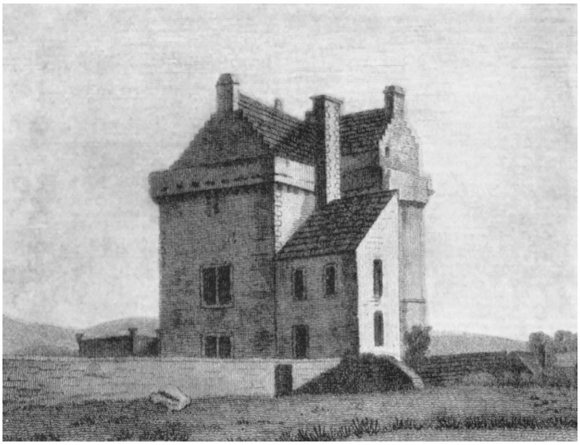
2

LOGARITHMS

2.1 NAPIER'S FIRST THREE TABLES

The mists of sixteenth-century Scotland frequently engulfed Merchiston Tower, near Edinburgh. But today Merchiston Tower is in downtown Edinburgh, right at the heart of the Merchiston Campus of Napier University. In the sixteenth



MERCHISTON TOWER IN THE EIGHTEENTH CENTURY
The lower building with slanted roof and the chimney are eighteenth-century additions.
From Knott, *Napier tercentenary memorial volume*, 1915, p. 54.

E.A. González-Velasco, *Journey through Mathematics: Creative Episodes in Its History*,
DOI 10.1007/978-0-387-92154-9_2,

century its seventh laird was Archibald Napier (Neper, Nepier, Nepair, Napeir, Nappier, Napper?). John Napier (Nepero in Latin, as we shall see later) was born in 1550 to Sir Archibald and his first wife, Janet Bothwell. At that time, Mary Tudor (later Stuart) was queen of the Scots and Edward VI king of the English. Three years earlier—to put things in perspective—Henry VIII had died, Cervantes was born, and Ivan IV (the Terrible) crowned himself tsar of Russia, the first with that title.

John Napier, an amateur Calvinist theologian who had studied religion at Saint Andrews, predicted that the end of the world would occur in the years between 1688 and 1700, in his book *A plaine discouery of the whole Reuelation of Saint Iohn* of 1593. Not a huge success at prediction, it must be pointed out. Napier's reputation may not have reached the present time either based on his theological studies or on his activities as a designer of weapons of mass destruction, intended for use against the enemies of the true religion. His lasting reputation is due to his third hobby: mathematical computation. As a landowner, he never held a job, and his time and energy could be entirely

POSSIBLY JOHN NAPIER, COMPUTING
Portrait by Francesco Delarame, engraving by Robert Cooper (c. 1810).

devoted to intellectual pursuits.

He developed an interest in reducing the labor required by the many tedious computations that were necessary in astronomical work, involving operations with the very large values of the trigonometric lengths given in the tables of his times. His interest in these matters might have been rekindled in 1590 on the occasion of a visit by Dr. John Craig, the king's physician. James VI of Scotland had set out to fulfill an important royal duty, to find a wife, and had selected Princess Anne of Denmark. In 1590, on the consent of her brother Christian IV, James set sail to Copenhagen to pick her up. Bad weather, however, caused his ship to land first on the island of Hveen near Copenhagen, which was the location of *Uraniborg*, the astronomical observatory of Tycho Brahe.

On his return from Denmark, Dr. John Craig paid the aforementioned visit to Napier to inform him of his findings while visiting Tycho Brahe. There were new, ingenious ways to perform some of those tedious computations required by astronomical calculations, and foremost was the use of *prosthaphæresis*. This imposing name, coined from the Greek words for addition (*προσθέσις*) and subtraction (*άφαίρεσις*), refers to the use of the trigonometric identity

$$\sin A \cdot \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)],$$

brought to Hveen by Paul Wittich (c. 1546–1586), an itinerant mathematician who visited *Uraniborg* for four months in 1580.[1] So, why should this be all the rage in computation? Let us say that we want to evaluate the product of two 10-digit numbers a and b. We know that tables of sine and cosine values were available at that time, and it was not a difficult matter to find numbers A and B, at least approximately, such that $a = \sin A$ and $b = \sin B$ (it may be necessary to divide or multiply a and b by a power of 10, but this can easily be managed). Then perform the simple operations $A - B$ and $A + B$, find their cosines in the table, subtract them as indicated in the previous formula, and divide by 2, and that is the product ab (except for an easy adjustment by a power of 10). The big problem of multiplying many-digit numbers was thus reduced to the simpler one of addition and subtraction.

But what about quotients, exponentiations, and roots? Napier sought a general method to deal with these computations, eventually found it, and

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[1] Its European origin goes back to Johannes Werner, who probably discovered it about 1510, but a similar formula for the product of cosines is usually credited to the Egyptian Abu'l-Hasan Ali ibn Abd al-Rahman ibn Ahmad ibn Yunus al-Sadafi al-Misri (c. 950–1009). Wittich's contribution was his realization that it can be used for multiplication of any two numbers. See Thoren, *The Lord of Uraniborg*, 1990, p. 237.

gave it to the world in his book *Mirifici logarithmorum canonis descriptio* (Description of the admirable table of logarithms) of 1614 (he had become eighth laird of Merchiston in 1608). The English translation's author's preface starts with a statement of his purpose [p. A_5^2]:

> Seeing there is nothing (right well beloued Students in the Mathematickes) that is so troublesome to Mathematicall practise, nor that doth more molest and hinder Calculators, then the Multiplications, Diuisions, square and cubical Extractions of great numbers, which besides the tedious expence of time, are for the most part subject to many slippery errors. I began therefore to consider in my minde, by what certaine and ready Art I might remoue those hindrances.[2]

This is a small volume of 147 pages, 90 of which are devoted to mathematical tables containing a list of numbers, mysteriously called *logarithms*, whose use would facilitate all kinds of computations. Before the tables, some geometrical theorems are given in the *Descriptio* about their properties, and examples are provided of their usefulness. About the use of logarithms we shall talk presently, but first we should say that no explanation was given in this book about how they were computed. Instead, there is the following disclaimer in an *Admonition* in Chapter 2 of *The first Booke* [pp. 9–10]:

> Now by what kinde of account or method of calculating they may be had, it should here bee shewed. But because we do here set down the whole Tables, . . . , we make haste to the vse of them: that the vse and profit of the thing being first conceiued, the rest may please the more, being set forth hereafter, or else displease the lesse, being buried in silence.

In short, if his Tables were well received Napier would be happy to explain how the *logarithms* were constructed; otherwise, let them go into oblivion. It happened that the *Descriptio* was a huge editorial success; a book well received and frequently used by scientists all over the world. Then an explanatory book became necessary but, although it was probably written before the *Descriptio*, it was published only posthumously (Napier died in 1617, one year after Shakespeare and Cervantes) under the title *Mirifici logarithmorvm canonis constrvctio*, when his son Robert included it with the 1619 edition of the *Descriptio*.[3]

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[2] From Wright's translation as *A description of the admirable table of logarithmes* ("all perused and approued by the Author"), 1616. Page references are to this edition.

[3] Page references are given by article number so that any of the available sources (see the bibliography) can be used to locate them. Quotations are from Macdonald's translation.

From Knott, *Napier tercentenary memorial volume*, Plate IX facing page 181.

The *Constrvctio* gives a glimpse into Napier's mind and his possible sources of inspiration. It starts as follows:

> 1. A logarithmic table [*tabula artificialis*] is a small table by the use of which we can obtain a knowledge of all geometrical dimensions and motions in space, by a very easy calculation.
>
> > ... very easy, because by it all multiplications, divisions and the more difficult extractions of roots are avoided ...
>
> It is picked out from numbers progressing in continuous proportion.

> 2. Of continuous progressions, an arithmetical is one which proceeds by equal intervals; a geometrical, one which advances by unequal and proportionately increasing or decreasing intervals.

It will be easier to interpret what Napier may have had in mind using modern notation. To us, a geometric progression is one of the form $a, ar, ar^2, ar^3, \ldots,$ where a and r are numbers, and it is clear that the exponents of r form an arithmetic progression with step 1. It was common knowledge in Napier's time that the product or quotient of two terms of a geometric progression contains a power of r whose exponent is the sum or difference of the original exponents.[4]

To take a specific instance, Michael Stifel (1487–1567) had published a list of powers of 2 and their exponents in his *Arithmetica integra* of 1544, with the exponents in the top row (*in superiore ordine*) and the powers in the

0.	1.	2.	3.	4.	5.	6.	7.	8.
1.	2.	4.	8.	16.	32.	64.	128.	256.

bottom row (*in inferiore ordine*), and right below he taught his readers how to use the table to multiply and divide (translated from *Liber* III, p. 237):

> As the addition (in the top row) of 3 and 5 makes 8, thus (in the bottom row) the multiplication of 8 and 32 makes 256. Also 3 is the exponent of eight itself, & 5 is the exponent of the number 32. & 8 is the exponent of the number 256. Similarly just in the top row, from of the subtraction of 3 from 7, remains 4, thus in the bottom row the division of 128 by 8, makes 16.

That is, to multiply $8 = 2^3$ and $32 = 2^5$ just add the exponents to obtain $2^8 = 256$, and to divide $128 = 2^7$ by $8 = 2^3$ just subtract the exponents to obtain $2^4 = 16$. Thus multiplication and division are easily reduced to additions and subtractions of exponents.

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[4] When writing r^n I am using current notation, which did not exist in Napier's time, for it was introduced by Descartes in *La Géométrie*, 1637. On page 299 of the original edition (as an appendix to the *Discours de la méthode*, *La Géométrie* starts on page 297) Descartes stated: *Et aa, ou a^2, pour multiplier a par soy mesme*; *Et a^3, pour le multiplier encore vne fois par a, & ainsi a l'infini* (And aa, or a^2, to multiply a by itself; And a^3, to multiply it once more by a, and so on to infinity). Quoted from Smith and Latham, *The geometry of René Descartes*, 1954, p. 7. This statement can also be seen, in the English translation by Smith and Latham only, in Hawking, ed., *God created the integers, The mathematical breakthroughs that changed history*, 2005, p. 293.

But what about 183/11? There are no such table entries, not even approximately. The trouble is that the values of 2^n are very far apart. For values of r^n to be close to each other, r must be close to 1. Napier realized at this point that the computations of powers of such an r may not be easy (think of, say, $r = 1.00193786$), expressing himself as follows in the *Constrvctio*:

> 13. The construction of every arithmetical progression is easy; not so, however, of every geometrical progression. ... Those geometrical progressions alone are carried on easily which arise by subtraction of an easy part of the number from the whole number.

What Napier meant is that if r is cleverly chosen, multiplication by r can be reduced to subtraction. In fact, he proposed—in his own words in Article 14—to choose r of the form $r = 1 - 10^{-k}$. Then multiplication of a term of the progression by r to obtain the next one is equivalent to a simple subtraction:

$$ar^{n+1} = ar^n(1 - 10^{-k}) = ar^n - ar^n 10^{-k},$$

and the last term on the right is easily obtained by a shifting of the decimal point. With such an r, it is not necessary to evaluate its successive powers by multiplication to obtain the terms of the progression.

On this basis Napier made his choice of a and r. First he chose $a = 10{,}000{,}000$ because at this point in his life he was mainly interested in computing with sines, and the best tables at that time (c. 1590), those of Regiomontanus and Rhaeticus, took the whole sine (*sinus totus*) to be 10,000,000. Then, in order for the terms of his geometric progression to be sines (that is, parts of the whole sine), he had to choose $r < 1$; and for ease of computation he selected

$$r = 1 - 0.0000001.$$

We have talked about shifting the decimal point as a simple operation. Was such a procedure, or even the decimal point, available in Napier's time? The use of decimal fractions can be found in China, the Islamic world, and Renaissance Europe. It eventually replaced the use of sexagesimal fractions, then in vogue, after Simon Stevin (1548–1620), of Bruges, explained the system clearly in his book in Flemish *De thiende* ("The tenth") of 1585. Their use had already been urged in the strongest terms by François Viète in his *Vniversalivm inspectionvm* of 1579 [p. 17]:

> Finally, sexagesimals & sixties are to be used sparingly or not at all in Mathematics, but thousandths & thousands, hundredths & hundreds, tenths & tens,

and their true families in Arithmetic, increasing & decreasing, are to be used frequently or always.[5]

Not only was Napier aware of decimal fractions and points, but, through his continued use of decimal notation in the *Constrvctio*, he may have been instrumental in making this notation popular in the world of mathematics. He explained it clearly at the beginning of the *Constrvctio*:

5. In numbers distinguished thus by a period in their midst, whatever is written after the period is a fraction, the denominator of which is unity with as many ciphers [zeros] after it as there are figures after the period.

Thus 10000000·04 is the same as $10000000\frac{4}{100}$; also 25·803 is the same as $25\frac{803}{1000}$; also 9999998·0005021 is the same as $9999998\frac{5021}{10000000}$ and so of others.

With his use of decimal notation thus clearly settled and his choice of a and r, Napier then continued his construction in Article 16 as follows:

Thus from the radius, with seven ciphers added for greater accuracy [that is, with seven zeros after the decimal point], namely, 10000000.0000000, subtract 1.0000000, you get 9999999.0000000; from this subtract .9999999, you get 9999998.0000001; and proceed in this way until you create a hundred proportionals, the last of which, if you have computed rightly, will be 9999900.0004950.

To sum up in present-day notation, Napier had applied the formula

$$10^7 r^{n+1} = 10^7 r^n (1 - 10^{-7}) = 10^7 r^n - r^n$$

[5] But Viète did not use the decimal point. Instead, he wrote the decimal part of a number as a fraction with an empty denominator, giving the semicircumference of a circle whose *sinus totus* is 100,000 as $314{,}159,\frac{265{,}36}{}$ [p. 15]. Viète used this notation through most of the book, but later he modified it, using first a vertical bar to separate the decimal part, which was also written in smaller or lighter type. For instance, he gave the *sinus* of 60° as 86,602|540,37 [p. 64]. Later, he replaced the vertical bar with another comma, writing now the semicircumference of the given circle as 314,159,265,36 [p. 69].

with $n = 0, \ldots, 100$ to perform the following computations:[6]

$$\begin{aligned}
10000000 \times 0.9999999^0 &= 10000000\\
10000000 \times 0.9999999^1 &= 9999999\\
10000000 \times 0.9999999^2 &= 9999998.0000001\\
10000000 \times 0.9999999^3 &= 9999997.0000003\\
&\cdots\cdots\cdots\\
10000000 \times 0.9999999^{100} &= 9999900.0004950.
\end{aligned}$$

This completes his *First table*, which, writing in Latin, he called "canon." Of course, the entries on the right-hand sides of the fourth line and those below are only approximations, as shown by the fact that we have only subtracted 0.9999998 from the third entry to obtain the fourth, but Napier had already remarked in Article 6: *When the tables are computed, the fractions following the period may then be rejected without any sensible error*.

This first table is just the start of Napier's work, and these entries do not even appear in the *Descriptio*. The use of this table in computation is limited in at least two ways (we will mention a third later on). The first is that it contains very few entries, and the second is that all the numbers on the right-hand sides are very close to 10,000,000 due to the choice of r. But these problems are easy to deal with by computing more entries and choosing a different value of r.

Thus Napier started another table, but this time, to move a little further from 10,000,000, he chose r to be the last number in the first canon (omitting the decimals, for dealing with such large numbers one need not trouble with them) over the first one,

$$r = \frac{9999900}{10000000} = 0.99999 = 1 - \frac{1}{100000},$$

[6] At the risk of delving into tedium by harping on mathematical notation, the $\times$ sign for multiplication did not exist in Napier's time. It was introduced for the first time in an anonymous appendix to the 1618 edition of Wright's translation of the *Descriptio* entitled "An appendix to the logarithmes," pp. 1–16, probably written by William Oughtred. It is reprinted in Glaisher, "The earliest use of the radix method for calculating logarithms, with historical notices relating to the contributions of Oughtred and others to mathematical notation," 1915. It also appears in Oughtred, *Clavis mathematicæ* (The key to mathematics), 1631. However, the $=$ sign that we use below did exist; it had been introduced by Robert Recorde (1510–1558) in his algebra book *The whetstone of witte* of 1557. The original symbol was longer than the present one, representing two parallel lines "bicause noe. 2. thynges, can be moare equalle." This book has unnumbered pages, but is divided into named parts. This quotation is from the third page of "The rule of equation, commonly called Algebers Rule." This is page 222 of the text proper, not counting the front matter.

and computed 51 entries by the same method as before, but now using the equation $10^7 r^{n+1} = 10^7 r^n - 100 r^n$ to obtain

$$\begin{aligned} 10000000 \times 0.99999^0 &= 10000000 \\ 10000000 \times 0.99999^1 &= 9999900 \\ &\cdots\cdots \\ 10000000 \times 0.99999^{50} &= 9995001.222927 \end{aligned}$$

(he erred: the last entry should be 9995001.224804). This is his second table.

Then he started again with the ratio

$$r = \frac{9995000}{10000000} = 0.9995 = 1 - \frac{1}{2000}$$

and computed 21 entries:

$$\begin{aligned} 10000000 \times 0.9995^0 &= 10000000 \\ &\cdots\cdots \\ 10000000 \times 0.9995^{20} &= 9900473.57808. \end{aligned}$$

He placed the numbers so obtained in the first column of his third canon. Finally, from each of these values he computed 68 additional entries using the ratio

$$0.99 \approx \frac{9900473}{10000000},$$

and these he located horizontally from each first value. Thus, his third table had 69 columns, and the entries in each column are 0.99 of those in the preceding column. The last entry of the last column is

$$9900473.57808 \times 0.99^{68} \approx 4998609.4034,$$

"roughly half the original number" (again, the last two decimal digits are in error).

From Napier's point of view, interested as he was at the time in computations with sines and applications to astronomy, this was enough, since—other than the magnification by 10,000,000—the right-hand sides above are the sines of angles from 90° to 30°. He would easily deal with smaller angles in Articles 51 and 52 of the *Constrvctio*.

First Column.	Second Column.		69th Column.
10000000.0000	9900000.0000		5048858.8900
9995000.0000	9895050.0000		5046334.4605
9990002.5000	9890102.4750		5043811.2932
9985007.4987	9885157.4237		5041289.3879
9980014.9950	9880214.8451	. . .	5038768.7435
.	.		.
.	.		.
.	.		.
9900473.5780	9801468.8423		4998609.4034

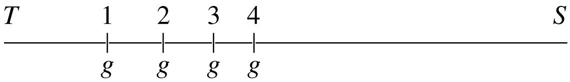
Now let us take stock of what Napier had and had not achieved to this point. He had elaborated several tables and subtables based on the idea of using the arithmetic progressions of what we now call exponents to perform computations with the table numbers. These contain about 1600 entries, not quite a sufficient number; and, what is more, the various tables are based on different choices of r, so that the product or quotient of numbers from different tables is impossible. Furthermore, there is a third problem in that his table entries are laden with decimals, while the sines in the tables of his time were all whole numbers (for simplicity, we shall refer to a *sinus* as a sine from now on). Something else had to be done.

2.2 NAPIER'S LOGARITHMS

What Napier needed is the reverse of what he had. In his first three tables the exponents are integers but the sines are laden with decimals. What he needed was to start with sines that are whole numbers, available in published tables, and then compute the corresponding exponents whatever they may be.

His next idea was based on a graphical representation, and once polished he gave it to us as follows in the *Constrvctio*. First he chose a segment ST, of length 10000000, to represent the *Sinus Totus* [Art. 24], and let 1, 2, 3, and 4 be points located in the segment TS as shown. Assume that TS, $1S$, $2S$, $3S$, $4S, \dots$ are sines in continued proportion

$$\frac{1S}{TS} = \frac{2S}{1S} = \frac{3S}{2S} = \frac{4S}{3S} = \cdots,$$



such as those of the First table, in which case the stated ratio is 0.9999999, $1S = 9999999$, and so on. Then he envisioned a point g that, starting at T with a given initial velocity, proceeds toward S with decreasing velocity, in such a manner that the moving point [*Descriptio*, p. 2]

> in equall times, cutteth off parts continually of the same proportion to the lines [segments] from which they are cut off.

This means, referring to the figure above and noting that subtracting each of the preceding fractions from 1 and simplifying gives

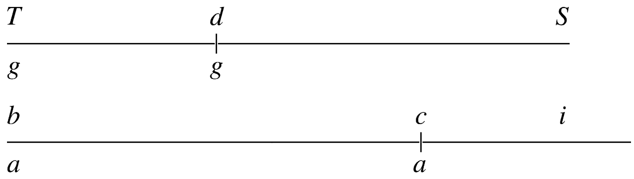
$$\frac{T1}{TS} = \frac{12}{1S} = \frac{23}{2S} = \frac{34}{3S} = \cdots,$$

that g traverses each of the segments $T1$, 12, 23, 34, ... in equal times. Napier referred to the described motion of g as *geometrical* (hence his use of the letter g). Differential equations were not known in Napier's time, and he could not explain this motion as clearly as we might wish.

Assume next that $1S$, $2S$, $3S$, $4S$, ... are the sines in the First table, and imagine an infinite half-line bi (drawn in the next figure), on which we choose coordinates 0, 1, 2, 3, 4, ... as follows. First, the origin is at b, and then let a new point a move on this line from b to the right with constant velocity, that of g when at T. We choose the coordinates 1, 2, 3, 4, ... (not shown in Napier's figure) to be the points on bi at which a arrives when g arrives at the points with labels 1, 2, 3, 4, ... Thus, as g passes through the points marking the sines of the First table in equal amounts of time, a reaches the exponents 1, 2, 3, 4, ... that generate these sines.

Napier chose to introduce this second line in a general situation, considering the case of an arbitrary sine as follows [Art. 26]:

> Let the line *TS* be radius, and dS a given sine in the same line: let g move geometrically from T to d in certain determinate moments of time. Again, let bi be another line, infinite towards i, along which, from b, let a move arithmetically [constant velocity] with the same velocity as g had at first when at T: and from the fixed point b in the direction of i let a advance in just the same moments of time [as it took g to advance to d] to the point c.



Then, in general, if the coordinates on the line bi are as we have just described, if dS is an arbitrary sine, if g and a start with the same initial velocity, and if a arrives at a point c in the same amount of time as g arrives at d, the distance bc represents the exponent—not necessarily an integer—corresponding to the sine dS.

In the *Constrvctio* [Article 26], Napier called the distance bc the "artificial number" of the "natural number" dS, but he later changed his terminology. Since the sines in Napier's basic tables are in continued proportion or ratio, he later coined the word logarithm, from the Greek words *λόγων* (*logon*) = ratio and *άριθμός* (*arithmos*) = number, to refer to each of these exponents. In the *Descriptio*, written a few years after the *Constrvctio*, he gave the definition as follows [pp. 4–5]:

> The Logarithme therefore of any sine is a number very neerely expressing the line, which increased equally in the meane time, while the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.

By the time the translation of the *Constrvctio* was prepared, the word logarithm was already in common use, and that motivated the translator to use it there. We shall do so from now on.

At this point, Napier faced the task of computing the logarithms of sines given by whole numbers. The start is easy:

> 27. Whence nothing [zero] is the logarithm of radius.

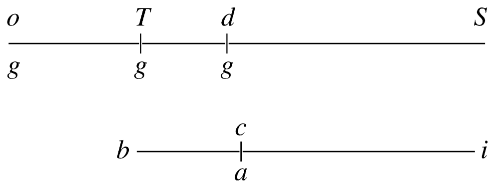
Quite clearly: if g and a do not move they remain at T and b, respectively. But the rest, the exact computation of logarithms of whole numbers, was far from trivial with the tools available to Napier (the already mentioned lack of differential equations was responsible). On the face of that impossibility, he made a giant leap by realizing that it may be good enough to find lower and upper bounds for such logarithms. With a certain amount of ingenuity he found such bounds.

28. Whence also it follows that the logarithm of any given sine is greater than the difference between radius and the given sine, and less than the difference between radius and the quantity which exceeds it in the ratio of radius to the given sine. And these differences are therefore called the limits of the logarithm.

This can be explained, and then his proof abbreviated, as follows. Let dS be the given sine and let oS be the quantity that exceeds the radius in the ratio

$$\frac{oS}{TS} = \frac{TS}{dS}.$$

According to the geometric motion of g and the arithmetical motion of a, "oT, Td, and bc are distances traversed in equal times." Then, since the velocities



of g and a are equal when g is at T and a at b, and since the velocity of g is decreasing but that of a is constant,

$$oS - TS = oT > bc \ (= \log dS) > Td = TS - dS,$$

which was to be proved.

It goes without saying that a present-day reader would like a restatement of Napier's result in current notation.[7] That is not difficult if we denote the given sine dS by x. Then $oS = TS^2/x$, and we can write the inequalities of Article 28 in the form

$$TS - x < \log x < \frac{TS^2}{x} - TS = \frac{TS(TS - x)}{x},$$

to which we have appended an equality from Article 29. It was this result that enabled Napier to find the logarithms of many sines, starting with those in the

[7] Napier stated all of his results in narrative form. The *Constrvctio* does not contain a single modern-looking formula, nor is any notation used for "artificial number" or logarithm. I have, however, adopted log as a natural choice, since there is no possibility of confusing this with any current mathematical function at this point.

First table (they are not all whole numbers, but their logarithms will be useful nevertheless). Starting with $x = 9999999$ we have $TS - x = 1.0000000$ and

$$\frac{TS(TS-x)}{x} = \frac{10000000}{9999999} = 1.00000010000001.$$[8]

Then, since these limits differ insensibly, Napier took [Art. 31]

$$\log 9999999 = 1.00000005.$$

Now, if we denote once more the sines in the First table—that is, the numbers on the right of the equal signs—by TS, $1S$, $2S$, $3S$, ..., we have already seen that as g goes through them in equal times, their logarithms follow the motion of a and we have [Art. 33]

$$\log 2S = 2 \log 1S, \quad \log 3S = 3 \log 1S,$$

and so on. In particular, for the last sine in the First table we have

$$\log 9999900.0004950 = 100 \log 1S = 100 \log 9999999,$$

and then, from the bounds on $\log 9999999$,

$$100 < \log 9999900.0004950 < 100.00001000.$$

We can pick $\log 9999900.0004950 = 100.000005$.

This method works well because the sines in the First table are very close to 10000000. But the theorem in Article 28 is not sufficiently accurate further down the line. For instance, if $x = 9000000$, which is still a lot closer to 10000000 than to zero, it would give

$$10000000 - 9000000 < \log 9000000 < \frac{10000000(10000000 - 9000000)}{9000000}.$$

That is, $1000000 < \log 9000000 < 1111111$, which is too wide a range to select $\log 9000000$ with any accuracy.

Thus Napier needed a better theorem, and actually found two. The first is:

36. The logarithms of similarly proportioned sines are equidifferent.

[8] A reader who is concerned that the application of this formula requires long division is invited to count the number of such divisions that will be necessary to elaborate Napier's complete table of logarithms. The total count will turn out to be extremely small.

This means that if v, x, y, and z are sines such that

$$\frac{x}{v} = \frac{z}{y},$$

then

$$\log x - \log v = \log z - \log y.$$

Napier stated [Art. 32]: "This necessarily follows from the definitions of a logarithm and of the two motions." This is because g traverses the distance $x - v$ in the same amount of time as $z - y$, and then a moves from $\log x$ to $\log v$ in the same amount of time as from $\log z$ to $\log y$.

His second theorem is a little more involved:

> 39. The difference of the logarithms of two sines lies between two limits; the greater limit being to radius as the difference of the sines to the less sine, and the less limit being to radius as the difference of the sines to the greater sine.

After providing a geometric proof similar to that in Article 28 but a little longer and requiring Article 36, he rephrased this result in a verbal form [Art. 40] that can be rewritten in today's terminology as follows. If x and y are the two sines and y is the larger, then

$$\frac{TS(y-x)}{y} < \log x - \log y < \frac{TS(y-x)}{x}.$$

Then he immediately put this result to good use by showing how to compute the logarithm of any number that is not a sine in the First table, but near one of them [Art. 41].

For example, let $x = 9999900$. In the First table the nearest sine is $y = 9999900.0004950 = 10000000 \times 0.9999999^{100}$ and, as shown in Article 33 (page 92),

$$100 < \log y < 100.0000100.$$

Then, with x and y as above, we have $TS(y - x) = 4950$, and the inequalities in Article 40 (stated here after quoting Article 39) give the approximate inequalities

$$0.00049500495002 < \log x - \log y < 0.00049500495005.$$

Napier did not state these inequalities, but only concluded that we can take $\log x - \log y = 0.0004950$, and then

$$\log x = \log y + 0.0004950.$$

Hence, from the bounds for $\log y$,

$$100.0004950 < \log x < 100.0005050.$$

But $x = 9999900$ appears in the Second table as $10000000 \times 0.99999^1$, and the sines in this table are in continued proportion, just as TS, $1S$, $2S$, $3S, \ldots$ are in Napier's general formulation. As g goes through them in equal times their logarithms are found from the motion of a in the same equal times. They are double, triple, etc., the logarithm of x. Thus, from the limits found above for $\log x$, the limits for the logarithms of the sines of the Second table are 200.0009900 and 200.0010100 for the second, 300.0014850 and 300.0015150 for the third, and so on until the limits for the logarithm of the last sine (the 50th after the first) are found to be

$$5000.0247500 < \log 9995001.222927 < 5000.0252500.$$

The actual logarithms of these sines can be chosen between the limits stated here [Art. 42]. Thus, the logarithms of the sines of the Second table have been quickly obtained by a cascading procedure identical to that giving those of the sines in the First table.

Next, in [Art. 43], "to find the logarithms of ... natural numbers ... not in the Second table, but near or between" the sines in this table, Napier used a new method. As above, start by naming a given sine and the one nearest to it in the Second table by x and y, with $y > x$. But now choose what Napier called a *fourth proportional*, that is, a number z such that (in our terminology)

$$\frac{z}{TS} = \frac{x}{y},$$

which is possible because the other three numbers are known. This places z within the bounds of the First table because $x \approx y$ implies that $z \approx TS$, and then the logarithm of z can be computed by the previous method. Then, the theorem in Article 36 and the fact that $\log TS = 0$ give

$$\log z = \log x - \log y.$$

In this equation, two of the logarithms are known and the third can then be found. Or, more cautiously, the limits for two of these logarithms can be determined, whence the limits for the third can be ascertained.

Napier applied this procedure to obtain the logarithm of $x = 9995000$, which is near the last entry, $y = 9995001.222927$, of the Second table. We have already found that

$$5000.0247500 < \log y < 5000.0252500,$$

and the fourth proportional is

$$z = TS\,\frac{9995000}{9995001.222927} = 9999998.7764614.$$

Then, replacing x with z and y with 9999999 in the limits of Articles 39 and 40, yields

$$\frac{TS(9999999 - z)}{9999999} = 0.2235386$$

and

$$\frac{TS(9999999 - z)}{z} = 0.2235386.$$

It follows that

$$\log z - \log 9999999 = 0.2235386.$$

Since we know that $1 < \log 9999999 < 1.0000001$, we obtain

$$1.2235386 < \log z < 1.2235387.$$

The equation $\log x = \log y + \log z$ and the limits found for the two logarithms on the right-hand side give (these are Napier's figures)

$$5001.2482886 < \log x < 5001.2487888.$$

From these limits Napier concluded as follows:

> Whence the number 5001.2485387, midway between them, is (by 31) taken most suitably, and with no sensible error, for the actual logarithm of the given sine 9995000.

Unfortunately, his result is a little off because the last entry of the Second table is in error, as mentioned above. On the good side of things, this got him into the Third table, whose second entry is precisely 9995000. Then the logarithms of the first two entries in the first column of the Third table are known, and the result in Article 36 allowed him to cascade down, computing the logarithms of all the entries in this column [Art. 44]. Then the logarithms of numbers that are near or between the entries of this column can be computed by the method of the fourth proportional illustrated above. Using this method, he computed the logarithm of 9900000, which is near the last entry of the first column of the Third table [Art. 45]. But this is the first entry in the second column. Now denote the first two sines in the first column by x and y and the first two sines in the second column by u and v. We know that

$$\frac{u}{x} = \frac{v}{y},$$

and from this that

$$\frac{v}{u} = \frac{y}{x}.$$

By Article 36, $\log v - \log u = \log y - \log x$, and, since the last difference is known, $\log v$ can be computed. Cascading down, the logarithms of all the sines in this second column can be found. Finally, once the logarithms of the numbers in the first two columns were obtained, the fact that the entries in each row of the Third table are in continued proportion means that their logarithms are equidifferent. This allowed Napier to compute the logarithms of all the entries in the Third table [Art. 46].

Having completed this second stage of his work, Napier now discarded the First and Second tables, keeping only the Third table with the logarithms of all its entries. This—the framework for the construction of the complete logarithmic table—he called *The Radical Table* [Art. 47]. A portion of it is shown on the next page. He considered it sufficient to leave just one decimal digit in the logarithms (*Artificiales*), while he had used seven in all preliminary computations to avoid error accumulation. It is not important to us that these logarithms are slightly erroneous due to the faulty last entry in the Second table (the last one of the 69th column should be 6934253.4), because we are interested in Napier's invention and how he constructed the table, but not in using it.

Then Napier embarked on the third stage of his work: the elaboration of a table of logarithms, which he called the *principal table*, for the sines of angles from 45° to 90° in steps of 1 minute. These can be obtained from any table of common sines. For each sine he found first the entry in the Radical table that is closest to it. But now that seven-decimal-digit accuracy was not needed, the method of the fourth proportional becomes irrelevant, and even the use of the inequalities in Article 38. What counted at this stage was speed, so that the principal table can actually be constructed. So, given a sine and its closest entry in the Radical table, x and y with $y > x$, compute $TS(y - x)$, and then, instead of dividing this product first by y and then by x to find the limits of $\log x - \log y$, just divide it by the easiest possible number between x and y. Then the result is an approximation of the true value of $\log x - \log y$, and the unknown logarithm can be approximately found from it.

RADICALIS TABVLÆ.

Columna prima.		*Columna secunda.*	
Naturales.	Artificiales	Naturales.	Artificiales
10000000.0000	0	9900000.0000	100503.3
9995000.0000	5001.2	9895050.0000	105504.6
9990002.5000	10002.5	9890102.4750	110505.8
9985007.4987	15003.7	9885157.4237	115507.1
9980014.9950	20005 0	9880214.8451	120508.3
&c. usque ad	usque ad	usque ad	usque ad
9900473.5780	100025.0	9801468.8423	200528.2

	Columna 69.	
	Naturales.	Artificiales.
& cæteri usque ad	5048858.8900	6834225.8
	5046333.4605	6839227.1
	5043811.2932	6844228.3
	5041289.3879	6849229.6
	3038768.7435	6854230.8
	& tandem	usque ad
	4998609.4034	6934250.8

Reproduced from page 25 of the 1620 edition of the *Constrvctio*.

For example [Art. 50], if $y = 7071068$, the nearest sine in the Radical table is $x = 7070084.4434$, and then $TS(y - x) = 9835566000$. Dividing this result by the easiest number between x and y, which Napier took to be 7071000, "there comes out" 1390.9. Thus, taking $\log x$ from the Radical table, gives us

$$\log y \approx \log x - 1390.9 = 3467125.4 - 1390.9 = 3465734.5.$$

"Wherefore 3465735 is assigned for the required logarithm of the given sine 7071068."

All it took from this point on was time and effort, and Napier was eventually able to compute the logarithms of the sines of all angles between 45° and 90° in steps of 1 minute. All of this by dipping his quill in ink and partly by candlelight, long division after long division.

The final stage would be to deal with the angles between 0° and 45°, since we know that what really matters is what happens in the first quadrant. To do this, Napier proposed two methods, both of which are based on the theorem in Article 36.

First, if we use the equations after the quotation of this theorem with $x = 2v$, $y = 5000000$, and $z = TS$, we have

$$\log 2v - \log v = \log 10000000 - \log 5000000,$$

and taking the last logarithm from the previously computed part of the principal table, but before rounding it as an integer, yields

$$\log v = \log 2v + \log 5000000 = \log 2v + 6931469.22.$$

Or as Napier put it [Art. 51],

> Therefore, also, 6931469.22 will be the difference of all logarithms whose sines are in the proportion of two to one. Consequently the double of it, namely 13862938.44, will be the difference of all logarithms whose sines are in the ratio of four to one; and the triple of it, namely 20794407.66, will be the difference of all logarithms whose sines are in the ratio of eight to one.

Not content with this, he showed, in a similar manner, that

> 52. All sines in the proportion of ten to one have 23025842.34 for the difference of their logarithms.

That this figure should have been 23025850.93 is neither here nor there. What matters is that he could now find the logarithms of the sines of angles below 45°. To quote his own statement [Art. 54]:

> This is easily done by multiplying the given sine by 2, 4, 8, 10, 20, 40, 80, 100, 200, or any other proportional number you please ... until you obtain a number within the limits of the Radical table.

The second method is based on a trigonometric identity that he expressed as follows:

> 55. As half radius is to the sine of half a given arc, so is the sine of the complement of the half arc to the sine of the whole arc.

He provided an ingenious geometric proof of this result, which, if we keep in mind that his sine is the product of the radius R and our sine, can be rewritten as follows:

$$\frac{R/2}{R\sin\frac{1}{2}\alpha} = \frac{R\sin\left(90^\circ - \frac{1}{2}\alpha\right)}{R\sin\alpha}.$$

Today we would write this result in the familiar form $\sin\alpha = 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$. Then he used the theorem in Article 36 again to obtain in verbal form a result that can be rewritten as

$$\log\frac{R}{2} - \log\left(R\sin\tfrac{1}{2}\alpha\right) = \log\left[R\sin\left(90^\circ - \tfrac{1}{2}\alpha\right)\right] - \log\left(R\sin\alpha\right).$$

Then, if $22.5^\circ < \frac{1}{2}\alpha \leq 45^\circ$, the two logarithms on the right-hand side as well as that of $R/2$ are in the principal table, and the remaining logarithm can be determined. Napier concluded [Art. 58]:

> From these, again, may be had in like manner the logarithms of arcs down to 11 degrees 15 minutes. And from these the logarithms of arcs down to 5 degrees 38 minutes. And so on, successively, down to 1 minute.

And now there is nothing left but the actual construction of the logarithmic table. Napier gave precise instructions about how to do it step by step, starting with the following advice [Art. 59]:

> Prepare forty-five pages, somewhat long in shape, so that besides margins at the top and bottom, they may hold sixty lines of figures.

A page of the resulting table is reproduced on the next page. The publisher did not seem to have pages that could accommodate 60 lines of figures, so he put two columns of 30 lines on each page. The first column of this sample page contains the angles from $28^\circ\,0'$ to $28^\circ\,30'$, the second column lists their sines (probably taken from Erasmus Rheinhold's table of sines, which Napier mentioned in particular), and the third their logarithms, computed as explained in the preceding articles. The rest of the angles corresponding to 28° would be on the next page, not shown. On arrival at the page containing the sines and logarithms for angles from $44^\circ\,30'$ to $44^\circ\,60'$ no new pages were used. Instead, the table continued on the right of that last page with angles increasing from $45^\circ\,0'$ at the bottom of the seventh column to $45^\circ\,30'$ at the top. The angles located on both ends of each horizontal line are complementary. Their sines and logarithms are listed in the sixth and fifth columns, respectively. Then the table continued in the same manner on the preceding page and then on those

Deg. 28 +|—

m.	Sines.	Logarith	Differen.	Logarit.	Sines.	
0	469472	756147	631658	124489	882948	60
1	469728	755600	630956	124644	882811	59
2	469985	955054	630255	124799	882674	58
3	470242	754508	629553	124954	882537	57
4	470499	653962	628853	125109	882401	56
5	470755	753416	628152	125264	882264	55
6	471012	752871	627452	125420	882127	54
7	471268	752327	626752	125575	881990	53
8	471525	751783	626052	125730	881853	52
9	471781	751239	625353	125886	881715	51
10	472038	750695	624654	126042	881578	50
11	472294	750152	623955	126197	881441	49
12	472551	749610	623256	126353	881303	48
13	472807	749067	622558	126509	881166	47
14	473063	748525	621860	126665	881028	46
15	473320	747984	621162	126822	880891	45
16	473576	747443	620465	126978	880753	44
17	473832	746902	619768	127134	880615	43
18	474088	746362	619071	127291	880477	42
19	474344	745822	618374	127448	880339	41
20	474600	745282	017677	127604	880201	40
21	474856	744743	616981	127761	880063	39
22	475112	744204	616285	127918	879925	38
23	475368	743665	615590	128075	879787	37
24	475624	743127	614894	128233	879649	36
25	475880	742589	614199	128390	879510	35
26	476136	742052	613506	128547	879372	34
27	476392	741515	112810	128705	879233	33
28	476647	740978	612115	128863	879095	32
29	476907	740442	611421	125020	878956	31
30	477159	739906	610727	129178	878817	30
						Min.

Deg. 61

Reproduced from Wright's translation of the *Descriptio*.

previously used, as shown on the sample page. The central column contains the differences between the logarithm on its left and that on its right. These are the logarithms of the sine of a given angle and of its cosine (which is equal to the sine of the complement), and it is useful in evaluating the logarithms of tangents when viewed as quotients of sine and cosine.

It must be said for the sake of fairness that the earliest discoverer of logarithms was Joost, or Jobst, Bürgi (1552–1632), a Swiss clockmaker, about 1588. He seems to have been inspired by Stifel's table, but he chose 1.0001 as his starting number close to 1. Then, if $N = 100000000 \times 1.0001^L$, Bürgi called $10L$, not L, the "red number" (*Rote Zahl*) corresponding to the "black number" (*Schwarze Zahl*) N. These are the colors in which these numbers were printed in his 1620 tables *Aritmetische vnd geometrische Progress Tabulen*. Thus, he lost publication priority to Napier.

Bürgi was in Prague as an assistant to the astronomer Johannes Kepler (1571–1630). Kepler had become an assistant to Tycho Brahe in 1598, and in 1599 Brahe moved to Prague, attracted by an offer from the emperor Rudolf II (who reigned over the *Sacrum Romanum Imperium* in the period 1576–1612) and Kepler followed the year after. On Brahe's death in 1601, Kepler became the official astronomer and went on to discover the three laws of planetary motion. His work, based on Brahe's astronomical observations, was assisted by the use of logarithms, and this helped spread their use in Europe. Bürgi remained an unknown not just because of questions of priority, but because most copies of his tables were lost during the Thirty Years' War. The crucial Battle of the White Mountain, in which 7,000 men lost their lives, was fought just outside Prague, in November 1620.

2.3 BRIGGS' LOGARITHMS

The first mathematics professorship in England, a chair in geometry, was endowed by Sir Thomas Gresham in 1596 at Gresham College in London, and Henry Briggs (1561–1631) was its first occupant. He was quite impressed by the invention of logarithms, and on March 10, 1615, he wrote to his friend James Usher:

> *Napper*, Lord of *Markinston*, hath set my Head and Hands a Work, with his new and admirable Logarithms. I hope to see him this Summer if it please God, for I never saw Book that pleased me better, and made me more wonder.[9]

The astrologer William Lilly (1602–1681) wrote his autobiography in 1667–1668 in the form of a letter to a friend. It contains an interesting—but rather fantastic—account of Briggs' arrival at Merchiston, stating that on meeting Napier, "almost one quarter of an hour was spent, each beholding the other almost with admiration, before one word was spoke."[10]

[9] Quoted from Parr, *The life of the Most Reverend Father in God, James Usher*, 1686, p. 36 of the collection of three hundred letters appended at the end of the volume.

[10] Lilly finished his book with an account of the first encounter between Napier and Briggs, pp. 235–238. This quotation is from p. 236. The entire account of Briggs' arrival is reprinted (almost faithfully) in Bell, *Men of mathematics*, 1937, p. 526.

Having overcome this mutual admiration hiatus, they set down to work. For this there was ample time, since Briggs remained at Merchiston for about a month. Both men had become aware of some serious shortcomings in the logarithm scheme published by Napier. For instance, if $x = yz$, then

$$\frac{x}{y} = \frac{z}{1},$$

and, according to Article 36, $\log x - \log y = \log z - \log 1$, or

$$\log yz = \log y + \log z - \log 1.$$

This is as true today as it was then, but today $\log 1 = 0$. In Napier's scheme, it was the logarithm of the *sinus totus* that was zero. The logarithm of 1, a number very close to S in the top segment TS on page 90, is an enormous number on the bottom line. It was computed by Napier to be 161180896.38 (while this is not entirely correct, because of the errors already mentioned, the true value in his own system is very close to it). Thus, the Neperian logarithm of a product is not the sum of the logarithms of the factors, and the Neperian logarithm of 1 must be remembered every time the logarithm of a product is to be computed. Similarly for a quotient: if $x = y/z$, then $\log x - \log 1 = \log y - \log z$ and

$$\log \frac{y}{z} = \log y - \log z + \log 1.$$

The second difficulty in using Napier's logarithms is in finding the logarithm of ten times a number whose logarithm is known. If that number is called x, the obvious equation

$$\frac{10x}{x} = \frac{10^7}{10^6}$$

leads to

$$\log 10x - \log x = \log 10^7 - \log 10^6,$$

and, since the logarithm of the *sinus totus* is zero, we obtain

$$\log 10x = \log x - \log 10^6.$$

What makes the computation of $\log 10x$ in this manner inconvenient is the fact that $\log 10^6 = 23025842$.

Both Napier and Briggs had been separately thinking of ways to solve these deficiencies, and each of them made some proposals to accomplish that. These involved, in either case, the construction of a new set of logarithms from scratch. But they differed about the key properties on which the new logarithms should be based. Here is Briggs' proposal (see below for reference):

> When I explained their [the logarithms'] doctrine publicly in London to my auditors at Gresham College; I noticed that it would be very convenient in the future, if 0 were kept for the Logarithm of the whole Sine (as in the *Canone Mirifico*) while the Logarithm of the tenth part of the same whole Sine ... were 1,00000,00000 ...

In short, Briggs proposed and started the construction of a new set of logarithms based on the values $\log 10^7 = 0$ and $\log 10^6 = 10^{10}$. This certainly simplifies the computation of $\log 10x$ from $\log x$.

Briggs wrote to Napier with this proposal, and later he journeyed to Edinburgh, where they met for the first time. It turns out that Napier had also been thinking about the change in the logarithms, and Briggs described Napier's counterproposal as follows:

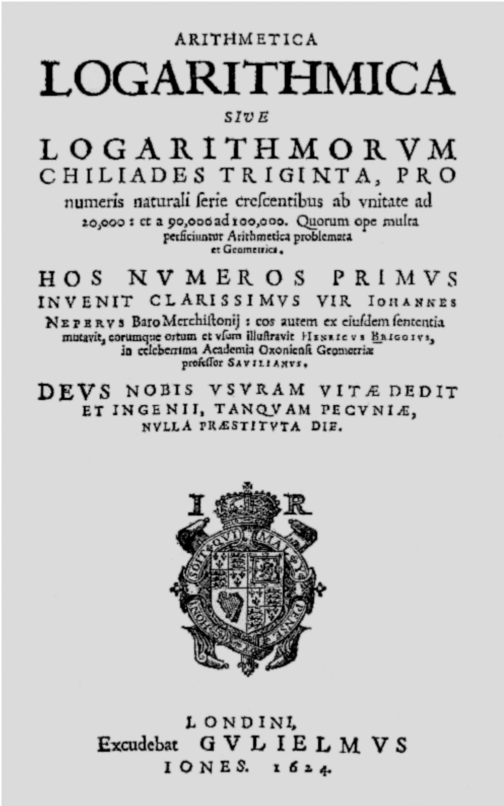
> But [he] recommended to therefore make this change, that 0 should be the Logarithm of Unity, & 1,00000,00000 [that of] the whole Sine: which I could not but acknowledge was most convenient by far.

Indeed, with $\log 1 = 0$, it follows from the preceding discussion that the logarithm of a product or quotient becomes the sum or difference of the logarithms. So Briggs put aside the logarithms that he had already computed, and in the following summer he again journeyed to Edinburgh and showed Napier the new recomputed logarithms. They appeared first in a small pamphlet of 16 pages, *Logarithmorum chilias prima* (The first thousand logarithms), whose preface is dated 1617, the year of Napier's death. In 1624 he published *Arithmetica logarithmica sive logarithmorvm chiliades triginta* (Logarithmic arithmetic or thirty thousands of logarithms), containing the new logarithms of the integers 1 to 20,000 and 90,000 to 100,000 (the reason for the large gap in the table will be given later). The quotations translated above are from the Preface to the Reader in this book.

However, we should not create the impression that he elaborated the new logarithms on the basis of Napier's proposal of the summer of 1615: that $\log 1 = 0$ and $\log 10^7 = 10^{10}$. The first of these equations he fully accepted [p. 2; 2–1],[11] for it makes the logarithm of a product or quotient the sum or difference of their logarithms [pp. 2,3; 2–2, 2–3].[12] Then it is easy to see

[11] References in square brackets are to the *Arithmetica logarithmica*. The first page reference is to the original in Latin; the second, after the semicolon, to Bruce's translation (see the bibliography) by chapter page (he does not have global page numbering).

[12] Briggs referred to these lemmas as axioms. These same properties, as well as the next one about the logarithm of a rational power, had been more clearly stated by Napier in the Appendix to the *Constrvctio*, pp. 50–51 of Macdonald's translation.


ARITHMETICA
LOGARITHMICA
SIVE
LOGARITHMORVM
CHILIADES TRIGINTA, PRO
numeris naturali ſerie creſcentibus ab vnitate ad
20,000 : et a 90,000 ad 100,000. Quorum ope multa
perficiuntur Arithmetica problemata
et Geometrica.
HOS NVMEROS PRIMVS
INVENIT CLARISSIMVS VIR IOHANNES
NEPERVS Baro Merchiſtonij : eos autem ex eiuſdem ſententia
mutavit, eorumque ortum et vſum illuſtravit HENRICVS BRIGGIVS,
in celeberrima Academia Oxonienſi Geometriæ
profeſſor SAVILIANVS.
DEVS NOBIS VSVRAM VITÆ DEDIT
ET INGENII, TANQVAM PECVNIÆ,
NVLLA PRÆSTITVTA DIE.

LONDINI,
Excudebat GVLIELMVS
IONES. 1624.

by repeated application of the product rule that if n is a positive integer, then $\log x^n = n \log x$. Also, if m is a positive integer and $y = x^{m/n}$, then $x^m = y^n$ and $m \log x = n \log y$, so that

$$\log x^{\frac{m}{n}} = \frac{m}{n} \log x,$$

which means that for a rational exponent $\log x^r = r \log x$. Thus, accepting the equation $\log 1 = 0$, we get these very convenient properties that Napier's own logarithms lack. However, Briggs rejected the second equation in favor of a new proposal made in the Appendix to the *Constrvctio*. This Appendix starts with the words [Macdonald translation, p. 48]:

> Among the various improvements of Logarithms, the more important is that which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the

Logarithm of either one tenth of unity or ten times unity.

This is the suggestion that Briggs actually quasi-accepted, in the form $\log 10 = 10^{14}$ [p. 3; 3–1]. The large number of zeros was intended to provide sufficient accuracy without resorting to numbers containing the still too new decimal point and digits. Soon thereafter, when decimal fractions were fully accepted, the best choice was $\log 10 = 1$. This results in the same logarithms that Briggs computed but with a decimal point in their midst.

But how are all these logarithms computed? At the time that Briggs was about to embark on the elaboration of his tables (Napier's health was failing in his 65th year, so it was up to Briggs to start a new series of computations)

JOHN NAPIER IN 1616
Engraving by Samuel Freeman from the portrait
in the possession of the University of Edinburgh.
From Smith, *Portraits of Eminent Mathematicians*, II.

there were two main methods to calculate logarithms, and both of them had been published in the Appendix to Napier's *Constrvctio*, as Briggs himself stated [p. 5; 5–2]. We do not know whether these are due to Napier or are a product of his collaboration with Briggs at Merchiston. The first method was based on the extraction of fifth roots and the second on the extraction of square roots. Napier felt that "though this method [of the fifth root] is considerably more difficult, it is correspondingly more exact" [Appendix, p. 50]. Briggs chose the method of the square root, and started evaluating (by hand, of course, which may take several hours) the square root of 10 [p 10; 6–2]:[13]

$$10^{1/2} = \sqrt{10} = 3.16227766016837933199889354,$$

so that $\log 3.16227766016837933199889354 = 0.5$. Then, by successive extraction of square roots, he evaluated $10^{1/4}$, $10^{1/8}$, and so on, down to

$$10^{1/2^{53}} = 1.00000000000000025563829864006470 7$$

and [p. 10; 6–3]

$$10^{1/2^{54}} = 1.00000000000000012781914932003235.$$

Briggs' values can be seen on the next page, reproduced from Chapter 6, page 10, of the second edition of *Arithmetica logarithmica*. Note that he did not use the decimal point, but inserted commas to help with counting spaces. On the logarithm side he omitted many zeros, which can be confusing. We know he chose the logarithm of 10 to be what we express as 10^{14}, but in this table it appears as 1,000. Down at the bottom, these logarithms are given with forty-one digits, making it difficult to know what an actual logarithm in this range is. To avoid possible misinterpretations, we follow the modern practice of using a decimal point for the numbers, but shall retain instead the first of Briggs' commas for their logarithms. If this comma is read as a decimal point, these are today's decimal logarithms, while the Briggsian logarithms are 10^{14} times larger. It must be pointed out that Briggs made a mistake in his computation of $10^{1/4}$. His digits are wrong from the twentieth on. This mistake trickles down through his entire table, but because the wrong digits are so far on the right, the error becomes smaller as it propagates, and his last two entries are almost in complete agreement with values obtained today using a computer.

The end result of this stage of Briggs' work is a table of logarithms, the table shown on page 107, displaying in the second column the logarithms of

[13] Briggs denoted the square root of 10 by $l.10$, meaning the *latus* (side) of 10; that is, the side of a square of area 10. The square root of the square root of 10 would then be $ll.10$, the eighth root $l.(8)10$, and so on. I have adopted our usual exponential notation, unknown at that time, as most convenient for our purposes.

10 *D* *ARITHMTICA* *E*

	Numeri continue Medij inter Denarium & Vnitatem.	*Logarithmi Rationales.*
	10	1,000
1	31622,77660,16837,93319,98893,54	0,50
2	17782,79410,03892,28011,97304,13	0,25
3	13335,21432,16332,40256,65389,308	0,125
4	11547,81984,68945,81796,61918,213	0,0625
5	10746,07828,32131,74972,13817,6538	0,03125
6	10366,32928,43769,79972,90627,3131	0,01562,5
7	10181,51721,71818,18414,73723,8144	0,00781,25
8	10090,35044,84144,74377,59005,1391	0,00390,625
9	10045,07364,25446,25156,64670,6113	0,00195,3125
10	10022,51148,29291,29154,65611,7367	0,00097,65625
11	10011,24941,39987,98758,85395,51805	0,00048,82812,5
12	10005,62312,60220,86366,18495,91839	0,00024,41406,25
13	10002,81116,78778,01323,99249,64325	0,00012,20703,125
14	10001,40548,51694,72581,62767,32715	0,00006,10351,5625
15	10000,70271,78941,14355,38811,70845	0,00003,05175,78125
16	10000,35135,27746,18566,08581,37077	0,00001,52587,89062,5
17	10000,17567,48442,26738,33846,78274	0,00000,76293,94531,25
18	10000,08783,70363,46121,46574,07431	0,00000,38146,97265,625
19	10000,04391,84217,31672,36281,88083	0,00000,19073,48632,8125
20	10000,02195,91867,55542,03317,07719	0,00000,09536,74316,40625
21	10000,01097,95873,50204,09754,72940	0,00000,04768,37158,20312,5
22	10000,00548,97921,68211,14626,60250,4	0,00000,02384,18579,10156,25
23	10000,00274,48957,07382,95091,25449,9	0,00000,01192,09289,55078,125
24	10000,00137,24477,59510,83282,69572,5	0,00000,00596,04644,77539,0625
25	10000,00068,62238,56210,25737,18748,2	0,00000,00298,02322,38769,53125
26	10000,00034,31119,22218,83912,75020,8	0,00000,00149,01161,19384,76562,5
27	10000,00017,15559,59637,84719,93879,1	0,00000,00074,50580,59692,38281,25
28	10000,00008,57779,79451,03051,17588,8	0,00000,00037,25290,29846,19140,625
29	10000,00004,28889,89633,54198,42901,3	0,00000,00018,62645,14923,09570,3125
30	10000,00002,14444,94793,77767,42970,4	0,00000,00009,31322,57461,54785,15625
31	10000,00001,07222,47391,14050,76926,8	0,00000,00004,65661,28730,77392,57812,5
32	10000,00000,53611,23694,13317,14831,4	0,00000,00002,32830,64365,38696,28906,25
33	10000,00000,26805,61846,70731,51508,7	0,00000,00001,16415,32182,69348,14453,125
34	10000,00000,13402,80923,26383,99277,7	0,00000,00000,58207,66091,34674,07226,5625
35	10000,00000,06701,40461,60946,55519,6	0,00000,00000,29103,83045,67337,03613,28125
36	10000,00000,03350,70230,79911,91730,0	0,00000,00000,14551,91522,83668,51806,64062,5
37	10000,00000,01675,35115,39815,61857,6	0,00000,00000,07275,95761,41834,25903,32031,25
38	10000,00000,00837,67557,69872,72426,9	0,00000,00000,03637,97880,70917,12951,66015,625-
39	10000,00000,00418,83778,84927,59087,9	0,00000,00000,01818,98940,35458,56475,83007,8125
40	10000,00000,00209,41889,42461,60262,5	0,00000,00000,00909,49470,17729,28237,91503,90625
41	10000,00000,00104,70944,71230,25311,0	0,00000,00000,00454,74735,08864,64118,95751,95312
42	10000,00000,00052,35472,35614,98950,4	0,00000,00000,00227,37367,54432,32059,47875,97656
43	10000,00000,00026,17736,17807,46048,9	0,00000,00000,00113,68683,77216,16029,73937,98828
44	10000,00000,00013,08868,08903,72167,8	0,00000,00000,00056,84341,88608,08014,86968,99414
45	10000,00000,00006,54434,04451,85869,75	0,00000,00000,00028,42170,94304,04007,43484,49707
46	10000,00000,00003,27217,02225,92881,337	0,00000,00000,00014,21085,47152,02003,71742,24853
47	10000,00000,00001,63608,51112,96427,283	0,00000,00000,00007,10542,73576,01001,85871,12426
48	10000,00000,00000,81804,25556,48210,295	0,00000,00000,00003,55271,36788,00500,92935,56213
49	10000,00000,00000,40902,12778,24104,311	0,00000,00000,00001,77635,68394,00250,46467,78106
50	10000,00000,00000,20451,06389,12051,946	0,00000,00000,00000,88817,84197,00125,23233,89053
51	10000,00000,00000,10225,53194,56025,921 *L*	0,00000,00000,00000,44408,92098,50062,61616,94526
52	10000,00000,00000,05112,76597,28012,947 *M*	0,00000,00000,00000,22204,46049,25031,30808,47263
53	10000,00000,00000,02556,38298,64006,470 *N*	0,00000,00000,00000,11102,23024,62515,65404,23631
54	10000,00000,00000,01278,19149,32003,235 *P*	0,00000,00000,00000,05551,11512,31257,82702,11815

the 54 weird numbers in the first column. But we would prefer to have a table of the logarithms of more ordinary numbers such as 2, 5, 17, or—to be systematic—what about the logarithms of the first 1000 positive integers? This was, precisely, Briggs' original goal, and the logarithms of such numbers can also be obtained by the successive extraction of square roots.

To see how this is done we can examine the last four rows of the reproduced table, those containing the entries labeled L to P. If we write P in the form $1 + p$ with $p = 0.00000000000000012781914932003235$, a short calculation shows that $N = 1{+}2p$, $M \approx 1{+}4p$, and $L \approx 1{+}8p$. The approximation loses accuracy as we progress through these numbers, in reverse alphabetical order, so we shall not go any further. Now, $\log P$ is the last number in the right column of the table, namely 0,00000000000000005551115123125782702118 15, and using the fact that $p \ll 1$ it is easy to see that

$$\log N = \log(1 + 2p) \approx \log(1 + p)^2 = 2 \log P,$$

$$\log M \approx \log(1 + 4p) \approx \log(1 + p)^4 = 4 \log P,$$

and

$$\log L \approx \log(1 + 8p) \approx \log(1 + p)^8 = 8 \log P.$$

This may be generously interpreted to mean that if a number of the form $1+r$ is "near the numbers L M N & P" then

$$\log(1 + r) = \log\left(1 + \frac{r}{p}p\right) \approx \frac{r}{p} \log P.$$

Briggs did not state any formulas and simply asserted that "the Logarithm of this number [a number in the stated range] is easily found, from the rule of proportion" [p. 11; 6–4], which he called the "golden rule" (*auream regulam*). He illustrated it with the number 1.00000000000000001, in which case

$$\frac{r}{p} = \frac{10000000000000000}{12781914932003235},$$

and asserted that its logarithm is 0,000000000000000043429448190325 1804 (the last three digits are in error).

It may be more interesting to find $\log 2$, although Briggs did not do it at this point in his book for reasons that will become apparent only too soon. We would start by computing 53 consecutive square roots of 2, ending with

$$2^{1/2^{53}} = 1.00000000000000007695479593116620.$$

This number is in the L to P range in the first column of Briggs' table, and then the golden rule (in which we have replaced $\approx$ with $=$ for convenience) and the stated value of $\log P$ give

$$\begin{aligned}\log 2^{1/2^{53}} &= \frac{07695479593116620}{12781914932003235} \log P \\ &= 0{,}00000000000000003342104322889629327919 31.\end{aligned}$$

Therefore,

$$\begin{aligned}\log 2 &= (2^{53})(0{,}0000000000000000334210432288962932791931) \\ &= 0{,}301029995663981169731 97.\end{aligned}$$

This method has two drawbacks. First, the accuracy is somewhat limited (the last six decimal digits are incorrect), although it may be considered acceptable for a table of logarithms. But the method is too labor-intensive.[14]

In Chapter VII, Briggs started exploring how to shorten his methods, and actually computed $\log 2$ as follows. First he noticed that it is sufficient to compute the logarithm of $1\frac{024}{1000}$ (since he did not use a decimal point, this is how he wrote 1.024) by the square root method, because, since 1024 is the tenth power of 2,

$$\log 2 = \frac{\log 1024}{10}$$

and

$$\log 1024 = \log 1000 + \log 1\tfrac{024}{1000} = 3 \log 10 + \log 1\tfrac{024}{1000}\cdot$$

Since 1.024 is closer to unity than 2, then 47 (as opposed to 53) square root extractions of 1.024 give the number 1.0000000000000001685160570539497 7, which is in the proportionality region L to P, and "the Logarithm of that number is found by the golden rule 0,0000000000000000731855936906239336" [p. 13; 7–3]. Thus,

$$\begin{aligned}\log 1.024 &= (2^{47})(0{,}0000000000000000731855936906239336) \\ &= 0{,}01029995663981195.\end{aligned}$$

Adding $3 \log 10$ and dividing by 10 yields $\log 2 = 0{,}30102999566398 1195$, as given in the *Arithmetica logarithmica* [p. 14; 7–4].

[14] Bruce has estimated that the computation of about fifty successive square roots is a process that may require between 100 and 200 hours of hard work [p. 7–6]. This is equivalent to several weeks if you have other commitments.

Then Briggs used the equation $5/10 = 1/2$ and the properties of logarithms to obtain $\log 5 - \log 10 = \log 1 - \log 2$, from which [p. 14; 7–5]

$$\begin{aligned}\log 5 &= \log 10 - \log 2 \\ &= 1{,}00000000000000000 - 0{,}30102999566398 1195 \\ &= 0{,}69897000433601 8805.\end{aligned}$$

Next, he gave a list of numbers whose logarithms can be easily computed from those previously computed using the properties of logarithms:

> From the multiplication of Two alone, by itself & into its factors, 4. 8. 16. 32. 64. &c. Likewise of Five by itself & into its factors, 25. 125. 625. 3125. &c, Two in factors of Five, 250. 1250. 6250. &c. Two into Ten 20. 200. 2000. 40. 400. 80. 800. &c.

The next prime after two is three, "whose Logarithm is most conveniently found from the Logarithm of Six," since $\log 3 = \log 6 - \log 2$. But six is very far from one, and it would take too many roots to bring it down to the $L - P$ proportionality region. Thus, Briggs decided to find the logarithm of

$$\frac{6^9}{10^7} = 1.0077696$$

instead, which he did again by the golden rule, and then obtained

$$\log 6 = \frac{7 + \log 1.0077696}{9}.$$

It should be evident that the elaboration of a complete table of logarithms by the square root method would be impossibly time-consuming. However, the logarithm of 6 was the last one that Briggs computed in this manner, for he made an additional discovery. He announced it at the start of Chapter 8 in the following terms [p. 15; 8–1]:

> We can find the Logarithm of any proposed number according to this method, by continued Means, which supplies our need by finding the square root laboriously enough, but the vexation of this enormous labor is considerably lessened through differences.

To illustrate this idea he chose the example of finding consecutive square roots of 1.0077696, the number involved in finding the logarithm of six. After computing some roots, as shown in the table reproduced on page 112 [p. 16;

8–2], he noticed that if each of them is written in the form $1 + A$, then each value of A is approximately twice the next value. For instance, the A part of the root in line 43 of this table (0.00048384026884662985492535) is about twice the A part in the top line of box 42 (0.00024189087882468563808727). Then Briggs decided to investigate the difference between half of each A and the next A, which he called B. The first of his B differences is seen in the third line of box 42, and it is a very small number in comparison with the A values under consideration. Using the next two A values, Briggs computed a second B, which is in the third line of box 41, and observed that it is about one quarter of the previous B. So he denoted by C the difference between this quarter and the second B, and noticed that C is very small in comparison to the B values.

The fact that Briggs—or any of his contemporaries for that matter—did not use subscripts, added to the fact that his table contains only one column, may make things difficult for the present-day reader, so another table has been included on page 113 to show things in current notation. In its first column, n indicates the order of the root; in the second these roots are given in the form $1 + A_n$, using the decimal point that Briggs omitted; and the remaining columns give the differences of several orders as computed by Briggs (some of his digits are incorrect, if we redo his work using a computer) and labeled by the headers. Leading zeros have been omitted, as originally done by Briggs, but without his vertical arrangement this may be confusing if not noted.

What is clear is that these successive differences become smaller and smaller, to the point that

$$F_9 = \frac{1}{32}E_8 - E_9 \approx 0$$

with the 26-decimal-digit accuracy used in these computations. At this point Briggs got the idea of working backward through the differences. If we take $F_{10} = 0$, since F_{10} is even smaller than F_9, then

$$0 = F_{10} = \frac{1}{32}E_9 - E_{10}$$

implies that $E_{10} = \frac{1}{32}E_9 = 0.0000,00000,00000,00000,00000,065$. In the same manner, $D_{10} = \frac{1}{16}D_9 - E_{10} = 0.0000,00000,00000,00000,02855,524$, and so on, as shown in the lower table on page 113, until we obtain $A_{10} =$ 0000,07558,20443,63012,14290,760, and then $1 + A_{10} = \sqrt{1 + A_9}$. It is evident that this method of evaluation of differences is much faster than directly finding the square root. Briggs illustrated this procedure again to compute A_{11}

	10077,696	
46	10038,77283,33696,24566,38465,51	
45	10019,36766,13694,66167,58702,29	
44	10009,67914,63909,90172,88907,20	
43	10004,83840,26884,66298,54925,35	A
42	10002,41890,87882,46856,38087,27	A
	2,41920,13442,33149,27462,67	$\frac{1}{2}A$
	29,25559,86292,89375,40	B
41	10001,20938,12639,71345,94391,94	A
	1,20945,43941,23428,19043,63	$\frac{1}{2}A$
	7,31301,52082,24651,69	B
	7,31389,96573,22343,85	$\frac{1}{4}B$
	88,44490,97692,15	C
40	10000,60467,23505,53096,80160,05	A
	60469,06319,85672,97195,97	$\frac{1}{2}A$
	1,82814,32576,17035,92	B
	1,82825,38020,56162,92	$\frac{1}{4}B$
	11,05444,39127,00	C
	11,05561,37211,52	$\frac{1}{8}C$
	116,98084,52	D
39	10000,30233,16050,56577,59647,94	A
	30233,61752,76548,40080,02	$\frac{1}{2}A$
	45702,19970,80432,08	B
	45703,58144,04258,98	$\frac{1}{4}B$
	1,38173,23826,90	C
	1,38180,54890,87	$\frac{1}{8}C$
	7,31063,97	D
	7,31130,28	$\frac{1}{16}D$
	66,31	E
38	10000,15116,46599,90567,29504,88	A
	15116,58025,28288,79823,97	$\frac{1}{2}A$
	11425,37721,50319,09	B
	‡ 11425,54992,70108,02	$\frac{1}{4}B$
	17271,19788,93	C
	17271,65478,36	$\frac{1}{8}C$
	45689,43	D
	45691,50	$\frac{1}{16}D$
	2,07	E
	2,07	$\frac{1}{32}E$

Hucuſque Differentiæ minores ſunt inventæ per ſubductionem majorum è partibus homogenearum præcedentium.

n	$1 + A_n$	$B_n = \frac{1}{2}A_{n-1} - A_n$	$C_n = \frac{1}{4}B_{n-1} - B_n$	$D_n = \frac{1}{8}C_{n-1} - C_n$	$E_n = \frac{1}{16}D_{n-1} - D_n$
0	1.0077,696				
1	1.0038,77283,33696,24566,38465,51				
2	1.0019,36766,13694,66167,58702,29				
3	1.0009,67914,63909,90172,88907,20				
4	1.0004,83840,26884,66298,54925,35				
5	1.0002,41890,87882,46856,38087,27	29,25559,86292,89375,40			
6	1.0001,20938,12639,71345,94391,94	7,31301,52082,24651,69	88,44490,97692,15		
7	1.0000,60467,23505,53096,80160,05	1,82814,32576,17035,92	11,05444,39127,00	116,98084,52	
8	1.0000,30233,16050,56577,59647,94	45702,19970,80432,08	1,38173,23826,90	7,31063,97	66,31
9	1.0000,15116,46599,90567,29504,88	11425,37721,50319,09	17271,19788,93	45689,43	2,07

n	$1 + A_n = 1 + \frac{1}{2}A_{n-1} - B_n$	$B_n = \frac{1}{4}B_{n-1} - C_n$	$C_n = \frac{1}{8}C_{n-1} - D_n$	$D_n = \frac{1}{16}D_{n-1} - E_n$	$E_n = \frac{1}{32}E_{n-1}$
10	1.0000,07558,20443,63012,14290,760	2856,32271,50461,680	2158,87118,092	2855,524	65
11	1.0000,03779,09507,73708,05241,254	714,07798,01904,126	269,85711,294	178,468	2

(his own table of backward differences, which was printed right below that of forward differences, shows a similar vertical arrangement of the A to D differences).[15]

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[15] Briggs gave a second method for computing the differences, which is based on evaluating successive powers. Fix a value of n and note that if $1 + A_n = \sqrt{1 + A_{n-1}}$, then

$$A_{n-1} = (1 + A_n)^2 - 1, \quad A_{n-2} = (1 + A_{n-1})^2 - 1 = (1 + A_n)^4 - 1,$$

and so on. Therefore,

$$B_n = \frac{1}{2}A_{n-1} - A_n = \frac{1}{2}\big[(1 + A_n)^2 - 1\big] - A_n = \frac{1}{2}A_n^2$$

and

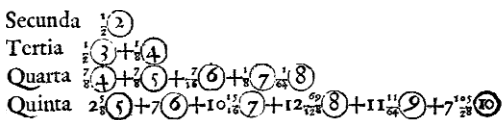
$$\begin{aligned} C_n &= \frac{1}{4}B_{n-1} - B_n = \frac{1}{4}\Big[\frac{1}{2}A_{n-2} - A_{n-1}\Big] - \frac{1}{2}A_n^2 \\ &= \frac{1}{8}\big[(1 + A_n)^4 - 1\big] - \frac{1}{4}\big[(1 + A_n)^2 - 1\big] - \frac{1}{2}A_n^2 \\ &= \frac{1}{2}A_n^3 + \frac{1}{8}A_n^4. \end{aligned}$$

Similarly, we would obtain

$$D_n = \frac{7}{8}A_n^4 + \frac{7}{8}A_n^5 + \frac{7}{16}A_n^6 + \frac{1}{8}A_n^7 + \frac{1}{64}A_n^8,$$

$$E_n = \frac{21}{8}A_n^5 + 7A_n^6 + \frac{175}{16}A_n^7 + \frac{1605}{128}A_n^8 + \frac{715}{64}A_n^9 + \frac{301}{28}A_n^{10} + \cdots,$$

and higher-order differences. We know that Briggs did not use subscripts, but in writing these equations he did not even use the letter A. The stated equations were printed in the *Arithmetica logarithmica* as


Secunda $\frac{1}{2}(2)$
Tertia $\frac{1}{2}(3)+\frac{1}{8}(4)$
Quarta $\frac{7}{8}(4)+\frac{7}{8}(5)+\frac{7}{16}(6)+\frac{1}{8}(7)\ \frac{1}{64}(8)$
Quinta $2\frac{5}{8}(5)+7(6)+10\frac{15}{16}(7)+12\frac{69}{128}(8)+11\frac{11}{64}(9)+7\frac{105}{28}(10)$

He went down to "Decima" but he never wrote beyond the tenth power of A_n.

This method is not faster than the previous one, but it is of some theoretical interest, if only because of what Briggs could have done with it but didn't. Using the equations in the header of the lower table on page 113, we see that

$$\begin{aligned} (1 + A_n)^{1/2} &= 1 + A_{n+1} = 1 + \tfrac{1}{2}A_n - B_{n+1} = 1 + \tfrac{1}{2}A_n - \big(\tfrac{1}{4}B_n - C_{n+1}\big) \\ &= 1 + \tfrac{1}{2}A_n - \tfrac{1}{4}B_n + \big(\tfrac{1}{8}C_n - D_{n+1}\big) \\ &= 1 + \tfrac{1}{2}A_n - \tfrac{1}{4}B_n + \tfrac{1}{8}C_n - \big(\tfrac{1}{16}D_n - E_{n+1}\big) \\ &= 1 + \tfrac{1}{2}A_n - \tfrac{1}{4}B_n + \tfrac{1}{8}C_n - \tfrac{1}{16}D_n + \tfrac{1}{32}E_n - \cdots. \end{aligned}$$

Giving a full account of the rest of Briggs' methods to save time in the computation of logarithms would be a very long story. In Chapter 9 he used the methods already explained and some clever ways of writing certain numbers as sums and products to evaluate the logarithms of all the primes from 2 to 97. Then the rules of logarithms provide those of whole composite numbers, and in Chapter 10 he dealt with fractions. In the next three chapters, Briggs presented several methods of interpolation, starting with simple proportion in Chapter 11. In Chapter 12 he considered a region of the table in which second-order differences for successive numbers whose logarithms are known remain nearly constant, and developed a method to find nine logarithms of equally spaced numbers between every two of those successive numbers. This is known today as *Newton's forward difference method* because it was later rediscovered by Newton (it is briefly described in Section 4.3).

With the aid of the method of finite differences of Chapter 8, Briggs computed a first table of the new logarithms of the numbers 1 to 1000 in *Logarithmorum chilias prima*. In 1620, between this publication and that of the *Arithmetica logarithmica*, he had become the first Savilian professor of geometry at Oxford—a chair endowed by Sir Henry Savile. As we have already mentioned, the 1624 edition of the *Arithmetica logarithmica* contains no logarithms of the numbers between 20,000 and 90,000. The reason for this is that the methods of Chapter 12, used to compute the printed logarithms, are not applicable to this range. Briggs proposed a new method in a longer Chapter 13, bearing the following very long title:

Bringing now to the right-hand side the values obtained for B_n to E_n by Briggs' second method, we obtain

$$\begin{aligned}(1+A_n)^{1/2} &= 1+\tfrac{1}{2}A_n-\tfrac{1}{8}A_n^2+\tfrac{1}{8}\left(\tfrac{1}{2}A_n^3+\tfrac{1}{8}A_n^4\right)\\ &\quad -\tfrac{1}{16}\left(\tfrac{7}{8}A_n^4+\tfrac{7}{8}A_n^5+\tfrac{7}{16}A_n^6+\tfrac{1}{8}A_n^7+\tfrac{1}{64}A_n^8\right)\\ &\quad +\tfrac{1}{32}\left(\tfrac{21}{8}A_n^5+7A_n^6+\tfrac{175}{16}A_n^7+\tfrac{1605}{128}A_n^8+\cdots\right)\\ &= 1+\tfrac{1}{2}A_n-\tfrac{1}{8}A_n^2+\tfrac{1}{16}A_n^3-\tfrac{5}{128}A_n^4+\tfrac{7}{256}A_n^5-\cdots\\ &= 1+\tfrac{1}{2}A_n+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}A_n^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}A_n^3+\cdots.\end{aligned}$$

Today we recognize this as the binomial series expansion of $(1+A_n)^{1/2}$. Although we may like to think that this particular case of the binomial theorem was implicitly contained in Briggs' work (this was first pointed out by Whiteside in "Henry Briggs: the binomial theorem anticipated," 1961), the fact is that Briggs, who had bigger fish to fry at that point, was not aware of it.

> It is desired to find the Logarithms of any Chiliad [that is, one thousand consecutive numbers]. Or given any set of equidistant numbers, together with the Logarithms of these four numbers, to find the Logarithms of intermediate numbers for each interval [between the given numbers].

The second paragraph begins as follows:[16]

> Take the first, second, third, fourth, &c. differences of the given Logarithms; & divide the first by 5, the second by 25, the third by 125, &c; ... the quotients are called the first, second, third, &c. mean differences, ...

He hoped to use it to compute the logarithms of the missing chiliads. He expressed this wish, four years later, in a letter to John Pell of 25 October 1628:[17]

> My desire was to have these Chiliades that are wanting betwixt 20 and 90 calculated and printed, and I had done them all almost by myselfe, and by some frends whom my rules had sufficiently informed, and by agreement the business was conveniently parted amongst us : but I am eased of that charge and care by one Adrian Vlacque ... But he hathe cut off 4 of my figures throughout : and hathe left out my dedication, and to the reader, and two chapters the 12 and 13, in the rest he hath not varied fromme at all.

What happened is that the large gap in Briggs' *Arithmetica logarithmica* of 1624 was filled by the Dutch publisher Adriaan Vlacq (c. 1600–1667) (or Ezechiel de Decker, a Dutch surveyor, assisted by Vlacq), but reducing the accuracy of the logarithms from 14 to 10 digits and eliminating Chapters 12 and 13, the summit of Briggs' work on finite and mean differences. Their work was published, without any notice to Briggs, in a second edition of the *Arithmetica logarithmica*.

Briggs turned his attention to a new project, the *Trigonometria britannica*. After expressing his displeasure about the omission of Chapter 13 in the Dutch edition of the *Arithmetica logarithmica*, he included his method of mean differences in Chapter 12 of this project. Briggs would work on it for the rest of his life but could not finish it. He left it in the hands of his friend Henry Gellibrand (1597–1636), professor of astronomy at Gresham College, who was able to complete it before his premature death and published it in 1633.

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[16] This and the preceding interpolation methods of Chapters 11 and 12 can be seen in modern notation in Goldstine, *A history of numerical analysis from the 16th through the 19th century*, 1977, pp. 23–32. See also Bruce's translation and his notes to these chapters.

[17] From Bruce, "Biographical Notes on Henry Briggs," p. 7.

2.4 HYPERBOLIC LOGARITHMS

The loss of Bürgi's tables was not the only effect of the Thirty Years' War on the development of logarithms. A Belgian Jesuit residing in Prague, Grégoire de Saint-Vincent (1584–1667), had previously made a discovery while inves-

GRÉGOIRE DE SAINT-VINCENT IN 1653
Engraving by Richard Collin.
From *Opus geometricum posthumum ad mesolabium*, Ghent, 1688.

tigating the area under the hyperbola that would eventually greatly facilitate the computation of logarithms. But its publication was delayed because Saint-Vincent fled from Prague (the Swedes are coming!) in 1631, at the start of the third phase of the Thirty Years' War.

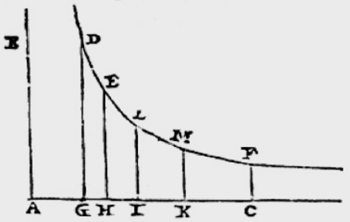
King Gustavus Adolphus II had landed with his troops on the coast of Pomerania in 1630 and set about to the invasion of central Europe. He won a brilliant victory at the Battle of Leipzig on September 17, 1631, but later died at the battle of Lützen on November 16, 1632. The Swedes were victorious, but Gustavus Adolphus was mortally wounded. In his haste to depart, Saint-Vincent left all his papers behind, but they were returned to him about ten years later, and his research on the squaring of the circle and the hyperbola was published in 1647 in a book of over 1250 pages: *Opvs geometricvm quadratvræ circvli et sectionvm coni.*

The result on the hyperbola mentioned above is contained in several propositions in Book VI, in particular in Proposition 109, on page 586, which is reproduced here. The text of the proposition can be translated as follows:

PROPOSITIO CIX.

Sint AB, AC aſymptoti hyperbolæ DEF : diuisâque AC, vt AG, AH, AI, AK, AC continuæ ſint proportionales, ponantur GD, EH, LI, MK, FC, ipſi AB æquidiſtantes.

Dico HD, IE , KL , CM ſegmenta eſſe æqualia.



Demonſtratio.

Qvoniam AG, AH, AI, &c. proportionales ſunt, DG, EH, LI, &c. eamdem quoque inter ſe continuant [e] rationem; vnde DH, EI [f] ſegmenta ſunt æqualia. Similiter cùm EH, LI, MK, FC in eadem ſint proportione, [g] æqualia quoque ſunt ſegmenta EI, LK, MC. Conſtat igitur veritas propoſitionis.

e 78. huius.
f 106. huius.
g Ibid.

Let AB, AC be the asymptotes of the hyperbola DEF : break up AC, as AG, AH, AI, AK, AC [*so that they*] *are in continued proportion, place GD, EH, LI, MK, FC, equidistant from* [*meaning parallel to*] *AB.*

I say that HD, IE, KL, CM are equal [*area*] *patches.*

In our terms, this means that if

$$\frac{AH}{AG} = \frac{AI}{AH} = \frac{AK}{AI} = \frac{AC}{AK} = \cdots,$$

then the hyperbolic areas $DGHE$, $EHIL$, $LIKM$, $MKCF$, ... are equal.

In other words, if we denote the ratio AH/AG by r, the abscissas AG, $AH = AGr$,

$$AI = AHr = AGr^2, \quad AK = AIr = AGr^3, \quad AC = AKr = AGr^4,$$

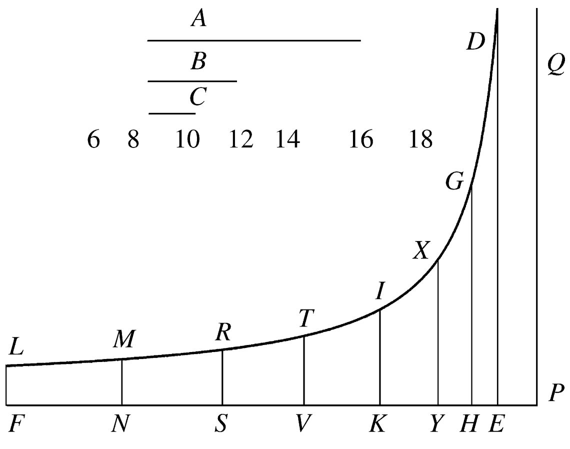
and so on, form a geometric progression, and then the hyperbolic areas $DGHE$, $DGIL$, $DGKM$, $DGCF$, ... form an arithmetic progression. If $AG = 1$, this is the same type of relationship that Michael Stifel had pointed out between successive powers of 2 and the corresponding exponents (page 83). We can conclude, in current terminology, that the relationship between the area under the hyperbola from $x = 1$ to an arbitrary abscissa $x > 1$ and the value of x is logarithmic. Saint-Vincent did not explicitly note this, but one of his students, Alfonso Antonio de Sarasa (1618–1667), did in his solution to a problem proposed by Marin Mersenne, a Minimite friar, in 1648. The next year, in his *Solvtio problematis a R P Marino Mersenno Minimo propositi*, de Sarasa stated the problem in this way:[18]

> Given three arbitrary magnitudes, rational or irrational, and given the logarithms of two, to find the logarithm of the third geometrically.

De Sarasa solved Mersenne's problem in developing his Proposition 10, which simply restates a particular case of the problem as follows:

> Given three magnitudes, A, B, and C, which can be shown in one and the same geometric progression, and given the logarithms of two of these magnitudes, say those of A and B, to determine the logarithm of the third, C, geometrically.

That is, de Sarasa assumed that A, B, and C are terms of a geometric progression, because this is the only case he thought he could solve. They are shown in the next figure as ordinates $GH = A$, $IK = B$, and $LF = C$ of a hyper-



bola, and we can think of the remaining shown ordinates as equally spaced

[18] This and the next two quotations are from Burn, "Alphonse Antonio de Sarasa and logarithms," 2001. This paper contains a detailed analysis of de Sarasa's *Solutio*. I follow a small part of Burn's presentation here since I have been unable to see the original work.

terms of the same geometric progression. If some ordinates of the hyperbola $y = 1/x$ (de Sarasa did not give an equation) are in geometric progression, then the corresponding abscissas are also in geometric progression. Then, by Saint-Vincent's Proposition 109, the area KL is four times the area NL, and the area GK is twice the area NL, at which point de Sarasa observed:

> Whence these areas can fill the place of the given logarithms (*Unde hae superficies supplere possunt locum logarithmorum datorum*).

This observation allowed de Sarasa to solve the modified Mersenne problem, which can be done quickly in current notation. Assume that there are real numbers $a > 0$ and $0 < r < 1$, and positive integers m and n such that

$$GH = ar^m, \quad XY = ar^{m+n}, \quad IK = ar^{m+2n}, \quad \text{and} \quad LF = ar^{m+6n}.$$

Let S denote the area under the hyperbola between any two of these consecutive ordinates. For example, $NL = S$. If we refer to the exponents m, $m + 2n$, and $m + 6n$ as the logarithms of $A = GH$, $B = IK$, and $C = LK$, respectively, then it is clear that the differences of these logarithms, $2n$ and $4n$, are equal to n times the area ratios GK/S and KL/S. We can write this as follows:

$$\log B - \log A = n\frac{GK}{S}$$

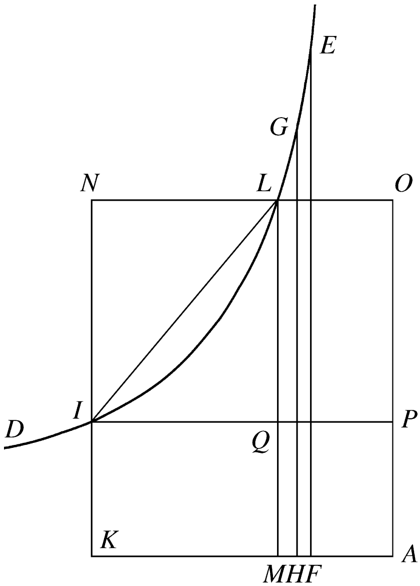
and

$$\log C - \log B = n\frac{KL}{S}\cdot$$

As an example, de Sarasa considered the case in which the numbers 6 to 18 shown in his figure are the logarithms of the ordinates GH to LF. Then $n = 2$, and the shown ordinates represent every other term of their progression. We have $\log A = 6$, $\log B = 10$, and $KL/S = 4$. It follows that $\log C - 10 = 8$ and $\log C = 18$.

What is important to us is de Sarasa's realization that areas between the hyperbola and the horizontal axis are like logarithms. While he kept things general, we need to specify. Thus, consider the hyperbola $y = 1/x$ and let $A(x)$ denote the area under it from 1 to some $x \geq 1$. It can be called, at least for now, the *hyperbolic logarithm* of x. If we denote this logarithm by Hlog, we have $A(x) = \text{H}\log x$. Since we have $A(1) = \text{H}\log 1 = 0$, then the hyperbolic logarithm shares this property with the Briggsian logarithm. Is it the Briggsian logarithm? We shall have the answer shortly.

The Scottish mathematician James Gregory (1638–1675) elaborated on the work of Saint-Vincent and de Sarasa and computed a number of hyperbolic logarithms in a 1667 short book published in Padua, where he resided at the time: *Vera circvli et hyperbolæ qvadratvra, in propria sua proportionis specie, inuenta, & demonstrata* (page references are to this original). In Proposition XXXII [p. 46], Gregory posed the problem of finding areas under the hyperbola DIL with asymptotes AO and AK. To do that he chose



the lengths $IK = 1000000000000$, $LM = 10000000000000 = 10IK$, and $AM = 1000000000000 = IK$ (Gregory's figure is clearly not to scale). Then he employed a series of polygons with an increasing number of sides, inscribed and circumscribed to the hyperbolic space $LIKM$ [pp. 47–48], and was able to approximate its area to be 23025850929940456240178700 [p. 49].

Gregory was, at the time, using very large numbers to avoid the still unfamiliar decimal point. If we choose a positive x-axis in the direction AO and a positive y-axis in the direction AK (a choice that is as good as the opposite, given that both are asymptotes of the hyperbola), we would write the equation of Gregory's hyperbola as $y = 10^{25}/x$. If instead we divide all the y-values by 10^{12} and all the x-values by 10^{13} (which is equivalent to the choice $IK = 1$, $LM = 10$, and $AM = 0.1$), then the equation of the hyperbola becomes the familiar $y = 1/x$, and the area stated above must be divided by 10^{25}.[19] Thus,

[19] Gregory himself made the choice $IK = 1$ and $LM = 10$ in Proposition XXXIII, entitled: *It is proposed to find the logarithm of any number whatever* [p. 49].

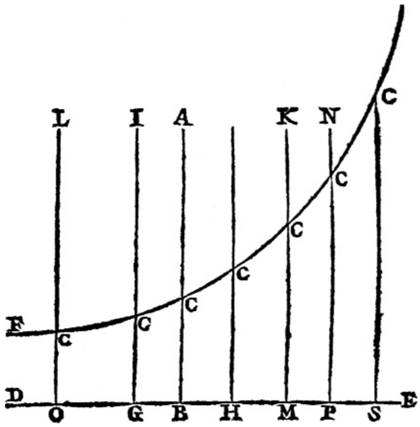
$LIKM = 2.302585092994045624017870$0. Notice that for $y = 1/x$ we have $AK = AP = 1$, and therefore each of the rectangles $IQMK$ and $LQPO$ has area 0.9. Then the hyperbolic areas $LIKM$ and $ILOP$ are equal. The figure clearly shows that $ILOP = \text{Hlog}\, AO$, that is, that $LIKM = \text{Hlog}\, LM$. Therefore, since $LM = 10$,

$$\text{Hlog}\, 10 = 2.3025850929940456240178700.$$

It is clear from a comparison of this value with the Briggsian logarithm of 10 that Hlog is a new logarithm. One, we must say in a spoiler mood, that would go on to a position of prominence in the future.

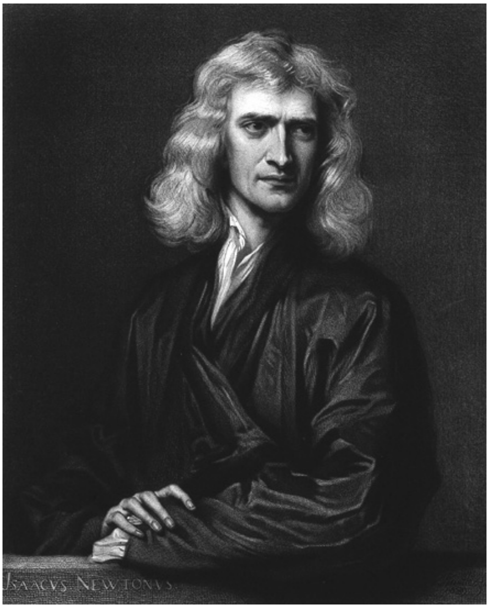
In the few remaining pages of this book Gregory showed how to evaluate additional hyperbolic logarithms [pp. 53–55] and considered the problem of finding a number from its logarithm [p. 56].

At the end of his stay in Italy, in 1668, Gregory published a second book: *Geometriæ pars vniversalis* (The universal part of geometry). In the preface, on pages six to eight, he gave the first graph ever of the hyperbolic logarithm (shown below, with the positive x-axis in the direction OL and $OC = 1$) and stated for the first time in print the properties of this curve.



2.5 NEWTON'S BINOMIAL SERIES

Sometime between the publication of de Sarasa's and Gregory's works, a young Isaac Newton (1642–1727) was fooling around with Pascal's triangle



ISAAC NEWTON IN 1689
Portrait by Sir Godfrey Kneller.
Farleigh House, Farleigh Wallop, Hampshire.

(which was known way before Pascal), giving the coefficients of the expansion of $(a+b)^n$ for $n = 0, 1, 2, \ldots$ This triangle can be rewritten in a rectangular arrangement, with all its entries moved to the left and zero-filled on the right, as shown in the next table:

$n=0$	1	0	0	0	0	0
$n=1$	1	1	0	0	0	0
$n=2$	1	2	1	0	0	0
$n=3$	1	3	3	1	0	0
$n=4$	1	4	6	4	1	0

This had already been done by Stifel in his *Arithmetica integra*, and Newton wrote it also as a rectangular array, but turning the rows into columns, in some manuscripts now preserved at the University Library, Cambridge, but never published in his lifetime. The second of these manuscripts, which started as a redraft of the first and was possibly written in the autumn of 1665,[20] is reproduced on page 126, and we shall present first the result obtained on the top third of the page: the computation of the area under the hyperbola

$$be = \frac{1}{1+x}$$

from the origin at d (it looks like ∂ in Newton's hand) to an arbitrary point e at some $x > 0$. Newton made the following observation about his vertical arrangement of Pascal's triangle:

> The composition of w$^{\text{ch}}$ table may be deduced from hence, viz: The sume of any figure & y$^{\text{e}}$ figure above it is equall to y$^{\text{e}}$ figure following it.

Referring instead to the horizontal arrangement by Stifel, which is more familiar to us, this can be translated as follows: any entry m, other than the first, in a given row is the sum of two entries in the previous row: the one above m and the one to its left. Viewing things in this way is important because it allowed Newton to construct a new column to the left of the one for $n = 0$ with the same property. If we insist on a horizontal arrangement, this gives us a new row above the one for $n = 0$, as follows:

$n = -1$	1	-1	1	-1	1	-1
$n = 0$	1	0	0	0	0	0
$n = 1$	1	1	0	0	0	0
$n = 2$	1	2	1	0	0	0
$n = 3$	1	3	3	1	0	0
$n = 4$	1	4	6	4	1	0

Notice that now there is no zero-fill on the right in the new first row. It continues indefinitely as an unending string of alternating 1's and -1's. If the new row is valid, it gives the following expansion:

$$(a+b)^{-1} = a^{-1} - a^{-2}b + a^{-3}b^2 - a^{-4}b^3 + a^{-5}b^4 - a^{-6}b^5 + \cdots$$

and, in the particular case $a = 1$ and $b = x$, we obtain the infinite series

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

[20] It appears under the modern title "Further development of the binomial expansion" in Whiteside, *The mathematical papers of Isaac Newton*, I, 1967, pp. 122–134. The table in question is on p. 122.

This is the first instance of what we call today *Newton's binomial series*, and it is known to be a valid expansion for small values of x. Newton did not write this expansion explicitly in the manuscript under discussion, but it is clear that this is what he had in mind when he computed the area under the hyperbola $be = (1+x)^{-1}$.

To do that, he considered first the graphs of the polynomials

$$1, \quad 1+x, \quad 1+2x+x^2, \quad 1+3x+3xx+x^3, \quad \&c.$$

(in the style of his time, Newton frequently wrote xx instead of x^2), whose coefficients are given by the other rows of the table above, and stated that the areas under these graphs from 0 to $x > 0$ are given by

$$x, \quad x+\frac{xx}{2}, \quad x+\frac{2xx}{2}+\frac{x^3}{3}, \quad x+\frac{3xx}{2}+\frac{3x^3}{3}+\frac{x^4}{4}, \quad \&c.^{21}$$

And then, making a leap of faith, he assumed that the same procedure would apply to finding the area under $(1+x)^{-1}$, represented by the infinite series whose coefficients are given by the entries in the top row of the extended table. Newton concluded with the following statement:

> By w$^{\text{ch}}$ table [this is where he implicitly assumes the series expansion for the hyperbola] it may appeare y$^{\text{t}}$ y$^{\text{e}}$ area of the hyperbola *abed* [meaning the area under $y = (1+x)^{-1}$ from 0 to $x > 0$] is
>
> $$x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}+\frac{x^9}{9}-\frac{x^{10}}{10} \quad \&\text{c.}$$

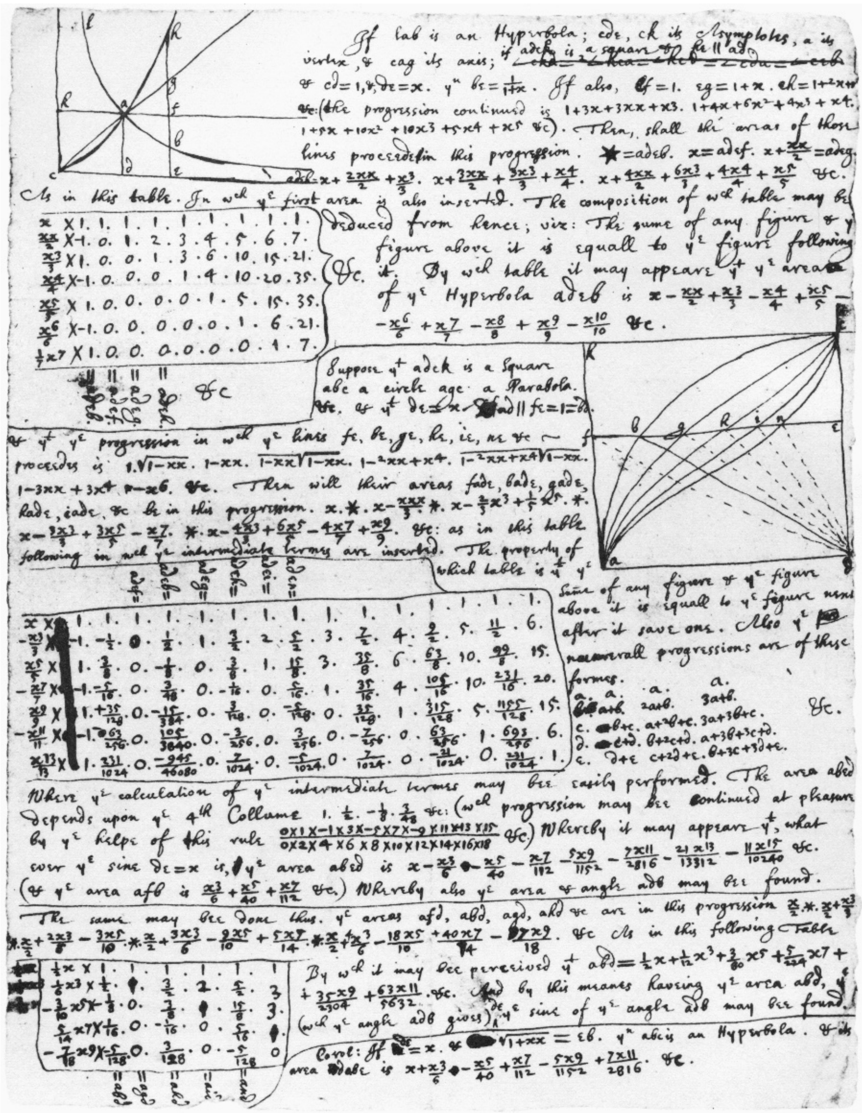
If we note that Newton's hyperbola, $y = 1/(1+x)$, is a translation of $y = 1/x$ to the left by one unit, then the area that Newton obtained is, in the notation introduced in Section 2.4, the same as the area $A(1+x)$ under $y = 1/x$ from $x = 1$ to $1+x > 1$. Thus, we can rewrite Newton's discovery as

$$\text{H}\log(1+x) = A(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\&c.$$

Newton obtained this unpublished result

> between the years 1664 & 1665. At w$^{\text{ch}}$ time I found the method of Infinite

[21] Newton had already figured out (although he was not the first to do so) that for $n = 0, 1, 2, \ldots$ the area under $y = x^n$ between 0 and $x > 0$ is $x^{n+1}/(n+1)$.

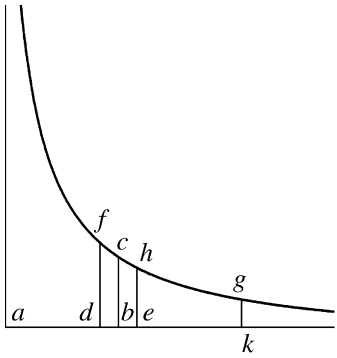


If lab is an Hyperbola; cde, ck its Asymptotes, a its vertex, & cag its axis; if adck is a square & fe || ad & cd = 1, & de = x. y^n be $= \frac{1}{1+x}$. If also, cf = 1. eg = 1 + x. eh = 1 + 2x + xx &c (the progression continued is 1 + 3x + 3xx + x3. 1 + 4x + 6x2 + 4x3 + x4. 1 + 5x + 10x2 + 10x3 + 5x4 + x5 &c). Then, shall the areas of those lines proceed in this progression. * = adeb. x = adef. $x + \frac{xx}{2}$ = adeg. adeh $= x + \frac{2xx}{2} + \frac{x3}{3}$. $x + \frac{3xx}{2} + \frac{3x3}{3} + \frac{x4}{4}$. $x + \frac{4xx}{2} + \frac{6x3}{3} + \frac{4x4}{4} + \frac{x5}{5}$ &c. As in this table. In wch y^e first area is also inserted. The composition of wch table may bee deduced from hence; viz: The sume of any figure & y^e figure above it is equall to y^e figure following it. By wch table it may appeare y^t y^e area of y^e Hyperbola adeb is $x - \frac{xx}{2} + \frac{x3}{3} - \frac{x4}{4} + \frac{x5}{5} - \frac{x6}{6} + \frac{x7}{7} - \frac{x8}{8} + \frac{x9}{9} - \frac{x10}{10}$ &c.

From Whiteside, *The mathematical papers of Isaac Newton*, I.
Facing the title page.

> series. And in summer 1665 being forced from Cambridge by the Plague[22] I computed y^{e} area of y^{e} Hyperbola at Boothby in Lincolnshire to two & fifty figures by the same method.[23]

Then, in a manuscript probably written in 1667,[24] Newton considered again the area under the hyperbola $y = (1 + x)^{-1}$, represented in the next figure with the origin at b and in which $ab = bc = 1$. After restating the area



under this hyperbola as an infinite series and making some calculations, he made the following statement [p. 186]:

> Now since the lines ad, ae, &c: beare such respect to y^{e} superficies [areas] $bcfd$, $bche$, &c: as numbers to their logarithmes; (viz: as y^{e} lines ad, ae, &c: increase in Geometricall Progression, so y^{e} superficies $bcfd$, $bche$, &c: increase in Arithmeticall Progression): Therefore if any two or more of those lines multiplying or dividing one another doe produce some other like ak, their correspondent superficies, added or subtracted one to or from another shall produce y^{e} superficies $bcgk$ correspondent to y^{t} line ak.

To interpret this statement in today's notation let x and y be the abscissas of the points d and e, respectively, so that the lengths ad and ae are $1 + x$ and $1 + y$.

[22] This was the great plague that took nearly 70,000 lives in London alone, and Cambridge University was closed.

[23] Newton made this statement on July 4, 1699, in one of his notebooks containing old annotations on John Wallis' work. It is quoted from Whiteside, *The mathematical papers of Isaac Newton*, I, p. 8.

[24] Reproduced in Whiteside, *The mathematical papers of Isaac Newton*, II, 1968, pp. 184–189. Page references are to this printing.

In view of the previous interpretation of areas under this hyperbola as values of H log, the areas $bcfd$ and $bche$ are H log $(1+x)$ and H log $(1+y)$. Then, if ak is the product of the lengths ad and ae, the area $bcgk$ is H log $(1+x)(1+y)$, and Newton's statement is that

$$\mathrm{H}\log(1+x)(1+y) = \mathrm{H}\log(1+x) + \mathrm{H}\log(1+y).$$

Similarly,

$$\mathrm{H}\log\frac{1+x}{1+y} = \mathrm{H}\log(1+x) - \mathrm{H}\log(1+y).$$

Thus, what Newton gave us in this manuscript is a statement (possibly the first ever) of the properties of hyperbolic logarithms. He illustrated these rules by the computation of a number of hyperbolic logarithms to 57 decimal figures [pp. 187–188]. Finally, to show that he could trust these rules, he computed the hyperbolic logarithm of 0.9984 in two ways: first by repeatedly using the stated rules with

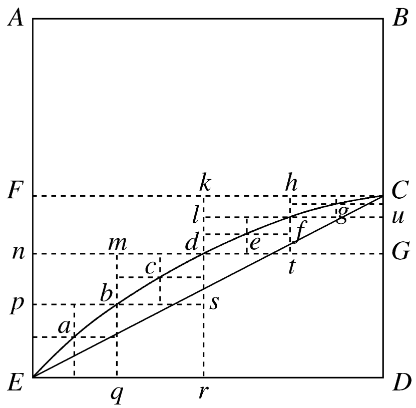
$$0.9984 = \frac{2\times2\times2\times2\times2\times2\times2\times2\times3\times13}{10000},$$

and then writing it as $0.9984 = 1 + (-0.0016)$ and using the infinite series. These two results (the first contains a minor error) agreed "in more y$^{\mathrm{n}}$ 50 figures" [p. 189].

Newton, who did not publish the preceding work, may have been disappointed in 1668, when other mathematicians published their work on the quadrature of the hyperbola using infinite series. The first, in April, was William, Viscount Brouncker (1620–1684), at the time the first president of the Royal Society.[25] The hyperbola $y = 1/x$ is represented by the curve EC in the next figure, in which AB is a segment of the x-axis from $x = 1$ to $x = 2$, and the positive y-axis, not shown, is directed down. The area $ABCdEA$ is bounded between the sum of the areas of the parallelograms with diagonals CA, dF, bn, fk, ap, cm, el, gh, &c. and the area of the parallelogram $ABDE$ minus the sum of the areas of the triangles (not explicitly drawn) EDC, EdC, Ebd, dfC, Eab, bcd, def, fgC, &c. In this manner Brouncker was able to conclude that

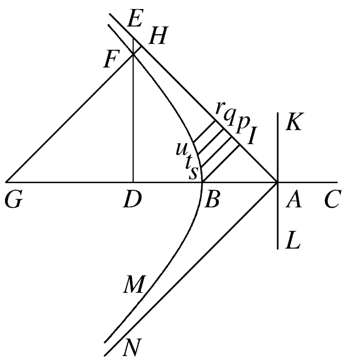
$$ABCdEA = \frac{1}{1\times2} + \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10}\ \ \&c.$$

[25] "The squaring of the hyperbola, by an infinite series of rational numbers, together with its demonstration, by that eminent mathematician, the Right Honourable the Lord Viscount Brouncker," 1668.



in infinitum [p. 646]. This is what we now call the logarithm of two. Brouncker's method can be generalized, but it is sufficiently unappealing to discourage such a course of action.

Next, Nicolaus Mercator (1620–1687)—born in the province of Holstein, Denmark (now Germany), with the last name Kaufmann but working in England—published the quadrature of the hyperbola in his 1668 book *Logarithmotechnia: sive methodus construendi logarithmos nova, accurata, & facilis*. Like Newton, but independently, Mercator based his quadrature on the infinite series for the quotient $1/(1+a)$, which he gave on page 30 using a as the variable. Using this series and referring to the next figure, in which



$AI = 1$ and IE is"divisa in partes æquales innumeras" of length a, he found the start of a series for the sum of the altitudes $ps + qt + ru + \cdots$, which multiplied by a gives the area $BIru$ [Proposition XVII, pp. 31–32]. Mercator did not explicitly give the logarithmic series, but it can be readily obtained from this, and two men showed how to do it. The first was John Wallis,[26] who showed that, defining $A = Ir$, the hyperbolic space $BIru$ is equal to the sum of the series

$$A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^6 + \tfrac{1}{4}A^5, \quad \text{\&c.}$$

[p. 754].[27] The second commentary on Mercator's work was published later in the same year by James Gregory in his *Exercitationes geometricæ*, 1668. He devoted the second *Exercitatio*, entitled *N. Mercatoris quadratura hyperbolæ geometricè demonstrata*, to prove Mercator's quadrature of the hyperbola using term-by-term integration. Here, in Consectario 2 (what we call corollaries he called *consectaria*, meaning conclusions or inferences) to Proposition IIII [p. 11] he gave the logarithmic series as follows. Referring to x as the "primus terminus" of his expansion, he stated it as: "primus terminus $-\ \frac{1}{2}$ secundi [the second power of x] $+\ \frac{1}{3}$ tertii $-\ \frac{1}{4}$ quartii $+$ &c. in infinitum."

The next advance in the computation of logarithms and in determining the exact nature of this elusive "Hlog" was also based on work of Newton: his generalization of the binomial series to fractional exponents. He stated his discovery in a letter of June 13, 1676, to Henry Oldenburg, secretary of the Royal Society, in response to an indirect request from Leibniz, who wished to have information on Newton's work on infinite series. It was in this letter that Newton introduced the notation $a^{\frac{m}{n}}$ for a number raised to a fractional power. In a subsequent letter of October 24,[28] he endeavored to explain how he arrived at the statement of his theorem, for which he never provided a proof, recalling his original work with the aid of some old manuscript.

He explained [p. 130] how at the beginning of his study of mathematics, he "happened on the works of our most Celebrated Wallis," in particular, on his *Arithmetica infinitorvm* of 1655.[29] In this work Wallis had managed to

[26] "Logarithmotechnia Nicolai Mercatoris," 1668.

[27] An abbreviated explanation in English was given by Edwards in *The historical development of the calculus*, 1979, pp. 162–163.

[28] Page references given below for this letter refer to the English translation in Turnbull, *The correspondence of Isaac Newton*, II. See the bibliography for additional sources.

[29] Wallis was the first to use of the symbol ∞ for infinity. It is in Proposition 1 of the First Part of *De sectionibus conicis* of 1655, p. 4, where he stated *esto enim* ∞ *nota numeri infiniti* (let ∞ be the symbol for an infinite number); reproduced in *Opera mathematica*, **1**, 1695, p. 29. It also appeared in Proposition XCI of *Arithmetica infinitorvm* of 1655, p. 70, in which he stated ... *erit* ∞ *vel infinitus* (... will be ∞ or infinity); reproduced in *Opera mathematica*, **1**, p. 405. Also in Stedall, *The arithmetic of infinitesimals. John Wallis 1656*, 2004, p. 71.

JOHN WALLIS IN 1658
Savilian Professor of Geometry at Oxford.
Portrait by Ferdinand Bol.
Photograph by the author from the original
at the *Musée de Louvre*, Paris.

find the areas under the curves that we now express by the equations

$$y = (1 - x^2)^0, \quad y = (1 - x^2)^1, \quad y = (1 - x^2)^2, \quad y = (1 - x^2)^3,$$

and so on, from the origin to $x > 0$, which are, in current notation,

$$x, \quad x - \tfrac{1}{3}x^3, \quad x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5, \quad x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7,$$

and so on.[30] To find the area under the circle $y = (1-x^2)^{\frac{1}{2}}$, he noticed that the exponent is the mean between 0 and 1, and attempted to use his new method

[30] In Wallis' notation, these can be seen in Proposition CXVIII of the *Arithmetica infinitorvm*. In *Opera mathematica*, **1**, p. 415. In Stedall, *The arithmetic of infinitesimals. John Wallis 1656*, p. 88.

of intercalation to find it between the first two areas stated above. This led him to a most interesting formula for π,[31] but in the end he was unable to do the intercalation.

Newton's power of observation, when he happened on Wallis' work, made him notice that in all the expressions for the areas given above [p. 130],[32]

> the first term was x, and that the second terms $\frac{0}{3}x^3$. $\frac{1}{3}x^3$. $\frac{2}{3}x^3$. $\frac{3}{3}x^3$ &c were in Arithmetic progression, and hence

the first two terms of the areas to be intercalated, those under the graphs of $y = (1-x^2)^{\frac{1}{2}}$, $y = (1-x^2)^{\frac{3}{2}}$, $y = (1-x^2)^{\frac{5}{2}}$, and so on (why stop with the circle), "should be

$$x - \frac{\frac{1}{2}x^3}{3}. \quad x - \frac{\frac{3}{2}x^3}{3}. \quad x - \frac{\frac{5}{2}x^3}{3}. \quad \text{\&c.''}$$

From his study of the area under the hyperbola, Newton was already sure that these areas would be given by infinite series, so he had to find the rest of the terms. Wallis' denominators from the third on are 5, 7, etc., which Newton kept for his intercalated series. Next he looked at the numerators of Wallis' coefficients, including that of the x term, and discovered that

> these were the figures of the powers of the number 11, namely of these 11^0. 11^1. 11^2. 11^3. 11^4. that is first 1. then 1,1. third 1,2,1. fourth 1,3,3,1. fifth 1,4,6,4,1, &c.[33]

The first figure, to use Newton's own word, in a power of 11 is always 1, and what he sought was a method to determine all the remaining figures if given the first two. Here is his own description of the discovery that he made:

---

[31] Proposition CXCI of the *Arithmetica infinitorvm*. In *Opera mathematica*, **1**, p. 469.

[32] I shall closely follow Newton's own recollections in his October 24 letter but will replace his expression $\overline{1-xx}|$ with $(1-xx)$. In his original work of 1665, already quoted at the start of this section, Newton expressed himself less clearly than in his recollections.

[33] Newton used the notation $\overline{11}|^0$, rather than 11^0, and so on for the other powers. Note that the word "figure" cannot be interpreted to mean "digit" starting with $11^5 = 161051$; but it means each of the coefficients of the powers of 10 in the expansion $11^5 = 1 \times 10^5 + 5 \times 10^4 + 10 \times 10^3 + 10 \times 10^2 + 5 \times 10^1 + 1 \times 10^0$, that is, the Pascal triangle coefficients in the expansion of $(10+1)^5$.

> ... and I found that on putting m for the second figure [in a power of 11], the rest would be produced by continual multiplication of the terms of this series.
> $$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \quad \&c.$$
> E. g. let $m = 4$, and $4 \times \dfrac{m-1}{2}$ that is 6 will be the third term [figure], & $6 \times \dfrac{m-2}{3}$ that is 4 the fourth, and $4 \times \dfrac{m-3}{4}$ that is 1 the fifth, & $1 \times \dfrac{m-4}{5}$ that is 0 the sixth, at which place in this case the series ends.

This procedure gives the figures of 11^4.

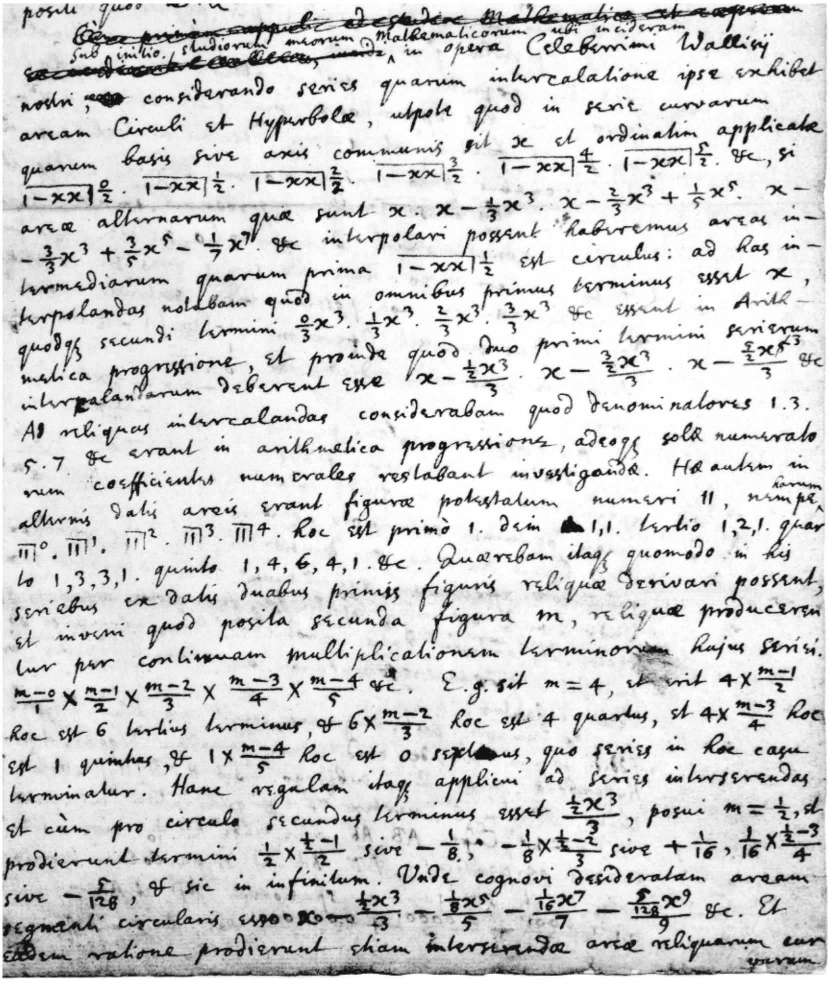
Assuming that the same rule applies to fractional exponents, which is a stretch to say the least, Newton considered the case $m = \frac{1}{2}$, and stated [p. 131]:

> ... since for a circle the second term was $\dfrac{\frac{1}{2}x^3}{3}$, I put $m = \frac{1}{2}$, and the terms appearing were
> $$\frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \text{ or } -\frac{1}{8}, \quad -\frac{1}{8} \times \frac{\frac{1}{2}-2}{3} \text{ or } +\frac{1}{16}, \quad +\frac{1}{16} \times \frac{\frac{1}{2}-3}{4} \text{ or } -\frac{5}{128},$$
> & so to infinity. From which I learned that the desired area of a segment of a circle is
> $$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} \quad \&c.$$
> And by the same reasoning the areas of the remaining curves to be inserted came forth ...

The area under the hyperbola was relevant to our discussion of logarithms, but would the remaining areas be equally relevant? Not really; it is Newton's next insight that matters:

> But when I had learnt this I soon considered that the terms
> $$(1-xx)^{\frac{0}{2}}. \quad (1-xx)^{\frac{2}{2}}. \quad (1-xx)^{\frac{4}{2}}. \quad (1-xx)^{\frac{6}{2}}. \quad \&c$$
> that is 1. $1 - xx$. $1 - 2xx + x^4$. $1 - 3xx + 3x^4 - x^6$ &c could be interpolated in the same way as the areas generated by them [the ones found by Wallis] : and that nothing else was required [for this purpose] but to omit the denominators 1, 3, 5, 7, &c ...

He forgot to say that we must also divide by x. This is what we now call term-by-term differentiation. Newton applied this procedure to his newly found areas to obtain the curves themselves.



Sub initio studiorum meorum Mathematicorum ubi incideram in opera Celeberrimi Wallisij nostri, considerando series quarum intercalatione ipse exhibet aream Circuli et Hyperbolae, utpote quod in serie curvarum quarum basis sive axis communis sit x et ordinatim applicatae $\overline{1-xx}|^{\frac{0}{2}}$. $\overline{1-xx}|^{\frac{1}{2}}$. $\overline{1-xx}|^{\frac{2}{2}}$. $\overline{1-xx}|^{\frac{3}{2}}$. $\overline{1-xx}|^{\frac{4}{2}}$. $\overline{1-xx}|^{\frac{5}{2}}$. &c, si areae alternarum quae sunt x. $x-\frac{1}{3}x^3$. $x-\frac{2}{3}x^3+\frac{1}{5}x^5$. $x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7$. &c interpolari possent haberemus areas intermediarum quarum prima $\overline{1-xx}|^{\frac{1}{2}}$ est circulus: ad has interpolandas notabam quod in omnibus primus terminus esset x, quodque secundi termini $\frac{0}{3}x^3$. $\frac{1}{3}x^3$. $\frac{2}{3}x^3$. $\frac{3}{3}x^3$ &c essent in Arithmetica progressione, et proinde quod duo primi termini serierum interpolandarum deberent esse $x-\frac{\frac{1}{2}x^3}{3}$. $x-\frac{\frac{3}{2}x^3}{3}$. $x-\frac{\frac{5}{2}x^3}{3}$ &c. Ad reliquas intercalandas considerabam quod denominatores 1. 3. 5. 7 &c erant in arithmetica progressione, adeoque solae numeratorum coefficientes numerales restabant investigandae. Hae autem in alternis datis areis erant figurae potestatum numeri 11, nempe harum 11^0. 11^1. 11^2. 11^3. 11^4. hoc est primo 1. deinde 1,1. tertio 1,2,1. quarto 1,3,3,1. quinto 1,4,6,4,1. &c. Quaerebam itaque quomodo in his seriebus ex datis duabus primis figuris reliquae derivari possent, et inveni quod posita secunda figura m, reliquae producerentur per continuam multiplicationem terminorum hujus seriei. $\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}$ &c. E.g. sit $m=4$, et erit $4\times\frac{m-1}{2}$ hoc est 6 tertius terminus, & $6\times\frac{m-2}{3}$ hoc est 4 quartus, et $4\times\frac{m-3}{4}$ hoc est 1 quintus, & $1\times\frac{m-4}{5}$ hoc est 0 sextus, quo series in hoc casu terminatur. Hanc regulam itaque applicui ad series interserendas. Et cum pro circulo secundus terminus esset $\frac{\frac{1}{2}x^3}{3}$, posui $m=\frac{1}{2}$, et prodierunt termini $\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}$ sive $-\frac{1}{8}$, $-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}$ sive $+\frac{1}{16}$, $\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}$ sive $-\frac{5}{128}$, & sic in infinitum. Unde cognovi desideratam aream segmenti circularis esse $x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-\frac{\frac{5}{128}x^9}{9}$ &c. Et eadem ratione prodierunt etiam interserendae areae reliquarum curvarum

A portion of the first page of Newton's October 1676 letter to Oldenburg.
From Turnbull, *The correspondence of Isaac Newton*, II.

Thus as *e.g.* $(1-xx)^{\frac{1}{2}}$ would have the value

$$1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6 \quad \&c$$

and $(1-xx)^{\frac{3}{2}}$ would have the value

$$1-\tfrac{3}{2}xx+\tfrac{3}{8}x^4+\tfrac{1}{16}x^6 \quad \&c$$

and $(1-xx)^{\frac{1}{3}}$ would have the value

$$1 - \tfrac{1}{3}xx - \tfrac{1}{9}x^4 - \tfrac{5}{81}x^6 \quad \&c.$$

But are these valid expansions or is it all pie in the sky? By way of demonstration, Newton just offered the following:

> To prove these operations I multiplied
>
> $$1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 \quad \&c$$
>
> into itself, & it became $1 - xx$, the remaining terms vanishing into infinity by the continuation of the series. On the other hand,
>
> $$1 - \tfrac{1}{3}xx - \tfrac{1}{9}x^4 - \tfrac{5}{81}x^6 \quad \&c$$
>
> twice multiplied into itself also produced $1 - xx$.

From these examples Newton was able to infer and state (in the June 13, 1676, letter) a general theorem. If m/n "is integral, or (so to speak) fractional" (what we now call a rational number), Newton's theorem stated that [34]

$$(1+x)^{m/n} = 1 + \frac{m}{n}x + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{1\cdot 2}x^2 + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{1\cdot 2\cdot 3}x^3 + \&c.$$

This is now called the *binomial theorem*. Newton used it, as well as other series expansions, in the development of what has become known as the calculus. The matter of which logarithm is the one previously referred to as "Hlog" will be taken up in the next section.

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[34] It is stated here in modern notation. Newton wrote his equation as

$$\overline{P+PQ}\rceil^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c,$$

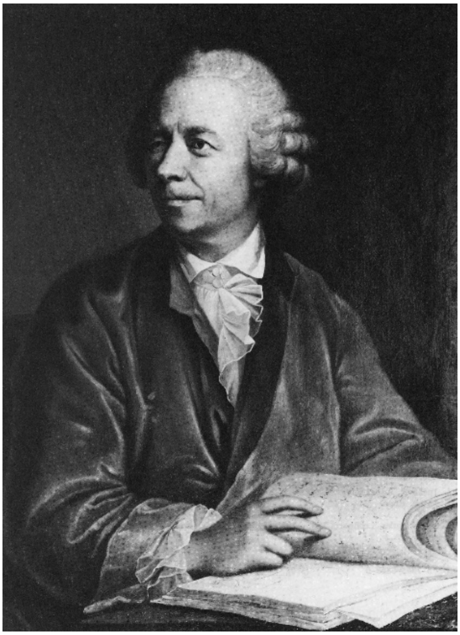
in which "I employ ... A for the first term, $P^{\frac{m}{n}}$; B for the second, $\frac{m}{n}AQ$, & so on." Putting $P=1$, $Q=x$, and rearranging yields the stated result.

Newton did not state the complete theorem until he wrote this letter of 1676, but he had already given the form of the general coefficient (which amounts to the same thing) at the end of the 1665 manuscript already quoted at the beginning of this section. Using x/y instead of m/n, he wrote this coefficient as

$$\frac{1 \times x \times \overline{x-y} \times \overline{x-2y} \times \overline{x-3y} \times \overline{x-4y} \times \overline{x-5y} \times \overline{x-6y}}{1 \times y \times \quad 2y \times \quad 3y \times \quad 4y \times \quad 5y \times \quad 6y \times \quad 7y} \quad \&c.$$

2.6 THE LOGARITHM ACCORDING TO EULER

The identification of the logarithm temporarily labeled "Hlog" in the preceding discussion was made by Leonhard Euler (1707–1783) of Basel. The son

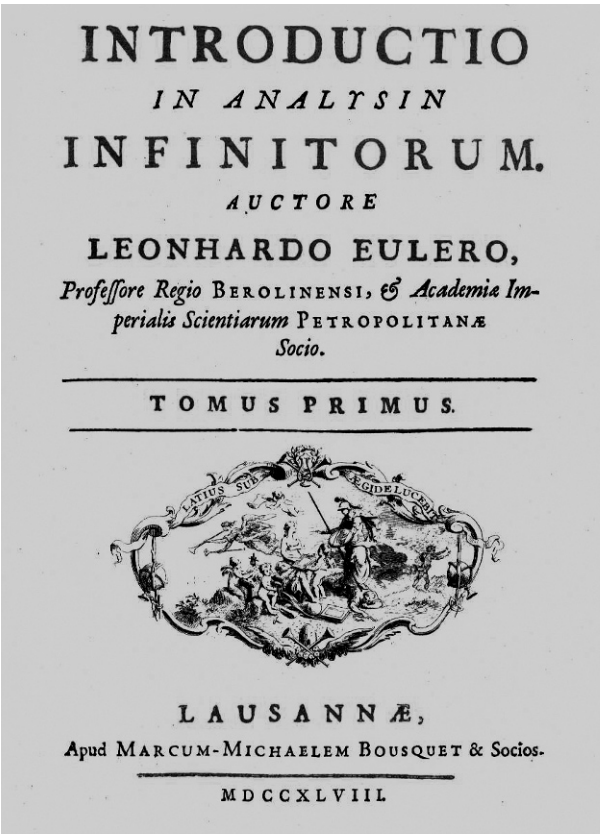


LEONHARD EULER IN 1756
Portrait by Jakob Emanuel Handmann.

of a preacher and destined to enter the ministry, his ability in mathematics soon convinced his father to let him switch careers, and he went on to become the most prolific mathematics writer of all time. In 1727, the year of Newton's death, he was invited to join the newly founded Academy of Saint Petersburg, in Russia, and soon began producing first-rate research. It was the next year, in a manuscript on the firing of cannon, that he introduced a soon to become famous number as follows: "Write for the number whose logarithm is unity, e," but he did not give a reason for this choice of letter.[35] By that time, he

[35] In "Meditatio in Experimenta explosione tormentorum nuper instituta," published posthumously in 1862, p. 800. The earliest printed appearance of the number e is in Euler's

had already defined the exponential and logarithmic functions, but the mathematical community at large had to wait until Euler was ready to publish. In 1741 he accepted a position at the Academy of Berlin, where he would remain for twenty-five years, and in 1744 he wrote his enormously influential treatise *Introductio in analysin infinitorum.* Published in Lausanne in 1748, it became


INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

the standard work on analysis during the second half of the eighteenth century.

In the first volume of this treatise Euler considered the definition of a^z (at this point he chose z as the symbol for a real variable) a trivial matter "easy

Mechanica sive motvs scientia analytice exposita, I, 1736 = *Opera omnia*, Ser. 2, **1**, p. 68. For English translations of the relevant passages of these works, see Smith, *A source book in mathematics*, pp. 95–96.

to understand from the nature of Exponents" [Art. 101].[36] It is not; while the meaning of $a^{\frac{m}{n}}$ was clear after Newton, what does $a^{\sqrt{7}}$ mean? Euler simply said "a certain value comprised between the limits a^2 and a^3" [Art. 97]. In short, he was not too clear about it but seemed to have in mind the existence of what we now call a continuous extension $y = a^z$ of $y = a^{\frac{m}{n}}$. Accepting this as a fact, then he defined its inverse function [Art. 102]:

> In the same manner as, given the number a, it is possible to find the value of y from any value of z, conversely, given any affirmative [positive] value of y, there is a convenient value of z, such that $a^z = y$; this value of z, regarded as a Function of y, is usually called LOGARITHM of y.

There is, of course, no such thing as *the* logarithm of y. What there is, instead, is one logarithm of y for each choice of the number a, "which, for this reason is called *base* of the logarithms" [Art. 102]. Then, in our present notation, $a^z = y$ if and only if $\log_a y = z$. However, logarithms as exponents were not new when the *Introductio* was published. They had been introduced by Euler in the unpublished manuscript cited at the beginning of this section, but the first systematic exposition of logarithms as exponents had already been printed—without knowledge of Euler's work—in the introduction to William Gardiner's *Tables of Logarithms* of 1742, "collected wholly from the papers" of William Jones.

The success of the *Introductio* rests on the amount and importance of the mathematical discoveries that Euler included in it, making it one of the most significant mathematics books of all times. Its readers might have been bewildered about the fact that there is one logarithm for each base a [Art. 107], but Euler easily showed that all logarithms of y are multiples of each other [Art. 108]. Indeed, if $z = \log_a y$ then $y = a^z$ and

$$\log_b y = \log_b a^z = z \log_b a = \log_a y \log_b a,$$ [37]

so that any two logarithmic functions, as we would say today, are constant multiples of each other. Thus it appears that we need retain only one logarithm, and the question is: what should be its base?

[36] References to the *Introductio* are by Article number rather than by page. In this way the reader can refer to any of the available editions cited in the bibliography.

[37] This is a modernized version. What Euler actually did is to show that if M and N are two numbers whose logarithms in base a are m and n, and whose logarithms in base b are μ and ν, then $m/n = \mu/\nu$. Taking $M = a$, so that $m = 1$, gives $\nu = n\mu$, and with $N = y$ this is the equation stated above.

The answer came from Euler's work in expanding both the exponential function $y = a^z$, $a > 1$, and the corresponding logarithm in infinite series. To do that, "let ω be an infinitely small number" [Art. 114]. Then, $a^0 = 1$ means that $a^\omega = 1 + \psi$, where ψ is also infinitely small. Write $\psi = k\omega$, where, as Euler remarked, "k is a finite number that depends on the value of the base a," and then for any number i [Art. 115],

$$a^{i\omega} = (1 + k\omega)^i = 1 + \frac{i}{1} k\omega + \frac{i(i-1)}{1 \cdot 2} k^2\omega^2 + \frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3} k^3\omega^3 + \text{\&c.}$$

Euler did not give a reason, but we know this to be true by Newton's binomial theorem if i is an integer or a quotient of integers. Now, for any number z let $i = z/\omega$. For Euler i was "infinitely large," but we may prefer to think of ω as a very small number chosen so that i is a very large quotient of integers. Putting $\omega = z/i$, the previous equations become

$$a^z = \left(1 + \frac{kz}{i}\right)^i = 1 + \frac{1}{1} kz + \frac{1(i-1)}{1 \cdot 2i} k^2z^2 + \frac{1(i-1)(i-2)}{1 \cdot 2i \cdot 3i} k^3z^3 + \text{\&c.}$$

Insisting on the fact that i is infinitely large, Euler stated that [Art. 116]

$$\frac{i-1}{i} = 1, \qquad \frac{i-2}{i} = 1,$$

and so on, which is approximately true if i is very large. Therefore,

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1 \cdot 2} + \frac{k^3z^3}{1 \cdot 2 \cdot 3} + \frac{k^4z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{\&c.}$$

For $z = 1$, he obtained the following relationship between a and k:

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \frac{k^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{\&c.},$$

and then we can ask the question: for which particular base a is $k = 1$? Clearly, this is true for [Art. 122]

$$a = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \text{\&c.}$$

Euler found the sum of this series to be 2.71828182845904523536028 &c. Later [Art. 123] he denoted this number by e—the first letter of the word exponential—and gave the series

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{\&c.}$$

Having expanded the general exponential function in an infinite series, Euler turned to the logarithmic function, which he denoted by l regardless of its base. To obtain its series expansion, he noted [Art. 118] that the equation $a^\omega = 1 + k\omega$ yields $\omega = l(1 + k\omega)$ and, consequently,

$$i\omega = il(1 + k\omega) = l(1 + k\omega)^i.$$

"It is clear, that the larger the number chosen for i, the more the Power $(1 + k\omega)^i$ will exceed unity;" that is, for any $x > 0$ (Euler switched from z to x at this point) we can choose i so that $x = (1 + k\omega)^i - 1$. Thus,

$$1 + x = (1 + k\omega)^i$$

and

$$l(1 + x) = l(1 + k\omega)^i = i\omega,$$

and i must be infinitely large because ω is infinitely small. From the definition of x it follows that [Art. 119]

$$i\omega = \frac{i}{k}(1 + x)^{1/i} - \frac{i}{k},$$

and then we can use Newton's binomial theorem to obtain

$$\begin{aligned}(1 + x)^{1/i} &= 1 + \frac{1}{i}x + \frac{\frac{1}{i}\left(\frac{1}{i} - 1\right)}{1 \cdot 2}x^2 + \frac{\frac{1}{i}\left(\frac{1}{i} - 1\right)\left(\frac{1}{i} - 2\right)}{1 \cdot 2 \cdot 3}x^3 \\ &\qquad + \frac{\frac{1}{i}\left(\frac{1}{i} - 1\right)\left(\frac{1}{i} - 2\right)\left(\frac{1}{i} - 3\right)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \&\text{c.} \\ &= 1 + \frac{1}{i}x - \frac{1(i - 1)}{i \cdot 2i}x^2 + \frac{1(i - 1)(2i - 1)}{i \cdot 2i \cdot 3i}x^3 \\ &\qquad - \frac{1(i - 1)(2i - 1)(3i - 1)}{i \cdot 2i \cdot 3i \cdot 4i}x^4 + \&\text{c.}\end{aligned}$$

(Euler did not write the first of these two series, but gave only the simplified form.) Since i is infinitely large,

$$\frac{i - 1}{2i} = \frac{1}{2}; \quad \frac{2i - 1}{3i} = \frac{2}{3}; \quad \frac{3i - 1}{4i} = \frac{3}{4}, \quad \&\text{c.};$$

from which

$$i(1+x)^{1/i} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

According to the last equation in Article 19, $i\omega$ is obtained by dividing this result by k and subtracting i/k. Then the equation $l(1+x) = i\omega$, established at the end of Article 18, shows that

$$l(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.\right).$$

In the particular case in which $a = e$ and $k = 1$ this becomes [Art. 123]

$$l(1+x) = \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$$

This is the same series obtained by Newton and Gregory, which shows that the elusive "Hlog" is actually l with base e. Euler called the values of this l "Logarithmi *naturales* seu *hyperbolici*" (natural or hyperbolic logarithms) [Art. 122]. From now on we shall reserve the letter l for natural logarithms.

Since every logarithm is a constant times l, the values of any logarithm can be computed from this series. So, what about $\log_{10} 2$ and $\log_{10} 5$? Is the use of this series faster than evaluating 54 square roots of 10 and then 52 square roots of 2, as we did when discussing the work of Briggs? In Chapter VI of the *Introductio* [Art. 106], Euler had used a variant of the square root method to find $\log_{10} 5$, obtaining the value 0.6989700, but then, at the very end of Article 106, he mentioned the discovery of "extraordinary inventions, from which logarithms can be computed more expeditiously." This refers to the use of the series derived above, which he would obtain in Chapter VII. However, it is not possible just to plug in, for instance, $x = 4$, because then, as Euler observed, the terms of this series "continually get larger" [Art. 120]. Mathematicians knew at that time—or they felt in their bones, in the absence of a theory of convergence—that if an infinite series is to have a finite sum, then its terms must decrease to zero, so that x cannot exceed 1 in this case. Euler found his way around this obstacle as follows. Replacing x with $-x$ in the equation giving $l(1+x)$ and subtracting the result from that equation [Art. 121] yields

$$l(1+x) - l(1-x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{&c.}$$
$$- \left(-\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{&c.} \right)$$
$$= 2\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \text{&c.} \right)$$

Using the properties of logarithms, this is equivalent to [Art. 123]

$$l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \text{&c.},$$ [38]

"which Series converges strongly, if x is replaced by an extremely small fraction." This device allows us to compute the logarithms of numbers larger than 1 from values of x smaller than one. Thus, for $x = 1/5$, one has $(1+x)/(1-x) = 3/2$, and Euler obtained (except for the numbers in parentheses)

$$l\frac{3}{2} = \frac{2}{1 \cdot 5} + \frac{2}{3 \cdot 5^3} + \frac{2}{5 \cdot 5^5} + \frac{2}{7 \cdot 5^7} + \frac{2}{9 \cdot 5^9} + \text{&c.}$$
$$(= 0.405465108108164381978013l0).$$

Next, substituting first $x = 1/7$ and then $x = 1/9$,

$$l\frac{4}{3} = \frac{2}{1 \cdot 7} + \frac{2}{3 \cdot 7^3} + \frac{2}{5 \cdot 7^5} + \frac{2}{7 \cdot 7^7} + \frac{2}{9 \cdot 7^9} + \text{&c.}$$
$$(= 0.2876820724517809274392l901),$$

and

$$l\frac{5}{4} = \frac{2}{1 \cdot 9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} + \frac{2}{7 \cdot 9^7} + \frac{2}{9 \cdot 9^9} + \text{&c.}$$
$$(= 0.2231435513142097557662950 9).$$

[38] This formula was found first by James Gregory in his *Exercitationes geometricæ*, as Consectario 4 to Proposition IIII [p. 12], in which Gregory denoted by H and 4 the abscissas that we denote by $1 - x$ and $1 + x$ and by S and 3 the corresponding ordinates. Once again referring to x (which he viewed as the area of a certain parallelogram) as "primus terminus," he stated his result as follows: "spatium Hyperbolicum $SH43 =$ duplo primi termini $+ \frac{2}{3}$ tertii $+ \frac{2}{5}$ quintii $+ \frac{2}{7}$ septimi $+ \frac{2}{9}$ nonii $+$ &c. in infinitum." Variants of the same formula were also used in logarithmic computation by Newton in his work on fluxions and infinite series of 1670–1671 (see Whiteside, *The mathematical papers of Isaac Newton*, III, 1969, p. 227) and by Edmund Halley in "A most compendious and facile Method for Constructing the Logarithms," 1695, pub. 1697.

Then, by the properties of logarithms,

$$l2 = l\frac{3}{2} + l\frac{4}{3} = 0.6931471805599453094172321,$$

$$l5 = l\frac{5}{4} + 2l2 = l\frac{5}{4} + 2\left(l\frac{3}{2} + l\frac{4}{3}\right) = 1.6094379124341003746007593,$$

and

$$l10 = l5 + l2 = 2.3025850929940456840179914.$$

This value, which Euler gave as stated here, may be compared to that obtained by Gregory at the end of Section 2.4. Finally [Art. 124], Euler returned to the computation of common logarithms and showed that if the hyperbolic logarithms are divided by the hyperbolic logarithm of 10, the common logarithms will be obtained. He did not give an example, but we shall show one, while simplifying his explanation (and reducing his accuracy) in the process. Using the already established equation $\log_b y = \log_a y \log_b a$ with $b = e$, $y = 2$, and $a = 10$, we obtain

$$\log_{10} 2 = \frac{l(2)}{l(10)} = \frac{0.693147180559945309}{2.302585092994045684} = 0.301029995663981195.$$

This is approximately the same value provided by the square root method, and it was in this manner that the natural logarithm became indispensable.

Now that the logarithm is a function, we can ask whether it has a derivative and how to find it. The series found above allowed Euler to find it as follows in his book on differential calculus [Article 180]:[39]

> We put $x + dx$ in place of x, so that y is transformed into $y + dy$; whereby we have
>
> $$y + dy = l(x + dx) \quad \& \quad dy = l(x + dx) - l(x) = l\left(1 + \frac{dx}{x}\right).$$
>
> As above the hyperbolic logarithm of this kind of expression $1 + z$ can be expressed by an infinite series, as
>
> $$l(1 + z) = \frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \&\text{c}.$$

[39] *Institutiones calculi differentialis*, 1755. This quotation is from Blanton's translation.

> Therefore if we substitute $\frac{dx}{x}$ for z, we obtain:
>
> $$dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \text{\&c.}$$
>
> Since all the terms of the series vanish in front of [meaning: compared to] the first term, it will be
>
> $$d.lx = dy = \frac{dx}{x}\cdot$$

In short, the derivative of lx is $1/x$. Of all the logarithms this is the only one with such a neat derivative. For if

$$\log_b x = \log_e x \log_b e = lx \log_b e,$$

then it is clear that

$$\log_b' x = \frac{1}{x} \log_b e$$

is not quite as good-looking as the derivative of lx. We conclude that the logarithm to keep is the natural logarithm, which is a very appropriate name for it.

So, what about the logarithms of negative numbers? In 1712 and 1713 a dispute over this had flared up in the correspondence between Gottfried Wilhelm Leibniz (1646–1716) and Johann Bernoulli (1667–1748). Leibniz had published an article[40] expressing his opinion on the logarithm of -1 [p. 167]:

> Indeed it is not positive, for such numbers are the Logarithms of positive numbers larger than unity. And yet it is not negative; because such numbers are the Logarithms of positive numbers smaller than unity. Therefore the Logarithm of -1 itself, which is not positive, nor negative, is left out as not true but imaginary.

An unfortunate choice of word (*imaginarius*), because it has a clear meaning in today's mathematics. What Leibniz meant is that $l(-1)$ does not exist. Bernoulli disagreed, stating his opinion that

$$lx = l(-x)$$[41]

[40] "Observatio, quod rationes sive proportiones non habeant locum circa quantitates nihilo minores, & de vero sensu methodi infinitesimalis," 1712.

[41] I have inserted parentheses where he had none, and will continue this practice to the end of this chapter for the benefit of the modern reader. However, the original notation of Euler, who did not use parentheses around expressions such as $-x$ and $+x$, is kept in the quotations.

in a letter to Leibniz of May 25, 1712. He gave four reasons for it, the first being that for $x > 0$ the differentials of both lx and $l(-x)$ are identical, and then the stated equation follows [pp. 886–887]. This was not accepted by Leibniz in his reply of June 30 because he believed that differentiating lx is legitimate for $x > 0$ only [p. 888]. Euler had also exchanged some correspondence with Bernoulli on this subject from 1727 to 1731, but, although he remained unconvinced by Bernoulli's arguments, he had no alternative theory of his own to propose at that time.

The controversy between Leibniz and Bernoulli resurfaced in 1745 when their correspondence was first published. By this time Euler had found the solution: $l(-a) = la + \pi(1 \pm 2n)\sqrt{-1}$, where n is any positive integer, and presented it in a letter of December 29, 1746, to the French mathematician Jean le Rond d'Alembert (1717–1783).[42] But d'Alembert responded:

> However, although your reasons are very formidable and very learned, I admit, Sir, that I am not yet completely convinced, because . . .

at which point he stated three of his own reasons for this lack of conviction, but we know now that they are invalid.[43] In fact, d'Alembert sided with Bernoulli, convinced by Bernoulli's fourth stated reason: that $(-x)^2 = x^2$ implies that $2l(-x) = 2lx$ and, once more,

$$l(-x) = l(+x).$$

Euler replied on April 15, 1747, but d'Alembert remained unconvinced and Euler eventually gave up the argument. Instead, on August of that year he sent an article, *Sur les logarithmes des nombres négatifs et imaginaires* (On the logarithms of negative and imaginary numbers), to the Academy of Berlin to give his solution; but he may have withdrawn it later because it was not published until 1862!

However, he presented another complete solution in 1749.[44] With both correspondents now dead, Euler felt free to express his own opinion. First, he explained how Bernoulli's first reason was wrong because it was [p. 144; 200]

[42] In a previous letter of September 24, Euler had communicated the value of $l(-1)$ to Gabriel Cramer as $(\pi \pm 2m\pi) \cdot \sqrt{-1}$. See Euler, *Opera omnia*, Ser. 4a, **1**, Birkhäuser Verlag, Basel, 1975, R. 469, pp. 93–94.

[43] Euler's solution and d'Alembert's reply can be seen in Euler, *Opera omnia*, Ser. 4a, **5**, Birkhäuser Verlag, Basel, 1980, pp. 251–253 and 256–259.

[44] "De la controverse entre Messrs. Leibnitz et Bernoulli sur les logarithmes négatifs et imaginaires," 1749, pub. 1751. Page references are to the original paper first and, after a semicolon, to the *Opera omnia*. Quotations are from the original paper.

> clear, that since the differential of $l{-}x$ & of $l{+}x$ is the same $\frac{dx}{x}$, the quantities $l{-}x$ and $l{+}x$ differ from one another by a constant quantity, which is equally evident, in view [of the fact] that $l{-}x = l{-}1 + l{+}x$.

Then he pronounced himself against Leibniz' belief that the logarithm of -1 does not exist [p. 154; 208]:

> Because, if $l{-}1$ were imaginary, its double, i.e. the logarithm of $(-1)^2 = +1$, would be too, which does not agree with the first principle of the doctrine of logarithms, according to which it is assumed that $l{+}1 = 0$.

However, Euler had a greater difficulty disposing of Bernoulli's fourth reason, because it rests on the belief that for any power p^n it is true that $l(p^n) = nlp$. But accepting this leads to contradiction [p. 147; 202]:[45]

> Because it is certain that $(a\surd - 1)^4 = a^4$, thus we'll also have $l(a\surd - 1)^4 = la^4$, & furthermore $4\,l(a\surd - 1) = 4\,la$, consequently $l(a\surd - 1) = la$.

Putting $a = 1$ leads to $l\sqrt{-1} = 0$, and this is impossible because Euler was already aware of the fact that (this will be explained in Section 3.5)

$$\frac{1}{2}\pi = \frac{l\sqrt{-1}}{\sqrt{-1}}.$$

But rejecting Bernoulli's belief that $l(-1) = l(+1) = 0$ also leads to contradiction [p. 148; 203]:

> To make this more evident, let $l{-}1 = \omega$, & if isn't $\omega = 0$, its double 2ω will not be $= 0$ either, but 2ω is the logarithm of the square of -1, and this being $+1$, the logarithm of $+1$ will no longer be $= 0$, which is a new contradiction.

These contradictions cannot be allowed to stand or the "enemies of Mathematics," as Euler called them [p. 154; 209], would have a field day. Not to fear. He found the source of these contradictions in a very insidious assumption that mathematicians, including himself, implicitly made [pp. 155–156; 210]:

> it is that one ordinarily assumes, almost without noticing, that each number has a unique logarithm ... Therefore I say, to make all these difficulties & contradictions disappear, that just as a consequence of the given definition each number has an infinitude of logarithms; which I will prove in the following theorem.

---

[45] The square root of -1 is shown here in Euler's original notation.

The theorem just restated that each number has an infinitude of logarithms. And how would he find the additional logarithms? By widening the search. Now that $\sqrt{-1}$ has become involved, it is not just the logarithms of negative numbers that we must seek, but those of complex numbers as well.

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