# Chapter 1
# The Inception of the Noether Theorems

Emmy Noether's two theorems on the relation between symmetries and conservation laws were a response to the mathematical problems that arose when Einstein proposed the generally covariant equations of general relativity, and when Hilbert and Klein pursued research related to the new physical theory. They served both to elucidate the problem of the conservation of the energy-momentum tensor in that new theory, and to reconcile formulations of the law of conservation of energy that had appeared, *a priori*, to be quite distinct. Her first theorem also offered a vast generalization of the conservation theorems in mechanics and in the special theory of relativity that had been known at the time. Using Lie's theory of continuous groups of transformations, she presented remarkably general results for the problem of applying the theory of differential invariants to the variational equations of physics.

## 1.1 From the Theory of Invariants to Special Relativity

The mid-nineteenth century was the period when the theory of invariants was created. Its origin is to be found in a problem in projective geometry, the search for a polynomial function, more generally, for a quantity defined on projective space, that would be invariant under any change of projective coordinates, which is to say, the search for a polynomial function or, more generally, a quantity that has an intrinsic geometric meaning.[1] The prototype of an algebraic invariant is the discriminant of a quadratic polynomial which remains identical to itself under a unimodular change of coordinates, i.e., one that conserves volumes. The vanishing of this discriminant corresponds to the degeneration of the associated quadratic equation. For Weyl, Arthur Cayley's "Mémoire sur les hyperdéterminants" [1846] was the founding paper for

[1] See Weitzenböck [1923], Study [1923], Weyl [1939], Dieudonné and Carrell [1971], Hawkins [1998], Procesi [1999], and Olver [1999]. The latter work contains 240 references to papers on invariants of which more than fifty were published before 1900. For the history of the theory of invariants, see, for example, the articles by Charles S. Fisher [1966] and Karen Hunger Parshall [1989].

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the theory of algebraic invariants.[2] Sylvester [1851] formulated the context in which one sought invariants. Given a "form," i.e., a homogeneous polynomial in several variables, and an "associated form," i.e., the polynomial such that its value on the variables which have undergone a linear or projective transformation is equal to that of the original polynomial evaluated on the nontransformed variables, he proposed that one seek quantities that remained unchanged under such a transformation, i.e., invariants. He introduced the concepts of covariant and contravariant substitutions to express the two ways in which the coefficients of a given form may be transformed into an associated form.[3] Thus defined, the search for the invariants of a form of given degree became a purely formal problem. Given a special class of forms, for example the binary quadratic forms, i.e., the homogeneous quadratic polynomials in two variables, the question was to find a complete list of all the algebraic invariants of a form of that class as functions of its coefficents. As early as 1858, Siegfried Aronhold and then Alfred Clebsch in 1861, Paul Gordan in 1868 and, after them, Heinrich Maschke [1900][1903] among other mathematicians, especially in Italy, developed an algorithmic method, called the symbolic method, based on the consideration of the decomposable elements in tensor products,[4] with the objective of obtaining from a known invariant for a form of a given class all that form's other invariants.

The research then turned toward the invariants of differential forms, in which case the coefficients are functions. Since the coefficients of those forms are not constant, their derivatives figure in the transformed expressions, and the invariants that are sought were called *differential invariants*. The symbolic method also worked for this type of invariant[5] but, because it appeared to be entirely calculatory, did not clarify the significance of the problem or reveal the new avenues that it in fact opened. On the one hand, it led naturally to the "absolute differential calculus," the tensor calculus and the covariant derivation of Gregorio Ricci-Curbastro and Tullio Levi-Civita[6] on manifolds,[7] because in fact, defining a tensor on a manifold amounts to

[2] Weyl [1939], p. 27.

[3] These two ways depend on whether one chooses to consider the coefficients of that "form" as, in modern terms, the components of a covariant or a contravariant tensor. Weitzenböck, in the preface to his book [1923], writes that a "tensor is finally nothing more than another name for what had hitherto been called a 'form'" ("Tensor is ja schließlich nur ein anderer Name für das, was man bisher 'Form' genannt hat"), and, in chapter 5, §15, he defines "covariants" and "contravariants." Tensors had been introduced by Waldemar Voigt in 1898 in his studies on crystallogrophy.

[4] Weyl [1939], p. 20. A modern description of the symbolic method may be found in Howe [1988], and see the indications in Hawkins [1998]. For examples of this method, see the papers and books cited above and, in particular, Weitzenböck [1923], chapter 1, §8, 10 and 13, and see Study [1923].

[5] See Wright [1908].

[6] An article by Ricci which gave a summary of his previous publications appeared in 1892. There subsequently appeared an article by Levi-Civita [1896], cited by Wright, and then the long article by Ricci and Levi-Civita in the *Mathematische Annalen* [1900]. See Weitzenböck [1923], chapter 13.

[7] Poincaré [1899], p. 6, note 1, wrote, "The word *variété* [translated here as 'manifold'] is now sufficiently well known so that I do not think it necessary to recall its definition. That is how one refers to a continuous set of points (or of systems of values): thus it is that in three-dimensional space,

defining it locally in a formulation that is invariant under a change of charts. Subsequently, this method was adapted for the determination of Poincaré's and Élie Cartan's integral invariants[8] to which the techniques of the variational calculus apply.[9] On the other hand, the search for methods that determine differential invariants led to differential equations that were invariant under the action of a group; one could therefore apply to this search the theory of continuous Lie groups of transformations[10] which permits expressing the invariance of an equation with respect to such a group, or even with respect to a local group. Lie had indeed devised a method for expressing such invariance by the vanishing of the directional derivatives, which have since been called Lie derivatives,[11] in the directions that are determined by the

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any surface is a two-dimensional manifold and any line a one-dimensional manifold" ("Le mot *variété* est maintenant assez usité pour que je n'aie pas cru nécessaire d'en rappeler la définition. On appelle ainsi tout ensemble continu de points (ou de systèmes de valeurs) : c'est ainsi que dans l'espace à trois dimensions, une surface quelconque est une variété à deux dimensions et une ligne quelconque, une variété à une dimension"). But Élie Cartan, who had studied with Poincaré, gave a definition of an abstract manifold in 1925 and reproduced it in his *Leçons sur la géométrie des espaces de Riemann* (1928) where he wrote: "The general concept of a manifold is rather difficult to define precisely" ("La notion générale de variété est assez difficile à définir avec précision"). For the history of the concept of manifold, going back to Bernhard Riemann, see Scholz [1999a].

[8] Poincaré [1899], Cartan [1922]. (See also, *infra*, Chap. 4, p. 99, note 34.) In the introduction to his book on integral invariants, a published version of the course that he gave at the Sorbonne in Paris in 1920–1921, Cartan wrote (p. ix), "Several chapters are devoted to the rules for the calculus of the differential forms which appear under the symbols for multiple integration. [...] I propose to call them differential forms with exterior multiplication or, in short, exterior differential forms, because they obey the rules of H. Grassmann's exterior multiplication." ("Plusieurs chapitres sont consacrés aux règles de calcul des formes différentielles qui se présentent sous les signes d'intégration multiple. [...] Je propose de les appeler formes différentielles à multiplication extérieure, ou, plus brièvement, formes différentielles extérieures, parce qu'elles obéissent aux règles de la multiplication extérieure de H. Grassmann.").

[9] See Weitzenböck [1923], chapter 14.

[10] Lie and Engel [1893]. The continuous groups are now called Lie groups. Léon Autonne (1859–1916) entitled a note to the *Comptes rendus* of the Paris Academy of Sciences, "On an application of the groups of Mr. Lie" [1891]. To the best of our knowledge, the first printed mention in French of the expression "groupes de Lie" is to be found in the thesis of Arthur Tresse [1893], "On the differential invariants of continuous groups of transformations," defended 30 November of that year at the University of Paris. Tresse, who had been a student of Lie in Leipzig, wrote in his introduction, "I recall the general propositions of M. Lie regarding the groups defined by systems of partial differential equations, groups that I call *Lie groups*." ("Je rappelle les propositions générales de M. Lie, sur les groupes définis par des systèmes d'équations aux dérivées partielles, groupes que j'appelle *groupes de Lie*.") Letters from Tresse to Lie from 1892 have been conserved in which he had already proposed that term. (See Stubhaug [2000], English translation, p. 370.) In English, the expression *Lie groups* was not yet current when Tresse was writing. Wright [1908] still referred to "the theory of groups of Lie." On the emergence of the theory of Lie groups, see Hawkins [2000].

[11] Noether refers to the vanishing of a Lie derivative as "Lie's differential equation" ("die Lie'sche Differentialgleichung"). Jan Arnoldus Schouten [1954], p. 104, note 1, defines the Lie derivatives and asserts that the term was used for the first time by David van Dantzig in two notes published in the transactions of the Amsterdam Academy of Science [1932]. In fact, in his second note, van Dantzig defines the operation of Lie derivation on tensors, but he attributes the first use of the term to Władisław Ślebodziński [1931] and adds that he owes the definition that he is presenting

underlying infinitesimal group, i.e., the Lie algebra of the Lie group.[12] This point of view was used by Joseph Edmund Wright [1908] in his search for the invariants of quadratic differential forms.

The connection between the search for differential invariants and that for the quantities conserved in the time-evolution of physical systems appeared gradually and, in its complete generality, only in the article of Noether that is the subject of this study. It is a consequence of a particular case of her first theorem, that of a differential equation deriving from a variational principle with a single independent variable. Edmund T. Whittaker (1873–1956), in his treatise on dynamics [1904], attributed the discovery of the laws of conservation of both linear momentum (2nd ed., 1917, p. 59) and angular momentum (p. 60) to Newton who, on the one hand, had already observed that, in the absence of exterior forces, the center of mass of a mechanical system is either at rest or displaced in a uniform rectilinear motion and, on the other, had generalized Kepler's law of areas. Concerning the law of conservation of energy, Whittaker recognized the role of Joseph-Louis Lagrange (p. 62) who, according to Aurel Wintner,[13] knew the consequences of the Galilean invariance of the equations of motion as early as 1777. Indeed, Lagrange proposed a new method in his "General remarks on the motion of several bodies that attract one another following the law of inverse squared distances" in order to obtain those laws of conservation that were already known.[14]

Lagrange wrote in the "Avertissement" of his *Méchanique Analitique* [1788], "This treatise [. . . ] will collect and present from a unified point of view the various principles that have been used until now to permit the solution of questions in mechanics."[15] He stated two fundamental principles of the calculus of variations,

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to Schouten and Egbert R. van Kampen who introduced it in an article which would in fact be published in Warsaw, in the *Prace Matematyczno-Fizycze*, in 1934 (vol. 41, pp. 1–19). We should remark that his article III, successor to the two articles which appeared in 1932, appeared in the same journal in 1934, but in English rather than in German, which demonstrates ever so clearly the impact that the Nazi seizure of power had upon the scientific community.

[12] The elements of the Lie algebra of a Lie group are the infinitesimal generators of its one-parameter subgroups. It is well known that the "infinitesimal group" introduced by Lie did not receive its modern name, "Lie algebra," until the 1930s. Nathan Jacobson writes, in the preface to his book [1962], p. v, that "it should be noted also that in these lectures [at the Institute for Advanced Study at Princet0on in 1933–1934] Professor Weyl, although primarily concerned with the theory of continuous Lie groups, set the subject of Lie algebras on its own independent course by introducing for the first time the term "Lie algebra" as a substitute for "infinitesimal group," which had been used exclusively until then." According to A. John Coleman [1997], this term, which had in fact been proposed by Jacobson and adopted by Weyl after some hesitation, had first been used by Richard Brauer in his edition of the notes of Weyl's 1934–1935 course, but was not immediately adopted. Weyl wrote, "In homage to Sophus Lie such an algebra is nowadays called a *Lie algebra*" ([1939], p. 260). In the bibliography of Jacobson's book one finds the expressions *Lie Ring* and *Liescher Ring* for the articles written in German after 1935, by Walter Landherr in that year, by Ernst Witt in 1937 and by Hans Zassenhaus in 1939.

[13] Wintner [1941], p. 426.

[14] Lagrange [1777], p. 162; *Œuvres de Lagrange*, vol. 4, p. 406.

[15] "Cet ouvrage [. . . ] réunira et présentera sous un même point de vue les différents Principes trouvés jusqu'ici pour faciliter la solution des questions de Méchanique," Lagrange [1788], p. v.

one of which is that "the known operation of integration by parts"[16] permits the elimination of the differentials of the variation.

He claimed that his analytical method for deriving "a general formula for the motion of bodies" ("une formule générale pour le mouvement des corps") yields "the general equations that contain the principles, or theorems known by the names of the conservation of kinetic energy, of the conservation of the motion of the center of mass, of the conservation of the momentum of rotational motion, or the principle of areas, and of the principle of least action."[17] He ascribed the first to Huygens (p. 183, and also p. 171), the second to Newton and to d'Alembert for a generalization (p. 185), the third to Euler, Daniel Bernoulli, and the Chevalier d'Arcy (1725–1779) (p. 186) and the fourth, founded on the principle of Maupertuis (1698–1759), to Euler for isolated bodies [1744], then to himself for interacting bodies (p. 188). While, before Lagrange, the various conservation results had been taken to be first principles belonging to the foundations of dynamics, Lagrange viewed them as consequences of the equations of dynamics, an important shift of point of view. But there was still no explicit link with invariance properties in this first edition, although on page 415, for the equations of the top in what is now called "the Lagrange case," he derived a first integral from the consideration of what would later be called an ignorable variable.

Lagrange proposed "The simplest method to obtain the equations which determine the movement of an arbitrary system of bodies subject to arbitrary accelerating forces,"[18] and he concluded that the equation he obtained "is entirely analogous to those found by the method of variations for the determination of maxima and minima of integral formulas, and will have to be treated according to the same rules."[19] The method of maxima and minima had already figured prominently in Euler's treatise "Method for the determination of curves enjoying a property of maximum or minimum" [1744], where he wrote, in the chapter on elastic curves, that, just as the center of mass must rest at the lowest point, "the curvature of rays traveling through a transparent medium of varying density is also, *a priori*, determined by the principle that they must reach a given point in the shortest possible time."[20] Euler applied his methods to many problems and asserted that "the methods described in this book are not only of great use in analysis, but are also most helpful for the solution of

[16] "L'opération connue des intégrations par parties," *ibid.*, p. 56.

[17] "Les équations générales qui renferment les Principes, ou théorèmes connus sous les noms de *conservation des forces vives*, de *conservation du mouvement du centre de gravité*, de *conservation du moment de mouvement de rotation*, ou *Principe des aires*, et de *principe de la moindre quantité d'action*," *ibid.*, p. 182.

[18] "Méthode la plus simple pour parvenir aux équations qui déterminent le mouvement d'un système quelconque de corps animés par des forces accélératrices quelconques," *ibid.*, p. 216.

[19] "Cette équation est entièrement analogue à celles que l'on trouve par la méthode des variations pour la détermination des maxima et minima des formules intégrales, et il faudra la traiter suivant les mêmes règles," *ibid.*, p. 231.

[20] "Similiter curvatura radiorum per medium diaphanum variæ densitatis transeuntium, tam a priori est determinata, quam etiam ex hoc principio, quod tempore brevissimo ad datum locum pervenire debeant," Euler [1744], p. 246.

problems in physics."[21] In Euler's notation, the equation expressing the fact that an integral is stationary takes the form, first, $N - \frac{P' - P}{dx} = 0$, and then $N - \frac{dP}{dx} = 0$,[22] an equation to be found later also in his "Elements of the calculus of variations" [1766], and which would be generalized by Lagrange. In particular, in his letter of 1756 to Euler, Lagrange considered the variation of double integrals for the first time.

It is only in the second edition, *Mécanique Analytique* [1811], that Lagrange observed a correlation between symmetries and the principles of conservation of certain quantities, in particular energy. In the first section of the second part of his treatise, concerning dynamics, he presented a detailed history of the diverse "principles or theorems" discovered by Galileo, Huygens, Newton, Daniel Bernoulli, Euler, d'Alembert and several other physicists. Concerning the conservation of the angular momenta he wrote, "Regarding the movement of several bodies about a fixed center, the sum of the products of the mass of each of those bodies by the velocity of its motion about that center, and by its distance from that center [...] is constant so long as there is no other action nor any exterior obstacle."[23] In article 7 of the fourth section, Lagrange introduced (p. 288) the kinetic energy, $T = \frac{1}{2} m \left( \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right)$, and, in the case where the force derives from a potential,[24] which he denoted by $V$, he wrote, for the "Lagrangian" $T - V$, the "Euler–Lagrange equations" (article 10, p. 290) using the method of the calculus of variations which he had introduced as early as 1760 to serve as the fundamental method of dynamics.[25] Then he asserted (article 14),

> An integration which can always be performed when the forces are functions of distances and the functions $T$, $V$, $L$, $M$, etc.[26] do not contain the finite variable $t$ is the one that yields the principle of the conservation of kinetic energy.[27]

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[21] "Methodi in hoc libro traditæ, non solum maximum esse usum in ipsa analysi, sed etiam eam ad resolutionem prolematum physicorum amplissimum subsidium afferre," *ibid.*, p. 245.

[22] Setting $N = \frac{\partial L}{\partial y}$ and $P = \frac{\partial L}{\partial y'}$, this equation takes the usual form of the case of a one-dimensional variational problem. The literature on the history of the calculus of variations is vast. See Goldstine [1980], Kreyszig [1994], and René Taton on the relations of Euler and Lagrange [1983].

[23] "Dans le mouvement de plusieurs corps autour d'un centre fixe, la somme des produits de la masse de chaque corps par sa vitesse de circulation autour du centre, et par sa distance au même centre [...] se conserve la même tant qu'il n'y a aucune action ni aucun obstacle extérieur." We cite the first volume of the 1965 edition, p. 227. We thank Professors Jean-Marie Souriau and Patrick Iglésias-Zemmour for calling our attention to several passages in Lagrange's work. We also benefited from unpublished research on Lagrange by Alain Albouy. For this aspect of Lagrange's work, see Vizgin [1972]. See also Marsden and Ratiu [1999], pp. 231–234.

[24] Modern notational practice has retained Lagrange's $V$ for the potential which is the opposite of the force function.

[25] Lagrange [1760].

[26] $L = 0$, $M = 0$, etc. represent the constraint equations.

[27] "Une intégration qui a toujours lieu lorsque les forces sont des fonctions de distances [i.e., ne dépendent pas des vitesses], et que les fonctions $T$, $V$, $L$, $M$, etc., ne contiennent point la variable finie $t$, est celle qui donne le principe de la conservation des forces vives," p. 295.

By means of the formula for integration by parts, he then demonstrated this "principle," that is, the theorem asserting that the total energy of a system, $T+V$, remains constant.[28] Concerning the other first integrals that were already known, Lagrange was less precise, merely saying, "The other integrals will depend on the nature of the differential equations of each problem, and one cannot provide a general method for finding them."[29]

Some thirty years later, Carl Gustav Jacobi (1804–1851), in his "Lectures on Dynamics," a course given at the Universiy of Königsberg in 1842–1843,[30] dealt with the relation between the Euclidean invariance of the Lagrangian in mechanics under the action of translations and rotations, and the laws of the conservation of linear and angular momenta. The third, fourth, fifth and sixth lectures of this course deal respectively, with the principle of the conservation of the motion of the center of mass, of the kinetic energy, of areas, and with the principle of least action. [31]

In 1897, Ignaz R. Schütz, then a member of the Institute for Theoretical Physics at Göttingen,[32] studied the principle of the conservation of energy and showed that it was largely independent of the principle of the equality of action and reaction asserted by Newton, and then derived the law of conservation of energy from the equations of motion, first for an isolated massive point particle, and then for a system of particles.

It was by using the theory of Lie groups and, in particular, the concept of infinitesimal transformation, that Georg Hamel[33] proposed establishing relations between mechanics and several domains of mathematics including, in particular, the calculus of variations. He published his habilitation thesis [1904a] and then an article, "On virtual displacements in mechanics" [1904b], where he studied the equivalence of various forms of the equations of mechanics and how they would change under virtual displacements. To that end he used the Lie brackets of infinitesimal symmetries (p. 425), which he called "the Jacobi symbols" ("die Jacobischen Symbole"), as well as the structure constants of the Lie group with which he was dealing

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[28] Concerning the meanings attributed to the conservation of energy before Hermann von Helmholtz (1821–1894) [1887] and especially Lagrange's concept of energy, consult Elkana [1974].

[29] "Les autres intégrales dépendront de la nature des équations différentielles de chaque problème ; et l'on ne saurait donner de règle générale pour les trouver," p. 297.

[30] Jacobi [1866]. This series of lectures was published posthumously by Clebsch.

[31] "Das Princip der Erhaltung der Bewegung des Schwerpunkts, der lebendigen Kraft, der Flächenräume, der kleinsten Wirkung (des kleinsten Kraftaufwandes)." The French "forces vives" and the German "lebendige Kraft" are translations of the Latin term "vis viva," introduced by Leibniz. The kinetic energy is one-half of the *vis viva*. In his book on the stability of motion [1877], Routh called the kinetic energy the "semi vis viva.'

[32] For Schütz, see Scott Walter's thesis, "Hermann Minkowski et la mathématisation de la relativité restreinte, 1905–1915," Nancy, 1996, or Rowe [2009]. Schütz, who was assistant to Ludwig Boltzmann (1844–1906) in Munich from 1891 to 1894, died in 1926. Schütz's article [1897] would be cited by Hermann Minkowski in his lecture in Cologne in 1908, translated in Lorentz *et al.* [1923].

[33] Hamel (1877–1954) was a student of Hilbert who defended his thesis in 1901. He was the author of several important treatises on mechanics. On p. 4, note 4, of [1904a], and on p. 417 of [1904b] he wrote of *der Lieschen Gruppentheorie*.

(p. 428). Ultimately, he asserted the equivalence of two forms of the equations of mechanics in the case of $n$ virtual displacements corresponding to the infinitesimal transformations of an $n$-parameter group.

Next, it was Gustav Herglotz (1881–1953) who studied various questions in the mechanics of solid bodies from the point of view of the special theory of relativity [1911]. He considered the ten-parameter invariance group[34] which acts on the four-dimensional space-time, now called Minkowski space-time. In his section 9 (pp. 511–513), using a method of the calculus of variations that would be used by Noether seven years later, he derived ten first integrals associated to the ten infinitesimal transformations of the Poincaré group. This section would be cited by Noether [1918c] and by Klein [1927].

In 1916 there appeared in the *Göttinger Nachrichten* a letter that Friedrich Engel[35] had addressed to Klein in which he remarked that, working from Herglotz's result and letting the speed of light tend to infinity, one could recover the ten well-known integrals of nonrelativistic mechanics. He then proposed to obtain the same result directly, without passing to the limit, by means of Lie's theory.[36] Using the Hamiltonian formalism and the invariance of the Hamiltonian under the action of the ten infinitesimal transformations of the ten-parameter group, called the Galilean group, he obtained the ten first integrals of the $n$-body problem, and in particular he recovered Schütz's 1897 result on the conservation of the total energy of the system. In a second letter [1917], Engel showed how to use the conserved quantities to integrate the equations of mechanics by the method of Lie, but he did not use a variational method in either paper.

Finally, on 15 August 1918, while Noether was completing the definitive version of her manuscript for the *Invariante Variationsprobleme*, Alfred Kneser[37] submitted an article to the *Mathematische Zeitschrift*, "Least action and Galilean

[34] This 10-dimensional Lie group, which is the semi-direct product of the 6-dimensional Lorentz group and the 4-parameter group of translations, was called by Herglotz "the 10-term group of 'motions"' ("die zehn gliedrige Gruppe der ‚Bewegungen'"). It is now called the Poincaré group, a term used for the first time by Wigner in 1939 (see Mehra [1974], p. 70). Wigner wrote ([1967], p. 18), "I like to call the group formed by these invariables [*sic*] the Poincaré group," and referred to Poincaré's publications of the years 1905 and 1906. According to Klein (in a letter to Pauli in 1921, see Appendix III, pp. 159–160), it was Poincaré who had perceived that the transformations introduced by Lorentz form a group, and, according to Wigner ([1967], p. 5) and Pais ([1982], p. 21), it was also Poincaré who gave their name to the Lorentz transformations. In the physics literature, the 10-dimensional Poincaré group is also often called the inhomogeneous Lorentz group or, sometimes, the Lorentz group.

[35] Engel (1861–1941) had written his *Habilitationsschrift* with Lie in Leipzig in 1885 and continued publishing on group theory. He is mostly known for his work with Lie on what became the three-volume treatise, Lie and Engel [1893]. See Hawkins [2000], pp. 77–78.

[36] Engel [1916]. See Mehra [1976], pp. 70–71, note 130.

[37] Kneser (1862–1930) was a well-known specialist in integral equations and the calculus of variations. (See Thiele [1997].) The author of a monograph on the calculus of variations [1900] that was re-issued in 1925, he was also the author of the first part of the chapter on this topic in Klein's *Encyklopädie der mathematischen Wissenschaften*. He was a *privatdozent* in Breslau [present-day Wrocław], then a professor from 1886 to 1889 at the University of Dorpat (now Tartu in Estonia), and later in Berlin, returning eventually to Breslau.

relativity" [1918] in which he developed Schütz's results [1897] using Lie's infinitesimal transformations and, as Noether would do, emphasized the relevance of Klein's Erlangen program, but did not treat questions of invariance. Slightly earlier, he had published another article [1917] where he applied the theory of Lie and Georg Scheffers to a study of variational equations and of the Hamilton–Jacobi equation, but in neither article did he touch on the problem of conserved quantities.

One can thus say that scattered results in classical and relativistic mechanics tying together properties of invariance and conserved quantities had already appeared in the publications of Noether's predecessors, without any of them having discovered the general correspondence principle. Noether supplied this general theory and consequently, after 1918, the earlier results became special cases of her first theorem. In the conclusion of his 1916 letter, Engel emphasized that the detour effected by considering the inhomogeneous Lorentz group was necessary to justify the existence of "the integral of kinetic energy and of the second integrals of the center of mass" ("das Integral des lebendigen Kraft und die zweiten Schwerpunktsintegrale") which had previously appeared "to have fallen from the heavens" ("wie vom Himmel gefallen"). Noether showed on the contrary that considering a symmetry group that was well adapted to the problem would render the known conservation laws natural, and also provided a general method for calculating conservation laws from invariances of a variational integral, and conversely, for calculating the symmetries of a variational problem from its known conservation laws.

## 1.2 The General Theory of Relativity and the Problem of the Conservation of Energy

The history of the discovery of general relativity has been amply studied, most recently in volumes of the series *Einstein Studies* and in the articles cited above. We shall therefore summarize only the elements of that history that are essential for an understanding of the role that Noether played in it.

In an article on the consequences of the principle of relativity, Einstein [1907] already observed that the laws of physics did not permit a distinction between a reference frame in a constant gravitational field and a uniformly accelerated reference frame, and he considered the question of the extension of the principle of relativity to this more general situation. After 1912 he sought an expression for the laws of gravitation that would be invariant under a group of transformations that would be larger than the group composed of the Lorentz transformations and translations, and would be invariant with respect to an arbitrary change of coordinates.

After several attemps in this direction and exchanges with Max Abraham and Gunnar Nordström in particular,[38] Einstein undertook, with the help of his friend, the mathematician Marcel Grossmann, a study of Ricci's and Levi-Civita's absolute

[38] For a detailed account of this period in Einstein's career, see Mehra [1974], Pais [1982], pp. 208–216 and 229, Rowe [1999] [2001], and the numerous references which are cited there.

differential calculus in order to supply a mathematical framework for the extension of the principle of relativity that he was seeking. He tried to formulate the laws of gravity in the form of generally covariant, second-order differential equations, which is to say, independently of the coordinate system that may be chosen, in terms of a nonconstant metric, $g_{\mu\nu}$, that would describe the gravitational potential. Einstein then temporarily abandoned the requirement that the equations of gravitation be generally covariant, because such a formulation did not yield a conservation law for energy.[39] At first he restricted his search to linear transformations; then he introduced the idea of systems of adapted coordinates which turned out to be systems of coordinates related by unimodular transformations, that is, transformations whose Jacobian equals 1 and which thus conserve volumes. This first version of general relativity is known as the *Entwurf* and is only a sketch of the eventual theory.

By restricting his search to these changes of coordinates, Einstein succeeded in November 1915 in establishing equations for gravitation. Still better, he recognized that, with a slight modification, these equations would be tensorial, thus generally covariant. On 4, 11 and 18 November 1915, he presented his conclusions before the Royal Prussian Academy of Sciences in Berlin [1915].

These new equations, however, created a grave problem because the law of the conservation of energy implied that when one adopted a suitably adapted system of coordinates, which was permitted by general covariance, the energy-momentum tensor vanished at every point in space.[40] This further implied that the scalar energy was constant. But that hypothesis was satisfied only in the case of a homogeneous gravitational field. In fact, these equations still lacked the trace term that Einstein introduced in his article of 25 November 1915, in which the equations of gravitation would find their definitive form. However, there still remained one point that was not satisfactory. The law of conservation of energy did not seem to be a direct consequence of the equations describing gravitation, nor did it seem to have mathematical justification.

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[39] The question of the conservation of energy was among the most important of Einstein's concerns throughout his career, as can be seen from his *Annalen der Physik* articles of 1906 and 1907 dealing with the inertia of energy, as well as from his letters to Michele Besso (Einstein and Besso [1972]). In particular, see the letters written during a visit to Ahrenshoop in Pomerania, 29 July 1918, no. 45, p. 129 (*Collected Papers* 8B, no. 591, pp. 835–837; 8 (English), pp. 613–614), where he writes that the total energy of a system is "an *integral* invariant without a corresponding *differential* invariant" ("*Integral*invariante, der keine *Differential*invariante entspricht"), and 20 August 1918, no. 46, p. 132 (*Collected Papers* 8B, no. 604, pp. 858–861; 8 (English), pp. 629–630), where he argues against one of Weyl's hypotheses and returns to the question of energy by insisting on the necessity of introducing "the tension tensor for the static gravitational field" ("das Spannungstensor für das statische Gravitationsfeld"). In his introduction to this correspondence, Pierre Speziali also mentions (p. li) the letters of 28 July 1925 (from Geneva), no. 76, p. 209, and 2 August 1925 (from Berne), no. 77, p. 211, but in fact Einstein wrote about the energy tensor as early as the end of 1913 or the beginning of 1914 (letter from Zurich, no. 9, p. 51; *Collected Papers* 5, no. 499, pp. 588–589; 5 (English), pp. 373–374). In the following letter, no. 10, p. 53 (*Collected Papers* 5, no. 514, pp. 603–604; 5 (English), pp. 381–382), written from Zurich in early March 1914, Einstein evokes the "law of conservation" (*Erhaltungssatz*) together with the gravitation equations to obtain conditions on the coefficients of the metric.

[40] See Earman and Glymour [1978].

Other papers, some by such highly reputed physicists as Paul Ehrenfest (1880–1933) and Hendrik A. Lorentz (1853–1928), contributed to a clarification of the question of the conservation of energy,[41] and there were many publications related to this problem. In 1916 an article by Ehrenfest [1916] appeared in the *Proceedings* of the Royal Academy of Sciences in Amsterdam in which he calculated the invariants of a variational problem. In the same volume Lorentz proposed a Lagrangian and established the equations of general relativity from the corresponding variational principle, then derived from them the law of the conservation of momentum and energy, but this was still in the framework of the preliminary version of the general theory of relativity. In 1917, Lorentz's student Adriaan Daniel Fokker published an invariant method for obtaining those results [1917], and discussed the consequences of the variational principle. This was shortly before Weyl [1917] succeeded in deriving the theorem of energy-momentum from Hamilton's principle. Still in 1917, Nordström, citing Einstein [1916a], Herglotz [1916] and the publications of Lorentz in 1915, calculated the "tension-energy tensor of matter" ("spannings-energietensor der materie"). From March to June 1916, Lorentz delivered a series of lectures in Leiden on Einstein's theory, and published in that year and in early 1917 a series of four articles in which he presented an invariant geometric theory of general relativity [1916].[42]

Noether was to refer to "Lorentz and his students (for example Fokker)," and would explicitly cite the latter's 1917 article. She was also to refer to Weyl, but without a precise reference to any of his publications. Her second theorem unifies certain of the results of the research of her predecessors, and it is she who brought to the fore the existence of identities satisfied by the Euler–Lagrange equations which appear with an infinite-dimensional symmetry group such as the group of all transformations of the manifold of general relativity.

## 1.3 The Publications of Hilbert and Klein on General Relativity

Since mid 1915 Hilbert had been working intensely to understand Einstein's papers and had sought to deduce the laws of physics in a generally covariant form from a limited number of axioms by combining Gustav Mie's (1868–1957) theory of electromagnetism (1912) with Einstein's theory of gravitation.[43] Hilbert was interested in these problems because he had already proved several fundamental

[41] For the historical context, see Pais [1987], Sauer [1999], Cattani and De Maria [1993], and see Trautman [1962] for a very clear exposition of the difficulties posed by the problem of the conservation of energy in general relativity. (See, *infra*, Chap. 6, p. 126.) For subsequent developments, see, for example, Havas [1990].

[42] Histories of these discoveries, together with analyses of the articles in which they were announced, have been published by Michel Janssen [1992] and Anne J. Kox [1992].

[43] On the events of 1915–1918 and the scientific relations among Einstein, Hilbert, Klein, and Noether, see Rowe [1999], who provides a detailed analysis based on archival documents. See also Einstein's correspondence in *Collected Papers* 8A.

theorems concerning invariants, and because relativity entered into the outstanding questions about geometry that had perplexed both Klein and himself. Already in the 1872 Erlangen program, Klein had defined a geometry as the data of a manifold and a group of transformations of that manifold, in modern terminology, a group of diffeomorphisms, thus identifying the study of a geometry with the search for the invariants of that group. Hilbert clearly saw a connection between, on the one hand, the theory of invariants and geometry, and, on the other, the problem of extending the special theory of relativity.

In late June–early July 1915, Einstein came to Göttingen at Hilbert's invitation[44] to deliver a series of lectures on the general theory of relativity—which was still the preliminary version which he would discard in November of that year. He was so enthusiastic about Hilbert and his reception of his theory that he wrote to his friend Heinrich Zangger upon his return on 7 July, "I was one week in Göttingen and learnt to know and like him. I delivered there six two-hour lectures on the now well clarified theory of gravitation, and I had the pleasure of completely convincing the mathematicians there [in Göttingen],"[45] and to Arnold Sommerfeld on 15 July, "In Göttingen I had the great pleasure to see that everything was understood to the last detail. I am most delighted with Hilbert."[46]

Hilbert and Einstein conducted an intense correspondence during the months of October and November 1915 in which they developed their closely related theories. Hilbert's approach was different from Einstein's because he used a variational principle to obtain the field equations.[47] In fact, in his article dated 20 November 1915, Hilbert introduced two axioms and a generally invariant function from which he deduced ten gravitational equations and four electromagnetic equations, all of which were covariant with respect to any change of coordinates. As the study of the proofs of Hilbert's article [1915] has demonstrated, his gravitational equations were Einstein's equations [1915] of which he had been apprised in a letter that he received when his article was still in proof, so that Einstein indeed had priority in

[44] From Einstein's correspondence, we know the dates of his stay in Göttingen, from 26 or 27 June to 5 July, since he wrote to Hilbert on 24 June that he would call on him on Monday morning (28 June 1915) (*Collected Papers* 8A, no. 91, p. 142; 8 (English), p. 107). A letter of 6 July mentions that he had returned from Göttingen the previous night (letter to Wander and Geertruida de Haas, *ibid.*, no. 92, pp. 142–143; 8 (English), p. 108).

[45] "Ich war eine Woche in Göttingen wo ich ihn kennen und lieben lernte. Ich hielt dort sechs zweistündige Vorträge über die nun schon sehr geklärte Gravitationstheorie und erlebte die Freude, die dortigen Mathematiker vollständig zu überzeugen," *Collected Papers* 8A, no. 94, pp. 144–145; 8 (English), pp. 109–110.

[46] "In Göttingen hatte ich die grosse Freude, alles bis ins Einzelne verstanden zu sehen. Von Hilbert bin ich ganz begeistert," *Collected Papers* 8A, no. 96, p. 147; 8 (English), p. 111. This quotation is also translated by Mehra [1974], p. 25. See Pais [1982], p. 259.

[47] One of Einstein's manuscripts, entitled "Appendix: Formulation of a theory on the basis of a variational principle," has now been published in the *Collected Papers* 6, no. 31, pp. 340–346. According to the editors, this text, which was written before 20 March 1916, may have been intended to serve as the last section of, or as an appendix to, his long article [1916a]. It was at the end of 1916 that Einstein published an article on a variational formulation of general relativity [1916b]. For more details regarding variational formulations of Einstein's equations, see Kishenassamy [1993]. Cf. also Noether [1918c], pp. 249–250, note 1 (pp. 15–16, note 20, in the above translation).

the discovery of the equations that bear his name.[48] By applying a theorem that he stated without proof, Hilbert obtained a conservation law for the energy-momentum tensor which, at first glance, was different from Einstein's. That theorem would be proved three years later by Noether.[49]

During the years 1917 and 1918, Klein and Einstein corresponded frequently, and the problem of the conservation of energy was the subject of numerous comments and requests for explanations that preceded and followed the publication of their several articles.[50] Klein and Hilbert also exchanged letters in 1918 about the conservation of energy and related topics. It is known that Klein discussed the problem of the conservation of energy with Noether and also with Carl Runge[51] in the spring of 1918, and that, together with Runge, he undertook a systematic study of the bibliography of the subject. Klein wrote to Hilbert on 5 March 1918,[52] informing him that he had spoken before the Royal Scientific Society in Göttingen (*Königliche Gesellschaft der Wissenschaften zu Göttingen*) on 25 February, advocating that one consider only the energy tensor of matter, and not that of gravitation, in the energetic balance of a field, that Runge had further developed his, i.e., Klein's, idea of the energetic balance of the gravitational field, that Runge would develop it "very well" ("sehr schön") the coming Friday (8 March 1918) in a lecture before the Scientific Society, and inviting him to attend, "Do come on Friday evening to the Scientific Society."[53] He added that Runge had put this theorem in a regular form by a suitable choice of coordinates for each particular case. Hilbert replied on 7 March, sending proofs of his "first note," in which he "worked out directly Runge's ideas."[54] In fact, on 8 March, Runge delivered a lecture before the Scientific Society, "On the Theorem of the Conservation of Energy in Gravitational Theory" ("Über den Satz von der Erhaltung der Energie in der Gravitationstheorie"). As early as 12 March, Noether wrote to Klein criticizing Runge's ideas.[55] In his letter to Einstein of

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[48] See Corry, Renn and Stachel [1997]. The problems of priority of discovery and of the relations between Hilbert and Einstein were first studied by Mehra [1974], then by Earman and Glymour [1978], Pais [1982], pp. 257ff. and 274–275, and Vizgin [1994], chapter 2. Also see Rowe [1999], pp. 199–205, and [2001], and the historical notes in *The Collected Papers of Albert Einstein*, 8A.

[49] See Section 6 of Noether's article (pp. 19–22 in the above translation) and, *infra*, Chap. 2, pp. 63–64.

[50] Klein [1918a, b and c] and Einstein [1916b] and [1918].

[51] Runge (1856–1927) published on very diverse subjects during his career. He had been a full professor of applied mathematics at Göttingen since 1904.

[52] The two letters concerning the law of conservation of energy that Klein wrote to Hilbert on 5 February and 5 March 1918, and Hilbert's brief answer, have been published in Hilbert and Klein [1985], nos. 126, 128 and 129, pp. 140–144.

[53] "Kommen Sie doch ja am Freitag Abend noch in die Gesellschaft der Wissenschaften," *ibid.*, no. 128, p. 142.

[54] "[. . . ] meiner ersten Mitteilung, in der ich gerade die Ideen von Runge auch ausgeführt hatte," *ibid.*, no. 129, p. 144. This "first note" does not correspond to any of Hilbert's published articles and may have been a draft. Hilbert refused to attend Runge's lecture to protest the presence of Edward Schröder (a professor of German philology at Göttingen) on the board of directors of the Scientific Society (*ibid.*).

[55] See Appendix II, pp. 153–157, and in particular note 3.

20 March 1918,[56] Klein mentioned the results that Runge had obtained, and informed him that his paper and Runge's were nearly ready for publication but, on 24 March, Einstein argued against Runge's ideas,[57] and that convinced both Klein and Runge not to publish the papers that they had been preparing "until [they] had arrived at a better perspective on the entire literature" dealing with the subject.[58] In early June, Klein proposed to speak about the article that Einstein was about to publish, "The theorem of the conservation of energy in general relativity,"[59] and, in his letter of 9 June 1918, Einstein wrote to Klein, "I am very pleased that you will talk about my article on energy. I shall now give you a complete proof of the tensorial character (for linear transformations) of $J_\sigma$."[60] But Klein finally abandoned the projected lecture because he was not convinced of the validity of Einstein's argument.

In early 1918 Klein published an article [1918a][61] in the form of an exchange of letters with Hilbert in which he simplified the argument that Hilbert had published in his article on "The foundations of physics" [1915], and offered a discrete criticism of that article. The uncertainties about the relationship beween the theories of Einstein and Hilbert, in particular regarding the associated laws of conservation, were finally dissipated by Klein in his later articles of 1918, "On the differential laws for the conservation of momentum and energy in Einstein's theory of gravitation" [1918b] (19 July) and "On the integral form of conservation laws and the theory of the spatially closed universe" [1918c] (6 December), where he elucidated, with Noether's theorems playing an essential role,[62] the derivation of Einstein's and Hilbert's laws of conservation and the vectorial nature of the quantities that Hilbert had defined. But the principal difficulty, that of explaining the difference in the nature of the conservation laws in classical mechanics and special relativity on the one hand, and in general relativity on the other hand, had, in fact, already been resolved by Noether in March 1918, and explained in the article [1918c] that was presented by Klein at the Scientific Society in July and submitted for publication in September of that year. Now, Einstein was evidently not aware of this immediately because he could still write to Klein, on 13 March, "The relations here [in general relativity] are exactly

[56] Einstein, *Collected Papers* 8A, no. 487, pp. 685–690; 8 (English), pp. 503–507.

[57] *Collected Papers* 8B, no. 492, pp. 697–699; 8 (English), pp. 512–514.

[58] "[...] wenn wir die volle Uebersicht über die jezt vorliegende Literatur haben," letter of 18 May 1918 from Klein to Einstein (*Collected Papers* 8B, no. 540, pp. 761–762; 8 (English), p. 559). Preliminary versions of the paper that Klein was preparing but which he chose not to publish as well as notes of his discussions with Runge have been conserved in the Göttingen archives (see Einstein, *Collected Papers* 7, p. 76, note 5, on Einstein's article [1918]). Runge never returned to this question after learning of Einstein's criticisms of his project.

[59] Einstein [1918].

[60] "Es Freut mich sehr, dass Sie über meine Energie-Arbeit vortragen werden. Ich teile Ihnen nun den Beweis für den Tensorcharakter (bez. linearer Transformationen) von $J_\sigma$ vollständig mit," *Collected Papers* 8B, no. 561, p. 791; 8 (English), p. 581.

[61] Even though it appeared in the volume dated 1917, Klein's article was actually submitted to the journal 25 January 1918.

[62] See, *infra*, for a more detailed analysis of the chronology of Noether's discoveries, pp. 46–48, and for Hilbert's and Klein's acknowledgments of Noether's contribution, in Chap. 3, pp. 66–71.

analogous to those of nonrelativistic theories."[63] We shall see (p. 47) that around that date, Noether, who was then visiting in Erlangen, had already written to Klein about this very point, which explains why, on 20 March, Klein could assert to Einstein that what he had claimed was far from being true. Einstein replied once again that one could consider the fact that the integrals $\int (\mathfrak{T}_\sigma^4 + \mathfrak{t}_\sigma^4)\, dV$ are constant with respect to time "as being entirely analogous and equivalent to the conservation law for the energy-momentum in the classical mechanics of continua."[64] What Noether had contributed to the question of Hilbert's energy vector was essential, as Klein would write to Einstein on 10 November 1918,[65] the only time in his correspondence with Einstein that he mentions Noether and the importance of her contribution.

Much later, in 1924, it was Schouten and Dirk Struik who observed that, in the special case of the Lagrangian of general relativity, the identities obtained by Noether's second theorem were also consequences of the Bianchi identities, which were well known in Riemannian geometry. They express the vanishing of the covariant differential of the curvature of the Levi-Civita connection associated to a metric.[66]

The difficult problem of the conservation of energy in general relativity began to be understood much later when the gravitation theory was put into Hamiltonian form by Richard Arnowitt, Stanley Deser and Charles W. Misner in 1962. There remained the problem of proving the positivity of the energy, which was eventually achieved by Edward Witten in 1981.[67]

## 1.4 Emmy Noether at Göttingen

Emmy Amalie Noether (1882–1935), "of Bavarian nationality and Israelite confession,"[68] was the daughter of the mathematician Max Noether (1844–1921). She

[63] "Es liegen hier genau analoge Verhältnisse vor wie bei den nicht-relativistischen Theorien," *Collected Papers* 8B, no. 480, p. 673; 8 (English), p. 494.

[64] "[...] welche dem Impuls-Energie-Satz der klassichen Mechanik der Kontinua als durchaus gleichartig und gleichwertig an die Seite gestellt werden kann," letter of 24 March 1918 cited in note 57. The symbol $\mathfrak{t}_\sigma^4$ denotes the time-components of the quantities $\mathfrak{t}_\sigma^\nu$ which Einstein had introduced in his article [1916b] and about which he complained to Hilbert in a letter of 12 April 1918, "everybody rejects my $\mathfrak{t}_\sigma^\nu$ as though they were not kosher"! ("Meine $\mathfrak{t}_\sigma^\nu$ werden als unkoscher von allen abgelehnt," *Collected Papers* 8A, no. 503, p. 715; 8 (English), p. 525). For the "pseudo-tensor" $\mathfrak{t}_\sigma^\nu$ , see, *infra*, Chap. 6, p. 127.

[65] Einstein, *Collected Papers* 8B, no. 650, p. 942; 8 (English), p. 692. See, *infra*, Chap. 3, p. 70.

[66] These identities were named after the Italian geometer Luigi Bianchi (1856–1928). See Levi-Civita [1925], p. 182, where the history of the Bianchi identities is sketched, and see Pais [1982], chapter 15c, pp. 274–278. In fact, in a 1917 article where he introduced the idea of parallel displacement, Levi-Civita had already applied the contracted Bianchi identities to the theory of gravitation, and had corresponded with Einstein on the subject. See Cattani and De Maria [1993] and Rowe [2002].

[67] See Faddeev [1982], Choquet-Bruhat [1984].

[68] "[...] bayerische-Staatsangehörigkeit und israelitische-Konfession," as she described herself in the beginning of a manuscript *curriculum vitæ* written around 1917 and reproduced on the first

wrote her doctoral thesis at Erlangen in 1907 under Paul Gordan (1837–1912), one of the most distinguished specialists in the theory of invariants. In Erlangen she also came under the influence of Ernst Fischer[69] (1875–1954). Her thesis, "On the Construction of the System of Forms of a Ternary Biquadratic Form," which dealt with the search for the invariants of a ternary biquadratic form, i.e., of a homogeneous polynomial of degree 4 in 3 variables, was published in "Crelle's Journal" [1908], while an extract had appeared a year earlier [1907]. In her next article, "On the theory of invariants of forms of *n* variables" [1911], which had been announced the year before its publication (Noether [1910]), she extended the arguments of her thesis to the case of forms in *n* variables. Then she studied the fields of rational functions in "Fields and systems of rational functions" [1915] which she had announced in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Noether [1913]). She had joined the German Mathematical Society (*Deutsche Mathematiker-Vereinigung*, or DMV) in 1909.

In 1916, in volume 77 of the *Mathematische Annalen*, Noether published a series of three articles [1916a, b, c] and then a fourth [1916d] on algebraic invariants. Regarding her articles on the invariants of finite groups[70] and on the search for bases of invariants that furnish expansions with integral or rational coefficients [1916a, b], Weyl wrote in 1935,

> The proof of finiteness is given by her for the invariants of a finite group (without using Hilbert's general basis theorem for ideals), for invariants with restriction to integral coefficients, and finally she attacks the same question along with the question of a minimum basis consisting of independent elements, for the fields of rational functions.[71]

In his book on the classical groups that appeared four years later, Weyl gave a summary of the proof contained in Noether [1916a],

> An elementary proof [of the first main theorem] for finite groups not depending on Hilbert's general theorem on polynomial ideals was given by E. Noether.[72]

And still later, in his analysis of Hilbert's work, he cited that article in a footnote once more.[73]

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page of her *Gesammelte Abhandlungen / Collected Papers*. A mention of religious affiliation was normally part of one's national identity in Germany in that period. For the biography of Noether, see Dick [1970] [1981], Kimberling [1981] and Srinivasan and Sally [1983]. A relatively complete electronic bibliography of materials relating to her life and works with links to other pertinent sites may be found at the web-site of the association « femmes et mathématiques ».

[69] On this point, see Weyl [1935a].

[70] For a modern version of the results of Noether [1916a] and an account of developments in the theory of invariants of finite groups, see Smith [2000], and for an extension to the case of prime charateristic of her results on a bound for the degrees of the generators of the ring of polynomial invariants for finite groups, see Fogarty [2001]. Noether herself had considered the case of prime charateristic in 1926.

[71] Weyl [1935a], p. 206, *Gesammelte Abhandlungen*, vol. 3, p. 430. This eulogy by Weyl, in English, was quoted in its entirety by Dick [1970], pp. 53–72, and [1981], pp. 112–152. See, *infra*, Chap. 3, pp. 77–78.

[72] Weyl [1939], p. 275. The reference to Noether's article is on p. 314, note 19 of chapter 8.

[73] Weyl [1944], p. 621, and *Gesammelte Abhandlungen*, vol. 4, p. 139, note 2. In Reid [1970], pp. 245–283, Weyl's 1944 text is abridged but includes the note referring to Noether (p. 249).

Noether's next publication [1918a] dealt with equations that admit a prescribed Galois group, a study that extends her 1915 article.[74]

In 1915 Klein and Hilbert invited Noether to Göttingen to help them in the development of the implications of general relativity theory, and she arrived in the spring. Research in the Göttingen archives[75] has shown that Noether took an active part in Klein's seminar. The seminars in Berlin in those years, despite Einstein's presence there, were much less oriented toward mathematical physics and, in particular, toward the mathematics of relativity theory.[76] The list of themes treated in Klein's seminar has been published in the Supplement to volume 3 of his *Gesammelte mathematische Abhandlungen*, p. 11. We extract from it the following titles:

– Summer 1916, Theory of invariants of linear transformations,

– Winter 1916/17, Theory of special relativity on an invariant basis,

– Summer 1917, Theory of invariants of general point transformations,

– Summer 1918–Winter 1918/19 until Christmas, General theory of relativity on an invariant basis, [. . . ]

– Winter 1920/21 until Christmas, Variational principles of classical mechanics and of general relativity.

Shortly after her arrival in Göttingen, Noether began work on the problem of the invariants of differential equations, and, in 1918, she published two articles on the subject, "Invariants of arbitrary differential expressions" [1918b] and "Invariant variational problems," *Invariante Variationsprobleme* [1918c], the article that will be studied here.[77] In it she takes up the work initiated by Hamel [1904a, b] and Herglotz [1911]. At the request of Hilbert, some time before May 1916, she had begun to study the various problems that resulted from the formulation of the general theory of relativity, and it is clear from a letter from Hilbert to Einstein of 27 May 1916 that she had already written some notes on the subject, notes that have not yet been identified and may not have been conserved. Hilbert wrote, "My law [of conservation] of energy is probably linked to yours; I have already given Miss Noether this question to study." In the next sentence he explained why the vectors $a^l$ and $b^l$ that had been considered by Einstein could not vanish in the limiting case in which the coefficients of the metric are constant, and he added that, to avoid a long explanation, he had appended to his letter "the enclosed note of Miss Noether."[78] On

[74] According to the algebraist Paul Dubreil [1986], this problem had been posed by Richard Dedekind (1831–1916). Modern work utilizing Noether's results and conjectures on this question have been analyzed by Richard G. Swan in the section "Galois Theory" of the chapter "Noether's Mathematics" in Brewer and Smith [1981], pp. 115–124.

[75] Rowe [1999].

[76] "Berlin is no match for Göttingen, in what concerns the liveliness of scientific interest, at all events in this area" ("Berlin kann sich, was Lebhaftigkeit des wissenschaftlichen Interesses anbelangt, wenigstens auf diesem Gebiete mit Göttingen nicht messen," letter of 7 July 1915 from Einstein to Heinrich Zangger, cited in note 45 above).

[77] The 1918 volume of the *Göttinger Nachrichten* which contains these two articles is available at the site http://www.emani.org (SUB Göttingen).

[78] "Mein Energiesatz wird wohl mit dem Ihrigen zusammenhängen: ich habe Frl. Nöther diese Frage schon übergeben. [. . . ] Ich lege der Kürtze [Wegen] den beiliegenden Zettel von Frl. Nöther bei," *Collected Papers* 8A, no. 222, pp. 290–292; 8 (English), pp. 215–216.

30 May 1916 Einstein answered him in a brief letter, "[...] I now understand everything in your article except the energy theorem." He then derived from the equation that Hilbert had proposed an apparently absurd consequence "which would deprive the theorem of its sense," and then asked, "How can this be clarified?" and continued, "Of course it would be sufficient if you asked Miss Noether to clarify this for me."[79] This exchange shows that Noether's expertise in this area of the discussions concerning general relativity was conceded by both Hilbert and Einstein as early as her first year in Göttingen.

Although her work on the energy vector that had been introduced by Hilbert began in 1916,[80] it was in the winter and spring of 1918 that Noether discovered the profound reason for the difficulties that had arisen in the interpretation of the conservation laws in general relativity. These considerations would be clearly stated in the *Invariante Variationsprobleme*, which contains two theorems on the relationship between the group of transformations that leave invariant the action integral of a Lagrangian system and the conservation laws, the one in the case of an invariance group with a finite number of parameters, the situation in classical mechanics and special relativity, and the other in the case of an invariance group of the same type as the group that figures in general relativity, a generally covariant theory, which is to say, one whose field equations are invariant under any change of coordinates. Thus what distinguishes the two cases is the presence in the second case of an invariance group depending on arbitrary functions.

It is on the *verso* of a postcard that Noether addressed to Klein from Erlangen, 15 February 1918,[81] that she sketched her second theorem. The formula in her line 8,

$$\delta f - \frac{\partial}{\partial x_1}\sum_i \frac{\partial f}{\partial \frac{\partial z_i}{\partial x_1}}\,\delta z_i - \cdots \frac{\partial}{\partial x_n}\sum_i \frac{\partial f}{\partial \frac{\partial z_i}{\partial x_n}}\,\delta z_i = -\sum_i \psi_i(z)\,\delta z_i,$$

is, except for some slight changes in notation and the sign convention adopted for the quantities $\psi_i$, identical to formula (5) of her article. In that article, equation (5) is preceded by equation (3) which contains the definition of the components $\psi_i$ of the Euler–Lagrange derivative—she calls them the "Lagrangian expressions" ("die Lagrangeschen Ausdrücke")—of the Lagrangian $f$ and which introduces the divergence term Div $A$. Then formula (5) provides the explicit expression of the quantity $A$ in the case of $n$ independent variables and a first-order Lagrangian.

Further on, the long equation that occupies two lines corresponds to the case of invariance under each of the translations of an $n$-dimensional space which, in the case of special relativity, is the 4-dimensional Minkowski space. Noether therefore considers, for every $\kappa = 1, 2, \ldots, n$, the variation $\delta z_i = \dfrac{\partial z_i}{\partial x_\kappa}$ which implies that, if $f$

[79] "In Ihrer Arbeit ist mir nun verständlich ausser dem Energiesatz. [...] was dem Satze seinen Sinn rauben würde. Wie klärt sich dies? Es genügt ja, wenn Sie Frl. Nöther beauftragen, mich aufzuklären," *Collected Papers* 8A, no. 223, pp. 293–294; 8 (English), pp. 216–217.

[80] See also the passages in Klein and Hilbert that are cited *infra*, Chap. 3, p. 65, as well as Mehra [1974], p. 70, note 129a, and Rowe [1999], p. 213.

[81] See a reproduction, as well as the transcription and a translation in Appendix I, pp. 149–151.

does not depend explicitly on $x_\kappa$, the variation of $f$ is the total derivative of $f$ with respect to $x_\kappa$. She then obtains "the $n$ identities" which appear on two lines in the middle of the page,

$$\frac{\partial}{\partial x_1}\Big(\sum_i \frac{\partial f}{\partial \frac{\partial z_i}{\partial x_1}}\frac{\partial z_i}{\partial x_\kappa}\Big)+\cdots+\frac{\partial}{\partial x_\kappa}\Big(\sum_i \frac{\partial f}{\partial \frac{\partial z_i}{\partial x_\kappa}}\frac{\partial z_i}{\partial x_\kappa}-f\Big)+\cdots\frac{\partial}{\partial x_n}\Big(\sum_i \frac{\partial f}{\partial \frac{\partial z_i}{\partial x_n}}\frac{\partial z_i}{\partial x_\kappa}\Big)$$
$$=\sum_i \psi_i(z)\frac{\partial z_i}{\partial x_\kappa}\,; \quad (\kappa = 1,2\ldots n).$$

She has thus determined the $n$ components of the $n$ conserved currents, i.e., the $n$ vector fields whose divergence vanishes when the Euler–Lagrange equations are satisfied, associated with the $n$ spatial directions.[82] In the case of special relativity, these $n^2 = 16$ components are those of the energy-momentum tensor.

But, in a generally covariant theory on an $n$-dimensional space which, in the case of general relativity, is a curved space-time with $n = 4$ dimensions, the space-time admits all the changes of coordinates where $x'_\kappa$ is an arbitrary function of the $x_\lambda$'s, which corresponds to an infinitesimal symmetry where $\dfrac{\partial}{\partial x_\kappa}$ is multiplied by an arbitrary function of the $x_\lambda$'s. From that Noether deduces that, in the generally covariant case, the identities

$$\sum_i \psi_i(z)\frac{\partial z_i}{\partial x_\kappa} = 0\,; \quad (\kappa = 1,2\ldots n)$$

are satisfied by the Lagrangian expressions, which shows that "the $\rho$ equations, $\psi_i = 0$, are equivalent to $\rho - n$ [equations]." Those identities appear four lines above the end of the text of Noether's postcard. As we shall emphasize (Chap. 2, p. 61), these identities are special cases of the general formula (16) that she would prove in the fourth section of her article. She writes here that she "hopes to be able to prove the general case, where the scalars $z_\alpha$ are replaced by the tensors $g_{\mu\nu}$, in an analogous manner," which shows that a solution of the problem posed by the general theory of relativity was already in view.

A month later, in her letter to Klein of 12 March 1918,[83] Noether formulated the fundamental idea that the lack of a theorem concerning energy in general

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[82] If one introduces the shorthand notation $z^i_\lambda$ for $\dfrac{\partial z_i}{\partial x_\lambda}$, the components of the conserved current associated with the infinitesimal symmetry $\dfrac{\partial}{\partial x_\kappa}$ are thus $N_1^{(\kappa)},\ldots,N_n^{(\kappa)}$, where

$$N_\lambda^{(\kappa)} = -\sum_{i=1}^{i=n}\frac{\partial f}{\partial z^i_\lambda} z^i_\kappa + f\delta_{\kappa\lambda}\,,$$

with $\delta_{\kappa\lambda} = 1$ if $\kappa = \lambda$ and 0 otherwise. In [1918c], Noether would introduce the variation $\bar{\delta} z_i$ which is, in this case, $-\dfrac{\partial z_i}{\partial x_\kappa}$, because the vector field $\dfrac{\partial}{\partial x_\kappa}$ has components $\delta_{\kappa\lambda}$, $\lambda = 1,2,\ldots,n$.

[83] See the reproduction of this letter, its transcription and a translation in Appendix II, p. 153–157.

relativity is due to the fact that the invariance groups that were considered were in fact subgroups of an infinite group, and therefore led to identities that are satisfied by the Lagrangian expressions, "by my additional research, I have now established that the [conservation] law for energy is not valid in the case of invariance *under any extended group generated by the transformation induced by the z's*."[84] Here $z$ designates the set of dependent variables, and the last words of the emphasized sentence should be understood as "invariance under the transformations of the $z$'s induced by *all* the transformations of the independent variables." A comparison of this sentence with the wording in Noether's section 6 shows that this is a preliminary formulation of an essential consequence of what would become her second theorem.

On 23 July 1918 Noether delivered a paper before the Göttingen Mathematical Society (*Mathematische Gesellschaft zu Göttingen*)[85] entitled, like the eventual article, *Invariante Variationsprobleme*, and whose summary begins, "In connection with research related to Hilbert's energy vector, the speaker [*die Referentin*, the feminine form of the word] stated the following general theorems [. . . ],"[86] and Klein at the 26 July 1918 session of the Royal Scientific Society in Göttingen presented a communication by Noether that bears the same title concerning the invariants of systems of equations that derive from a Lagrangian, which is further testimony to the importance he attributed to Noether's results and to her collaboration. The *Invariante Variationsprobleme* [1918c] would appear in the *Göttinger Nachrichten*[87] with the mention, "the definitive version of the manuscript was prepared only at the end of September." Noether published her own summary of the article in the *Jahrbuch über die Fortschritte der Mathematik*, a yearly collection of abstracts that was the ancestor of the *Zentralblatt* and of *Mathematical Reviews*, now *MathSciNet*. This summary consists of a statement of the two theorems and bears the same title as that article.[88]

Noether submitted the *Invariante Variationsprobleme* to the university with the support of Hilbert and Klein to obtain a habilitation which was awarded in 1919, after the war,[89] after the proclamation of the Weimar Republic and a favorable de-

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84 This passage is quoted by Rowe [1999], p. 218, in a different translation.

85 See Appendix V, p. 167.

86 "Im Zusammenhang mit der Untersuchungen über den Hilbertschen Energievektor hat die Referentin folgende allgemeine Sätze aufgestellt [. . . ]," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 27, Part 2 (1918), p. 47. See Dick [1970], p. 15, and [1981], p. 33, and Rowe [1999], p. 221.

87 A digitalized version of the *Göttinger Nachrichten* of 1918 is now available at the web site http://gdz.sub.uni-goettingen.de/en/gdz/.

88 *Jahrbuch über die Fortschritte der Mathematik*, 46 (1916–1918), p. 770, section Analysis, chapter Calculus of variations.

89 One can infer from the list of Hilbert's students in his *Gesammelte Abhandlungen*, vol. 3, p. 433, what would have seemed likely, that the war had largely interrupted the presence at the university of the male students, and also delayed the research of those who returned after the war, because none defended theses between 21 December 1914 and 5 June 1918, while the next thesis defense took place 7 July 1920. Judging from the list in Klein's *Gesammelte mathematische Abhandlungen*, vol. 3, pp. 11–13, none of his students defended his thesis during the war, but of course for a different reason. Klein retired in 1913 and had not directed doctoral students since 1911.

cision of the new "Ministry of Science, Arts and Education," and long after the strange incident immortalized in a well-known story about Hilbert's unsuccessful attempt to convince his colleagues to make an exception to the rules barring women from obtaining a habilitation, the first step toward an appointment to the faculty.[90] After having sketched the contents of her earlier publications, she gave the following summary of the article that she had submitted for her habilitation:

> The last two studies that we shall mention concern the differential invariants and the variational problems and are, in part, the result of the assistance that I provided to Klein and Hilbert in their work on Einstein's general theory of relativity. [...] The second study, *Invariante Variationsprobleme*, which I have chosen to present for my habilitation thesis, deals with arbitrary, continuous groups, finite or infinite, in the sense of Lie, and derives the consequences of the invariance of a variational problem under such a group. These general results contain, as particular cases, the known theorems concerning first integrals in mechanics and, in addition, the conservation theorems and the identities among the field equations in relativity theory, while, on the other hand, the converse of these theorems is also given [...].[91]

In the list of habilitations in the 1919 volume of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* we find, "Miss Dr. Emmy Noether has been awarded a habilitation as a *Privatdozentin* in mathematics at the University of Göttingen."[92]

Noether returned to the theory of invariants, though this time they were algebraic invariants, in a paper delivered before the Göttingen Mathematical Society, 5 November 1918, on the invariants of binary forms,[93] and a year later she submitted an article on this subject [1919].

In 1922, there appeared volume III.3 of the *Encyklopädie der mathematischen Wissenschaften*,[94] which was devoted to differential geometry and contained a

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[90] See the detailed study by Cordula Tollmien [1991] and the article by Tilman Sauer [1999].

[91] "Schließlich sind noch zwei Arbeiten über Differentialinvarianten und Variationsprobleme zu nennen, die dadurch mitveranlaßt sind, daß ich die Herren Klein und Hilbert bei ihrer Beschäftigung mit der Einsteinschen allgemeinen Relativitätstheorie unterstützte. [...] Die zweite Arbeit ‚*Invariante Variationsprobleme*', die ich als Habilitationsschrift bezeichnet hatte, beschäftigt sich mit beliebigen endlichen oder unendlichen kontinuierlichen Gruppen, im Lieschen Sinne und zieht die Folgerungen aus der Invarianz eines Variationsproblems gegenüber einer solchen Gruppe. In den allgemeinen Resultaten sind als Spezialfälle die in der Mechanik bekannten Sätze über erste Integrale, die Erhaltungssätze und die in der Relativitätstheorie auftretenden Abhängigkeiten zwischen den Feldgleichungen enthalten, während andererseits auch die Umkehrung dieser Sätze gegeben wird." This text is an extract from the curriculum vitae (*Lebenslauf*) accompanying her habilitation. The original, manuscript, German text is transcribed in Dick [1970], p. 16. It was translated into English in Dick [1981], p. 36, and, with some inaccuracies, in Kimberling [1981], p. 15.

[92] "Fräulein Dr. Emmy Noether hat sich als Privatdozentin der Mathematik an der Universität Göttingen habilitiert." *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 28, Part 2 (1919), p. 36. We note the feminine title, *Privatdozentin*. Appointment as a *Privatdozent* was equivalent to appointment as an assistant professor, but that position implied no remuneration by the university, rather direct remuneration by the students.

[93] *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 28, Part 2 (1918–1919), p. 29.

[94] This encyclopedia had been launched in 1898 under Klein's direction. It was translated into French and published by Gauthier-Villars under the title, *Encyclopédie des sciences mathématiques*

section 10, also designated by III E 1, "New work in the theory of algebraic invariants. Differential invariants" ("Neuere Arbeiten der algebraischen Invariantentheorie. Differentialinvarianten"), written by Weitzenböck and completed in March 1921. In subsection 7, "Differential invariants of infinite groups," he wrote (p. 36), "Recently, differential invariants of infinite groups in connection with a variational principle were considered by E. Noether, using a somewhat more general type of group,"[95] and he referred to subsection 27 (*sic* for 28) of that section. In the second part, "Differentialinvarianten," Section C, "Theorie der Differentialformen," this last subsection (no. 28, pp. 68–71) is entitled "Formal calculus of variations and differential invariants" ("Formale Variationsrechnung und Differentialinvarianten") and contains the footnote, "Diese Nr. rührt von E. Noether her," literally "This subsection originates from E. Noether," and was understood after her death as meaning, "This subsection was contributed by E. Noether." Even though her name appears neither in the table of contents (pp. 1–2), nor in the bibliography on page 3, and although it is written in the third person ("E. Noether shows that ..."), this two-page subsection was included in the list of Noether's publications which appeared at the end of her eulogy by van der Waerden.[96] It was subsequently included by Auguste Dick[97] in her bibliography of Noether's writings [1970] [1981] and was reprinted in Noether's *Gesammelte Abhandlungen / Collected Papers*, probably in both cases on the basis of van der Waerden's testimony. In the fifteen-line final paragraph of this short summary, we find references to earlier work that is also cited in the *Invariante Variationsprobleme*, with an additional reference to Klein's paper [1918a], then a restatement of her two theorems which had been published three years earlier:

> The fundamental version of E. Noether shows that to the invariance of $J$ under a group $G_\rho$ (a finite group with $\rho$ essential parameters), there correspond $\rho$ linearly independent divergences; to the invariance under an infinite group which contains $\rho$ arbitrary functions and their derivatives up to order $\sigma$, there correspond $\rho$ identities between the Lagrangian expressions and their derivatives up to order $\sigma$. In both cases, the converse is valid.[98]

At the end of the above paragraph there is a summary of section 5 of Noether's article: "Given the fact that the Lagrangian expressions are (relative) invariants of the group, one also has a process that generates invariants."[99] This subsection, which

*pures et appliquées*, as the volumes appeared in Germany but, because of the war, the translation was interrupted after 1916, which is to say, before the publication of vol. III.3.

[95] "Neuerdings wurden von E. Noether unter Verwendung eines etwas allgemeineren Gruppebegriffes Differentialinvarianten von unendliche Gruppen in Zussammenhang mit einem Variationsprinzip betrachted," p. 36.

[96] See, *infra*, p. 78. This list appears on p. 475 of van der Waerden [1935].

[97] Dick (1910–1993) held a doctorate in mathematics from the University of Vienna and taught in a high school. She published a book and several articles on Noether, and collaborated in the edition of the works of Erwin Schrödinger (1984).

[98] "Die prinzipielle Fassung bei *E. Noether* zeigt, daß der Invarianz gegenüber einer unendlichen Gruppe, die $\rho$ willkürliche Funktionen bis zur $\sigma^{\text{ten}}$ Ableitung enthält, entsprechen $\rho$ Abhängigkeiten zwischen den Lagrangeschen Ausdrücken und ihren Ableitungen bis zur $\sigma^{\text{ten}}$ Ordnung. In beiden Fällen gilt die Umkehrung," *Encyclopädie*, III.3, p. 71.

[99] "Da die Lagrangeschen Ausdrücke (relative) Invarianten der Gruppe werden, hat man zugleich einen Invarianten erzeugenden Prozeß," *ibid.*

is indeed in Noether's style, may have been written by Noether herself, but this is not entirely clear. In any case, apart from the reference contained in this subsection, we have not found any mention of the *Invariante Variationsprobleme* article in any of Noether's subsequent published works. She did not direct the research of any of her doctoral students toward topics related to variational problems.[100] That suggests that, after having submitted it for her habilitation thesis, she no longer attached great importance to its results.

In Leipzig in 1922, on the occasion of the annual meeting of the German Mathematical Society, she delivered a survey of "Algebraic and Differential Invariants" ("Algebraische und Differentialinvarianten"),[101] and she treated these questions for the last time in her career in an article with the same title in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* [1923]. In the beginning of this paper[102] she remarked that the "naïve and formal" period of research on algebraic invariants had concluded with Hilbert and his utilization of arithmetic methods in algebra, and that, for differential invariants, "this critical period is characterized [...] by the name of Riemann, or, more concretely, [...] by the methods of the formal calculus of variations,"[103] but she cited among her previous works only the articles of 1915 on the existence of rational bases, of 1916 on the existence of a finite basis of invariants for finite groups [1916a], of 1918 on the invariants of differential equations [1918b], and of 1919 on the invariants of binary forms, omitting the *Invariante Variationsprobleme*. On the last page,[104] she referred to Weyl, and to Schouten, whose later papers deal with differential concomitants.

While, as we observed above, Noether never again mentioned her results of 1918 on the variational calculus in print after the 1922 encyclopedia article, if she is indeed its author, she had one occasion to cite the *Invariante Variationsprobleme* when she urged the rejection of a poorly written manuscript submitted to the *Mathematische Annalen* by the physicist Gawrillov Rashko Zaycoff[105] that reproduced and claimed to generalize her results. In a letter of 10 January 1926, written from Blaricum, a village in North Holland,[106] to Einstein,[107] who had evidently asked her

[100] See the list of the doctoral theses she directed in Dick [1970], p. 42, and [1981], pp. 185–186.

[101] See Dick [1970], p. 10, and [1981], p. 20.

[102] Noether [1923], p. 177, *Abhandlungen*, p. 436.

[103] "Und diese kritische Periode ist für die algebraischen Invarianten charakterisiert durch den Namen Riemann—oder in sachlicher Hinsicht: [...] durch die Methoden der formalen Variationsrechnung," *ibid.*

[104] Noether [1923], p. 184, *Abhandlungen*, p. 443.

[105] G. R. Zaycoff (1901–1982) studied in Sofia, Göttingen and Berlin, and published articles on relativity and on quantum mechanics. From 1935 on, he worked as a statistician at the University of Sofia.

[106] Blaricum was the residence of the intuitionist mathematician Luitzen Egbertus Brouwer (1881–1966) whom Noether had come to visit for a month in the middle of December 1925 (Alexandrov [1979], cited by Roquette [2008], p. 292). It was also from Blaricum that Weitzenböck had dated the preface of his book [1923].

[107] See the reproduction of this letter, its transcription and a translation in Appendix IV, pp. 161–165.

to evaluate the paper, she justifies her recommendation to reject the article on the grounds that:

> It is first of all a restatement that is not at all clear of the principal theorems of my "Invariante Variationsprobleme" (Göttinger Nachrichten, 1918 or 19), with a slight generalization—the invariance of the integral up to a divergence term—which can actually already be found in Bessel-Hagen (Math. Annalen, around 1922).[108]

Obviously she clearly remembered her work—but not its exact date of publication—and was well aware of Bessel-Hagen's. In the next paragraph she points out that the credit for this generalization is due to Bessel-Hagen and adds a disclaimer that highlights her honesty and lack of ambition: "citing me here [in Zaycoff's second paragraph] is an error" ("daß er mich hier zitiert ist irrtümlich"). After criticizing the nearly incomprehensible computations contained in this paper, she concludes that it does not represent real progress, while her own intent in writing her article had been "to state in a rigorous fashion the significance of the principle and, above all, to state the converse which does not appear here."[109] Then she suggests that a part of the paper might be suitable for some physics journal, and she further suggests that a reference could be made to the statement of her theorems in "Courant–Hilbert," i.e., the recently published book of Courant and Hilbert [1924].[110] Thus, in her own modest way, Noether was conscious of the value of her work. The abstract, rigorous and general point of view that is the mark of all her mathematics is evident in her words, "to state in a rigorous fashion the significance of the principle."

*The Klein Jubilee* — Noether's correspondence shows great respect for Klein and she dedicated the *Invariante Variationsprobleme* to him on the occasion of his academic jubilee.[111] It used to be a frequent practice in German universities to celebrate the fiftieth anniversary of an eminent professor's doctorate, *das goldene Doktorjubiläum*. In 1916 Hilbert had written an article for the jubilee of Hermann Amandus Schwarz, which was reprinted in the same volume of the *Göttinger Nachrichten* as Noether's [1916b]. Max Noether's jubilee was celebrated 5 March 1918. Klein's doctorate having been awarded 12 December 1868 at the University of Bonn, his academic jubilee was celebrated in Göttingen, at the university on 10 December 1918, and at the Mathematical Society two days later with a lecture on his scientific work delivered by Paul Koebe.[112]

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108 "Es handelt sich zuerst um eine nicht allzu durchsichtige Wiedergabe der Hauptsätze meiner 'Invarianten Variationsprobleme' (Göttinger Nachrichten 1918 oder 19), mit einer geringen Erweiterung—Invarianz des Integrals bis auf Divergenzglied—die sich schon bei Bessel-Hagen findet (Math. Annalen etwa 1922)." It was Bessel-Hagen's article [1921], analyzed below, in Chap. 4, p. 91, that formally introduced the symmetries up to divergence.

109 "Mir kam es in den 'Invarianten Variationsproblemen' nur auf die scharfe Formulierung der Tragweite des Prinzips an, und vor allem auf die Umkehrung[,] die hier nicht herein spielt."

110 See, *infra*, Chap. 4, p. 95.

111 Jubilee, from the Hebrew *yovel*, horn, which became a metonymy for a fiftieth year because, in the biblical calendrical cycle, every fiftieth year was to be inaugurated by sounding such a horn.

112 "On the scientific work of F. Klein, in particular on the theory of automorphic functions" ("Über F. Kleins wissenschaftliche Arbeiten, insbesondere die die Theorie der automorphen Funktionen

## 1.5 After Göttingen

After the period 1915–1918, Noether directed her research toward abstract algebra, the theory of ideals and the representation theory of algebras, and became one of the most important mathematicians of her time. She was deprived of her employment by the Nazis and compelled to leave Göttingen where her forceful personality and great talent had attracted many students. Since she was not a civil servant, she could not be dismissed directly, but she was put on leave with full pay on 25 April 1933.[113] On 13 September, paragraph 3 of the law of 7 April 1933 that excluded all persons of non-aryan descent from the civil service was applied to all the Jews who taught in the universities, civil servants or not, with few exceptions. On that day, she wrote to Richard Brauer:

> Since presently paragraph 3 comes into effect—I was notified today that my permission to teach has been rescinded in accordance with this paragraph [. . . ].[114]

In the next lines she asks whether Brauer has any prospect of employment, then discusses her own possibilities for the coming academic year, and, in the last part of her letter, she gives news of the mathematical results of three young Göttingen mathematicians, among whom Max Deuring, who had defended in 1931 his doctoral thesis written under her direction, and Ernst Witt, who had joined the Nazi party in May and defended his thesis in July. Noether left Göttingen shortly thereafter, visited Russia briefly, but preferred refuge in the United States where, until her premature death in 1935, she taught at Bryn Mawr College[115] outside Philadelphia, a women's undergraduate school with a small graduate school to which a number of male students had been admitted since 1931. She also participated very actively in the mathematical life of the Institute for Advanced Study at Princeton, a short train ride from Philadelphia. After her death she was replaced at Bryn Mawr by Nathan Jacobson for the 1935–1936 academic year.[116] Numerous articles and books have discussed her life and her work as an algebraist.[117]

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betreffenden"). See *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 28, Part 2 (1919), p. 30. This ceremony is mentioned in the preface to Klein's *Gesammelte mathematische Abhandlungen*, vol. 1, p. iii, and the (unsigned) text of an address delivered on that occasion is printed on the pages that follow the preface. (We observe that the editors of the *Gesammelte mathematische Abhandlungen*, vol. 1 (1920), acknowledged (p. v) the assistance of Miss E. Noether for the correction of the proofs.) Also see, in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 27, Part 2 (1918), pp. 59–60, a letter of congratulations from the DMV, and on p. 63, the announcement of the formation of a foundation by Klein's friends and students and another letter of congratulations by the proponents of this foundation, which was delivered by Robert Fricke.

113 Segal [2003], p. 125.

114 "Da augenblicklich §3 in Aktion tritt—ich habe heute die Mitteilung der entzogenen Lesebefügnis nach diesem [. . . ]," letter from Emmy Noether to Richard Brauer, in the Bryn Mawr archives, partially translated in Curtis [1999], pp. 213–214.

115 Some of her German mathematics books can still be found in the mathematics department.

116 *Notices of the American Mathematical Society*, October 2000, p. 1061. Jacobson (1910–1999) had attended her lectures in Princeton. He later was the editor of her *Collected Papers*.

117 See Dick [1970] [1981], Kimberling [1981], Srinivasan and Sally [1983], Teicher [1999], Curtis [1999], etc.

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