Preface

The Fourier transform is known as one of the most powerful and useful methods for finding fundamental solutions of operators with constant coefficients. However, in the general case this method has its own limitations. This book presents several other methods that can be used concurrently with the Fourier transform method to obtain heat kernels for elliptic and sub-elliptic operators. The text contains a large number of examples which facilitate understanding.

An Overview for the Reader. The theory of parabolic operators describes the distribution of heat on a given manifold as well as evolution phenomena and diffusion processes. The solution of an initial value problem for a parabolic partial differential equation depends on its heat kernel, which is the fundamental solution of the associated parabolic operator. Hence the importance of finding explicit formulas for these kernels.

This monograph presents several theories for finding explicit formulas for heat kernels for both elliptic and sub-elliptic operators. These methods are treated in distinct chapters. We shall find heat kernels for classical operators by several different methods. Some methods come from stochastic processes, others come from quantum physics, and others are purely mathematical. Depending on the symmetry, geometry and ellipticity, some methods are more suited for certain operators rather than others.

This book is a perfect reference material for graduate students, researchers in pure and applied mathematics as well as theoretical physicists interested in understanding different ways of approaching evolution operators.

Scientific Outline. Heat kernels arise naturally from probabilistic properties of stochastic processes. The transition density of a stochastic process provides the heat kernel for the associated generator operator, which is a second-order PDE operator. For instance, the generator of a Brownian motion in a plane is the operator $\frac{1}{2}(\partial_x^2 + \partial_y^2)$. On the other side, since the x- and y-components of the Brownian motion are independent, the joint transition density, given that the motion starts at (x_0, y_0) at $t = 0$, is the product of two transition densities:

$$\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2t}} \times \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2t}} = \frac{1}{2\pi t} e^{-\frac{1}{2t}[(x-x_0)^2+(y-y_0)^2]},$$

which is the heat kernel for the aforementioned generator operator.

One of the large classes of operators studied in this book is the sum-of-squares operators. These operators might be either elliptic or sub-elliptic. The methods for finding the heat kernel depend on the commutativity condition of the operators. If the operators commute, then the heat kernel is the product of the heat kernels of the operators. If the operators do not commute, then the Trotter formula applies and the heat kernel is computed using the path integral method. This method was borrowed from quantum physics, and it was used to compute propagators for the Schrödinger equation. This technique of path integrals was initiated by Feynman in the early 1940s.

Another class of operators investigated by this book is the sum between a second partial differential operator and a smooth potential. The case of linear and quadratic potentials can be solved explicitly by the path integral method, by van Vleck's formula, or by geometric methods that encounter classical action and volume function. Finally, they can also be solved by means of pseudo-differential operators. For instance, the aforementioned operator $\frac{1}{2}(\partial_x^2 + \partial_y^2)$ has a heat kernel of the type

$$K(x_0, y_0, x, y; t) = V(t)e^{-S_{cl}(x_0,y_0,x,y;t)}, \tag{0.0.1}$$

where $V(t) = \frac{1}{2\pi t}$ describes the density of geodesics emerging from (x_0, y_0), and

$$S_{cl}(x_0, y_0, x, y; t) = \frac{1}{2t}[(x - x_0)^2 + (y - y_0)^2] = \frac{1}{2t}\mathrm{dist}^2\big((x_0, y_0), (x, y)\big)$$

is the classical action on the Euclidean plane from (x_0, y_0) to (x, y) at time t. The key idea here is that the heat propagates mainly along the geodesics of the associated (sub-)Riemannian space. Then the heat density at a certain point in the space, which is the heat kernel of a certain operator, depends on the action along the geodesic and the density of geodesics at that point, given that all geodesics start at a given initial point. The density of geodesics is described by the volume function, which in certain special cases is given by a van Vleck determinant.

This idea can be applied in general for elliptic operators to obtain locally closed-form expressions for the heat kernels for certain operators. However, there are some limitations to the applicability of the method, and it depends on the nature of the potential function. The operators involving a power potential of degree greater than 2 do not in general have a closed-form solution. Geometrically speaking, this reduces to infinitely many geodesics between two points whose energies cannot be obtained in a closed-form solution in terms of the boundary points. None of the methods presented in this book can be applied to obtain exact solutions for these types of operators. For instance, the well-known problem of nonsolvability of the quartic oscillator problem is one of them.

In the case of sub-elliptic operators, formula (0.0.1) no longer holds, due to the degeneracy of the operator, and another formula will take its place. This new formula will require a fiber integration along the characteristic variety (which for elliptic operators degenerate to a point). We note that pseudo-differential techniques and the path integration method do not, in general, provide easy ways of obtaining heat kernels for sub-elliptic operators.

The novelty of this work is the diversity of methods aimed at computing heat kernels for elliptic and sub-elliptic operators. It is interesting that apparently distinct branches of mathematics, such as stochastic processes, differential geometry, special functions, quantum mechanics and PDEs, have all a common concept – the heat kernel. This concept unifies the aforementioned domains of mathematics, and hence deserves us dedicating our study to it.

It is worth noting the relation of the material of this book with other previous books on the subject. One of the long-standing textbooks in the field is the well-known book of David Widder [111], which appeared in the mid-1970s. This treats the heat equation mainly in dimension 1, discussing boundary value problems, Green's functions, integral transforms, theta-functions, the Huygens property, series expansions, heat polynomials, and other miscellaneous topics. Unlike the classical tract of the aforementioned reference, the present monograph covers the heat equation in several other contexts, such as geometric, stochastic, and quantic. However, the present work does not intend to replace Widder's book, but to complement it with newer facts regarding the geometric and analytic contexts.

This book contains most of the heat kernels computable by means of elementary functions. Future developments in this field can consider the possibility of closed-form expressions of heat kernels involving elliptic functions and hyperelliptic functions. These types of special functions have already appeared in the explicit computation of geodesics on certain sub-Riemannian manifolds, and we treated them in the monograph [27]. Similar types of functions also appeared in considering the heat kernel of operators with polynomial potential of degree greater than 2, one of the most famous being the "quartic oscillator" example. In general, the heat kernels of sub-elliptic operators associated with some sub-Riemannian manifolds of step larger than 2 may lead to the need for special functions. However, our present restrictive knowledge of the subject of hyperelliptic function theory indicates today's limit of explicit computability of these types of heat kernels.

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