4

Applications of Bergman Geometry

In this chapter, results will be presented that arise by combining geometric arguments with the asymptotic curvature constancy at the boundary (discussed in the previous chapter) and other aspects of the geometry of the Bergman metric. The completeness of the Bergman metric of strongly pseudoconvex domains (Theorem 3.4.2) fits the whole situation into the framework of global Riemannian geometry, the basic idea of which is that the global geometry of a complete Riemannian manifold is controlled by curvature. Without completeness, this property fails entirely (cf. [Gromov 1969]). But, with completeness in hand, one expects curvature information to control the geometry in many respects.

4.1 Applications of Stability near the Boundary

The first result to be discussed has to do with small perturbations of the unit ball in $\mathbb{C}^n$, $n \geq 2$. A perturbation of the unit disc in $\mathbb{C}$ that is small in the C^∞ sense produces a domain that is still biholomorphic to the unit disc, by the Riemann mapping theorem. But in $\mathbb{C}^n$, $n \geq 2$, perturbations of the unit ball are generically not biholomorphic to the unit ball. This can be seen from Tanaka-Chern–Moser theory, but it can also be established by using more elementary arguments involving only counting the parameters in biholomorphic mappings and in representations of the boundary. There are, at high jet levels, more parameters in boundary choice than in germs of biholomorphic mappings. Details of this idea, which goes back to Poincaré, can be found in [Fefferman 1974] or [Greene/Krantz 1981].

Theorem 4.1.1. *There is a neighborhood $\mathcal{U}$ of the unit ball in $\mathbb{C}^n$ in the C^∞ topology on domains such that every $\Omega \in \mathcal{U}$ is either*

(1) *biholomorphic to the unit ball*

R. E. Greene et al., *The Geometry of Complex Domains,* Progress in Mathematics,
DOI 10.1007/978-0-8176-4622-6_4, © Springer Science+Business Media, LLC 2011

or else

(2) *$Aut(\Omega)$ has a fixed point, which is to say, there is an $x \in \Omega$ such that $\gamma(x) = x$ for every $\gamma \in Aut(\Omega)$.*

Proof. To begin with, choose $\mathcal{U}$ so that if $\Omega \in \mathcal{U}$, then Ω is (C^∞) strongly pseudoconvex. By Corollary 3.4.4, Ω is biholomorphic to the unit ball if $\mathrm{Aut}(\Omega)$ is noncompact. Now impose on $\mathcal{U}$ the additional conditions (via Theorem 3.6.2) that, if $\Omega \in \mathcal{U}$, then the Bergman metric has negative sectional curvatures and that, if $\Omega \in \mathcal{U}$, then Ω is diffeomorphic to the ball and hence simply connected. [This latter condition is taken for granted in general by our discussion of C^∞ topology. We reiterate it here for emphasis.]

With $\mathcal{U}$ satisfying these conditions, if $\Omega \in \mathcal{U}$ and Ω is not biholomorphic to the unit ball, then $\mathrm{Aut}(\Omega)$ is a compact group of isometries of a complete, simply connected manifold of everywhere negative sectional curvature—first, Ω with its Bergman metric. It is a standard theorem of E. Cartan (cf. [Klingenberg 1982], for example) that a compact group of isometries of a complete manifold of nonpositive sectional curvature has a fixed point. [The fixed point is obtained as the "center of gravity" of the orbit of any arbitrary point.] □

The fixed point theorem of E. Cartan that was applied to establish Theorem 4.1.1 is usually proved using the strict convexity of the square of the distance function. first, on a complete, simply connected Riemannian manifold with all sectional curvatures nonpositive, the function $\mathrm{dis}^2(\cdot, p_0)$ is C^∞, strongly convex for each point $p_0 \in M$. Indeed, its second derivative along each arclength-parameter geodesic is ≥ 2. This is an aspect of the Hessian comparison ideas developed in a more general context in [Greene/Wu 1977]. [It is also related to H. Karcher's proof ([Karcher 1989]; see also, e.g., [Klingenberg 1982], p. 226 ff) of the Toponogov comparison theorem ([Toponogov 1959]). But there the inequalities go the other way: nonnegative sectional curvature implies second derivatives ≤ 2.] But in the specific instance at hand, a direct proof by the second variation Formula is easy and standard.

With this convexity in mind, one establishes the existence of a fixed point for a compact group G of isometries of M as follows. Choose p_0 in M arbitrarily. Define $F : M \to \mathbb{R}$ by, for each $p \in M$, $F(p) = \int_{g \in G} \mathrm{dis}^2(g(p), p_0)\, dg$, where dg is the invariant measure on G. The function F is C^∞ and strongly convex; indeed, its second derivative along each arclength-parameter geodesic is ≥ 2, as one sees by differentiation under the integral sign. Moreover, completeness, the compactness of G, and the triangle inequality combine to show that F is proper. If p is far from p_0, then $F(p)$ is large because p is far from the compact set $\{g(p_0) : g \in G\}$. So F goes to infinity as p tends to infinity. Thus F has a unique minimum, say at the point q_0. But, because the function F is G-invariant—$F(g(x)) = F(x)$ for all $x \in M$, $g \in G$—this unique minimum is fixed by the elements of G. [Note that there is no claim that q_0 is the unique fixed point of the G action. A different choice of p_0 could potentially yield a different fixed point, and indeed the G action might have many fixed points.]

This argument admits a variant in which differentiability is brought less to the fore. This is a considerable digression, but it will make possible in a moment an equally considerable generalization of Theorem 4.1.1. In this variant, one considers, instead of the function F, convex sets associated to the situation.

Each closed ball $\mathrm{cl}(B(p_0, r)) \equiv \{q : \mathrm{dis}(q, p_0) \leq r\}$, $p_0 \in M$, is convex, because $\mathrm{dis}^2(\cdot, p_0)$ is a convex function. [The notion of convexity is unambiguous here since geodesic connections are unique on such manifolds.] Now, if G is not the one-element group, then, for small r, the set $\bigcap_{g \in G} \mathrm{cl}(B(g(p_0), r))$ is empty. On the other hand, if r is large, then, since G is compact, this intersection is definitely nonempty. Thus there is an $r_0 > 0$ such that the intersection is empty for $r < r_0$ and nonempty for $r > r_0$. One sees easily that $\bigcap_{g \in G} \mathrm{cl}(B(g(p_0), r_0))$ is nonempty but has empty interior. This set is clearly G-invariant.

At this point, one can bring into play a familiar "trick" of Riemannian geometry (cf. [Cheeger/Gromoll 1971]): a closed, convex set with empty interior (as a subset with possibly nonempty boundary) lies in a totally geodesic submanifold of M of lower dimension, which dimension can be chosen to be minimal. The group G acts on this unique, minimal-dimensional submanifold, so the argument can be repeated. Repetition yields eventually (since dimension drops at each stage) a compact, G-invariant, totally geodesic submanifold of M. But, for our particular M, such a submanifold must be a point: This follows from the strong convexity of $\mathrm{dis}^2(\cdot, q)$ for any point q chosen arbitrarily in M. [*Detail:* If N is a compact, totally geodesic submanifold of M with no boundary, then, for any $q \in M$, $\mathrm{dis}^2(\cdot, q)$ has a maximum value on N, say at $x \in N$. But then $\mathrm{dis}^2(\cdot, q)$ has a maximum at x along each geodesic through x. Thus $\mathrm{dis}^2(\cdot, q)$ is constant along such geodesics, contradicting strong convexity of $\mathrm{dis}^2(\cdot, q)$. This contradiction can be averted only if N consists of the point x alone.] We have gone into this matter in some detail because in fact this alternative line of reasoning enables Theorem 4.1.1 to be extended considerably. first, L. Lempert has proved the following (personal communication to the third author).

Theorem 4.1.2 (Lempert). *If G is a compact group of automorphisms of a convex, bounded, open domain Ω (convex in the usual Euclidean sense of the word), then G has a fixed point.*

The proof of this result is obtained by first showing that the balls in Ω relative to the Kobayashi metric are convex in the Euclidean sense ([Lempert 1981]). Then one can apply the geometric reasoning just discussed. In more detail: On a strongly convex domain with C^6 boundary, consider the convex sum of two extremal discs for the Kobayashi metric. The sum defines a holomorphic disc contained in the domain due to convexity. From this follows the Euclidean convexity of the Kobayashi distance ball for the strongly convex domain. Then the exhaustion of a bounded convex domain by strongly convex domains implies the Euclidean convexity for the Kobayashi distance ball for general convex domains. To obtain a fixed point of the compact subgroup G, consider the G-orbit of a point. As before in the Riemannian case, for a positive number r, the intersection, say S_r, of the closed balls of radius r centered at a point in the orbit is nonempty for some sufficiently large r. Take the smallest r for which S_r is nonempty. Then this S_r is convex and has empty interior. Thus it has dimension strictly less than that of the original domain. Equip S_r with the restricted Kobayashi distance. Then continue this process with S_r. This ends with a G-invariant 0-dimensional set which is convex and hence a single point. This is a fixed point of G.

To put Theorems 4.1.1 and 4.1.2 into context, one needs to know that, in general, a compact group of automorphisms of a C^∞ strongly pseudoconvex domain can be free of fixed points, even when the domain is homeomorphic or diffeomorphic to the ball. This is *not* obvious! Most compact topological group actions on balls that come to mind are conjugate to linear actions and hence have fixed points. And, *a fortiori*, examples of compact automorphism groups of domains homeomorphic to balls without fixed points are even less accessible.

Here, however, is a way to produce examples:

There is a finite group, say Γ, acting smoothly on S^7 with exactly one fixed point ([Stein 1976]; see also, for more on the general situation, [Petrie 1982]). This action can in fact be taken to be real analytic: this possibility is a general feature, once the existence of such a smooth action is known ([Illman 1994]). For any such (real analytic) action by Γ, a Γ-invariant Riemannian metric g_0 can be found by the usual averaging process. Then the complement in S^7 of every sufficiently small closed g_0-ball around the fixed point is real analytically diffeomorphic to a (standard) ball in $\mathbb{R}^7$ on which the finite group Γ acts real analytically and acts without fixed point. In this way, one obtains a bounded domain W in $\mathbb{R}^7$, diffeomorphic to the ball, such that W is real analytically acted upon by the finite group, without fixed points, and the closure of W is contained in a larger bounded domain V to which the group action extends real analytically, also without fixed points. The domain W (as well as V at the same time) can be taken to be real analytically equivalent to a standard ball. In fact, W can be taken to be a standard ball in $\mathbb{R}^7$.

By averaging, there is a group-invariant function $F : V \to \mathbb{R}$ such that F is real analytic and $W = \{p \in V \colon F(p) < 1\}$ and such that dF is nowhere zero on $\{p \in V \colon F(p) = 1\}$.

Now each element γ of the finite group Γ extends to be a biholomorphic map of some neighborhood V_γ of the closure of W in $\mathbb{C}^7$ into some open neighborhood of the closure of V. The intersection $\widehat{W} := \bigcap_{\gamma \in \Gamma} V_\gamma$ is a neighborhood in $\mathbb{C}^7$ of the closure of W.

Consider the function $y_1^2 + y_2^2 + \cdots + y_7^2$ on $\mathbb{C}^7$, where $z_j = x_j + \sqrt{-1}y_j$. By averaging and shrinking $\widehat{W}$ if necessary (while still keeping it a neighborhood of the closure of W), we obtain a group-invariant C^∞ function $\varphi \colon \widehat{W} \to \mathbb{R}$, say, such that $\varphi \geq 0$ and $\{p \in \widehat{W} \colon \varphi(p) = 0\}$ is the set where $y_j = 0$ for all

$j = 1, \ldots, 7$ and such that φ is strictly plurisubharmonic (since $y_1^2 + y_2^2 + \cdots + y_7^2$ is). Here, "group-invariant" does not mean that the set $\widehat{W}$ is invariant under the action by Γ but only that $\varphi(p) = \varphi(\gamma(p))$ for each $\gamma \in \Gamma$ and each $p \in \widehat{W}$.

Next, note that we can also average the function

$$(z_1, \ldots, z_7) \mapsto F(x_1, \ldots, x_7)$$

over the Γ-action, when $z = (z_1, \ldots, z_7)$ is in a neighborhood in $\mathbb{C}^7$ of the closure of W. This yields a group-invariant function $\widehat{F}$ on a small enough such neighborhood in $\mathbb{R}^7 \subset \mathbb{C}^7$.

Now consider $\widehat{F} + M\varphi$, where M is a (large) positive constant to be determined and let

$$\widetilde{W}_M := \{p \colon \widehat{F}(p) + M\varphi(p) < 1\}.$$

Then $W \subset \widetilde{W}_M$, since $F = \widehat{F} < 1$ on W and $\varphi = 0$ on W. Moreover, for M large enough, $\widetilde{W}_M$ is C^∞ strongly pseudoconvex because φ is C^∞ strictly plurisubharmonic. The nonvanishing of the gradient of $\widehat{F} + M\varphi$ at the boundary of $\widetilde{W}_M$ is easily checked. Finally, the domain $\widetilde{W}_M$ is group-invariant—the group Γ acts on it—because the defining function is group-invariant.

When M again is large enough, the group action on $\widetilde{W}_M$ is without fixed point. For, otherwise a limiting argument would produce a fixed point for the group action on W, since, as $M \to +\infty$, the domains $\widetilde{W}_M$ collapse to W.

This construction is of course quite general. It would apply to any finite group acting smoothly on a sphere with exactly one fixed point: the specific reference to S^7 is only an historical tribute to [Stein 1976]. Indeed, one could similarly deal with compact groups in general acting smooth on a sphere with one fixed point. Note also that the domain $\widetilde{W}_M$ cannot be biholomorphic to the ball, since every finite (or indeed compact) subgroup of automorphism group of the ball has a fixed point. Thus Aut $(\widetilde{W}_M)$ is a compact group (see Corollary 3.4.4) acting without fixed points on $\widetilde{W}_M$.

Now we explore results from the paper [Greene/Krantz 1981] that are based on Theorem 3.5.1, on the stability of Bergman metric curvature near the boundary of a C^∞ strongly pseudoconvex domain.

The following lemma will be pivotal to the considerations in this subsection.

Lemma 4.1.3. *Let Ω_0 be a fixed strongly pseudoconvex domain with C^∞ boundary that is not biholomorphic to the ball. Then there are a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology on domains, a number $\delta > 0$, and a point $p \in \Omega_0$ such that if $\Omega \in \mathcal{U}$ then $p \in \Omega$ and*

$$dis(f(p), \partial\Omega) \geq \delta$$

for all $f \in$ Aut (Ω).

Proof. According to Theorem 4.2.2, the holomorphic sectional curvature of the Bergman metric of Ω_0 is not constant. (Theorem 4.2.2 will be proved later by an argument independent of the present Lemma 4.1.3.) In particular, there is a constant $\lambda > 0$, a point $p \in \Omega_0$ and a J-invariant 2-plane P such that the sectional curvature $\kappa(P)$ of the Bergman metric of Ω_0 at p satisfies

$$\left|\kappa(P) + \frac{4}{n+1}\right| > \lambda.$$

From the stability result Theorem 3.5.2, there is a neighborhood $\mathcal{U}_1$ of Ω_0 in the C^∞ topology on domains such that $p \in \Omega$ if $\Omega \in \mathcal{U}_1$ and

$$\left|\kappa_\Omega(P) + \frac{4}{n+1}\right| > \frac{\lambda}{2}$$

for all $\Omega \in \mathcal{U}_1$, where $\kappa(P)$ = the sectional curvature of the 2-plane P at p for the Bergman metric of Ω. By Theorem 3.5.1, there is a C^∞ neighborhood $\mathcal{U}_2$ of Ω_0 and a constant $\delta > 0$ such that if $\Omega \in \mathcal{U}_2$, if $q \in \Omega$ with dis $(q, \mathbb{C}^n \setminus \Omega) < \delta$, and if Q is a J-invariant 2-plane at q, then

$$\left|\kappa_\Omega(Q) + \frac{4}{n+1}\right| > \frac{\lambda}{2}.$$

Now sectional curvature is invariant under isometries, and hence sectional curvatures of a Bergman metric are invariant under biholomorphic maps. Moreover, (the differentials of) biholomorphic maps take J-invariant 2-planes to J-invariant 2-planes. It follows that if $\Omega \in \mathcal{U}_1 \cap \mathcal{U}_2$, then the orbit of the point p under Aut (Ω) contains no points x with dis $(x, \partial\Omega) < \delta$. □

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We say that Ω is *rigid* if Aut $(\Omega) = \{\text{id}\}$. In other words, Ω is rigid if the only biholomorphic mapping of Ω to itself is the identity mapping.

Theorem 4.1.4. *Let Ω_0 be a smoothly bounded, strongly pseudoconvex domain that is rigid. Then any sufficiently small C^∞ perturbation of Ω_0 is also rigid. In other words, the set of rigid, strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary is open in the C^∞ topology of domains.*

Remark. It follows from [Burns/Shnider/Wells 1978] (which uses the theory of Tanaka/Chern/Moser invariants [Chern/Moser 1974], [Tanaka 1965]) that the collection of all smoothly bounded, rigid, strongly pseudoconvex domains is dense in the collection of all smoothly bounded, strongly pseudoconvex domains. Actually, this density can be established without the use of invariant theory, just by parameter counting, by using systematically that the number of parameters at a given jet level for a hypersurface is larger than the number of parameters for local biholomorphic maps, as already discussed. Coupled with the result of the theorem, this implies that the collection of smoothly bounded strongly pseudoconvex domains with nontrivial automorphism group is residual—in the sense of the Baire category theory. The rigid domains are an open dense set (in the C^∞ topology on domains). Rigidity is "generic."

Proof of Theorem 4.1.4. The proof will be by contradiction: Suppose there is a sequence $\{\Omega_j\}_{j=1}^{\infty}$ of C^∞ strongly pseudoconvex domains converging in the C^∞ topology to a C^∞ strongly pseudoconvex domain Ω_0 with $\mathrm{Aut}\,(\Omega_0) = \{\mathrm{id}\}$ but such that, for each $j \geq 1$, $\mathrm{Aut}\,(\Omega_j) \neq \{\mathrm{id}\}$. Observe that if $\alpha_j : \Omega_j \to \Omega_j$ is a sequence of holomorphic mappings then, by standard normal families arguments, there is a subsequential limit mapping $\alpha_0 : \Omega_0 \to \mathrm{cl}(\Omega_0)$. Choose, for each j, $\alpha_j \in \mathrm{Aut}\,(\Omega_j)$, $\alpha_j \neq \mathrm{id}_{\Omega_j}$.

The domain Ω_0 is certainly not biholomorphic to the ball. So Lemma 4.1.3 tells us that there is a point $p \in \Omega_0$ and a number $\delta > 0$ such that the points $\{\alpha_j(p)\}$ lie in $\{z \in \Omega_j : \mathrm{dis}(z, \partial\Omega_j) > \delta\}$ for all sufficiently large j. In particular, we can be sure that $\{\alpha_j(p)\}$ lie in $\{z \in \Omega_0 : \mathrm{dis}(z, \partial\Omega_0) > \delta\}$ as long as j is sufficiently large. As a result, the mapping $\alpha_0 : \Omega_0 \to \mathrm{cl}(\Omega_0)$ must itself be an automorphism. (See Theorem 1.3.4.)

Since $\mathrm{Aut}\,(\Omega_0) = \{\mathrm{id}\}$, we conclude that $\alpha_0 = \mathrm{id}$. In order for us to obtain a contradiction, it suffices to show that the sequence $\{\alpha_j\}$ could have been selected to be bounded away from the identity, for all large j, on some compact subset of Ω_0. In so constructing the sequence α_j, we will (discarding a finite number of domains and mappings if necessary) take $p \in \Omega_j$ and $\mathrm{dis}(p, \partial\Omega_j) > \delta$ for all j.

We first claim that there is an $\epsilon > 0$ such that, if the orbit of p under $\mathrm{Aut}\,(\Omega_j)$ is contained in the Bergman metric ball on Ω_j of size ϵ around p, then there is a fixed point of $\mathrm{Aut}\,(\Omega_j)$ contained in this ball. To prove this claim, notice that the group $\mathrm{Aut}\,(\Omega_j)$ will be compact if the orbit of p is bounded in the Bergman metric; and if the orbit of p is contained in a sufficiently small ball about p, then that compact orbit will also have a unique Riemannian center of mass in the ball, which will be a fixed point of the group action. The required smallness of this ball is stable under C^∞ perturbation of the metric, hence under C^∞ perturbation of the domain. Hence that smallness can be chosen uniformly in j. This stability and consequent uniformity comes from the C^∞ interior stability of the Bergman metric and the usual conditions for existence of a unique Riemannian center of mass (cf. [Grove/Karcher 1973]).

Now, suppose that it is not possible to select $\alpha_j \in \mathrm{Aut}\,(\Omega_j)$ which are bounded away from the identity on the Euclidean ball of radius $\delta/4$ around p. Passing to a subsequence if necessary, we may assume that $\mathrm{Aut}\,(\Omega_j)$ restricted to this ball converges to the identity. Then, as we have previously noted, for all large j there will be a fixed point—call it p_j—for $\mathrm{Aut}\,(\Omega_j)$ with p_j in the Bergman metric ball of radius ϵ about p. [Here we are assuming, without loss of generality, that the Bergman metric balls of radius 2ϵ around p for the Bergman metrics of the Ω_j are all contained in the Euclidean ball of radius $\delta/4$ about p.]

Thus, for all large j, $\mathrm{Aut}\,(\Omega_j)$ is isomorphic to a subgroup H_j of the unitary group via the mapping $\alpha \mapsto d\alpha|_{p_j}$, as usual. Now here is the crux of the argument: since the unitary group does not contain arbitrarily small nontrivial subgroups, there is a positive constant $\eta > 0$ such that, for each sufficiently large j, there is an element $\beta_j \in \mathrm{Aut}\,(\Omega_j)$ with the distance of $d\beta_j\big|_{p_j}$ to the identity exceeding η (where distance is relative to some fixed bi-invariant metric on the unitary group). But this fact, together with the facts that the Bergman metrics of the Ω_j converge C^∞ to that of Ω_0 uniformly on the Euclidean ball of radius $3\delta/8$ about p and that the p_j lie in the fixed compact closed ball of Euclidean radius $\delta/4$ about p, implies that the action of the elements β_j does not converge to the identity on the Euclidean ball about p of radius $3\delta/8$. This contradiction completes the proof. $\square$

A similar, but simpler, argument establishes the following result. We refer the reader to [Greene/Krantz 1981] for the details.

Theorem 4.1.5. *Each biholomorphic equivalence class is closed in the C^∞ topology on the set of C^∞ strongly pseudoconvex domains.*

4.2 Bergman Representative Coordinates

The Bergman kernel function gives rise not only to the Bergman metric, as already discussed, but also to some special local holomorphic coordinate systems which play a significant role in the study of biholomorphic mappings and in particular will be heavily used here. These local coordinate systems, known as Bergman representative coordinates, share certain properties with the geodesic normal coordinates of Riemannian geometry. In particular, biholomorphic mappings are linear when expressed in representative coordinates, in analogy with isometries being linear in geodesic normal coordinates. But geodesic normal coordinates are never holomorphic unless the (Kähler) metric is flat, that is, locally isometric to $\mathbb{C}^n$, while the Bergman representative coordinates are holomorphic in all cases where they are defined.

As we shall see, the Bergman representative coordinates provide a natural way to analyze, among other things, smoothness to the boundary of biholomorphic mappings. But this possibility was overlooked for some time by the mathematical community as a whole. Bergman himself suggested this use for representative coordinates at the 1975 AMS Summer Institute on Several Complex Variables in Williamstown, Massachusetts. This suggestion was treated with respect by the several hundred people who heard it there, as befitted Bergman's venerable age and his stature in the field. But the remark was almost, it seems, completely misunderstood. This is somewhat surprising in view of the great interest at that time in simplifying the latter part of Fefferman's then new paper [Fefferman 1974], in which the asymptotic expansion for the Bergman kernel obtained in the first part is shown by an intricate argument involving geodesics to imply boundary smoothness. As we shall see below, Bergman's suggested use of representative coordinates was exactly *a propos*: these coordinates provide precisely the right tool to obviate the analysis of geodesics and to go directly to smoothness to the boundary. [The later paper [Webster 1979] gives one method for implementing Bergman's idea, though without attribution to Bergman and hence, one supposes, independently.]

Bergman's representative coordinates are also involved in the proof of Lu Qi-Keng's theorem (Theorem 4.2.2) on bounded domains with Bergman metrics of constant holomorphic sectional curvature. This result will be stated in detail and proved in the present section.

We turn first to the definition of Bergman representative coordinates.

Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be a point of Ω. The "diagonal" Bergman kernel $K_\Omega(q,q)$ is of course real and positive so that there is a neighborhood of q such that, for all z, w in the neighborhood, $K_\Omega(z,w) \neq 0$. Then for all z, w in that neighborhood, we define

$$b_j(z) = b_{j,q}(z) = \frac{\partial}{\partial \overline{w}_j} \log \frac{K(z,w)}{K(w,w)}\bigg|_{w=q}.$$

It is actually certain constant-coefficient linear combinations of these that will be the ultimate "Bergman representative coordinates," but we begin with the functions just defined. Note that these coordinates are well defined, independent of the choice of logarithmic "branches." Each $b_j(z)$ is clearly a holomorphic function of z.

Notice that some restriction on z to be in a neighborhood of q may be actually necessary, since it may be that $K_\Omega(z,w)$ vanishes for some pairs $(z,w) \in \Omega \times \Omega$.[1] In any event, the mapping

$$z \longmapsto \big(b_1(z), \ldots, b_n(z)\big) \in \mathbb{C}^n$$

is defined and holomorphic in a neighborhood of the point q. Note also that $(b_1(q), \ldots, b_n(q)) = (0, \ldots, 0)$.

We are hoping to use these functions, and later certain special linear combinations of them, as holomorphic local coordinates in a neighborhood of q. By the holomorphic inverse function theorem, these functions give local coordinates if the holomorphic Jacobian

$$\det \left(\frac{\partial b_j}{\partial z_k}\right)_{j,k=1,\ldots,n}$$

is nonzero at q.

But in fact the nonvanishing of this determinant at q is an immediate consequence of a fact that we have established already, first, that the Bergman metric is positive definite. To see this relationship, notice that

$$\begin{aligned} \frac{\partial b_j}{\partial z_k}\bigg|_{z=q} &= \frac{\partial}{\partial z_k}\left(\frac{\partial}{\partial \overline{w}_j} \log K(z,w)\right)\bigg|_{z=w=q} \\ &= \frac{\partial^2}{\partial z_k \partial \overline{z}_j} \log K(z,z)\bigg|_{z=q}. \end{aligned}$$

[1] The point w is involved only very near q, but variation of z over all of Ω might lead to zeros of $K(z,w)$. In fact the zeros of $K_\Omega(z,w)$ do actually arise, even when Ω is required to be topologically a ball; see, e.g., [Boas 1986].

This last term is of course the Hermitian inner product $\left\langle \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_j} \right\rangle\Big|_q$ with respect to the Bergman metric. Thus the expression

$$\det \left(\frac{\partial b_j}{\partial z_k} \right)\Bigg|_q$$

is the determinant of the inner product matrix of a positive definite Hermitian inner product. Hence this determinant is positive and, in particular, nonzero.

The utility of the new coordinates in studying biholomorphic mappings comes from the following lemma.

Lemma 4.2.1. *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{C}^n$ with $q_1 \in \Omega_1$ and $q_2 \in \Omega_2$. Denote by $b_1^1, \ldots, b_n^1$ the Bergman coordinates as defined near q_1 in Ω_1 (using the Bergman kernel for Ω_1) and $b_1^2, \ldots, b_n^2$ the Bergman coordinates defined in the same way near q_2 in Ω_2 (using the Bergman kernel for Ω_2). Suppose that there is a biholomorphic mapping $F : \Omega_1 \to \Omega_2$ with $F(q_1) = q_2$. Then the function defined near $0 \in \mathbb{C}^n$ by*

$$(\alpha_1, \ldots, \alpha_n) \longmapsto \text{the } b^2\text{-coordinates of the } F\text{-image of the point of } \Omega_1 \text{ with } b^1\text{-coordinates } (\alpha_1, \ldots, \alpha_n)$$

is a $\mathbb{C}$-linear transformation.

In short form, we say that biholomorphic mappings are linear when expressed in the Bergman representative coordinates b^j.

Proof of the lemma. To avoid confusion, we write $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ for the $\mathbb{C}^n$-coordinates in Ω_1 and $(Z_1, \ldots, Z_n)$ and $(W_1, \ldots, W_n)$ for the $\mathbb{C}^n$-coordinates in Ω_2. In addition, we write K_1 for K_{Ω_1} and K_2 for K_{Ω_2}. Now observe that, for each $j = 1, \ldots, n$,

$$\frac{\partial}{\partial \overline{w}_j} \log \frac{K_2(F(z), F(w))}{K_2(F(w), F(w))} = \frac{\partial}{\partial \overline{w}_j} \log \frac{K_1(z, w)}{K_1(w, w)}.$$

The reason for this identity is

$$\begin{aligned}\frac{K_2(F(z), F(w))}{K_2(F(w), F(w))} = \frac{K_1(z, w)}{K_1(w, w)} &\times (\text{a holomorphic function of } z) \\ &\times (\text{a holomorphic function of } w).\end{aligned}$$

This last follows from the transformation law—the factors that are conjugate holomorphic in w cancel out, since they are the same in numerator and denominator. Thus we obtain (from the complex chain rule) that

$$\begin{aligned} b_j^1(z) &\stackrel{\text{def}}{=} \frac{\partial}{\partial \overline{w}_j} \log \frac{K_1(z, w)}{K_1(w, w)} \Bigg|_{w=q_1} \\ &= \frac{\partial}{\partial \overline{w}_j} \log K_2(F(z), F(w)) - \log K_2(F(w), F(w)) \Bigg|_{w=q_1} \\ &= \sum_k \left[\frac{\partial \overline{F}^k}{\partial \overline{w}_j} \cdot \frac{\partial}{\partial \overline{W}_k} \cdot \log \frac{K_2(F(z), W)}{K_2(W, W)} \right] \Bigg|_{W=F(q_1)}, \end{aligned}$$

where F^k is the k-th coordinate of $F(w_1, \ldots, w_k)$. But this last expression is exactly

$$\sum_k \frac{\partial \overline{F}^k}{\partial \overline{w}_j}\bigg|_{w=q_1} \cdot b_k^2(F(z)).$$

Hence

$$b_j^1(z) = \sum_k \frac{\partial \overline{F}^k}{\partial \overline{w}_j}\bigg|_{w=q_1} \cdot b_k^2(F(z)).$$

Since the Jacobian matrix $(\partial F^k/\partial w_j)$ of F is invertible at q, it follows that the $b_k^2(F(w))$ are linear functions of the $b_j^1(z)$ coordinates, as required. $\square$

The lemma is sufficiently surprising to justify looking at an explicit example. Let $\Omega_1 = \Omega_2 =$ the unit disc in $\mathbb{C}$. Set $q_1 = a$ in the disc, and take $q_2 = 0$. Define

$$F(z) = \lambda \cdot \frac{z-a}{1-\overline{a}z}$$

for some complex λ of unit modulus. Then the b^1-coordinates at $q = a$ are the evaluation at $w = a$ of

$$\begin{aligned}\frac{\partial}{\partial \overline{w}} \log \frac{1/(1-z\overline{w})^2}{1/(1-w\overline{w})^2} &= -2\frac{\partial}{\partial \overline{w}}\left[\log(1-z\overline{w}) - \log(1-w\overline{w})\right] \\ &= -2\left(\frac{-z}{1-z\overline{w}} + \frac{w}{1-w\overline{w}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}b^1(z) &= 2\left(\frac{z}{1-z\overline{a}} - \frac{a}{1-a\overline{a}}\right) \\ &= 2\left(\frac{z - za\overline{a} - a + za\overline{a}}{(1-z\overline{a})(1-a\overline{a})}\right) \\ &= \left(\frac{z-a}{1-\overline{a}z}\right) \cdot \frac{2}{1-a\overline{a}}.\end{aligned}$$

To get b^2-coordinates, we do the same calculations, but evaluate at 0 to obtain

$$b^2(z) = 2z.$$

Thus the biholomorphic map F takes the point z with b^1-coordinate α (equaling $2(z-a)/[(1-\overline{a}z)(1-\overline{a}a)]$) to the point with z-coordinate $\lambda(z-a)/(1-\overline{a}z)$ and hence with b^2-coordinate $2\lambda(z-a)/(1-\overline{a}z) = (1-\overline{a}a)\lambda \cdot b^1(z)$.

The mapping is indeed linear. The computationally inclined reader is invited now to see how the b^1-, b^2-coordinate setup enables one to regenerate the formula for the automorphisms (found in Section 3.3) of the ball in $\mathbb{C}^n$ that take, e.g., $(a, 0, \ldots, 0)$ to $(0, \ldots, 0)$; one need only be armed with the knowledge that the Bergman kernel for the unit ball in $\mathbb{C}^n$ is $c_n(1 - z \cdot \overline{w})^{-(n+1)}$. Of course, in practice, we used these biholomorphic mappings originally to actually compute the Bergman kernel, but it is still a matter of some interest to watch this regeneration of the maps in action.

The coordinates we have been discussing can be pushed one step further towards being truly "canonical," that is, dependent only on the complex structure. Let $q \in \Omega$ and let $V_1, \ldots, V_n$ denote holomorphic vector fields defined in an open neighborhood of q satisfying

$$\langle V_j, V_k \rangle \Big|_q = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

Then for each fixed $z \in \Omega$, we define

$$\beta_j = \overline{V_j} \log \frac{K_\Omega(z, w)}{K_\Omega(w, w)} \Big|_{w=q}.$$

[Hence the V_js act as differential operators only on the w-variables.] The proof that $(b_1, \ldots, b_n)$ defines a coordinate system at q can be easily modified to show that this map $z \mapsto (\beta_1(z), \ldots, \beta_n(z))$ is a well-defined holomorphic coordinate system at q.

Again, given a biholomorphic mapping $F : \Omega \to \widetilde{\Omega}$ and the respective Bergman representative coordinate systems $(\beta_1, \ldots, \beta_n)$ at $q \in \Omega$ and $(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n)$ at $F(q) \in \widetilde{\Omega}$, the map F takes expression in these coordinate systems as follows: there is a nonsingular complex matrix A_{jk} such that

$$\widetilde{\beta}_j = \sum_{k=1}^{n} A_{jk} \beta_k.$$

There are further properties. At the "center" q, the β-coordinate vector fields are orthonormal relative to the Bergman metric. (The same holds, of course, for $\widetilde{\beta}$-coordinates at $F(q)$.) It is these coordinates that we shall hereinafter call the *Bergman representative coordinates* of Ω at q. It of course remains true that biholomorphic mappings are linear in these coordinates. But in addition they are *unitary* linear mappings![2]

Notice that these coordinates themselves are unique up to a unitary rotation. Generally, one could not expect any further canonical aspect than that: Since unitary rotations act as biholomorphic maps on the unit ball, one cannot expect coordinates that are canonical beyond up-to-a-unitary-rotation. The Bergman representative coordinates are as canonical as holomorphic coordinates could be.

[2] On the other hand, the ordinary holomorphic (but nonbiholomorphic) mappings do not show any particular characteristic property in this coordinate system.

The Bergman representative coordinates are, as already noted, in some ways similar to geodesic normal coordinates, but with the additional property of being holomorphic. Further extraordinary properties will develop as we continue our discussion. Note, meanwhile, that the whole concept of representative coordinates extends essentially automatically to complex manifolds for which the Bergman metric construction for $(n, 0)$ forms already discussed above yields a positive definite metric. The construction can still be done locally, using general local holomorphic coordinates, and it remains true that the Bergman coordinates linearize holomorphic mappings. And again, the coordinates can be made more nearly canonical by using a basis for the differentiation that is orthonormal relative to the Bergman metric. A new point arises in that the quotient $K(z, w)/K(w, w)$ is not defined as such: it becomes defined only after a local coordinate choice around w and separately around z, if z is far from w. This turns out not to matter: this whole matter is discussed further in Chapter 11.

Our first application of Bergman representative coordinates is to the proof of a remarkable theorem of Lu Qi-Keng on domains with a Bergman metric of constant holomorphic sectional curvature.

Theorem 4.2.2 (Lu Qi-Keng). *If Ω is a bounded domain in $\mathbb{C}^n$, the Bergman metric of which is complete and has constant holomorphic sectional curvature, then Ω is biholomorphic to the unit ball.*

Notice that this result is certainly specific to the Bergman metric. For example, the annulus $\{\zeta \in \mathbb{C} : 1 < |\zeta| < R\}$, $R > 1$, admits a complete metric of constant (holomorphic) sectional curvature (see Section 2.3). But it is not even homeomorphic to the unit disc, much less biholomorphic to it.

This theorem has a complex manifold generalization: this is presented in Chapter 11.

Proof of Theorem 4.2.2. We first observe that the holomorphic sectional curvature, say c, must be negative. For, if c were positive, then Ω would be a complete Riemannian manifold with all sectional curvatures greater than or equal to $c/4 > 0$ (see Section 3.5).[3] Hence Ω would be compact by standard Riemannian geometry. [This is Myers's theorem: A complete Riemannian manifold with sectional curvature everywhere $\geq \epsilon > 0$ has diameter $\leq \pi/\sqrt{\epsilon}$ and is hence compact (cf. e.g., [Petersen 2006]).]

If c were zero, then the universal cover of Ω would be a complete, simply connected Kähler manifold of sectional curvature 0 and hence would be biholomorphically isometric to $\mathbb{C}^n$. But then, since Ω is bounded, the covering map into Ω would be constant by Liouville's theorem. This would contradict surjectivity of the covering map (to say the least!).

[3] This follows by the formula for Riemannian sectional curvature in case the holomorphic sectional curvature is constant. See Section 3.6 for the negative case: the positive case is the same up to the sign change.

It remains to discuss the case $c < 0$. If g_Ω is the Bergman metric of Ω (with constant negative holomorphic sectional curvature c), then the metric

$$g := -\frac{c(n+1)}{4} g_\Omega$$

has constant (negative) holomorphic sectional curvature $-4/(n+1)$ (cf. the remarks on scaling by constant factors at the end of Subsection 3.3.1). Thus the simply connected covering space $\widehat{\Omega}$ of Ω with the pullback $\widehat{g}$ of the metric g is a complete simply connected Kähler manifold with constant holomorphic sectional curvature $-4/(n+1)$. By standard Kähler geometry (cf. [Kobayashi/Nomizu 1963]), $(\widehat{\Omega}, \widehat{g})$ is biholomorphically isometric to B^n with its Bergman metric. Thus we obtain a holomorphic covering map $F : B^n \to \Omega$ which is locally isometric for the Bergman metric on B^n and g on Ω, respectively.

To prove the theorem, we need only show that F is in fact injective.

For this let $q = F(0)$. Since F is a covering map, it is locally invertible. first, there exists an open neighborhood U of q and a neighborhood V of 0 such that $F|_V : V \to U$ is a biholomorphism. Denote by H_0 the inverse of $F|_V$.

With $z, w \in U$, let

$$K_0(z,w) := K_{B^n}(H_0(z), H_0(w)).$$

Then

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_0(z,z) = g_{j\bar{k}} = \lambda\ g_{\Omega j\bar{k}}$$

by the condition on F above, where $\lambda = -\frac{c(n+1)}{4}$. This implies that

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_0(z,z) - \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \lambda \log K_\Omega(z,z) = 0$$

for every $z \in U$, and furthermore that

$$\log K_0(z,w) - \lambda \log K_\Omega(z,w) = \varphi(z) + \overline{\varphi(w)}$$

for every $z, w \in U$, for some holomorphic function $\varphi : U \to \mathbb{C}$. Actually for this one may need to replace U by a smaller, simply connected neighborhood; but that can be done without loss of generality, here and in what follows. Consequently one obtains

$$\frac{\partial}{\partial \bar{w}_j} \log \frac{K_0(z,w)}{K_0(w,w)} - \frac{\partial}{\partial \bar{w}_j} \lambda \log \frac{K_\Omega(z,w)}{K_\Omega(w,w)} = 0$$

for every $z, w \in U$.

This last gives rise to the direct computation with Bergman's representative coordinate systems $b^1 : V \to \mathbb{C}^n$ and $b^2 : U \to \mathbb{C}^n$. As in the introduction for Bergman's representative coordinate systems, one obtains that

$$H_0(\zeta) = (F|_V)^{-1} = (b^1)^{-1} \circ A \circ b^2(\zeta) \qquad (\star)$$

for every $\zeta \in U$. Here, of course, A is the linear map represented by the matrix with the (j,k)-th entry

$$\lambda \overline{\frac{\partial F_k}{\partial z_j}\Big|_0}.$$

Now look at the expressions in $(\star)$. The map b^1 is in fact a constant multiple of the Euclidean coordinate system. Therefore it extends to all of $\mathbb{C}^n$ holomorphically, needless to say. So does the linear map A. The map $\zeta \to b^2(\zeta)$ extends to a holomorphic mapping of $\Omega \setminus Z_q$, where

$$Z_q = \{\zeta \in \Omega \mid K_\Omega(\zeta, q) = 0\}.$$

Since $K_\Omega(\cdot, q)$ is a holomorphic function on Ω with $K_\Omega(q,q) \neq 0$, the set Z_q is an analytic variety whose complex codimension in Ω is 1. Hence $\Omega \setminus Z_q$ is a connected, dense, and open subset of Ω. Therefore, using the expression of H_0 in $(\star)$, the map H_0 extends to a holomorphic mapping of $\Omega \setminus Z_q$ into $\mathbb{C}^n$. Let H denote this extension.

Now, let $X := F^{-1}(Z_q)$. Then one immediately sees that

$$X = \{z \in B^n \mid K_\Omega(F(z), q) = 0\}.$$

Since $K_\Omega(F(0), q) = K_\Omega(q,q) \neq 0$, we see that X is again a complex analytic subvariety of B^n with complex codimension 1. Thus $B^n \setminus X$ is a connected, dense, and open subset of B^n. Furthermore, $H \circ F : B^n \setminus X \to \mathbb{C}^n$ is holomorphic with $H \circ F(z) = z$ for every $z \in V$, as $H = H_0$ on V. This means that $H \circ F(z) = z$ for every $z \in B^n \setminus X$. Now, for every $\zeta \in \Omega \setminus Z_q$, choose $x \in B^n$ such that $F(x) = \zeta$. Then

$$H(\zeta) = H(F(x)) = x.$$

This implies that $H(\Omega \setminus Z_q) \subset B^n$.

We see that H is holomorphic on $\Omega \setminus Z_q$. The removable singularity theorem for bounded holomorphic maps (the Riemann extension theorem) yields that H extends to a holomorphic mapping of Ω into $\mathbb{C}^n$. Since H continues to play the role of left inverse of F, it follows easily that F has to be injective. This completes the proof. $\square$

It is worthwhile to look back to see the exact role of completeness in this proof. Completeness in fact played no role in the construction of the local inverse which turned out to be a global, one-sided inverse. But completeness was used to get the holomorphic, locally isometric covering map from B to Ω in the first place. Without completeness, one would have only a locally defined covering map, and the subsequent arguments would not apply to inverting this map, it not being defined on all of B.

4.3 Equivariant Embedding and Concrete Realization of Abstract Complex Structures

Suppose that Ω is a bounded domain in $\mathbb{C}^n$ that contains the origin 0. There may be (nonidentity) elements of $\mathrm{Aut}(\Omega)$ that act on Ω as the restrictions to Ω of unitary linear transformations of $\mathbb{C}^n$, that is, as elements of $U(n)$. The set of such elements of $\mathrm{Aut}(\Omega)$ is clearly the set of restrictions to Ω of those elements α of $U(n)$ such that $\alpha(\Omega) = \Omega$. If every element of the isotropy group $I_0 = \{f \in \mathrm{Aut}(\Omega) : f(0) = 0\}$ arises in this fashion, then we say that Ω is *equivariantly embedded* in $\mathbb{C}^n$ at 0.

In this case, the mapping of I_0 into $U(n)$ defined by $f \mapsto df|_0$, with $f \in I_0$, is an injective, continuous isomorphism of I_0 into a compact subgroup of $U(n)$, with each element of this subgroup mapping Ω to itself. Thus the isomorphism of Corollary 1.3.7 becomes a concrete matter: the group of differentials, always isomorphic for any bounded Ω to the isotropy at a point, is literally the group of mappings itself. The obvious examples of this kind of behavior are balls and polydiscs centered at the origin. In fact, by Corollary 1.3.2, any complete circular domain has this equivariant embedding property. The following surprising result gives in effect an equivariant "re-embedding" of any domain close to the ball.

Theorem 4.3.1 (Greene–Krantz). *There is a neighborhood $\mathcal{U}$, in the C^∞ topology on domains, of the unit ball in $\mathbb{C}^n$ such that, for every $\Omega \in \mathcal{U}$, there is a biholomorphic map $F : \Omega \to \mathbb{C}^n$ with $F(0) = 0$ and with $F(\Omega)$ equivariantly embedded at 0.*

In the case $n = 1$, this result expresses the familiar fact (the Riemann mapping theorem) that a domain that is C^∞ close to the disc is biholomorphic to the disc via a biholomorphic mapping taking 0 to 0. The disc itself is of course equivariantly embedding at 0. But, for $n \geq 2$, the theorem is startling, just because the Riemann mapping theorem fails entirely even for domains C^∞ close to the ball. In general, Ω will definitely not be biholomorphic to the ball; also $F(\Omega)$ will be *not* the ball but some other domain that somehow exhibits the "abstract" symmetries of Ω around 0 as concrete geometric symmetries of $F(\Omega)$ that extend to be unitary rotations of $\mathbb{C}^n$ itself.

Proof of Theorem 4.3.1. It has already been observed that the expression of an automorphism in Bergman representative coordinates (around a point and its image) is a unitary linear transformation. Thus, taking F to be the Bergman representative coordinate map at 0 of Ω will do the job for the theorem, provided that the Bergman representative map is defined on all of Ω and is injective and nonsingular everywhere. These properties are *not* automatic; for example, for general bounded domains Ω, $K_\Omega(z, w)$ can have zeros even in cases where Ω is homeomorphic to the ball ([Boas 1986]). However, it turns out that the Bergman representative coordinate map $F_\Omega : \Omega \to \mathbb{C}^n$ at $0 \in \Omega$ is in fact an everywhere-defined holomorphic diffeomorphism onto a bounded, open set in $\mathbb{C}^n$ for all Ω that are sufficiently close in the C^∞ sense to the unit ball B^n.

To establish this last fact, note first that F_{B^n} is indeed a diffeomorphism. Indeed, it is the identity map of the ball to the ball (up to a dilation constant). This one checks by direct calculation. In particular, F_{B^n} extends to be a diffeomorphism of the closed ball $\mathrm{cl}(B^n)$ into $\mathbb{C}^n$, in the sense that it extends to the closure to be an injective C^∞ map with everywhere nonzero (real) Jacobian determinant.

The next step of the proof is to recall from basic differential topology (cf. [Munkres 1966]) that the property of being a diffeomorphism of a compact manifold-with-boundary into a Euclidean space is stable in the C^1 topology. In particular, a C^∞ mapping of the closed unit ball that is C^1 close to the identity will be such a diffeomorphism.

In our case, we are interested in a C^∞ mapping, the mapping via Bergman representative coordinates, not of the ball but of a domain Ω that is C^∞ close to the ball. But, following the usual terminology of differential topology, we say that a map $F : \mathrm{cl}(\Omega) \to \mathbb{C}^n$ is C^1 (or C^∞) close to a map $G : \mathrm{cl}(B^n) \to \mathbb{C}^n$ if there is a diffeomorphism $H : \mathrm{cl}(B^n) \to \mathrm{cl}(\Omega)$, H itself close to the identity, with $F \circ H$ close to the map G on B^n. Then it remains true that if F is C^1 close to a diffeomorphism in this sense, then it is itself a diffeomorphism (of $\mathrm{cl}(\Omega)$) into $\mathbb{C}^n$.

Thus the question of $F : \Omega \to \mathbb{C}^n$ being a diffeomorphism can be dealt with by showing that F extends C^∞ to $\mathrm{cl}(\Omega)$ and that $F : \mathrm{cl}(\Omega) \to \mathbb{C}^n$ is C^1 close to the G on B^n in the sense indicated.

At first sight this might seem difficult to establish: There are two direct approaches to the Bergman kernel. One is by its definition via the "reproducing property", that inner product with $K(z,w)$ gives the value at w for elements of $A^2(\Omega)$. The other is the formula for $K(z,w)$ in terms of an orthonormal basis for $A^2(\Omega)$. But neither of these seems amenable to producing information on the behavior of $K(z,w)$ with w fixed, z approaching the boundary. Interior behavior is more reasonably expected to be stable. (See Theorem 3.5.3, as well as Theorem 10.1.4.) But it turns out that the behavior of $K_\Omega(z,w)$, with w fixed in Ω, and z going to the boundary, can be effectively analyzed via the solution of the $\overline{\partial}$ problem as follows.

With $w \in \Omega$ fixed, let r be a positive number that is less than the distance of w to $\mathbb{C}^n \setminus \Omega$. Choose a nonnegative function $\rho : \mathbb{C}^n \to \mathbb{R}$ with $\rho(z)$ depending on $\|z\|$ only, and with $\rho(z) = 0$ if $\|z\| \geq r$ and finally with $\int_{\mathbb{C}^n} \rho(z)\, dV(z) = 1$. Then by the mean value property for each $f \in A^2(\Omega)$ this formula holds:

$$f(w) = \int_\Omega f(z)\rho(z-w)\, dV(z).$$

In particular, the reproducing (Bergman) kernel $K(z,w)$ with defining property

$$f(w) = \int_\Omega f(z)\overline{K(z,w)}\, dV(z)$$

is the L^2 projection of $\rho(z-w)$ onto $A^2(\Omega)$, with z being the variable and w fixed. This projection can be thought of as obtained via the solution of a $\overline{\partial}$ problem. first, let u be the solution (in $L^2(\Omega)$) of $\overline{\partial}u(z) = \overline{\partial}(\rho(z-w))$ which is orthogonal in $L^2(\Omega)$ to $A^2(\Omega)$. Then $K(z,w) = \rho(z-w) - u(z)$.

The solutions of $\overline{\partial}u = f$, where $\overline{\partial}f = 0$, with u orthogonal in $L^2(\Omega)$ to $A^2(\Omega)$, are of course the standard topics in the study of the $\overline{\partial}$-Neumann problem. In particular, in our case, when Ω is C^∞ close to B and hence strongly pseudoconvex, the indicated solution u of $\overline{\partial}u(z) = \overline{\partial}(\rho(z-w))$ is C^∞ on $\mathrm{cl}(\Omega)$. This is the usual smoothness-to-the-boundary result ([Folland/Kohn 1972]): note that $\overline{\partial}(\rho(z-w))$ is compactly supported in Ω and hence is itself obviously smooth on $\mathrm{cl}(\Omega)$.

Of course this method of finding $K(z,w)$ applies when $\Omega = B$ in particular. Thus the kind of C^1 closeness of $K_\Omega(z,w)$ to $K_B(z,w)$ that we are looking for can be considered from the viewpoint of the stability of the solution for the $\overline{\partial}$-Neumann problem under variation of the domain on which the solution is occurring. This stability seems eminently plausible. Indeed, it is assumed without further comment in Kohn's classic work on the $\overline{\partial}$-Neumann problem [Folland/Kohn 1972], where it is used to deduce the Newlander–Nirenberg theorem on integrable almost complex structures. But a completely explicit discussion of the stability issue can be found in [Greene/Krantz 1981], as part of the general discussion of the stability of the nondiagonal Bergman kernel and of the asymptotic expansion of the diagonal kernel function at the boundary.

There it is shown that, if Ω is sufficiently C^∞ close to a fixed, strongly pseudoconvex domain Ω_0, and if a $(0,1)$ form ω on $\mathrm{cl}(\Omega)$ with $\overline{\partial}\omega = 0$ is sufficiently C^∞ close to a (fixed) $(0,1)$ form ω_0 on $\mathrm{cl}(\Omega_0)$ with $\overline{\partial}\omega_0 = 0$, then the $\overline{\partial}$-Neumann solution of $\overline{\partial}u = \omega$ on Ω is C^∞ close on $\mathrm{cl}(\Omega)$ (i.e., in a given C^∞ neighborhood of) to the $\overline{\partial}$-Neumann solution of $\overline{\partial}u_0 = \omega_0$ on $\mathrm{cl}(\Omega_0)$. This is established via a detailed study of the standard proof of the regularity of the $\overline{\partial}$-Neumann problem.

This result implies the needed C^1 stability of Bergman representative coordinates to show that the Bergman map $F : \Omega \to \mathbb{C}^n$ via representative coordinates is a diffeomorphism. For Ω close to the unit ball and w close to 0, the $(0,1)$ form $\overline{\partial}_\Omega(\rho(z-w))$, w fixed, $\overline{\partial}$ calculated relative to z, is C^∞ close to $\overline{\partial}_{B^n}(\rho(z))$ if w is sufficiently close to 0. Our previous observation on the relationship between the $\overline{\partial}$ solution and the Bergman kernel implies that $K_\Omega(z,w)$ is uniformly C^∞ close to $K_{B^n}(z,w)$ for Ω which is C^∞ close to B^n and w in some fixed neighborhood of 0. Since $K_\Omega(z,w)$ is conjugate holomorphic in w, Cauchy estimates give that

$$\frac{\partial}{\partial \overline{w}} \log K_\Omega(w,w)\bigg|_{w=0} \quad \text{is uniformly close to} \quad \frac{\partial}{\partial \overline{w}} \log K_{B^n}(w,w)\bigg|_{w=0}$$

and that

$$\frac{\partial}{\partial \overline{w}} \log K_\Omega(z,w)\bigg|_{w=0} \quad \text{is } C^\infty \text{ close to} \quad \frac{\partial}{\partial \overline{w}} \log K_{B^n}(z,w)\bigg|_{w=0} \quad \text{on } \mathrm{cl}(\Omega).$$

Thus the Bergman representative coordinate map F_Ω for Ω at 0 is C^1 close to the Bergman representative coordinate map for the ball B^n, which is the identity (up to a constant factor). Thus the Bergman representative coordinate map F_Ω is a holomorphic diffeomorphism of $\mathrm{cl}(\Omega)$ into $\mathbb{C}^n$, and the proof of the theorem is complete. $\square$

The stability of the $\bar\partial$-Neumann solution under perturbation of the boundary of a strongly pseudoconvex bounded domain is a special case of a more general situation: Suppose that $\Omega_0 \cup \partial\Omega_0$ is a C^∞ manifold-with-boundary and that J_0 is an almost complex structure that is C^∞ on $\Omega_0 \cup \partial\Omega_0$ and integrable on Ω_0. In this situation, it makes sense to take as an hypothesis that $\partial\Omega_0$ is strongly pseudoconvex (cf. [Folland/Kohn 1972])—assume now that $\partial\Omega_0$ is indeed C^∞ strongly pseudoconvex. Suppose also that $\Omega_0 \cup \partial\Omega_0$ is given a C^∞ Hermitian metric. Then, if f is a C^∞ function on $\Omega_0 \cup \partial\Omega_0$, we may conclude that there is a unique function $u : \Omega_0 \to \mathbb{C}$ with $\bar\partial u = \bar\partial f$ on Ω and with u orthogonal to $A^2(\Omega)$ (in the inner product relative to the given Hermitian metric). Also u is C^∞ on $\Omega_0 \cup \partial\Omega_0$. [One can in fact so solve $\bar\partial u = \omega$, where ω is a $(0,1)$ form satisfying $\bar\partial\omega = 0$ and with ω having 0 harmonic representative. But the special situation where $\omega = \bar\partial f$, as indicated, suffices for our purposes, the harmonic representative being 0 following automatically in this instance.]

This setup has, as shown in [Greene/Krantz 1981] (and implied already in [Folland/Kohn 1972]), a stability similar to the stability associated to the stability under perturbation of a strongly pseudoconvex domain in $\mathbb{C}^n$ already discussed. first, let J be another almost complex structure on $\Omega_0 \cup \partial\Omega_0$ and let f be a C^∞ function on $\Omega_0 \cup \partial\Omega_0$ and J an almost complex structure tensor that is C^∞ close to J_0. If now f is C^∞ close to f_0 on $\Omega_0 \cup \partial\Omega_0$, then the $\bar\partial$-Neumann solution of $\bar\partial_J u = \bar\partial_J f$ is C^∞ close to the $\bar\partial$-Neumann solution of $\bar\partial_{J_0} u_0 = \bar\partial_{J_0} f_0$, provided that the $\bar\partial_J$ solution is determined for a J-Hermitian metric which is C^∞ close to the given J_0-Hermitian metric on $\Omega_0 \cup \partial\Omega_0$. This latter condition can always be arranged by setting $h =$ the J-symmetrization of h_0, i.e.,

$$h(\cdot, \cdot\cdot) = \frac{1}{2}\left(h_0(\cdot, \cdot\cdot) + h_0\left(J(\cdot), J(\cdot\cdot)\right)\right).$$

One could add into this picture the C^∞ perturbation of $\Omega_0 \cup \partial\Omega_0$ itself, but this would not actually increase the generality, since such a perturbation could be absorbed into perturbation of J_0 and f_0.

This more abstract form of $\bar\partial$-stability has an important application: it yields a proof of the perturbation result of Hamilton asserting that all perturbations of the complex structure of a bounded, strongly pseudoconvex domain can be realized by embedding ([Hamilton 1977]). This result was originally established by Hamilton using the Nash–Moser implicit function theorem. But the proof based on $\bar\partial$-stability in [Greene/Krantz 1981] is easier and more natural, and is also rather brief.

Theorem 4.3.2 ([Hamilton 1977]; cf. [Greene/Krantz 1981]). *If Ω_0 is a C^∞ bounded domain in $\mathbb{C}^n$ with strongly pseudoconvex boundary and if J is an almost complex structure defined and C^∞ on $cl(\Omega_0)$ which is integrable on Ω_0 and C^∞ close to the almost complex structure J_0 of $\mathbb{C}^n$ on $\Omega_0 \cup \partial\Omega_0$, then there is a domain Ω, C^∞ close to Ω_0 in the C^∞ topology on domains, such that (Ω_0, J) is biholomorphic to (Ω, J_0).*

In particular, every "abstract" perturbation of the ball is realized by a perturbation of the ball as a geometric object in $\mathbb{C}^n$.

Proof of Theorem 4.3.2. Let $f_1, \ldots, f_n$ be the coordinate functions on Ω_0, i.e.,

$$f_j(z) = \text{the } z_j \text{ coordinate function in } \mathbb{C}^n \text{ evaluated at the point } z.$$

Then $\overline{\partial}_{J_0} f_j \equiv 0$ for each $j = 1, \ldots, n$. If J is C^∞ close to J_0, then $\overline{\partial}_J f_j$ is C^∞ small on Ω_0. The stable $\overline{\partial}$ estimates then give that, if $\overline{\partial}_J u_j = \overline{\partial}_J f_j$ and u_j is the $\overline{\partial}_J$-Neumann solution of this equation, then each u_j is C^1 small in particular. [Here we use the construction described earlier for the automatic manufacture of a stably varying Hermitian metric for (Ω_0, J).] In particular, the n-tuple of functions $f_j - u_j$, $j = 1, \ldots, n$, gives a mapping which is C^1 close on $\Omega_0 \cup \partial\Omega_0$ to the mapping given by the f_js themselves, first the identity injection of Ω_0 into $\mathbb{C}^n$. In particular, the $f_j - u_j$, $j = 1, \ldots, n$, are coordinates of a diffeomorphism of $\Omega_0 \cup \partial\Omega_0$ onto an open set with smooth boundary in $\mathbb{C}^n$, by the C^1 stability of diffeomorphisms.

But the function $f_j - u_j$, each j, is J-holomorphic since $\overline{\partial}_J(f_j - u_j) = \overline{\partial}_J f_j - \overline{\partial}_J u_j \equiv 0$ on Ω_0. $\square$

The idea of this last proof was originally proposed by M. Kuranishi and communicated to the first author (Greene) by J. Eells (private communication).

The uniqueness of the $\overline{\partial}$-Neumann solution, once a Hermitian metric is chosen, together with the proof method just used, makes possible an equivariant extension of Hamilton's embedding theorem. This result generalizes Theorem 4.3.1 to cases where equivariant embedding via Bergman representative coordinates cannot in general be obtained.

Theorem 4.3.3 ([Greene/Krantz 1982]). *Suppose that Ω_0 is a C^∞ strongly pseudoconvex domain in $\mathbb{C}^n$ and that G is a compact subgroup of* $\mathrm{Aut}\,(\Omega_0)$. *Suppose further that Ω_0 is equivariantly embedded for G in the sense that G acts on Ω_0 as the restrictions of holomorphic isometries of $\mathbb{C}^n$. Let J be an almost complex structure on $\Omega_0 \cup \partial\Omega_0$ which is integrable on Ω_0 and is C^∞ close to the $\mathbb{C}^n$ complex structure J_0 on $\Omega_0 \cup \partial\Omega_0$ and let $\Gamma : G \times \Omega_0 \to \Omega_0$ be a G-action on Ω_0 which is J-holomorphic and C^∞ close to the original G-action on Ω_0. Then there is a diffeomorphism $F : \Omega_0 \cup \partial\Omega_0 \to \mathbb{C}^n$ such that:*

(1) *The mapping F is holomorphic as a map from (Ω_0, J) to $(\mathbb{C}^n, J_0)$.*
(2) *The mapping F is C^∞ close to the injection of Ω_0 into $\mathbb{C}^n$.*
(3) *The mapping $F \circ \Gamma \circ F^{-1}$, which is the G-action on $F(\Omega_0)$, is the restriction to $F(\Omega_0)$ of a G-action on $\mathbb{C}^n$ by holomorphic isometries of $\mathbb{C}^n$.*

(4) *The G-action on $\mathbb{C}^n$ given in* **(3)** *is C^∞ close to the original G-action on $\mathbb{C}^n$ attached to that equivariant embedding of Ω_0.*

Proof (outline). Let h_0 be the $\mathbb{C}^n$ Hermitian metric restricted to Ω_0 so that h_0 is invariant under the original G-action, say $\Gamma_0 \times \Omega_0 \to \mathbb{C}^n$, on Ω_0. Since Γ is C^∞ close to this original G-action, the average $\widehat{h}$ of h_0 with respect to the Γ-action is C^∞ close to h_0. Note that this is also C^∞ close to h_0 since Γ is C^∞ close to an action isometric for h_0. Observe further that $\widehat{h}$ may not be J-Hermitian, even though Γ acts by J-holomorphic maps, since h_0 is likely not J-Hermitian. but the J-symmetrization of $\widehat{h}$ already discussed, call it h, *is* J-Hermitian, and it is C^∞ close to h_0 since J is C^∞ close to J_0. This metric h is thus Γ-invariant, J-Hermitian, and C^∞ close to h_0.

Now let $f_1, \ldots, f_n$ be the coordinate functions on Ω_0 so that G acts linearly on them, if we choose a suitable new origin in $\mathbb{C}^n$ (a compact group of isometries of $\mathbb{C}^n$ has a fixed point and we choose such a fixed point as origin). Let u_j be the $\overline{\partial}$-Neumann solution of $\overline{\partial}_J u_j = \overline{\partial}_J f_j$ determined by the Γ-invariant metric h. Since Γ acts almost linearly on the f_js, the mapping Γ acts almost linearly on the u_j s as well, because the $\overline{\partial}_J$ solution process is Γ-invariant. So Γ acts almost linearly on the holomorphic functions $f_j - u_j$ which, moreover, determine an embedding of $\Omega_0 \cup \partial\Omega_0$.

A standard process of making an almost-linear action linear, which will preserve J-holomorphicity, completes the construction of the desired equivariant J-holomorphic embedding. [The process involves replacing the functions $F_j = f_j - u_j$ by functions, which are C^∞ close, defined by

$$\big(\widehat{F}_1(z), \ldots, \widehat{F}_n(z)\big) = \int_G \Gamma_0\big(g^{-1}, (F_1(gz), \ldots, F_n(gz))\big)\, dg,$$

where $\int_G$ is the invariant (Haar) integral with total measure 1.][4] □

4.4 Semicontinuity of Automorphism Groups

Symmetry is easily destroyed but not so easily created. To make the straight crooked requires only an arbitrarily small effort, while to make the crooked straight requires a definite action.

These intuitions, that symmetry is unstable but an increase in symmetry requires a substantial change, holds with precision in a variety of circumstances. The goal of this section is a result of this type for the automorphism groups of C^∞ strongly pseudoconvex domains. This result will depend for its

[4] The reader unfamiliar with this process of converting close-to-linear to actually linear actions by way of re-embedding might find it instructive to consider the example in which G is the two-element group $\{1, g\}$ and $F(g(z))$ is close to $-F(z)$. Then the map $\widehat{F}$ defined by $z \mapsto [1/2](F(z) - F(g(z)))$ satisfies precisely $\widehat{F}(g(z)) = -F(z)$ so that G acts linearly indeed on the $\widehat{F}$ embedding, which really is an embedding since $\widehat{F}$ is in fact close to F.

proof on a theorem similar in spirit concerning compact Riemannian manifolds ([Ebin 1968]).

Theorem 4.4.1 (Ebin). *If (M, g_0) is a C^∞ compact Riemannian manifold, then there is a neighborhood $\mathcal{G}$ of g_0 in the C^∞ topology on C^∞ Riemannian metrics such that: If $g \in \mathcal{G}$ then there is a diffeomorphism $F : M \to M$ (C^∞ close to the identity) such that the set*

$$\left\{F \circ \alpha \circ F^{-1} : \ \alpha : M \to M \text{ is an isometry for } g\right\}$$

is a subset of, and hence a subgroup of

$$\{\beta : \ \beta : M \to M \text{ is an isometry for } g_0\} .$$

In particular, the group of isometries of M relative to g is isomorphic to a subgroup of the group of isometries of g_0.

Ebin's original proof of the theorem just stated involved infinite-dimensional manifolds and the construction of "slices" in the Lie group sense for the action of the diffeomorphism group on the manifold M. However, the result can in fact be established by finite-dimensional methods and ordinary Lie group theory. We outline the argument now.

Let

$$V_\Lambda = \text{the finite-dimensional linear span of all eigenfunctions of the Laplacian for } g_0 \text{with eigenvalues} < \Lambda.$$

[We use here the differential geometer's Laplacian $-\sum_j \partial^2/\partial x_j^2$ at the center of a geodesic normal coordinate system, so that the spectrum of the Laplacian is nonnegative.] If we equip V_Λ with the standard L^2 inner product on functions determined by the measure M for g_0, then the compact group of isometries for g_0 acts on V_Λ orthogonally. Moreover, if we choose an orthonormal basis $f_1, \ldots, f_N$ for V_Λ, then the map $E_0 : M \to \mathbb{R}^N$ defined by

$$M \ni p \mapsto (f_1(p), \ldots f_N(p))$$

is an embedding if Λ is chosen sufficiently large. This is an historic theorem of S. Bochner ([Bochner 1937], cf. [Greene/Wu 1975a] and [Greene/Wu 1975b] for a contemporary context and the noncompact manifold situation). With Λ so chosen, the embedding E_0 is equivariant in the sense that there is an injective homomorphism H_0 : [Isometry group of g_0] $\to O(N)$ such that, for each isometry α of g_0 and $p \in M$, $H_0(\alpha)$ applied to $E_0(p)$ equals $E_0(\alpha(p))$.

Now assume further that Λ is not in the spectrum of the Laplacian Δ_0 of g_0: this choice of course is possible consistently with the sufficient largeness of Λ of the previous paragraph, since the spectrum of Δ_0 is discrete. With Λ thus chosen, both sufficiently large and not in the spectrum of Δ_0, there is a "spectral stability" property of the equivariant embedding situation as follows.

Let g_j, $j = 1, 2, 3, \ldots$ be a sequence of C^∞ Riemannian metrics converging to g_0 in the C^∞ topology. Let $V_{\Lambda,j}$ = (the span of the eigenfunctions for the g_j-Laplacian Δ_j with eigenvalues $< \Lambda$). Then, for all j sufficiently large, the dimension of the finite-dimensional vector space $V_{\Lambda,j}$ = the dimension N of the space V_Λ defined earlier. Moreover, again for each j sufficiently large, there is a basis $(f_1^j, \ldots, f_N^j)$ for $V_{\Lambda,j}$, orthogonal with respect to the g_j-measure on M. These bases can be chosen so that, for each fixed $k \in \{1, \ldots, N\}$, the function f_k^j, $j = 1, 2, 3, \ldots$ converges to the function f_k in the C^∞ topology. This "spectral stability" result is part of the perturbation theory of linear operators; it is proved in detail in Kato's well-known book [Kato 1966] on that subject. [At first sight, these spectral stability results seem not only appealing but almost obvious, since the eigenfunctions of Δ_j are competitors, after suitable correction, for the minimization of Dirichlet integrals—the Rayleigh method—that gives eigenfunctions of Δ. But subtleties arise in any attempt to reason in the opposite direction, to control the eigenfunctions of Δ_j from those of Δ. These difficulties are treated in [Kato 1966] by the method of resolvents.]

From this we obtain embeddings $E_j : M \to \mathbb{R}^N$, for each j sufficiently large, which are equivariant for the isometry group of g_j. Moreover, the E_j's as constructed converge to E_0 in the C^∞ topology.

Let G_0 (the isometry group of g_0) be equal to the subgroup of $O(N)$ obtained by the equivariant embedding E_0, and G_j = the subgroup arising in the same way from the isometry group of g_j and the equivariant embedding E_j.

Now, for any sequence $\{\alpha_j : M \to M\}$ such that α_j is an isometry of g_j for each $j = 1, 2, 3, \ldots$, there is a subsequence $\{\alpha_{j_k}\}$ which converges in the C^∞ topology to an isometry of g_0: this follows from a standard normal families argument. [Convergence to a g_0-distance-preserving map is immediate, and the limit must be a C^∞ isometry for g_0 by the Myers–Steenrod theorem [Myers/Steenrod 1939]. That the convergence is then in the C^∞ topology is a matter of standard differential geometry, using the facts that the isometries are determined by a single point image and differential at that point and that geodesics, which are preserved, depend C^∞ on the metric.] Thus, combining this with the C^∞ convergence of the E_j to E_0, we obtain the following.

If $\mathcal{U}$ is a neighborhood in $O(N)$ of G_0, then $G_j \subset \mathcal{U}$ for all j sufficiently large. By a standard result in Lie group theory ([Montgomery/Zippin 1942]), G_j is isomorphic to a subgroup of G_0 for each j sufficiently large, and this isomorphism is given by conjugation by an element A_j of $O(N)$. Here the A_j's can be taken to converge to the identity. Modifying the E_j's themselves by conjugation, we can assume that the A_j's are equal to the identity and $G_j \subset G_0$. Since E_j and E_0 are equivariant embeddings into $O(N)$, the desired diffeomorphism of M to M (to conjugate isometries of g_j into isometries of g_0) can be obtained by sending $p \in M$ to the $\mathbb{R}^N$-closest point to $E_j(p)$ in $E_0(M)$. □

The possibility of averaging over compact groups gives a useful corollary about group actions as such. For the statement of the corollary, we say that a sequence of C^∞ group actions $G_j \times M \to M$ *sub-converges* in the C^∞ topology to an action $G_0 \times M \to M$ if every sequence α_j of G_j-action elements has a subsequence α_{j_k} which converges in the C^∞ topology to a G_0-action element.

Corollary 4.4.2. *If $G_j \times M \to M$ is a sequence of actions on a compact manifold M by compact Lie groups G_j and if the G_j-actions sub-converge in the C^∞ topology to a compact Lie group action $G_0 \times M \to M$, then for all j sufficiently large, there is a diffeomorphism $F_j : M \to M$ such that the conjugation by F_j of the G_j-action is a subgroup of the G_0-action. Moreover, the F_j may be chosen to converge to the identity map of M in the C^∞ topology.*

This corollary follows from the proof of Ebin's theorem (Theorem 4.4.1) by averaging a fixed Riemannian metric over the group actions to produce G_j-invariant metrics g_j converging in C^∞ topology to a G_0-invariant metric g_0.

Generically, that is for a dense open set of metrics, the isometry group is in fact the identity alone (see [Ebin 1968]). Our interest here, however, is in the metrics which have a nontrivial isometry group.

The main goal of this section is to prove the statement analogous to Ebin's theorem (Theorem 4.4.1) for C^∞, strongly pseudoconvex domains.

Theorem 4.4.3 ([Greene/Krantz 1982a]). *If Ω_0 is a bounded, C^∞, strongly pseudoconvex domain in $\mathbb{C}^n$ that is not biholomorphic to the ball, then there is a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology (on bounded domains with C^∞ boundary) such that, if $\Omega \in \mathcal{U}$, then there is a real diffeomorphism $F : \Omega \to \Omega_0$ such that F is C^∞ close to the identity and*

$$\left\{F \circ \alpha \circ F^{-1} : \alpha \in Aut(\Omega)\right\} \subset \mathrm{Aut}\,(\Omega_0).$$

In particular, $Aut(\Omega)$ is isomorphic to a subgroup of $Aut(\Omega_0)$.

The essential idea of the proof of this theorem is to note, from Lu Qi-Keng's theorem (Theorem 4.2.2), that the Bergman metric of Ω_0 does not have constant holomorphic sectional curvature, while at the same time the holomorphic sectional curvature is asymptotically constant at the boundary. So far, this is just a recapitulation of the curvature proof of Bun Wong's theorem (Corollary 3.4.4, Theorem 9.2.1). Noting further that these curvature estimates are stable under C^∞ perturbations of $\partial\Omega_0$, one expects to find that the smooth extension to the closure $\mathrm{cl}(\Omega_0)$ of $\mathrm{Aut}\,(\Omega_0)$, guaranteed by Fefferman's result on smoothness to the boundary ([Fefferman 1974]) will also be stable under perturbation of $\partial\Omega_0$ in the following sense: If Ω is C^∞ close to Ω_0, then $\mathrm{Aut}\,(\Omega)$ on $\mathrm{cl}(\Omega)$ is C^∞ close to $\mathrm{Aut}\,(\Omega_0)$ on $\mathrm{cl}(\Omega_0)$ in the sense that each element of $\mathrm{Aut}\,(\Omega)$ belongs to some pre-chosen C^∞ neighborhood of $\mathrm{Aut}\,(\Omega)$ on $\mathrm{cl}(\Omega_0)$. Of course $\mathrm{cl}(\Omega_0)$ is a compact manifold with boundary so that Ebin's theorem (Theorem 4.4.1) as just stated and proved (for manifolds without boundary) does not apply as such. But, by passing to the "metric double" and introducing suitable automorphism-invariant metrics, we can apply Ebin's theorem on manifolds without boundary. We now turn to a more detailed version of the outline just given.

The detailed proof will be based on two propositions:

Proposition 4.4.4. *If Ω_0 is a C^∞ strongly pseudoconvex domain and if Ω_0 is not biholomorphic to the ball, then there are a point p in Ω_0, a compact set $K_0 \subset \Omega_0$, and a C^∞ neighborhood $\mathcal{V}$ of Ω_0 in the C^∞ topology on domains such that, if $\Omega \in \mathcal{V}$, then $\Omega \supset K_0 \cup \{p\}$ and the* $\mathrm{Aut}\,(\Omega)$*-orbit of p lies in K_0.*

This proposition has already been in effect established and is restated here only for convenience and clarity.

Proposition 4.4.5. *If Ω_0 is a C^∞ strongly pseudoconvex domain not biholomorphic to the unit ball then, for each $\ell = 1, 2, \ldots$, there are a C^∞ neighborhood $\mathcal{V}$ of Ω_0 and a positive constant C_ℓ such that, for each $\Omega \in \mathcal{V}$ and each $f \in \mathrm{Aut}\,(\Omega)$, the Euclidean derivatives of order $\leq \ell$ of f at points $p \in \Omega$ have absolute value $\leq C_\ell$.*

For brevity, we shall summarize this last statement by saying that

> The derivatives of order $\leq \ell$ of elements in $\mathrm{Aut}\,(\Omega)$ are stably uniformly bounded.

(where "stably" refers to variation of Ω near Ω_0 and "uniformly" refers to variation over the points of the domain Ω).

This proposition, which is in effect a stable version of the smoothness-to-the-boundary theorem by Fefferman, will be established later.

Armed with these propositions, we can now establish the following lemma of normal families type.

Lemma 4.4.6. *If Ω_j, $j = 1, 2, \ldots$, converge in the C^∞ topology to Ω_0 (with Ω_0 being C^∞, strongly pseudoconvex, and not biholomorphic to the ball), and if $g_j \in \mathrm{Aut}\,(\Omega_j)$, then there are subsequences Ω_{j_k}, g_{j_k}, $k = 1, 2, \ldots$, such that g_{j_k} converges in the C^∞ topology to an element $g_0 \in \mathrm{Aut}\,(\Omega_0)$.*

See the definition in Section 3.5 for C^∞ topology on the collection of domains in $\mathbb{C}^n$. Hereinafter, we write $G_j = \mathrm{Aut}\,(\Omega_j)$ and $G_0 = \mathrm{Aut}\,(\Omega_0)$. The lemma then says in effect that, for j large, the action of each element of G_j is close to the action of an element of G_0.

Proof of the lemma. Fix a point p and a compact set K_0 as in Proposition 4.4.4. Then, for j large, $g_j(p) \in K_0 \subset \Omega_j$. By normal families, there is a subsequence g_{j_k} which converges uniformly on each compact subset of Ω_0, and the limit of this subsequence is an element g_0 of G_0 (this follows from a straightforward modification of Theorem 1.3.4). Proposition 4.4.5 then implies the C^∞ convergence of $\{g_{j_k}\}$ on $\mathrm{cl}(\Omega_{j_k})$ (respectively to g_0 on $\mathrm{cl}(\Omega_0)$).

To check this last assertion in detail, it suffices to show that $\{g_{j_k}\}$ on $\mathrm{cl}(\Omega_{j_k})$ is a Cauchy sequence in the $C^{\ell+1}$ norm for each fixed $\ell = 1, 2, \ldots$. For this, suppose that $\epsilon > 0$ is given. Choose a compact set $K \subset \Omega_0$ such that, for all Ω which are C^∞ close enough to Ω_0 and $x \in \partial\Omega$, there is a polygonal arc in Ω, of length not exceeding $\epsilon/[3C_{\ell+1}]$, from some point $s \in K$ to the point x. [Here $C_{\ell+1}$ is the constant from Proposition 4.4.5.] The possibility of choosing K in this fashion is elementary: Simply let the set K be the $\epsilon/[4C_\ell]$ normal "push-in" of Ω_0.

Now choose k_0 so large that (from Cauchy estimates), $g_{j_{k_1}} - g_0$ and $g_{j_{k_2}} - g_0$ have C^ℓ-norm on K bounded above by $\epsilon/3$ if $k_1, k_2 \geq k_0$. For such k_1, k_2, the C^ℓ-norm of the difference $g_{j_{k_1}} - g_{j_{k_2}}$ is $\leq \epsilon$ on $\mathrm{cl}(\Omega_{k_1})$, $\mathrm{cl}(\Omega_{k_2})$ provided that k_1, k_2 are also required to be so large that Ω_{k_1}, Ω_{k_2} are sufficiently C^∞ close to Ω_0 and hence to each other. $\square$

Lemma 4.4.7. *There is a neighborhood $\mathcal{V}$ of Ω_0 in the C^∞ topology on domains and a family g_Ω, $\Omega \in \mathcal{V}$, with g_Ω a C^∞ Riemannian metric on cl(Ω) such that,* **(1)** *if* $\mathrm{Aut}\,(\Omega)$ *acts isometrically on g_Ω and* **(2)** *if $\{\Omega_j\}$ is a sequence in $\mathcal{V}$ converging C^∞ to Ω_0, then $\{g_{\Omega_j}\}$ converges C^∞ to g_{Ω_0}.*

Proof. Set g_{Ω_0} equal to the average with respect to $\mathrm{Aut}\,(\Omega_0)$ of the Euclidean metric on $\mathrm{cl}(\Omega_0)$. For each $\Omega \neq \Omega_0$, choose diffeomorphisms $F_\Omega : \mathrm{cl}(\Omega) \to \mathrm{cl}(\Omega_0)$ such that F_Ω converges as Ω tends to Ω_0 in the C^∞ topology. Set g_Ω equal to the average over the compact (for $\mathcal{V}$ small enough) group $\mathrm{Aut}\,(\Omega)$ of the pullback metric $F_\Omega^* g_{\Omega_0}$. By Lemma 4.4.6, each element of $\mathrm{Aut}\,(\Omega)$ acts nearly isometrically on $F_\Omega^* g_{\Omega_0}$, in the C^∞ sense of "nearly," on $\mathrm{cl}(\Omega)$. This is because g_{Ω_0} is $\mathrm{Aut}\,(\Omega_0)$-invariant and each element of $\mathrm{Aut}\,(\Omega)$ is C^∞ close to an element of $\mathrm{Aut}\,(\Omega_0)$. The conclusion of the lemma concerning convergence follows. $\square$

Lemma 4.4.8. *The metrics g_Ω in Lemma 4.4.7 can be chosen to be product metrics near the boundary.*

Here "the product metric" near the boundary of Ω means precisely that, for each boundary point x of $\mathrm{cl}(\Omega)$, there is a real local coordinate system $(x_1, x_2, \ldots, x_{2n})$ in a neighborhood of x with

- *the boundary* $\mathrm{cl}(\Omega) \setminus \Omega$ equaling $\{(x_1, x_2, \ldots, x_{2n-1}, 0)\}$;
- the points of Ω in the neighborhood of x satisfying $x_{2n} < 0$ (and vice versa);
- the metric in the given neighborhood having at $(x_1, x_2, \ldots, x_{2n})$ the form

$$dx_{2n}^2 + \big(\text{a positive definite quadratic formin } dx_1, dx_2, \ldots, dx_{2n-1} \\ \text{with coefficients depending only on } (x_1, x_2, \ldots, x_{2n-1})\big).$$

Proof of the lemma. An $\mathrm{Aut}\,(\Omega)$ product metric of this sort at and near the boundary is easily obtained using the map

$$\partial\Omega \times [0, \delta) \to \Omega$$

defined by

$$(b, t) \mapsto \exp_p(tN),$$

where N is the inward-pointing normal at b relative to the previous g_Ω-metric and $\exp_p$ is the g_Ω-exponential map. Choose δ so small that the map is a diffeomorphism and define the metric by declaring this diffeomorphism to be isometric for (the metric on $\partial\Omega$)$+dt^2$. This construction is $\operatorname{Aut}(\Omega)$-invariant. Using an $\operatorname{Aut}(\Omega)$-invariant partition of unity to make a transition to the previous g_Ω will provide all properties: the partition of unity function is taken to depend only on the t variable. $\square$

The proof of Theorem 4.4.3 can now be completed as follows: With the metrics g_Ω chosen as in Lemma 4.4.8, in particular as product metrics near the boundary, we form compact Riemannian manifolds $(\widehat{\Omega}, \widehat{g}_\Omega)$ by taking $\widehat{\Omega}$ to be the manifold "double" of Ω and $\widehat{g}_\Omega$ to be the natural metric on $\widehat{\Omega}$, equal to g_Ω on each copy of Ω and fitting together to form a C^∞ metric across the (one copy of) $\partial\Omega$ on account of the product metric. Let G_Ω be the group generated by $\operatorname{Aut}(\Omega)$ and the interchange operation I_Ω that interchanges the two copies of Ω that are "glued" to form $\widehat{\Omega}$. We now apply Ebin's theorem (Theorem 4.4.1) to deduce that the isometry group of $\widehat{\Omega}$ is diffeomorphism-conjugate (via a diffeomorphism close to the identity) to a subgroup H_Ω of the isometry group of $\widehat{\Omega}_0$. Now, by our previous analysis via normal families, H_Ω lies in a small neighborhood of G_{Ω_0} in the isometry group of $\widehat{\Omega}_0$. This isometry group is a compact Lie group and G_{Ω_0} is a compact, hence closed, subgroup and H_Ω is also compact and therefore closed. Standard Lie group theory yields that H_Ω is conjugate to a subgroup of $G_{\widehat{\Omega}_0}$ by way of an isometry of $\widehat{\Omega}_0$ close to the identity. Thus the diffeomorphism conjugation together with this second conjugation gives a close-to-the-identity diffeomorphism $F : \Omega \to \Omega_0$ conjugating $G_{\widehat{\Omega}}$ to $G_{\widehat{\Omega}_0}$.

Now $G_{\widehat{\Omega}_0}$ contains I_{Ω_0}. Also, the only possible fixed points of an element of $G_{\widehat{\Omega}}$ that is not preserving each copy of Ω are lying in $\partial\Omega$. It follows that F in fact maps $\partial\Omega$ diffeomorphically to $\partial\Omega$, and thus F, being close to the identity, maps Ω to Ω_0. As a result,

$$F\big|_{\mathrm{cl}(\Omega)} : \mathrm{cl}(\Omega) \to \mathrm{cl}(\Omega_0)$$

is the conjugating diffeomorphism called for in the theorem. $\square$

The reader with a mind towards maximum generality will have noticed that complex analysis really played no role in the latter part of this proof. In particular, the proof technique gives rise to the following results.

Theorem 4.4.9 (Ebin's Theorem for Manifolds with Boundary). *If (M, g_0) is a compact, C^∞ Riemannian manifold with boundary, then there is a neighborhood $\mathcal{U}$ of g_0 in the C^∞ topology on Riemannian metrics such that, for each $g \in \mathcal{U}$, there is a diffeomorphism $F : M \to M$ (which can be chosen to be C^∞ close to the identity) such that, for each g-isometry $f : M \to M$, the mapping $F^{-1} \circ f \circ F$ is a g_0-isometry.*

Theorem 4.4.10. *If G_0 is a compact subgroup of the diffeomorphism group of a compact manifold (possibly with boundary), then there is a neighborhood $\mathcal{V}$ of G_0 in the C^∞ topology on the diffeomorphism group such that every compact subgroup G of the diffeomorphism group, with $G \subset \mathcal{V}$, is conjugate to* *a subgroup of* G_0 *via a diffeomorphism (which may be taken* C^∞ *close to the identity).*

The proofs of these results are obtained by extracting suitable portions of the proof of Theorem 4.4.3.

4.5 Obtaining a Stable Extension

Let K be a compact subset of Ω_0. Let ℓ be a positive integer. The Cauchy estimates then imply that there is a constant $C > 0$ such that

$$\left|\nabla^j \alpha(z)\right| \leq C$$

for all $\alpha \in \operatorname{Aut}(\Omega_0)$ and all $z \in K$. Thus the essential point in establishing Proposition 4.4.4 is to consider points near the boundary of Ω_0.

Lemma 4.5.1. *Let* $\epsilon > 0$ *be a positive number. Then*

$$\inf\{\mathit{dis}(\alpha(q), \partial\Omega_0) : \alpha \in \operatorname{Aut}(\Omega_0), q \in \Omega_0, \mathit{dis}(q, \partial\Omega_0) \geq \epsilon\}$$

is a positive number. [Here, as usual, dis denotes Euclidean distance.]

Proof. Suppose the contrary. Then there are a sequence $\{q_j\}$ of points in Ω_0 with $\operatorname{dis}(q_j, \partial\Omega_0) \geq \epsilon$ and a sequence of automorphisms $\alpha_j \in \operatorname{Aut}(\Omega_0)$ with

$$\lim_{j\to\infty} \operatorname{dis}(\alpha_j(q_j), \partial\Omega_0) = 0.$$

The sequence $\{\alpha_j\}$ is a normal family. By reasoning that has already been explained in detail, there is a subsequence $\{\alpha_{j_k}\}$ that converges normally to an automorphism $\alpha_0 \in \operatorname{Aut}(\Omega_0)$. Passing again to a subsequence, we may assume that $\{q_{j_k}\}$ converges to a point $q_0 \in \overline{\Omega_0}$.

But clearly $\operatorname{dis}(q_0, \partial\Omega_0) \geq \epsilon$, so q_0 actually lies in Ω_0 itself. As a result, $\alpha_0(q_0)$ is in Ω_0. But $\alpha_0(q_0)$ is the limit of the sequence $\alpha_{j_k}(q_{j_k})$ and also $\lim_{k\to\infty} \operatorname{dis}(\alpha_{j_k}(q_{j_k}), \partial\Omega_0) = 0$. In conclusion, $\operatorname{dis}(\alpha_0(q_0), \partial\Omega_0) = 0$ (since the distance function is continuous). This last statement contradicts the fact that $\alpha_0(q_0)$ lies in the interior of Ω_0. That is a contradiction. □

Lemma 4.5.2. *If* ϵ *is a positive number, then there is a* $\delta > 0$ *such that*

$$\sup\{\mathit{dis}(\alpha(q), \partial\Omega_0) : \alpha \in \operatorname{Aut}(\Omega_0), q \in \Omega_0, \mathit{dis}(q, \partial\Omega_0) \leq \delta\} < \epsilon.$$

Proof. The proof is similar to that of the last lemma, with a normal families argument now being applied to the inverses of the automorphisms. The details are left to the reader. □

Lemma 4.5.3. *Let* Ω_0 *be a strongly pseudoconvex domain with* C^∞ *boundary. Fix a point* $p_0 \in \partial\Omega_0$*. Then there are numbers* $\epsilon, \eta > 0$ *such that if* $z, w \in \Omega_0$*,* $\mathit{dis}(z, w) < \epsilon$*, and* $\mathit{dis}(w, p_0) < \epsilon$*, then* $|K_{\Omega_0}(z, w)| \geq \eta$*.*

Proof. This is an immediate consequence of the Fefferman asymptotic expansion (3.4) in Section 3.4. The details are again left to the reader. □

In the next lemma $\mathcal{J}_\Phi(z)$ denotes the complex Jacobian determinant of the mapping Φ at the point z.

Lemma 4.5.4. *If Ω_0 is a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$, then there is a constant $C > 0$ such that*

$$\sup\{|\mathcal{J}_\alpha(z)| : \alpha \in \mathrm{Aut}\,(\Omega_0), z \in \Omega_0\} \le C$$

and

$$\inf\{|\mathcal{J}_\alpha(z)| : \alpha \in \mathrm{Aut}\,(\Omega_0), z \in \Omega_0\} \ge C^{-1}.$$

Proof. The first estimate follows from the second by applying the result to α^{-1}. So we concentrate on the second.

Suppose that no such C exists. Then there are a sequence of automorphisms $\alpha_j \in \mathrm{Aut}\,(\Omega_0)$ and a sequence of points $q_j \in \Omega_0$ such that $\lim_{j\to\infty} \mathcal{J}_{\alpha_j}(q_j) = 0$. Passing to a subsequence if necessary, we may assume that the q_j converge to a point $q_0 \in \overline{\Omega_0}$.

We claim that $q_0 \in \partial\Omega_0$. For, if it were the case that $q_0 \in \Omega_0$, then Lemma 4.5.1 tells us that $\{\alpha_j(q_j)\}$ is bounded away from $\partial\Omega_0$. Hence, by the Cauchy estimates, $\{|\mathcal{J}_{\alpha_j^{-1}}(\alpha_j(q_j))|\}$ is bounded as $j \to +\infty$. This last is impossible since $\mathcal{J}_{\alpha_j^{-1}}(\alpha_j(q_j)) = 1/\mathcal{J}_{\alpha_j}(q_j)$ and $\lim \mathcal{J}_{\alpha_j}(q_j) = 0$.

So $q_0 \in \partial\Omega_0$, and there are, by Lemma 4.5.3, positive numbers ϵ and η such that $|K_{\Omega_0}(z,w)| \ge \eta$ if $z, w \in \Omega_0$ are within distance η of q_0. Therefore $|K_{\Omega_0}(q_0, r_0)| \ge \eta$ for any $r_0 \in \Omega_0$ with $\mathrm{dis}(q_0, r_0) < \epsilon$. Choose a fixed such r_0. It follow from Lemma 4.5.1 that $\liminf_{j\to\infty} \mathrm{dis}(\alpha_j(r_0), \partial\Omega_0) > 0$. Then, by the Cauchy estimates, it follows that $\limsup_{j\to\infty} |\mathcal{J}_{\alpha_j}(r_0)|$ is finite. But we also know that $\limsup_{j\to\infty} |K_{\Omega_0}(\alpha_j(q_j), \alpha_j(r_0))|$ is finite.

Now $K_{\Omega_0}(q_j, r_0) = \mathcal{J}_{\alpha_j}(q_j)\overline{\mathcal{J}_{\alpha_j}(r_0)}K_{\Omega_0}(\alpha_j(q_j), \alpha_j(r_0))$. Since $\lim_{j\to\infty} \mathcal{J}_{\alpha_j}(q_j) = 0$, the finiteness of the two limits-suprema just established now implies that $\lim K_{\Omega_0}(q_j, r_0) = 0$. But $\lim_{j\to\infty} K_{\Omega_0}(q_j, r_0) = K_{\Omega_0}(q_0, r_0) \ne 0$. This contradiction completes the proof of the lemma. □

Lemma 4.5.5. *If Ω_0 is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary, then there exist $\epsilon, \eta > 0$ such that: If $w \in \Omega_0$ and dis$(w, \partial\Omega_0) <$ ϵ and if $z \in \Omega_0$ and dis$(z, w) < [3/2]\,$dis$(w, \partial\Omega_0)$, then $|K_{\Omega_0}(z,w)| \ge \eta$ and $|det(\partial b_{i,w}/\partial z_j)| \ge \eta$, where the determinant is that of the complex Jacobian of the Bergman representative coordinate map $(b_{1,w}, \ldots, b_{n,w})$ at w.*

Proof. The basic bound $|K_{\Omega_0}(z,w)| \ge \eta$ can be deduced from Lemma 4.5.3 by a compactness argument. For the moment, it guarantees that the functions $b_{i,w}(z)$ are in fact defined for the z-values in question.

In order to study the Jacobian determinant $\det(\partial b_{i,w}/\partial z_j)$, notice first that

$$\begin{aligned}\frac{\partial}{\partial z_j} b_{i,w} &= \frac{\partial^2}{\partial z_j \partial \overline{w}_i}\left[\log \frac{K_{\Omega_0}(z,w)}{K_{\Omega_0}(w,w)}\right] \\ &= \frac{\partial^2}{\partial z_j \partial \overline{w}_i}[\log(K_{\Omega_0}(z,w)],\end{aligned}$$

because the expression $K_{\Omega_0}(w,w)$ has no z-dependence. Thus the relevant quantities can be calculated by substituting the asymptotic expansion for $K_{\Omega_0}(z,w)$ into the formula given. The following version of this substitution, and the subsequent calculations, is motivated by the somewhat simpler calculation when Ω_0 is the unit ball.

In order to calculate the boundary behavior of $\det[\partial b_{i,w}/\partial z_j]$ for a general strongly pseudoconvex domain Ω_0, and thus to complete the proof of Lemma 4.5.5, we shall use some standard notation as follows.

- $\psi : \mathbb{C}^n \to \mathbb{R}$ is a C^∞ function such that $\Omega_0 = \{z \in \mathbb{C}^n : \psi(z) > 0\}$ and $\nabla\psi$ is nonzero at every point of $\partial\Omega_0$,
- $X(z,w)$ represents the "Levi polynomial" of ψ, first,

$$\begin{aligned}X(z,w) := \psi(w) &+ \sum_{j=1}^{n} \frac{\partial \psi}{\partial w_j}\bigg|_w (z_j - w_j) \\ &+ \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial w_j \partial w_k}\bigg|_w (z_j - w_j)(z_k - w_k),\end{aligned}$$

  and
- $\delta(w) := \operatorname{dis}(w, \partial\Omega_0)$.

Let $p_0 \in \partial\Omega_0$. For the moment, we restrict ourselves to the situation that $z, w \in \Omega$ satisfy:

$$|w - p_0| < \epsilon$$

and

$$|z - w| < \frac{3}{2}\delta(w).$$

Note that this implies $|z - p_0| \le 3\epsilon$. Choose ϵ sufficiently small so that, by a complex affine linear change of the coordinates in $\mathbb{C}^n$,

$$p_0 = (0, \ldots, 0); \qquad \frac{\partial \psi}{\partial x_1}\bigg|_{p_0} = 1,$$

$$\frac{\partial \psi}{\partial y_1}\bigg|_{p_0} = \frac{\partial \psi}{\partial y_i}\bigg|_{p_0} = \frac{\partial \psi}{\partial x_i}\bigg|_{p_0} = 0, \qquad i \ge 2,$$

and

$$\frac{\partial^2 \psi}{\partial w_i \partial \overline{w}_j}\bigg|_{w=p_0} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases},$$

where $i = 1, \ldots, n$. (first, $\psi(w) = \mathrm{Re}\, w_1 - |w_1|^2 - \ldots - |w_n|^2 +$ higher order terms.)

A term which has its absolute value not exceeding $C\delta^r$ for some constant C, as $\delta \to 0$, will be written $\lesssim \delta^r$. A term which is uniformly comparable in absolute value to δ^r (i.e., which has absolute value $\leq C\delta^r$ and $\geq C^{-1}\delta^r$ for some positive constant C) as $\delta \to 0$ will be written $\sim \delta^r$. And, if the limit (as $\delta \to 0$) of the term divided by δ is 1, then the term will be written $\cong \delta$.

With this notation and $\delta = \delta(w)$:

1. $\psi(w) = (\cong\delta) = (\cong\mathrm{Re}\, w_1)$;
2. $\frac{\partial\psi}{\partial w_1} = \frac{1}{2} + (\lesssim\delta)$;
3. $\frac{\partial\psi}{\partial w_i} = (\lesssim\delta),\ i \geq 2$.

Therefore, for such w and z in Ω_0 with $|z - w| < \frac{3}{2}\delta(w)$, we see that

$$\begin{aligned} |X(z,w)| &= |(\cong\delta) + \frac{1}{2}(z_1 - w_1) + (\lesssim\delta^2)| \\ &\geq |(\cong\delta)| - \frac{3}{4}\delta - |(\lesssim\delta^2)| \\ &\geq \frac{1}{4}|(\cong\delta)| - |(\lesssim\delta^2)|. \end{aligned}$$

In particular, $X(z,w) = (\sim\delta)$ (the bound above is obvious).

The determinant $\det(\partial^2 b_{i,w}/\partial z_j \partial \overline{w}_i)$ becomes, upon substitution of the expansion

$$X^{-(n+1)}(z,w)[\varphi(z,w) + X^{(n+1)}(z,w) \cdot \widetilde{\varphi}(z,w) \log X(z,w)]$$

for $K_{\Omega_0}(z,w)$,

$$\begin{aligned} &(-1)^n (n+1)^n \det\Bigg[\frac{\partial^2}{\partial z_j \partial \overline{w}_i}(\log X(z,w)) \\ &\quad -(n+1)^{-1} \frac{\partial^2}{\partial z_j \partial \overline{w}_i} \log(\varphi + X^{n+1}(z,w)\widetilde{\varphi} \log X(z,w))\Bigg]_{i,j=1}^{n}. \end{aligned}$$

Now

$$\frac{\partial^2}{\partial z_j \partial \overline{w}_i} \log X(z,w) = X^{-1} \cdot \frac{\partial^2 X}{\partial z_j \partial \overline{w}_i} - \frac{\partial X}{\partial z_j} \frac{\partial X}{\partial \overline{w}_i} \cdot X^{-2}.$$

Thus, up to a nonvanishing absolute constant factor, the determinant to be evaluated is

$$\begin{aligned} X^{-2n} \det\Bigg[&X \cdot \frac{\partial^2 X}{\partial z_j \partial \overline{w}_i} - \frac{\partial X}{\partial z_j} \frac{\partial X}{\partial \overline{w}_i} \\ &- (n+1)^{-1} X^2 \cdot \frac{\partial^2}{\partial z_j \partial \overline{w}_i} \log(\varphi + X^{n+1}\widetilde{\varphi} \log X)\Bigg]_{i,j=1}^{n}. \end{aligned} \tag{4.5.2}$$

The terms in the determinant can be easily checked to have the following order-of-magnitude behavior:

$$X^2 \frac{\partial^2}{\partial z_j \partial \overline{w}_i} \log(\varphi + X^{n+1} \widetilde{\varphi} \log X) = (\lesssim \delta^2)$$

[since $\varphi(p_0, p_0) \neq 0$]. Also,

$$\begin{aligned}
X \frac{\partial^2 X}{\partial z_j \partial \overline{w}_i} &= (\lesssim \delta^2), \qquad i \neq j \\
X \frac{\partial^2 X}{\partial z_i \partial \overline{w}_i} &= -X + (\lesssim \delta^2) = (\sim \delta), \qquad i = 1, \ldots, n \\
\frac{\partial X}{\partial z_1} \frac{\partial X}{\partial \overline{w}_1} &= (\sim 1) \\
\frac{\partial X}{\partial z_1} \frac{\partial X}{\partial \overline{w}_i} &= (\lesssim \delta), \qquad i \neq 1 \\
\frac{\partial X}{\partial z_j} \frac{\partial X}{\partial \overline{w}_1} &= (\lesssim \delta), \qquad j \neq 1 \\
\frac{\partial X}{\partial z_j} \frac{\partial X}{\partial \overline{w}_i} &= (\lesssim \delta^2), \qquad i \neq 1, j \neq 1.
\end{aligned}$$

Thus the entire expression (4.5.2) becomes

$$(\sim \delta)^{-2n} \det \begin{bmatrix}
(\sim 1) & (\lesssim \delta) & \cdots & \cdots & \cdots & (\lesssim \delta) \\
(\lesssim \delta) & (\sim \delta) & & & & \\
\vdots & & (\sim \delta) & & (\lesssim \delta^2) & \\
\vdots & & & \ddots & & \\
\vdots & & (\lesssim \delta^2) & & \ddots & \\
(\lesssim \delta) & & & & & (\sim \delta)
\end{bmatrix}$$

[The diagonal entries are of size $(\sim \delta)$ except the $(1,1)$-entry; the off-diagonal entries except the first row and the first column are of size $(\lesssim \delta^2)$.] Thus, the determinant of the Jacobian of the Bergman representative coordinate map at w p_0 is of $(\sim \delta(w)^{-(n+1)})$.

It is time to establish Lemma 4.5.5. By compactness of $\partial \Omega_0$, one can choose finitely many boundary points and associated ϵ-balls around them and corresponding ws from each ball to end up with an ϵ-neighborhood of the boundary $\partial \Omega_0$ for which the Jacobian determinant of the Bergman representative coordinate map is bounded away from zero. $\square$

Proof of Proposition 4.4.5. Now we give (at long last) the proof of Proposition 4.4.5. The basic idea is to exploit the fact that, in Bergman representative coordinates, an automorphism is given by a linear map. Thus estimation of its derivatives can be accomplished by estimating (1) its differential and (2) the relationship between representative coordinates and Euclidean coordinates.

Now the proof proceeds by contradiction. If the conclusion is false, then there are

(i) a sequence of domains Ω_ν converging in the C^∞ topology to a limit domain Ω_0;
(ii) a sequence $\{\alpha_\nu : \Omega_\nu \to \Omega_\nu\}$ of automorphisms;

and

(iii) a sequence of points $\{p_\nu \in \Omega_\nu\}$ and a Euclidean differential operator

$$\mathcal{D} = \left(\frac{\partial}{\partial z_1}\right)^{j_1} \left(\frac{\partial}{\partial z_2}\right)^{j_2} \cdots \left(\frac{\partial}{\partial z_n}\right)^{j_n}, \quad j_1, \ldots, j_k > 0,$$

with

$$\lim_{\nu \to \infty} |\mathcal{D}\alpha_\nu(p_\nu)| = +\infty.$$

Passing to a subsequence, we may assume that the sequences $\{p_\nu\}$ and $\{\alpha_\nu(p_\nu)\}$ converge to points $p_0, q_0 \in \mathrm{cl}(\Omega_0)$, respectively. We also may assume that both $\{\alpha_\nu\}$ and $\{\alpha_\nu^{-1}\}$, respectively, converge uniformly on compact subsets of Ω_0 to an automorphism α_0 of Ω_0 and its inverse α_0^{-1}, respectively (the possibility of establishing this last assertion was treated in Section 4.1 as well as in [Greene/Krantz 1981]). Now repeat the reasoning used in the proof of Lemma 4.5.4 to show that $p_0 \in \partial\Omega_0$. The same reasoning implies (because the inverse sequence $\{\alpha_\nu^{-1}\}$ converges to α_0^{-1}) that q_0 is also in $\partial\Omega_0$.

Select, by Lemma 4.5.5, a point $w_0 \in \Omega_0$ with these properties:

(A) $K_{\Omega_0}(p_0, w_0) \neq 0$;
(B) If $d_0(z)$ = the Jacobian determinant $\det(\partial b_{j,w_0}/\partial z_k)\big|_z$, $j, k = 1, \ldots, n$, then

$$\liminf_{z \to p_0} |d_0(z)| > 0.$$

[Here b_{j,w_0} are the Bergman representative coordinate functions that we introduced earlier.]

Because $K_{\Omega_0}(\cdot, w_0)$ extends to be a C^∞ function on the set

$$\left\{ z \in \mathrm{cl}(\Omega_0) \colon \mathrm{dis}(z, w) < \frac{3}{2}\mathrm{dis}(w_0, \partial\Omega_0) \right\},$$

property (A) implies that the Bergman representative coordinate functions b_{j,w_0} have C^∞ extensions to a neighborhood of p_0 in $\mathrm{cl}(\Omega_0)$. Property (B) is thus equivalent to the assertion that $d_0(p_0) \neq 0$. In particular, there is a number $\epsilon > 0$ such that the functions b_{j,w_0}, $j = 1, \ldots, n$, form a C^∞ coordinate system (holomorphic in Ω_0) on

$$\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}.$$

[Notice that we are *not* claiming that the functions b_{j,w_0} are holomorphic across $\partial\Omega_0$; rather, these functions extend to be C^∞ across $\partial\Omega_0$ in the sense that their real and imaginary parts are C^∞ as real functions. In general they will only be holomorphic on Ω_0 itself.]

By Lemma 4.5.5, the Bergman representative coordinate functions b^ν_{j,w_0}, for Ω_ν, $j = 1, \ldots, n$, and $\nu = 1, 2, \ldots, \infty$, on

$$\mathrm{cl}(\Omega_\nu) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}$$

converge in the C^∞ sense to the b_{j,w_0} on

$$\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}.$$

In particular, for all ν sufficiently large, the functions b^ν_{j,w_0}, $j = 1, \ldots, n$ form a C^∞ coordinate system on

$$\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}.$$

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, $\alpha : \Omega \to \Omega$ be an automorphism with Euclidean components $(\alpha_1, \ldots, \alpha_n)$, and $\mathcal{J}_\alpha(z)$ denote the Jacobian determinant of α at z. Recall the following transformation formulas:

$$K_\Omega(z, w) = \overline{\mathcal{J}_\alpha(w)} \cdot \mathcal{J}_\alpha(z) \cdot K_\Omega(\alpha(z), \alpha(w)), \tag{1}$$

$$b_{j,w}(z) = \sum_{\ell=1}^{n} \overline{\left(\frac{\partial \alpha_\ell}{\partial w_j}\right)}\bigg|_w b_{\ell,\alpha(w)}(\alpha(z)), \tag{2}$$

$$\left(\frac{\partial b_{j,w}}{\partial z_k}\right)\bigg|_z = \sum_{\ell,m=1}^{n} \overline{\left(\frac{\partial \alpha_\ell}{\partial w_j}\right)}\bigg|_w \cdot \left(\frac{\partial \alpha_m}{\partial z_k}\right)\bigg|_z \cdot \left(\frac{\partial b_{\ell,\alpha(w)}}{\partial z_m}\right)\bigg|_{\alpha(z)}, \tag{3}$$

$$\det\left(\frac{\partial b_{j,w}}{\partial z_k}\right)\bigg|_z = \overline{\mathcal{J}_\alpha(w)} \cdot \mathcal{J}_\alpha(z) \cdot \det\left(\frac{\partial b_{\ell,\alpha(w)}}{\partial z_m}\right)\bigg|_{\alpha(z)}. \tag{4}$$

Formula (1) is the standard transformation formula for the Bergman kernel; formulas (2) and (3) follow from (1) by differentiation; and formula (4) can by derived from (2) by using a little algebra.

The next observation is that $\det(\partial b_{j,\alpha_0(w_0)}/\partial z_k)\big|_{w_0} \neq 0$. To prove this assertion, notice that, by Lemma 4.5.5, the determinant equals

$$\lim_{\nu\to\infty} \det(\partial b^\nu_{j,\alpha_\nu(w_0)}/\partial z_k)\big|_{\alpha_\nu(p_\nu)};$$

this expression in turn equals, by formula (4),

$$\lim_{\nu\to\infty} \big(\mathcal{J}_{\alpha_\nu}(w_0)\big)^{-1} \cdot \big(\overline{\mathcal{J}_{\alpha_\nu}(p_\nu)}\big)^{-1} \cdot \det\big(\partial b^\nu_{j,w_0}/\partial z_k\big)\big|_{p_\nu}.$$

Since, by Lemma 4.5.4, the expression $|\mathcal{J}_{\alpha_\nu}|$ is bounded above on $\mathrm{cl}(\Omega_\nu)$ (uniformly in ν) and since

$$\lim_{\nu\to\infty} \det\left(\frac{\partial b^\nu_{j,w_0}}{\partial z_k}\right)\bigg|_{p_\nu} = \det\left(\frac{\partial b_{j,w_0}}{\partial z_k}\right)\bigg|_{p_0} \neq 0,$$

it follows that indeed $\det\big(\partial b_{j,\alpha_0(w_0)}/\partial z_k\big)\big|_{q_0} \neq 0$.

From the nonvanishing of this last determinant, it follows that the functions $b_{j,\alpha(w_0)}$ form a C^∞ coordinate system in some neighborhood in $\mathrm{cl}(\Omega_0)$ of q_0. In particular, there is a positive number η such that these functions form a C^∞ coordinate system on $\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, q_0) \leq \eta\}$. Lemma 4.5.5 then implies that, for all sufficiently large ν, the functions $b^\nu_{j,\alpha(w_0)}$ form a C^∞ coordinate system on $\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, \nu_0) \leq \eta\}$; moreover, this coordinate system converges in the C^∞ topology to the coordinate system $b_{j,\alpha(w_0)}$ on $\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, q_0) \leq \eta\}$.

For any ν sufficiently large, $\mathrm{dis}(p_\nu, p_0) \leq \epsilon$ and $\mathrm{dis}(\alpha(p_\nu), q_0) \leq \eta$. Thus, for all sufficiently large ν, the mapping α_ν in a neighborhood of p_ν is completely determined—in w_0-Bergman coordinates (of Ω_ν) going to $\alpha_\nu(w_0)$-Bergman coordinates (of Ω_ν)—by formula (3). This mapping is linear with bounded differential. But, since both w_0-Bergman coordinates (of Ω_ν) and $\alpha_\nu(w_0)$-Bergman coordinates of Ω_ν are converging in the C^∞ topology to C^∞ coordinate systems (independent of ν), it follows by the chain rule that the Euclidean derivatives of each fixed order α_ν at $\alpha_\nu(p_\nu)$ are bounded above uniformly in ν as $\nu \to \infty$. This contradiction completes the proof of Proposition 4.4.5. $\square$

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The Geometry of Complex Domains
Greene, R.E.; Kim, K.-T.; Krantz, S.G.
2011, XIV, 303 p. 14 illus., Hardcover
ISBN: 978-0-8176-4139-9
A product of Birkhäuser Basel