
Contents

Preface	xvii
1 GEOMETRY OF MOTION	1
1 Trajectories in $\mathbb{R}^3$ and Intrinsic Triads	1
2 Areolar Velocity and Central Motions	3
3 Geometry of Rigid Motions	5
4 The Euler Angles	6
5 Rotation Matrices and Angular Velocity	7
6 Velocity and Acceleration	10
6.1 Translations	10
6.2 Precessions	10
6.3 Rototranslations	11
7 The Axis of Motion	11
8 Relative Rigid Motions and Coriolis's Theorem	12
9 Composing Rigid Motions	13
9.1 Connecting the Euler Angles and ω	15
10 Fixed and Moving Axodes	16
10.1 Precessions: Fixed and Moving Cones	17
11 Plane Rigid Motions	18
11.1 Center of Instantaneous Rotation	19
11.2 Centroides	20
Problems and Complements	20
2c Areolar Velocity and Central Motions	20
2.1c Cycloidal Trajectories	20
2.2c The Brachistochrone and Tautochrone	20
2.3c Hypocycloidal and Epicycloidal Trajectories	21
2.4c Kepler's Gravitational Laws	22
2.5c Apsidal Points	22
2.6c Elliptic Trajectories of Some Central Motions	23

8c	Relative Rigid Motions and Coriolis's Theorem	24
9c	Composing Rigid Motions	25
9.1c	Connecting the Euler Angles and ω	25
10c	Fixed and Moving Axodes	25
10.1c	Cone with Fixed Vertex and Rolling without Slipping . .	25
10.2c	Cylinder Rolling without Slipping	26
11c	Plane Rigid Motions	27
11.1c	Parametric Equations of Fixed and Moving Centroides .	27
11.2c	Centroides for Hypocycloidal Motions	29
11.3c	The Cardano Device	29
11.4c	More on the Cardano Device	30
2	CONSTRAINTS AND LAGRANGIAN COORDINATES .	33
1	Constrained Trajectories	33
2	Constrained Mechanical Systems	35
2.1	Actual and Virtual Displacements	36
2.2	Holonomic Constraints	37
2.3	Unilateral Constraints	37
3	Intrinsic Metrics and First Fundamental Form	38
4	Geodesics	39
5	Examples of Geodesics	41
5.1	Geodesics in a Plane	41
5.2	Geodesics on a Sphere	41
5.3	Geodesics on Surfaces of Revolution	42
	Problems and Complements	43
1c	Constrained Trajectories	43
1.1c	Elliptic Coordinates	43
1.2c	Parabolic Coordinates	44
1.3c	Spherical Coordinates	46
1.4c	Cylindrical Coordinates	46
2c	Constrained Mechanical Systems	46
2.1c	Holonomic and Nonholonomic Constraints	46
2.2c	Disk Rolling without Slipping on a Line	47
2.3c	Sphere Rolling without Slipping in a Plane	48
2.4c	Rigid Rod with Constrained Extremities	49
3c	Intrinsic Metrics and First Fundamental Form	51
3.1c	A Parameterization of the Torus	51
4c	Geodesics	52
5c	Examples of Geodesics	52
5.1c	The Clairaut Theorem [29]	52

3	DYNAMICS OF A POINT MASS	55
1	Newton's Laws and Inertial Systems	55
2	Mathematical Formulations of (1.1)–(1.2)	57
3	General Theorems of Point-Mass Dynamics	57
3.1	Positional and Central Forces	58
3.2	Conservative Forces	59
4	The Two-Body Problem	60
4.1	Gravitational Trajectories	61
5	Newton's and Kepler's Laws and Inertial Systems	62
6	Dynamics of a Point Mass Subject to Gravity [133]	64
6.1	On the Notion of Weight, Vertical Axis, and Gravity ...	65
6.2	Gravitational Motion near the Surface of Earth	66
7	Motion of a Constrained Point Mass	67
7.1	Smooth Constraints and Relative Energy	68
7.2	Rough Constraints and Relative Energy	70
7.3	Remarks on Fixed Constraints	71
7.4	Remarks on the Case of $\mathbf{F}$ Conservative	72
8	The Mathematical Pendulum	72
	Problems and Complements	75
3c	General Theorems of Point-Mass Dynamics	75
3.1c	Elastic Forces	75
3.2c	Point Mass Moving in a Fluid	75
3.3c	Elastic Motions in Viscous Media	76
3.4c	Forced Oscillation	77
3.5c	Phase Delay	77
3.6c	Amplitude of Forced Oscillations	78
4c	The Two-Body Problem	78
4.1c	Resolving Kepler's Motions	78
4.2c	Stable Circular Orbits	79
4.3c	Radial Potentials	80
4.4c	Closed Orbits	82
4.5c	The n -Body Problem	83
6c	Dynamics of a Point Mass Subject to Gravity [133]	84
6.1c	Motion of $\{P; m\}$ in the Air: Viscous Regime	85
6.2c	Motion of $\{P; m\}$ in the Air: Hydraulic Regime	85
6.3c	Upward Initial Velocity ($v_o > 0$)	85
6.4c	Downward Initial Velocity ($v_o < 0$)	86
7c	The Mathematical Pendulum	86
7.1c	The String Pendulum	86
7.2c	The Cycloidal Pendulum (Huygens [81])	87
7.3c	The Spherical Pendulum (Lagrange [101])	88
7.4c	Small Oscillations of a Spherical Pendulum	89
7.5c	The Foucault Pendulum [11, 57]	90

	7.6c	Point Mass Sliding on a Circle	91
	7.7c	Point Mass Sliding on a Sphere	92
4		GEOMETRY OF MASSES	93
	1	Material Systems and Measures	93
	1.1	Continuous Distribution of Masses	93
	1.2	Discrete Distribution of Masses	94
	2	Center of Mass and First-Order Moments	94
	3	Second-Order Moments and Huygens's Theorem	94
	4	Ellipsoid of Inertia and Principal Axes	96
	5	Miscellaneous Remarks	99
	5.1	Center of Mass and Principal Axes of Inertia	99
	5.2	Planar Systems	100
	6	Computing Some Moments of Inertia	100
	6.1	Homogeneous Material Segment	100
	6.2	Homogeneous Material Rectangle	101
	6.3	Homogeneous Material Parallelepiped	101
	6.4	Homogeneous Material Circle and Disk	102
	6.5	Homogeneous Material Right Circular Cylinder	102
	6.6	Homogeneous Material Sphere and Ball	103
		Problems and Complements	103
	2c	Center of Mass and First-Order Moments	103
	3c	Second-Order Moments and Huygens's Theorem	104
	3.1c	Inertia Tensor for a Homogeneous Ellipsoid	104
	5c	Miscellaneous Remarks	105
	5.1c	Finding the Principal Axes of a Planar System	105
	5.2c	Planar Systems and Mohr's Circle	106
	6c	Computing Some Moments of Inertia	107
	6.1c	Right Circular Cone	107
	6.2c	Homogeneous Material Triangle	108
	6.3c	Homogeneous Material Right Triangle	110
	6.4c	Spiral Ramp	110
5		SYSTEMS DYNAMICS	111
	1	Flow Map and Derivatives of Integrals	111
	2	General Theorems of System Dynamics	112
	2.1	D'Alembert's Principle	112
	2.2	Momentum	113
	2.3	Angular Momentum	114
	2.4	Energy	114
	3	Cardinal Equations	115
	3.1	Cardinal Equations in Noninertial Systems	116
	3.2	Motion Relative to the Center of Mass	116
	3.3	König's Theorem [24, 96]	117

4	Mechanical Quantities of Rigid Systems	117
4.1	Kinetic Energy	118
5	Workless Constraints: Discrete Systems	120
5.1	Proof of Lagrange's Theorem	121
6	The Principle of Virtual Work	122
6.1	The Principle of Virtual Work for Rigid Systems	122
	Problems and Complements	123
3c	Cardinal Equations	123
3.1c	Disk Rolling on a Slanted Guide as in Figure 3.1c	123
3.2c	Rod Moving in a Plane with a Fixed Extreme	124
3.3c	Ring Sliding on a Material Spinning Segment	125
3.4c	A Material Segment with Extremities on Rectilinear Guides	126
3.5c	Vibrations of a Flywheel	129
4c	Mechanical Quantities of Rigid Systems	131
4.1c	Disk Rolling without Slipping on a Line	131
4.2c	Disk Rolling in a Plane	131
4.3c	Rigid Rod with Constrained Extremities	132
4.4c	Rigid Rod Moving in a Plane	132
4.5c	The Double Pendulum	132
4.6c	Cone of §10.1c of Chapter 1	134
4.7c	Square Plate with a Vertex on a Rectilinear Guide	135
4.8c	Two Hinged Rods with Constrained Extremities	137
4.9c	Triangle Rotating about an Axis	140
6	THE LAGRANGE EQUATIONS	141
1	Kinetic Energy in Terms of Lagrangian Coordinates	141
1.1	Kinetic Energy for Rigid-Body Motion	143
2	The Principle of Virtual Work	143
3	The Lagrange Equations	145
3.1	Fixed Constraints and the Energy Integral	146
3.2	Motion along Geodesics	147
4	Lagrangian Function for Conservative Fields	147
4.1	Further Integrals of Motion and Kinetic Momenta	148
5	Lagrangian and Hamiltonian	149
5.1	Canonical Form of the Lagrange Equations	149
5.2	Partial and Total Time Derivative of the Hamiltonian	150
6	On Motion in Phase Space	150
6.1	Integrals of Motion	151
7	Equilibrium Configurations	151
7.1	Equilibrium for Rigid Systems	152
8	Canonical Form of the n -Body Problem	153
9	Lagrangian Coordinates Relative to $\{P_o; m_o\}$	154
10	Inertial Systems and Further Integrals of Motion	155

11	The Planar 3-Body Problem (Lagrange [99])	156
12	Configuration of the Three Bodies	157
12.1	Configuration of an Equilateral Triangle	159
12.2	Configuration of a Segment	159
13	Collapse of the n Bodies	160
13.1	The Lagrange–Jacobi Identity	161
13.2	The Sundman Inequality [142]	161
13.3	A Collapse Would Occur in Only Finite Time	161
13.4	A Total Collapse Would Occur Only If $\mathbf{M} = 0$	162
	Problems and Complements	163
1c	Kinetic Energy in Lagrangian Coordinates	163
1.1c	Homogeneous Functions	163
2c	The Principle of Virtual Work	164
2.1c	An Example of Nonconservative Lagrangian Components of Forces	164
4c	Lagrangian Function for Conservative Fields	164
4.1c	Rigid Rod with Extremities Sliding on a Circle	164
4.2c	Points Sliding on Rectilinear Guides	165
4.3c	Pulleys and Weights	166
4.4c	Homogeneous Material Frame	168
4.5c	Disk Rolling on a Slanted Guide	169
5c	Lagrangian and Hamiltonian	170
5.1c	The Legendre Transform of the Lagrangian	170
7c	Equilibrium Configurations	171
7	PRECESSIONS AND GYROSCOPES	173
1	The Euler Equations	173
2	Precessions by Inertia or Free Rotators	174
3	Moving and Fixed Polhodes	176
3.1	Equations of the Moving Polhode	176
3.2	Geometry of Moving Polhodes	177
3.3	Ellipsoids of Rotation	178
3.4	Principal Axes of Inertia as Axes of Permanent Rotation	179
4	Integrating Euler’s Equations of Free Rotators (Jacobi [88])	179
4.1	Integral of (4.3) when $\mathbf{I} \neq \mathcal{I}_i$, $i = 1, 2, 3$	180
4.2	Integral of (4.3) when $\mathbf{I} = \mathcal{I}_i$ for Some $i = 1, 2, 3$	180
5	Rotations about a Fixed Axis	181
5.1	Rotations about a Principal Axis of Inertia	181
6	Gyroscopes	182
7	Spinning Top Subject to Gravity (Lagrange)	184
7.1	Choosing the Initial Data: Sleeping Top	184
7.2	First Integrals and Constant Solutions	186
8	Precession with Zero Initial Velocity	186

8.1	Oscillatory Nutation and Its Period	187
8.2	Precession about the Fixed Vertical Axis $\mathbf{e}_3$	188
8.3	Visualizing the Motion	189
9	Precession with Arbitrary Initial Velocity	190
10	Precessions of Constant Direction ($g(\theta_o) \geq 0$)	191
11	Precessions That Invert Their Direction ($g(\theta_o) < 0$)	192
12	Spinning Top Subject to Friction	193
	Problems and Complements	195
1c	The Euler Equations	195
1.1c	Precession Relative to the Center of Mass	195
1.2c	Precession of Earth	195
1.3c	Poinsot and Astronomical Precessions of Earth	196
1.4c	Poinsot Precession of Earth	196
1.5c	Astronomical Precession of Earth	197
5c	Rotations about a Fixed Axis	197
5.1c	The Compound Pendulum (Huygens [81])	197
6c	Gyroscopes	199
7c	Spinning Top Subject to Gravity	200
7.1c	Intrinsic Equation of a Gyroscope	200
7.2c	The Gyroscopic Compass or Gyrocompass	201
7.3c	Integrating the Intrinsic Equations	203
10c	Precessions of Constant Direction ($g(\theta_o) \geq 0$)	204
10.1c	Asymptotic Amplitude of the Nutation	204
10.2c	Asymptotic Period of the Nutation	205
8	STABILITY AND SMALL OSCILLATIONS	207
1	Notion of Stability in Phase Space	207
1.1	Equilibrium Configurations	208
2	Lyapunov Stability Criteria	208
3	Criteria of Instability	210
3.1	Some Generalizations and the Četaev Theorem	210
4	Dirichlet Stability Criteria	211
5	Stability and Instability of the Poinsot Precessions	212
6	Linearized Motions	214
6.1	Stability and Instability of Linearized Motions	215
7	Small Oscillations	216
8	Vibrations of Masses Subject to Elastic Forces	217
9	Normal Coordinates	219
10	Degenerate Vibrations	220
11	Fundamental Vibrations	221
	Problems and Complements	222
2c	Lyapunov Stability Criteria	222
2.1c	Harmonic Oscillator and Exponential Decay	222
2.2c	Damped Oscillator	222

4c	Dirichlet Stability Criteria	223
4.1c	Point Mass Constrained on a Curve	223
4.2c	On the Dirichlet Instability Criterion	224
4.3c	Rigid Rod with Extremities on a Parabola	224
4.4c	Miscellaneous Problems	225
4.5c	Material Rod and Point Mass $\{P; m\}$ Connected by a Spring	225
5c	Stability and Instability of Poincot Precessions	226
5.1c	Instability of Rotations about the Intermediate Axis	226
5.2c	Ellipsoids of Rotation	227
6c	Small Oscillations	227
6.1c	Small Oscillations of the System in §4.5c	227
6.2c	Computing the Normal Frequencies	228
6.3c	Double Pendulum in §4.5c of the Complements of Chapter 5	228
7c	Degenerate Vibrations	229
9	VARIATIONAL PRINCIPLES	231
1	Maxima and Minima of Functionals	231
1.1	Stationary Points of Functionals	232
2	The Least Action Principle	233
2.1	Reduced Action and Maupertuis Principle	234
3	The Hamilton–Jacobi Equation	236
4	The Functional of Geometrical Optics	237
5	Huygens Systems	238
6	Canonical Form of Huygens Systems	240
6.1	Contact Virtual Differential Forms	241
7	Wave Fronts	242
7.1	First Definition of Wave Front	242
7.2	Second Definition of Wave Front	243
7.3	Equivalence of the Two Definitions	243
7.4	Velocity of the Light Ray and of the Wave Front	244
7.5	Normal Velocity and Normal “Slowness” of the Wave Front	244
8	The Huygens Principle	244
8.1	Optical Length	245
	Problems and Complements	246
1c	Maxima and Minima of Functionals	246
1.1c	A Functional with Stationary Points and No Extrema	246
1.2c	A Functional with a Minimum That Is Not Achieved	247
1.3c	The Brachistochrone	247
1.4c	Motion along Geodesics	249
1.5c	An Isoperimetric Problem	250

2c	The Least Action Principle	252
2.1c	On the Two-Point Boundary Value Problem (2.1)	252
2.2c	Stationary Points of the Action Cannot Be Maxima ...	252
2.3c	Critical Points of the Action Are Local Minima	253
4c	The Functional of Geometrical Optics	255
10	CANONICAL TRANSFORMATIONS	257
1	Changing the Variables of Motion	257
1.1	Pointwise Transformations	258
2	Variational and Canonical Transformations	258
2.1	Canonical Transformations	259
3	Constructing Classes of Canonical Transformations	260
4	Constructing Canonical Transformations by Other Pairs of Independent Variables	261
5	Examples of Canonical Transformations	262
5.1	The Flow Map Is a Canonical Transformation	262
5.2	On the Primitives of $d\omega$	263
5.3	Jacobi Integration of Hamilton Equations [87]	263
6	Symplectic Product in Phase Space and Symplectic Matrices ..	264
6.1	Symplectic Linear Transformations	264
7	Characterizing Canonical Transformations by Symplectic Jacobians	265
8	Poisson Brackets [131]	266
8.1	Invariance of the Poisson Brackets by Canonical Transformations	266
9	The Jacobi Identity [89]	267
10	Generating First Integrals of Motion by the Poisson Brackets ..	268
11	Infinitesimal Canonical Transformations	269
11.1	Variations for Infinitesimal Canonical Transformations ..	270
12	Variational and Canonical Transformations	271
12.1	Proof of Lemma 12.1	272
	Problems and Complements	273
4c	Constructing Canonical Transformations by Other Pairs of Independent Variables	273
5c	Examples of Canonical Transformations	274
5.8c	Linear Canonical Transformations	276
6c	Symplectic Product in Phase Space and Symplectic Matrices ..	276
7c	Characterizing Canonical Transformations by Symplectic Jacobians	277
7.1c	Linear Canonical Transformations by Symplectic Matrices	277
7.2c	The Poincaré Recurrence Theorem	277
7.3c	A Note on Liouville's Theorem	278
7.4c	Integral Invariants	279

8c	Poisson Brackets	280
8.1c	Lagrange Brackets [103]	280
8.2c	Invariance of Lagrange Brackets by Canonical Transformations	281
11	INTEGRATING HAMILTON–JACOBI EQUATIONS AND CANONICAL SYSTEMS	283
1	Complete Integrals of Hamilton–Jacobi Equations	283
1.1	Examples of Complete Integrals of (1.1)	283
1.2	Hamiltonian Independent of t	284
2	Separation of Variables	284
2.1	An Example of Complete Separability	286
2.2	Cyclic Variables	286
3	Reducing the Rank of a Canonical System	286
3.1	Canonical Systems with the Energy Integral	287
3.2	Cyclic Variables	288
4	The Liouville Theorem of Quadratures	288
4.1	Poisson Brackets in $2(N + 1)$ Variables	289
4.2	Proof of Liouville’s Theorem	289
	Problems and Complements	291
1c	Complete Integrals of Hamilton–Jacobi Equations	291
1.1c	Envelopes of Solutions	291
2c	Separation of Variables	292
2.1c	Spherical Coordinates	292
2.2c	Parabolic Coordinates	294
2.3c	Elliptic Coordinates	295
2.4c	Force-Free $\{P; m\}$ in $\mathbb{R}^3$	297
2.5c	The Cycloidal Pendulum	297
12	INTRODUCTION TO FLUID DYNAMICS	299
1	Geometry of Deformations	299
1.1	Incompressible Deformations	300
1.2	The Equation of Continuity	300
2	Cardinal Equations	301
3	The Stress Tensor and Cauchy’s Theorem	302
3.1	Symmetry of the Stress Tensor	304
3.2	Miscellaneous Remarks	304
4	Perfect Fluids and Cardinal Equations	305
4.1	Barotropic Fluids	305
4.2	Fluid in Uniform Rotation	306
5	Rotations and Deformations	307
6	Vortex Filaments and Vortex Sheets	308
6.1	Circulation and Helmholtz Theorem	309
7	Equations of Motion of Ideal Fluids	310

7.1	Flow past an Obstacle	311
8	Barotropic Flows with Conservative Forces	311
8.1	Incompressible Barotropic Fluids Subject to Gravity	313
9	Material Lines and Surfaces	313
10	Transport of the Vorticity in Barotropic Fluids Subject to Conservative Forces	314
11	Barotropic Potential Flows	316
11.1	Barotropic, Potential, Stationary Flows	318
11.2	Stationary, Potential, Incompressible Flows	318
12	Stationary, Incompressible Potential Flow past an Obstacle	319
12.1	The Paradox of D'Alembert	319
13	Friction Tensor for Newtonian Viscous Fluids	320
14	The Navier–Stokes Equations	321
14.1	Conservation and Dissipation of Energy	322
14.2	Dimensionless Formulation, Reynolds Number, and Similarities	322
	Problems and Complements	324
7c	Equation of Motion of Ideal Fluids	324
7.1c	Transmission of Sound Waves	324
7.2c	Continuity Equation in Cylindrical Coordinates	325
8c	Barotropic Flows with Conservative Forces	325
8.1c	Torricelli's Theorem	325
11c	Barotropic Potential Flows	326
11.1c	Bidimensional Incompressible Flows	326
12c	Stationary, Incompressible Potential Flow past an Obstacle	327
12.1c	Proof of Proposition 12.1	327
13c	Friction Tensor for Newtonian Viscous Fluids	329
13.1c	Isotropic Tensors of the Fourth Order	329
14c	The Navier–Stokes Equations	332
14.1c	A Paradox of Ideal Fluids	332
14.2c	Viscous Fluids: Solving the Paradox	333
	References	335
	Index	343

Classical Mechanics

Theory and Mathematical Modeling

DiBenedetto, E.

2011, XX, 351 p. 63 illus., Hardcover

ISBN: 978-0-8176-4526-7

A product of Birkhäuser Basel