Chapter 2
Some Fixed Point Theorems

In this chapter we discuss the classical Banach contraction principle and a fixed point theorem for increasing operators that will be used in connection to sub- and super-solutions of elliptic boundary value problems.

2.1 The Banach Contraction Principle

Let X be a complete metric space. An operator $T : X \to X$ is a *contraction* if there exists $\alpha \in (0, 1)$ such that

$$d_X(T(u), T(v)) \le \alpha\, d_X(u, v), \quad \forall\, u, v \in X, \tag{2.1}$$

where $d_X(u, v)$ denotes the distance from u to v in X.

Remark 2.1.1 From (2.1) it immediately follows that T is continuous.

Theorem 2.1.2 *If X is a complete metric space and T is a contraction on X which maps X into itself, then there exists a unique $z \in X$ such that $T(z) = z$.*

Proof *Existence*. For any fixed $u_0 \in X$ let us define the sequence u_k by setting

$$u_{k+1} = T(u_k), \quad k \in \mathbb{N}.$$

One has that for every $j \ge 1$

$$d_X(u_{j+1}, u_j) = d_X(T(u_j), T(u_{j-1})) \le \alpha d_X(u_j, u_{j-1})$$

and this, by induction, implies

$$d_X(u_{j+1}, u_j) \le \alpha^j d_X(u_1, u_0).$$

Then, it follows that

$$d_X(u_{k+1}, u_h) \le \sum_{j=h}^{k} d_X(u_{j+1}, u_j) \le \left[\sum_{j=h}^{k} \alpha^j\right] d_X(u_1, u_0).$$

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Since $0 < \alpha < 1$, u_k is a Cauchy sequence. Let $z \in X$ be such that $u_k \to z$. Passing to the limit into $u_{k+1} = T(u_k)$ and using the fact that T is continuous, it follows that $z = T(z)$.

Uniqueness. Let $z_1, z_2 \in X$ be fixed points of T. From this and (2.1) we infer

$$d_X(z_1, z_2) = d_X(T(z_1), T(z_2)) \le \alpha d_X(z_1, z_2).$$

Since $\alpha < 1$, it follows that $z_1 = z_2$. $\square$

As a typical application of the Banach contraction principle we can prove the existence and uniqueness of solutions of the Cauchy problem for a first order differential equation. This will be achieved by transforming the differential problem into an equivalent integral equation.

Let (x_0, y_0) be a point in a domain $\Omega \subset \mathbb{R}^2$. For a continuous function $f : \Omega \longrightarrow \mathbb{R}$, we consider the Cauchy problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \tag{2.2}$$

By a (local) solution of (2.2) we mean a C^1 function $y(x)$ defined in some interval $(a, b) \subset \mathbb{R}$ such that $(x, y(x)) \in \Omega$ and $y'(x) = f(x, y(x))$ for every $x \in (a, b)$ which passes by the point (x_0, y_0), i.e. $y(x_0) = y_0$.

Lemma 2.1.3 *The Cauchy problem* (2.2) *is equivalent to the integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2.3}$$

Proof If $y(x)$ satisfies (2.3) then, clearly, $y(x_0) = y_0$. Moreover, differentiating one finds

$$y'(x) = f(x, y(x)).$$

Hence $y(x)$ is a solution of (2.2). Conversely, let $y(x)$ be a solution of (2.2). Integrating from x_0 to x the identity $y'(x) \equiv f(x, y(x))$ we get

$$\int_{x_0}^{x} y'(t)dt = \int_{x_0}^{x} f(t, y(t))dt.$$

Using the initial condition $y(x_0) = y_0$ we deduce (2.3). $\square$

Definition 2.1.4 We say that $f(x, y)$ is locally Lipschitzian with respect to y at (x_0, y_0) if there exist a neighborhood U of (x_0, y_0) and $L > 0$ such that

$$|f(x, y) - f(x, y_1)| \le L\,|y - y_1|, \quad \forall\,(x, y), (x, y_1) \in U. \tag{2.4}$$

If the preceding relationship is valid in all the domain Ω of f we say that f is (globally) Lipschitzian on Ω with respect to y.

Obviously, any function f which is C^1 with respect to y in Ω is locally Lipschitzian on Ω with respect to y. On the other hand, any Lipschitzian function with respect to y is continuous in the variable y. But the converse is not true. For example, $f(x, y) = \sqrt{|y|}$ is not Lipschitzian at $(0, 0)$.

Theorem 2.1.5 *Suppose that $f(x, y)$ is continuous and locally Lipschitzian with respect to y at (x_0, y_0). Then the Cauchy problem (2.2) has a unique solution $y(x)$ defined in a neighborhood of x_0.*

Proof Let $I = [x_0 - \delta, x_0 + \delta]$ with

$$0 < \delta < \min \left\{ \frac{1}{\mathrm{L}}, \frac{\mathrm{a}}{\mathrm{M}} \right\},$$

where $a, L > 0$ are chosen in such a way that (2.4) holds in $U = [x_0 - a, x_0 + a] \times [y_0 - a, y_0 + a]$ and $M = \sup_{(x,y)\in U} |f(x, y)|$. We will use the Banach contraction principle to show that the equivalent integral Eq. (2.3) has a unique solution in I. Let also denote by X the Banach space $C(I)$ endowed with the sup norm

$$||y|| = \sup_{x\in I} |y(x)|$$

and consider the ball B in X of radius a centered at y_0, that is,

$$B = \{y \in X : ||y - y_0|| \le a\}.$$

Define the operator $T : X \mapsto X$ by setting

$$T[y](x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2.5}$$

First of all, let us show that $T(B) \subset B$. Actually,

$$|T[y](x) - y_0| \le M\,\delta < a.$$

Taking the supremum in I, we find $||T[y] - y_0|| < a$ and hence $T[y] \in B$. Next, we show that T is a contraction on B. Actually, using the fact that f is locally Lipschitzian we get

$$\begin{aligned} |T[y](x) - T[y_1](x)| &\le \int_{x_0}^{x} |f(t, y(t)) - f(t, y_1(t))|dt \\ &\le \int_{x_0}^{x} L\,|y(t) - y_1(t))|dt \le \delta L\,||y - y_1||. \end{aligned}$$

Taking again the supremum in I,

$$||T[y] - T[y_1]|| \le \delta L\,||y - y_1||.$$

Since $\delta L < 1$, T is a contraction. Using the Banach contraction principle, we infer that T has a unique fixed point y^* on B. From $T[y^*] = y^*$ we deduce

$$y^*(x) = T[y^*](x) = y_0 + \int_{x_0}^{x} f(t, y^*(t))dt.$$

Therefore y^* is the (unique) solution of (2.3) we were looking for. □

Remark 2.1.6 Observe that the existence result proved above is local. Indeed, the interval of existence $I = [x_0 - \delta, x_0 + \delta]$ depends on L, M, and on the initial condition. The following example shows that the local result is the only one we can hope for. Consider the Cauchy problem

$$\begin{cases} y' = y^2, \\ y(0) = p > 0. \end{cases}$$

One checks that

$$y(x) = \frac{p}{1 - px}$$

satisfies the Cauchy problem. The maximal interval of definition of this solution is $(0, p^{-1})$ and depends on the initial condition. Let us point out that $f(y) = y^2$ is not globally Lipschitzian.

Remark 2.1.7 If Ω is a strip $\Omega = \{(x, y) : a < x < b,\ y \in \mathbb{R}\}$ and f is globally Lipschitzian on this strip, then (2.2) has a unique solution defined on all (a, b) (a can be $-\infty$ and/or b can be $+\infty$).

Remark 2.1.8 If f is not Lipschitzian, but is merely continuous, it is possible to prove that (2.2) has a solution, defined locally near x_0 (Peano's theorem, see Exercise 18), though the uniqueness can fail. For example, the problem

$$\begin{cases} y' = \sqrt{|y|}, \\ y(0) = 0, \end{cases}$$

has infinitely many solutions: one is $y \equiv 0$; in addition for any $a > 0$ any function

$$y(x) = \begin{cases} 0, & \text{for } |x| < a, \\ \frac{1}{4}\,(x - a)|x - a|, & \text{for } |x| \geq a, \end{cases}$$

is also a solution.

In the next chapter, as a second application of the Banach contraction principle, we will prove the local inversion theorem (see Theorem 3.1.1).

2.2 Increasing Operators

In this section we will discuss another iteration scheme on ordered Banach spaces.

Let X be a Banach space endowed with an *ordering* $\leq$ such that (*linear ordering*)

$$v \leq w \Rightarrow \alpha v + z \leq \alpha w + z, \quad \forall\, v, w, z \in X, \quad \forall \alpha \geq 0.$$

We write $w \geq v$ if and only if $v \leq w$. We will also suppose that the norm in X is related to the ordering by the fact that there exists $C > 0$ such that

$$0 \leq v \leq w \Rightarrow \|v\| \leq C\|w\|. \tag{2.6}$$

We say that an operator $T : X \to X$ is increasing if

$$v \le w \Rightarrow T(v) \le T(w), \quad \forall\, v, w \in X.$$

If $v \in X$ satisfies $v \le T(v)$, it is called a sub-solution of the fixed point equation of T, $T(u) = u$. Similarly, $w \in X$ is a super-solution if $T(w) \le w$.

Given a sub-solution $v \in X$, we define an iteration scheme by setting

$$\begin{cases} u_0 = v \\ u_{k+1} = T(u_k), \quad k = 1, 2, \dots \end{cases} \tag{2.7}$$

Lemma 2.2.1 *Let $T : X \to X$ be an increasing operator and suppose that there exist a sub-solution $v \in X$ and a super-solution $w \in X$ of the fixed point equation of T such that $v \le w$. Then the sequence u_k given by* (2.7) *satisfies $v \le u_k \le u_{k+1} \le w$, for all $k = 0, 1, \dots$.*

Proof We argue by induction. By the definition of sub-solution, for $k = 0$ one has $u_1 = T(u_0) = T(v) \ge v$. Moreover, from $u_k \ge u_{k-1}$ and the fact that T is increasing we infer that $T(u_k) \ge T(u_{k-1})$ and hence

$$u_{k+1} = T(u_k) \ge T(u_{k-1}) = u_k.$$

Similarly, one has that $u_0 = v \le w$ and, if $u_k \le w$, the fact that T is increasing and the definition of super-solution yield $u_{k+1} = T(u_k) \le T(w) \le w$. □

Theorem 2.2.2 *Let $T \in C(X, X)$ be compact and increasing and assume that there exist a sub-solution $v \in X$ and a super-solution $w \in X$ of the fixed point equation of T satisfying $v \le w$. Then the sequence u_k given by* (2.7) *converges to some $u \in X$ such that $T(u) = u$. Moreover, $v \le u \le w$.*

Proof Since, by Lemma 2.2.1, $0 \le u_k - v \le w - v$, the property (2.6) implies that

$$\|u_k\| \le \|u_k - v\| + \|v\| \le C\|w - v\| + \|v\| \le C_1.$$

Since T is a compact operator, the sequence $T(u_k)$ is relatively compact and, up to a subsequence, it converges to some $u \in X$ (actually by the monotonicity property of u_k, the whole sequence converges). From $u_{k+1} = T(u_k)$ and the continuity of T, we infer that $u = T(u)$. Moreover, again using Lemma 2.2.1, it follows that $v \le u \le w$. □

Remark 2.2.3 By the definition of u_k, $u = \lim_{k\to\infty} u_k$ is the minimal fixed point of T in $\{z \in X : v \le z \le w\}$.

Later on, Theorem 2.2.2 will be applied to the study of the existence of solutions of nonlinear elliptic boundary value problems via sub- and super-solutions (see Sect. 7.2).

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