

Chapter 2

Partitions of Graphs

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Abstract Many difficult optimization problems on graphs become tractable when restricted to some classes of graphs, usually to hereditary classes. A large part of these problems can be expressed in the vertex partitioning formalism, i.e., by partitioning of the vertex set of a given graph into subsets V_1, \dots, V_k called colour classes, satisfying certain constraints either internally or externally, or both internally and externally. These requirements may be conveniently captured by the symmetric k -by- k matrix M . Concepts which are modeled by M -partitions fall naturally into the three types; each is represented in this work by some problem.

Any minimal reducible bound for a hereditary property is in some sense the best possible partition. A number of such partitions are given.

Clustering is a central optimization problem (among many others) with applications in various disciplines, e.g., computational biology, communications networks, image processing, pattern analysis [41, 53, 57, 60], and numerous other fields. Some new results on k -clustering of graphs are proved.

Another type of M -partition is a matching cutset. The main known results on this subject are collected.

The last part of this work is devoted to acyclic partitions of graphs where we consider important classes of graphs and their acyclic reducible bounds.

For each partition type the complexity of considered problems is given. Also a number of open problems are presented.

Keywords Colourings · Partitions · Hereditary properties · k -clustering · Domination · Cut matching · Complexity

MSC2000: Primary 05C15, 05C70; Secondary 05C12, 05C69, 05C85, 05C90

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2.1 Introduction and Notation

We present some of the basic definitions, notation, and terminology used in this chapter. Other terminology will be introduced as it naturally occurs in the text and those concepts not defined can be found in [4, 30, 59].

We consider finite undirected graphs without loops or multiple edges. The vertex set and the edge set of graph G are denoted by $V(G)$ and $E(G)$, respectively, and \mathcal{I} is used to denote the class of these graphs.

Many difficult (NP-hard) optimization problems on graphs become tractable when restricted to some classes of graphs, usually to hereditary classes. A large part of these problems can be expressed in the vertex partitioning formalism, i.e., by partitioning of the vertices of a given graph into subsets V_1, \dots, V_k called *colour classes*, satisfying certain constraints either internally or externally, or both internally and externally. These requirements may be conveniently captured by the symmetric k -by- k matrix \mathbf{M} in which the diagonal entries $m_{ii} = \mathcal{P}_i$ encode the internal restrictions on the sets V_i and the off-diagonal entries $m_{ij} = \mathcal{P}_{ij}$ ($i \neq j$) encode the restriction on the edges between V_i and V_j . Formally, it can be defined as follows.

Definition 2.1.1. Let \mathbf{M} be a fixed symmetric $k \times k$ matrix with entries $m_{ii} = \mathcal{P}_i$ and $m_{ij} = \mathcal{P}_{ij} \subseteq \mathcal{B}$ for $i \neq j$, where \mathcal{B} is the class of all bipartite graphs.

An \mathbf{M} -colouring (partition) of a graph G is a partition of vertices of G into k subsets V_1, \dots, V_k corresponding to the rows (and columns) of the matrix \mathbf{M} such that the subgraph of G induced by V_i has the property \mathcal{P}_i for $i = 1, \dots, k$. Vertices of the set V_i are said to be *i -coloured*. For every two distinct colours i and j , the subgraph induced by all the edges linking an i -coloured vertex and a j -coloured vertex has the property \mathcal{P}_{ij} , $1 \leq i, j \leq k$.

Notice that properties \mathcal{P}_{ij} always form some classes of bipartite graphs. A graph G is *bipartite* if it admits a vertex partition $V(G) = V_1 \cup V_2$ such that every edge of G joins two different V_i 's.

Graph-theoretical concepts which are modeled by \mathbf{M} -partitions fall naturally into three types:

(T1) $m_{ii} = \mathcal{P}_i$, $m_{ij} = \mathcal{P}_{ij} = \mathcal{B}$ ($i \neq j$), $i, j = 1, \dots, k$; i.e., no restrictions between colour classes.

(T2) $m_{ii} = \mathcal{I}$, $m_{ij} = \mathcal{P}_{ij} \subseteq \mathcal{B}$ ($i \neq j$), $i, j = 1, \dots, k$; i.e., no internal restrictions.

(T3) $m_{ii} = \mathcal{P}_i$, $m_{ij} = \mathcal{P}_{ij} \subseteq \mathcal{B}$ ($i \neq j$), $i, j = 1, \dots, k$.

2.1.1 Examples of \mathbf{M} -Partitions

A $(\mathcal{P}_1, \dots, \mathcal{P}_k)$ -partition of G is defined as an \mathbf{M} -partition of type (T1).

In the case when all $\mathcal{P}_i = \mathcal{O}$, where \mathcal{O} is the class of edgeless graphs, we have a k -colouring of G . Thus a k -colouring of G is an \mathbf{M} -partition of G , where the matrix \mathbf{M} has \mathcal{O} on the main diagonal and \mathcal{B} everywhere else.

The smallest k for which there is a k -colouring of G is called the *chromatic number* of G and is denoted by $\chi(G)$.

Karp [40] has shown that the k -colouring problem ($k \geq 3$) is NP-complete for graphs and up to this time no polynomial-time algorithm is known for this problem. However, the k -colouring problem becomes much easier when we restrict the inputs to special classes of graphs.

An \mathbf{M} -partition of type (T1) of G is called k -clustering, if

$$m_{ii} = \mathcal{P} = \{G \in \mathcal{I} : \text{diam}(G) \leq k\}.$$

Let (V_1, V_2) be an \mathbf{M} -partition of G . Denote by \mathcal{M} the property of graphs for which the edges between the set V_1 and V_2 induce a matching, i.e., any two of them have no common endvertex.

An \mathbf{M} -partition of G such that $m_{ii} = \mathcal{I}$, $m_{12} = m_{21} = \mathcal{M} \subseteq \mathcal{B}$ is called a *matching cutset* of G .

A *path* in G is an alternating sequence of distinct vertices and edges beginning and ending with vertices in which each edge joins the vertex before it to the one following it. The first and the last vertex of this sequence is called the *endvertex* of the path. The path with n vertices is denoted by P_n . A *cycle* in G is a path with at least three vertices together with an edge joining its endvertices. A cycle with n vertices is denoted by C_n . The *length* of a path (cycle) is the number of edges in it. A graph G is called *acyclic* if G does not contain a cycle. Let us denote by \mathcal{D}_1 the class of all acyclic graphs.

An \mathbf{M} -partition of a graph G such that $m_{ii} = \mathcal{O}$ and $m_{ij} = \mathcal{D}_1$ for $i \neq j$, $1 \leq i, j \leq k$ is called an *acyclic colouring* of G .

The minimum k such that G has an acyclic k -colouring is the *acyclic chromatic number* of G , denoted by $\chi_a(G)$.

Similarly, for a class \mathcal{P} of graphs, the *acyclic chromatic number* of \mathcal{P} , denoted by $\chi_a(\mathcal{P})$, is defined as the maximum $\chi_a(G)$ over all graphs $G \in \mathcal{P}$, assuming that $\chi_a(G)$ is finite for all $G \in \mathcal{P}$.

2.1.2 Hereditary Properties

Two graphs G and H are *isomorphic* if there is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

A *property of graphs* is any nonempty class of graphs from \mathcal{I} which is closed under isomorphisms. We use the terms *class of graphs* and *property of graphs* interchangeably.

Graph H is a *subgraph* of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and is denoted by $H \subseteq G$. A subgraph H of G is *induced* if every pair of vertices in H which are adjacent in G are also adjacent in H . This fact is denoted by $H \leq G$.

Definition 2.1.2. A property \mathcal{P} of graphs is said to be *induced hereditary* (*hereditary*, also called *monotone*) if whenever $G \in \mathcal{P}$ and $H \leq G$ ($H \subseteq G$), then also $H \in \mathcal{P}$ and *additive* if it is closed under disjoint union, i.e., if every component of G has property \mathcal{P} , then $G \in \mathcal{P}$.

Obviously any hereditary property is induced hereditary, too.

Following [11] we list some examples and notations of hereditary properties of graphs.

$$\begin{aligned}
 \mathcal{O} &= \{G \in \mathcal{I} : E(G) = \emptyset\}, \\
 \mathcal{O}^k &= \{G \in \mathcal{I} : \chi(G) \leq k\}, \\
 \mathcal{S}_k &= \{G \in \mathcal{I} : \text{the maximum degree } \Delta(G) \leq k\}, \\
 \mathcal{O}_k &= \{G \in \mathcal{I} : \text{each component of } G \text{ has order at most } k + 1\}, \\
 \mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}; \\
 &\quad \text{i.e., the minimum degree } \delta(H) \leq k \text{ for each } H \subseteq G\}, \\
 \mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or} \\
 &\quad K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, \\
 \mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}.
 \end{aligned}$$

Observe that:

\mathcal{O} is the class of edgeless graphs,
 \mathcal{O}^k is the class of k -colourable graphs,
 \mathcal{S}_k is the class of graphs with degree bounded by k ,
 \mathcal{D}_1 is the class of *acyclic* graphs,
 \mathcal{T}_2 is the class of *outerplanar* graphs,
 \mathcal{T}_3 is the class of *planar* graphs.

Additionally, let us denote by $\mathcal{LF} = \mathcal{D}_1 \cap \mathcal{S}_2$, the class of *linear forests*.

A hereditary property \mathcal{P} can be uniquely determined by the set of *minimal forbidden subgraphs* which can be defined as follows.

Definition 2.1.3. $F_{\subseteq}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P}, \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$

Let \mathcal{F} be a family of graphs, $Forb_{\subseteq}(\mathcal{F})$ is defined to be the property of all graphs having no subgraph isomorphic to any graph from \mathcal{F} . Thus $\mathcal{P} = Forb_{\subseteq}(F_{\subseteq}(\mathcal{P}))$. In such a manner is defined, for example, the property $\mathcal{I}_k = Forb_{\subseteq}(\{K_{k+2}\})$, also called the class of K_{k+2} -free graphs.

Similarly, the set of minimal forbidden subgraphs $F_{\leq}(\mathcal{P})$ for an induced hereditary property \mathcal{P} is defined.

Let us denote by \mathbb{L} the set of all hereditary and by \mathbb{L}_{\leq} induced hereditary properties of graphs, and let the corresponding sets of additive properties be denoted by \mathbb{L}^a and \mathbb{L}_{\leq}^a , respectively. The sets \mathbb{L} , \mathbb{L}^a , \mathbb{L}_{\leq} and \mathbb{L}_{\leq}^a partially ordered by the set inclusion, form complete distributive lattices with the set intersection as the meet operation [17]. Obviously (\mathbb{L}, \subseteq) is a proper sublattice of $(\mathbb{L}_{\leq}, \subseteq)$.

We now consider some examples of partitions of outerplanar and planar graphs.

In [21] it has been proved that each outerplanar graph has an $(\mathcal{LF}, \mathcal{LF})$ -partition. An algorithmic proof of that fact is given in [18]. Each outerplanar graph also has an $(\mathcal{O}, \mathcal{D}_1)$ -partition. A natural question arises: does a property $\mathcal{P} \subset \mathcal{LF}$ exist such that each outerplanar graph has a $(\mathcal{P}, \mathcal{LF})$ -partition? In other words: is the $(\mathcal{LF}, \mathcal{LF})$ -partition the best possible for the class of outerplanar graphs? An answer is given later on.

It is well known [26] that every planar graph has vertex arboricity at most 3; i.e., it has a $(\mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1)$ -partition.

Stein [55] (see also [8, 37]) strengthened this result by proving that every planar graph can be vertex partitioned into two forests and one edgeless graph; i.e., an planar graph has an $(\mathcal{O}, \mathcal{D}_1, \mathcal{D}_1)$ -partition.

Another strengthening was obtained by Poh [51] and independently by Goddard [36]. They proved that every planar graph has an $(\mathcal{LF}, \mathcal{LF}, \mathcal{LF})$ -partition.

2.1.3 Reducibility

To more precisely analyse different partitions and compare them, we need some new notation.

A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_k$ is defined as the set of all graphs having a $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k)$ -partition. If $\mathcal{P}_1 = \dots = \mathcal{P}_k = \mathcal{P}$, then we write $\mathcal{P}_1 \circ \dots \circ \mathcal{P}_k = \mathcal{P}^k$.

It is easy to see that $\mathcal{P}_1 \circ \dots \circ \mathcal{P}_k$ is (induced) hereditary and additive whenever $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ are (induced) hereditary and additive, respectively.

An (induced) hereditary property \mathcal{R} is said to be *reducible* if there exist two (induced) hereditary properties \mathcal{P}_1 and \mathcal{P}_2 such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ and *irreducible*, otherwise.

Definition 2.1.4. (Mihók and Toft, see [39]) For a given irreducible property $\mathcal{P} \in \mathbb{L}$, a reducible property $\mathcal{R} \in \mathbb{L}$ is called a *minimal reducible bound* for \mathcal{P} if $\mathcal{P} \subset \mathcal{R}$ and for each reducible property $\mathcal{R}' \subset \mathcal{R}$, $\mathcal{P} \not\subset \mathcal{R}'$.

We consider reducibility and minimal reducible bounds only in the lattice \mathbb{L}^a . The family of all minimal reducible bounds for \mathcal{P} in this lattice is denoted by $\mathbf{B}(\mathcal{P})$.

Example 1. The class \mathcal{O}^2 of bipartite graphs is the smallest reducible property in the lattice \mathbb{L}^a .

Obviously it is the unique minimal reducible bound for \mathcal{P} if and only if $\mathcal{P} \subset \mathcal{O}^2$. In general, finding the set of minimal reducible bounds for a given irreducible property is a difficult problem. The existence of the set $\mathbf{B}(\mathcal{P})$ for any property \mathcal{P} is proved in [5].

2.1.4 Examples of Some Reducible Bounds

From the previously mentioned results for outerplanar and planar graphs we can write:

1. $\mathcal{T}_2 \subset \mathcal{O} \circ \mathcal{D}_1$,
2. $\mathcal{T}_2 \subset \mathcal{LF}^2$,
3. Four Colour Theorem implies: $\mathcal{T}_3 \subset (\mathcal{O}^2 \cap \mathcal{T}_3)^2$,
4. [55]: $\mathcal{T}_3 \subset \mathcal{O} \circ \mathcal{D}_1^2$ was improved by Thomassen [56]: $\mathcal{T}_3 \subset \mathcal{D}_1 \circ (\mathcal{D}_2 \cap \mathcal{T}_3)$,
5. [36, 51]: $\mathcal{T}_3 \subset \mathcal{LF}^3$.

2.1.5 Minimal Reducible Bounds for Outerplanar and Planar Graphs

Theorem 2.1.5 ([47]). $\mathbf{B}(\mathcal{T}_2) = \{\mathcal{O} \circ \mathcal{D}_1, \mathcal{LF} \circ \mathcal{LF}\}$.

From this theorem it follows that for each reducible property $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ such that $\mathcal{T}_2 \subset \mathcal{R} \subset \mathcal{LF}^2$ there is an outerplanar graph G which is not $(\mathcal{P}, \mathcal{Q})$ -partitionable. The same holds for $\mathcal{O} \circ \mathcal{D}_1$. In this sense, the bounds given above are best possible.

For a class of planar graphs the problem of minimal reducible bounds is much harder. Until now any minimal reducible bound for \mathcal{T}^3 is not known. In [12] a few minimal reducible bounds for some interesting subclasses of planar graphs are given.

Theorem 2.1.6 ([12]).

1. $\mathbf{B}(\mathcal{T}_3 \cap \mathcal{D}_2) = \{\mathcal{O} \circ \mathcal{D}_1\}$,
2. $\mathbf{B}(\mathcal{T}_3 \cap \mathcal{D}_3) \ni \mathcal{D}_1 \circ \mathcal{D}_1$,
3. $\mathbf{B}(\mathcal{T}_3 \cap \mathcal{O}^3) \ni \mathcal{O} \circ (\mathcal{O}^2 \cap \mathcal{T}_3)$.

We are not able to find other minimal reducible bounds for 3-colourable planar graphs. We can only show:

Theorem 2.1.7 ([12]). *Barnette's Conjecture, if true, gives a minimal reducible bound \mathcal{D}_1^2 for $\mathcal{T}_3 \cap \mathcal{O}^3$.*

Theorem 2.1.7 gives an unexpected relation to a well-known conjecture of Barnette which says that whenever G is a 3-connected bipartite 3-regular planar graph, then G has a Hamiltonian cycle. This conjecture is true if and only if each 3-colourable planar graph has a vertex partition into two subsets such that each of them induces a forest.

2.1.6 Minimal Reducible Bounds for Some Other Classes of Graphs

In [15] a subclass of planar graphs (slightly wider than the class of outerplanar graphs) called 1-*nonouterplanar* was considered. For this class of graphs, in contrast with the class of outerplanar graphs (see Theorem 2.1.5), there is an infinite number of minimal reducible bounds.

Let us define a few properties:

$\mathcal{UC}_i = \{G \in \mathcal{I} : \text{each component of } G \text{ contains at most one cycle of length } i \text{ and no cycle of any other length}\},$

$\mathcal{UC}_i^k = \{G \in \mathcal{UC}_i : \text{if } G \text{ contains a cycle (of length } i), \text{ then the minimum degree in } G \text{ of the vertices of this cycle is at most } k + 2\}.$

$\nabla_k = \{G \in \mathcal{I} : \text{each component of } G \text{ belongs to } \mathcal{UC}_3^k \cup \bigcup_{i \geq k+2} \mathcal{UC}_{2i+1}\}.$

For convenience, let $\nabla_\infty = \mathcal{UC}_3$.

For a plane graph G let $\text{Int}(G)$ denote the set of vertices not belonging to the external face. If G is a connected planar graph, we define $\text{int}(G)$ to be the minimum value of $|\text{Int}(G)|$ over all plane embeddings of G .

If G is a planar graph with r components H_1, \dots, H_r then we define

$$\text{int}(G) = \max\{\text{int}(H_i) : 1 \leq i \leq r\}.$$

If $\text{int}(G) \leq k$ then G is said to be k -*nonouterplanar* and we denote this property by \mathcal{NOP}_k , i.e., $\mathcal{NOP}_k = \{G \in \mathcal{T}_3 : \text{int}(G) \leq k\}.$

It is easy to see that $\mathcal{NOP}_0 = \mathcal{T}_2$ and $\mathcal{NOP}_\infty = \mathcal{T}_3$.

Theorem 2.1.8 ([15]).

$$\mathcal{B}(\mathcal{NOP}_1) = \{\mathcal{LF} \circ \mathcal{LF}, \mathcal{O}_1 \circ \mathcal{D}_1\} \cup \{\mathcal{O} \circ \nabla_k : k = 0, 1, \dots, \infty\}.$$

The minimal reducible bound for the class of triangle free graphs is trivial and it follows from the theorem given by Nešetřil and Rödl.

Theorem 2.1.9 ([50]). *Let $\mathcal{F}(\mathcal{P})$ be a finite set of 2-connected graphs. Then for every graph G of property \mathcal{P} there is a graph H of property \mathcal{P} such that for any partition (V_1, V_2) of $V(H)$ there is an i , $i = 1$ or $i = 2$, for which the subgraph $H[V_i]$ contains G .*

From this theorem we immediately have the following.

Corollary 2.1.10 ([48]). *Let $\mathcal{F}(\mathcal{P})$ be a finite set of 2-connected graphs. Then the property \mathcal{P} has exactly one minimal reducible bound. $\mathcal{O} \circ \mathcal{P}$.*

Corollary 2.1.10 implies that the class \mathcal{I}_k of K_{k+2} -free graphs has only one trivial minimal reducible bound $\mathcal{O} \circ \mathcal{I}_k$.

For the class of graphs with a bounded order of components and the class of k -degenerate graphs we have the following sets of minimal reducible bounds.

Theorem 2.1.11 ([48]). *For any positive integer k ,*

$$B(\mathcal{O}_k) = \{\mathcal{O}_p \circ \mathcal{O}_q : p + q + 1 = k\},$$

$$B(\mathcal{D}_k) = \{\mathcal{D}_p \circ \mathcal{D}_q : p + q + 1 = k\}.$$

A k -tree is a graph defined inductively as follows. A clique of order k is a k -tree. If G is a k -tree of order n , $n \geq k$, and K is a clique of G of order k , then the graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k -tree of order $n + 1$. Any subgraph of a k -tree is a *partial k -tree*. Let us denote the class of all partial k -trees by \mathcal{PT}_k . Obviously this class is hereditary.

Similar relations as above hold for the class of partial k -trees.

Theorem 2.1.12 ([52]). *For any positive integer k ,*

$$B(\mathcal{PT}_k) = \{\mathcal{PT}_p \circ \mathcal{PT}_q : p + q + 1 = k\}.$$

Partitions and minimal reducible bounds for minor hereditary properties were considered in [19].

2.1.7 Complexity of Some Selected Graph Partition Problems

We begin with complexity of partition problems for the class of planar graphs.

Theorem 2.1.13. *The following partition problems are NP-complete:*

1. [38]: $(\mathcal{O}, \mathcal{D}_1)$ -partition for planar graphs G with $\Delta(G) \leq 4$,
2. [38]: $(\mathcal{D}_1, \mathcal{D}_1)$ -partition even for maximal planar graphs,
3. [35]: $(\mathcal{O}, \mathcal{O}, \mathcal{O})$ -partition for planar graphs G with $\Delta(G) \leq 4$,
4. [38]: $(\mathcal{O}, \mathcal{O}, \mathcal{D}_1)$ -partition for planar graphs in which each face has size 3 or 4.

Conjecture 2.1.14 ([38]). $(\mathcal{O}, \mathcal{O}, \mathcal{D}_1)$ -partition is NP-complete for maximal planar graphs.

Below are presented partition problems for some interesting classes of graphs. General discussions on partitions with respect to hereditary or induced hereditary properties can be found in [2, 33].

Theorem 2.1.15. *The following partition problems for graphs (in general) are NP-complete:*

1. [35]: $(\mathcal{O}, \mathcal{O}^2)$ -partition,
2. [35]: $(\mathcal{O}, \mathcal{D}_1)$ -partition,
3. [46]: $(\mathcal{O}, \mathcal{S}_1)$ -partition,
4. [20]: $(\mathcal{O}, \text{Forb}_{\leq}(\{C_4, 2K_2, C_5\}))$ -partition,

5. [20]: $(\mathcal{O}, \text{Forb}_{\leq}(\{P_4\}))$ -partition,
6. [20]: $(\mathcal{O}, \text{Forb}_{\leq}(\{P_4, C_4\}))$ -partition,
7. [20]: $(\mathcal{O}, \mathcal{I}_1)$ -partition.

2.2 k -Clustering

Another type of (T1) partition of graphs with properties \mathcal{P}_i , $i = 1, \dots, k$, which are not hereditary is k -clustering. Clustering is a central optimization problem (among many others) with applications in various disciplines, e.g., computational biology, communications networks, image processing, pattern analysis [41, 53, 57, 60], and numerous other fields.

From a general point of view the goal of clustering is to find the groups (*clusters*) that are both homogeneous and well separated; i.e., elements within the same cluster should be similar and elements in different clusters dissimilar. The process of generating the clusters is called *clustering*. In the graph-theoretic approach to clustering, one builds from the data a *similarity graph* in which vertices correspond to objects and edges connect two vertices with similarity values above some predefined threshold. Typical objectives include: minimizing the maximum diameter of a cluster (k -clustering), minimizing the average distance between pairs of clustered points (k -clustered sum), and many others.

2.2.1 Minimum k -Clustering

Problem 2.2.1. k -CLUSTERING

Given a graph $G = (V, E)$ and a positive integer l , determine whether there is a partition of V into at most l subsets such that each of these subsets induces a subgraph of G with diameter at most k .

Problem 2.2.2. MINIMUM k -CLUSTERING

Given a graph $G = (V, E)$, find the smallest integer l such that there is a partition of V into l subsets each inducing a subgraph of G with diameter at most k .

The number of clusters in a minimum k -clustering of a graph G is denoted by $cl_k(G)$.

Theorem 2.2.3 ([23]). *Problem k -CLUSTERING is NP-complete for $l \geq 3$.*

Linear time algorithms for k -clustering on trees and some classes of perfect graphs were given in [1, 32].

The 1-CLUSTERING problem is equivalent to the problem PARTITION INTO CLIQUES which is NP-complete for any fixed $l \geq 3$. A graph G has 1-CLUSTERING into at most l clusters if and only if the complement of G has an l -colouring.

The graph 2-CLUSTERING problem is related to the NP-complete problem DOMINATING SET. A subset $D \subseteq V$ is said to be a *dominating set* of the graph $G = (V, E)$ if for every vertex $u \in V - D$ there exists a vertex $v \in D$ such that $uv \in E$. The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$.

Problem 2.2.4. DOMINATING SET

Given a graph $G = (V, E)$ and a positive integer k , decide whether G has a dominating set S with $|S| \leq k$.

Theorem 2.2.5 ([29]). *For any graph G*

$$cl_2(G) \leq \gamma(G).$$

Proof. Let $D = \{w_1, \dots, w_l\}$ be a dominating set of G . 2-clustering $\{Q_1, \dots, Q_l\}$ of G with $w_i \in Q_i$ for all $i = 1, \dots, l$ can be generated by assigning remaining vertices v to Q_i for which $vw_i \in E$. Thus a dominating set of cardinality $\gamma(G)$ induces a clustering of G into the smallest number of subsets for which each subgraph induced by Q_i has a dominating vertex. \square

2.2.2 Strongly Chordal Graphs

A class of *chordal graphs*, denoted by \mathcal{C} , is defined by the set of minimal forbidden induced subgraphs as follows:

$$\mathbf{F}_{\leq}(\mathcal{C}) = \{C_n : n \geq 4\}.$$

A graph $G = (V, E)$ is said to be *strongly chordal* if every induced subgraph contains a simple vertex, where a vertex v of a graph G is *simple* if the set $\{N[u] : u \in N[v]\}$ can be linearly ordered by inclusion; i.e., the closed neighbourhoods form a chain under inclusion; see Fig. 2.1. A class of strongly chordal graphs is denoted by \mathcal{SC} .

A *Sun* S_r is a chordal graph on $2r$ vertices, for some $r \geq 3$, whose vertex set can be partitioned into two sets: $U = \{u_1, \dots, u_r\}$ and $W = \{w_1, \dots, w_r\}$ such that:

- (a) $S_r[U]$ is a complete graph (r -clique) and W is an independent set in S_r
- (b) for each i and j , w_i is adjacent to u_j if and only if $i = j$ or $i \equiv j + 1 \pmod{r}$.

Theorem 2.2.6 ([31]). *A graph is strongly chordal if and only if it does not contain as an induced subgraph a cycle of length greater than three or Sun S_r , $r \geq 3$.*

Theorem 2.2.7 ([29]). *Let G be a strongly chordal graph. Then*

$$cl_2(G) = \gamma(G).$$

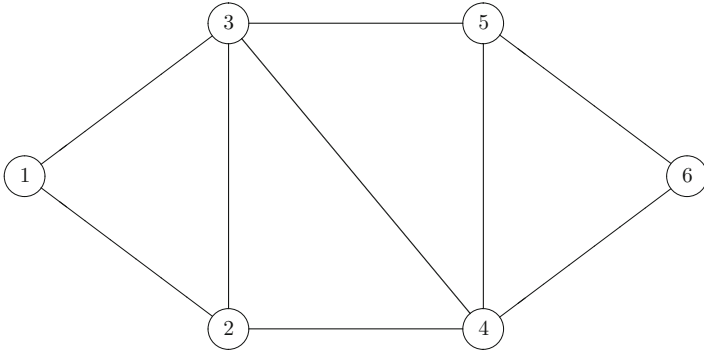


Fig. 2.1 A strongly chordal graph with a strong ordering (1, 2, 3, 4, 5, 6). The vertex 1 is simple: $N[1] = \{1, 2, 3\} \subseteq N[2] = \{1, 2, 3, 4\} \subseteq N[3] = \{1, 2, 3, 4, 5\}$

Consequently, there is a linear time algorithm to compute an optimal 2-clustering of a strongly chordal graph that is given with a strong elimination ordering. This algorithm applies a linear time algorithm to compute a minimum dominating set for strongly chordal graphs [31].

2.2.3 Hereditary Clique-Helly Graphs

A set family \mathcal{F} has the *Helly property* if every collection of pairwise-intersecting sets from \mathcal{F} has a common element.

In other words, if $\{S_1, \dots, S_m\} \subseteq \mathcal{F}$, then

$$(\forall i, j \ S_i \cap S_j \neq \emptyset) \Rightarrow \left(\bigcap_{k=1}^m S_k \neq \emptyset \right).$$

A *clique* of a graph G is a maximal complete subgraph of G . We also use “*clique*” to mean “*vertex set of a clique*”.

A graph G is *clique-Helly* if the cliques of G have the Helly property, and G is *hereditary clique-Helly* if every induced subgraph of G is clique-Helly. Let us denote the class of hereditary clique-Helly graphs by \mathcal{HCH} . The set of minimal forbidden subgraphs of \mathcal{HCH} is given by $\mathbf{F}_{\leq}(\mathcal{HCH}) = \{H_0 = S_3, H_1, H_2, H_3\}$, where H_1 is obtained from S_r by adding the edge $w_1 w_2$, H_2 from H_1 by adding the edge $w_2 w_3$, and H_3 from H_2 by adding the edge $w_1 w_3$.

Let us denote the class of hereditary clique-Helly chordal graphs by \mathcal{HCHC} ; i.e., let

$$\mathcal{HCHC} = \mathcal{HCH} \cap \mathcal{C}.$$

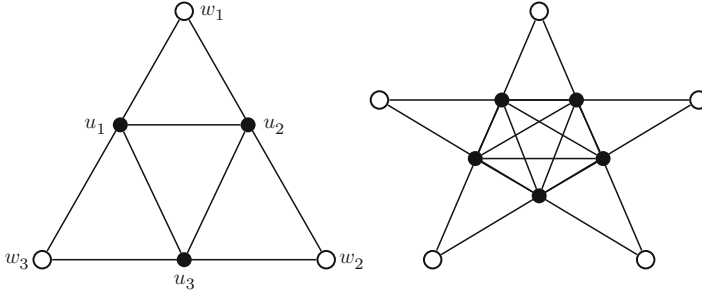


Fig. 2.2 Hajós graph: S_3 and Sun: S_5

Since each graph H_i , $i = 1, 2, 3$, from $\mathbf{F}_{\leq}(\mathcal{HCH})$ contains C_4 as an induced subgraph, then the minimal forbidden subgraphs of \mathcal{HCHC} are the Hajós graph S_3 and cycles of length greater than 3.

Consequently, it follows that

$$SC \subset \mathcal{HCHC} \subset \mathcal{C}.$$

Since $S_5 \in \mathcal{HCHC} - SC$ and $S_3 \in \mathcal{C} - \mathcal{HCHC}$, the above inclusions are proper.

Theorem 2.2.8. *There is a polynomial-time recognition algorithm for \mathcal{HCHC} graphs.*

2.2.4 Minimum k -Clustering in \mathcal{HCHC}

Theorem 2.2.9. *Let $G \in \mathcal{HCHC}$. Then*

$$cl_2(G) = \gamma(G).$$

Proof. By Theorem 2.2.5 we have $cl_2(G) \leq \gamma(G)$. We prove that for $G \in \mathcal{HCHC}$, $cl_2(G) \geq \gamma(G)$.

Let $G \in \mathcal{HCHC}$ and $\{Q_1, \dots, Q_l\}$ be 2-clusters of G with $l = cl_2(G)$. By heredity, $Q_i \in \mathcal{HCHC}$ and $\text{diam}(Q_i) \leq 2$ for $i = 1, \dots, l$. Now, it is enough to prove that each Q_i has a universal vertex. Consider two cases.

Case 1. Q_i does not contain the Sun S_r , $r \geq 4$, $i = 1, \dots, l$.

Hence each cluster Q_i is strongly chordal and Q_i has a universal vertex x_i , $i = 1, \dots, l$. Thus the set $\{x_1, \dots, x_l\}$ is dominating in G ; i.e., $\gamma(G) \leq l = cl_2(G)$.

Case 2. For some i , $1 \leq i \leq l$, Q_i contains a Sun S_r , $r \geq 4$.

Let us denote Q_i and S_r briefly by Q and S , respectively, and let $V(S) = \{u_1, \dots, u_r, w_1, \dots, w_r\}$ with $G[\{u_1, \dots, u_r\}] = K_r$. Consider three vertices in W with consecutive labels: w_t, w_{t+1}, w_{t+2} . It is easy to see that $d_S(w_t, w_{t+2}) = 3$

and therefore in S there is an induced path $w_t u_{t+1} u_{t+2} w_{t+2}$ of length three. Since $d_Q(w_t, w_{t+2}) = 2$, then there is $x \in V(Q) - V(S)$, a common neighbour of both w_t and w_{t+2} . Since Q is chordal, the vertex x has to be adjacent to u_{t+1} and u_{t+2} . If $w_{t+1}x \notin E$, then the set $\{w_t, u_{t+1}, w_{t+1}, u_{t+2}, w_{t+2}, x\}$ induces the Hajós graph, a contradiction. Thus $w_{t+1}x \in E$. It holds for any vertices with consecutive labels in the independent set W , thus x is adjacent to all vertices of S . Hence for each pair of vertices w_i and w_{i+2} there is a common neighbour $y \in V(Q) - V(S)$ with the same properties as above, possible $x = y$.

Let $X = \{x_1, \dots, x_t\}$ be a set of vertices of $V(Q) - V(S)$ each of which dominates $V(S)$. If $x_i x_j \notin E$, then any two vertices of W with consecutive labels, say w_1, w_2 together with x_i, x_j , induce in Q a cycle C_4 , which contradicts the chordality. It follows that the subgraph induced in Q by the set $\{x_1, \dots, x_t\}$ is a clique.

Let $Y = V(Q) - (V(S) \cup X)$. If $|Y| \leq 1$, then any vertex of X is a universal vertex of Q . Assume that $|Y| \geq 2$. Let $y_1, y_2 \in Y$ and suppose that y_1, y_2 do not have a common neighbour in Y . Then either $d(y_1, y_2) \geq 3$ or $y_1 y_2 \in E$. In the first case, we have a contradiction; in the second, there is in Q an induced cycle C_4 , which contradicts chordality of Q . Hence any two vertices of Y have a common neighbour in X . Since $Q[X \cup Y] \in \mathcal{HCHC}$, the Helly property implies that there is a vertex $x \in X$ such that $xy \in E$ for all $y \in Y$. Thus the vertex x is a universal vertex of Q , which completes the proof. \square

Problem 2.2.10. VERTEX COVER

Given a graph $G = (V, E)$ and an integer k , $1 \leq k \leq |V|$. Is there a vertex cover of cardinality $\leq k$ for G , i.e., a subset V' of V with $|V'| \leq k$ such that V' contains at least one vertex from every edge in E ?

Theorem 2.2.11. *Problem DOMINATING SET is NP-complete for \mathcal{HCHC} graphs.*

Proof. We transform the VERTEX COVER problem which remains NP-complete even when restricted to planar triangle free graphs [35] to DOMINATION SET in \mathcal{HCHC} , in the following way: Let $G = (V, E)$ be a planar triangle-free graph. Construct the graph $H = (U, F)$ with the vertex set $U = V \cup E$ and the edge set

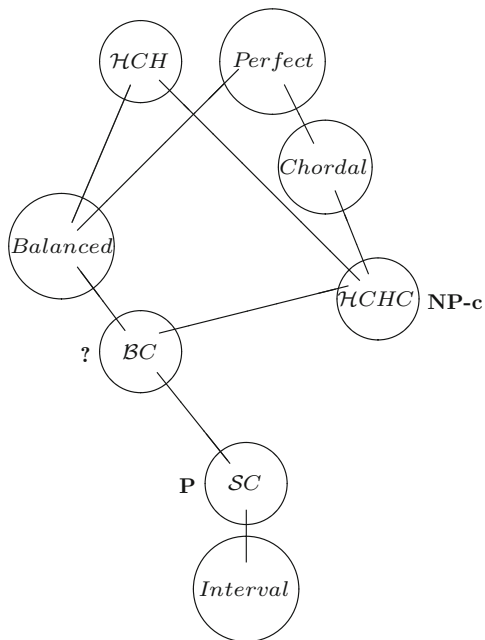
$$F = \{vv' : v, v' \in V, v \neq v'\} \cup \{ve : v \in V, e \in E \text{ and } v \in e\}.$$

Claim 1. $H \in \mathcal{HCHC}$.

Proof. It is easy to see that H is chordal. Now it is enough to prove that H does not contain the Sun S_3 as an induced subgraph. On the contrary, suppose that $S_3 \leq H$. It implies that w_1, w_2, w_3 and u_1, u_2, u_3 of S_3 correspond to some vertices, say e_1, e_2, e_3 and v_1, v_2, v_3 , respectively, in H . By the above and the construction of H it follows that the graph G contains a triangle induced by the set $\{v_1, v_2, v_3\}$, a contradiction. \square

Claim 2. G has a vertex cover of cardinality d if and only if H has a dominating set of cardinality d .

Fig. 2.3 Graph classes and domination set complexity



Proof. Let C be a vertex cover of G with $|C| = d$. Since $H[V]$ is a clique, every vertex of V is dominated by each vertex of C . Every $e \in E$ is covered by some v in C , thus e is dominated by v in H . Therefore C is a dominating set of cardinality d in H .

Suppose now that H has a dominating set D of cardinality d . If $D \subseteq V$, then D is also a vertex cover of G . (Each vertex $e \in E$ in H is dominated by some element of D , thus each $e \in E(G)$ has at least one vertex in D .) If $D \cap E \neq \emptyset$, i.e., say $e = vv' \in E$ is in D we may replace e by v or v' and get a new dominating set D' of the same cardinality. Applying this procedure to all $e \in D \cap E$ we will finally get a dominating set $D^* \subseteq V$ with $|D^*| = d$. \square

Hence the NP-completeness of the DOMINATING SET problem in \mathcal{HCHC} follows from the VERTEX COVER problem for planar triangle-free graphs. \square

2.2.5 *Balanced Graphs*

A hypergraph H is said to be *balanced* if every odd cycle has an edge containing three vertices of the cycle; see [4]. In other words, H is balanced if and only if its incidence matrix contains no square submatrix of an odd cycle.

A graph G is called *balanced* if its clique hypergraph is balanced.

Theorem 2.2.12 ([28]). *There is a polynomial-time recognition algorithm for balanced graphs.*

Let us denote the class of balanced chordal graphs by \mathcal{BC} .

Theorem 2.2.13 ([45]). *A graph $G \in \mathcal{BC}$ if and only if it is odd Sun-free chordal.*

From Theorem 2.2.9 we have

Corollary 2.2.14. *Let $G \in \mathcal{BC}$. Then*

$$cl_2(G) = \gamma(G).$$

Open Problem 2.2.15. *Problem DOMINATING SET remains open for the class \mathcal{BC} .*

2.3 Matching Cutset

A set M of independent edges in a graph G is called a *matching*. A set $F \subseteq E(G)$ is a cutset in G if $G - F$ has more components than G . If a cutset is a matching of G then it is called a *matching cutset*. The problem of recognizing graphs with a matching cutset is well studied in the literature. For a survey of what is known to date, see [20, 27, 44, 58]. Historically, the first theorem for matching cutset complexity was given by Chvátal [25].

We list some important results concerning this problem. Some of them deal with well-known classes of graphs.

Theorem 2.3.1 ([25]). *It is NP-complete to recognize graphs with a matching cutset even if the input is restricted to graphs with $\Delta = 4$.*

The next two results yield conditions on vertex degrees of bipartite graphs sufficient to guarantee NP-completeness of matching cutsets.

Theorem 2.3.2 ([49]). *MATCHING CUTSET is NP-complete, even if the input is restricted to bipartite graphs of minimum degree 2.*

Theorem 2.3.3 ([44]). *MATCHING CUTSET is NP-complete, even if the input is restricted to bipartite graphs with one colour class consisting only of vertices of degree 3 and the other colour class consisting only of vertices of degree 4.*

Below are presented a few graph classes for which the MATCHING CUTSET problem is polynomial. The *line graph* of G , denoted by $L(G)$, is the graph the vertex set of which is the edge set of G and two vertices of $L(G)$ are adjacent if and only if, as edges in G , they are adjacent. A graph H is called a *line graph* if there is a graph G such that H is isomorphic to $L(G)$.

Theorem 2.3.4 ([49]). *Let $G = (V, E)$ be a line graph. Then we can determine in $O(|E|)$ time whether G has a matching cutset.*

Theorem 2.3.5 ([49]). *Let $G = (V, E)$ be a graph without induced cycles of length ≤ 4 . Then we can determine in $O(|V|^3|E|)$ time whether G has a matching cutset.*

Theorem 2.3.6 ([16]). *Let G be a graph with $\text{diam}(G) = 2$. Then the MATCHING CUTSET problem for G can be solved in polynomial time.*

Open Problem 2.3.7. *Find k which is a boundary that separates NP-complete instances of diameter k of the MATCHING CUTSET problem from polynomially solvable ones.*

2.4 Acyclic Colourings

Acyclic colourings have been studied extensively over the past 30 years. Several authors have been able to determine $\chi_a(\mathcal{P})$ for some classes \mathcal{P} of graphs such as graphs of maximum degree 3, considered by Grünbaum in [37] and of maximum degree 4, studied by Burstein in [24]. The acyclic chromatic number of planar graphs was determined by Borodin in 1979; see [8] for details. Planar graphs with “large” girth, outerplanar, and 1-planar graphs were also considered; see, for instance, [9, 10].

2.4.1 Selected Results

Theorem 2.4.1 ([42]). *It is an NP-complete problem to decide for a given graph G and $k \geq 3$ if the acyclic chromatic number of G is at most k .*

In 2004 Skulrattanakulchai [54] proved that there is a linear time algorithm that acyclically colours any graph of maximum degree 3 in four colours.

Theorem 2.4.2 ([34]). *For any graph G of maximum degree 5, $\chi_a(G) \leq 9$ and there exists a linear time algorithm to acyclically colour G in at most nine colours.*

Authors suspect that the upper bound of nine colours in the case $\Delta(G) = 5$ is not tight.

Theorem 2.4.3 ([37]).

$$\chi_a(\mathcal{S}_3) \leq 4.$$

Theorem 2.4.4 ([8]).

$$\chi_a(\mathcal{T}_3) \leq 5.$$

The bound $\chi_a(\mathcal{S}_3) \leq 4$ proved by Grünbaum in Theorem 2.4.3 is the best possible. Moreover, Kostochka and Mel'nikov [43] proved that there are bipartite 2-degenerate planar graphs which are not acyclically 4-colourable. Acyclic colourings turned out to be useful for obtaining results about other types of colourings, see [39].

2.4.2 Brooks-Type Results

The well-known theorem of Brooks [22] relates the chromatic number of a graph to its maximum degree.

Theorem 2.4.5. *For any connected graph G ,*

$$\chi(G) \leq \Delta(G) + 1$$

with equality if and only if either $\Delta(G) = 2$ and G is an odd cycle or $\Delta(G) \neq 2$ and G is a complete graph.

There are many generalisations of the Brooks theorem. These theorems are called Brooks-type results. For the acyclic chromatic number finding a sharp upper bound as a function of maximum degree seems to be an extremely hard problem.

Theorem 2.4.6 ([3]).

$$\chi_a(\mathcal{S}_\Delta) \leq C\Delta^{4/3}.$$

Theorem 2.4.7 ([34]). *For a graph G of maximum degree $\Delta \geq 5$ we have*

$$\chi_a(G) \leq \frac{\Delta(\Delta - 1)}{2}.$$

Open Problem 2.4.8. *Find a sharp upper bound for $\chi_a(G)$ as a function of $\Delta(G)$.*

2.4.3 Improper Acyclic Colouring

Studies have begun in [7] of acyclic colourings of graphs with respect to hereditary properties of graphs. Namely, they have considered outerplanar, planar graphs, and graphs with bounded degree; see [6, 7]. They call such acyclic colouring improper.

Formally, an *improper acyclic colouring* of graphs is an \mathbf{M} -partition with $m_{ii} = \mathcal{P}_i$ and $m_{ij} = \mathcal{D}_1$ for $i \neq j$, $1 \leq i, j \leq k$. We denote it briefly by $((\mathcal{P}_1, \dots, \mathcal{P}_k), \mathcal{D}_1)$.

The class of graphs having $((\mathcal{P}_1, \dots, \mathcal{P}_k), \mathcal{D}_1)$ -partition is denoted by $\mathcal{P}_1 \odot \dots \odot \mathcal{P}_k$.

An additive hereditary property \mathcal{R} is said to be *acyclic reducible* in \mathbb{L}^a if there are nontrivial additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2$ and *acyclic irreducible* in \mathbb{L}^a , otherwise.

It is easy to see that the smallest acyclic reducible property in \mathbb{L}^a is the property $\mathcal{O} \odot \mathcal{O} = \mathcal{D}_1$. Obviously, \mathcal{D}_1 is the unique minimal acyclic reducible bound for \mathcal{P} if and only if $\mathcal{P} \subset \mathcal{D}_1$.

A maximal outerplanar graph G with at least three vertices is called a *2-path of order $n = 2p$* , if G consists of two paths $P_1 = (x_1, x_2, \dots, x_p)$, $P_2 = (y_1, y_2, \dots, y_p)$ and additional edges: $x_i y_i$, $i = 1, \dots, p$ and $x_j y_{j+1}$ for $j = 1, \dots, p-1$. For an odd $n = 2p-1$ a 2-path H is defined as $H = G - x_p$, where G is a 2-path of even order. A 2-path of order n is denoted by P_n^2 .

A maximal outerplanar graph G with at least three vertices is called a *fan of order n* , if G is obtained from a star $K_{1,n-1}$ by joining all vertices of degree one by a path. A fan of order n is denoted by F_n .

Additionally we assume that the graph K_1 and K_2 is a *trivial 2-path* and a *trivial fan*. For each $n \leq 5$ there is exactly one (up to isomorphism) maximal outerplanar graph which is a 2-path and a fan.

Some acyclic reducible bounds for the class of outerplanar graphs can be found in [13].

Theorem 2.4.9 ([13]).

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot \text{Forb}(S_3, P_6^2),$$

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot \text{Forb}(S_3, F_6).$$

We have an interesting proposition which characterises fans and 2-paths in the class of maximal outerplanar graphs. It was used in the proof of the next corollary and theorem and gives some light on an open problem formulated at the end of this section.

Proposition 2.4.10 ([13]). *Let G be a maximal outerplanar graph of order $n \geq 3$. Then*

- (a) *G is a fan if and only if neither $S_3 \subseteq G$ nor $P_6^2 \subseteq G$.*
- (b) *G is a 2-path if and only if neither $S_3 \subseteq G$ nor $F_6 \subseteq G$.*

Let us recall that a *block* of a given graph G is defined to be a maximal connected subgraph of G without a cutvertex.

A *fan (2-path) tree* is a connected graph G every block of which is a fan (2-path).

Let us define the property \mathcal{FT} (\mathcal{PT}) as the class of all fan (2-path) trees and their subgraphs. Both classes are hereditary and form a proper subclass of all outerplanar graphs.

From the definition of \mathcal{FT} it follows that S_3 and P_6^2 do not belong to \mathcal{FT} . Similarly, S_3 and F_6 do not belong to \mathcal{PT} . It implies the following corollary.

Corollary 2.4.11.

$$\mathcal{FT} \subset \text{Forb}(S_3, P_6^2),$$

$$\mathcal{PT} \subset \text{Forb}(S_3, F_6).$$

Because of the above corollary, the next theorem gives two acyclic reducible bounds for outerplanar graphs which are better than those in Theorem 2.4.9.

Theorem 2.4.12.

$$\mathcal{T}_2 \subset \mathcal{O} \odot \mathcal{FT},$$

$$\mathcal{T}_2 \subset \mathcal{O} \odot \mathcal{PT}.$$

Open Problem 2.4.13. Find at least one minimal acyclic reducible bound for the class of outerplanar graphs.

Theorem 2.4.14 ([7]).

$$\mathcal{S}_3 \subseteq \mathcal{S}_1 \odot \mathcal{S}_1 \odot \mathcal{S}_1.$$

Conjecture 2.4.15 ([7]).

$$\mathcal{S}_3 \subseteq \mathcal{S}_2 \odot \mathcal{S}_2.$$

Theorem 2.4.16 ([14]).

1. Boiron, Sopena, and Vignal's conjecture is true; i.e., $\mathcal{S}_3 \subseteq \mathcal{S}_2 \odot \mathcal{S}_2$ holds.
2. There is a polynomial time algorithm to acyclically $(\mathcal{S}_2, \mathcal{S}_2)$ -colour each graph $G \in \mathcal{S}_2$.
3. It is NP-complete to determine whether a graph $G \in \mathcal{S}_4$ has an acyclic $(\mathcal{S}_2, \mathcal{S}_2)$ -colouring.

The first statement of Theorem 2.4.16 cannot be generalized for $k = 4$. It was observed by Hałuszczak [Hałuszczak M, 2007, private communication]. But this observation can be extended for all $k \geq 4$.

Theorem 2.4.17. $\mathcal{S}_k \not\subseteq \mathcal{S}_{k-1} \odot \mathcal{S}_{k-1}$, for $k \geq 4$.

Proof. Let $G = C_k + D_{k-2}$, where by D_n the edgeless graph of order n is denoted; i.e., the graph G is obtained by joining each vertex of cycle of order k with each vertex of edgeless graph of order $k - 2$. Obviously, the graph $G \in \mathcal{S}_k$ but $G \notin \mathcal{S}_{k-1} \odot \mathcal{S}_{k-1}$. We left an easy proof of that fact to a reader. \square

Some other type results on acyclic colouring of graphs are presented in [6].

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<http://www.springer.com/978-0-8176-4788-9>

Structural Analysis of Complex Networks

Dehmer, M. (Ed.)

2011, XIV, 486 p. 85 illus., Hardcover

ISBN: 978-0-8176-4788-9

A product of Birkhäuser Basel