Delta t$  is the conditional expectation of  $N_{t+\Delta t} - N_t$  given  $\mathcal{F}_t$ ; and, hence, the term

intensity at  $t$  for the random variable  $R_t$ . Also, especially when  $R_t$  depends on the past  $\{N_s : s < t\}$ , the process  $N$  is said to be *self-exciting*.

Another interpretation for 6.7 is to re-read 6.8 when  $\mathbb{E} N_t < \infty$  for every  $t$ : Then,

$$6.9 \quad \tilde{N}_t = N_t - \int_{(0,t]} ds R_s, \quad t \in \mathbb{R}_+,$$

is a martingale, and for every bounded predictable process  $F$ ,

$$6.10 \quad L_t = \int_{(0,t]} F_s d\tilde{N}_s, \quad t \in \mathbb{R}_+,$$

is again a martingale, both with respect to  $\mathcal{F}$  obviously.

Given that the counting process  $N$  admits  $R$  as its intensity process, the condition 6.8 and Theorem V.6.18 show that  $N$  is a Poisson process relative to  $\mathcal{F}$  if and only if  $R$  is deterministic, that is,  $R_t = r(t)$  for some positive Borel function  $r$  on  $\mathbb{R}_+$  with finite integral over bounded intervals. For the process  $N$  constructed in Exercise 2.36, where  $N$  is conditionally Poisson,  $R_t$  is random but is not affected by  $N$ , that is,  $R$  is exogeneous to  $N$  (of course,  $N$  depends on  $R$ ). The following suggests a construction for  $N$  in all cases. Recall that  $I(z, B)$  is one if  $z \in B$  and is zero otherwise.

6.11 THEOREM. *Let  $M$  be a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\mu = \text{Leb} \times \text{Leb}$ , and suppose that it is Poisson relative to  $\mathcal{F}$  in the sense of 6.1. Suppose that  $N$  is a counting process that satisfies*

$$6.12 \quad N_t(\omega) = \int_{[0,t] \times \mathbb{R}_+} M(\omega; ds, dz) I(z, (0, R_s(\omega)]), \quad \omega \in \Omega, \quad t \in \mathbb{R}_+,$$

for some positive  $\mathcal{F}$ -predictable process  $R$ . Then,  $N$  is adapted to  $\mathcal{F}$  and has  $R$  as its intensity relative to  $\mathcal{F}$ . Moreover, for every increasing function  $f$  on  $\mathbb{N} = \{0, 1, \dots\}$ ,

$$6.13 \quad \mathbb{E} f \circ N_t = f(0) + \mathbb{E} \int_{[0,t]} ds [f \circ (N_s + 1) - f \circ N_s] R_s, \quad t \in \mathbb{R}_+.$$

6.14 REMARK. In practice, 6.13 is more useful as a differential equation:

$$6.15 \quad \frac{d}{dt} \mathbb{E} f \circ N_t = \mathbb{E} R_t [f \circ (N_t + 1) - f \circ N_t], \quad t > 0.$$

In particular, taking  $f(n) = 1 - x^n$  for fixed  $x$  in  $[0, 1]$ , we obtain a differential equation for the generating function of  $N_t$ :

$$\frac{d}{dt} \mathbb{E} x^{N_t} = -(1 - x) \mathbb{E} x^{N_t} R_t, \quad x \in [0, 1], \quad t \in \mathbb{R}_+.$$

Also, taking  $f(n) = n$  and  $f(n) = n^2$  yield

$$\frac{d}{dt} \mathbb{E} N_t = \mathbb{E} R_t, \quad \frac{d}{dt} \mathbb{E} N_t^2 = \mathbb{E} R_t + 2 \mathbb{E} R_t N_t.$$

**Proof of Theorem 6.11**

Let  $g(\omega, s, z)$  be the integrand appearing on the right side of 6.12. The function  $g$  is the composition of the mappings  $(\omega, s, z) \mapsto R_s(\omega, z)$  and  $(r, z) \mapsto I(z, (0, r])$ ; the first mapping is measurable with respect to  $\mathcal{F}^p \otimes \mathcal{B}_{\mathbb{R}_+}$  and  $\mathcal{B}_{\mathbb{R}_+ \times \mathbb{R}_+}$  by the assumed  $\mathcal{F}$ -predictability of  $R$ ; the second is obviously measurable with respect to  $\mathcal{B}_{\mathbb{R}_+ \times \mathbb{R}_+}$  and  $\mathcal{B}_{\mathbb{R}_+}$ . Thus,  $g$  is in  $\mathcal{F}^p \otimes \mathcal{B}_{\mathbb{R}_+}$ .

Let  $F$  be a positive process in  $\mathcal{F}^p$ . Then,  $G(\omega, t, z) = F_t(\omega)g(\omega, t, z)$  defines a positive process  $G$  in  $\mathcal{F}^p \otimes \mathcal{B}_{\mathbb{R}_+}$ . It follows from 6.12 and the definitions of  $g$  and  $G$  that

$$\begin{aligned} \int_{\mathbb{R}_+} F_t dN_t &= \int_{\mathbb{R}_+ \times \mathbb{R}_+} M(dt, dz) G(t, z) \\ \int_{\mathbb{R}_+ \times \mathbb{R}_+} dt dz G(t, z) &= \int_{\mathbb{R}_+} dt F_t \int_{\mathbb{R}_+} dz I(z, (0, R_t)) = \int_{\mathbb{R}_+} dt F_t R_t. \end{aligned}$$

Taking expectations and using Theorem 6.2, we see that

$$6.16 \quad \mathbb{E} \int_{\mathbb{R}_+} F_t dN_t = \mathbb{E} \int_{\mathbb{R}_+} dt F_t R_t,$$

which proves the claim that  $R$  is the  $\mathcal{F}$ -intensity of  $N$ .

There remains to prove 6.13 for  $f$  positive increasing on the integers. We start by observing that, since  $N$  is a counting process,

$$\begin{aligned} f(N_t) &= f(0) + \sum_{s \leq t} [f(N_s) - f(N_{s-})] \\ &= f(0) + \int_{[0, t]} [f(N_{s-} + 1) - f(N_{s-})] dN_s. \end{aligned}$$

On the right side, the integrand is left-continuous in  $s$  and adapted and thus predictable, and it is positive since  $f$  is increasing. Thus, we may take the integrand to be  $F_s 1_{[0, t]}(s)$  in 6.16, which yields

$$\begin{aligned} \mathbb{E} f(N_t) &= f(0) + \mathbb{E} \int_{[0, t]} ds [f(N_{s-} + 1) - f(N_{s-})] R_s \\ &= f(0) + \mathbb{E} \int_{[0, t]} ds [f(N_s + 1) - f(N_s)] R_s, \end{aligned}$$

where the last equality is justified by noting that the value of a Lebesgue integral does not change when the integrand is altered at countably many points (by replacing  $N_{s-}$  by  $N_s$ ).  $\square$

In the setting of the preceding theorem, it follows from 6.12 that  $N$  depends on  $R$  and, hence,  $R$  depends on  $N$ . However, in non-mathematical terms, it is possible that “ $N$  depends on  $R$  but  $R$  is independent of  $N$ ,” as in the phrase “economy depends on the weather but the weather is independent of the economy.” That is the situation if  $R$  is independent of  $M$ , as in

Exercise 2.36 for instance, and then  $N$  is conditionally Poisson given  $R$ . In the more interesting case where  $R$  depends on  $M$  and  $N$ , the process  $N$  appears implicitly on the right side of 6.12; hence, then, 6.12 is an integral equation to be solved for  $N$ . The term *self-exciting* is used in such cases where the past of  $N$  over  $(0, t)$  defines, or affects, the value  $R_t$ .

### Example: Branching with immigration

This is about the evolution of a system as follows. Primary particles arrive into the system according to a Poisson process with rate  $a$ ; they form the original,  $k = 0$ , generation. The particles of the  $k^{\text{th}}$  generation give births to the particles of the  $(k+1)^{\text{th}}$  generation. Each particle, of whatever generation, gives births according to a Poisson process with rate  $b$  independent of the doings of all other particles. We are interested in the size  $N_t$  of the total population at time  $t$ . The following is the mathematical model for this story.

6.17 MODEL. Let  $M$  be a standard Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  (with mean  $\text{Leb} \times \text{Leb}$ ). Let  $\mathcal{F}$  be the augmented filtration generated by  $M$ . Define  $N$  and  $R$  by setting  $R_0 = N_0 = 0$  and

$$6.18 \quad R_t = a + bN_{t-}, \quad N_t = \int_{(0,t] \times \mathbb{R}_+} M(ds, dz) I(z, (0, R_s]), \quad t > 0,$$

where  $a$  and  $b$  are constants in  $(0, \infty)$ . Note that  $N$  is right-continuous and adapted to  $\mathcal{F}$ , and  $R$  is left-continuous and predictable.  $\square$

We describe the solution  $N(\omega) : t \mapsto N_t(\omega)$  for a typical  $\omega$  belonging to the almost sure event described in Theorem 2.18: the set  $D_\omega$  of atoms of the measure  $M_\omega$  is such that no two atoms arrive at the same time and there are only finitely many atoms in any bounded rectangle. We start with  $N_0(\omega) = 0$ ; we look for the first  $t$  such that  $(t, z)$  is an atom in  $D_\omega$  with size  $z$  under  $a$ ; that  $t$  is the time  $T_1(\omega)$  of the first jump for  $N(\omega)$ . We continue recursively: having picked  $T_k(\omega)$ , the time of the  $k^{\text{th}}$  jump, we look for the first  $t$  after  $T_k(\omega)$  such that  $(t, z)$  is in  $D_\omega$  and  $z$  is under  $a + kb$ ; that  $t$  is  $T_{k+1}(\omega)$ . In short, putting  $T_0(\omega) = 0$  for convenience, we define

$$6.19 \quad T_{k+1}(\omega) = \inf\{t > T_k(\omega) : (t, z) \in D_\omega, z \leq a + kb\}, \quad k \in \mathbb{N}.$$

Then,  $T_1(\omega) < T_2(\omega) < \dots$  are the successive jump times of  $N(\omega)$ , and

$$6.20 \quad N_t(\omega) = k \iff T_k(\omega) \leq t < T_{k+1}(\omega).$$

It follows from 6.19 and the Poisson nature of  $M$  that  $T_1, T_2 - T_1, T_3 - T_2, \dots$  are independent and exponentially distributed with respective parameters  $a, a + b, a + 2b, \dots$

To show that  $N$  is a counting process, there remains to show that it does not explode in finite time, that is,  $N_t < \infty$  almost surely for each  $t$  (which

implies that, for almost every  $\omega$ , we have  $N_t(\omega) < \infty$  for all  $t < \infty$ ). This can be shown by showing that  $\lim T_k = +\infty$  almost surely. It is easier and has more value to take a direct approach and show that  $N_t$  has finite expectation. We use 6.14 to this end:

$$\frac{d}{dt} \mathbb{E} N_t = \mathbb{E} R_t = \mathbb{E} (a + bN_{t-}) = a + b \mathbb{E} N_t$$

since  $N_t = N_{t-}$  almost surely. Solving the differential equation with the initial condition  $\mathbb{E} N_0 = 0$ , we get

$$6.21 \quad \mathbb{E} N_t = \frac{a}{b}(e^{bt} - 1), \quad t \in \mathbb{R}_+.$$

Finally, we consider the distribution of  $N_t$  for fixed  $t$ . To this end, we use 6.15 of Remark 6.14 with a well-chosen  $f$  to get a recursive formula for

$$6.22 \quad p_k(t) = \mathbb{P}\{N_t = k\}, \quad k \in \mathbb{N}, \quad t \in \mathbb{R}_+.$$

Fix  $k$ . Let  $f$  be the indicator of the set  $\{k+1, k+2, \dots\}$ . Then,  $f(N_t)$  becomes the indicator of the event  $\{N_t > k\}$ , and  $f(N_t + 1) - f(N_t)$  the indicator of  $\{N_t = k\}$ . And, on the event  $\{N_t = k\}$ , we have  $R_t = a + bk$  almost surely since  $N_t = N_{t-}$  almost surely. Thus 6.15 becomes

$$\frac{d}{dt} \mathbb{P}\{N_t > k\} = (a + bk)\mathbb{P}\{N_t = k\}, \quad k \in \mathbb{N}.$$

Equivalently,

$$\begin{aligned} \frac{d}{dt} p_0(t) &= -ap_0(t), \\ \frac{d}{dt} p_k(t) &= -(a + bk)p_k(t) + (a + bk - b)p_{k-1}(t), \quad k \geq 1, \end{aligned}$$

with the obvious initial conditions stemming from  $p_0(0) = 1$ . This system can be solved recursively:

$$6.23 \quad p_0(t) = e^{-at}, \quad p_k(t) = (a + kb - b) \int_0^t e^{-(a+bk)(t-s)} p_{k-1}(s) ds, \quad k \geq 1.$$

### Example: Self-exciting shot processes

These are similar to processes described in the preceding example, except that the immigration rate is  $ae^{-ct}$  at time  $t$  and that each particle in the system gives births at rate  $be^{-cu}$  when the particle's age is  $u$ . The preceding example is the particular case where  $c = 0$ . We are interested in the number  $N_t$  of particles in the system at time  $t$ .

6.24 **MODEL.** Let  $M$  be a standard Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}_+$ . Let  $\mathcal{F}$  be the augmented filtration generated by it. Define  $N$  and  $R$  by setting  $N_0 = 0$  and  $R_0 = a$  and

$$6.25 \quad R_t = ae^{-ct} + \int_{(0,t)} be^{-c(t-s)} dN_s, \quad t > 0,$$

$$6.26 \quad N_t = \int_{[0,t] \times \mathbb{R}_+} M(ds, dz) I(z, (0, R_s]), \quad t > 0,$$

where  $a, b, c$  are constants in  $(0, \infty)$ . Note that  $N$  is right-continuous,  $R$  is left-continuous, and both are adapted to  $\mathcal{F}$ .  $\square$

The process  $R$  of 6.25 is a shot-noise process driven by  $N$ ; it resembles that in Example 2.12, but now  $N$  is driven by  $R$  in turn. It is clear from 6.26 that  $N$  increases by jumps of size one. To show that it is a counting process, there remains to show that  $N_t < \infty$  almost surely. This last point is easy: note that  $R$  and  $N$  of Model 6.24 are dominated by the respective ones of the Model 6.17; hence,  $\mathbb{E} N_t \leq \frac{a}{b} e^{bt}$  by 6.21; in view of Remark 6.14, the exact value can be obtained by integrating 6.35 below.

The solution to the coupled system 6.25–6.26 is as follows for the typical “good”  $\omega$  of Theorem 2.18: The first jump of  $N(\omega)$  occurs at the first time  $t$  where  $(t, z)$  is an atom of  $M_\omega$  with  $z \leq ae^{-bt}$ ; for  $u$  in  $(0, t]$  we have  $R_u(\omega) = ae^{-cu}$ , and  $N_t(\omega) = 1$  obviously. Assuming that  $s$  is the  $k^{\text{th}}$  jump time and  $R_s(\omega) = r$  and  $N_s(\omega) = k$ , the time of next jump is the smallest  $t$  in  $(s, \infty)$  where  $(t, z)$  is an atom of  $M_\omega$  having  $z \leq (r+b)e^{-c(t-s)}$  and, then,  $R_u(\omega) = (r+b)e^{-c(u-s)}$  for all  $u$  in  $(s, t]$ , and  $N_u(\omega) = k$  for  $u$  in  $[s, t)$ , and  $N_t(\omega) = k+1$ .

6.27 **Markov property.** The process  $R$  is a Markov process, that is, for every time  $t$ , the future process  $\hat{R} = \{R_{t+u}; u \in \mathbb{R}_+\}$  is conditionally independent of the past  $\mathcal{F}_t$  given the present state  $R_t$ . Moreover, given that  $R_t = x$ , the conditional law of  $\hat{R}$  is the same as the law of  $R$  starting from  $a = x$ .

To see this we re-write 6.25 and 6.26 for the time  $t+u$ . Define  $\hat{M}(A) = M(\hat{A})$  with  $\hat{A} = \{(t+u, z) : (u, z) \in A\}$ , and observe that  $\hat{M}$  is independent of  $\mathcal{F}_t$  and has the same law as  $M$ , that is, Poisson on  $\mathbb{R}_+ \times \mathbb{R}_+$  with mean  $\text{Leb} \times \text{Leb}$ . Define

$$6.28 \quad \hat{R}_u = R_{t+u}, \quad \hat{N}_u = N_{t+u} - N_t, \quad u \in \mathbb{R}_+.$$

It follows from 6.25 and 6.26 after some re-arrangement that

$$6.29 \quad \hat{R}_u = \hat{R}_0 e^{-cu} + \int_{(0,u)} be^{-c(u-s)} d\hat{N}_s$$

$$6.30 \quad \hat{N}_u = \int_{[0,u] \times \mathbb{R}_+} \hat{M}(ds, dz) I(z, (0, \hat{R}_s]).$$



These show that the pair  $(\hat{N}, \hat{R})$  satisfies the equations 6.25 and 6.26 with  $a = \hat{R}_0$  and  $M$  replaced with  $\hat{M}$ . Since the solution to 6.25–6.26 is unique, and since  $\hat{M}$  is standard Poisson independent of  $\mathcal{F}_t$ , we conclude that  $(\hat{N}, \hat{R})$  is conditionally independent of  $\mathcal{F}_t$  given  $\hat{R}_0 = R_t$  and  $\hat{N}_0 = 0$ .

Indeed we have shown more than what was listed in 6.27: The pair  $(N, R)$  is a Markov process as well as the process  $R$ .  $\square$

6.31 *Distribution of  $R_t$ .* Let  $f$  be a bounded function that is differentiable and assume that its derivative  $f'$  is also bounded. Define

$$6.32 \quad Gf(x) = -cx f'(x) + x[f(x+b) - f(x)].$$

Then,

$$6.33 \quad \mathbb{E} f(R_t) = f(a) + \int_{(0,t)} ds \mathbb{E} Gf(R_s).$$

To prove this, we start by noting that each jump of  $R$  is of size  $b$  and that, between the jumps,  $R$  decays exponentially at rate  $c$ . It follows that

$$f(R_t) = f(R_0) - \int_{(0,t)} cR_s f'(R_s) ds + \int_{(0,t)} [f(R_s+b) - f(R_s)] dN_s.$$

Within the last integral, the integrand is predictable, because it is a continuous function of a left-continuous adapted process. Thus, by the meaning of intensity (see 6.6)

$$\mathbb{E} f(R_t) = f(a) - \mathbb{E} \int_{(0,t)} cR_s f'(R_s) ds + \mathbb{E} \int_{(0,t)} R_s [f(R_s+b) - f(R_s)] ds;$$

here, the boundedness of  $f$  and  $f'$  is used to ensure that the expectations are well-defined. This last formula is exactly the result that was to be shown.  $\square$

In practice, it is generally easier to use 6.33 in its differential form:

$$6.34 \quad \frac{d}{dt} \mathbb{E} f(R_t) = \mathbb{E} Gf(R_t), \quad \mathbb{E} f(R_0) = f(a).$$

Here are several quick uses: Taking  $f(x) = x$  and solving 6.34,

$$6.35 \quad \mathbb{E} R_t = ae^{-(c-b)t}, \quad t \in \mathbb{R}_+.$$

Taking  $f(x) = e^{-px}$  and noting that

$$Gf(x) = (1 - e^{-pb} + cp) \frac{\partial}{\partial p} e^{-px},$$

we see that

$$6.36 \quad u(t, p) = \mathbb{E} e^{-pR_t}, \quad t, p \in \mathbb{R}_+,$$

is the solution to

$$6.37 \quad \frac{\partial}{\partial t} u(t, p) = (1 - e^{-pb} + cp) \frac{\partial}{\partial p} u(t, p)$$

with boundary conditions  $u(t, 0) = 1$  and  $u(0, p) = e^{-ap}$ .  $\square$

Going back to 6.35, we observe that  $\mathbb{E} R_t = a$  for all  $t$  if  $c = b$ . Otherwise, if  $c > b$ , the expectation goes to 0 as  $t \rightarrow \infty$ , and, if  $c < b$ , it goes to  $+\infty$ . Indeed, these observations can be sharpened:

6.38 *Martingales.* The process  $R$  is a martingale if  $c = b$ , supermartingale if  $c > b$ , and submartingale if  $c < b$ . If  $c \geq b$ ,

$$R_\infty = \lim_{t \rightarrow \infty} R_t$$

exists and is an integrable random variable with  $0 \leq \mathbb{E} R_\infty \leq a$ . In particular, when  $c > b$ , we have  $R_\infty = 0$  almost surely.

To show these claims, we go back to the Markov property 6.27 for  $R$  to conclude that the conditional expectation of  $R_{t+u}$  given  $\mathcal{F}_t$  is equal to  $\mathbb{E} R_u$  with  $a$  replaced by  $R_t$ , that is, in view of 6.35,

$$6.39 \quad \mathbb{E}_t R_{t+u} = R_t e^{-(c-b)u}, \quad t, u \in \mathbb{R}_+.$$

This shows that  $R$  is a martingale if  $c = b$ , supermartingale if  $c > b$ , and submartingale if  $c < b$ . In the first two cases, that is, if  $c \geq b$ , we have a positive supermartingale, which necessarily converges almost surely to some integrable random variable  $R_\infty \geq 0$ ; that is by the convergence theorems for such. Moreover, by Fatou's lemma,

$$\mathbb{E} R_\infty = \mathbb{E} \liminf R_t \leq \liminf \mathbb{E} R_t,$$

which shows that  $0 \leq \mathbb{E} R_\infty \leq a$  via 6.35. In particular, this becomes  $\mathbb{E} R_\infty = 0$  when  $c > b$ , which implies that  $R_\infty = 0$ .

## Compensators

Not every counting process has an intensity process. This is to compensate for this lack and introduce the general ideas involved.

Let  $N$  be a counting process adapted to some filtration  $\mathcal{F}$  and suppose that  $\mathbb{E} N_t < \infty$  for every  $t$  in  $\mathbb{R}_+$ . Then, it is known that there exists an increasing  $\mathcal{F}$ -predictable process  $C = (C_t)$  such that

$$6.40 \quad \tilde{N}_t = N_t - C_t, \quad t \in \mathbb{R}_+,$$

is an  $\mathcal{F}$ -martingale; this follows from the continuous time version of Doob's decomposition applied to the submartingale  $N$ . The process  $C$  is called the *compensator* for  $N$  relative to  $\mathcal{F}$ , or *dual-predictable projection* of  $N$ . The

term *compensator* may be justified by noting that  $\tilde{N}$  is a martingale if and only if

$$6.41 \quad \mathbb{E} \int_{\mathbb{R}_+} F_t dN_t = \mathbb{E} \int_{\mathbb{R}_+} F dC_t$$

for every positive (or bounded)  $\mathcal{F}$ -predictable process  $F$ .

Comparing 6.41 with Definition 6.6 of intensities, we conclude the following: An  $\mathcal{F}$ -predictable process  $R$  is the intensity of  $N$  relative to  $\mathcal{F}$  if and only if

$$6.42 \quad C_t = \int_0^t ds R_s, \quad t \in \mathbb{R}_+.$$

is the compensator of  $N$  relative to  $\mathcal{F}$ . Thus, existence of an intensity has to do with the absolute continuity of  $C$ .

The following is the counterpart to the construction of Theorem 6.11. It is analogous to Theorem 6.18 of Chapter V on non-stationary Poisson processes. When time is reckoned with the clock  $C$ , the process  $N$  appears to be Poisson with unit rate.

6.43 THEOREM. *Let  $N$  be a counting process adapted to  $\mathcal{F}$  and with  $\mathbb{E} N_t < \infty$  for all  $t$ . Let  $C$  be its compensator relative to  $\mathcal{F}$ . Suppose that  $C$  is continuous and  $\lim_{t \rightarrow \infty} C_t = +\infty$ . Define, for  $u$  in  $\mathbb{R}_+$ ,*

$$6.44 \quad S_u = \inf\{t : C_t > u\}, \quad \hat{N}_u = N_{S_u}, \quad \hat{\mathcal{F}}_u = \mathcal{F}_{S_u}.$$

*Then,  $\hat{N}$  is a Poisson process with unit rate with respect to  $\hat{\mathcal{F}}$ , and for each  $t$ , almost surely,*

$$6.45 \quad N_t = \hat{N}_{C_t}.$$

REMARK. We regard  $C_t$  as time shown on a rigged clock when the standard time is  $t$ . Then,  $S_u$  becomes the standard time when the clock shows  $u$ . If  $N$  is an arrival process with standard time parameter, then  $\hat{N}$  is the same arrival process in clock time. In general  $C$  and  $\hat{N}$  are dependent. In the special case that  $C$  and  $\hat{N}$  are independent, the formula 6.45 is very convenient: for instance, then, the conditional probability that  $N_t - N_s = k$  given  $\mathcal{F}_s$  is equal to, for  $0 \leq s < t$  arbitrary,

$$\mathbb{E}_s e^{-(C_t - C_s)} (C_t - C_s)^k / k!, \quad k \in \mathbb{N}.$$

In other words, if  $\hat{N}$  and  $C$  are independent, then the conditional law of  $N$  given  $C$  is that of a non-stationary Poisson process with mean function  $C$ . The further special case where  $C$  is deterministic yields a non-stationary Poisson process  $N$ .

*Proof.* Let  $S$  and  $T$  be stopping times with  $S \leq T < \infty$ . Let  $H \in \mathcal{F}_S$  and define  $F_t = 1_H 1_{(S, T]}(t)$ . Then,  $F$  is predictable, and using 6.41 with this  $F$  shows that

$$6.46 \quad \mathbb{E}_S(N_T - N_S) = \mathbb{E}_S(C_T - C_S).$$

Fix  $0 \leq u < v$  and let  $S = S_u$  and  $T = S_v$ ; these are finite stopping times in view of 6.44 and the assumption that  $\lim C_t = \infty$  while  $C_t < \infty$  almost surely for every  $t$  in  $\mathbb{R}_+$  (since  $\mathbb{E}C_t = \mathbb{E}N_t < \infty$ ). Now,  $N_T - N_S = \hat{N}_v - \hat{N}_u$  by the definition of  $\hat{N}$ , and  $C_T - C_S = v - u$  by 6.44 and the continuity of  $C$ . It now follows from 6.46 that  $\hat{N}_v - \hat{N}_u$  is independent of  $\hat{\mathcal{F}}_u$  and has mean equal to  $v - u$ . It now follows from the characterization theorem 5.5 (or Theorem 6.18 of Chapter V) that  $\hat{N}$  is Poisson with respect of  $\hat{\mathcal{F}}$  with unit rate.

There remains to show that, for each  $t$ , 6.45 holds almost surely. If  $C$  is strictly increasing, then  $S_{C_t} = t$  and the equality 6.45 is without question. If  $C$  is not such, then  $S_{C_t} \geq t$  and the strict inequality may hold for some outcomes  $\omega$ . For such  $\omega$ , however,  $C(\omega)$  remains flat and equal to  $C_t(\omega)$  over the interval  $(t, S_{C_t}(\omega)]$ . In view of 6.46, almost surely on the set of all such  $\omega$ , the process  $N$  can have no jumps over the interval  $[t, S_{C_t}]$ . Thus,  $N_t = \hat{N}_{C_t}$  almost surely.  $\square$

## Exercises

6.47 *Shot process.* For the process  $R$  of model 6.24, show that

$$\text{Var } R_t = ab^2 \int_t^{2t} e^{-(c-b)s} ds.$$

6.48 *Continuation.* Show that, for  $f$  as in 6.31 and  $g : \mathbb{N} \mapsto \mathbb{R}$  arbitrary bounded,

$$\frac{d}{dt} \mathbb{E} f(R_t)g(N_t) = \mathbb{E} h(R_t, N_t)$$

where

$$h(x, n) = -cx f'(x)g(n) + x[f(x+a)g(n+1) - f(x)g(n)].$$

Use this to derive a partial differential equation for

$$u(t, p, q) = \mathbb{E} \exp_{-}(pR_t + qN_t).$$

6.49 *Covariance density for  $N$  of 6.24.* We switch to regarding  $N$  as a random measure on  $\mathbb{R}_+$ . Notationally,  $dN_t$  becomes  $N(dt)$ , and  $d\hat{N}_u$  becomes  $N(t+du)$  when  $\hat{N}_u = N_{t+u} - N_t$ . Notice that 6.8 and 6.39 may be presented as

$$\mathbb{E}_t N(t+du) = du \mathbb{E}_t R_{t+u} = du R_t e^{-(c-b)u}.$$

Show that, for  $t, u > 0$ ,

$$\frac{1}{dt du} [\mathbb{E} N(dt)N(t+du) - \mathbb{E} N(dt) \mathbb{E} N(t+du)] = e^{-(c-b)u} \text{Var } R_t.$$

The left side is called the covariance density at  $(t, t+u)$ .

**6.50 Branching formulation.** This is another formulation of the model 6.24. We view the population as the sum of generations  $0, 1, \dots$ ; The original generation consists of particles that arrived into the system according to a Poisson process  $N_0$  with deterministic intensity process  $R_0(t) = ae^{-ct}$ ,  $t \in \mathbb{R}_+$ . Having described the generation  $k$ , we recall that each particle of generation  $k$ , if it starts life at time  $s$ , gives births to generation  $k+1$  particles at the rate  $be^{-cu}$  at the time  $s+u$ . Thus, the births of  $(k+1)^{\text{th}}$  generation particles form a conditionally Poisson process  $N_{k+1}$  given the intensity process

$$R_{k+1}(t) = \int_{(0,t)} N_k(ds) be^{-c(t-s)}.$$

The process  $N$  is the sum of all generations,

$$N(A) = \sum_{k=0}^{\infty} N_k(A), \quad \text{Borel } A \subset \mathbb{R}_+.$$

The advantage of this formulation is that we have a chain  $N_0, N_1, N_2, \dots$  of random counting measures on  $\mathbb{R}_+$ , each one determines the intensity process for the next, each one being conditionally Poisson given its own intensity.

**6.51 Hawkes processes.** These are processes much like  $N$  of the model 6.24, but with some slight generality:  $N$  is still defined by 6.26, but 6.25 is replaced with

$$R_t = a + \int_{(0,t)} g(t-s) dN_s, \quad t > 0,$$

where  $g$  is some deterministic positive Borel function.

a) Suppose that  $g$  is bounded. Show that, then,  $\mathbb{E} N_t < \infty$  for all  $t$  and there is a unique counting process  $N$  with this intensity process. Hint: Compare this with the model 6.17.

b) Compute  $\mathbb{E} R_t$  and  $\mathbb{E} N_t$  exactly.

c) Discuss the limiting behavior of  $R_t$  as  $t \rightarrow \infty$  in the case when  $g$  is Lebesgue-integrable.

**6.52 Continuation.** Consider the processes  $R$  and  $N$  of the preceding exercise for the special case

$$g(u) = 1_{[b,c)}(u), \quad u \in \mathbb{R}_+,$$

where  $0 < b < c$  are constants. This corresponds to the case each particle gives births to new ones at unit rate starting when it is at age  $b$  and ending when it dies at age  $c$ . Describe the solution  $N$ . Compute  $\mathbb{E} N_t$  explicitly.

**6.53 Departures from an  $M/M/\infty$  queue.** Let  $L$  be a Poisson process with rate  $a$ . Let  $M$  be as in Theorem 6.11. Suppose that  $L$  and  $M$  are independent and let  $\mathcal{F}$  be the augmented filtration generated by them. Define  $N$  by 6.12 with

$$R_t = b \cdot (L_{t-} - N_{t-})^+, \quad t > 0,$$

with  $R_0 = N_0 = 0$ . Note that  $R_t = 0$  when  $L_{t-} \leq N_{t-}$ , which implies that  $L_t \geq N_t$  for all  $t$ .

a) Show that

$$\mathbb{P}\{L_{t+u} - L_t = N_{t+u} - N_t = 0 | L_t - N_t = k\} = e^{-(a+bk)u}.$$

b) Show that  $Q = L - N$  can be regarded as the queue size process in Exercise 3.35 with the further assumption that each service lasts an exponentially distributed time with parameter  $b$ .



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