

Chapter 2

Function Spaces and the First Renormalized Functional

Suppose $\mathcal{M}_0$ is as in the introduction, and the gap condition

$$\text{there are adjacent } v_0, w_0 \in \mathcal{M}_0 \text{ with } v_0 < w_0 \quad (*)_0$$

holds. Our goal is to show there are solutions of (PDE) heteroclinic in x_1 from v_0 to w_0 and 1-periodic in the remaining variables. This requires introducing a class of admissible functions and an appropriate functional on this class whose minima will be the desired solutions of (PDE). As a first attempt, take the class of $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$ functions that are asymptotic to v_0 and w_0 as $x_1 \rightarrow \pm\infty$ in some reasonable way and

$$\text{minimize } \int_{\mathbb{R} \times \mathbb{T}^{n-1}} L(u) dx \quad (2.1)$$

over this class. By writing $u \in W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$, we mean that $u(x + e_i) = u(x)$, $2 \leq i \leq n$. Unfortunately, this functional may not be bounded from below. In addition, if $F > 0$ on $\mathbb{T}^{n-1}$, the functional will be infinite for any admissible u . Thus a more careful approach is required, and the functional in (2.1) must be modified. Such a “renormalized” functional that is bounded from below will be introduced. Toward that end, let $v, w \in \mathcal{M}_0$, $v < w$, and define

$$\widehat{\Gamma}_1 = \widehat{\Gamma}_1(v, w) = \{u \in W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1}) \mid v \leq u \leq w\}.$$

For $i \in \mathbb{Z}$, set $T_i = [i, i + 1] \times \mathbb{T}^{n-1}$. Now for $u \in \widehat{\Gamma}_1$ and $i \in \mathbb{Z}$, define

$$J_{1,i}(u) = \int_{T_i} L(u) dx - c_0$$

with c_0 as in (1.5). For $p, q \in \mathbb{Z}$ with $p \leq q$ and $u \in \widehat{\Gamma}_1$, set

$$J_{1;p,q}(u) = \sum_{i=p}^q J_{1,i}(u).$$

It is easily seen that $J_{1;p,q}(u)$ is bounded from below, but its lower bound may depend on $q - p$. The next result helps us obtain a better lower bound.

Proposition 2.2. *Let $\ell \in \mathbb{N}^n$ and*

$$\Gamma_0(\ell) = \{u \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid u(x + \ell_i e_i) = u(x), 1 \leq i \leq n\}.$$

Set

$$J_0^\ell(u) = \int_0^{\ell_1} \cdots \int_0^{\ell_n} L(u) dx$$

and

$$c_0(\ell) = \inf_{u \in \Gamma_0(\ell)} J_0^\ell(u). \quad (2.3)$$

Then

$$\mathcal{M}_0(\ell) = \{u \in \Gamma_0(\ell) \mid J_0^\ell(u) = c_0(\ell)\} \neq \emptyset.$$

Moreover, $\mathcal{M}_0(\ell) = \mathcal{M}_0$ and $c_0(\ell) = (\prod_{i=1}^n \ell_i) c_0$.

Proof. The proof of Proposition 2.2 is contained in Moser's work [1]. Some of the arguments will be required repeatedly in this paper, so it is convenient to give the proof in the current simple setting. Since J_0^ℓ is weakly lower semicontinuous on $\Gamma_0(\ell)$, $\mathcal{M}_0(\ell) \neq \emptyset$. Moreover, standard elliptic regularity arguments show that $u \in \mathcal{M}_0(\ell)$ implies that u is a classical solution of (PDE).

Next it will be shown that $\mathcal{M}_0(\ell)$ is an ordered set. If not, there exist $v, w \in \mathcal{M}_0(\ell)$ and $\xi, \eta \in \prod_{i=1}^n (\ell_i \mathbb{T}^1)$ such that $v(\xi) = w(\eta)$ and $v(\eta) < w(\eta)$. Set $\varphi = \max(v, w)$ and $\psi = \min(v, w)$. Then $\varphi, \psi \in \Gamma_0(\ell)$ and

$$J_0^\ell(\varphi) + J_0^\ell(\psi) = J_0^\ell(v) + J_0^\ell(w) = 2c_0(\ell). \quad (2.4)$$

Since

$$J_0^\ell(\varphi), J_0^\ell(\psi) \geq c_0(\ell),$$

(2.4) implies $J_0^\ell(\varphi) = J_0^\ell(\psi) = c_0(\ell)$, so $\varphi, \psi \in \mathcal{M}_0(\ell)$. Therefore φ and ψ are classical solutions of (PDE) with $\varphi \geq \psi$, $\varphi(\xi) = \psi(\xi)$, and $\varphi(\eta) > \psi(\eta)$. Thus $f \equiv \varphi - \psi \geq 0$ and satisfies the linear elliptic partial differential equation

$$-\Delta f + af = -bf, \quad x \in \mathbb{R}^n, \quad (2.5)$$

where $a = \max(A, 0)$, $b = \min(A, 0)$, and

$$A = \begin{cases} \frac{F_u(x, \varphi(x)) - F_u(x, \psi(x))}{\varphi(x) - \psi(x)} & \text{if } \varphi(x) > \psi(x), \\ F_{uu}(x, \varphi(x)) & \text{if } \varphi(x) = \psi(x). \end{cases}$$

Since $a \geq 0$ and $b \leq 0$ are continuous, the elliptic maximum principle applies to (2.5) and shows that either $f \equiv 0$ or $f > 0$ in $\mathbb{R}^n$. But $f(\xi) = 0$ and $f(\eta) > 0$, a contradiction. Hence no such v and w exist and $\mathcal{M}_0(\ell)$ is an ordered set.

Now to prove the final assertions of Proposition 2.2, let $u \in \mathcal{M}_0(\ell)$. If each $u \in \mathcal{M}_0(\ell)$ satisfies

$$u(x + e_i) = u(x), \quad 1 \leq i \leq n, \quad (2.6)$$

then $\mathcal{M}_0(\ell) = \mathcal{M}_0$ and $c_0(\ell) = (\prod_{i=1}^n \ell_i) c_0$. To verify (2.6), suppose $u \in \mathcal{M}_0(\ell)$. Since $u(x + e_i) \in \mathcal{M}_0(\ell)$, $i = 1, \dots, n$, and $\mathcal{M}_0(\ell)$ is ordered, either (2.6) holds or

$$(i) \ u(x + e_i) > u(x) \quad \text{or} \quad (ii) \ u(x + e_i) < u(x) \quad (2.7)$$

for each i . But if (2.7) (i) is satisfied,

$$u(x) = u(x + \ell_i e_i) \geq \dots \geq u(x + e_i) > u(x),$$

a contradiction. A similar argument when (2.7) (ii) holds shows that (2.6) is valid, and the proposition is proved.

Now a better lower bound for $J_{1;p,q}(u)$ can be obtained.

Proposition 2.8. *There is a constant $K_1 \geq 0$, depending on v and w but independent of $p, q \in \mathbb{Z}$ and $u \in \widehat{\Gamma}_1$, such that*

$$J_{1;p,q}(u) \geq -K_1.$$

Proof. Let $u \in \widehat{\Gamma}_1$. Then

$$\begin{aligned} J_{1,p}(u) &= \int_{T_p} \left(\frac{1}{2} |\nabla(u-v)|^2 + \nabla(u-v) \cdot \nabla v \right. \\ &\quad \left. + \frac{1}{2} |\nabla v|^2 + F(x, u) - F(x, v) + F(x, v) \right) dx - c_0 \\ &= \frac{1}{2} \|\nabla(u-v)\|_{L^2(T_p)}^2 + \int_{T_p} (\nabla(u-v) \cdot \nabla v + F(x, u) - F(x, v)) dx. \end{aligned} \quad (2.9)$$

Now

$$\left| \int_{T_p} (F(x, u) - F(x, v)) dx \right| \leq M_1 \|w - v\|_{L^\infty(T_0)}, \quad (2.10)$$

where $M_1 = \max_{\mathbb{T}^{n+1}} |F_u(x, u)|$. Also

$$\int_{T_p} \nabla(u-v) \cdot \nabla v dx = \int_{\partial T_p} (u-v) \frac{\partial v}{\partial \nu} dS - \int_{T_p} (u-v) \Delta v dx, \quad (2.11)$$

where ν denotes the outward-pointing normal. Since $u \in \widehat{\Gamma}_1$ and v is a solution of (PDE),

$$\left| \int_{T_p} (u - v) \Delta v \, dx \right| \leq \|F_u(\cdot, v)\|_{L^\infty(T_p)} \int_{T_p} (w - v) \, dx \leq M_1 \|w - v\|_{L^\infty(T_0)}. \quad (2.12)$$

The boundary term in (2.11) can be estimated by

$$\left| \int_{\partial T_p} (u - v) \frac{\partial v}{\partial \nu} \, dS \right| \leq 2 \left\| \frac{\partial v}{\partial x_1} \right\|_{L^\infty(T_0)} \|w - v\|_{L^\infty(T_0)}. \quad (2.13)$$

Combining (2.9)–(2.13) yields

$$\left| J_{1,p}(u) - \frac{1}{2} \|\nabla(u - v)\|_{L^2(T_p)}^2 \right| \leq M_2 \|w - v\|_{L^\infty(T_0)}, \quad (2.14)$$

where $M_2 = 2M_1 + 2\left\|\frac{\partial v}{\partial x_1}\right\|_{L^\infty(T_0)}$. This proves Proposition 2.8 for $q = p, p + 1$, or $p + 2$ with $K_1 = 3M_2\|w - v\|_{L^\infty(T_0)}$. Thus suppose that $q > p + 2$. Define χ via

$$\chi = \begin{cases} v, & x_1 \leq p, \\ (x_1 - p)u + (p + 1 - x_1)v, & p \leq x_1 \leq p + 1, \\ u, & p + 1 \leq x_1 \leq q, \\ (x_1 - q)v + (q + 1 - x_1)u, & q \leq x_1 \leq q + 1, \\ v, & q + 1 \leq x_1. \end{cases} \quad (2.15)$$

Then χ extends naturally to $\widehat{\Gamma}_1$ as a $(q + 1 - p)$ -periodic function of x_1 . Hence by Proposition 2.2,

$$0 \leq J_{1;p,q}(\chi) = J_{1,p}(\chi) + J_{1;p+1,q-1}(u) + J_{1,q}(\chi),$$

or

$$J_{1;p+1,q-1}(u) \geq -J_{1,p}(\chi) - J_{1,q}(\chi). \quad (2.16)$$

Next observe that

$$\chi - v = (x_1 - p)(u - v), \quad p \leq x_1 \leq p + 1,$$

so

$$\begin{aligned} |\nabla(\chi - v)|^2 &= (x_1 - p)^2 |\nabla(u - v)|^2 + (u - v)^2 + 2(x_1 - p)(u - v) \frac{\partial}{\partial x_1} (u - v) \\ &= (x_1 - p)^2 |\nabla(u - v)|^2 + \frac{\partial}{\partial x_1} ((x_1 - p)(u - v)^2) \end{aligned}$$

and

$$\|\nabla(\chi - v)\|_{L^2(T_p)}^2 \leq \|\nabla(u - v)\|_{L^2(T_p)}^2 + \|w - v\|_{L^\infty(T_0)}^2. \quad (2.17)$$

Hence by (2.14) and (2.17),

$$J_{1,p}(\chi) \leq \frac{1}{2} \|\nabla(u - v)\|_{L^2(T_p)}^2 + M_2 \|w - v\|_{L^\infty(T_0)} + \frac{1}{2} \|w - v\|_{L^\infty(T_0)}^2. \quad (2.18)$$

Finally,

$$\begin{aligned} J_{1;p,q}(u) &= J_{1,p}(u) + J_{1;p+1,q-1}(u) + J_{1,q}(u) \\ &\geq -4M_2 \|w - v\|_{L^\infty(T_0)} - \|w - v\|_{L^\infty(T_0)}^2 \\ &\equiv -K_1. \end{aligned} \quad (2.19)$$

Remark 2.20. If $v = v_0$ and $w = w_0$, $\|w - v\|_{L^\infty(T_0)} \leq 1$.

The lower bound for $J_{1;p,q}(u)$ provided by Proposition 2.8 suggests defining

$$J_1(u) = \varliminf_{\substack{p \rightarrow -\infty \\ q \rightarrow \infty}} J_{1;p,q}(u) \quad (2.21)$$

for $u \in \widehat{\Gamma}_1$. For J_1 so defined, there is also an upper bound for $J_{1;p,q}(u)$:

Lemma 2.22. *If $u \in \widehat{\Gamma}_1$ and $p, q \in \mathbb{Z}$ with $p \leq q$,*

$$J_{1;p,q}(u) \leq J_1(u) + 2K_1. \quad (2.23)$$

Proof. By (2.21) and Proposition 2.8,

$$\begin{aligned} J_1(u) &= \varliminf_{s \rightarrow -\infty} J_{1;s,p-1}(u) + J_{1;p,q}(u) + \varliminf_{t \rightarrow \infty} J_{1;q+1,t}(u) \\ &\geq J_{1;p,q}(u) - 2K_1. \end{aligned}$$

Define

$$\begin{aligned} \Gamma_1 \equiv \Gamma_1(v, w) \equiv & \{u \in \widehat{\Gamma}_1 \mid \|u - v\|_{L^2(T_i)} \rightarrow 0, i \rightarrow -\infty, \\ & \|u - w\|_{L^2(T_i)} \rightarrow 0, i \rightarrow \infty\}. \end{aligned}$$

Fortunately, the expression for J_1 simplifies when we are dealing with $u \in \Gamma_1$, since the $\underline{\lim}$'s in (2.21) become limits. The next result shows this and more:

Proposition 2.24. *If $u \in \Gamma_1$ and $J_1(u) < \infty$, then*

$$J_{1,i}(u) \rightarrow 0, \quad |i| \rightarrow \infty, \quad (2.25)$$

$$\|\tau_{-i}^1 u - v\|_{W^{1,2}(T_0)} \rightarrow 0, \quad i \rightarrow -\infty, \quad (2.26)$$

$$\|\tau_{-i}^1 u - w\|_{W^{1,2}(T_0)} \rightarrow 0, \quad i \rightarrow \infty, \quad (2.27)$$

$$J_1(u) = \lim_{\substack{p \rightarrow -\infty \\ q \rightarrow \infty}} J_{1;p,q}(u). \quad (2.28)$$

Proof. By (2.23) with $p = q = i$, $J_{1,i}(u)$ is bounded from above independently of $i \in \mathbb{Z}$. Hence by (2.14), $\|\nabla(\tau_{-i}^1 u - v)\|_{L^2(T_0)}$ is bounded independently of $i \in \mathbb{Z}$. Since

$$\|\tau_{-i}^1 u - v\|_{L^2(T_0)} \leq \|w - v\|_{L^\infty(T_0)}, \quad (2.29)$$

$\tau_{-i}^1 u - v$ is bounded in $W^{1,2}(T_0)$. Therefore there is a $\varphi \in W^{1,2}(T_0)$ such that $\tau_{-i}^1 u - v \rightarrow \varphi$ weakly in $W^{1,2}(T_0)$ and strongly in $L^2(T_0)$ for a subsequence of i 's $\rightarrow -\infty$. But since $u \in \Gamma_1$, $\|\tau_{-i}^1 u - v\|_{L^2(T_0)} \rightarrow 0$ as $i \rightarrow -\infty$. Hence $\varphi = 0$, and it readily follows that $\tau_{-i}^1 u \rightarrow v$ weakly in $W^{1,2}(T_0)$ and strongly in $L^2(T_0)$ as $i \rightarrow -\infty$ along the full sequence. By the weak convergence in $W^{1,2}(T_0)$,

$$\int_{T_0} \nabla v \cdot \nabla(\tau_{-i}^1 u - v) dx \rightarrow 0, \quad i \rightarrow -\infty,$$

and by the convergence in $L^2(T_0)$,

$$\int_{T_0} (F(x, \tau_{-i}^1 u) - F(x, v)) dx \rightarrow 0, \quad i \rightarrow -\infty.$$

These observations and (2.9) show that

$$\underline{\lim}_{i \rightarrow -\infty} J_{1,i}(u) = \underline{\lim}_{i \rightarrow -\infty} \frac{1}{2} \|\nabla(\tau_{-i}^1 u - v)\|_{L^2(T_0)}^2 \geq 0. \quad (2.30)$$

If $\underline{\lim}_{i \rightarrow -\infty} J_{1,i}(u)$ is positive, $J_1(u) = \infty$, contrary to hypothesis. Hence $\underline{\lim}_{i \rightarrow -\infty} J_{1,i}(u) = 0$. Providing a similar argument for $i \rightarrow \infty$, (2.25) follows with $\lim$ replaced by $\underline{\lim}$. Then (2.30) yields (2.26)–(2.27) along a subsequence. Next it will be shown that

$$(i) \lim_{p \rightarrow -\infty} J_{1;p,0}(u) \quad \text{and} \quad (ii) \lim_{q \rightarrow \infty} J_{1;1,q}(u) \quad (2.31)$$

exist. Then (2.25) and (2.28) are valid, and returning to (2.9) again shows that (2.26)–(2.27) hold. A slight variant of an argument from [7] – see also Bosetto and Serra [25] – will be employed.

Their proofs being the same, the existence of (2.31) (i) will be verified. Set

$$\mathcal{P} = \{p \in \mathbb{Z} \mid p < 0 \text{ and } J_{1,p}(u) \leq 0\}.$$

If $\mathcal{P}$ is a finite set, $J_{1;p,0}(u)$ is a monotone nondecreasing sequence with $J_{1;p,0}(u) \leq J_1(u) + 2K_1$. Hence (2.31) (i) follows. If $\mathcal{P}$ is infinite, (2.9) shows that

$$\lim_{i \rightarrow -\infty, i \in \mathcal{P}} \|\tau_{-i}^1 u - v\|_{W^{1,2}(T_0)} = 0. \quad (2.32)$$

Suppose $J_{1;p,0}(u)$ does not converge as $p \rightarrow -\infty$. Set

$$\ell^- = \varliminf_{p \rightarrow -\infty} J_{1;p,0}(u), \quad \ell^+ = \varlimsup_{p \rightarrow -\infty} J_{1;p,0}(u),$$

so $-K_1 \leq \ell^- < \ell^+$. Choose

$$0 < \varepsilon < (\ell^+ - \ell^-)/5. \quad (2.33)$$

The following technical lemma is useful at this point.

Lemma 2.34. *For any $\gamma > 0$, there is a $\delta = \delta(\gamma) > 0$ such that if $u \in \Gamma_1(v, w)$, $p, q \in \mathbb{Z}$, with $p < q$ and*

$$\|u - v\|_{W^{1,2}(T_j)} \leq \delta \text{ or } \|u - w\|_{W^{1,2}(T_j)} \leq \delta \quad (2.35)$$

for $j = p$ and q , then

$$J_{1;p+1,q-1}(u) \geq -\gamma. \quad (2.36)$$

Assuming Lemma 2.34 for the moment, choose $\gamma = \varepsilon$ and $\delta = \delta(\varepsilon)$. By (2.30) and (2.32), there is a $p_0 \in \mathcal{P}$ such that

$$\begin{cases} J_{1,p}(u) \geq -\varepsilon & \text{for all } p \leq p_0, \\ \|\tau_{-p}^1 u - v\|_{W^{1,2}(T_0)} \leq \delta, & p \leq p_0, p \in \mathcal{P}. \end{cases} \quad (2.37)$$

Hence by Lemma 2.34,

$$J_{1;p+1,q-1}(u) \geq -\varepsilon, \quad (2.38)$$

whenever $p, q \in \mathcal{P}$ and $p < q \leq p_0$. Choose sequences $(p_k), (q_k) \subset -\mathbb{N}$ such that $q_{k+1} < p_k < q_k < p_0$ and

$$J_{1;p_k,0}(u) \rightarrow \ell^-; \quad J_{1;q_k,0}(u) \rightarrow \ell^+, \quad k \rightarrow \infty. \quad (2.39)$$

Therefore there is a k_0 such that for $k \geq k_0$,

$$J_{1;p_k,0}(u) \leq \ell^- + \varepsilon; \quad J_{1;q_k,0}(u) \geq \ell^+ - \varepsilon. \quad (2.40)$$

Next let $\widehat{q}_k$ be the largest $q \in \mathcal{P}$ such that $q < q_k$ and let $\widehat{p}_k$ be the smallest $p \in \mathcal{P}$ such that $p \geq p_k$. Then

$$J_{1;p_k,\widehat{p}_k-1}(u) \geq 0 \quad (2.41)$$

(where this term is not present if $\widehat{p}_k = p_k$). Thus by (2.40),

$$J_{1;\widehat{p}_k,0}(u) \leq \ell^- + \varepsilon. \quad (2.42)$$

Similarly

$$J_{1;\widehat{q}_k+1,q_k-1}(u) \geq 0, \quad (2.43)$$

so by (2.40) again,

$$J_{1;\widehat{q}_k+1,0}(u) \geq \ell^+ - \varepsilon. \quad (2.44)$$

Consequently, by (2.42), (2.44), and (2.33),

$$J_{1;\widehat{p}_k,\widehat{q}_k}(u) = J_{1;\widehat{p}_k,0}(u) - J_{1;\widehat{q}_k+1,0}(u) \leq \ell^- + \varepsilon - (\ell^+ - \varepsilon) < -3\varepsilon. \quad (2.45)$$

On the other hand, by (2.38),

$$J_{1;\widehat{p}_k+1,\widehat{q}_k-1}(u) \geq -\varepsilon, \quad (2.46)$$

which combined with (2.37) with $p = p_k, q_k$ yields

$$J_{1;\widehat{p}_k,\widehat{q}_k}(u) \geq -3\varepsilon, \quad (2.47)$$

contrary to (2.45). Thus $\ell^+ = \ell^-$ and the proof of Proposition 2.24 is complete modulo the:

Proof of Lemma 2.34. Suppose, e.g., (2.35) holds with the v term. Take χ as in (2.15). Then (2.16) implies the result, provided that

$$|J_{1,p}(\chi)| + |J_{1,q}(\chi)| \leq \gamma. \quad (2.48)$$

But (2.48) follows from (2.35), the form of χ , and the continuity of $J_{1,i}$ (in $\|\cdot\|_{W^{1,2}(T_i)}$) for $i \in \mathbb{Z}$.

Corollary 2.49. *Suppose $u \in \widehat{\Gamma}_1(v, w)$, $J_1(u) < \infty$, and $u \leq \tau_{-1}^1 u$. Then either (i) $u \in \mathcal{M}_0$, or (ii) there are $\varphi, \psi \in \mathcal{M}_0$ with $v \leq \varphi < \psi \leq w$ such that $u \in \Gamma_1(\varphi, \psi)$.*

Proof. Set $u_k = \tau_{-k}^1 u$. Since $J_1(u) < \infty$ and by (2.23) (u_k) is bounded in $W^{1,2}(T_0)$, there is a $\varphi \in W^{1,2}(T_0)$ such that $u_k \rightarrow \varphi$ as $k \rightarrow -\infty$ along a subsequence, weakly in $W^{1,2}(T_0)$ and strongly in $L^2(T_0)$. Since $\tau_{-1}^1 u_k = u_{k+1} \geq u_k$, the entire sequence

converges to φ in $L^2(T_0)$ and $\tau_{-1}^1 \varphi = \varphi$, i.e., $\varphi \in \Gamma_0$. If $\varphi \notin \mathcal{M}_0$, $J_0(\varphi) > c_0 + \varepsilon$ for some $\varepsilon > 0$. Since J_0 is weakly lower semicontinuous,

$$c_0 + \varepsilon < J_0(\varphi) \leq \liminf_{k \rightarrow \infty} J_0(u_k).$$

But then $J_1(u) = \infty$, a contradiction. Thus φ and similarly ψ , the weak limit of u_k as $k \rightarrow \infty$, belong to $\mathcal{M}_0$. If $\varphi = \psi$, $u \leq \tau_{-1}^1 u$ implies $u = \varphi$ and (i) holds. Otherwise $\varphi < \psi$ and (ii) is valid.

Having established some convergence results for J_1 , next a compactness property of minimizing sequences will be studied. It represents, in the current setting, the analogue of the Palais–Smale condition in other contexts involving critical point theory and is modeled on similar results in [7].

Proposition 2.50. *Let $\mathcal{Y} \subset \widehat{\Gamma}_1(v, w)$. Suppose $\mathcal{Y}$ possesses the following property:*

(Y_1^1) *Let $u \in \mathcal{Y}$, $p \in \mathbb{N}$, and let U be a sequential weak limit (in $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$) of $(u_k) \subset \mathcal{Y}$. Define $\chi_p \equiv \chi_p(u, U)$ by*

$$\chi_p = \begin{cases} u, & x_1 \leq -p, \\ U, & -p + 1 \leq x_1 \leq p, \\ u, & p + 1 \leq x_1, \end{cases}$$

and extend χ_p to the intermediate intervals as in (2.15). Then $\chi_p(u, U) \in \mathcal{Y}$ for all large p (independently of u).

Define

$$c(\mathcal{Y}) = \inf_{u \in \mathcal{Y}} J_1(u). \quad (2.51)$$

If $c(\mathcal{Y}) < \infty$ and (u_k) is a minimizing sequence for (2.51), then there is a $U \in \widehat{\Gamma}_1$ such that along a subsequence, $u_k \rightarrow U$ in $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$.

Proof. Since (u_k) is a minimizing sequence for (2.51), there is an $M > 0$ such that

$$J_1(u_k) \leq M. \quad (2.52)$$

By Lemma 2.22, (u_k) is bounded in $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$. Therefore passing to a subsequence, it can be assumed that there is a $U \in W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$ such that $u_k \rightarrow U$ weakly in $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$, strongly in $L_{\text{loc}}^2(\mathbb{R} \times \mathbb{T}^{n-1})$, and pointwise a.e. Thus $U \in \widehat{\Gamma}_1$. It remains to show that convergence is in $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$. For $i \in \mathbb{Z}$, set

$$\delta_i = \lim_{s \rightarrow \infty} J_{1,i}(u_s) - J_{1,i}(U). \quad (2.53)$$

Since $J_{1,i}$ is weakly lower semicontinuous, $\delta_i \geq 0$. By Proposition 2.8, Lemma 2.22, and (2.53), for any $p \in \mathbb{N}$,

$$\begin{aligned} -K_1 &\leq J_{1;-p,p}(U) = \sum_{-p}^p \left(\lim_{s \rightarrow \infty} J_{1,i}(u_s) - \delta_i \right) \leq \lim_{s \rightarrow \infty} J_{1;-p,p}(u_s) - \sum_{-p}^p \delta_i \\ &\leq \lim_{s \rightarrow \infty} J_1(u_s) + 2K_1 - \sum_{-p}^p \delta_i. \end{aligned} \quad (2.54)$$

Therefore by (2.52) and (2.54),

$$\sum_{i \in \mathbb{Z}} \delta_i \leq M + 3K_1. \quad (2.55)$$

Consequently $\delta_i \rightarrow 0$ as $|i| \rightarrow \infty$. Next observe that by (2.53), (2.9), and the convergence already established for u_k ,

$$\delta_i = \frac{1}{2} \lim_{s \rightarrow \infty} \left(\|\nabla(u_s - v)\|_{L^2(T_i)}^2 - \|\nabla(U - v)\|_{L^2(T_i)}^2 \right). \quad (2.56)$$

Since

$$\begin{aligned} \|\nabla(u_s - U)\|_{L^2(T_i)}^2 &= \|\nabla(u_s - v)\|_{L^2(T_i)}^2 + \|\nabla(U - v)\|_{L^2(T_i)}^2 \\ &\quad - 2 \int_{T_i} \nabla(u_s - v) \cdot \nabla(U - v) dx, \\ \lim_{s \rightarrow \infty} \|\nabla(u_s - U)\|_{L^2(T_i)}^2 &= \lim_{s \rightarrow \infty} \|\nabla(u_s - v)\|_{L^2(T_i)}^2 - \|\nabla(U - v)\|_{L^2(T_i)}^2. \end{aligned} \quad (2.57)$$

Thus combining (2.56)–(2.57) yields

$$2\delta_i = \lim_{s \rightarrow \infty} \|\nabla(u_s - U)\|_{L^2(T_i)}^2. \quad (2.58)$$

By (Y_1^1) , $\chi_{k,p} \equiv \chi_p(u_k, U) \in \mathcal{Y}$ for large p . Therefore

$$\begin{aligned} c(\mathcal{Y}) &\leq J_1(\chi_{k,p}) = J_{1;-\infty,-p}(u_k) + J_{1;-p+1,p-1}(U) + J_{1;p,\infty}(u_k) \\ &\quad + J_{1,-p}(\chi_{k,p}) - J_{1,-p}(u_k) + J_{1,p}(\chi_{k,p}) - J_{1,p}(u_k). \end{aligned} \quad (2.59)$$

Passing to a subsequence of (u_k) for which (2.58) holds as a limit, it follows that there is an $\alpha_p \rightarrow 0$ as $p \rightarrow \infty$ such that

$$|J_{1,-p}(\chi_{k,p}) - J_{1,-p}(u_k)| + |J_{1,p}(\chi_{k,p}) - J_{1,p}(u_k)| \leq \alpha_p \quad (2.60)$$

for $k \geq k_0(p)$. Thus by (2.59)–(2.60),

$$\begin{aligned} c(\mathcal{Y}) &\leq J_1(u_k) + J_{1;-p+1,p-1}(U) - J_{1;-p+1,p-1}(u_k) + \alpha_p \\ &\leq J_1(u_k) + \lim_{s \rightarrow \infty} J_{1;-p+1,p-1}(u_s) - J_{1;-p+1,p-1}(u_k) - \sum_{-p+1}^{p-1} \delta_i + \alpha_p. \end{aligned} \quad (2.61)$$

Letting $k \rightarrow \infty$ gives

$$\sum_{-p+1}^{p-1} \delta_i \leq \alpha_p, \quad (2.62)$$

and then letting $p \rightarrow \infty$ shows that $\delta_i = 0$ for all $i \in \mathbb{Z}$, completing the proof of Proposition 2.50.

Remark 2.63. For the results of Chapters 3–5, the choice of $\mathcal{Y}$ is such that a milder version of (Y_1^1) suffices: There is an $R > 0$ such that whenever $u \in \mathcal{Y}$ and $\chi \in \widehat{\Gamma}_1$ with $\chi(x) = u(x)$ for $|x_1| \geq R$, then $\chi \in \mathcal{Y}$. However, for the results of the later sections involving multitransition solutions, the sets $\mathcal{Y}$ used there involve additional integral constraints. These constraints are also satisfied by the weak $W_{\text{loc}}^{1,2}$ limits of sequences of $\mathcal{Y}$, and the full strength of (Y_1^1) is needed for these settings.

In applications of Proposition 2.50 in later sections, the members of $\mathcal{Y}$ will satisfy some asymptotic conditions as in the definition of Γ_1 . The convergence of u_k to U is merely in $W_{\text{loc}}^{1,2}, (\mathbb{R} \times \mathbb{T}^{n-1})$ so a priori U need not possess this asymptotic behavior. Consequently, U may not belong to $\mathcal{Y}$. Nevertheless, if an additional condition is satisfied by the minimizing sequence, U will satisfy (PDE), as the next result shows.

Proposition 2.64. *Under the hypothesis of Proposition 2.50, suppose*

(Y_2^1) there is a minimizing sequence (u_k) for (2.51) such that for some $r \in (0, \frac{1}{2})$, some $z \in \mathbb{R}^n$, all smooth φ with support in $B_r(z) = \{x \in \mathbb{R}^n \mid |x - z| < r\}$ and associated $t_0(\varphi) > 0$,

$$c(\mathcal{Y}) \leq J_1(u_k + t\varphi) + \delta_k \quad (2.65)$$

for all $|t| \leq t_0(\varphi)$, where $\delta_k = \delta_k(\varphi) \rightarrow 0$ as $k \rightarrow \infty$.

Then the weak limit U of u_k satisfies (PDE) in $B_r(z)$.

Proof. Suppose (u_k) is the minimizing sequence for (2.51) satisfying (2.65). Define ε_k via

$$J_1(u_k) = c(\mathcal{Y}) + \varepsilon_k, \quad (2.66)$$

so $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. By (2.65),

$$c(\mathcal{Y}) \leq J_1(u_k) = c(\mathcal{Y}) + \varepsilon_k \leq J_1(u_k + t\varphi) + \delta_k + \varepsilon_k,$$

or

$$J_1(u_k) \leq J_1(u_k + t\varphi) + \delta_k + \varepsilon_k. \quad (2.67)$$

Now $B_r(z) \subset [p, q + 1] \times \mathbb{T}^{n-1}$ for some $p, q \in \mathbb{Z}$, $p \leq q$. Then by (2.67),

$$J_{1;p,q}(u_k) \leq J_{1;p,q}(u_k + t\varphi) + \delta_k + \varepsilon_k. \quad (2.68)$$

Letting $k \rightarrow \infty$ and using the $W_{\text{loc}}^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1})$ convergence of u_k to U , (2.68) shows that

$$J_{1;p,q}(U) \leq J_{1;p,q}(U + t\varphi),$$

or

$$\int_{B_r(z)} L(U) dx \leq \int_{B_r(z)} L(U + t\varphi) dx. \quad (2.69)$$

for all smooth φ with support in $B_r(z)$ and $|t| \leq t_0(\varphi)$. Hence standard elliptic regularity arguments imply that U is a solution of (PDE) in $B_r(z)$.

The final result in this section provides a useful tool for comparison arguments that will be used repeatedly later. For $v \in \mathcal{M}_0$, set

$$\Gamma_1(v) = \{u \in \widehat{\Gamma}_1(v-1, v+1) \mid \|u - v\|_{L^2(T_i)} \rightarrow 0, |i| \rightarrow \infty\}.$$

Remark 2.70. It is readily verified that the conclusions of Proposition 2.24 hold for $\Gamma_1(v)$, (2.27) being deleted and (2.26) valid for $|i| \rightarrow \infty$.

Define

$$c_1(v) = \inf_{u \in \Gamma_1(v)} J_1(u) \quad (2.71)$$

and set

$$\mathcal{M}_1(v) = \{u \in \Gamma_1(v) \mid J_1(u) = c_1(v)\}.$$

Theorem 2.72. *If F satisfies (F_1) – (F_2) , then $c_1(v) = 0$ and $\mathcal{M}_1(v) = \{v\}$.*

Proof. Since $v \in \Gamma_1(v)$ and $J_1(v) = 0$,

$$c_1(v) \leq 0. \quad (2.73)$$

To obtain the reverse inequality, it suffices to show that

$$J_1(u) \geq 0 \quad (2.74)$$

for all $u \in \Gamma_1(v)$. Thus suppose $u \in \Gamma_1(v)$ and $J_1(u) < \infty$. In the definition of χ_p in (Y_1^1) of Proposition 2.50, replace u by v , U by u , and denote the resulting function by χ_p . Thus $\chi_p \in \Gamma_1(v)$. Set $\varphi_p = \chi_p|_{[-p-1, p+1] \times \mathbb{T}^{n-1}}$ and extended as a $(2p+2)$ -periodic function of x_1 . Then $\varphi_p \in \Gamma_0(\ell)$ with $\ell = (2p+2, 0, \dots, 0)$, so by Proposition 2.2,

$$0 \leq J_{1;-p-1,p}(\varphi_p) = J_{1;-p,p}(\varphi_p) = J_{1;-p,p}(\chi_p) = J_1(\chi_p). \quad (2.75)$$

Now

$$\begin{aligned} J_1(\chi_p) &= J_1(u) + J_{1,-p}(\chi_p) - J_{1,-p}(u) \\ &\quad + J_{1,p}(\chi_p) - J_{1,p}(u) - J_{1;-\infty,-p-1}(u) - J_{1;p+1,\infty}(u) \\ &\equiv J_1(u) - R_p(u), \end{aligned}$$

so by (2.75),

$$R_p(u) \leq J_1(u). \quad (2.76)$$

Now to prove (2.74), it will be shown that $R_p(u) \rightarrow 0$ as $p \rightarrow \infty$. By Remark 2.70 and Proposition 2.24, the tails $J_{1;-\infty,-p-1}(u)$, $J_{1;p+1,\infty}(u)$ approach 0 as $p \rightarrow \infty$ and likewise the differences

$$J_{1,-p}(\chi_p) - J_{1,-p}(u), \quad J_{1,p}(\chi_p) - J_{1,p}(u)$$

go to 0 as $p \rightarrow \infty$, since $\tau_{\pm p}^1 \chi_p$, $\tau_{\pm p}^1 u \rightarrow v$ in $W^{1,2}(T_0)$ via (2.26).

Remark 2.77. The above argument holds equally well if $v \pm 1$ is replaced by $v \pm j$ for any $j \in \mathbb{N}$.

It remains to prove that $\mathcal{M}_1(v) = \{v\}$. Let $u \in \mathcal{M}_1(v)$. Then $v-1 \leq u \leq v+1$, so for any $z \in \mathbb{R}^n$, $r \in (0, \frac{1}{2})$, φ smooth with support in $B_r(z)$, and $|t|$ small (depending on φ), $v-2 \leq u + t\varphi \leq v+2$. Hence with the aid of Remark 2.77, and $u_k = u$, note that (Y_2^1) of Proposition 2.64 (with $\delta_k = 0$) is satisfied. Consequently, u satisfies (PDE) for all $z \in \mathbb{R}^n$. By (F_2) , $u \in \mathcal{M}_1(v)$ implies $\tau_{-1}^1 u \in \mathcal{M}_1(v)$. If $\tau_{-1}^1 u = u$, u is 1-periodic in x_1 , and $\|u - v\|_{L^2(T_i)} \rightarrow 0$ as $|i| \rightarrow \infty$ then implies $u \equiv v$, completing the proof. Thus suppose $u \neq \tau_{-1}^1 u$. An argument like that of Proposition 2.2 (and essentially due to Moser [1]) then leads to a contradiction. We claim that

$$(i) \ u < \tau_{-1}^1 u \quad \text{or} \quad (ii) \ u > \tau_{-1}^1 u. \quad (2.78)$$

Otherwise, set $\varphi = \max(u, \tau_{-1}^1 u)$ and $\psi = \min(u, \tau_{-1}^1 u)$. Then $\varphi \geq \psi$ and there are points ξ and η such that $\varphi(\xi) = \psi(\xi)$ and $\varphi(\eta) > \psi(\eta)$. Note that for any $i \in \mathbb{Z}$,

$$\int_{T_i} (L(\varphi) + L(\psi)) dx = \int_{T_i} (L(u) + L(\tau_{-1}^1 u)) dx,$$

or

$$J_{1,i}(\varphi) + J_{1,i}(\psi) = J_{1,i}(u) + J_{1,i}(\tau_{-1}^1 u). \quad (2.79)$$

Therefore summing over i leads to

$$J_1(\varphi) + J_1(\psi) = J_1(u) + J_1(\tau_{-1}^1 u) = 0. \quad (2.80)$$

Since $\varphi, \psi \in \Gamma_1(v)$, $J_1(\varphi), J_1(\psi) \geq 0$. Hence by (2.80), $\varphi, \psi \in \mathcal{M}_1(v)$ and thus they satisfy (PDE). Consequently their difference $f = \varphi - \psi$ is nonnegative and

satisfies (2.5). Hence a contradiction as in the proof of Proposition 2.2 obtains, yielding (2.78). The remaining argument is the same for (i) or (ii) in (2.78), so suppose (i) holds. Then for all $j \in \mathbb{N}$,

$$\tau_j^1 u < u < \tau_{-j}^1 u. \quad (2.81)$$

Letting $j \rightarrow \infty$ gives

$$v \leq u \leq v, \quad (2.82)$$

and the proof of Theorem 2.72 is complete.

Remark 2.83. Suppose $(*)_0$ holds. Set

$$\widetilde{\Gamma}_1(v_0) = \{u \in \Gamma_1(v_0) \mid v_0 \leq u \leq w_0\}$$

and

$$\widetilde{c}_1(v_0) = \inf_{u \in \widetilde{\Gamma}_1(v_0)} J_1(u).$$

Then since $\widetilde{\Gamma}_1(v_0) \subset \Gamma_1(v_0)$,

$$0 = c_1(v_0) \leq \widetilde{c}_1(v_0) \leq J_1(v_0) = 0, \quad (2.84)$$

so $\widetilde{c}_1(v_0) = 0$ and likewise

$$\widetilde{\mathcal{M}}_1(v_0) = \{u \in \widetilde{\Gamma}_1(v_0) \mid J_1(u) = \widetilde{c}_1(v_0)\} = \{v_0\}$$

via Theorem 2.72.

Remark 2.85. Suppose condition (F_3) holds, i.e., F is even in $x_1, \dots, x_n$. Then as was shown in [9],

$$c_0 = \inf_{u \in W^{1,2}([0,1]^n)} J_0(u)$$

and any $u \in \mathcal{M}_0$ is even in $x_1, \dots, x_n$. Therefore if $u \in \widehat{\Gamma}_1(v_0, w_0)$,

$$J_{1,i}(u) = \int_{T_i} L(u) dx - c_0 \geq 0$$

for all $i \in \mathbb{Z}$ and $J_1(u) \geq 0$ on this set of functions. This fact allows us to obtain several of the results of this section and in the sequel much more simply. See, e.g., [13, 14] for a treatment of (PDE) under this additional hypothesis.



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