Chapter 2

Local Theory of Fourier Integrals

2.1. Symbols

In this section we generalize the classes of amplitude functions encountered in the Introduction and in Section 1.2, and we collect some useful properties of these "symbol spaces."

Definition 2.1.1. A *conic manifold* is a C^∞ paracompact manifold V together with a proper and free C^∞ action of $\mathbb{R}_+$ on V. ($\mathbb{R}_+$ is regarded as a multiplicative group.) It follows that the orbit space $V' = V/\mathbb{R}_+$ has a C^∞ structure making V into a fiber bundle over V' with $\mathbb{R}_+$ as fiber, and the mapping α assigning to each $v \in V$ the orbit through v as projection. The orbit $\alpha(v)$ is also called the *cone axis* through v.

Piecing together local sections by means of a partition of unity in V' one can always construct a global section $V' \to V$, making the bundle trivial. For this, we observe that if s_α, s_β are local sections, then s_α/s_β is a strictly positive function. So, if φ_α is the partition of unity, then

$$s = \prod_\alpha (s_\alpha/s_\beta)^{\varphi_\alpha} \cdot s_\beta$$

is independent of β and defines the desired global section s. However, we avoid writing $V = V' \times \mathbb{R}_+$ because there may be no preferred choice of the unit section.

A function f on V is called *homogeneous of degree* μ if $\tau^* f = t^\mu f$ for all $\tau \in \mathbb{R}_+$. Here the pullback $\tau^* f$ of f by means of τ is defined by

$$(\tau^* f)(v) = f(\tau v), \qquad v \in V. \tag{2.1.1}$$

If L is a smooth vector field on V and f is a smooth function on V, then $\tau^*(Lf) = (\tau^* L)(\tau^* f)$, where

$$(\tau^* L)(v) = D\tau_v^{-1}(L(\tau v)), \qquad v \in V. \tag{2.1.2}$$

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Here τ is regarded as a diffeomorphism: $V \to V$, so $D\tau_v$ is a linear mapping: $T_v(V) \to T_{\tau v}(V)$. A vector field L on V will be called homogeneous of degree ν if $\tau^* L = \tau^\nu \cdot L$ for all $\tau \in \mathbb{R}_+$, it follows that Lf is homogeneous of degree $\mu + \nu$ if f is homogeneous of degree μ and L is homogeneous of degree ν. Note that homogeneous vector fields of degree 0 induce flows in V which commute with the action of $\mathbb{R}_+$ on V.

Definition 2.1.2. Let V be a conic manifold, $\mu, \rho \in \mathbb{R}$, $0 \le \rho \le 1$. A *symbol on V of order μ and type ρ* is a function $a \in C^\infty(V)$ such that

$$\tau^*(L_k \cdot \ldots \cdot L_1 a) = O(\tau^{\mu - k\rho}) \quad \text{for} \quad \tau \to \infty, \tag{2.1.3}$$

locally uniformly in V and for all C^∞ vector fields $L_1, \ldots, L_k$ in V that are homogeneous of degree -1. The space of these symbols is denoted by $S^\mu_\rho(V)$.

Note that $a \in S^\mu_1(V)$ if a is homogeneous of degree μ. In most applications we will only use symbols of type 1 and we will write $S^\mu_1(V) = S^\mu(V)$. Note also that $a \in S^\mu_\rho(V)$ if and only if

$$\tau^*(L_k \cdot \ldots \cdot L_1 a) = O(\tau^{\mu + k\delta}) \quad \text{for} \quad \tau \to \infty, \tag{2.1.4}$$

locally uniformly in V and for all homogeneous C^∞ vector fields L_j of degree 0, $\delta = 1 - \rho$. From an a priori point of view vector fields of degree 0 are more natural, but we have chosen $\rho = 1 - \delta$ as type number rather than δ because we shall make extensive use of homogeneous vector fields of degree -1 later on. The following assertions follow immediately from the definition.

Proposition 2.1.1. *$S^\mu_\rho(V)$ is a linear space. $S^\mu_\rho(V) \subset S^{\mu'}_{\rho'}(V)$ if $\mu \le \mu'$, $\rho \ge \rho'$. If $a \in S^\mu_\rho(V)$ and L is a homogeneous vector field of degree ν, then $La \in S^{\mu+\nu+(1-\rho)}_\rho(V)$. If $a \in S^\mu_\rho(V)$, $b \in S^{\mu'}_\rho(V)$, then $a \cdot b \in S^{\mu+\mu'}_\rho(V)$. Finally, if W is another conic manifold and χ is a C^∞ mapping: $V \to W$ commuting with the actions of $\mathbb{R}_+$ on V, respectively, W, then $\chi^*: a \to a \circ \chi$ maps $S^\mu_\rho(W)$ into $S^\mu_\rho(V)$.*

The following propositions are the analogues of Theorems 2.7 and 2.9 in Hörmander [41]. We denote $S^{-\infty}(V) = \bigcap_{\mu \in \mathbb{R}} S^\mu_\rho(V)$, which does not depend on ρ.

Proposition 2.1.2. *Suppose $a_j \in S^{\mu_j}_\rho(V)$, $j = 0, 1, 2, \ldots$, and $\mu_j \searrow -\infty$ for $j \to \infty$. Then there exists $a \in S^{\mu_0}_\rho(V)$ such that*

$$a - \sum_{j<k} a_j \in S^{\mu_k}_\rho(V) \quad \textit{for all} \;\; k = 1, 2, \ldots . \tag{2.1.5}$$

In this case we say $a \sim \sum a_j$.

Proof. We may write $V = V' \times \mathbb{R}_+$. Let K_j be an increasing sequence of compact subsets of V' such that every compact subset of V' is contained in one of them. Let $\mathcal{L}_j$ be a finite set of C^∞ vector fields on V' such that the $L(x)$, $L \in \mathcal{L}_j$ span $T_x(V')$ for every $x \in K_j$. Choose $\varphi \in C^\infty(\mathbb{R}_+)$ equal to 0 when $\tau \leq 1/2$ and equal to 1 when $\tau \geq 1$. We can then select a sequence $\tau_j \to \infty$ increasing so rapidly that

$$\left| L_\ell \cdots L_1 \Big(\frac{\partial}{\partial \tau}\Big)^k \varphi(\tau_j^{-1}\tau) a_j(x,\tau) \right| \leq 2^{-j}\, \tau^{\mu_{j-1} - \rho k + (1-\rho)\ell} \tag{2.1.6}$$

for $x \in K_i$, $L_1, \ldots, L_\ell \in \mathcal{L}_1$, $k + \ell + i \leq j$, $\tau \geq 1$.

In fact, since there is only a finite number of conditions for given j, we only need to use the fact that $\tau^k \frac{\partial^k}{\partial \tau^k} \varphi(\tau_j^{-1}\tau)$ is uniformly bounded for each j. We can now take

$$a(x,\tau) = \sum_{j=0}^{\infty} \varphi(\tau_j^{-1}\tau) \cdot a_j(x,\tau) \tag{2.1.7}$$

which is a locally finite sum since $\varphi(\tau_j^{-1}\tau) = 0$ for $\tau \leq \frac{1}{2}\tau_j$. The estimates (2.1.3) and (2.1.5) follow by remarking that any vector field on a neighborhood of K_j can be written as a linear combination on K_j of the $L \in \mathcal{L}_j$, with smooth coefficients. So the estimates follow by induction on the number of applied vector fields from the estimates for the $L \in \mathcal{L}_j$. $\square$

Proposition 2.1.3. *Let $c \in C^\infty(V)$ be such that for every set of homogeneous vector fields $L_1, \ldots, L_k$ and every compact $K \subset V$ there are constants C, μ such that*

$$|(L_k \cdot \ldots \cdot L_1 c)(\tau v)| \leq C \cdot \tau^\mu, \qquad v \in K, \quad \tau \geq 1. \tag{2.1.8}$$

Then $c \in S^{-\infty}(V)$ if for any $\nu \in \mathbb{R}$:

$$c(\tau v) = O(\tau^\nu) \qquad \textit{for} \;\; \tau \to \infty, \tag{2.1.9}$$

locally uniformly in $v \in V$.

We conclude that $b \in S_\rho^{\mu_0}(V)$ and $b \sim \sum a_j$ if the a_j are as in Proposition 2.1.2, b satisfies (2.1.8), *and if there exists a sequence $\mu_k \searrow -\infty$ for $k \to \infty$ such that*

$$\Big(b - \sum_{j<k} a_j\Big)(\tau v) = O(\tau^{\mu_k}) \qquad \text{for } \tau \to \infty \tag{2.1.10}$$

locally in $v \in V$, for all k.

Proof. If K, K' are compact sets, K in the interior of K', then we have an a priori estimate of the form

$$\|f\|_{1,K} \le C\sqrt{\|f\|_{0,K'} \cdot \|f\|_{2,K'}}\,. \tag{2.1.11}$$

Here $\|\ \|_{j,K}$ denotes some fixed C^j-norm taken over K. It suffices to prove (2.1.11) locally for C^∞ functions with compact support and there it follows from the following observation of E. Landau [52]:

If f is $C^2\colon \mathbb{R} \to \mathbb{R}$, $|f(x)| \le P$, $|f''(x)| \le Q$ for all $x \in \mathbb{R}$, then $|f'(x)| \le \sqrt{2PQ}$ for all $x \in \mathbb{R}$.

Proof.

$$\begin{aligned} f(x) - f(x-\varepsilon) &= \varepsilon f'(x) + \tfrac{1}{2}\varepsilon^2 f''(\xi_1), \qquad & x - \varepsilon < \xi_1 < x \\ f(x+\varepsilon) - f(x) &= \varepsilon f'(x) + \tfrac{1}{2}\varepsilon^2 f''(\xi_2), \qquad & x < \xi_2 < x + \varepsilon, \end{aligned}$$

so

$$f(x+\varepsilon) - f(x-\varepsilon) = 2\varepsilon f'(x) + \tfrac{1}{2}\varepsilon^2 (f''(\xi_1) + f''(\xi_2)).$$

This leads to $|f'(x)| \le P/\varepsilon + \frac{1}{2}\varepsilon Q$, take the minimum of the right-hand side for $\varepsilon > 0$. □

So because c decreases faster than any power of τ and $L_2 L_1 c$ is bounded by some power of τ as $\tau \to \infty$ for any homogeneous vector fields L_1, L_2, and we see that Lc decreases faster than any power of τ as $\tau \to \infty$ for any homogeneous vector field L. By induction it follows that $L_k \ldots L_1 c$ decreases faster than any power of τ as $\tau \to \infty$ for any homogeneous vector fields $L_1, \ldots, L_k$, which means that $c \in S^{-\infty}(V)$.

For the second assertion, note that Proposition 2.1.2 implies that $a \sim \sum a_j$ for some $a \in S_\rho^{\mu_0}(V)$. Then $c = b - a$ satisfies (2.1.8), (2.1.9) in view of (2.1.10), so $b - a \in S^{-\infty}(V)$. This implies that $b \in S_\rho^{\mu_0}(V)$ and $b \sim \sum a_j$. □

If E is a smooth N-dimensional vector bundle over a paracompact manifold X, then $E \setminus 0$ is a conic manifold with respect to the multiplications with $\tau \in \mathbb{R}_+$ in the fibers. Introducing inner products in the fibers E_x depending smoothly on $x \in X$ (this can be done locally and then globally by piecing together with a partition of unity in X), we see that the bundle of unit spheres is a smooth section: $(E \setminus 0)/\mathbb{R}_+ \to E \setminus 0$, for this reason $(E \setminus 0)/\mathbb{R}_+$ is called the *sphere bundle* SE of E.

$S^\mu_\rho(E)$ will be defined as the set of $a \in C^\infty(E)$ such that $a\big|_{E\setminus 0} \in S^\mu_\rho(E \setminus 0)$. If $V = U \times \mathbb{R}^N \setminus \{0\}$, U open in $\mathbb{R}^n$ (the local model for $E \setminus 0$), then a vector field

$$L = \sum_{j=1}^n a_j(x,\theta)\frac{\partial}{\partial x_j} + \sum_{k=1}^N b_k(x,\theta)\frac{\partial}{\partial \theta_k}$$

on V is homogeneous of degree ν if and only if the a_j and b_k are homogeneous of degree ν and $\nu+1$, respectively.

It follows that $S^\mu_\rho(U \times \mathbb{R}^N)$ is precisely the space of all $a \in C^\infty(U \times \mathbb{R}^N)$ such that for any compact subset K of U and any multi-indices α, β we have an estimate of the form:

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial \theta}\right)^\alpha a(x,\theta)\right| \le C_{\alpha,\beta,K}(1+|\theta|)^{\mu-\rho|\alpha|+(1-\rho)|\beta|} \tag{2.1.12}$$

for $x \in K$, $\theta \in \mathbb{R}^N \setminus \{0\}$. This is the original form in which Hörmander introduced the symbol spaces $S^\mu_{\rho,\delta}$, $\delta = 1-\rho$. The inequalities (2.1.12) are invariant under changes of the local trivialization of E in view of Proposition 2.1.1.

The space $S^\mu_\rho(E)$ will be topologized by taking the best possible constants $C_{\alpha,\beta,K}$ in (2.1.12) as semi-norms. With this topology $S^\mu_\rho(E)$ is a Fréchet space. (A similar topology can be introduced on $S^\mu_\rho(V)$ for an arbitrary conic manifold V but we will not use this in the sequel.) A subset M of $S^\mu_\rho(E)$ is bounded if all seminorms are bounded on M. Because of the theorem of Ascoli, the topology of pointwise convergence and that of $S^{\mu'}_\rho(E)$, $\mu' > \mu$ are identical on bounded subsets of $S^\mu_\rho(E)$. This leads to

Proposition 2.1.4. *Let $a \in S^\mu_\rho(E)$, $\chi \in C^\infty(E)$, $\chi(x,\theta) = 1$ for $\theta = 0$ and $\chi(x,\theta) = 0$ for $|\theta| \ge 1$. Define $a_\varepsilon(x,\theta) = \chi(x,\varepsilon\theta) \cdot a(x,\theta)$. Then $a_\varepsilon \in S^{-\infty}(E)$, $a_\varepsilon \to a$ in $S^{\mu'}_\rho(E)$ for $\varepsilon \to 0$, any $\mu' > \mu$.*

Proof. The functions $\chi_\varepsilon(x,\theta) = \chi(x,\varepsilon\theta)$, $\varepsilon \in [0,1]$ form a bounded subset in $S^0(E)$. Because multiplication with a is continuous: $S^0(E) \to S^\mu_\rho(E)$, then a_ε, $\varepsilon \in [0,1]$ form a bounded subset of $S^\mu_\rho(E)$. The proof is completed by remarking that $a_\varepsilon \to a$ pointwise. $\square$

Corollary 2.1.5. *Let A be a linear mapping from the space $f \in C^\infty(E)$ that vanish for large $|\theta|$ to a Fréchet space F, which is continuous for the $S^\mu_\rho(E)$-topology for every $\mu \in \mathbb{R}$. Then there is a unique extension of A to $S^\infty_\rho(E) := \bigcup_{\mu\in\mathbb{R}} S^\mu_\rho(E)$ that is continuous: $S^\mu_\rho(E) \to F$ for all $\mu \in \mathbb{R}$.*

The generalization of this section to symbol densities of order α is left to the reader. It should be remarked however that the standard density ω_0 of order α in $\mathbb{R}^N$ is homogeneous of degree αN, in the sense that $\tau^*\omega_0 = \tau^{\alpha N} \cdot \omega_0$ for all $\tau \in \mathbb{R}_+$. The relation

$$a \cdot \omega_0 \leftrightarrow a \tag{2.1.13}$$

is an identification between $S^\mu_\rho(\mathbb{R}^n \times \mathbb{R}^N, \Omega_\alpha)$ and $S^{\mu-\alpha N}_\rho(\mathbb{R}^n \times \mathbb{R}^N)$.

2.2. Distributions defined by oscillatory integrals

Let X be open in $\mathbb{R}^n$. The integral

$$I_\varphi(au) = \iint e^{i\varphi(x,\theta)} a(x,\theta) u(x)\, dx\, d\theta, \qquad u \in C^\infty_0(X) \tag{2.2.1}$$

is absolutely convergent if φ is real, $a \in S^\mu_\rho(X \times \mathbb{R}^N)$, and $\mu + N < 0$. In this case $u \to I_\varphi(au)$ is continuous on $C^0_0(X)$ and therefore defines a distribution A in X of order 0. φ will be called a *phase function* if it is homogeneous of degree 1 and has no critical points as a function of (x,θ). In this case the condition on the order of a can be dropped.

Theorem 2.2.1. *Suppose $\varphi \in C^\infty(X \times \mathbb{R}^N \setminus \{0\})$ is real-valued, homogeneous in θ of degree 1 and $d_{(x,\theta)}\varphi(x,\theta) \neq 0$ for all $(x,\theta) \in X \times \mathbb{R}^N \setminus \{0\}$. Suppose $\rho > 0$. Then the mapping $a \to I_\varphi(au)$, defined for symbols a that vanish for large $|\theta|$, can for every $u \in C^\infty_0(X)$ be extended to $S^\infty_\rho(X \times \mathbb{R}^N)$ such that it is continuous on $S^\mu_\rho(X \times \mathbb{R}^N)$ for every μ. Moreover, for every $a \in S^\mu_\rho(X \times \mathbb{R}^N)$ the linear form $A\colon u \to I_\varphi(au)$ is a distribution of order k if $\mu - k\rho + N < 0$.*

Proof. Let L be a homogeneous C^∞ vector field of degree -1 on $X \times \mathbb{R}^N \setminus \{0\}$ such that $L\varphi = 1$. This exists locally on the sphere bundle $|\theta| = 1$. With a partition of unity the local L can be glued together to a C^∞ vector field on $|\theta| = 1$ such that $L\varphi = 1$. L extends in a unique way to a homogeneous C^∞ vector field of degree -1 on $X\times\mathbb{R}^N\setminus\{0\}$, and because $L\varphi$ is homogeneous of degree 0 we conclude that $L\varphi=1$ on $X\times\mathbb{R}^N\setminus\{0\}$.

Now let $\chi\in C^\infty(X\times\mathbb{R}^N)$, $\chi=0$ for $|\theta|\le 1/2$ and $\chi=1$ for $|\theta|\ge 1$. Then $M=\frac{1}{i}\chi L+(1-\chi)$ is a first order differential operator with smooth coefficients on $X\times\mathbb{R}^N$ and $Me^{i\varphi}=e^{i\varphi}$. Moreover its transposed operator ${}^tM$ maps $S^\mu_\rho(X\times\mathbb{R}^N)$ into $S^{\mu-\rho}_\rho(X\times\mathbb{R}^N)$ for all μ. We obtain

$$I_\varphi(au)=\iint e^{i\varphi}({}^tM)^k(au)\,dx\,d\theta,\qquad \text{any } k, \tag{2.2.2}$$

for any a that vanishes for large $|\theta|$, using repeated partial integrations. So in view of Corollary 2.1.5 the mapping $a\to I_\varphi(au)$ has a continuous extension to $S^\mu_\rho(X\times\mathbb{R}^N)$, which is equal to the absolutely convergent integral (2.2.2) if $m-k\rho+N<0$. Because at most k derivatives of u appear in (2.2.2) this defines a distribution $u\to I_\varphi(au)$ of order k in X. $\square$

If φ,a depend continuously on a parameter t in $C^\infty(X\times\mathbb{R}^N\setminus\{0\})$ and in $S^\mu_\rho(X\times\mathbb{R}^N)$, respectively, then the corresponding distribution A depends continuously on t in view of (2.2.2). This can also be used to justify differentiations with respect to t under the integral sign. Note that we have continuous dependence of a_t in $S^{\mu'}_\rho(X\times\mathbb{R}^N)$ for all $\mu'>\mu$ as soon as the a_t depend continuously on t pointwise and are bounded in $S^\mu_\rho(X\times\mathbb{R}^N)$. See the remark before Proposition 2.1.4.

Theorem 2.2.2. *Let $a\in S^m_\rho$, $\rho>0$ and φ be as in Theorem 2.2.1. Then $WF(A)$ is contained in the closed conic subset*

$$\{(x,d_x\varphi(x,\theta))\in T^*(X)\setminus 0;\quad (x,\theta)\in \operatorname{ess\,supp} a,\ d_\theta\varphi(x,\theta)=0\}$$

of $T^*(X)\setminus 0$. Here ess supp a *is defined as the smallest conic subset of $X\times\mathbb{R}^N\setminus\{0\}$ outside of which a is of class $S^{-\infty}$.*

Proof. We have to prove that the integral

$$\begin{aligned}\langle e^{-i\tau\psi}u,A\rangle &= \iint e^{i[\varphi(x,\theta)-\tau\psi(x,\sigma)]}a(x,\theta)u(x)\,dx\,d\theta\\ &= \tau^N\iint e^{i\tau[\varphi(x,\theta)-\psi(x,\sigma)]}a(x,\tau\theta)u(x)\,dx\,d\theta.\end{aligned} \tag{2.2.3}$$

is rapidly decreasing as $\tau\to\infty$, uniformly for σ in a neighborhood of σ_0, if supp u is contained in a sufficiently small neighborhood U of x_0 and

$$\begin{aligned}&d_x\psi(x_0,\sigma_0)\ne d_x\varphi(x_0,\theta) \text{ for any } \theta \text{ such that}\\ &(x_0,\theta)\in \operatorname{ess\,supp} a, d_\theta\varphi(x_0,\theta)=0.\end{aligned}$$

But this means that $d_{(x,\theta)}\chi(x_0,\theta,\sigma_0) \neq 0$ for all θ with $(x_0,\theta) \in$ ess supp a, if we write

$$\chi(x,\theta,\sigma) = \varphi(x,\theta) - \psi(x,\sigma). \tag{2.2.4}$$

So the proof follows from an application of Proposition 2.1.1, or rather its proof, to (2.2.3). Write (2.2.3) in the form (2.2.2) to be sure of having absolutely convergent integrals and to justify the partial integrations. □

2.3. Oscillatory integrals with nondegenerate phase functions

Theorem 2.3.1. *Let a, φ be as in Theorem 2.2.1, $\rho > 1/2$. Let $\psi \in C^\infty(X \times \Sigma)$, $\xi_0 = d_x\psi(x_0,\sigma_0) \neq 0$,*

$$d_{(x,\theta)}[\varphi - \psi](x_0,\theta_0,\sigma_0) = 0 \tag{2.3.1}$$

and finally

$$d^2_{(x,\theta)}[\varphi - \psi](x_0,\theta_0,\sigma_0) \tag{2.3.2}$$

is nondegenerate.

Then there exists a neighborhood X_0 of x_0, Σ_0 of σ_0, and a conic neighborhood Γ_0 of (x_0,θ_0) in $X \times \mathbb{R}^N \setminus \{0\}$ such that if $u \in C_0^\infty(X)$, supp *$u \subset X_0$, and* ess supp *$a \subset \Gamma_0$, we have the following asymptotic development:*

$$\begin{aligned}\langle e^{-i\tau\psi(\cdot,\sigma)}u, A\rangle \sim e^{-i\tau\psi(x(\sigma),\sigma)} &\cdot \tau^{\frac{1}{2}(N-n)} \\ &\cdot (2\pi)^{\frac{1}{2}(N+n)} \cdot |\det Q(\sigma)|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}\operatorname{sgn} Q(\sigma)} \\ &\cdot \sum_{k=0}^{\infty} \frac{1}{k!}(R(\sigma)^k g(y,\sigma,\tau))_{y=0} \cdot \tau^{-k}\end{aligned} \tag{2.3.3}$$

for $\tau \to \infty$, uniformly in $\sigma \in \Sigma_0$.

Here $(x(\sigma),\theta(\sigma))$ is the unique solution in Γ_0 of

$$d_{(x,\theta)}[\varphi - \psi](x(\sigma),\theta(\sigma),\sigma) = 0, \tag{2.3.4}$$

$$Q(\sigma) = d^2_{(x,\theta)}[\varphi - \psi](x(\sigma),\theta(\sigma),\sigma), \tag{2.3.5}$$

$$g(y(x,\theta,\sigma),\sigma,\tau) \cdot |\det d_{(x,\theta)}y(x,\theta,\sigma)| = a(x,\tau\theta) \cdot u(x), \tag{2.3.6}$$

where $(x,\theta) \to y(x,\theta,\sigma)$ is a diffeomorphism such that

$$y(x(\sigma),\theta(\sigma),\sigma) = 0, \qquad d_{(x,\theta)}y(x(\sigma),\theta(\sigma),\sigma) = I, \tag{2.3.7}$$

and

$$[\varphi - \psi](x, \theta, \sigma) = -\psi(x(\sigma), \sigma) + \tfrac{1}{2}\langle Q(\sigma)y, y\rangle. \tag{2.3.8}$$

Finally we have used the second-order partial differential operator

$$R(\sigma) = \tfrac{1}{2}i\Big\langle Q(\sigma)^{-1}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big\rangle. \tag{2.3.9}$$

Proof. Let $\alpha \in C_0^\infty(X \times \mathbb{R}^N \setminus \{0\})$ be equal to 1 on a neighborhood of (x_0, θ_0) and vanish outside such a small neighborhood of (x_0, θ_0) such that $\varphi - \psi$ has no critical points as a function of $(x, \theta) \in \operatorname{supp}(1 - \alpha)$ if $\sigma \in \Sigma_0$. Then

$$\langle e^{-i\tau\psi}u, A\rangle = I_1(\tau) + I_2(\tau),$$

where

$$I_1(\tau) = \tau^N \iint e^{i\tau(\varphi-\psi)}\, \alpha a u\, dx\, d\theta$$

$$I_2(\tau) = \tau^N \iint e^{i\tau(\varphi-\psi)}(1 - \alpha)\, a u\, dx\, d\theta.$$

Using the proof of Theorem 2.2.2 we see that $I_2(\tau)$ is rapidly decreasing as $\tau \to \infty$, uniformly in $\sigma \in \Sigma_0$. The integral $I_1(\tau)$ has its amplitude supported in a fixed compact subset of (x, θ)-space so (2.3.3) now immediately follows from Lemma 1.2.2 and Proposition 1.2.4. Note that $d_{(x,\theta)}(\varphi - \psi) = 0$ implies $d_\theta\varphi(x, \theta) = 0$ so $\varphi(x, \theta) = \langle\theta, d_\theta\varphi(x, \theta)\rangle = 0$ in view of the homogeneity of φ. $\square$

The asymptotic development (2.3.3) characterizes the distribution A modulo $C^\infty(X)$ in the following sense: If $\tilde{A} \in \mathcal{D}'(X)$ also satisfies (2.3.3) then $x_0 \notin \operatorname{sing\,supp}(A - \tilde{A})$. We now analyze the conditions (2.3.1) and (2.3.2). Let Γ be a cone in $X \times \mathbb{R}^N \setminus \{0\}$. The phase function φ is called *nondegenerate in* Γ if

$$d_\theta\varphi(x, \theta) = 0,\ (x, \theta) \in \Gamma \quad \Rightarrow \quad d_{(x,\theta)}\frac{\partial\varphi(x, \theta)}{\partial\theta_j} \text{ are linearly independent for } j = 1, \dots, N. \tag{2.3.10}$$

The condition (2.3.10) implies that for some open cone $\tilde{\Gamma} \supset \Gamma$ the manifold

$$C_\varphi = \{(x, \theta) \in \tilde{\Gamma};\ d_\theta\varphi(x, \theta) = 0\} \tag{2.3.11}$$

is a conic C^∞ submanifold of $X \times \mathbb{R}^N \setminus 0$ of dimension $(n + N) - N = n$. This is a direct application of the implicit function theorem.

Lemma 2.3.2. *If φ is a nondegenerate phase function, then*

$$T^{(\varphi)} : (x, \theta) \mapsto (x, d_x\varphi(x, \theta)) \tag{2.3.12}$$

is an immersion: $C_\varphi \to T^*(X) \setminus 0$, commuting with the multiplication with positive real numbers in the fibers. So its image Λ_φ is an immersed n-dimensional conic submanifold of $T^*(X) \setminus 0$.

Proof. We have to prove that

$$(\delta x, \delta\theta) \in T_{(x,\theta)}(C_\varphi),\ DT^{(\varphi)}_{(x,\theta)}(\delta x, \delta\theta) = 0 \quad \Rightarrow \quad (\delta x, \delta\theta) = 0.$$

Now

$$(\delta x, \delta\theta) \in T_{(x,\theta)}(C_\varphi) \quad \Leftrightarrow \quad d_x d_\theta\varphi\, \delta x + d_\theta d_\theta\varphi\, \delta\theta = 0$$

and

$$DT^{(\varphi)}_{(x,\theta)}(\delta x, \delta\theta) = (\delta x, d_x d_x\varphi\, \delta x + d_\theta d_x\varphi\, \delta\theta)$$

so we have to prove that $d_\theta d_x\varphi\, \delta\theta = 0$, $d_\theta d_\theta\varphi\, \delta\theta = 0$ implies that $\delta\theta = 0$. But this is exactly the condition that φ is nondegenerate. □

Lemma 2.3.3. *The following assertions are equivalent.*

(i) *$\varphi - \psi$ has a nondegenerate stationary point as a function of (x, θ) in $x = x_0$, $\theta = \theta_0$, $\sigma = \sigma_0$.*

(ii) a) *φ is a nondegenerate phase function in a conic neighborhood of (x_0, θ_0) and*

b) *The graph of $d_x\psi$ intersects Λ_φ transversally in the point $(x_0, \xi_0) \in T^*(X) \setminus 0$, where*

$$\xi_0 = d_x\varphi(x_0, \theta_0) = d_x\psi(x_0, \sigma_0).$$

Proof. First note that $d_{(x,\theta)}(\varphi - \psi)(x_0, \theta_0, \sigma_0) = 0 \Leftrightarrow (x_0, d_x\psi(x_0, \sigma_0)) = (x_0, d_x\varphi(x_0, \theta_0)) \in \Lambda_\varphi$.

Second, the stationary point is nondegenerate if and only if:

$$\begin{gathered} d_x(d_x\varphi - d_x\psi) \cdot \delta x + d_\theta d_x\varphi \cdot \delta\theta = 0 \\ \text{and} \\ d_x d_\theta\varphi \cdot \delta x + d_\theta d_\theta\varphi \cdot \delta\theta = 0 \end{gathered} \tag{2.3.13}$$

imply that $\delta x = 0$, $\delta\theta = 0$. In particular (2.3.13), taking $\delta x = 0$, implies

$$d_\theta d_x\varphi \cdot \delta\theta = 0,\ d_\theta d_\theta\varphi \cdot \delta\theta = 0 \quad \Rightarrow \quad \delta\theta = 0,$$

that is, φ is a nondegenerate phase function.

If conversely φ is a nondegenerate phase function, then Λ_φ has tangent space in $(x_0, d_x\varphi(x_0,\theta_0))$ equal to

$$\big\{(\delta x, d_x d_x\varphi \cdot \delta x + d_\theta d_x\varphi \cdot \delta\theta);\ d_x d_\theta\varphi \cdot \delta x + d_\theta d_\theta\varphi \cdot \delta\theta = 0\big\}.$$

The tangent space to the graph of $d_x\psi$ is equal to $\{(\delta x, d_x d_x\psi \cdot \delta x);\ \delta x \in T_{x_0}(X)\}$. The condition (2.3.13) means precisely that these spaces have zero intersection, that is, are transversal. (Note that $d\psi$ and Λ_φ have dimension equal to n and $\dim T^*(X) = 2n$.) $\square$

Lemma 2.3.3 implies that if φ is a nondegenerate phase function on ess supp a, then the condition (2.3.2) is satisfied for the general function ψ satisfying (2.3.1). Indeed for any $(x_0,\xi_0) \in T^*(X)$ and any symmetric $n\times n$ matrix B there is a C^∞ function ψ on X such that $d\psi(x_0) = \xi_0$, $d^2\psi(x_0) = B$. Here we use that the set of symmetric B such that $\{(\delta x, B\delta x);\ \delta x \in T_{x_0}(X)\}$ is transversal to $T_{(x_0,\xi_0)}(\Lambda_\varphi)$ is open and dense in the space of all symmetric B. (See Theorem 3.3.7.)

From (2.3.3) we see therefore that if $\varphi(x,\theta)$ and $\tilde\varphi(x,\tilde\theta)$ are nondegenerate phase functions at $(x_0,\theta_0) \in X\times\mathbb{R}^N\setminus\{0\}$ and $(x_0,\tilde\theta_0) \in X\times\mathbb{R}^{\tilde N}\setminus\{0\}$, respectively, then any distribution A, defined by the phase function φ and some amplitude a such that ess supp a is contained in a conic neighborhood of (x_0,θ_0), is modulo $C^\infty(X)$ equal to a distribution defined by the phase function $\tilde\varphi$ and some other amplitude $\tilde a$, only if the corresponding manifolds Λ_φ, $\Lambda_{\tilde\varphi}$ coincide. The following theorem shows that also the converse is true.

Theorem 2.3.4. *Suppose $\varphi(x,\theta)$ and $\tilde\varphi(x,\tilde\theta)$ are nondegenerate phase functions at $(x,\theta_0) \in X\times\mathbb{R}^N\setminus\{0\}$ and at $(x_0,\tilde\theta_0) \in X\times\mathbb{R}^{\tilde N}\setminus\{0\}$, respectively. Let Γ and $\tilde\Gamma$ be open conic neighborhoods of (x_0,θ_0) and $(x_0,\tilde\theta_0)$ such that $T_\varphi\colon C_\varphi \to \Lambda_\varphi$ and $T_{\tilde\varphi}\colon C_{\tilde\varphi} \to \Gamma_{\tilde\varphi}$ are injective, respectively. If $\Lambda_\varphi = \Lambda_{\tilde\varphi}$ then any Fourier integral A, defined by the phase function φ and an amplitude $a \in S^\mu_\rho(X\times\mathbb{R}^N)$, $\rho > 1/2$, with* ess supp *a contained in a sufficiently small conic neighborhood of (x_0,θ_0), is equal to a Fourier integral defined by the phase function $\tilde\varphi$ and an amplitude $\tilde a \in S_\rho^{\mu+\frac{1}{2}(N-\tilde N)}(X\times\mathbb{R}^{\tilde N})$.*

Proof. We start by reducing the number of θ-variables as far as possible. Let $(x_0,\theta_0) \in \Gamma$. Applying an orthogonal transformation in the θ-variables we can assume that

$$
\begin{aligned}
&\theta = (\theta', \theta''), \quad \theta' = (\theta_1, \dots, \theta_k), \quad \theta'' = (\theta_{k+1}, \dots, \theta_N), \\
&\qquad d^2_{\theta'}\varphi(x_0, \theta_0) = 0, \quad d_{\theta'}d_{\theta''}\varphi(x_0, \theta_0) = 0, \\
&\qquad\quad \text{and finally } d^2_{\theta''}\varphi(x_0, \theta_0) \text{ is nondegenerate.}
\end{aligned}
\tag{2.3.14}
$$

Without loss of generality we may also assume that $\theta''_0 = 0$. (2.3.14) implies that we can solve $\theta'' = \theta''(x, \theta')$ from the equation $d_{\theta''}\varphi(x, \theta', \theta'') = 0$, $\theta''(x_0, \theta'_0) = 0$. Write $\varphi(x, \theta', \theta'') = \varphi_1(x, \theta') + \psi(x, \theta', (\psi(x, \theta', \theta''))\theta'')$ where $\varphi_1(x, \theta') = \varphi\big(x, \theta', \theta''(x, \theta')\big)$. With these notations, we now write

$$
\begin{aligned}
\langle u, A\rangle &= \iiint e^{i\varphi(x,\theta',\theta'')} a(x, \theta', \theta'')\, u(x)\, dx\, d\theta' d\theta'' \\
&= \iint e^{i\varphi_1(x,\theta')} b(x, \theta')\, u(x)\, dx\, d\theta',
\end{aligned}
\tag{2.3.15}
$$

where

$$
b(x, \theta') = \int e^{i\psi(x,\theta',\theta'')} a(x, \theta', \theta'')\, d\theta''. \tag{2.3.16}
$$

Here $d^2_{\theta''}\psi(x_0, \theta_0)$ is nondegenerate, so if $a \in S^\mu_\rho(X \times \mathbb{R}^N)$, $\rho > 1/2$, and ess supp a contained in a sufficiently small conic neighborhood of (x_0, θ_0) we see from Section 1.2 that $b \in S^{\mu + \frac{1}{2}(N-k)}_\rho(X \times \mathbb{R}^k)$.

Now $d_{\theta'}d_{\theta''}\varphi(x_0, \theta_0) = 0$ implies that φ_1 is a nondegenerate phase function on a conic neighborhood of (x_0, θ'_0) and $\Lambda_{\phi_1} = \Lambda_\phi$ (locally).

$d^2_{\theta'}\varphi(x_0, \theta_0) = 0$ implies that $\{x_0\} \times \mathbb{R}^{N-k}$ is contained in $T_{(x_0,\theta'_0)}(C_{\varphi_1})$, so the differential of the projection $C_{\varphi_1} \ni (x, \theta') \to x$ has in (x_0, θ'_0) rank equal to $n - k$. This rank is equal to the rank of the projection $\Lambda_{\varphi_1} \ni (x, \xi) \to x \in X$ in (x_0, ξ_0). So the kernel of the latter projection has rank equal to k and we have proved

Lemma 2.3.5. *The number of θ-variables is $\geq$ the dimension k of the intersection of the tangent spaces of Λ_φ, and the fiber of $T^*(X)$, at (x_0, ξ_0). Moreover, every Fourier integral defined by the phase function φ and amplitude $a \in S^\mu_\rho(X \times \mathbb{R}^N)$,* ess supp *$a$ in a sufficiently small conic neighborhood of (x_0, θ_0), can also be defined by a phase function in k variables and amplitude $b \in S^{\mu + \frac{1}{2}(N-k)}_\rho(X \times \mathbb{R}^k)$ given by* (2.3.16).

Conversely we can always raise the number of frequency variables from k to an arbitrary $\tilde{N} \geq k$, because any $b \in S^{\mu + \frac{1}{2}(N-k)}_\rho(X \times \mathbb{R}^k)$ arises by (2.3.16) from some $\tilde{a} \in S^{\mu + \frac{1}{2}(N - \tilde{N})}(X \times \mathbb{R}^{\tilde{N}})$, if we replace φ by $\tilde{\varphi}$ in the definition preceding (2.3.16). So we now may assume that $N = \tilde{N}$ $(= k)$. In view of the formula

$$\iint e^{i\tilde{\varphi}(x,\tilde{\theta})} a(x,\tilde{\theta})u(x)\,dx\,d\tilde{\theta} = \iint e^{i\varphi(x,\theta)} a(x,\tilde{\theta}(x,\theta)) \cdot |\det d_\theta\tilde{\theta}(x,\theta)| \cdot u(x)\,dx\,d\theta, \tag{2.3.17}$$

the proof of Theorem 2.3.4 is therefore completed by

Lemma 2.3.6. *If $\Lambda_\varphi = \Lambda_{\tilde{\varphi}} = \Lambda$, $N = \tilde{N} =$ minimal, then there exists a C^∞ mapping $(x,\theta) \to \tilde{\theta}(x,\theta)$: $\Gamma \to \mathbb{R}^{\tilde{N}} \setminus \{0\}$, homogeneous in θ of degree 1, such that*

$$\tilde{\theta}(x_0,\theta_0) = \tilde{\theta}_0, \qquad \tilde{\varphi}(x,\tilde{\theta}(x,\theta)) = \varphi(x,\theta) \quad \text{on } \Gamma. \tag{2.3.18}$$

Proof of Lemma 2.3.6. The diffeomorphisms $T_\varphi : C_\varphi \to \Lambda$, $T_{\tilde{\varphi}} : C_{\tilde{\varphi}} \to \Lambda$ induce a diffeomorphism $T := T_{\tilde{\varphi}}^{-1} \circ T_\varphi$: $C_\varphi \to C_{\tilde{\varphi}}$. If π denotes the projection $(x,\theta) \mapsto x$, then we have $\pi\big(T(x,\theta)\big) = x$ on C_φ, so $T(x,\theta) = \big(x, \tilde{\theta}(x,\theta)\big)$ for some C^∞ function $\tilde{\theta}(x,\theta)$ on C_φ. $\tilde{\theta}$ is homogeneous in θ of degree 1 because T_φ and $T_{\tilde{\varphi}}$ are, and we have

$$\begin{aligned} \tilde{\varphi}(x,\tilde{\theta}(x,\theta)) &= \varphi(x,\theta) = 0, \\ d_{\tilde{\theta}}\tilde{\varphi}(x,\tilde{\theta}(x,\theta)) &= d_\theta(x,\theta) = 0, \\ d_x\tilde{\varphi}(x,\tilde{\theta}(x,\theta)) &= d_x\varphi(x,\theta), \quad \text{on } C_\varphi. \end{aligned} \tag{2.3.19}$$

Now extend $\tilde{\theta}$ to a homogeneous C^∞ function of degree 1 on a conic neighborhood in $X \times (\mathbb{R}^N \setminus \{0\})$ of C_φ, define $\psi(x,\theta) = \tilde{\varphi}(x,\tilde{\theta}(x,\theta))$ there. Then ψ is a nondegenerate phase function, $\Lambda_\psi = \Lambda$ and $\psi - \varphi$ vanishes of order 2 on $C_\varphi = C_\psi$. We are ready if we can find a homogeneous C^∞ function $\theta'(x,\theta)$ of degree 1 such that $\psi(x,\theta'(x,\theta)) = \varphi(x,\theta)$.

Because C_ψ is equal to the manifold $d_\theta\psi = 0$ and we can choose $\frac{\partial\psi}{\partial\theta_1}, \dots, \frac{\partial\psi}{\partial\theta_N}$ as the first N coordinates in a coordinization, Taylor development at C_ψ gives

$$(\psi - \varphi)(x,\theta) = B(x,\theta)(d_\theta\psi(x,\theta), d_\theta\psi(x,\theta)) \tag{2.3.20}$$

for some symmetric bilinear form $B(x,\theta)$, depending C^∞ on (x,θ) and being homogeneous of degree 1 in θ. Analogously

$$\psi(x,\theta+\mu) = \psi(x,\theta) + \langle d_\theta\psi(x,\theta), \mu\rangle + C(x,\theta)(\mu,\mu) \tag{2.3.21}$$

for some other symmetric bilinear form $C(x,\theta)$, homogeneous in θ of degree -1.

Now try $\theta'(x,\theta) = \theta + W(x,\theta)d_\theta\psi(x,\theta)$ for some W. This leads to the equation

$$W + W'CW = B \tag{2.3.22}$$

on C_ψ. This equation has the solution $W = 0$ if $B = 0$, so the implicit function theorem gives a unique C^∞ solution $W(x,\theta)$ on a conic neighborhood of C_ψ, homogeneous in θ of degree 1. Note that $B(x_0,\theta_0) = 0$ because at (x_0,θ_0) we have $d_\theta^2\varphi = d_\theta^2\psi = 0$, which also implies that $d_x d_\theta\varphi = d_x d_\theta\psi$. □

The above theorems show that one should rather speak of distributions A *defined by a conic manifold* Λ in $T^*(X)\setminus 0$, which locally is equal to Λ_φ, φ a nondegenerate phase function, instead of distributions defined by some phase function φ. A differential geometric characterization of such manifolds Λ will be given in Chapter 3, after we have studied the differential geometric structure of the cotangent bundle $T^*(X)$ in more detail. After Chapter 3 we will also be able to give an invariant (that is, independent of the choice of the phase function) characterization of the principal symbol of A. In a primitive way the principal symbol of A at (x_0,ξ_0) can be defined as the mapping assigning to each $\psi \in C^\infty(X)$, with $\xi_0 = d_x\psi(x_0) \neq 0$ the top order term in the asymptotic development of $e^{i\tau\psi(x_0)}\cdot\langle e^{-i\tau\psi}u, A\rangle$ for $\tau \to \infty$, that is, the quantity

$$\begin{aligned}\tau^{\frac{1}{2}(N-n)}(2\pi)^{\frac{1}{2}(N+n)}\cdot|\det Q|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}\operatorname{sgn} Q}\\ \cdot a(x_0,\tau\theta_0)\cdot u(x_0) + O(\tau^{\mu+\frac{1}{2}(N-n)+(1-2\rho)})\quad\text{for } \tau\to\infty.\end{aligned} \tag{2.3.23}$$

Since the dependence on ψ only involves $d_x^2\psi(x_0)$ we see that the principal symbol s is a function on the space of symmetric matrices B such that $\{(\delta x, B\cdot\delta x)\}$ is transversal to $T_{(x_0,\xi_0)}(\Lambda_\varphi)$. We see that

$$s(B') = t(B',B)\cdot s(B) \tag{2.3.24}$$

where the factor $t(B',B)$ does not depend on the amplitude a. The collection of all functions s satisfying (2.3.24) therefore is a complex one-dimensional space $L_\varphi(x_0,\xi_0)$, which only depends on Λ_φ. The $L_\varphi(x_0,\xi_0)$, $(x_0,\xi_0)\in\Lambda_\varphi$ form a complex line bundle L over Λ_φ of which the principal symbol of A becomes a section. Also this complex line bundle L can only be understood better after we have obtained more insight into the differential geometric structure of $T^*(X)$.

2.4. Fourier integral operators (local theory)

A *Fourier integral operator* is defined as an operator $A: C_0^\infty(Y) \to \mathcal{D}'(X)$ such that the distribution kernel $K_A \in \mathcal{D}'(X \times Y)$ is a Fourier integral defined by a nondegenerate phase function in an open cone Γ in $T^*(X \times Y) \setminus 0$ and some amplitude $a \in S_\rho^\mu(X \times Y \times \mathbb{R}^N)$, ess supp $a \subset \Gamma$. We have $WF'(A) \subset \Lambda'_\varphi$ if we write

$$\Lambda' = \{((x,\xi),(y,\eta)) \in T^*(X) \times T^*(Y); (x,y,\xi,-\eta) \in \Lambda\}. \tag{2.4.1}$$

for any subset Λ of $T^*(X \times Y)$. Conversely each $(x,\xi) \in \Lambda'_\varphi$ can be made to be an element of $WF'(A)$ by a suitable choice of the amplitude, using Theorem 2.3.1.

According to Theorem 1.4.1 the operator A can be extended to $\mathcal{D}'_V(Y) \cap \mathcal{E}'(Y)$ (and to $\mathcal{D}'_V(Y)$ if suitable assumptions are made on supp K_A) if there are no $((x,\xi),(y,\eta)) \in \Lambda'_\varphi$ such that $\xi = 0$ and $(y,\eta) \in V$. In particular A can be extended to $\mathcal{E}'(Y)$ (and to $\mathcal{D}'(Y)$, respectively) if $d_x\varphi \neq 0$ when $d_\theta\varphi = 0$, that is, if φ has no stationary points as a function of (x,θ). On the other hand A maps $C_0^\infty(Y)$ into $C^\infty(X)$ if φ has no stationary points as a function of (y,θ). Note that it was included in the definition of a phase function that φ has no stationary points as a function of (x,y,θ).

Examples (more will follow in Chapters 4 and 5).

(1) Fourier integrals in X can be regarded as Fourier integral operators by taking $Y = \{\text{point}\}$.

(2) Let κ be a smooth map from X to Y. Then

$$(\kappa^* u)(x) = u(\kappa(x)) = (2\pi)^{-n} \iint e^{i\langle \kappa(x)-y,\eta\rangle} u(y)\, dy\, d\eta,$$

and it follows that $\kappa^*: C^\infty(Y) \to C^\infty(X)$ is a Fourier integral operator defined by a nondegenerate phase function φ such that

$$\Lambda'_\varphi = \{((x,\xi),(y,\eta)); y = \kappa(x), \xi = {}^tD\kappa_x \cdot \eta\}. \tag{2.4.2}$$

If κ is a diffeomorphism, then Λ'_φ is the graph of the induced transformation $\tilde{\kappa}: T^*(X) \setminus 0 \to T^*(Y) \setminus 0$ defined by

$$\tilde{\kappa}(x,\xi) = (\kappa(x), ({}^tD\kappa_x)^{-1}(\xi)). \tag{2.4.3}$$

If X is a submanifold of Y, $\dim X < \dim Y$ and κ is the identity: $X \to Y$ then κ^* is the restriction operator $\rho: C^\infty(Y) \to C^\infty(X)$. In this case

$$\Lambda'_\varphi = \{((x,\xi),(y,\eta)); y = x, \xi = \eta|_{T_x(X)}\} \tag{2.4.4}$$

which is far from the graph of a map. ρ can be extended continuously to $\mathcal{D}'_\Gamma(Y)$, for any closed cone Γ in $T^*(Y) \setminus 0$ that does not meet the set

$$\{(y,\eta) \in T^*(Y) \setminus 0; y \in X, \eta|_{T_x(X)} = 0\}, \tag{2.4.5}$$

that is, the *normal bundle in* $T^*(Y) \setminus 0$ *of the submanifold* X.

(3) *Pseudodifferential operators in* X are defined as Fourier integral operators with $Y = X$ and

$$\begin{aligned} \Lambda'_\varphi &\subset \text{diagonal in } T^*(X) \setminus 0 \times T^*(X) \setminus 0 \\ &= \text{graph of the identity: } T^*(X) \setminus 0 \to T^*(X) \setminus 0. \end{aligned} \tag{2.4.6}$$

These operators will be considered in more detail in Section 2.5.

In view of Theorem 1.3.7 the following product theorem seems the natural one. We denote

$$\operatorname{diag} V = \{(v,v) \in V \times V; v \in V\} \tag{2.4.7}$$

for any set V.

Theorem 2.4.1. *Let X, Y, Z be open in $\mathbb{R}^{n_X}$, $\mathbb{R}^{n_Y}$, $\mathbb{R}^{n_Z}$ respectively. Let A_1 be a Fourier integral operator: $C_0^\infty(Y) \to \mathcal{D}'(X)$ defined by a nondegenerate phase function φ_1 in an open cone Γ_1 in $X \times Y \times \mathbb{R}^{N_1} \setminus \{0\}$ and an amplitude $a_1 \in S_\rho^{\mu_1}(X \times Y \times \mathbb{R}^{N_1})$,* ess supp $a_1 \subset \Gamma_1$. *Similarly A_2 is a Fourier integral operator: $C_0^\infty(Z) \to \mathcal{D}'(Y)$ defined by a nondegenerate phase function φ_2 in an open cone Γ_2 in $Y \times Z \times \mathbb{R}^{N_2} \setminus \{0\}$ and an amplitude $a_2 \in S_\rho^{\mu_2}(Y \times Z \times \mathbb{R}^{N_2})$,* ess supp $a_2 \subset \Gamma_2$. *Assume that $\rho > 1/2$ and:*

$$\begin{aligned} &\textit{The projection from } \pi_{X\times Y}(\operatorname{supp} a_1) \times \pi_{Y\times Z}(\operatorname{supp} a_2) \\ &\cap X \times (\operatorname{diag} Y) \times Z \textit{ into } X \times Z \textit{ is a proper mapping,} \end{aligned} \tag{2.4.8}$$

$$\eta \neq 0 \;\; \textit{if} \;\; (x,\xi,y,\eta) \in \Lambda'_{\varphi_1} \;\; \textit{or} \;\; (y,\eta,z,\xi) \in \Lambda'_{\varphi_2}, \tag{2.4.9}$$

$$\xi \neq 0 \textit{ or } \zeta \neq 0 \textit{ if } (x,\xi,y,\eta) \in \Lambda'_{\varphi_1} \;\; \textit{and} \;\; (y,\eta,z,\zeta) \in \Lambda'_{\varphi_2} \tag{2.4.10}$$

(2.4.11) $\quad \Lambda'_{\varphi_1} \times \Lambda'_{\varphi_2}$ *intersects* $T^*(X) \times (\operatorname{diag} T^*(Y)) \times T^*(Z)$ *transversally.*

Then $A_1 \circ A_2$ *is well defined and modulo an operator with* C^∞ *kernel equal to a Fourier integral operator:* $C_0^\infty(Z) \to \mathcal{D}'(X)$ *defined by a nondegenerate phase function* φ *in an open cone* Γ *in* $X \times Z \times \mathbb{R}^N \setminus \{0\}$, $N = N_1 + N_2 + n_Y$, *and an amplitude* $a \in S^\mu_\rho(X \times Z \times \mathbb{R}^N)$, $\mu = \mu_1 + \mu_2 - n_Y$, ess supp $a \subset \Gamma$. *Moreover,*

$$\Lambda'_\varphi = \Lambda'_{\varphi_1} \circ \Lambda'_{\varphi_2}. \tag{2.4.12}$$

Proof. $A_1 \circ A_2$ is well defined in view of (2.4.8), (2.4.9) and Theorem 1.4.1, and we have

$$\begin{aligned} K_{A_1 \circ A_2}(x, z) = \iiint & e^{i[\varphi_1(x,y,\theta)+\varphi_2(y,z,\sigma)]} a_1(x, y, \theta) \\ & \cdot a_2(y, z, \sigma)\, dy\, d\theta\, d\sigma \end{aligned} \tag{2.4.13}$$

in the distribution sense. If we let x, z vary in a compact subset of $X \times Z$ then the corresponding $(x, z, \theta, \sigma, y)$ such that $|(\theta, \sigma)| = (|\theta|^2 + |\sigma|^2)^{1/2} = 1$, $(x, y, \theta) \in \operatorname{supp} a_1$, $(y, z, \sigma) \in \operatorname{supp} a_2$, vary in a compact set in view of (2.4.8). What follows will only refer to such $(x, z, \theta, \sigma, y)$.

Because $d_{(y,\theta)}\varphi_1(x, y, \theta) \neq 0$ if $\theta \neq 0$ in view of (2.4.9) there exists $0 < \varepsilon < 1$ such that $|\sigma| \leq \varepsilon|\theta|$,

$$|(\theta, \sigma)| = 1 \Rightarrow d_{(y,\theta)}[\varphi_1 + \varphi_2](x, z, \theta, \sigma, y) \neq 0.$$

Let χ_1 be a homogeneous C^∞ function of degree 0 in (θ, σ) such that $\chi_1 = 1$ for $|\sigma| \leq \frac{1}{2}\varepsilon|\theta|$ and $\chi_1 = 0$ for $|\sigma| > \varepsilon|\theta|$ if $|(\theta, \sigma)| = 1$. It follows, using partial integration in (x, θ) and using the bound $|\sigma| \leq \varepsilon|\theta|$ for σ, that

$$\iiint e^{i(\varphi_1+\varphi_2)} \chi_1 \cdot a_1 \cdot a_2\, dy\, d\theta\, d\sigma \tag{2.4.14}$$

is a C^∞ function of (x, z).

Similarly there is a homogeneous C^∞ function of degree 0 in (θ, σ) such that $\chi_2 = 1$ for $|\theta| \leq \frac{1}{2}\varepsilon|\sigma|$, $\chi_2 = 0$ for $|\theta| > \varepsilon|\sigma|$ if $|(\theta, \sigma)| = 1$ and

$$\iiint e^{i(\varphi_1+\varphi_2)} \chi_2 \cdot a_1 \cdot a_2\, dy\, d\theta\, d\sigma \tag{2.4.15}$$

is a C^∞ function of (x, z). So we are left with the integral

$$\iiint e^{i[\varphi_1+\varphi_2]} b\, dy\, d\theta\, d\sigma, \quad \text{where} \quad b = (1 - \chi_1 - \chi_2) a_1 a_2. \tag{2.4.16}$$

Notice that

$$\tfrac{1}{2}\varepsilon|\sigma| \le |\theta| \le 2\varepsilon^{-1}|\sigma| \quad \text{on} \quad \operatorname{supp} b. \tag{2.4.17}$$

We claim that (2.4.16) is a Fourier integral with phase

$$\varphi(x,z,(\theta,\sigma,\tilde{y})) = \varphi_1(x,\tilde{y}/|(\theta,\sigma)|,\theta) + \varphi_2(\tilde{y}/|(\theta,\sigma)|,z,\sigma) \tag{2.4.18}$$

(frequency variables $(\theta,\sigma,\tilde{y})$), and amplitude

$$a(x,z,(\theta,\sigma,\tilde{y})) = b(x,z,\theta,\sigma,\tilde{y}/|(\theta,\sigma)|)\cdot|(\theta,\sigma)|^{-n_Y}. \tag{2.4.19}$$

Indeed, because of (2.4.17) we stay away from the boundaries $\theta = 0$, $\sigma \ne 0$ and $\theta \ne 0$, $\sigma = 0$ of the region where φ is a C^∞ function. Moreover (2.4.10) implies that $d_x\varphi_1 \ne 0$ or $d_z\varphi_2 \ne 0$ if $d_\theta\varphi_1 = 0$, $d_\sigma\varphi_2 = 0$, $d_y(\varphi_1+\varphi_2) = 0$, so φ is a phase function as defined in Section 2.2.

Inequality (2.4.17) also implies that $a \in S^\mu_\rho(X \times Z \times \mathbb{R}^N)$. Indeed, derivation with respect to, say, θ improves by a factor $(1+|\theta|)^{-\rho}$, which in fact is an improvement by a factor $(1+|\theta|+|\sigma|)^{-\rho}$ in view of $|\theta| \ge \frac{1}{2}\varepsilon|\sigma|$.

We now investigate (2.4.11) and (2.4.12). $\Lambda'_{\varphi_1}\times\Lambda'_{\varphi_2}$ intersects $T^*(X)\times(\operatorname{diag} T^*(Y))\times T^*(Z)$ precisely at the points $(x, d_x\varphi_1, y, -d_y\varphi_1, y, d_y\varphi_2, z, -d_z\varphi_2)$ where $-d_y\varphi_1 = d_y\varphi_2$, $d_\theta\varphi_1 = 0$, $d_\theta\varphi_2 = 0$, which proves (2.4.12).

The tangent space of Λ'_{φ_1} consists of the vectors

$$\begin{aligned} &(\delta x, d(d_x\varphi_1)u,\ \delta y,\ -d(d_y\varphi_1)u),\\ &\qquad \text{such that}\quad d(d_\theta\varphi_1)u = 0, \quad u = (\delta x,\delta y,\delta\theta), \end{aligned} \tag{2.4.20}$$

and the tangent space of Λ'_{φ_2} consists of the vectors

$$\begin{aligned} &(\delta y, d(d_y\varphi_2)v,\ \delta z,\ -d(d_z\varphi_2)v),\\ &\qquad \text{such that}\quad d(d_\sigma\varphi_2)v = 0, \quad v = (\delta y,\delta z,\delta\sigma), \end{aligned} \tag{2.4.21}$$

The intersection of $T(\Lambda'_{\varphi_1}\times\Lambda'_{\varphi_2}) = T(\Lambda'_{\varphi_1})\times T(\Lambda'_{\varphi_2})$ with $T\big(T^*(X)\times(\operatorname{diag} T^*(Y))\times T^*(Z)\big)$ therefore has the same dimension as the kernel of $d\, d_{(\theta,\sigma,y)}[\varphi_1+\varphi_2]$. The intersection is transversal if and only if this dimension is equal to

$$\begin{aligned} ((n_X+n_Y)+(n_Y+n_Z)) &+ (2n_X+2n_Y+2n_Z)\\ &- (2n_X+4n_Y+2n_Z) = n_X+n_Z, \end{aligned}$$

that is, if and only if rank $d\, d_{(\theta,\sigma,y)}[\varphi_1+\varphi_2] = N_1+N_2+n_Y$. But this means precisely that φ is a nondegenerate phase function. □

We conclude this section by a little philosophy on the question: what should we call the *order* of a Fourier integral operator defined by a phase φ and an amplitude $a \in S^\mu_\rho(X \times Y \times \mathbb{R}^N)$? Suppose that the order is a function of μ, $n_X + n_Y$, and N, and suppose that the order of a product (when defined) is equal to the sum of the orders. In view of Lemma 2.3.5 it must be of the form $\mu + \frac{1}{2}N + f(n_X + n_Y)$, and the additivity of the orders means that

$$f(n_X + n_Z) = f(n_X + n_Y) + f(n_Y + n_Z) + \tfrac{1}{2}n_Y$$

in view of Theorem 2.4.1. So we are led to define the order of a Fourier integral operator $A\colon C_0^\infty(Y) \to \mathcal{D}'(X)$ defined by an amplitude of order μ as

$$\text{order}\,(A) = \mu + \tfrac{1}{2}N - \tfrac{1}{4}(n_X + n_Y). \tag{2.4.22}$$

With this choice Theorem 2.4.1 can be supplemented by:

$$\text{order}\,(A_1 \circ A_2) = \text{order}\, A_1 + \text{order}\, A_2. \tag{2.4.23}$$

2.5. Pseudodifferential operators in $\mathbb{R}^n$

A *pseudodifferential operator of order m and type ρ* on $\mathbb{R}^n$ is defined as a Fourier integral operator A with phase

$$\varphi(x, y, \eta) = \langle x - y, \eta \rangle, \tag{2.5.1}$$

and amplitude $a(x, y, \eta) \in S^m_\rho((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^n)$. Observe that the order m coincides with the order defined at the end of Section 2.4. Writing $X = \mathbb{R}^n$, we see that $\Lambda'_\varphi = \text{diag}\, T^*(X) \setminus 0 =$ graph of the identity: $T^*(X) \setminus 0 \to T^*(X) \setminus 0$, and in view of Theorem 2.3.4 we could replace (2.5.1) by any nondegenerate phase function $\varphi(x, y, \theta)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N$ such that $d_\theta \varphi(x, y, \theta) = 0 \Rightarrow x = y$, $d_x \varphi(x, y, \theta) = -d_y \varphi(x, y, \theta)$.

Because of $\Lambda'_\varphi = \text{diag}\, T^*(X) \setminus 0$ we can write

$$WF'(A) = \text{diag}\, WF(A) \tag{2.5.2}$$

for a uniquely defined closed conic subset $WF(A)$ of $T^*(\mathbb{R}^n) \setminus 0$, called the *wave front set of the pseudodifferential operator A* (by abuse of language). The space of all pseudodifferential operators of order m and type ρ will be denoted by $L^m_\rho(\mathbb{R}^n)$.

Definition 2.5.1. Let $A \in L^m_\rho(\mathbb{R}^n)$, $\rho > 1/2$, be properly supported. The *complete symbol* σ_A of A is defined by

$$\sigma_A(x,\eta) = e^{-i\langle x,\eta\rangle} A(e^{i\langle\cdot,\eta\rangle})(x). \tag{2.5.3}$$

We now compare this with the definition of the symbol in the introduction:

Theorem 2.5.1. *Let A be as in Definition* 2.5.1. *Then*

$$(Au)(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\eta\rangle} \sigma_A(x,\eta) u(y)\, dy\, d\eta \tag{2.5.4}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. If A is given by the amplitude $a(x,y,\eta)$, then

$$\sigma_A(x,\eta) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{j=1}^{n} \frac{1}{i} \frac{\partial^2}{\partial y_j \partial \eta_j} \Big)^k a(x,y,\eta) \Big|_{y=x}. \tag{2.5.5}$$

Proof. Write $u(y) = (2\pi)^{-n} \int e^{i\langle y,\eta\rangle} \cdot (\mathcal{F}u)(\eta) d\eta$. In view of the continuity of A this implies

$$(Au)(x) = (2\pi)^{-n} \int A(e^{i\langle\cdot,\eta\rangle})(x) \cdot (\mathcal{F}u)(\eta)\, d\eta,$$

that is, (2.5.4). For (2.5.5) we remark that

$$\begin{aligned}(Ae^{i\langle\cdot,\eta\rangle})(x) &= (2\pi)^{-n} \iint e^{i\langle x-y,\theta\rangle} a(x,y,\theta) e^{i\langle y,\theta\rangle}\, dy\, d\theta \\ &= e^{i\langle x,\eta\rangle} (2\pi)^{-n} \iint e^{-i\langle y,\theta\rangle} a(x, x+y, \eta+\theta)\, dy\, d\theta.\end{aligned}$$

Now apply Proposition 1.2.4. □

Theorem 2.5.2. *Let $A_1 \in L^{m_1}_\rho(\mathbb{R}^n)$, $A \in L^{m_2}_\rho(\mathbb{R}^n)$ be properly supported, $\rho > 1/2$. Then $A_1 \circ A_2 \in L^{m_1+m_2}_\rho(\mathbb{R}^n)$ is properly supported, and its symbol is given by*

$$\sigma_{A_1 \circ A_2}(x,\zeta) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\sum_{j=1}^{n} \frac{1}{i} \frac{\partial^2}{\partial y_j \partial \eta_j} \Big)^k \sigma_{A_1}(x,\eta) \cdot \sigma_{A_2}(y,\zeta), \tag{2.5.6}$$

the differentiations taken at $\eta = \zeta$, $y = x$.

Proof. Apply the reduction of frequency variables of Lemma 2.3.5 to the product formula (2.4.13). □

Corollary 2.5.3. *If we define the principal symbol a of A by $a = \sigma_A$ mod $S_\rho^{m+(1-2\rho)}(\mathbb{R}^n)$, then the principal symbol of $A_1 \circ A_2$ is equal to the product of the principal symbols a_1 and a_2 of A_1 and A_2, respectively.*

Moreover, $[A_1, A_2] = A_1 \circ A_2 - A_2 \circ A_1 \in L_\rho^{m_1+m_2+1-2\rho}(\mathbb{R}^n)$ and its principal symbol of order $m_1 + m_2 + 1 - 2\rho$ is equal to $\frac{1}{i}\{a_1, a_2\}$. Here the Poisson brackets are defined by

$$\{f, g\} = \sum \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \sum \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} \tag{2.5.7}$$

for any differentiable functions f, g on $\mathbb{R}^n \times \mathbb{R}^n = T^*(\mathbb{R}^n)$.

(Such a relation between operators on X and functions on $T^*(X)$, assigning to the commutator of the operators $\frac{1}{i}$ × Poisson brackets of the functions, is familiar in quantum mechanics.)

Theorem 2.5.4. *If $A \in L_\rho^m(\mathbb{R}^n)$ then A is continuous: $H^s_{\mathrm{comp}}(\mathbb{R}^n) \to H^{s-m}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.*

Proof. $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ is the set of $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $(1 + |D|^2)^{s/2}(\varphi u) \in L^2(\mathbb{R}^n)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. We may assume that A is compactly supported. Let $u \in H^s_{\mathrm{comp}}(\mathbb{R}^n)$, then $(1 + |D|^2)^{s/2}u \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, so

$$\begin{aligned}(1 + |D|^2)^{(s-m)/2}Au = (1 + |D|^2)^{(s-m)/2}&A(1 + |D|^2)^{-s/2} \\ &\cdot (1 + |D|^2)^{s/2}u \in L^2_{\mathrm{loc}}(\mathbb{R}^n).\end{aligned}$$

Here $B = (1+|D|^2)^{(s-m)/2}A(1+|D|^2)^{-s/2}$ is of order 0 and such operators map $L^2_{\mathrm{comp}}(\mathbb{R}^n)$ into $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

A proof of the latter statement can be given as follows. One first verifies that if $P \in L_\rho^m(\mathbb{R}^n)$, then $P^* \in L_\rho^m(\mathbb{R}^n)$, if P^* denotes the adjoint of P with respect to the L^2 inner product. This follows from Theorem 2.5.1, in combination with the observation that P^* is defined by the amplitude

$$a(x, y, \eta) = \overline{\sigma_P(y, \eta)}.$$

(One can also use Theorem 4.4.1 in combination with the definition of $L_\rho^m(X)$ following Proposition 4.2.4.)

Next, $B \in L_\rho^0(\mathbb{R}^n)$ implies that $B^*B \in L_\rho^0(\mathbb{R}^n)$, which has a bounded principal symbol. This means that there is a positive constant C, a pseudo-differential operator $P \in L_\rho^0(\mathbb{R}^n)$, and an integral operator R with smooth kernel, such that $B^*B = C - P^*P + R$. Because $(P^*Pu, u) = (Pu, Pu) \geq 0$, we get that

$$\|Bu\|^2 = (B^*Bu, u) \leq C\|u\|^2 + (Ru, u)$$

and the L^2-boundedness of B follows from the L^2-boundedness of R. $\square$

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