Preface

Grand visions in mathematics can begin with simple observations. It is hardly more than a homework exercise to prove that what we nowadays call the Poincaré metric on the unit disc is invariant under the biholomorphic maps of the unit disc to itself. But this easily established fact, when combined with the (profound) uniformization theorem of Poincaré and Koebe, yields the striking conclusion that, with a small number of exceptions, every Riemann surface has a canonical complete Hermitian metric of constant Gauss curvature -1. This result became a basic tool for the study of Riemann surfaces. From this result also grew the whole subject of canonical metrics, an area which has become central in transcendental algebraic geometry and in the topology of low-dimensional manifolds.

It is natural to ask what analogue there might be in higher complex dimensions of the Poincaré metric on the unit disc. Indeed, this was asked not long after the era in the early 1900s of the uniformization theorem (Theorem 2.5.1) and the canonical metric idea for Riemann surfaces. The higher dimensional situation is inevitably different from the situation in complex dimension 1 because the Riemann mapping theorem fails in higher dimensions. It was Poincaré again who showed that the unit ball in $\mathbb{C}^2$ was not biholomorphic to the product of the unit disc with itself. In a similar vein, it was understood around the same time that uniformization of algebraic surfaces was not possible in the same form as the Riemann surface result: there is no single simply connected cover for all the algebraic surfaces with only a few exceptions, no analogue to the unit disc being the universal cover of all but a few Riemann surfaces. But quite early on, in the 1920s, Stefan Bergman showed how to attach to each bounded domain in $\mathbb{C}^n$, $n \geq 1$, a canonical metric with the biholomorphic invariance properties of the Poincaré metric on the unit disc: each biholomorphic mapping of a bounded domain to itself was an isometry of the metric, and moreover, any biholomorphic mapping of one bounded domain to another was an isometry of their respective metrics. Uniformization was a failure, but invariant metrics were successful indeed.

The Bergman metric is only numerically computable in most instances, not given by formulas, and for some time it remained primarily an intriguing general idea rather than a specifically useful one. But the development of the detailed theory of the $\overline{\partial}$ operator by Hörmander, Andreotti-Vesentini, Kohn, and many others made accessible information about the behavior of the Bergman kernel and metric, especially on strongly pseudoconvex bounded domains with smooth boundary. The Bergman kernel is expressible directly in terms of the solutions of $\overline{\partial}$ that are orthogonal to holomorphic functions, and this expression means that the kernel and hence the metric can be analyzed in $\overline{\partial}$ terms. In particular, Fefferman's asymptotic expansion of the Bergman kernel (1974) near the boundary of a C^∞ bounded, strongly pseudoconvex domain opened up the possibility of realizing the grand vision of unifying complex function theory and geometry in this case.

This unification of function theory and geometry for domains in $\mathbb{C}^n$ is the subject of this book—hence its title. In particular, the use of geometric methods yields many results about biholomorphic mappings in general and especially about automorphisms, that is, biholomorphic maps of a domain to itself. The fact that a biholomorphic map is an isometry means that the curvature invariants of differential geometry are preserved by biholomorphic maps, and this provides a powerful method of studying the biholomorphic mappings themselves.

While the Bergman metric has become over the years a familiar item in several complex variables that occurs in many texts on the subject, the study via curvature of the geometry of the Bergman metric has been largely confined to research papers up to now. Thus it seemed to the authors that the body of information on this and related topics was both large enough and coherent enough to justify its treatment in a book. That it was large enough is clear from the length of this book. The question of being coherent we leave to the reader, with hope for the best.

This book is not self-contained: on occasion we use, without apology and sometimes without proof, standard results of several complex variables and in particular of the theory of the $\overline{\partial}$ operator. Even so, we have tried to make the book as accessible as possible to the nonspecialist. Most of the arguments can be followed convincingly by simply taking the unproved background results on faith, these being usually very specific and easily stated, if not easily proven. In this sense, the book will be accessible, we hope, to anyone with a basic background in complex analysis and differential geometry. We have also separated out the more technical aspects of the differential geometry so that the complex analyst can most appreciate the shape of the arguments involving curvature by simply knowing that somehow curvature attaches differential invariants to each point that must be preserved under isometries and hence preserved under biholomorphic maps. Really detailed information on differential geometry is rather seldom needed. Geodesics, for example, hardly occur in the book at all. We have tried, in short, to make almost everything accessible to as many readers as possible without short-changing the readers with more specific expertise. Brave words, but we did try.

This book is wide-ranging, though all the topics are related. And a description of the mathematical prerequisites of the book as a whole and of the various chapters specifically may be useful. All of the book presumes basic knowledge of complex analysis in several variables, with the exception of Chapter 2, which concerns one variable only. Especially important is some working knowledge of normal families. A quick summary of what is needed is given in Chapter 1. Chapter 1 also provides a summary of what is known and needed about automorphism groups being Lie groups. These results can be taken on faith if need be. Chapter 1 also begins to talk about Riemannian metrics. Not much depth is needed here nor will be needed later about Riemannian geometry, but the reader is presumed to have in mind what a Riemannian metric is, at least. Chapter 2 is about automorphisms of Riemann surfaces. The results there provide motivation for later developments, but as it happens, the contents of this chapter are not explicitly used anywhere else in the book. Again, metric concepts are used but at a quite elementary level—Gauss curvature and some ideas about geodesics suffice. In Chapter 3, the idea of the Bergman metric is introduced, and the geometry of the Bergman metric is systematically exploited. The Bergman metric is by nature a Kähler metric, but rather little is needed here about Kähler geometry in detail. Indeed, it is not really necessary to know what a Kähler metric is. What is needed is the realization that attached to a metric structure, a Riemannian metric in general, are some second-order differential invariants which are preserved by mappings that preserve the metric itself. Of course, the deeper meaning of these curvature invariants, if known, will enhance the reader's appreciation of the power and elegance of their application to complex analysis. But in the strictly logical sense, one could think of them as simply formulas, which happened to have certain important invariance properties. The same remarks apply to the continuation of these developments in Chapter 4.

Chapter 5 involves some considerable background in Lie group theory, especially in its second half, on the Bedford–Dadok argument. But Chapter 5 is not needed for the later parts of the book, and the reader who is so inclined can simply take as answered the question of which compact Lie groups occur as the automorphism group of a smoothly bounded strongly pseudoconvex domain in Euclidean space—first all of them do—and skip this chapter altogether.

Chapter 6 is similarly not needed for subsequent developments. It answers a natural and interesting (in the authors' view) question, and the argument in the noncompact case is not far outside the usual ways of thinking in several complex variables. The compact case involves some ideas from further afield, in algebraic geometry, and can be omitted without penalty if desired.

Chapter 7 reviews some metric ideas more general than the smooth Riemannian metrics that were used earlier. These more general metrics are of fundamental importance in several complex variables and are likely somewhat familiar to complex analysts in any case. References are given to further details about these metrics. This material is of central importance to the whole subject, though it is not needed in subsequent chapters as such. Automorphisms of Reinhardt domains, the subject of Chapter 8, require some information about Lie algebras if they are to be studied in detail, but the reader can gain a good impression without this.

Chapters 9 and 10 are in fact the natural continuation of Chapters 3 and 4 and can be read effectively immediately after Chapter 4, with the intervening chapters skipped. Chapters 9 and 10 introduce what is known as the scaling method at a rather more leisurely pace than is followed in the rest of the book, since this material is both very important and not so widely available in systematic form. Indeed some of the material here is new. Chapter 11 looks back on the whole book and discusses where the results could have been stated and proved more generally. For ease of reading, many of the results in the earlier parts of the book were stated in special cases—e.g., for domains in Euclidean spaces rather than complex manifolds—and Chapter 11 clarifies what additional generality holds without the introduction of fundamentally new arguments.

This book has been under construction for some considerable time. The authors have benefited during this effort from interactions with many colleagues. We thank them all. In particular, the third named author (Krantz) thanks Alexander Isaev for his collaboration and for many helpful ideas over the years. Several institutions have offered us mathematical hospitality during the writing. In addition to our home institutions, we thank MSRI, the Technical University of Denmark, the American Institute of Mathematics, and l'École Polytechnique de France (Palaiseau). We thank Ms. Ae-Ryoung Seo of POSTECH and Mr. Felipe Garcia Hernandez of UCLA, who each read the whole manuscript and made helpful suggestions. It goes without saying that any remaining errors are the authors' sole responsibility.

Some mathematical subjects begin slowly, by accumulation of many small contributions, like a river forming from many small streams. The general idea of the deep relationship between function theory and geometry does indeed have many historical sources in the nineteenth century, as indicated briefly in the opening paragraphs of this preface. But the specific subject of this book began definitely and quite suddenly with the work of Stefan Bergman. Without his work, this book would not have existed. We dedicate it to his memory.

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http://www.springer.com/978-0-8176-4139-9

The Geometry of Complex Domains
Greene, R.E.; Kim, K.-T.; Krantz, S.G.
2011, XIV, 303 p. 14 illus., Hardcover
ISBN: 978-0-8176-4139-9
A product of Birkhäuser Basel