Existence and Uniqueness of Disturbed Open-Loop Nash Equilibria for Affine-Quadratic Differential Games

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Abstract In this note, we investigate the solution of a disturbed quadratic open loop (OL) Nash game, whose underlying system is an affine differential equation and with a finite time horizon. We derive necessary and sufficient conditions for the existence/uniqueness of the Nash/worst-case equilibrium. The solution is obtained either via solving initial/terminal value problems (IVP/TVP, respectively) in terms of Riccati differential equations or solving an associated boundary value problem (BVP). The motivation for studying the case of the affine dynamics comes from practical applications, namely the optimization of gas networks. As an illustration, we applied the results obtained to a scalar problem and compare the numerical effectiveness between the proposed approach and an usual Scilab BVP solver.

1 Introduction

The problem of solving games with quadratic performance criterion and a linear differential equation defining the constraints has been addressed by many authors (see e.g. [4] and [5] and references therein), since this is often used as a benchmark to evaluate game outcomes/strategies. In [8], a disturbed linear quadratic differential game, where each player chooses his strategy according to a modified Nash equilibrium model under OL information structure, has been analyzed. Conditions for the existence and uniqueness of such an equilibrium were given, and it was also shown how these conditions are related to certain Riccati differential equations.

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However, some applied problems exist where the underlying system of the game is affine, for instance, in the modeling of the optimization of gas networks as a dynamic game, where the unknown offtakes from the gas net are modeled as a disturbance (see for instance [2]). In [3], the payoffs obtained by the players of a linear disturbed game and the equivalent undisturbed setting have been compared.

In this paper, we consider a quadratic performance criterion for every player, a finite planning horizon, an affine differential equation as constraint, and following [8] use an operator approach.

In [6], affine LQ Nash games equilibria on infinite planning horizon are studied and conditions for existence and uniqueness of an equilibrium are obtained. Moreover, in [5] and [7] it has been remarked that a particular transformation applied to make the game a standard linear quadratic one, which then can be solved by standard methods, cannot be applied since it may corrupt the OL information structure. In our case, also exists a simple transformation of the form

$$x(t) = y(t) + Z(t),$$

where $y(t)$ fulfills a linear differential equation while $x(t)$ solves the IVP (1) and $Z(t)$ is an adequately chosen function depending on the affine term $F(t)$. But, as the aforementioned observation clearly also applies to the finite time horizon situation, independently of a disturbance being present (or not), the OL information structure may be corrupted as well. Therefore, it is the aim of this paper to extend directly the work of Jank and Kun in [8] to the affine case, as well as the affine LQ game stated in [4] to the disturbed situation.

In this note, we generalize the procedure to calculate the equilibrium controls proposed in [8], where the controls are obtained in terms of the solution of certain Riccati differential equations, to the case when the underlying system is affine. The outcome is an interesting procedure from the applications point of view, since only IVPs have to be solved instead of a BVP. We ought to mention that usual routines for solving BVPs cannot be applied to the vectorial case.

The outline is as follows. Section 2 states the problem and introduces the notation. In Sect. 3, sufficient conditions for the existence/uniqueness of an OL equilibrium are investigated. Moreover, more explicit representations for such solutions, in terms of certain Riccati differential equations, are obtained. In Sect. 4, we provide a scalar example, where the equilibrium controls are calculated in two different manners: (a) using the transformation stated in Theorem 4, and (b) solving directly the BVP of Theorem 3. In the Appendix, the approach stated in Theorem 4 is fully described as an algorithm.

2 Preliminaries

We discuss disturbed noncooperative affine differential games with quadratic costs, defined on a finite time horizon $\left[t_0, t_f\right] \subset \mathbb{R}$. This means that the underlying system is affine, with each state being controlled by N players and a disturbance term:

$$\begin{aligned} \dot{x} &= A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t) + C(t)w(t) + F(t), \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, piecewise continuous and bounded functions, $A(t) \in \mathbb{R}^{n\times n}$, $B_i(t) \in \mathbb{R}^{n\times m_i}$, $C(t) \in \mathbb{R}^{n\times m}$, and $F(t) \in \mathbb{R}^n$. Furthermore, $u_i(\cdot) \in \mathscr{U}_i$ denotes the control of the i^{th} player, and $w \in \mathscr{W}$ the disturbance. $\mathscr{U}_i$ and $\mathscr{W}$ denote, respectively, the Hilbert spaces $\mathscr{L}_2^{m_i}\left[t_0, t_f\right]$ and $\mathscr{L}_2^{m}\left[t_0, t_f\right]$. We also set $\mathscr{U} = \prod_{i=1}^{N} \mathscr{U}_i$.

For $i = 1, \ldots, N$, the cost functional has the form:

$$\begin{aligned} J_i\left(u_1(\cdot), \ldots, u_N(\cdot), w\right) &= x^T(t_f)K_{if}x(t_f) \\ &+ \int_{t_0}^{t_f} \left(x^T(t)Q_i(t)x(t) + \sum_{j=1}^{N} u_j^T(t)R_{ij}u_j(t) + w^T(t)P_i(t)w(t) \right) \mathrm{d}t \end{aligned} \tag{2}$$

with symmetric matrices $K_{if} \in \mathbb{R}^{n\times n}$, and symmetric, piecewise continuous and bounded matrix functions $Q_i(t) \in \mathbb{R}^{n\times n}$, $R_{ij}(t) \in \mathbb{R}^{m_i\times m_j}$, and $P_i(t) \in \mathbb{R}^{m\times m}$, $i, j = 1, \ldots, N$.

We recall next the definition of Nash/worst-case strategy:

Definition 1. We define the Nash/worst-case equilibrium in two stages. Consider $u = (u_1, \ldots, u_N) \in \mathscr{U}$, then:

1. $\hat{w}_i(u) \in \mathscr{W}$ is the *worst-case disturbance from the point of view of the i^{th} player* according to these controls if

$$J_i\left(u, \hat{w}_i(u)\right) \geq J_i\left(u, w\right)$$

   holds for each $w \in \mathscr{W}, i \in \{1, \ldots, N\}$.
2. The controls $(\tilde{u}_1, \ldots, \tilde{u}_N) \in \mathscr{U}$ form a *Nash/worst-case equilibrium* if for all $i = 1, \ldots, N$
   (a) There exists a worst-case disturbance from the point of view of the i^{th} player according to all controls $(u_1, \ldots, u_N) \in \mathscr{U}$ and
   (b)
$$J_i\left(\tilde{u}_i, \bar{\tilde{u}}_i, \hat{w}_i(\tilde{u}_i, \bar{\tilde{u}}_i)\right) \leq J_i\left(u_i, \bar{\tilde{u}}_i, \hat{w}_i(u_i, \bar{\tilde{u}}_i)\right),$$
   where $\bar{u}_i = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$, holds for each worst-case disturbance $\hat{w}_i(u) \in \mathscr{W}$ and admissible control function $u_i \in \mathscr{U}_i$.

We now define some necessary notation.

Let $\mathscr{H}^n_{t_f}$ be the Hilbert space of the set of square integrable $\mathbb{R}^n$–valued functions on $\left[t_0, t_f\right] \subset \mathbb{R}$, with the scalar product:

$$\langle f, g\rangle_{\mathscr{H}^n_{t_f}} = \langle f, g\rangle = f^T(t_f)g(t_f) + \int_{t_0}^{t_f} f^T(t)g(t)\mathrm{d}t, \qquad f, g \in \mathscr{H}^n_{t_f}. \quad (3)$$

Denote by $\Phi(t,\tau)$ the solution of the IVP:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\tau) = A(t)\Phi(t,\tau), \qquad \Phi(t,t) = I_n, \quad (4)$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

For $i, j = 1, \ldots, N$, we define the following linear operators:

$$\begin{aligned}
\Phi : \mathbb{R}^n &\longrightarrow \mathscr{H}^n_{t_f}\\
x_0 &\rightsquigarrow \Phi(\cdot, t_0)\, x_0,\\[1ex]
\mathscr{B}_i : \mathscr{U}_i &\longrightarrow \mathscr{H}^n_{t_f}\\
u_i &\rightsquigarrow \textstyle\int_{t_0}^{\cdot} \Phi(\cdot,\tau)\, B_i(\tau)\, u_i(\tau)\, \mathrm{d}\tau,\\[1ex]
\mathscr{C} : \mathscr{W} &\longrightarrow \mathscr{H}^n_{t_f}\\
w &\rightsquigarrow \textstyle\int_{t_0}^{\cdot} \Phi(\cdot,\tau)\, C(\tau)\, w(\tau)\, \mathrm{d}\tau,\\[1ex]
\mathscr{F} : \mathscr{L}_2\left[t_0, t_f\right] &\longrightarrow \mathscr{H}^n_{t_f}\\
F &\rightsquigarrow \textstyle\int_{t_0}^{\cdot} \Phi(\cdot,\tau)\, F(\tau)\, \mathrm{d}\tau.
\end{aligned}$$

And we define finally the operators:

$$\begin{aligned}
\bar{Q}_i : \mathscr{H}^n_{t_f} &\longrightarrow \mathscr{H}^n_{t_f}\\
x(\cdot) &\rightsquigarrow \left(t \longmapsto \begin{cases} Q_i(t)x(t), & t \neq t_f\\ K_{if}\,x(t), & t = t_f\end{cases}\right)\\
\bar{R}_{ij} : \mathscr{H}^{m_i}_{t_f} &\longrightarrow \mathscr{H}^{m_j}_{t_f}\\
x(\cdot) &\rightsquigarrow \left(t \longmapsto \begin{cases} R_{ij}(t)x(t), & t \neq t_f\\ 0, & t = t_f\end{cases}\right)\\
\bar{P}_i : \mathscr{H}^m_{t_f} &\longrightarrow \mathscr{H}^m_{t_f}\\
x(\cdot) &\rightsquigarrow \left(t \longmapsto \begin{cases} P_i(t)x(t), & t \neq t_f\\ 0, & t = t_f\end{cases}\right).
\end{aligned}$$

Using the above definitions, we write the solution of the differential equation (1) on $\mathscr{H}^n_{t_f}$ as:

$$x(\cdot) = \Phi x_0 + \sum_{j=1}^{N} \mathscr{B}_i u_i + \mathscr{C}w + \mathscr{F}. \quad (5)$$

Similarly, the cost functionals (2) are rewritten in terms of scalar products:

$$J_i\left(u_1(\cdot),\ldots,u_N(\cdot),w\right) = \left\langle x, \bar{Q}_i x\right\rangle_{\mathscr{H}_{t_f}^n} + \sum_{j=1}^{N}\left\langle u_j, \bar{R}_{ij} u_j\right\rangle_{\mathscr{H}_{t_f}^{m_j}} + \left\langle w, \bar{P}_i w\right\rangle_{\mathscr{H}_{t_f}^m}, i = 1,\ldots,N. \tag{6}$$

3 Sufficient Existence Conditions for OL Equilibrium Controls

In this section, we state the modified theorems for the affine linear-quadratic case, generalizing the work presented in [8].

Theorem 1. *For $i = 1,\ldots,N$, let us define the operators*

$$F_i : \mathscr{U}_i \longrightarrow \mathscr{U}_i, \qquad G_i : \mathscr{W} \longrightarrow \mathscr{W}, \qquad H_i : \mathscr{W} \longrightarrow \mathscr{U}_i,$$
$$F_i := \mathscr{B}_i^* \bar{Q}_i \mathscr{B}_i + \bar{R}_{ii}, \quad G_i := \mathscr{C}^* \bar{Q}_i \mathscr{C} + \bar{P}_i, \quad H_i := \mathscr{B}_i^* \bar{Q}_i \mathscr{C},$$

where $-^*$ denotes the adjoint of an operator.

1. *There exists a unique worst-case disturbance $\hat{w}_i \in \mathscr{W}$ from the point of view of the i^{th} player if and only if $G_i < 0$. This disturbance, is then given by:*

$$\hat{w}_i\left(u_1,\ldots,u_N\right) = -G_i^{-1}\left(H_i^* u_i + \mathscr{C}^* \bar{Q}_i\left(\Phi x_0 + \sum_{j\neq i}\mathscr{B}_j u_j + \mathscr{F}\right)\right) \tag{7}$$

for all $(u_1,\ldots,u_N) \in \mathscr{U}$.

2. *Moreover, for $i = 1,\ldots,N$, let $G_i < 0$ and $F_i > 0$. Then $(\tilde{u}_1,\ldots,\tilde{u}_N) \in \mathscr{U}$ form an OL Nash/worst case equilibrium if and only if for each $i = 1,\ldots,N$*

$$\tilde{u}_i = \left(F_i - H_i G_i^{-1} H_i^*\right)^{-1}\left(H_i G_i^{-1}\mathscr{C}^* - \mathscr{B}_i^*\right)\bar{Q}_i\left(\Phi x_0 + \sum_{j\neq i}\mathscr{B}_j u_j + \mathscr{F}\right). \tag{8}$$

Proof. The proof follows just as for the linear case, see [8]. □

Remark 1. From the proof, one can see that only the positive definiteness of $\left(F_i - H_i G_i^{-1} H_i^*\right)$ is required to ensure a unique best reply. This seems to be a weaker condition than for the undisturbed case, where $F_i > 0$ is required. Although in this article we use the assumptions in Theorem 1, this problem needs to be further investigated.

The next theorem describes the Nash/worst-case controls in a feedback form for a "virtual" worst-case state trajectory for the i^{th} player. Please note at this point that every player may have a different "virtual" worst case trajectory, which in general is not the state trajectory.

Theorem 2. *Suppose that matrices P_i and R_{ii}, $i = 1,\ldots,N$, are negative and positive definite, respectively. Suppose also that the operators G_i and F_i, $i = 1,\ldots,N$, are negative and positive definite, respectively. Thence, $\tilde{u}_1,\ldots,\tilde{u}_N$ form an OL Nash/worst-case equilibrium if and only if the following equations are fulfilled:*

$$\tilde{u}_i = -R_{ii}^{-1}\mathscr{B}_i^*\bar{Q}_i\hat{x}_i \tag{9}$$

$$\hat{w}_i = -P_i^{-1}\mathscr{C}^*\bar{Q}_i\hat{x}_i, \tag{10}$$

where

$$\hat{x}_i = \Phi x_0 + \sum_{j\neq i}\mathscr{B}_j\tilde{u}_j + \mathscr{C}\hat{w}_i + \mathscr{F}. \tag{11}$$

Proof. The proof follows as for the linear case, see [8]. □

In this theorem, the controls are described in terms of adjoint operators. For convenience of the reader in the following lemma, we present the general construction of such adjoint operators for our particular function spaces.

Lemma 1. *Let $L : [t_0, t_f] \longrightarrow \mathbb{R}^{n\times k}$ be a piecewise continuous and bounded mapping for some $k, n \in \mathbb{N}$. Supposing that $\mathscr{L}$ denotes the linear operator*

$$\begin{aligned} \mathscr{L} : \mathscr{L}_2^k\left[t_0, t_f\right] &\longrightarrow \mathscr{H}_{t_f}^n \\ u \quad &\rightsquigarrow \int_{t_0}^{\cdot} \Phi(\cdot,\tau)L(\tau)u(\tau)\mathrm{d}\tau, \end{aligned} \tag{12}$$

the adjoint operator $\mathscr{L}^* : \mathscr{H}_{t_f}^n \longrightarrow \mathscr{L}_2^k\left[t_0, t_f\right]$ is given by

$$\mathscr{L}^* y := L^T(\cdot)\left[\Phi^T\left(t_f,\cdot\right)y(t_f) + \int_{t_0}^{\cdot}\Phi^T(t,\cdot)y(t)\mathrm{d}t\right]. \tag{13}$$

Proof. Obvious from the definition of adjoint operator (see for instance [8]). □

Using this lemma, we can express the equilibrium controls of Theorem 2, as well as the worst-case disturbance, as:

$$\tilde{u}_i = -R_{ii}^{-1}(t)B_i^T(t)\left(\Phi^T(t_f,t)K_{if}\hat{x}_i(t_f) + \int_{t_0}^{t_f}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\mathrm{d}\tau\right) \tag{14}$$

$$\hat{w}_i = -P_i^{-1}(t)C^T(t)\left(\Phi^T(t_f,t)K_{if}\hat{x}_i(t_f) + \int_{t_0}^{t_f}\Phi^T(\tau,t)Q_i(\tau)\hat{x}_i(\tau)\mathrm{d}\tau\right). \tag{15}$$

Theorem 3. *Consider the assumptions on the matrices R_{ii} and P_i, as well as on the operators F_i, G_i, of Theorem 2 fulfilled. Assume also that the solution of set of TVPs:*

$$\dot{E}_i(t) = -A^T(t)E_i(t) - E_i(t)A(t) - Q_i(t) + E_i(t)\left(S_i(t) + T_i(t)\right)E_i(t)$$

$$E_i\left(t_f\right) = K_{if}, \tag{16}$$

where $E_i(t) \in \mathbb{R}^{n\times n}, i = 1, \ldots, N$, is a symmetric matrix, and $S_i := B_i R_{ii}^{-1} B_i^T$ and $T_i := C P_i^{-1} C^T$, exist in the interval $\left[t_0, t_f\right]$.

1. *The BVP, for $i = 1, \ldots, N$,*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} e_i(t) = \left(A^T(t) - E_i(t)\left(S_i(t) + T_i(t)\right)\right) e_i(t) \\ \qquad\qquad + E_i(t) \sum_{j\neq i} S_j(t)\left(E_j(t)\hat{x}_j(t) + e_j(t)\right) - E_i(t)F(t) \\ e_i\left(t_f\right) = 0, \end{cases} \tag{17}$$

with $e_i(t) \in \mathbb{R}^n$ and $t \in \left[t_0, t_f\right]$, and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{x}_i(t) = A(t)\hat{x}_i(t) - \sum_{j=1}^N S_j(t)\left(E_j(t)\hat{x}_j(t) + e_j(t)\right) \\ \qquad\qquad - T_i(t)\left(E_i(t)\hat{x}_i(t) + e_i(t)\right) + F(t) \\ \hat{x}_i\left(t_0\right) = x_0, \end{cases} \tag{18}$$

with $\hat{x}_i(t) \in \mathbb{R}^n$, is equivalent to (9), (10) and (11).

2. *The control functions*

$$\tilde{u}_i(t) = -R_{ii}^{-1}(t)B_i^T(t)\left(E_i(t)\hat{x}_i(t) + e_i(t)\right) \tag{19}$$

form an OL Nash/worst-case equilibrium if and only if e_i and $\hat{x}_i$ are solutions of (17) and (18), respectively. Moreover, the corresponding worst-case disturbance of the i^{th} player is given by

$$\hat{w}_i = -P_i^{-1}(t)C^T(t)\left(E_i(t)\hat{x}_i(t) + e_i(t)\right). \tag{20}$$

3. *The Nash/worst-case equilibrium represented by (19) is unique if and only if the BVPs (17) and (18) have a unique solution.*

Proof. The proof follows as for the linear case, see [8]. □

For simplicity of exposition, we are going to consider $N = 2$. We also consider that every coefficient, the solution of the standard Riccati equation, as well as the BVP, depend on t.

Theorem 4. *Suppose that for the 2-player game the assumptions on the matrices R_{ii} and P_i and on the operators $F_i, G_i, i = 1, 2$, in Theorem 2 are fulfilled. Suppose further that the symmetric matrix Riccati differential equations (16), for $i = 1, 2$, and the following nonsymmetric matrix Riccati differential equation*

$$\dot{W} = \begin{pmatrix} 0 & E_1 S_2 E_2 \\ E_2 S_1 E_1 & 0 \end{pmatrix} + \begin{pmatrix} -A^T + E_1(S_1 + T_1) & E_1 S_2 \\ E_2 S_1 & -A^T + E_2(S_2 + T_2) \end{pmatrix} W$$

$$-W \begin{pmatrix} A - (S_1 + T_1)E_1 & -S_2 E_2 \\ -S_1 E_1 & A - (S_2 + T_2)E_2 \end{pmatrix} - W \begin{pmatrix} -S_1 - T_1 & -S_2 \\ -S_1 & -S_2 - T_2 \end{pmatrix} W$$

$$W\left(t_f\right) = 0 \in \mathbb{R}^{2n \times 2n} \tag{21}$$

admit bounded solutions over the interval $\left[t_0, t_f\right]$. *Consider* $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$. *Then, the BVP, (17) and (18), is uniquely solvable. Moreover, using these solutions the (unique) optimal Nash/worst-case control function for each player can be obtained in the following form:*

$$\tilde{u}_1 = -R_{11}^{-1} B_1^T \left((E_1 + W_{11})\hat{x}_1 + W_{12}\hat{x}_2 + W_{11}y_1 + W_{12}y_2 - y_3\right)$$

$$\tilde{u}_2 = -R_{22}^{-1} B_2^T \left((E_2 + W_{22})\hat{x}_2 + W_{21}\hat{x}_1 + W_{21}y_1 + W_{22}y_2 - y_4\right), \tag{22}$$

as well as the worst-case disturbance:

$$\hat{w}_i = -P_i^{-1} C^T \left((E_i + W_{ii})\hat{x}_i + W_{ij}\hat{x}_j + W_{ii}y_i + W_{ij}y_j - y_{i+2}\right), i, \quad j = 1, 2, \quad j \neq i, \tag{23}$$

where $\hat{x}_i$ *denotes the worst-case trajectory from the point of view of the* i^{th} *player, and* $Y = (y_1, y_2, y_3, y_4)^T$ *is the solution of the following TVP:*

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{M(t)} Y(t) - \begin{pmatrix} F \\ F \\ -E_1 F \\ -E_2 F \end{pmatrix}$$

$$Y(t_f) = 0 \tag{24}$$

Proof. The BVP, (17) and (18), can be written in matrix form as a nonhomogeneous differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = M(t)X(t) + \begin{pmatrix} F \\ F \\ -E_1 F \\ -E_2 F \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} \hat{x}_1(t_0) \\ \hat{x}_2(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0 \end{pmatrix}, \tag{26}$$

$$\begin{pmatrix} e_1(t_f) \\ e_2(t_f) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{27}$$

where $X(t) = \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ e_1(t) \\ e_2(t) \end{pmatrix}$ and

$$M = \begin{pmatrix} A-(S_1+T_1)\,E_1 & -S_2E_2 & -(S_1+T_1) & -S_2 \\ -S_1E_1 & A-(S_2+T_2)\,E_2 & -S_1 & -(S_2+T_2) \\ 0 & E_1S_2E_2 & -A^T+E_1\,(S_1+T_1) & E_1S_2 \\ E_2S_1E_1 & 0 & E_2S_1 & -A^T+E_2\,(S_2+T_2) \end{pmatrix}.$$

Also consider the auxiliary TVP (24):

$$\dot{Y}(t) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} Y(t) - \begin{pmatrix} F \\ F \\ -E_1F \\ -E_2F \end{pmatrix},$$
$$Y(t_f) = 0.$$

Adding (24) and (25) yields a homogeneous differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(X(t)+Y(t)) = M(t)\,(X(t)+Y(t)). \tag{28}$$

Consider $z(t) = X(t)+Y(t)$, and applying Radon's lemma (see [1, Chap. 3]), we solve:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} A-(S_1+T_1)(E_1+W_{11})-S_2W_{21} & -(S_2(E_2+W_{22})-(S_1+T_1)W_{12}) \\ -S_1(E_1+W_{11})-(S_2+T_2)W_{21} & A-(S_2+T_2)(E_2+W_{22})-S_1W_{12} \end{pmatrix}$$
$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} x_0+y_1(0) \\ x_0+y_2(0) \end{pmatrix} \tag{29}$$

and

$$\begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix} = W(t)\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$
$$\begin{pmatrix} z_3(t_f) \\ z_4(t_f) \end{pmatrix} = 0. \tag{30}$$

Thence, calculate the virtual trajectory:

$$\begin{pmatrix} \hat{x}(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ e_1(t) \\ e_2(t) \end{pmatrix} = z(t) - Y(t) \tag{31}$$

after solving the TVP (24).

Now it is possible to calculate the equilibrium controls (22). □

4 Numerical Example

Consider that the players' cost functionals have the following form:

$$J_1 = x(t_f)^2 + \int_{t_0}^{t_f} \left(3u_1(t)^2 - 2w(t)^2\right) \mathrm{d}t \tag{32}$$

$$J_2 = 3x(t_f)^2 + \int_{t_0}^{t_f} \left(u_1(t)^2 - 4w(t)^2\right) \mathrm{d}t. \tag{33}$$

And the underlying one-dimensional system:

$$\dot{x}(t) = 3u_1(t) - u_2(t) + w(t) + 2 \tag{34}$$

$$x_0 = 30. \tag{35}$$

Similarly as for the linear case in [8], since $R_{ii} > 0$ and $\bar{Q}_i \geq 0$, we immediately obtain the positive definiteness of F_i. Also, by the operators' definition in Sect. 2, we have:

$$\left\langle w, \left(\mathscr{C}^* \bar{Q}_i \mathscr{C} + \bar{P}_i\right) w\right\rangle = \left\langle \mathscr{C}w, \bar{Q}_i \mathscr{C}w\right\rangle + \left\langle w, \bar{P}_i w\right\rangle \tag{36}$$

$$= \int_0^{0.5} P_i w(t)^2 \mathrm{d}t + K_{if} \left(\int_0^{0.5} w(t)\mathrm{d}t\right)^2 \tag{37}$$

$$\leq \int_0^{0.5} \left(K_{if} + P_i\right) w(t)^2 \mathrm{d}t < 0 \tag{38}$$

for $i = 1, 2$, which yields the negative definiteness of G_i.

Figure 1 shows the chosen disturbance.

To evaluate the reliability of our approach, we computed the solution of the BVP (17) and (18) first using the decoupled approach stated in Theorem 4, and the obtained equilibrium control profiles are represented in Fig. 2.

The same equilibrium control profiles obtained when using the Scilab routine `bvode` are described in Fig. 3.

In Fig. 2, e_1, e_2 and $\hat{x}_1$, $\hat{x}_2$ correspond to the unique solution of the BVP (17) and (18) obtained via Theorem 4 and calculated in (31), whereas in Fig. 3 e_1-BVP, e_2-BVP and $\hat{x}_1$-BVP, $\hat{x}_2$-BVP correspond to the solution of the BVP (17) and (18) obtained by direct calculation with the Scilab routine bvode.

From the observation of the equilibrium control profiles obtained with both approaches, one may observe that these profiles are similar. Moreover, the CPU time obtained when using the approach described in Theorem 4 is smaller.

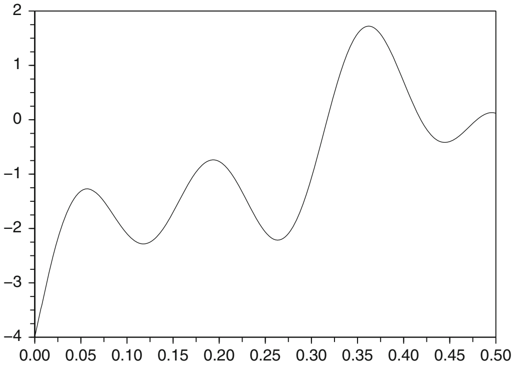

Fig. 1 Chosen disturbance

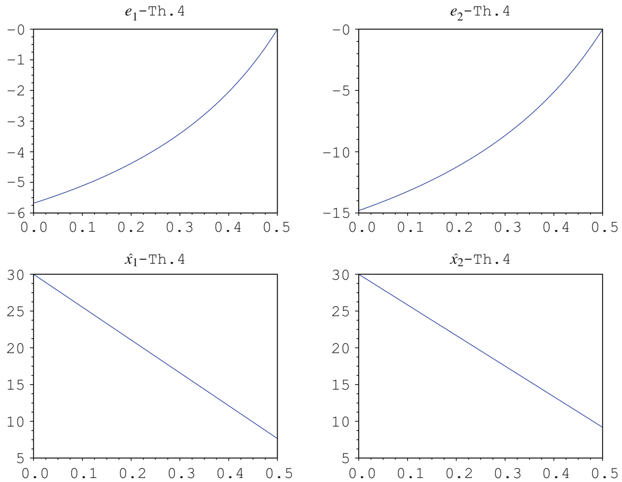

Fig. 2 CPU time = 0.1 s

5 Concluding Remarks

We investigated conditions for the existence/uniqueness of equilibrium controls of an affine disturbed differential game with finite planning horizon. The proposed algorithm to compute these equilibrium controls is based on solving initial and terminal value problems and has been applied to a case with a scalar differential equation. Its numerical performance has been compared to the performance of a Scilab BVP solver used to calculate the solution of the BVP stated in Theorem 3. The reason which makes the proposed algorithm amenable to practical cases is the following: To implement a recursive matricial BVP is not easy for problems whose number of players is $N \geq 2$, and even when $N = 2$ difficulties arise in the nonscalar

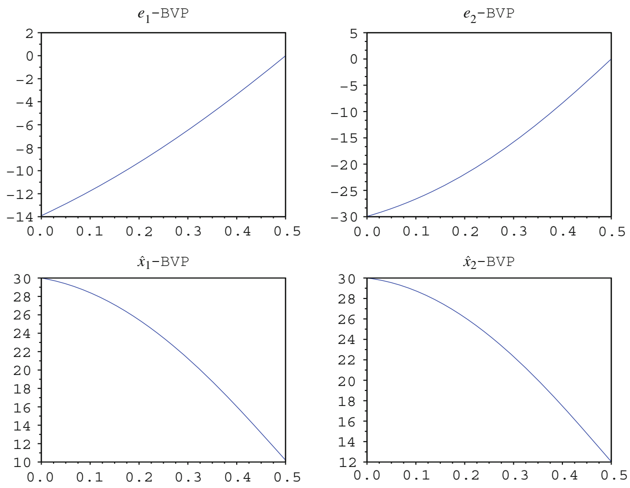

Fig. 3 CPU time = 0.41 s

case; Also the approach avoiding the solution of boundary value problems seems to have a better numerical performance in terms of CPU-times. But, notice that the BVP could still have a solution and hence an equilibrium exists, although condition (21) is violated and, moreover, the BVP need not have a unique solution yielding nonunique Nash equilibriums.

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Appendix: Algorithm

Assumptions

- All weighting matrices are symmetric
- P_i, R_{ii} are negative and positive definite, respectively
- G_i, F_i are negative and positive definite, respectively

Calculations

We understand that every coefficient varies with t, however we use a simpler notation for clarity sake.

1. Solve backward the Riccati differential equation

$$\dot{E}_i(t) = -A^T E_i - E_i A - Q_i + E_i(S_i + T_i)E_i$$
$$E_i(t_f) = K_{if}, \tag{39}$$

with $S_i = B_i R_{ii}^{-1} B_i^T$, and $T_i = C P_i^{-1} C^T$.

2. Solve backward the TVP

$$\dot{Y}(t) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} Y(t) - \begin{pmatrix} F \\ F \\ -E_1 F \\ -E_2 F \end{pmatrix}$$
$$Y(t_f) = 0. \tag{40}$$

3. Solve backward the TVP

$$\dot{W}(t) = \underbrace{\begin{pmatrix} 0 & E_1 S_2 E_2 \\ E_2 S_1 E_1 & 0 \end{pmatrix}}_{M_{21}}$$
$$+ \underbrace{\begin{pmatrix} -A^T + E_1(S_1 + T_1) & E_1 S_2 \\ E_2 S_1 & -A^T + E_2(S_2 + T_2) \end{pmatrix}}_{M_{22}} W(t)$$
$$-W(t) \underbrace{\begin{pmatrix} A - (S_1 + T_1)E_1 & -S_2 E_2 \\ -S_1 E_1 & A - (S_2 + T_2)E_2 \end{pmatrix}}_{M_{11}}$$
$$-W(t) \underbrace{\begin{pmatrix} -S_1 - T_1 & -S_2 \\ -S_1 & -S_2 - T_2 \end{pmatrix}}_{M_{12}} W(t), \quad W(t_f) = 0 \in \mathbb{R}^{2n \times 2n} \tag{41}$$

$M_{22} \neq M_{11}^T$.

4. Solve forward the IVP

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} A - (S_1 + T_1)(E_1 + W_{11}) - S_2 W_{21} & -(S_2(E_2 + W_{22}) - (S_1 + T_1)W_{12}) \\ -S_1(E_1 + W_{11}) + (S_2 + T_2)W_{21} & A - (S_2 + T_2)(E_2 + W_{22}) - S_1 W_{12} \end{pmatrix}$$

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} x_0 + y_1(0) \\ x_0 + y_2(0) \end{pmatrix}. \tag{42}$$

5. With $W(t) = \begin{pmatrix} W_{11}(t), & W_{12}(t) \\ W_{21}(t), & W_{22}(t) \end{pmatrix}$, $\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$, and $\begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix}$ calculate:

$$\begin{aligned} \begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix} &= W(t) \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \\ \begin{pmatrix} z_3(t_f) \\ z_4(t_f) \end{pmatrix} &= 0. \end{aligned} \tag{43}$$

6. Calculate the virtual trajectory

$$\begin{pmatrix} \hat{x}(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ e_1(t) \\ e_2(t) \end{pmatrix} = z(t) - Y(t). \tag{44}$$

7. Calculate the controls:

$$\begin{aligned} \tilde{u}_1(t) &= -R_{11}^{-1} B_1^T \left((E_1(t) + W_{11}(t))\, \hat{x}_1(t) + W_{12}(t)\hat{x}_2(t) + W_{11}(t) y_1(t) \right. \\ &\qquad \left. + W_{12}(t) y_2(t) - y_3(t) \right) \\ \tilde{u}_2(t) &= -R_{22}^{-1} B_2^T \left((E_2(t) + W_{22}(t))\, \hat{x}_2(t) + W_{21}(t)\hat{x}_1(t) + W_{21}(t) y_1(t) \right. \\ &\qquad \left. + W_{22}(t) y_2(t) - y_4(t) \right). \end{aligned} \tag{45}$$

8. Calculate the worst-case disturbance:

$$\begin{aligned} \hat{w}_i &= -P_i^{-1} C^T \Big((E_i(t) + W_{ii}(t))\, \hat{x}_i(t) + W_{ij}(t)\hat{x}_j(t) + W_{ii}(t) y_i(t) \\ &\qquad + W_{ij}(t) y_j(t) - y_{i+2}(t) \Big), i = 1, 2. \end{aligned} \tag{46}$$

9. Calculate the mixed trajectory

$$\dot{x}(t) = Ax(t) + B_1\tilde{u}_1(t) + B_2\tilde{u}_2(t) + C\bar{w}(t) + F(t), \tag{47}$$

where $\bar{w}$ is some chosen disturbance.

10. Calculate the values of the cost functionals at the equilibrium:

$$\begin{aligned} J_i\left(\tilde{u}_1(\cdot), \tilde{u}_2(\cdot), \bar{w}_i(\cdot)\right) &= x(t_f)^T K_{if} x(t_f) \\ &\quad + \int_0^{t_f} \left(x^T Q_i x + \tilde{u}_i^T R_{ii} \tilde{u}_i + \bar{w}^T P_i \bar{w} \right) \mathrm{d}t, \end{aligned} \tag{48}$$

where matrices K_{if}, Q_i, R_{ij}, P_i are symmetric.

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