# 2

# Background Material on Asymptotic Analysis of Extremal Problems

This chapter is intended to provide various facts, notions, and concepts which play a fundamental role in modern asymptotic analysis of optimization problems. We recall some main concepts and basic results of measure theory, Sobolev spaces, and boundary value problems which are used later. We include proofs only if the line of arguments is of importance for the understanding of subsequent remarks. For a deeper insight in the subject, we refer to the books of Adams [2], Bucur and Buttazzo [38], Evans and Gariepy [106], Kantorovich and Akilov [128], Lions and Magenes [173], Maz'ya [185], Yosida [251], Ziemer [267], and so on.

## 2.1 Measure theory and basic notation

Let $\Omega \subset \mathbb{R}^n$ be a nonempty set. We say that a collection $\mathcal{E}$ of subsets of $\Omega$ is a $\sigma$-algebra on $\Omega$ if

$$\emptyset \in \mathcal{E}, \quad \Omega \setminus A \in \mathcal{E} \ \text{ whenever } \ A \in \mathcal{E},$$
$$\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{E} \ \text{ whenever } \ A_k \in \mathcal{E} \ \text{ for every } \ k \in \mathbb{N}.$$

Given a $\sigma$-algebra on $\Omega$, we say that the pair $(\Omega, \mathcal{E})$ is a measure space.

We denote by $\mathcal{B}(\Omega)$ the intersection of all $\sigma$-algebras on $\Omega$ containing the open subsets of $\Omega$. It turns out that $\mathcal{B}(\Omega)$ is actually the smallest $\sigma$-algebra on $\Omega$ containing the open subsets of $\Omega$, and it is called the $\sigma$-algebra of Borel subsets of $\Omega$ and its elements are called Borel sets.

Let $(\Omega, \mathcal{B}(\Omega))$ be a Borel measure space. We define measures as set functions.

**Definition 2.1.** *A function $\mu : \mathcal{B}(\Omega) \to \mathbb{R}$ is a Borel measure on $\Omega$ (or simply a measure) if $\mu(\emptyset) = 0$ and $\mu$ is countably additive in the sense that*

$$A = \bigcup_{k\in\mathbb{N}} A_k, \ A_k \cap A_j = \emptyset \ \textit{ if } \ k \neq j \quad \Rightarrow \quad \mu(A) = \sum_{k\in\mathbb{N}} \mu(A_k). \tag{2.1}$$

P.I. Kogut, G.R. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8149-4_2,

*The set of such measures will be denoted by* $\mathcal{M}(\Omega)$. *We also say that a Borel measure is positive if it takes its values in* $[0,\infty)$. *The set of positive Borel measures will be denoted by* $\mathcal{M}_+(\Omega)$.

We observe that, in the case of measures, the series in (2.1) must necessary converge absolutely, since the union on the left-hand side of (2.1) does not depend on the order in which the sets $A_1, A_2, \dots$ are listed.

Let $\mu : 2^{\Omega} \to \mathbb{R}$ be a set function. We define the restriction $\mu \llcorner A$ of $\mu$ to $A \subset \Omega$ by $\mu \llcorner A(B) = \mu(A \cap B)$ for all $B \in 2^{\Omega}$.

**Definition 2.2.** *A positive Borel measure on* $\Omega$ *that is finite on each compact subset of* $\Omega$ *is said to be a Radon measure on* $\Omega$.

The restriction of the Lebesgue measure to $\mathcal{B}(\mathbb{R}^n)$ is a classical example of a Radon measure on $\mathbb{R}^n$. Note also that the Lebesgue measure on $\mathbb{R}^n$ can be defined as the unique positive Radon measure $\mathcal{L}_n$ on $\mathbb{R}^n$ satisfying

$$\mathcal{L}_n([0,1]^n) = 1 \quad \text{and} \quad \mathcal{L}_n(a + tA) = t^n \mathcal{L}_n(A) \quad \forall\, a \in \mathbb{R}^n,\ A \in \mathcal{B}(\mathbb{R}^n), t > 0. \tag{2.2}$$

We denote $|A| = \mathcal{L}_n(A)$.

**Definition 2.3.** *For* $\mu \in \mathcal{M}(\Omega)$ *and* $A \in \mathcal{B}(\Omega)$, *we define the total variation of* $\mu$ *on* $A$ *by*

$$|\mu|(A) = \sup \left\{ \sum_{k\in\mathbb{N}} |\mu(A_k)| \ :\ A = \bigcup_{k\in\mathbb{N}} A_k,\ A_k \cap A_j = \emptyset \quad \textit{if} \quad k \neq j \right\}.$$

It is well known that the total variation of a measure is a positive measure taking only finite values $|\mu(A)| \leq |\mu|(A)$ for all $A \in \mathcal{B}(\Omega)$, and the total variation can be viewed as a norm on the set of measures on $\Omega$.

We will indicate by $\mathcal{M}_b(\Omega)$ the space of Radon measures on $\Omega$ with finite total variation. Note that $\mathcal{M}_b(\Omega)$ is a Banach space (i.e., a complete linear normed space) when equipped with the norm $\|\mu\|_{\mathcal{M}_b(\Omega)} := |\mu|(\Omega)$.

**Definition 2.4.** *The support of* $\mu \in \mathcal{M}(\Omega)$ *is defined as*

$$\operatorname{spt} \mu = \{x \in \Omega \ :\ |\mu|(B_r(x)) > 0 \ \textit{for all open balls} \ \ B_r(x) \subset \Omega\}.$$

**Theorem 2.5.** *Every measure* $\mu \in \mathcal{M}_+(\Omega)$ *is regular in the following sense:*

$$\mu(A) = \inf \{\mu(B) \ :\ A \subset B, B \ \ \textit{open}\}, \tag{2.3}$$

$$\mu(A) = \sup \{\mu(C) \ :\ C \subset A, C \ \ \textit{closed}\} \tag{2.4}$$

*for all* $A \in \mathcal{B}(\Omega)$.

Note that approximating closed sets with compact sets, we also have

$$\mu(A) = \sup \{\mu(C) \ :\ K \subset A, K \ \ \text{compact}\}.$$

**Definition 2.6.** *Let $\mu \in \mathcal{M}_+(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$. We say that $\lambda$ is absolutely continuous with respect to $\mu$ (and we write $\lambda \ll \mu$) if $\lambda(A) = 0$ for every $A \in \mathcal{B}(\Omega)$ with $\mu(A) = 0$.*

*We say that $\lambda$ is singular with respect to $\mu$, if there exists a set $E \in \mathcal{B}(\Omega)$ such that $\mu(E) = 0$ and $\lambda(A) = 0$ for all $A \in \mathcal{B}(\Omega)$ with $A \cap E = \emptyset$ (in this case, we say that $\lambda$ is concentrated on $E$).*

For $\mu \in \mathcal{M}(\Omega)$ we adopt the usual notation $L^p(\Omega, \mathrm{d}\mu)$ to indicate the space of $p$-summable functions with respect to $\mu$ on $\Omega$. We omit $\mu$ if it is the Lebesgue measure. Let us observe that if $f \in L^1(\Omega, \mathrm{d}\mu)$ and $\mu \in \mathcal{M}(\Omega)$ then we can define the measure $f\mu \in \mathcal{M}(\Omega)$ by

$$f\mu(A) = \int_A f \,\mathrm{d}\mu.$$

Hence, $f\mu \ll |\mu|$ and $|f\mu| = |f||\mu|$. In particular, if $\lambda \ll \mu$, then $\lambda = f\mu$ for some $f \in L^1(\Omega, \mathrm{d}\mu)$.

**Theorem 2.7.** *(Radon–Nikodym) For $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}_+(\Omega)$ there exists a function $f \in L^1(\Omega, \mathrm{d}\mu)$ and a measure $\lambda^s$, singular with respect to $\lambda$, such that*

$$\lambda = f\mu + \lambda^s.$$

This relation is called the Radon–Nikodym decomposition of $\lambda$ with respect to $\mu$.

**Definition 2.8.** *Let $\mu = \mathcal{L}_n$ and let $f \in L^1(\Omega)$. Then each point $x \in \Omega$ with*

$$\lim_{r \to 0+} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \,\mathrm{d}\mu(y) = 0$$

*is called a Lebesgue point for $f$. Note that the set of Lebesgue points of $f$ depends on the particular choice of the representative in the equivalence class of $L^1(\Omega)$. Hence, we will always take a particular choice of the representative of $f$ whenever we consider Lebesgue points.*

The expression "$\mu$ a.e." means "almost everywhere with respect to the measure $\mu$" – that is, except possibly on a set $A$ with $\mu(A) = 0$.

### 2.1.1 Hausdorff measures

We next introduce certain "lower-dimensional" measures on $\mathbb{R}^n$, which allow one to measure certain "very small" subsets of $\mathbb{R}^n$. The idea is that $A$ is an "$s$-dimensional subset" of $\mathbb{R}^n$ if there is a so-called Hausdorff measure $\mathcal{H}^s$ such that $0 < \mathcal{H}^s(A) < +\infty$ even if $A$ is very complicated geometrically. To do so, we will construct a positive measure $\mathcal{H}^s$ on $\mathbb{R}^n$ from set functions, following a procedure due to Carathéodory, which we briefly recall.

**Definition 2.9.** *A set function* $\lambda : 2^{\Omega} \to [0,+\infty]$ *is an outer measure if* $\lambda(\emptyset) = 0$ *and* $\lambda$ *is countably subadditive, that is,*

$$\lambda(B) = \sum_{k\in\mathbb{N}} \lambda(B_k) \quad \textit{for} \quad B \subseteq \bigcup_{k\in\mathbb{N}} B_k.$$

*We say that a set* $M$ *is measurable for* $\lambda$ *if*

$$\lambda(B) = \lambda(B \cap M) + \lambda(B \setminus M) \quad \forall\, B \subseteq \Omega.$$

Let $\mathcal{M}_\lambda$ be a family of all measurable sets for $\lambda$. If $\lambda$ is an outer measure, then $\mathcal{M}_\lambda$ is a $\sigma$-algebra and $\lambda|_{\mathcal{M}_\lambda}$ is countably additive (see Evans [106]). So, if $\mathcal{B}(\Omega) \subseteq \mathcal{M}_\lambda$ and $\lambda(\Omega) < +\infty$, then $\mu = \lambda|_{\mathcal{M}_\lambda} \in \mathcal{M}_+(\Omega)$. In order to see that an outer measure generates a measure by the above construction, we have to prove that Borel sets are measurable. The following proposition provides a simple criterion for measurability.

**Proposition 2.10.** *Let* $\lambda$ *be an outer measure. Then Borel sets are measurable for* $\lambda$ *if and only if*

$$\operatorname{dist}(A,B) > 0 \quad \Rightarrow \quad \lambda(A) + \lambda(B) = \lambda(A \cup B) \tag{2.5}$$

*for all* $A, B \subseteq \Omega$*, where* $\operatorname{dist}(A,B) = \inf\{|x-y| \;:\; x \in A, y \in B\}$.

We now apply Carathéodory's construction to define the measure which will be of use in the sequel.

**Definition 2.11.** *Let* $\alpha \geq 0$ *and* $\delta > 0$*. For all* $E \subset \mathbb{R}^n$ *we define the pre-Hausdorff measure* $\mathcal{H}^\alpha_\delta$ *of* $E$ *as*

$$\mathcal{H}^\alpha_\delta(E) = \frac{\omega_\alpha}{2^\alpha} \inf\left\{ \sum_{k\in\mathbb{N}} (\operatorname{diam} E_k)^\alpha \;:\; \operatorname{diam} E_k < \delta,\; E \subseteq \bigcup_{k\in\mathbb{N}} E_k \right\},$$

*where* $\omega_\alpha = \pi^{\alpha/2}/\Gamma(\alpha/2+1)$ *and* $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}\exp(-s)\,\mathrm{d}s$ *is the Euler function, which coincides with the Lebesgue measure of the unit ball in* $\mathbb{R}^n$ *if* $\alpha$ *is integer.*

Note that $\mathcal{H}^\alpha_\delta(E)$ is decreasing in $\delta$. Consequently, the limit

$$\mathcal{H}^\alpha(E) = \lim_{\delta\to 0+} \mathcal{H}^\alpha_\delta(E) = \sup_{\delta>0} \mathcal{H}^\alpha_\delta(E)$$

exists and is in $[0,+\infty]$. The value $\mathcal{H}^\alpha(E)$ is called the $\alpha$-dimensional Hausdorff measure of $E$.

*Remark 2.12.* Let us note the following:

(a) $\mathcal{H}^\alpha$ is a regular Borel measure ($0 \leq \alpha < +\infty$).
(b) $\mathcal{H}^\alpha$ is the null measure if $\alpha > n$.

(c) $\mathcal{H}^\alpha(\lambda A) = \lambda^\alpha \mathcal{H}^\alpha(A)$ for all $\lambda > 0$, $A \subset \mathbb{R}^n$.
(d) If $\mathcal{H}^\alpha(A) < \infty$, then $\mathcal{H}^\beta(A) = 0$ for all $0 \le \alpha < \beta < \infty$.
(e) If $\mathcal{H}^\beta(A) > 0$, then $\mathcal{H}^\alpha(A) = +\infty$ for all $0 \le \alpha < \beta < \infty$.

Thus, if $\alpha < n$, then $\mathcal{H}^\alpha(A) = +\infty$ for all nonempty open sets $A \subset \mathbb{R}^n$, and $\mathcal{H}^\alpha(A)$ agrees with the ordinary "$k$-dimensional surface area" on nice sets $A \in \mathcal{B}(\mathbb{R}^n)$. Thus, $\mathcal{H}^\alpha$ is not a positive Radon measure on $\mathbb{R}^n$ if $\alpha < n$, since $\mathbb{R}^n$ is not $\sigma$-finite with respect to $\mathcal{H}^\alpha$. However, if $\mathcal{H}^\alpha(B) < +\infty$ for some $B \in \mathcal{B}(\Omega)$, then $\mathcal{H}^\alpha \llcorner B \in \mathcal{M}_+(\Omega)$. Moreover, it can be proved that $\mathcal{H}^n = \mathcal{L}_n$ in $\mathbb{R}^n$.

## 2.2 Sobolev spaces and boundary value problems

Here, we briefly outline some basic facts from the theory of Sobolev spaces and the theory of boundary value problems, which are widely used throughout the book. Most of these result are well known; therefore, we just formulate them without proof. For details and complete proofs, the reader may turn to the numerous textbooks on the subject – e.g., Adams [2], Kantorovich and Akilov [128], Lions and Magenes [173], Maz'ya [185], Sobolev [232], Smirnov [231], and so forth.

### 2.2.1 Weak derivatives

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, let $C^k(\Omega)$ be the space of $k$-times continuously differentiable functions $u : \Omega \to \mathbb{R}$, and let $C_0^k(\Omega)$ be the functions in $C^k(\Omega)$ with compact support in $\Omega$. Hence, $C_0^\infty(\Omega)$ is the set of all real-valued infinitely differentiable functions with compact support in $\Omega$. In this case, for any $x_0 \in \Omega$ and $\varepsilon > 0$ small enough, the function

$$\varrho_\varepsilon(x) = \varrho\left(\frac{x - x_0}{\varepsilon}\right), \text{ where } \varrho(x) = \begin{cases} \exp\left\{-\dfrac{1}{1 - \|x\|^2_{\mathbb{R}^n}}\right\} & \text{if } \|x\|_{\mathbb{R}^n} < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is in $C_0^\infty(\Omega)$. The support of $\varrho_\varepsilon$ is the ball with center $x_0$ and radius $\varepsilon$.

We define the set $C_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the uniform topology. It is a separable Banach space if equipped with the $\|\cdot\|_\infty$-norm.

Denote by $L^p(\Omega)$, $1 \le p < +\infty$, the space of functions defined on $\Omega$ and $p$th-power summable in the sense of Lebesgue. An element of $L^p(\Omega)$ is an equivalence class of functions different only on a set of zero $\mathcal{L}_n$-measure. The space $L^p(\Omega)$ equipped with norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{1/p}$$

is a Banach space.

Introduce the space $L^\infty(\Omega)$ of essentially bounded (i.e., bounded by a constant almost everywhere) real-valued functions in $\Omega$ that are Lebesgue measurable. The space $L^\infty(\Omega)$ is equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| = \inf_{A\subset\Omega,\ |A|=0} \sup_{x\in\Omega\setminus A} |u(x)|;$$

that is, "ess sup" means the supremum up to a set of zero measure. In the case of a bounded domain $\Omega$, we have

$$\|u\|_{L^\infty(\Omega)} = \lim_{p\to\infty} \|u\|_{L^p(\Omega)},$$

which justifies the notation $L^\infty(\Omega)$.

Suppose $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of order $|\alpha| = \alpha_1 + \dots + \alpha_n = k$. We call a function $\varphi$ belonging to $C_0^\infty(\Omega)$ a test function. We say that $v \in L^1(\Omega)$ is the $\alpha$th-weak partial derivative of $u \in L^1(\Omega)$, denoted $D^\alpha u = v$, provided

$$\int_\Omega u D^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\varphi \,\mathrm{d}x \tag{2.6}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. If a vector $\mathbf{v} = \{v_1, \dots, v_n\}$, $v_i \in L^1(\Omega)$, is the gradient of a function $u \in L^1(\Omega)$ in the weak sense, then we denote it either by $\nabla u$ or by $\partial u/\partial x$.

### 2.2.2 Sobolev spaces

Having fixed $1 \le p < +\infty$ and $k$, we denote by $W^{k,p}(\Omega)$ the Sobolev space formed by all functions $u \in L^p(\Omega)$ such that for each multi-index $\alpha$ with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left( \int_\Omega \sum_{|\alpha|\le k} |D^\alpha u(x)|^p \,\mathrm{d}x \right)^{1/p} \tag{2.7}$$

is finite. If $p = 2$, we usually write $H^k(\Omega) = W^{k,2}(\Omega)$ $(k = 0, 1, \dots)$. By convention we set $H^0(\Omega) = L^2(\Omega)$ and $D^0 v = v$. The letter $H$ is used, since $H^k(\Omega)$ is a Hilbert space equipped with the scalar product

$$(u_1, u_2)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u_1(x) D^\alpha u_2(x) \,\mathrm{d}x.$$

The space $W_0^{k,p}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. The space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is naturally associated with the Dirichlet problem, since the inclusion $u \in H_0^1(\Omega)$ represents an equivalent formulation of the boundary condition $u|_{\partial\Omega} = 0$. To clarify the presentation further, we will consider only the Sobolev space $W^{1,p}(\Omega)$, which is a separable Banach space for the norm (2.7) and reflexive if $1 < p < +\infty$.

We recall that a Banach space $X$ is said to be *compactly embedded* in a Banach space $Y$ provided (i) $\|x\|_Y \leq C\|x\|_X$ $(x \in X)$ for some constant $C$ and (ii) each bounded sequence in $X$ is precompact in $Y$. The following definition expresses in precise terms the intuitive notion of a regular boundary ($C^\infty$, $C^k$, or Lipschitz).

**Definition 2.13.** *(i) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that $\Omega$ is a bounded open set with $C^k$, $k \geq 1$, boundary, if for every $x \in \partial\Omega$, there exist a neighborhood $U \subset \mathbb{R}^n$ of $x$ and a one-to-one and onto map $T : Q \mapsto U$, where*

$$Q = \{x \in \mathbb{R}^n \;:\; |x_j| < 1,\; j = 1, 2, \ldots, n\},$$
$$T \in C^k(\overline{Q}), T^{-1} \in C^k(\overline{U}), T(Q_+) = U \cap \Omega, T(Q_0) = U \cap \partial\Omega$$

*with $Q_+ = \{x \in Q \;:\; x_n > 0\}$ and $Q_0 = \{x \in Q \;:\; x_n = 0\}$.*

*(ii) If $T$ is in $C^{k,\alpha}$, $0 < \alpha \leq 1$, we will say that $\Omega$ is a bounded open set with $C^{k,\alpha}$ boundary.*

*(iii) If $T$ is in $C^{0,1}$, we will say that $\Omega$ is a bounded open set with Lipschitz boundary.*

Thus, an open set $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary if for every $x \in \partial\Omega$, there exists a neighborhood $U \subset \mathbb{R}^n$ of $x$ such that $U \cap \partial\Omega$ is a graph, in a suitable coordinate system, of a Lipschitz continuous function whose epigraph contains $U \cap \Omega$; see Fig. 2.1. Note that every polyhedron has a Lipschitz boundary, whereas the unit ball in $\mathbb{R}^n$ has a $C^\infty$ boundary. If $\Omega$ is an open convex set, then $\Omega$ has a Lipschitz boundary as well. If $\Omega$ has a Lipschitz boundary, then for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$, there exists the outward unit vector normal to $\partial\Omega$, which we denote by $\mathbf{n}_\Omega$ (see Nečas [205]).

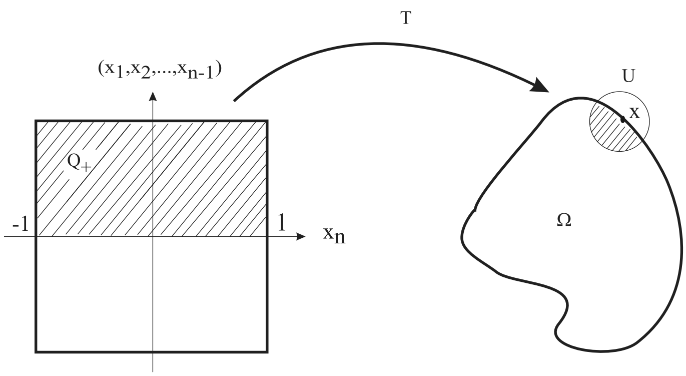


**Fig. 2.1.** A regular boundary

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. Then, by the Sobolev embedding theorem and the Rellich–Kondrachov theorem (see Adams [2], Maz'ya [185]), we have the following:

- If $1 \le p < n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with
  – compact embedding for $q \in [1, p^*)$, where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$, and
  – continuous embedding for $q = p^*$;
- If $p = n$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with compact embedding for $q \in [1, \infty)$;
- If $p > n$, $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ with compact embedding.

Hereafter we call the constant $p^* = np/(n-p)$ the Sobolev conjugate of $p$.

Note that in all cases (i.e., $1 \le p \le \infty$), the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact. These results are still valid for $W_0^{1,p}(\Omega)$ without any regularity assumption on $\partial\Omega$. However, examples of domains $\Omega$ between two spirals can be constructed where the embedding $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is not compact [91]. If $\Omega$ is unbounded, then the compactness of that embeddings is lost. Indeed, as in P. L. Lions [174, 175], we concentrate on the case when $\Omega = \mathbb{R}^n$.

**Theorem 2.14.** *Assume that for $n \ge 3$,*

$$f_k \to f \quad \text{strongly in} \quad L^2_{loc}(\mathbb{R}^n), \quad Df_k \rightharpoonup Df \quad \text{in} \quad \left[L^2(\mathbb{R}^n)\right]^n.$$

*Suppose further that*

$$|Df_k|^2 \rightharpoonup \mu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^n), \quad |f_k|^{2^*} \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^n).$$

*Then there exist an at most countable index set $J$, distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, and non-negative weights $\{\mu_j, \nu_j\}$ such that*

$$\nu = |f|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \ge |Df|^2 + \sum_{j\in J} \mu_j \delta_{x_j},$$

*where $\delta_{x_j} \in \mathcal{M}_b(\mathbb{R}^n)$ denotes the Dirac measure located at the point $x_j$.*

Note that a typical function $y \in W^{1,p}(\Omega)$ is not, in general, continuous and is only defined almost everywhere on $\Omega$. Since $\partial\Omega$ has $n$-dimensional Lebesgue measure 0, there is no direct meaning we can give to the expression "$u$ restricted to $\partial\Omega$." The notion of a *trace operator* resolves this problem – namely if $\partial\Omega$ is Lipschitz continuous, then there exists a unique linear continuous map

$$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that for any $y \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, one has $\gamma(y) = y|_{\partial\Omega}$. The function $\gamma(y)$ is called the trace of $y$ on $\partial\Omega$.

*Remark 2.15.* It is worth noting that when $\Omega$ is an open set with cusps on the boundary, then the existence of the trace operator with the above properties may fail. Indeed, consider, for instance, in $\mathbb{R}^2$ the domain

$$\Omega = \left\{(x_1, x_2) \;:\; 0 < x_1 < 1, 0 < x_2 < x_1^5\right\}.$$

Then for the function $u(x_1, x_2) = 1/x_1$, we have

$$\frac{\partial u}{\partial x_1} = -\frac{1}{x_1^2}, \quad \frac{\partial u}{\partial x_2} = 0 \text{ in } \Omega$$

in the distributional sense. So

$$\begin{aligned}\int_\Omega \left\{u^2 + |\nabla u|^2\right\} \mathrm{d}x &= \int_\Omega \left\{\frac{1}{x_2^2} + \frac{1}{x_1^4}\right\} \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_1^1 \mathrm{d}x_1 \int_0^{x_1^5} \left\{\frac{1}{x_2^2} + \frac{1}{x_1^4}\right\} \mathrm{d}x_2 \\ &= \int_0^1 \left\{x_1^3 + x_1\right\} \mathrm{d}x_1 = \frac{3}{4}.\end{aligned}$$

Thus, $u \in H^1(\Omega)$, but the trace of $u$ on $\partial\Omega$ does not belong to $L^2(\partial\Omega)$ since $\frac{1}{x_1} \notin L^2(0,1)$.

It is clear now that the space $H_0^1(\Omega)$ can be defined as

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega),\ \gamma(u) = 0\right\}.$$

However, we note that $\gamma$ is not onto $L^p(\partial\Omega)$ (i.e., there exist functions in $L^p(\partial\Omega)$ which are not traces of any element of $W^{1,p}(\Omega)$). In particular, if $p = 2$, then this leads us to the following set $H^{1/2}(\partial\Omega) := \gamma(H^1(\Omega))$, where $H^{1/2}(\partial\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{H^{1/2}(\partial\Omega)} = \left(\|u\|^2_{L^2(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}} \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$

with compact embedding $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

As mentioned earlier, if $\partial\Omega$ is Lipschitz continuous, then the unit outward normal vector $\mathbf{n} = (n_1, \ldots, n_n)$ to $\Omega$ is well defined almost everywhere. As a result, the well-known *Green formula* for smooth functions can be extended to Sobolev spaces. Indeed, in this case we have

$$\int_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x = -\int_\Omega v \frac{\partial u}{\partial x_i} \,\mathrm{d}x + \int_{\partial\Omega} \gamma(u)\gamma(v)\, n_i \,\mathrm{d}s, \quad i = 1, \ldots, n,$$

for any $u, v \in H^1(\Omega)$.

For any bounded domain $\Omega$, the Friedrichs inequality

$$\int_\Omega y^2 \,\mathrm{d}x \le C \int_\Omega |\nabla y|^2 \,\mathrm{d}x, \quad \forall\, y \in H_0^1(\Omega), \tag{2.8}$$

holds with a constant $C$ independent of $y$. Inequality (2.8) implies that the functional $\|y\|_1 = \left(\int_\Omega |\nabla y|^2 \,\mathrm{d}x\right)^{1/2}$ can be taken as an equivalent norm in $H_0^1(\Omega)$, and, indeed, we will always consider $\|y\|_1$ as a norm in this space.

*Remark 2.16.* It should be stressed that we need to impose a condition of the type $y = 0$ on $\partial\Omega$ (which comes from the hypothesis $y \in H_0^1(\Omega)$) to avoid constant functions $y$ (which imply $\nabla y = 0$), otherwise inequality (2.8) would be trivially false.

If $u \in H^1(\Omega)$ and $\Omega$ is a bounded connected open domain with Lipschitz boundary, then the Poincaré inequality

$$\int_\Omega y^2 \,\mathrm{d}x \le C \left\{ \left( \int_\Omega y \,\mathrm{d}x \right)^2 + \int_\Omega |\nabla y|^2 \,\mathrm{d}x \right\}, \quad \forall\, y \in H^1(\Omega), \tag{2.9}$$

is valid, and, by the Rellich–Kondrachov compactness theorem, the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Sometimes the Poincaré inequality appears in the following form. If $1 \le p \le \infty$ and if we set

$$\mathcal{M}_\Omega(y) = \frac{1}{|\Omega|} \int_\Omega y(x) \,\mathrm{d}x,$$

then there exists a constant $C(\Omega, p) > 0$ so that

$$\|y - \mathcal{M}_\Omega(y)\|_{L^p(\Omega)} \le C(\Omega, p) \|\nabla y\|_{L^p(\Omega)}, \quad \forall\, y \in W_{1,p}(\Omega).$$

The dual space of $H_0^1(\Omega)$ (i.e., the set of all continuous linear functionals on $H_0^1(\Omega)$) is denoted by $H^{-1}(\Omega)$. If $f$ is an element of $H^{-1}(\Omega)$, then $\langle f, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ stands for the value of the functional $f$ applied to the element $y \in H_0^1(\Omega)$. Note that for any element $F \in H^{-1}(\Omega)$, there exist $n+1$ functions $f_0,\ f_1, \dots,\ f_n$ such that $F = f_0 + \sum_{k=1}^n \partial f_k / \partial x_k$ in the sense of distributions, that is,

$$\langle F, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f_0 y \,\mathrm{d}x - \sum_{k=1}^n \int_\Omega f_k \frac{\partial y}{\partial x_k} \,\mathrm{d}x.$$

Moreover,

$$\|F\|^2_{H^{-1}(\Omega)} = \inf \sum_{k=0}^n \|f_k\|^2_{L^2(\Omega)},$$

where the infimum is taken over all vectors

$$(f_0, f_1, \dots, f_n) \in \left[ L^2(\Omega) \right]^{n+1}$$

such that the representation for $F$ given above holds true.

One can give an example of a linear functional on $H_0^1(\Omega)$, setting

$$\langle f_0, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f_0 \varphi \,\mathrm{d}x, \quad f_0 \in L^2(\Omega).$$

Similarly, if $\varphi \in H^{1/2}(\partial\Omega)$ and $f \in L^2(\partial\Omega)$, one also has

$$\langle f, y\rangle_{(H^{1/2}(\partial\Omega))^*, H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} f\varphi \, \mathrm{d}s.$$

Suppose that $\partial\Omega$ is Lipschitz continuous. Then one has $L^2(\Omega) \subset H^{-1}(\Omega)$ with compact injection. Hence, the following embeddings are compact (so-called Gelfand–Lions triplet of Sobolev spaces):

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

*Remark 2.17.* It is clear that the restriction of any element of $(H^1(\Omega))^*$ to $H_0^1(\Omega)$ is in $H^{-1}(\Omega)$. However, the dual space $(H^1(\Omega))^*$ is not contained in $H^{-1}(\Omega)$ since $(H^1(\Omega))^*$ can be identified with the direct sum $H^{-1}(\Omega) \oplus H^{-1/2}(\partial\Omega)$, where $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$. Moreover, if $\partial\Omega$ is Lipschitz continuous, $\mathbf{y} \in H(\Omega, \mathrm{div}) = \left\{\mathbf{y} \,:\, \mathbf{y} \in \left[L^2(\Omega)\right]^n, \mathrm{div}\,\mathbf{y} \in L^2(\Omega)\right\}$, and $w \in H^1(\Omega)$, then $\mathbf{y} \cdot \mathbf{n} \in H^{-1/2}(\partial\Omega)$, the map

$$H(\Omega, \mathrm{div}) \ni \mathbf{y} \mapsto \mathbf{y} \cdot \mathbf{n} \in H^{-1/2}(\partial\Omega)$$

is linear and continuous, and

$$-\int_\Omega (\mathrm{div}\mathbf{y}) w \, \mathrm{d}x = \int_\Omega \mathbf{y} \cdot \nabla w \, \mathrm{d}x + \langle \mathbf{y} \cdot \mathbf{n}, w\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \,. \tag{2.10}$$

For any vector $\mathbf{p} \in \mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^n$, the *divergence* is an element of the space $H^{-1}(\Omega)$ defined by

$$\langle \mathrm{div}\, \mathbf{p}, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = -\int_\Omega \mathbf{p} \cdot \nabla\varphi \, \mathrm{d}x, \quad \forall\, \varphi \in H_0^1(\Omega), \tag{2.11}$$

where "·" denotes the scalar product of two vectors. The following estimate is evident:

$$\|\mathrm{div}\, \mathbf{p}\|_{H^{-1}(\Omega)} = \sup_{\|\varphi\|_1 = 1} \int_\Omega \mathbf{p} \cdot \nabla\varphi \, \mathrm{d}x \le \|\mathbf{p}\|_{\mathbf{L}^2(\Omega)}. \tag{2.12}$$

A vector field $\mathbf{p}$ is said to be *solenoidal* if $\mathrm{div}\, \mathbf{p} = 0$. We say that a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ is *potential* if $\mathbf{v}$ can be represented in the form $\mathbf{v} = \nabla u$, where $u \in H_0^1(\Omega)$.

### 2.2.3 Vector-valued spaces of the type $\mathbf{L^p(a, b; X)}$

Let $\mathbf{X}$ be a Banach space, $\Omega \subset \mathbb{R}^n$, and $p$ such that $1 \le p \le +\infty$. We denote by $L^p(\Omega; \mathbf{X})$ the set of measurable functions $y : \Omega \to \mathbf{X}$ such that $\|u(\cdot)\|_{\mathbf{X}} \in L^p(\Omega)$. Similarly, one can also define the set of distributions $\mathcal{D}'(\Omega; \mathbf{X})$ on $\Omega$ with values in $X$. $L^p(\Omega; \mathbf{X})$ is a Banach space with respect to the norm

$$\|y\|_{L^p(\Omega;\mathbf{X})} = \left(\int_\Omega \|u(x)\|_{\mathbf{X}}^p \, \mathrm{d}x\right)^{1/p}.$$

If $\mathbf{X}$ is reflexive and $1 < p < \infty$, the space $L^p(\Omega; \mathbf{X})$ is reflexive, too. Moreover, if $\mathbf{X}$ is separable and $1 \leq p < \infty$, then $L^p(\Omega; \mathbf{X})$ is separable.

This type of spaces is well adapted to the study of problems where one of the variables plays a special role. For instance, this occurs for the variable "time" in time-dependent problems. For various results on vector-valued functions, we refer to Schwartz [224] and Lions and Magenes [173].

Let $B_0$ and $B$ be two Banach spaces such that the embedding $B_0 \hookrightarrow B$ is compact. A natural question is whether the embedding $L^p(a, b; B_0) \hookrightarrow L^p(a, b; B)$ is also compact. Actually, one can prove that this is not true, in general. However, if we have three Banach spaces $B_0 \hookrightarrow B \hookrightarrow B_1$ such that $B_0$ and $B_1$ are reflexive and the embedding $B_0 \hookrightarrow B$ is compact, then the embedding $W \hookrightarrow L^{p_0}(a, b; B)$ is compact too, where

$$W = \left\{ y \ : \ y \in L^{p_0}(a, b; B_0), \frac{\partial y}{\partial t} \in L^{p_1}(a, b; B_1) \right\}, \quad 1 < p_0, p_1 < +\infty,$$

is a Banach space with respect to the graph norm

$$\|y\|_W = \|y\|_{L^{p_0}(a,b;B_0)} + \left\| \frac{\partial y}{\partial t} \right\|_{L^{p_1}(a,b;B_1)}.$$

Here, the derivative $\partial y / \partial t$ is the distribution in $\mathcal{D}'(a, b; B_1)$ defined by

$$\frac{\partial y}{\partial t}(\varphi) = - \int_a^b y \frac{\partial \varphi}{\partial t} \, \mathrm{d}t, \quad \forall \, \varphi \in \mathcal{D}(a, b).$$

The following theorem plays an important role in the study of PDEs.

**Theorem 2.18.** *Let us define the Banach spaces*

$$\mathcal{W} = \left\{ y \ : \ y \in L^2(a, b; H_0^1(\Omega)), \frac{\partial y}{\partial t} \in L^2(a, b; H^{-1}(\Omega)) \right\},$$
$$\mathcal{W}_1 = \left\{ y \ : \ y \in L^2(a, b; L^2(\Omega)), \frac{\partial y}{\partial t} \in L^2(a, b; H^{-1}(\Omega)) \right\},$$

*equipped with the norm of the graph. Then the following properties holds true:*

*(a) The embeddings $\mathcal{W} \hookrightarrow L^2(a, b; L^2(\Omega))$, $\mathcal{W}_1 \hookrightarrow L^2(a, b; H^{-1}(\Omega))$ are compact.*

*(b) One has the embedding*

$$\mathcal{W} \hookrightarrow C([a, b]; L^2(\Omega)), \quad \mathcal{W}_1 \hookrightarrow C([a, b]; H^{-1}(\Omega)),$$

*where, for $\mathbf{X} = L^2(\Omega)$ or $\mathbf{X} = H^{-1}(\Omega)$, one denotes by $C([a, b]; \mathbf{X})$ the space of measurable functions on $[a, b] \times \Omega$ such that $y(t, \cdot) \in \mathbf{X}$ for any $t \in [a, b]$ and such that the map $t \in [a, b] \mapsto y(t, \cdot) \in \mathbf{X}$ is continuous.*

*(c) For any $u, v \in \mathcal{W}$, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u(t,x)v(t,x)\,\mathrm{d}x = \langle u'(t,\cdot), v(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle v'(t,\cdot), u(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,.$$

Let $y \in L^2\left(a,b;H_0^1(\Omega)\right) \cap C\left([a,b];L^2(\Omega)\right)$. Then the following density result holds: For any $\delta > 0$, there exists $\Phi \in C^\infty([a,b];\mathcal{D}(\Omega))$ such that

$$\|y-\Phi\|_{C([a,b];L^2(\Omega))} \le \delta, \quad \|\nabla y - \nabla\Phi\|_{L^2((a,b)\times\Omega)} \le \delta.$$

To end this subsection, we recall one property, useful in the sequel, concerning the space $L^2(\Omega; C_{\mathrm{per}}(Y))$, where $C_{\mathrm{per}}(Y)$ denotes the subset of $C(\overline{Y})$ of $Y$-periodic functions – namely the space $L^2(\Omega; C_{\mathrm{per}}(Y))$ is separable and dense in $L^2(\Omega; L^2(Y)) = L^2(\Omega \times Y)$.

### 2.2.4 Lax–Milgram's lemma

Consider a real Hilbert space $V$. Let $a(u,v)$ be a bilinear form on $V$ (i.e., $a : V \times V \to \mathbb{R}$ is linear with respect to each argument). Suppose that the form $a(\cdot,\cdot)$ is continuous and coercive – that is, the following inequalities are satisfied:

$$\begin{aligned} &a(u,v) \le \nu_1\|u\|_V\|v\|_V, \quad \forall\, u,v \in V,\ \nu_1 > 0,\\ &a(u,u) \ge \nu_2\|u\|_V^2, \quad \forall\, u \in V,\ \nu_2 > 0. \end{aligned} \tag{2.13}$$

Let $V^*$ be the dual space of $V$. For any fixed $u \in V$, the linear form $a(u,v)$ is continuous with respect to $v \in V$ and represents an element of $V^*$ denoted by $\mathcal{A}u$. Thus, we obtain a linear operator defined by the formula

$$\langle \mathcal{A}u, v\rangle_{V^*,V} = a(u,v), \quad \mathcal{A} : V \to V^*.$$

Inequalities (2.13) show that the operator $\mathcal{A}$ is bounded and coercive: $\|\mathcal{A}\| \le \nu_2$, $\langle \mathcal{A}u, u\rangle_{V',V} \ge \nu_1\|u\|_V^2$. For any given $f \in V^*$, consider the following problem: Find an element $u \in V$ such that

$$a(u,v) = \langle f, v\rangle_{V^*,V}\,, \quad \forall\, v \in V. \tag{2.14}$$

In fact, this problem is equivalent to establishing solvability of the equation $\mathcal{A}u = f$. The following assertion generalizes the Riesz representation theorem.

**Lemma 2.19.** *(Lax–Milgram) The problem (2.14) has a solution $u$ which is unique and satisfies the estimate $\|u\|_V \le \nu_1^{-1}\|f\|_{V^*}$. In other words, the bounded coercive operator $\mathcal{A}$ is an isomorphism between the spaces $V$ and $V^*$ and the norm of the inverse operator is bounded by $\nu_1^{-1}$.*

If the bilinear form $a(u,v)$ is symmetric, then $\mathcal{A}u = f$ is the Euler equation associated with the following variational problem:

$$E = \inf_{v \in V} F(v), \quad F(v) = \frac{1}{2} a(v,v) - \langle f, v \rangle_{V^*,V} \,.$$

Indeed, let $u = \mathcal{A}^{-1} f$. Then

$$\begin{aligned} 2(F(v) - F(u)) &= \langle \mathcal{A}v, v \rangle_{V^*,V} - \langle \mathcal{A}u, u \rangle_{V^*,V} - 2 \langle f, v \rangle_{V^*,V} \\ +2 \langle f, u \rangle_{V^*,V} &= \langle \mathcal{A}v, v \rangle_{V^*,V} - \langle \mathcal{A}u, u \rangle_{V^*,V} + 2 \langle f, u \rangle_{V^*,V} \\ -2 \langle \mathcal{A}u, v \rangle_{V^*,V} &= \langle \mathcal{A}v, v \rangle_{V^*,V} - 2 \langle \mathcal{A}u, v \rangle_{V^*,V} + \langle \mathcal{A}u, u \rangle_{V^*,V} \\ &= \langle \mathcal{A}(v-u), v-u \rangle_{V^*,V} > 0, \quad \forall\, v \neq u. \end{aligned}$$

So, $u = \mathcal{A}^{-1} f$ is the unique minimizer for $F(v)$.

### 2.2.5 General setting of the variational formulation of boundary value problems

To begin with, we recall the notation of a well-posed problem introduced by Hadamard. Let $\mathcal{P}$ be a boundary value problem and let $\mathcal{Y}$ and $\mathcal{F}$ be two Banach spaces. We say that $\mathcal{P}$ is well-posed (with respect to $\mathcal{Y}$ and $\mathcal{F}$) if the following hold:

(i) For any element $f \in \mathcal{F}$ there exists a solution $y \in \mathcal{Y}$ of $\mathcal{P}$.
(ii) The solution is unique.
(iii) The map $\mathcal{F} \ni f \mapsto y \in \mathcal{Y}$ is continuous.

Obviously, the well-posedness of a problem depends on the choice of the spaces $\mathcal{Y}$ and $\mathcal{F}$. As a matter of fact, the examples of boundary value problems with this property, which we treat in the sequel, are related to an equation of the form $\mathcal{A}y = f$, where the operator $\mathcal{A}$ is given as follows:

$$\mathcal{A} = -\mathrm{div}(\mathbf{A}(x)\nabla) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right). \tag{2.15}$$

Here, $\mathbf{A}(x) = \{a_{ij}(x)\}$ is a matrix (not necessarily symmetric) with bounded measurable elements (i.e., $\|a_{ij}\|_{L^\infty(\Omega)} \le \beta$) satisfying the ellipticity condition

$$\exists\, \alpha > 0 \text{ such that } \sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j \ge \alpha \sum_{i=1}^{n} \lambda_i^2 \text{ a.e. on } \Omega\ \forall \lambda \in \mathbb{R}^n. \tag{2.16}$$

The matrix will always be associated with the bilinear form

$$a(y,\varphi) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x) \nabla y \, \mathrm{d}x. \tag{2.17}$$

### The Dirichlet problem

Let $f \in H^{-1}(\Omega)$ and consider the problem

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \quad \text{in} \quad \Omega, \qquad y = 0 \text{ on } \ \partial\Omega. \tag{2.18}$$

The corresponding variational formulation is

$$\begin{cases} \quad \text{Find} \ \ y \in H_0^1(\Omega) \text{ such that} \\ a(y,\varphi) = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\, \varphi \in H_0^1(\Omega). \end{cases} \tag{2.19}$$

Because of the Friedrichs inequality (2.8), the form $a(y,\varphi)$ is coercive on $H_0^1(\Omega)$. Therefore, for any $f \in H^{-1}(\Omega)$, there exists a unique element $y \in H_0^1(\Omega)$ satisfying (2.19). Moreover, $\|y\|_{H_0^1(\Omega)} \le \alpha^{-1}\|f\|_{H^{-1}(\Omega)}$. This element is called a weak solution of the Dirichlet problem (2.18). It is clear that (2.18) is a well-posed problem (in Hadamard's sense) for the choice $\mathcal{Y} = H_0^1(\Omega)$ and $\mathcal{F} = H^{-1}(\Omega)$ (or $\mathcal{F} = L^2(\Omega)$).

Assume that $\partial\Omega$ is Lipschitz continuous. Suppose we are given $f$ in $H^{-1}(\Omega)$ and $g$ in $H^{1/2}(\partial\Omega)$. Consider the nonhomogeneous Dirichlet problem

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \quad \text{in} \quad \Omega, \qquad y = g \text{ on } \ \partial\Omega.$$

Using the trace notion, we say that $y$ is a weak solution of this problem iff

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \quad \text{in} \quad \mathcal{D}'(\Omega), \qquad \gamma(y) = g \text{ in } \ H^{1/2}(\partial\Omega). \tag{2.20}$$

Then the problem (2.20) has a unique solution $y$ in $H^1(\Omega)$ such that

$$\|y\|_{H^1(\Omega)} \le C\left[\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right],$$

where $C$ is a positive constant depending on $\Omega$, $\alpha$, and $\beta$.

Moreover, we have the following result (see Lions and Magenes [173]).

**Theorem 2.20.** *Suppose that $\mathbb{N} \ni s \ge 2$. Then for any $f \in H^{s-2}(\Omega)$ and $g \in H^{s-1/2}(\partial\Omega)$, there exists a unique solution $y \in H^s(\Omega)$ of the problem*

$$-\triangle\, y = f \quad \text{in} \quad \Omega, \qquad \gamma(y) = g.$$

*Moreover, in this case,*

$$\|y\|^2_{H^s(\Omega)} \le C\left[\|f\|^2_{H^{s-2}(\Omega)} + \|g\|^2_{H^{s-1/2}(\partial\Omega)}\right], \tag{2.21}$$

*where the constant $C$ is independent of $f$ and $g$.*

### The Neumann problem

Let $f \in (H^1(\Omega))^*$ and $g \in H^{-1/2}(\partial\Omega)$. Let us introduce the following notation:

$$\frac{\partial}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x) n_i \frac{\partial}{\partial x_j}, \tag{2.22}$$

where $n_i$ denotes $i$th component of the unit outward normal vector to $\Omega$. We consider the Neumann problem

$$-\operatorname{div}(\mathbf{A}\nabla y) + y = f \quad \text{in} \quad \Omega, \qquad \frac{\partial y}{\partial \nu_A} = 0 \text{ on } \quad \partial\Omega. \tag{2.23}$$

The corresponding variational formulation is

$$\begin{cases} \quad\text{Find} \quad y \in H^1(\Omega) \text{ such that} \\ \widetilde{a}(y,\varphi) = \displaystyle\int_\Omega f\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in H^1(\Omega), \end{cases} \tag{2.24}$$

where now $\widetilde{a}$ is defined by

$$\widetilde{a}(y,\varphi) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}x, \quad \forall y, \varphi \in H^1(\Omega).$$

Let us observe that if $y$ is a solution of the problem (2.24), then (2.23) holds in $\mathcal{D}'(\Omega)$. Then, due to Remark 2.17 (see (2.10)), $\mathbf{A}\nabla y$ belongs to $H^1(\Omega, \operatorname{div})$, and therefore, $\partial y/\partial \nu_A$ is well-defined as an element of $H^{1/2}(\partial\Omega)$. This is the sense to be given to the boundary condition in (2.23).

If $g = 0$ and the domain $\Omega$ is sufficiently smooth (for instance, such that the Poincaré inequality holds in $\Omega$), then for any $f \in (H^1(\Omega))^*$, there exists a unique solution $y \in H^1(\Omega)$ of the problem (2.24). Moreover,

$$\|y\|_{H^1(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \|f\|_{(H^1(\Omega))^*} \quad \text{and} \quad \|y\|_{H^1(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \|f\|_{L^2(\Omega)},$$

provided $f \in L^2(\Omega)$.

As for the nonhomogeneous Neumann problem in domains with a Lipschitz continuous boundary, namely

$$-\operatorname{div}(\mathbf{A}\nabla y) + y = f \quad \text{in} \quad \Omega, \qquad \frac{\partial y}{\partial \nu_A} = g \text{ on } \quad \partial\Omega, \tag{2.25}$$

we have the following result: For any $f \in L^2(\Omega)$ and for any $g \in H^{-1/2}(\partial\Omega)$ there exists a unique solution $y \in H^1(\Omega)$ of the problem (2.25). Moreover, in this case,

$$\|y\|_{H^1(\Omega)} \le \frac{1}{\min\{1,\alpha\}} \left( \|f\|_{L^2(\Omega)} + C(\Omega)\|g\|_{H^{-1/2}(\partial\Omega)} \right), \tag{2.26}$$

where $C(\Omega)$ is a positive constant.

If we consider, instead of (2.25), the nonhomogeneous Neumann problem

$$-\operatorname{div}(\mathbf{A}\nabla y) = f \quad \text{in} \quad \Omega, \qquad \frac{\partial y}{\partial \nu_A} = g \text{ on } \partial\Omega \tag{2.27}$$

under the same hypotheses on $f$ and $g$ as earlier, then the corresponding bilinear form $a(y, \varphi)$ is no longer coercive on $H^1(\Omega)$, but it is coercive in the quotient space $W(\Omega) = H^1(\Omega)/\mathbb{R}$. This space is the space of classes of equivalence with respect to the relation

$$y_1 \backsimeq y_2 \Leftrightarrow y_1 - y_2 \;\text{ is a constant,}\;\; \forall\, y_1, y_2 \in H^1(\Omega).$$

Let us denote by $\dot{y}$ the class of equivalence represented by $y$. Then the natural variational formulation of (2.27) is

$$\begin{cases} \qquad \text{Find } \;\dot{y} \in W(\Omega) \text{ such that } \;\forall\, \dot{\varphi} \in W(\Omega) \\ \dot{a}(\dot{y}, \dot{\varphi}) = \displaystyle\int_\Omega f\varphi \,\mathrm{d}x + \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}\,, \quad \forall\, \varphi \in \dot{\varphi}, \end{cases} \tag{2.28}$$

where $\dot{a}$ is defined by

$$\dot{a}(\dot{y}, \dot{\varphi}) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \,\mathrm{d}x, \quad \forall y \in \dot{y}, \varphi \in \dot{\varphi},\; \forall \dot{\varphi}, \dot{y} \in W(\Omega).$$

It is clear that this problem makes sense if the right-hand side of (2.28) is independent of $\varphi \in \dot{\varphi}$ – namely suppose that $f \in L^2(\Omega)$ and for any $g \in H^{-1/2}(\partial\Omega)$ satisfy the compatibility condition

$$\int_\Omega f\, dx + \langle g, 1\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0,$$

then there exists a unique solution $\dot{y} \in W(\Omega)$ of the problem (2.28) with the estimate

$$\|\dot{y}\|_{W(\Omega)} \leq \frac{1}{\min\{1, \alpha\}} \left(\|f\|_{L^2(\Omega)} + C(\Omega)\|g\|_{H^{-1/2}(\partial\Omega)}\right). \tag{2.29}$$

We refer to Lions and Magenes [173] for the following result.

**Theorem 2.21.** *Suppose that $\mathbb{N} \ni s \geq 2$ and that $f \in H^{s-2}(\Omega)$ and $g \in H^{s-1-1/2}(\partial\Omega)$ satisfy the compatibility condition:*

$$\int_\Omega f(x)\,\mathrm{d}x + \int_{\partial\Omega} g(s)\,\mathrm{d}\mathcal{H}^{n-1} = 0. \tag{2.30}$$

*Then, there exists a unique solution $\dot{y} \in H^s(\Omega) \setminus \mathbb{R}$ to the problem (2.27). Moreover,*

$$\|\dot{y}\|^2_{H^s(\Omega)\setminus\mathbb{R}} \leq C\left[\|f\|^2_{H^{s-2}(\Omega)} + \|g\|^2_{H^{s-1/2}(\partial\Omega)} + \|y\|^2_{L^2(\Omega)}\right],$$

*where the constant $C$ is independent of $f$ and $g$.*

### The Robin problem

As in the previous case, suppose that $\partial\Omega$ is Lipschitz continuous and let $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. The Robin boundary value problem we consider can be stated as follows:

$$-\operatorname{div}(\mathbf{A}\nabla y) + y = f \quad \text{in} \quad \Omega, \qquad \frac{\partial y}{\partial \nu_A} + dy = 0 \text{ on } \partial\Omega, \tag{2.31}$$

where $d \in \mathbb{R}$ is such that $d \geq 0$. The variational formulation of this problem is then

$$\begin{cases} \qquad \text{Find} \ \ y \in H^1(\Omega) \text{ such that} \\ \widetilde{a}(y,\varphi) = \displaystyle\int_\Omega f\varphi \, \mathrm{d}x + \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \quad \forall\, \varphi \in H^1(\Omega), \end{cases} \tag{2.32}$$

where for all $y, \varphi \in H^1(\Omega)$,

$$a(y,\varphi) = \int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \, \mathrm{d}x + \int_\Omega y\varphi \, \mathrm{d}x + d\int_{\partial\Omega} y\varphi \, \mathrm{d}s. \tag{2.33}$$

In view of our suppositions, the linear form

$$F(y) = \int_\Omega fy \, \mathrm{d}x + < g, \varphi >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$$

is bounded on $H^1(\Omega)$. Hence, $F \in \big(H^1(\Omega)\big)^*$. Then, having observed that the bilinear form $a(y,\varphi)$ given by (2.33) is continuous on $H^1(\Omega) \times H^1(\Omega)$ and coercive, since $d$ is positive, we can apply the Lax–Milgram Lemma 2.19 with $V = H^1(\Omega)$ to get a unique element $y \in H^1(\Omega)$ satisfying (2.32). Moreover, in this case we have the same estimate for $\|y\|_{H^1(\Omega)}$ as in (2.26).

To end this section, we consider the case where one has a Dirichlet condition on a part of the boundary $\partial\Omega$ and a homogeneous Robin one on the rest of the boundary. Let $\Omega$ be a bounded connected domain with a Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are two disjoint closed sets and $\Gamma_1$ is of positive measure. Let $V$ be the closure of $C_0^\infty(\mathbb{R}^n \setminus \Gamma_1)$ with respect to norm of $H^1(\Omega)$, that is,

$$V = \left\{y \mid y \in H^1(\Omega), y|_{\Gamma_1} = 0\right\}.$$

We say that $y \in V$ is a weak solution to the mixed Dirichlet–Robin problem

$$\begin{cases} -\operatorname{div}(\mathbf{A}\nabla y) = f \quad \text{in} \quad \Omega, \\ y = 0 \quad \text{on} \quad \Gamma_1, \quad \dfrac{\partial y}{\partial \nu_A} + dy = 0 \text{ on } \ \Gamma_2, \end{cases} \tag{2.34}$$

where $d \geq 0$ if

$$\int_\Omega \nabla\varphi \cdot \mathbf{A}(x)\nabla y \, \mathrm{d}x + d\int_{\Gamma_2} y\varphi \, \mathrm{d}s = \int_\Omega f\varphi \, \mathrm{d}x \quad \forall\, \varphi \in V.$$

The existence of a unique solution to this problem immediately follows from the Lax–Milgram lemma and the inequality

$$\int_{\Omega} \varphi^2(x)\,\mathrm{d}x \le C_0 \int_{\Omega} |\nabla\varphi|^2\,\mathrm{d}x, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^n \setminus \Gamma_1),$$

which, in analogy to (2.8), will also be called the Friedrichs inequality.

## 2.3 Spaces of periodic functions

In this section, we provide a notion of periodicity for functions in the Sobolev space $H^1$. Let us consider measurable functions defined on $\mathbb{R}^n$ and periodic in each argument $x_1, x_2, \ldots, x_n$ with periods $l_1, l_2, \ldots, l_n$, respectively. Let $\square$ be the parallelepiped in $\mathbb{R}^n$ defined by $\square = (0, l_1) \times \cdots \times (0, l_n)$. We will refer to $\square$ as the reference period. Then the function $f$ is called $\square$-periodic iff

$$f(x + kl_i\mathbf{e}_i) = f(x) \quad \text{a.e. on } \ \mathbb{R}^n, \ \forall\,k \in \mathbb{Z}, \ \forall\,i \in \{1, 2, \ldots, n\},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

The mean value of a periodic functions is essential when studying periodic oscillating functions. Let us recall that for any $\square$-periodic function $f$ the mean value of $f$ is the real number $\mathcal{M}_\square(f)$ given by

$$\mathcal{M}_\square(f) = \frac{1}{|\square|} \int_\square f(y)\,\mathrm{d}y,$$

where $|\square| = l_1 l_2 \cdots l_n$ is the volume of the parallelepiped $\square$. The Lebesgue space of periodic measurable functions with a finite norm $\mathcal{M}_\square^{1/\alpha}(|f|^\alpha)$ for $\alpha \ge 1$ we denote by $L^\alpha(\square)$. The following property of periodic functions is frequently used in the asymptotic analysis.

**A property of the mean value**. Let $f$ be a $\square$-periodic function, $f \in L^\alpha(\square)$, $\alpha \ge 1$. Then

$$\int_{\Omega} f(\varepsilon^{-1}x)\varphi(x)\,\mathrm{d}x \longrightarrow \mathcal{M}_\square(f) \int_{\Omega} \varphi(x)\,\mathrm{d}x \quad \text{as} \ \ \varepsilon \to 0 \tag{2.35}$$

for every $\varphi \in L^{\alpha'}(\Omega)$, where $1/\alpha + 1/\alpha' = 1$, $\alpha \in (0, \infty)$, and $\Omega$ is an arbitrary bounded domain in $\mathbb{R}^n$.

Let $C^\infty_{per}(\square)$ be the subset of $C^\infty(\mathbb{R}^n)$ of $\square$-periodic functions. We denote by $H^1_{per}(\square)$ the completion of the space $C^\infty_{per}(\square)$ with respect to the $H^1$-norm. It should be stressed that $H^1_{per}(\square)$ does not coincide with the entire Sobolev space $H^1(\Omega)$ for $\Omega = \square$. Functions in $H^1_{per}(\square)$ as well as all other periodic functions, are assumed to be defined on $\mathbb{R}^n$ – namely if $u \in H^1_{per}(\square)$, then $u$ is in $H^1(D)$ for any bounded open subset $D$ of $\mathbb{R}^n$ and $u$ satisfies the following condition:

$$u(x + kl_i\mathbf{e}_i) = u(x), \quad \forall\, k \in \mathbb{Z},\ \forall\, i \in \{1, 2, \dots, n\}.$$

Thus, we may write $H^1_{per}(\square) = \left\{u \in H^1_{loc}(\mathbb{R}^n),\ u \text{ is } \square\text{-periodic}\right\}$. The space $H^1_{per}(\square)$ possesses the following obvious property: If $u \in H^1_{per}(\square)$, then $u$ has the same trace on the opposite site faces of $\square$.

In the sequel, we will make use of the quotient space $\mathcal{W}_{per}(\square) = H^1_{per}(\square)/\mathbb{R}$ defined as the space of equivalent classes with respect to the relation

$$u \cong v \Leftrightarrow u - v \text{ is a constant, } \forall\, u, v \in H^1_{per}(\square).$$

As usual, we denote by $\dot{u}$ the equivalence class represented by $u$. Then the quantity

$$\|\dot{u}\|_{\mathcal{W}_{per}(\square)} = \|\nabla u\|_{L^2(\square)}, \quad \forall\, u \in \dot{u},\ \dot{u} \in \mathcal{W}_{per}(\square)$$

defines a norm on $\mathcal{W}_{per}(\square)$ for which $\mathcal{W}_{per}(\square)$ is a Banach space. Moreover, the dual space $\mathcal{W}^*_{per}(\square)$ can be identified with the set

$$\left\{ F \in \mathcal{W}^*_{per}(\square) \,\middle|\, F(c) = 0,\ \forall\, c \in \mathbb{R} \right\},$$

with $\langle F, \dot{u}\rangle_{\mathcal{W}^*_{per}(\square), \mathcal{W}_{per}(\square)} = \langle F, u\rangle_{(H^1_{per}(\square))^*, H^1_{per}(\square)}$ for all $u \in \dot{u}$ and $\forall\, \dot{u} \in \mathcal{W}_{per}(\square)$.

We define also the space of *solenoidal periodic vector fields*, by setting

$$\mathbf{L}^2_{sol}(\square) := \left\{\mathbf{p} \in \mathbf{L}^2(\square),\ \operatorname{div} \mathbf{p} = 0 \text{ in } \mathbb{R}^n\right\}. \tag{2.36}$$

Clearly, $\mathbf{L}^2_{sol}(\square)$ is a closed subspace of $\mathbf{L}^2(\square)$, and, by definition, $\mathbf{p} \in \mathbf{L}^2_{sol}(\square)$ if $\mathbf{p} \in \mathbf{L}^2(\square)$ and the identity

$$\int_\square \mathbf{p} \cdot \nabla\varphi \,\mathrm{d}x = 0, \quad \forall\, \varphi \in C^\infty_{per}(\square)$$

is valid (see [261]). Moreover, in this case, the following orthogonal representation holds:

$$\mathbf{L}^2(\square) = \mathbf{L}^2_{sol}(\square) \oplus \boldsymbol{\mathcal{V}}^2_{pot}(\square), \quad \boldsymbol{\mathcal{V}}^2_{pot}(\square) = \left\{\nabla u,\ u \in H^1_{per}(\square)\right\}.$$

It should be pointed out that because of the Poincaré inequality, $\boldsymbol{\mathcal{V}}^2_{pot}(\square)$ is a closed subspace of $\mathbf{L}^2(\square)$. We also introduce the space of potential periodic vector fields

$$\mathbf{L}^2_{pot}(\square) = \mathbb{R}^n \oplus \boldsymbol{\mathcal{V}}^2_{pot}(\square).$$

Then any vector field $\mathbf{v} \in \mathbf{L}^2_{pot}(\square)$ is potential by definition and can be represented in the form $\mathbf{v} = \mathcal{M}_\square(\mathbf{v}) + \nabla u$.

## 2.4 Weak and weak-∗ convergence in Banach spaces

In this section, we recall without proofs the basic facts from functional analysis concerning the weak and weak-∗ convergence in Banach spaces.

Let $F$ be a Banach space equipped with the norm $\|\cdot\|_F$ and let $F^*$ be its dual. We set $E = F^*$. The norm of $u$ in $E$ is defined by the formula

$$\|u\|_E = \sup_{\|f\|_F=1} \langle u, f\rangle_{E,F} \,,$$

where $\langle u, f\rangle_{E,F}$ denotes the value of $u \in E$ at $f \in F$.

**Definition 2.22.** *A sequence $\{u_k\}_{k=1}^{\infty} \subset E$ is said to be strongly convergent to $u \in E$ as $k \to \infty$ if $\|u_k - u\|_E \to 0$ as $k \to \infty$. In this case, we write*

$$u_k \to u \text{ strongly in } E \text{ as } k \to \infty.$$

**Definition 2.23.** *A sequence $\{u_k\}_{k=1}^{\infty} \subset E$ is said to converge weakly to $u$, written $u_k \rightharpoonup u$ in $E$, provided*

$$\langle u^*, u_k\rangle_{E^*,E} \to \langle u^*, u\rangle_{E^*,E} \,, \quad \forall\, u^* \in E^*.$$

For instance, let $\Omega$ be an open bounded domain in $\mathbb{R}^n$. We write $w_k \rightharpoonup w$ in $L^p(\Omega)$ if for any $\varphi \in L^{p'}(\Omega)$ $(1 \le p < \infty, p' = p/(p-1))$

$$\lim_{k\to\infty} \int_\Omega w_k \varphi \,\mathrm{d}x = \int_\Omega w\varphi \,\mathrm{d}x.$$

A weakly convergent sequence is necessarily bounded in the norm.

**Theorem 2.24.** *(Boundedness of weakly convergent sequences) Assume $u_k \rightharpoonup u$ in $E$. Then the following hold:*

*(i)* $\{u_k\}_{k=1}^{\infty}$ *is bounded in* $E$,
*(ii)* $\|u\|_E \le \liminf_{k\to\infty} \|u_k\|_E$,

*that is, the norm in $E$ is lower semicontinuous with respect to the weak convergence in $E$.*

Moreover, a refinement of (ii) holds: If $E$ is reflexive, $u_k \rightharpoonup u$ in $E$, and $\lim_{k\to\infty} \|u_k\|_E = \|u\|_E$, then

$$u_k \to u \text{ strongly in } E.$$

For the case of Lebesgue spaces, Brezis and Lieb [31] obtained the following: *Let $u_k \rightharpoonup u$ in $L^p(\Omega)$ $(1 \le p < \infty)$ and $u_k \to u$ almost everywhere in $\Omega$. Then*

$$\lim_{k\to\infty} \left( \|u_k\|^p_{L^p(\Omega)} - \|u_k - u\|^p_{L^p(\Omega)} \right) = \|u\|^p_{L^p(\Omega)}.$$

The following theorem states one of the main properties of the weak convergence in reflexive Banach spaces. For the proof, which is rather technical, we refer to Yosida [251] or Kantorovich and Akilov [128].

**Theorem 2.25.** *(Eberlein-Ŝmuljan) Assume that $E$ is a reflexive Banach space and let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in $E$. Then the following hold:*

*(i) There exist a subsequence* $\{u_{k_j}\}_{j=1}^{\infty}$ *of* $\{u_k\}_{k=1}^{\infty}$ *and an element* $u \in E$ *such that*

$$u_{k_j} \rightharpoonup u \quad \text{in } E \text{ as } k \to \infty.$$

*(ii) If each weakly convergent subsequence of* $\{u_k\}_{k=1}^{\infty}$ *has the same limit* $u$*, then the whole sequence* $\{u_k\}_{k=1}^{\infty}$ *weakly converges to* $u$*.*

This theorem is definitely false if we take $F = L^1(\Omega)$ and $E = L^\infty(\Omega)$ as a simple example illustrates. Indeed, let us consider the following sequence in $L^1(\Omega)$ ($\Omega = (-1,1)$):

$$x_k = \begin{cases} 0 & \text{if } t \in (-1,-1/n) \cup (1/n,1); \\ n^2 t + n & \text{if } t \in [-1/n, 0); \\ -n^2 t + n & \text{if } t \in [0, 1/n], \end{cases} \qquad k = 1,2,\ldots.$$

It is clear that $\sup_{k\in\mathbb{N}} \|x_k\|_{L^1(\Omega)} = 1$. However, this sequence is not compact with respect to the weak convergence in $L^1(\Omega)$. In order to prove it, we note that

$$\lim_{k\to\infty} \int_\Omega x_k(t)\varphi(t)\,\mathrm{d}t = \varphi(0), \quad \forall\,\varphi \in C_0^\infty(\Omega), \tag{2.37}$$

by the mean value theorem. Indeed, due to the continuity of $\varphi$, for any $\eta > 0$ there exists an $\varepsilon_0 > 0$ such that $|\varphi(t) - \varphi(0)| < \eta$, provided $|t| < \varepsilon_0$. Then

$$\begin{aligned} \left|\int_\Omega x_k(t)\varphi(t)\,\mathrm{d}t - \varphi(0)\right| &= \left|\int_{-1/n}^{1/n} x_k(t)\varphi(t)\,\mathrm{d}t - \frac{n}{2}\int_{-1/n}^{1/n} \varphi(0)\,\mathrm{d}t\right| \\ &\le \frac{n}{2}\int_{-1/n}^{1/n} |\varphi(t) - \varphi(0)|\,\mathrm{d}t < \frac{n}{2}\eta \int_{-1/n}^{1/n} \mathrm{d}t = \eta \end{aligned}$$

for all $\varepsilon < \varepsilon_0$.

Let us suppose that there is a function $x \in L^1(\Omega)$ such that

$$\int_\Omega x(t)\varphi(t)\,\mathrm{d}t = \varphi(0), \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{2.38}$$

Since $t\varphi(t) \in C_0^\infty(\Omega)$, from (2.37) it follows that

$$\int_\Omega x(t)t\varphi(t)\,\mathrm{d}t = t\varphi(t)|_{t=0} = 0 \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

Hence, by Raymond's lemma (see, for instance, [49]), we conclude that $tx(t) = 0$ a.e. in $\Omega$. So, $x(t) = 0$ for almost all $t \in \Omega$, and we come into conflict with (2.38). This means that the above sequence does not converge weakly in $L^1(\Omega)$.

We will also need a more general form of the weak convergence, which is usually called the weak-$*$ convergence.

**Definition 2.26.** *A sequence $\{u_k\}_{k=1}^\infty \subset E$ is said to converge weakly-$*$ to $u$ (written $u_k \overset{*}{\rightharpoonup} u$ in $E = F^*$), provided*

$$\langle u_k, u^* \rangle_{F^*,F} \to \langle u, u^* \rangle_{F^*,F}, \quad \forall\, u^* \in F.$$

In particular, if $u_k, u \in L^1(\Omega)$ and the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $L^1(\Omega)$, then $u_k \overset{*}{\rightharpoonup} u$, provided the relation

$$\lim_{k\to\infty} \int_\Omega u_k \varphi \,\mathrm{d}x = \int_\Omega u\varphi \,\mathrm{d}x$$

holds for any $\varphi \in C_0^\infty(\Omega)$.

The main properties of weak convergence are still valid for weak-$*$ convergence. In particular, any weakly-$*$ convergent sequence in $E$ is bounded and the norm in $E$ is lower semicontinuous with respect to the weak-$*$ convergence in $E$. It is very important for what follows that a weak-$*$ limit is uniquely defined. It is also clear that the weak convergence in $E$ (in $L^1(\Omega)$, in particular) implies the weak-$*$ convergence. However, in general, the converse statement does not hold. Indeed, let

$$\Omega = (0,1), \quad u_k = \frac{1}{\sqrt{\pi k}} \exp(-t^2 k^{-1}).$$

Then $u_k \overset{*}{\rightharpoonup} 0$ in $L^1(\Omega)$. However,

$$\int_\Omega u_k \,\mathrm{d}t = \langle 1, u_k \rangle_{L^\infty(\Omega), L^1(\Omega)} \longrightarrow \frac{1}{2} \quad \text{as} \quad k \to \infty.$$

The equivalent of Theorem 2.25 for weak-$*$ convergence read as follows.

**Theorem 2.27.** *Let $F$ be a separable Banach space and let $E = F^*$. If $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $E$, then following hold:*

*(i) There exist a subsequence $\{u_{k_j}\}_{j=1}^\infty$ of $\{u_k\}_{k=1}^\infty$ and an element $u \in E$ such that*

$$u_{k_j} \overset{*}{\rightharpoonup} u \quad \text{in } E \text{ as } k \to \infty.$$

*(ii) If each weakly-$*$ convergent subsequence of $\{u_k\}_{k=1}^\infty$ has the same limit $u$, then the whole sequence $\{u_k\}_{k=1}^\infty$ weakly-$*$ converges to $u$.*

One often has to find the limit of the product $\langle f_k, u_k \rangle_{E^*,E}$ as $k \to \infty$. In the trivial case, when either $u_k \rightharpoonup u$ in $E$ and $f_k \to f$ in $E^*$ or $u_k \to u$ in $E$ and $f_k \overset{*}{\rightharpoonup} f$ in $E^*$, we have $\langle f_k, u_k \rangle_{E^*,E} \to \langle f, u \rangle_{E^*,E}$. However, in general, one cannot pass to the limit in the product $\langle f_k, u_k \rangle_{E^*,E}$ as $k \to \infty$ when both of these sequences are only known to be weakly convergent.

The following classical example is very significant. Let $v(t)$ be the periodic function of period 1, defined on $\mathbb{R}$ by $v(t) = \sin(2\pi t)$, and set $\square = [0,1)$, $\Omega = (a,b)$, and

$$v_k(t) := v\,(kt) = \sin\left(2\pi kt\right), \quad t \in (a,b),$$

where $a, b \in \mathbb{R}$. Applying the mean value property (2.35), we have

$$v_k \rightharpoonup M_{\Box}(v) = \int_0^1 \sin(2\pi t)\,\mathrm{d}t = 0 \quad \text{in} \;\; L^2(a,b) \;\; \text{as} \;\; k \to \infty.$$

On the other hand, $\langle v_k, v_k \rangle_{L^2(\Omega),L^2(\Omega)}$ does not converge to 0. Indeed,

$$\begin{aligned}\langle v_k, v_k \rangle_{L^2(\Omega),L^2(\Omega)} &= \int_a^b \sin^2\left(2\pi kt\right)\,\mathrm{d}t = \frac{1}{2\pi k}\int_{2\pi ka}^{2\pi kb} \sin^2(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{2\pi k}\int_{2\pi ka}^{2\pi kb} \frac{1-\cos(2\tau)}{2}\,\mathrm{d}\tau \\ &= \frac{b-a}{2} + \frac{1}{8\pi k}\left[-\sin(4\pi kb) + \sin(4\pi ka)\right],\end{aligned}$$

so that, as $k \to \infty$,

$$\langle v_k, v_k \rangle_{L^2(\Omega),L^2(\Omega)} \longrightarrow \frac{b-a}{2} \neq 0.$$

To understand the ways in which a weakly convergent sequence of functions can fail to be strongly convergent, we consider a sequence $\{u_k\}_{k=1}^{\infty}$ in $L^p(\Omega)$ ($1 < p < \infty$), where $\Omega$ is a bounded smooth open subset of $\mathbb{R}^n$, such that $u_k \rightharpoonup u$ in $L^p(\Omega)$ as $k \to \infty$. Let us observe that even if we know the functions $\{u_k\}_{k=1}^{\infty}$ to be bounded in the supremum norm, so that $u_k$ converges weakly to $u$ in $L^q(\Omega)$ for all $1 \le q < \infty$, we still cannot deduce strong convergence in $L^q(\Omega)$ for any $1 \le q < \infty$. The difficulty is with the possibility of very rapid fluctuations in the functions $u_k$ (see the previous example). This is the problem of wild oscillations.

Second, observe that even if we know additionally that

$$u_k \to u \;\; \text{a.e. in} \;\; \Omega,$$

so that wild oscillations are excluded, we still cannot deduce strong convergence in $L^p(\Omega)$. The obstruction is that the mass of $|u_k - u|^p$ may somehow coalesce onto a set of zero Lebesgue measure. This is the problem of concentration. To characterize the concentration effects, let us suppose that $u_k \rightharpoonup u$ in $L^p(\Omega)$ and introduce the following measure:

$$\theta_k(E) = \int_E |u_k - u|^p\,\mathrm{d}t, \quad k = 1, 2, \ldots,$$

where $E$ is a Borel subset of $\Omega$. Thus, $\theta_k(E)$ controls how close the function $u_k$ is to $u$ in the $L^p$-norm restricted to the set $E$. Following DiPerna and Majda [95], we call the value $\theta(E) = \limsup_{k\to\infty} \theta_k(E)$ the reduced defect measure associated with the weak convergence $u_k \rightharpoonup u$ in $L^p(\Omega)$. The idea is that $\theta(E)$

encodes an information about the extent to which strong convergence fails. In particular,

$$u_k \to u \quad \text{in } \ L^p(E) \ \text{ if and only if } \ \theta(E) = 0 \text{ (see [105]).}$$

For instance, let $\Omega = (-1, 1)$, $u = 0$, and

$$u_k(t) = \begin{cases} k & \text{if } \ -k^{-1} \le t \le k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_k \rightharpoonup 0$ in $L^2(\Omega)$ and $\theta$ is concentrated on $E = \{0\}$. In this case $\theta(\Omega \setminus V) = 0$ for each open set $V \supset E$.

The following theorem, taken from Valadier's work [245], gives necessary and sufficient conditions under which weak convergence implies strong convergence.

**Theorem 2.28.** *Suppose that $u_k \rightharpoonup u$ in $L^1(\Omega)$. Then $u_k \to u$ strongly if and only if the following criterion is satisfied: $\forall \varepsilon > 0$, $\forall A \subset \Omega$ with $\operatorname{meas}(A) > 0$, $\exists N \in \mathbb{N}$, $\exists B \subset A$ with $\operatorname{meas}(B) > 0$, such that, $\forall k \ge N$, the inequality*

$$\frac{1}{\operatorname{meas}(B)} \int_B \left| u_k(t) - \frac{1}{\operatorname{meas}(B)} \int_B u_k(t)\,\mathrm{d}t \right| \mathrm{d}t < \varepsilon$$

*holds true.*

### 2.4.1 Weak convergence of measures

Let $\mathcal{M}(\Omega)$ be the set of Borel measures on $\Omega$. For all $\mu \in \mathcal{M}(\Omega)$, we define the functional

$$L_\mu(\varphi) = \int_\Omega \varphi\,\mathrm{d}\mu, \quad \forall\, \varphi \in L^1(\Omega, \mathrm{d}\mu). \tag{2.39}$$

It is clear that $L_\mu$ is linear and continuous on $C_0(\Omega)$. Following Riesz's theorem [28, 221], for any linear continuous functional $L_\mu : C_0(\Omega) \to \mathbb{R}$ there exists a unique measure $\mu \in \mathcal{M}(\Omega)$ such that the representation $L_\mu(\varphi) = \displaystyle\int_\Omega \varphi\,\mathrm{d}\mu$ holds true for every $\varphi \in C_0(\Omega)$. Thus, the map $\mu \mapsto L_\mu$ is a bijection between $M(\Omega)$ and $(C_0(\Omega))^*$. Moreover, in this case, we have $\|L_\mu\| = |\mu|(\Omega)$. Indeed,

$$\begin{aligned} \|L_\mu\| &= \sup\left\{ \int_\Omega \varphi\,\mathrm{d}\mu \ : \ \varphi \in C_0(\Omega), \ |\varphi| \le 1 \right\} \\ &\le \sup\left\{ \int_\Omega |\varphi|\,\mathrm{d}|\mu| \ : \ \varphi \in C_0(\Omega), \ |\varphi| \le 1 \right\} = \int_\Omega \mathrm{d}|\mu| = |\mu|(\Omega). \end{aligned}$$

Thus, measures can be identified as elements of the dual space of continuous functions vanishing on $\partial\Omega$. Hence, they inherit a notion of weak-∗ convergence which was defined earlier. In view of this, we can define a weak topology on $\mathcal{M}(\Omega)$ as follows.

**Definition 2.29.** *We say that a sequence* $\{\mu_k\}_{k=1}^{\infty} \subset \mathcal{M}(\Omega)$ *converges weakly to* $\mu$ *(and we write* $\mu_k \rightharpoonup \mu$*) if* $L_{\mu_k} \overset{*}{\rightharpoonup} L_\mu$ *in the weak-$*$ topology of* $C_0(\Omega)$*, that is,*

$$\lim_{k\to\infty} \int_\Omega \varphi \, \mathrm{d}\mu_k = \int_\Omega \varphi \, \mathrm{d}\mu, \quad \forall \varphi \in C_0(\Omega).$$

By the Banach–Steinhaus theorem, we have that

$$\text{if} \quad \mu_k \rightharpoonup \mu, \quad \text{then} \quad \sup_k |\mu_k|(\Omega) < +\infty.$$

Note, moreover, that by the lower semicontinuity of the dual norm with respect to weak-$*$ convergence, we have that $\mu \mapsto |\mu|(\Omega)$ is weakly lower semicontinuous (i.e., $|\mu|(\Omega) \le \liminf_{k\to\infty} |\mu_k|(\Omega)$, provided $\mu_k \rightharpoonup \mu$).

*Example 2.30.* Let $\square$ be the parallelepiped in $\mathbb{R}^n$ defined by

$$\square = (0, l_1) \times \cdots \times (0, l_n).$$

We will refer to $\square$ as the reference period. Let $F$ be a $\square$-periodic connected domain in $\mathbb{R}^n$. Let $\Omega$ be as usual a bounded smooth open subset of $\mathbb{R}^n$. So, we may always suppose that $\Omega$ is a measurable set in the sense of Jordan. Let us introduce the following measure on $\Omega$,

$$\mathrm{d}\mu = \rho(x)\mathrm{d}x, \quad \rho(x) = \begin{cases} |F \cap \Omega|^{-1} & \text{if } \ x \in \Omega, \\ 0 & \text{if } \ x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.40}$$

It is clear that $\mu \in \mathcal{M}(\Omega)$ as a periodic Borel positive measure with periodicity cell $\square$ and $\displaystyle\int_\square \mathrm{d}\mu = 1$. This measure is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}_n$. Let us consider the sequence $\{\mu_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\Omega)$, where

$$\mu_k(B) = k^{-n}\mu(kB), \quad \forall k \in \mathbb{N},$$

for every Borel set $B \subset \mathbb{R}^n$. Then $\mathrm{d}\mu_k = \rho(kx)\,\mathrm{d}x$ and

$$\int_{k^{-1}\square} \mathrm{d}\mu_k = k^{-n} \int_\square \mathrm{d}\mu = k^{-n}.$$

We wish to prove that $\mu_k \rightharpoonup \mathcal{L}_n$ as $k \to \infty$. Indeed, let $\square_{\mathbf{i}} = \square + \mathbf{i}$, where $\mathbf{i}$ is a vector in $\mathbb{R}^n$ with integer components. Then $k^{-1}\square_{\mathbf{i}}$ is a partition of $\mathbb{R}^n$ for every fixed $k \in \mathbb{N}$ and

$$\int_\Omega \varphi(x)\,\mathrm{d}\mu_k = \sum \int_{k^{-1}\square_{\mathbf{i}}} \varphi(x)\,\mathrm{d}\mu_k + \sum \int_{k^{-1}\square_{\mathbf{i}} \cap \Omega} \varphi(x)\,\mathrm{d}\mu_k \tag{2.41}$$

for every $\varphi \in C_0(\Omega)$, where the first sum is taken over all $\mathbf{i}$ such that $k^{-1}\square_{\mathbf{i}}$ is inside $\Omega$ and the second sum is over all $\mathbf{i}$ such that $k^{-1}\square_{\mathbf{i}}$ and $\partial\Omega$ have common points. Let us first consider the first sum in (2.41). Since $\varphi \in C_0(\Omega)$, there exist points $x_{\mathbf{i}} \in k^{-1}\square_{\mathbf{i}}$ such that

$$\int_{k^{-1}\square_{\mathbf{i}}} \varphi(x)\,\mathrm{d}\mu_k = \varphi(x_{\mathbf{i}}) \int_{k^{-1}\square_{\mathbf{i}}} \mathrm{d}\mu_k = \varphi(x_{\mathbf{i}}) k^{-n} \int_{\square} \mathrm{d}\mu = k^{-n}\varphi(x_{\mathbf{i}}).$$

This, together with the fact that $\sum k^{-n}\varphi(x_{\mathbf{i}})$ is the construction of a Riemann sum, implies that

$$\lim_{k\to\infty} \sum k^{-n}\varphi(x_{\mathbf{i}}) \to \int_{\Omega} \varphi(x)\,\mathrm{d}x. \tag{2.42}$$

Let us consider the second sum in (2.41). We have

$$\left|\sum \int_{k^{-1}\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_k\right| \le \max_{x\in\Omega} |\varphi(x)| k^{-n} M(k),$$

where $M(k)$ is the number of cubes $k^{-1}\square_{\mathbf{i}}$ containing the boundary of $\Omega$. Since $k^{-n}M(k) \to 0$ by the Jordan measurability property of $\Omega$, we conclude

$$\lim_{k\to\infty} \sum \int_{k^{-1}\square_{\mathbf{i}}\cap\Omega} \varphi(x)\,\mathrm{d}\mu_k = 0. \tag{2.43}$$

Thus, the desired property follows by taking (2.42)–(2.43) into account.

**Theorem 2.31.** *Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{M}(\Omega)$ with $\sup_k |\mu_k|(\Omega) < +\infty$. Then there exists a subsequence of $\{\mu_k\}_{k=1}^{\infty}$ weakly converging to some $\mu \in \mathcal{M}(\Omega)$.*

To characterize this result, which can be found in Federer [109] and Evans and Gariepy [106], let us consider a bounded sequence $\{u_k\}_{k=1}^{\infty}$ in $L^1(\Omega, d\mu)$, where $\mu$ is a positive Borel measure on $\Omega$. Then Theorem 2.31 applied with $\mu_k = u_k\mu$ yields the relative compactness of $\{\mu_k\}_{k=1}^{\infty}$ only in the weak-$\mathcal{M}(\Omega)$ topology. Consequently, in general, its cluster points need not be in $L^1(\Omega)$.

Let us list some general properties of the weak convergence in the space of Radon measures $\mathcal{M}_b(\Omega)$ which we apply below. We recall that a sequence $\{\mu_k\}_{k=1}^{\infty}$ of Radon measures on $\mathbb{R}^n$ is said to be bounded if

$$\sup_{k\in\mathbb{N}} \mu_k(K) < +\infty \quad \text{for each compact set} \quad K \subset \mathbb{R}^n.$$

**Lemma 2.32.** *Let $\{\mu_k\}_{k=1}^{\infty}$ and $\mu$ be Radon measures on $\Omega$ such that $\mu_k \rightharpoonup \mu$ in $\mathcal{M}_b(\Omega)$ as $k \to \infty$. Then (see Zhikov [258]) the following hold:*

1. *$\eta\mu_k \rightharpoonup \eta\mu$ in $\mathcal{M}_b(\Omega)$ for every positive function $\eta \in C(\overline{\Omega})$.*
2. *$\liminf_{k\to\infty} \mu_k(A) \ge \mu(A)$ for every open set $A \subset \Omega$.*
3. *$\limsup_{k\to\infty} \mu_k(K) \le \mu(K)$ for every compact set $K \subset \Omega$.*

*4. If* $\lim_{k\to\infty}\mu_k(\Omega)=\mu(\Omega)$*, then the weak convergence* $\mu_k \rightharpoonup \mu$ *implies the following convergence:*

$$\lim_{k\to\infty}\int_\Omega \varphi\,\mathrm{d}\mu_k=\int_\Omega \varphi\,\mathrm{d}\mu,\quad \forall\,\varphi\in C(\overline{\Omega}).$$

Since $\mathcal{M}_b(\Omega)$ is the dual space of $C_0(\Omega)$, it follows that every Radon measure on $\Omega$ can be identified with an element of the space of distributions $\mathcal{D}'(\Omega)$ in the usual way. Therefore, we say that a Radon measure $\mu$ belongs to $\mathcal{M}_b(\Omega)\cap W^{-1,q}(\Omega)$ if and only if there exist functions $f_0,f_1,\dots,f_n\in L^q(\Omega)$ such that

$$\int_\Omega \varphi\,\mathrm{d}\mu=\int_\Omega f_0\varphi\,\mathrm{d}x-\sum_{k=1}^n\int_\Omega f_k\frac{\partial\varphi}{\partial x_k}\,\mathrm{d}x,\quad \varphi\in C_0^\infty(\Omega).$$

In other words, we say that a Radon measure $\mu$ on $\Omega$ belongs to $W^{-1,q}(\Omega)$ if there exists $f\in W^{-1,q}(\Omega)$ such that

$$\langle f,\varphi\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}=\int_\Omega \varphi\,\mathrm{d}\mu,\quad \varphi\in C_0^\infty(\Omega).$$

In this case, we can identify $f\in W^{-1,q}(\Omega)$ and $\mu\in\mathcal{M}_b(\Omega)$. Note also that, by the Riesz theorem, every non-negative element of $W^{-1,q}(\Omega)$ is a Radon measure.

**Theorem 2.33 ([105]).** *Assume that a sequence of Radon measures* $\{\mu_k\}_{k=1}^\infty$ *is bounded in* $\mathcal{M}_b(\Omega)$*. Then* $\{\mu_k\}_{k=1}^\infty$ *is precompact in* $W^{-1,q}(\Omega)$ *for each* $1\le q<1^*=n/(n-1)$*.*

*Proof.* In view of Theorem 2.31, we may extract a subsequence $\{\mu_{k_j}\}_{j=1}^\infty\subset\{\mu_k\}_{k=1}^\infty$ so that $\mu_{k_j}\rightharpoonup\mu$ in $\mathcal{M}(\Omega)$ for some measure $\mu\in\mathcal{M}_b(\Omega)$. Let us set $q'=q/(q-1)$ and denote by $B$ the closed unit ball in $W_0^{1,q'}(\Omega)$. Since $1\le q<1^*$, we have $q'>n$. So, by the Sobolev embedding theorem, $B$ is a compact set in $C_0(\overline{\Omega})$. Hence, for a given $\delta>0$, there exist functions $\{\phi_i\}_{i=1}^{N(\delta)}\subset C_0(\overline{\Omega})$ such that

$$\sup_{1\le i\le N(\delta)}\|\varphi-\phi_i\|_{C(\overline{\Omega})}<\delta,\quad \forall\,\varphi\in B.$$

Thus, if $\varphi\in B$, then

$$\left|\int_\Omega\varphi\,\mathrm{d}\mu_{k_j}-\int_\Omega\varphi\,\mathrm{d}\mu\right|\le 2\delta\sup_j\left|\mu_{k_j}\right|(\Omega)+\left|\int_\Omega\phi_i\,\mathrm{d}\mu_{k_j}-\int_\Omega\phi_i\,\mathrm{d}\mu\right|$$

for some index $1\le i\le N(\delta)$. Consequently,

$$\lim_{j\to\infty}\sup_{\varphi\in B}\left|\int_\Omega\varphi\,\mathrm{d}\mu_{k_j}-\int_\Omega\varphi\,\mathrm{d}\mu\right|=0,$$

and so $\mu_{k_j}\to\mu$ in $W^{-1,q}(\Omega)$.

The notion of a Radon measure can be easily extended to the case when $\Omega$ is a locally compact Hausdorff space. For more results about weak convergence of measures and its consequences, we refer to Evans and Gariepy [106], Federer [109], and Rudin [221].

### 2.4.2 Weak convergence in $L^1(\Omega)$

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$. Let $u_k \in L^1(\Omega)$ ($k \in \mathbb{N}$) and $u \in L^1(\Omega)$ be given functions. The sequence $\{u_k\}_{k=1}^{\infty}$ is said to be weakly convergent to $u$ in $L^1(\Omega)$ if

$$\lim_{k\to\infty} \int_\Omega u_k \varphi \,\mathrm{d}x = \int_\Omega u\varphi \,\mathrm{d}x, \quad \forall\, \varphi \in L^\infty(\Omega).$$

Since $L^1(\Omega)$ cannot be characterized as the dual of some Banach space, the notion of weak-∗ convergence is not interesting in this space. However, as was mentioned earlier, for any bounded sequence $\{u_k\}_{k=1}^{\infty}$ in $L^1(\Omega)$ its cluster points need not be in $L^1(\Omega)$. Indeed, having applied Theorem 2.31 with $\mathrm{d}\mu_k = u_k \,\mathrm{d}x$, we obtain the relative compactness of $\{\mu_k\}_{k=1}^{\infty}$ only in the weak-$\mathcal{M}(\Omega)$ topology. At this point, one can ask under which conditions a bounded sequence in $L^1(\Omega)$ is weakly compact. To answer this question, we need the following definitions.

**Definition 2.34.** *A sequence $\{u_k\}_{k=1}^{\infty}$ in $L^1(\Omega)$ is said to be equi-integrable if, for any $\eta > 0$, there exists $\delta > 0$ such that*

$$\forall\, k \in \mathbb{N}, \quad \int_E |u_k(x)| \,\mathrm{d}x < \eta, \quad \text{for any } \; E \subset \Omega \; \text{ with } \; |E| < \delta,$$

*where $|E|$ stands for the Lebesgue measure of $E$.*

**Definition 2.35.** *A function $h(t)$ $(t \geq 0)$ is said to be coercive, if it is non-negative, non-decreasing, and satisfies the condition*

$$\lim_{t\to\infty} t^{-1} h(t) \,\mathrm{d}t = +\infty.$$

Then the answer to the above question is as in the following proposition.

**Proposition 2.36.** *(Dunford–Pettis) The following statements are equivalent:*

*(a) The sequence $\{u_k\}_{k=1}^{\infty}$ is weakly compact in $L^1(\Omega)$.*
*(b) The sequence $\{u_k\}_{k=1}^{\infty}$ is equi-integrable.*
*(c) There is a coercive function $h(t)$ such that* $\sup_{k\in\mathbb{N}} \int_\Omega h(|u_k|) \,\mathrm{d}x < +\infty$.
*(d) Given $\delta > 0$, there is $\lambda = \lambda(\delta)$ such that* $\sup_{k\in\mathbb{N}} \int_{\{|u_k|>\lambda\}} |u_k| \,\mathrm{d}x < \delta$.

**Theorem 2.37.** *(Generalized Lebesgue's Theorem) Let $\{u_k\}_{k=1}^{\infty}$ be an equi-integrable sequence in $L^1(\Omega)$ such that $u_k \to u$ almost everywhere on $\Omega$ as $k \to \infty$. Then $u \in L^1(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$.*

The proof of the above statements can be found in Dunford and Schwartz [99], Ekeland and Temam [104], and Natanson [198].

One of the useful implication of the condition (c) is that for any $u \in L^1(\Omega)$, there exists a coercive function $\varphi$ such that $\varphi(|u|) \in L^1(\Omega)$. Of course, this result is very simple and can be proved quite easily on the basis of the following observation: For any given numerical series $\sum_{k=1}^{\infty} a_k < +\infty$, where $a_k \geq 0$, there exists a sequence $\lambda_k \to \infty$ such that $\sum_{k=1}^{\infty} \lambda_k a_k < +\infty$.

## 2.5 Elements of capacity theory

In this section, we give the notion of capacity as a way to study certain "small" subsets of $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $1 < p < +\infty$. The $p$-capacity of a subset $E$ in $\Omega$ is

$$\mathrm{cap}_p\,(E, \Omega) = \inf\left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x \;:\; u \in \mathcal{U}_E \right\},$$

where $\mathcal{U}_E$ is the set of all functions of the Sobolev space $W_0^{1,p}(\Omega)$ such that $u \geq 1$ almost everywhere in a neighborhood of $E$.

We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as q.e.) in a set $E$ if it holds for all $x \in E$ except for a subset $N$ of $E$ with $\mathrm{cap}_p\,(N, E) = 0$. The expression *almost everywhere* refers, as usual, to the Lebesgue measure.

A subset $A$ of $\Omega$ is said to be $p$-quasi-open if for every $\varepsilon > 0$, there exists an open subset $A_\varepsilon$ of $\Omega$, such that $A \subseteq A_\epsilon$ and $\mathrm{cap}_p\,(A_\varepsilon \setminus A, \Omega) < \epsilon$. The class of all $p$-quasi-open subsets of $\Omega$ we denote by $\mathcal{A}(\Omega)$.

A function $f : \Omega \to \mathbb{R}$ is said to be $p$-quasi-continuous (resp., quasi-lower semicontinuous) if for every $\varepsilon > 0$, there exists a continuous (resp., lower semicontinuous) function $f_\varepsilon : \Omega \to \mathbb{R}$ such that $\mathrm{cap}_p\,(\{f \neq f_\epsilon\}, \Omega) < \varepsilon$, where $\{f \neq f_\epsilon\} = \{x \in \Omega \;:\; f(x) \neq f_\epsilon(x)\}$. It is well known that (see, e.g., Ziemer [267]) every function $u \in W_0^{1,p}(\Omega)$ has a $p$-quasi-continuous representative, which is uniquely defined up to a set of $p$-capacity 0. We will always identify the function $u$ with its quasi-continuous representative, so that a pointwise condition can be imposed on $u(x)$ for $p$-quasi-every $x \in \Omega$. Note that with this convention we can write

$$\mathrm{cap}_p\,(E, \Omega) = \inf\left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x \;:\; u \in W_0^{1,p}(\Omega), u \geq 1 \text{ q.e. on } E \right\} \tag{2.44}$$

for every subset $E$ of $\Omega$.

Since $p$ is fixed, the index $p$ may be dropped when speaking about $p$-quasi-open sets, $p$-quasi-continuity, and so forth. It is clear that a set of zero capacity has zero measure, but the converse is not true. Moreover, when $p > n$, the $p$-capacity of a point is strictly positive and every $W^{1,p}$-function has a continuous representative. Therefore, a property which holds $p$-quasi-everywhere, with $p > n$, holds in fact everywhere. We refer to [106, 120] for a review of the main properties of the $p$-capacity. We recall the following key results.

**Theorem 2.38.** *Assume $A, B \subset \Omega$. Then the following hold:*

1. $\mathrm{cap}_p\,(A, \Omega) = \inf\left\{\mathrm{cap}_p\,(U, \Omega)\ :\ U \text{ is open}, A \subset U \subseteq \Omega\right\}$.
2. $\mathrm{cap}_p\,(\lambda A, \mathbb{R}^n) = \lambda^{n-p}\mathrm{cap}_p\,(A, \mathbb{R}^n)$, $\lambda > 0$.
3. $\mathrm{cap}_p\,(B(x,r), \mathbb{R}^n) = r^{n-p}\mathrm{cap}_p\,(B(0,1), \mathbb{R}^n)$.
4. $\mathrm{cap}_p\,(A, \Omega) \le C\mathcal{H}^{n-p}(A)$ *for some constant $C$ depending only on $p$ and $n$.*
5. $\mathcal{L}_n(A) \le C\mathrm{cap}_p\,(A, \Omega)^{n/(n-p)}$ *for some constant $C$ depending only on $p$ and $n$.*
6. $\mathrm{cap}_p\,(A \cup B, \Omega) + \mathrm{cap}_p\,(A \cap B, \Omega) \le \mathrm{cap}_p\,(A, \Omega) + \mathrm{cap}_p\,(B, \Omega)$.

**Theorem 2.39.** *Let $u \in H^1(\mathbb{R}^n)$. Then for q.e. $x \in \mathbb{R}^n$,*

$$\lim_{\varepsilon \to 0} |B(x,\varepsilon)|^{-1} \int_{B(x,\varepsilon)} u(y)\,\mathrm{d}y = \widetilde{u}(x),$$

*where $\widetilde{u}$ is a quasi-continuous representative of $u$.*

**Theorem 2.40.** *Every strongly converging sequence in $H^1(\mathbb{R}^n)$ has a subsequence converging q.e. in $\mathbb{R}^n$.*

**Theorem 2.41.** *Let $A$ and $\Omega$ be two bounded open subsets of $\mathbb{R}^n$ such that $A \subset \Omega$ and consider an element $u$ of $H_0^1(\Omega)$. Then $u|_A \in H_0^1(A)$ if and only if $\widetilde{u} = 0$ quasi-everywhere on $\Omega \setminus A$, where $\widetilde{u} = 0$ is a quasi-continuous representative of $u$.*

In view of this, we note that the following two spaces:

$$H_o^1(A; \Omega) = \left\{\varphi \in H_0^1(\Omega)\ :\ \varphi = 0 \text{ a.e. in } \ \Omega \setminus A\right\} \text{ and}$$
$$H_0^1(A; \Omega) = \left\{\varphi \in H_0^1(\Omega)\ :\ \varphi = 0 \text{ q.e. in } \ \Omega \setminus A\right\}$$

are not equal, in general. Indeed, $\varphi \in H_0^1(A; \Omega)$ cannot be characterized by merely saying that this function and its derivatives are 0 almost everywhere in $\Omega \setminus A$. By definition, a function $\varphi \in H^1(\Omega)$ is said to be 0 quasi-everywhere in a subset $E$ of $\Omega$ if there exists a quasi-continuous representative of $\varphi$ which is 0 quasi-everywhere in $E$. This makes sense, since any two quasi-continuous representatives of an element $\varphi$ of $H^1(\Omega)$ are equal quasi-everywhere. So, we have $H_0^1(A; \Omega) \subset H_o^1(A; \Omega)$. However, as can be seen from the following example, in general the reverse inclusion is not true.

Let $B(0,r)$ be the open ball of radius $r > 0$ in $\mathbb{R}^n$. Let us set $\Omega = B(0,3)$ and $A = B(0,2) \setminus \partial B(0,1)$. It is well known that the circular crack $\partial B(0,1)$ in $A$ has nonzero capacity but zero Lebesgue measure $\mathcal{L}_n$. Since $\partial B(0,1)$ has zero measure, it follows that $H^1_o(A;\Omega)$ contains functions $\psi \in H^1_0(B(0,2))$ whose restriction to $B(0,2)$ are not 0 on the sphere $\partial B(0,1)$. Hence, those functions $\psi$ do not belong to $H^1_0(A,\Omega)$. So, $H^1_o(A;\Omega) \nsubseteq H^1_0(A;\Omega)$. However, if $A$ has a Lipschitz boundary, then the spaces $H^1_o(A;\Omega)$ and $H^1_0(A;\Omega)$ can be identified (see Delfour and Zolésio [91]). In view of this, it is natural to introduce the following terminology (see Rauch and Taylor [217]).

**Definition 2.42.** *A set $A$ is said to be stable with respect to $\Omega$ if $H^1_o(A;\Omega) = H^1_0(A;\Omega)$.*

Following Buttazzo and Dal Maso [45], let us denote by $\mathcal{M}^p_0(\Omega)$ the set of all non-negative Borel measures $\mu$ on $\Omega$ such that the following hold:

1. $\mu(B) = 0$ for every Borel set $B \subset \Omega$ with $\operatorname{cap}_p(B,\Omega) = 0$.
2. $\mu(B) = \inf\{\mu(U) \ : \ U \ \text{quasi-open}, \ B \subseteq U\}$ for every Borel set $B \subset \Omega$.

When $p = 2$, we use the notation $\mathcal{M}_0(\Omega)$ instead of $\mathcal{M}^2_0(\Omega)$.

As examples of measures in the class $\mathcal{M}^p_0(\Omega)$ we can quote the following:

(i) $\varphi\mathcal{L}_n \in \mathcal{M}^p_0(\Omega)$ for every $\varphi \in L^\infty(\Omega)$, where, as usual, $\mathcal{L}_n$ is the $n$-dimensional Lebesgue measure.

(ii) If $n-2 < \alpha \le n$, then the $\alpha$-dimensional Hausdorff measure $\mathcal{H}^\alpha$ belongs to $\mathcal{M}^p_0(\Omega)$. This is a consequence of the two following implications:

$$\mathcal{H}^{n-2}(B) < +\infty \Rightarrow \operatorname{cap}_p(B,\Omega) = 0,$$
$$\operatorname{cap}_p(B,\Omega) = 0 \Rightarrow \mathcal{H}^{n-2+\delta}(B) = 0, \quad \forall\delta > 0.$$

(iii) The measure

$$\infty_S(B) = \begin{cases} 0 & \text{if } \operatorname{cap}_p(B \cap S,\Omega) = 0, \\ +\infty & \text{otherwise} \end{cases} \tag{2.45}$$

belongs to $\mathcal{M}^p_0(\Omega)$ for every quasi-closed set S, and so does the measure $\mu_A = \infty_{\Omega\setminus A}$ for every open set $A \subset \Omega$, that is,

$$\mu_A(B) = \begin{cases} 0 & \text{if } \operatorname{cap}_p(B \setminus A,\Omega) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{B}^*(\Omega)$ be the $\sigma$-field generated by the Borel subsets of $\Omega$. It is well known that a subset $E$ of $\Omega$ belongs to $\mathcal{B}^*(\Omega)$ if and only if there exists $B \in \mathcal{B}(\Omega)$ with $\operatorname{cap}_p(E\triangle B) = 0$, where $\triangle$ denotes the symmetric difference of sets. Therefore, each measure $\mu \in \mathcal{M}^p_0(\Omega)$ can be extended in a unique way to a countably additive set function, still denoted by $\mu$, defined on the larger $\sigma$-field $\mathcal{B}^*(\Omega)$.

We say that $A(\mu)$ is a regular set for the measure $\mu \in \mathcal{M}^p_0(\Omega)$ if $A(\mu)$ is defined as the union of all open subsets $A$ of $\Omega$ such that $\mu(A) < +\infty$. The singular set $S(\mu)$ is defined as the complement of $A(\mu)$ in $\Omega$. It is easy to see that $A(\mu)$ is also open, and if $A$ is a open subset of $\Omega$ which intersects $S(\mu)$, then $\mu(A) = +\infty$.

## 2.6 On the space $W_0^{1,p}(\Omega) \cap L^p(\Omega, \mathrm{d}\mu)$ and its properties

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$. Let us fix $\mu \in \mathcal{M}_0^p(\Omega)$ and denote by $\mathbf{X}_\mu^p(\Omega)$ the vector space of all functions $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega |u|^p \,\mathrm{d}\mu < +\infty$. Note that this definition makes sense because $\mu$ vanishes on all sets of capacity 0 and every function $u \in W_0^{1,p}(\Omega)$ is defined up to a set of capacity 0. So, the integral $\int_\Omega |u|^p \,\mathrm{d}\mu$ is unambiguously defined. On $\mathbf{X}_\mu^p(\Omega)$ we consider the norm

$$\|u\|_{\mathbf{X}_\mu^p(\Omega)} = \left[\int_\Omega |Du|^p \,\mathrm{d}x + \int_\Omega |u|^p \,\mathrm{d}\mu\right]^{1/p}.$$

**Theorem 2.43.** $\mathbf{X}_\mu^p(\Omega)$ *is a Banach space.*

*Proof.* Let $\{u_i\}_{i=1}^\infty$ be a Cauchy sequence in $\mathbf{X}_\mu^p(\Omega)$. Then $\{u_i\}_{i=1}^\infty$ is a Cauchy sequence both in $W_0^{1,p}(\Omega)$ and in $L^p(\Omega, \mathrm{d}\mu)$. Therefore, $\{u_i\}_{i=1}^\infty$ converges to a function $u$ in $W_0^{1,p}(\Omega)$ and to a function $v$ in $L^p(\Omega, \mathrm{d}\mu)$. Taking into account the fact that every convergent sequence in $W_0^{1,p}(\Omega)$ is relatively compact with respect to the pointwise convergence q.e. on $\Omega$, we can extract a subsequence $\{u_{i_k}\}_{k=1}^\infty$ converging to $u$ q.e. in $\Omega$. Since $\mu$ vanishes on all sets with capacity 0, $\{u_{i_k}\}_{k=1}^\infty$ converges to $u$ $\mu$-a.e. in $\Omega$. On the other hand, a further subsequence of $\{u_{i_k}\}_{k=1}^\infty$ converges to $v$ $\mu$-a.e. in $\Omega$. Hence, $u = v$ $\mu$-a.e. in $\Omega$ and, therefore, $u \in \mathbf{X}_\mu^p(\Omega)$ and $\{u_i\}_{i=1}^\infty$ converges to $u$ both in $W_0^{1,p}(\Omega)$ and in $L^p(\Omega, \mathrm{d}\mu)$. This implies that $\{u_i\}_{i=1}^\infty$ converges to $u$ in $\mathbf{X}_\mu^p(\Omega)$. Thus, the normed space $\mathbf{X}_\mu^p(\Omega)$ is complete.

It is clear now that in the case when $p = 2$, $\mathbf{X}_\mu^2(\Omega) = H_0^1(\Omega) \cap L^2(\Omega, \mathrm{d}\mu)$ is a Hilbert space with respect to the scalar product,

$$(u, v)_{\mathbf{X}_\mu^2(\Omega)} = \int_\Omega DuDv \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}\mu. \tag{2.46}$$

Let us consider now some examples which illustrate the structure of the space $\mathbf{X}_\mu^p(\Omega)$ under some special assumptions on the measure $\mu$.

*Example 2.44.* Assume that $\mu = g\mathcal{L}_n$ with $g \in L^q(\Omega)$, where

$$\begin{cases} q \in [1, +\infty) & \text{if } \ p \geq n, \\ q \in [1, p^* = pn/(n-p)] & \text{if } \ p < n. \end{cases}$$

By the Sobolev embedding theorem, we have that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Hence, $\mathbf{X}_\mu^p(\Omega) = W_0^{1,p}(\Omega)$ with equivalent norm.

*Example 2.45.* Let $A$ be an open subset of $\Omega$ and let $S = \Omega \setminus A$. If $\mu$ is equal to the measure $\mu = \infty_S + g\mathcal{L}_n$, where $g \in L^q(\Omega)$, $q$ satisfies the conditions of the previous example, and $\infty_S$ is defined by (2.45), then $\mathbf{X}^p_\mu(\Omega) = W^{1,p}_0(A)$, and the norms in $\mathbf{X}^p_\mu(\Omega)$ and $W^{1,p}_0(\Omega)$ are equivalent by the Poincaré inequality.

Consider now a measure $\mu \in \mathcal{M}^p_0(\Omega)$. By $\big(\mathbf{X}^p_\mu(\Omega)\big)^*$ we denote the dual space of $\mathbf{X}^p_\mu(\Omega)$, with duality pairing $\langle \cdot, \cdot \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)}$. Note that even in the case when $p = 2$, the space $\mathbf{X}^2_\mu(\Omega)$ is not dense in $L^2(\Omega)$. Therefore, we do not identify the isomorphic spaces $\mathbf{X}^2_\mu(\Omega)$ and $\big(\mathbf{X}^2_\mu(\Omega)\big)^*$. Let us show that the spaces $L^{p'}(\Omega)$, $W^{-1,p'}(\Omega)$, and $L^{p'}(\Omega, \mathrm{d}\mu)$ can be viewed as linear subspaces of $(\mathbf{X}_\mu(\Omega))^*$ (here $p' = p/(p-1)$).

Let $i : \mathbf{X}^p_\mu(\Omega) \to W^{1,p}_0(\Omega)$ be the natural embedding defined by $i(u) = u$ for every $u \in \mathbf{X}^p_\mu(\Omega)$. The transpose map ${}^t i : W^{-1,p'}(\Omega) \to \big(\mathbf{X}^p_\mu(\Omega)\big)^*$ allows us to consider $W^{-1,p'}(\Omega)$ as a subspace of $(\mathbf{X}_\mu(\Omega))^*$. With a little abuse of notation, which is discussed in a moment, we write $f$ instead ${}^t i(f)$ for every $f \in W^{-1,p'}(\Omega)$. With this convention we have

$$\langle f, v \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)} = \langle f, v \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)}\,, \quad \forall\, v \in \mathbf{X}^p_\mu(\Omega). \tag{2.47}$$

In particular, for $f \in L^{p'}(\Omega)$, we have

$$\langle f, v \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)} = \int_\Omega f v \,\mathrm{d}x, \quad \forall\, v \in \mathbf{X}^p_\mu(\Omega).$$

The abuse in our notation consists in the fact that the map ${}^t i : W^{-1,p'}(\Omega) \to \big(\mathbf{X}^p_\mu(\Omega)\big)^*$ is, in general, not injective, because $\mathbf{X}^p_\mu(\Omega)$ is, in general, not dense in $W^{1,p}_0(\Omega)$. Therefore, there may exist two elements $f$ and $g$ of $W^{-1,p'}(\Omega)$ such that $f \neq g$ in $W^{-1,p'}(\Omega)$ but $f = g$ in $\big(\mathbf{X}^p_\mu(\Omega)\big)^*$, where the last equality means ${}^t i(f) = {}^t i(g)$, according to our convention (2.47).

*Example 2.46.* Assume that $\mu$ is the measure $\infty_\Omega$ defined in (2.45) taking $S = \Omega$. Then $\mathbf{X}^p_\mu(\Omega) = \{0\}$; hence, ${}^t i(f) = 0$ for every $f \in W^{-1,p'}(\Omega)$. Therefore, in view of (2.47), we have $f = 0$ in $\big(\mathbf{X}^p_\mu(\Omega)\big)^*$ for every $f \in W^{-1,p'}(\Omega)$.

Let $j : \mathbf{X}^p_\mu(\Omega) \to L^p(\Omega, \mathrm{d}\mu)$ be the natural embedding defined by $j(u) = u$ for every $u \in \mathbf{X}^p_\mu(\Omega)$. Then the transpose map ${}^t j : L^{p'}(\Omega, \mathrm{d}\mu) \to \big(\mathbf{X}^p_\mu(\Omega)\big)^*$ allows us to consider $L^{p'}(\Omega, \mathrm{d}\mu)$ as a subspace of $\big(\mathbf{X}^p_\mu(\Omega)\big)^*$. For every $g \in L^{p'}(\Omega, \mathrm{d}\mu)$, the image ${}^t j(g)$ is denoted by $g\mu$. With this convention we have

$$\langle g\mu, v \rangle_{(\mathbf{X}^p_\mu(\Omega))^*, \mathbf{X}^p_\mu(\Omega)} = \int_\Omega v g \,\mathrm{d}\mu, \quad \forall\, v \in \mathbf{X}^p_\mu(\Omega). \tag{2.48}$$

Since $\mathbf{X}^p_\mu(\Omega)$ is, in general, not dense in $L^p(\Omega, \mathrm{d}\mu)$, the map ${}^t j : L^{p'}(\Omega, \mathrm{d}\mu) \to \big(\mathbf{X}^p_\mu(\Omega)\big)^*$ is, in general, not injective. Therefore, there may exist two elements $f$ and $g$ of $L^{p'}(\Omega, \mathrm{d}\mu)$ such that $f \neq g$ in $L^{p'}(\Omega, \mathrm{d}\mu)$, that is,

$$\mu(\{x \in \Omega \;:\; f(x) \neq g(x)\}) > 0,$$

but $f\mu = g\mu$ in $\mathbf{X}^p_\mu(\Omega)$.

*Example 2.47.* Let $E$ be the set of all points $x = (x_1, x_2, \ldots, x_n)$ in $\Omega$ whose first coordinate $x_1$ is rational. Let $\mu = \infty_E + \mathcal{L}_n$. Then it is clear that $\mathbf{X}^p_\mu(\Omega) = \{0\}$. Therefore, taking $g = \chi_{\Omega \setminus E}$, where $\chi_{\Omega \setminus E}$ is the characteristic function of the set $\Omega \setminus E$, we have $g \in L^{p'}(\Omega, \mathrm{d}\mu)$ and $g \neq 0$ in $L^{p'}(\Omega, \mathrm{d}\mu)$, whereas $g\mu = {}^t j(g) = 0$ in $\left(\mathbf{X}^p_\mu(\Omega)\right)^*$.

Let us fix a measure $\mu \in \mathcal{M}^2_0(\Omega)$. Then by the Riesz–Fréchet representation theorem, for every $F \in \left(\mathbf{X}^2_\mu(\Omega)\right)^*$, there exists a unique $u \in \mathbf{X}^2_\mu(\Omega)$ such that

$$\langle F, v \rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)} = (u, v)_{\mathbf{X}^2_\mu(\Omega)}, \quad \forall\, v \in \mathbf{X}^2_\mu(\Omega). \tag{2.49}$$

By definition (2.46) of the scalar product in $\mathbf{X}^2_\mu(\Omega)$, (2.49) is equivalent to

$$\int_\Omega DuDv\,\mathrm{d}x + \int_\Omega uv\,\mathrm{d}\mu = \langle F, v \rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)}, \quad \forall\, v \in \mathbf{X}^2_\mu(\Omega).$$

However, according to our conventions (2.47) and (2.48), the last equality can be written in the form

$$\begin{aligned} \langle -\triangle u, v \rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)} &+ \langle u\mu, v \rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)} \\ &= \langle F, v \rangle_{\left(\mathbf{X}^2_\mu(\Omega)\right)^*, \mathbf{X}^2_\mu(\Omega)}, \quad \forall\, v \in \mathbf{X}^2_\mu(\Omega). \end{aligned} \tag{2.50}$$

This shows that each element $F$ of $\left(\mathbf{X}^2_\mu(\Omega)\right)^*$ can be represented as $F = f + g\mu$ with $f \in H^{-1}(\Omega)$ and $g \in L^2(\Omega, \mathrm{d}\mu)$. On the other hand, because of (2.50), we refer to the solution of (2.49) as the solution of the problem

$$u \in \mathbf{X}^2_\mu(\Omega), \quad -\triangle u + u\mu = F \quad \text{in} \quad \left(\mathbf{X}^2_\mu(\Omega)\right)^*.$$

## 2.7 Sobolev spaces with respect to a measure

Let $\Omega$ be an open domain in $\mathbb{R}^n$ and let $\mu$ be a finite positive (e.g., probability) measure of $\mathcal{M}^p_0(\Omega)$. We introduce the Sobolev space $W^{1,p}(\Omega, \mathrm{d}\mu)$ as follows.

**Definition 2.48.** *We say that a function $u$ belongs to $W^{1,p}(\Omega, \mathrm{d}\mu)$ if there exist a sequence $\left\{u_k \in C^\infty(\overline{\Omega})\right\}_{k=1}^\infty$ and a vector-function $\mathbf{z} \in \mathbf{L}^p(\Omega, \mathrm{d}\mu) := [L^p(\Omega, \mathrm{d}\mu)]^n$ such that*

$$u_k \to u \quad \text{in} \quad L^p(\Omega, \mathrm{d}\mu) \quad \text{and} \quad \nabla u_k \to \mathbf{z} \quad \text{in} \quad \mathbf{L}^p(\Omega, \mathrm{d}\mu). \tag{2.51}$$

*In this case we say that $\mathbf{z}$ is a gradient or $\mu$-gradient of $u$ and denote it by $\nabla^\mu u$.*

In other words, to define the space $W^{1,p}(\Omega, \mathrm{d}\mu)$, we construct the space $W = W(\Omega, \mathrm{d}\mu)$ as the closure in $L^p(\Omega, \mathrm{d}\mu) \times \mathbf{L}^p(\Omega, \mathrm{d}\mu)$ of the set of pairs $\{(u, \nabla u) \ : \ \forall\, u \in C^\infty(\overline{\Omega})\}$. Thus, the elements of $W$ are pairs $(u, \mathbf{z})$, where the vector $\mathbf{z}$ is denoted by $\nabla^\mu u$, and said to be a gradient of $u$. As a result, the collection of the first components $u$ is called the Sobolev space $W^{1,p}(\Omega, \mathrm{d}\mu)$. Note that if we put in (2.51) functions $u \in C_0^\infty(\Omega)$, we just obtain the definition of the Sobolev space $W_0^{1,p}(\Omega, \mathrm{d}\mu)$. Note also that we do not introduce a norm in these spaces. If $p = 2$, we usually write $H^1(\Omega, \mathrm{d}\mu) = W^{1,2}(\Omega, \mathrm{d}\mu)$.

*Remark 2.49.* In the above definition, the strong convergence in $L^p(\Omega, \mathrm{d}\mu)$ and $\mathbf{L}^p(\Omega, \mathrm{d}\mu)$ can be replaced by the weak convergence in the same spaces.

In general, the gradient of a $W^{1,p}(\Omega, \mathrm{d}\mu)$ function is not unique since a function $u$ in $W^{1,p}(\Omega, \mathrm{d}\mu)$ can have many gradients. Let us denote by $\Gamma^\mu(u)$ the set of all gradients of a fixed function $u \in W^{1,p}(\Omega, \mathrm{d}\mu)$. It is clear that $\Gamma^\mu(u)$ has the structure $\Gamma^\mu(u) = \nabla^\mu u + \Gamma^\mu(0)$, where $\nabla^\mu u$ is some gradient and $\Gamma^\mu(0)$ is the set of gradients of 0. By definition, $\mathbf{z} \in \Gamma^\mu(0)$ if there exist $u_k \in C^\infty(\overline{\Omega})$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to\infty} \int_\Omega |u_k|^p \, \mathrm{d}\mu = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega |\nabla u_k - \mathbf{z}|^p_{\mathbb{R}^n} \, \mathrm{d}\mu = 0.$$

Obviously, $\Gamma^\mu(0)$ is a closed subspace of the vector space $\mathbf{L}^p(\Omega, \mathrm{d}\mu)$. So, the gradient of an arbitrary $W^{1,p}(\Omega, \mathrm{d}\mu)$ function can be viewed as the corresponding equivalence class. As an illustration of nonuniqueness of the gradient, we consider the following example.

*Example 2.50.* **The case of a singular measure $\mu$ concentrated on the segment**. Let $I = \{x \mid a \le x_1 \le b; x_2 = 0\}$ be a segment in $\mathbb{R}^2$ and suppose that a bounded domain $\Omega \subset \mathbb{R}^2$ contains $I$. Let $\mu$ be a probability measure concentrated on this segment, uniformly distributed on it, and coinciding with 1D Lebesgue measure on $I$ – namely we set

$$\mathrm{d}\mu = \frac{1}{b-a} \chi(x_1) \, \mathrm{d}x_1 \times \delta(x_2), \tag{2.52}$$

where $\chi(t)$ is the characteristic function of the segment $[a, b]$ and $\delta(t)$ is the Dirac mass concentrated at 0. It is clear that $\mu$ is a singular measure with respect to $\mathcal{L}_2$, and, by definition of the class $\mathcal{M}_0(\Omega)$, we have $\mu \in \mathcal{M}_0(\Omega)$. Note also that $\mu(\Omega \setminus I) = 0$. Therefore, any functions taking the same values on the segment $I$ coincide as elements of $L^2(\Omega, \mathrm{d}\mu)$.

By Definition 2.48, a function $u$ is an element of $H^1(\Omega, \mathrm{d}\mu)$ if there are a sequence of smooth functions $u_k \in C^\infty(\overline{\Omega})$ and $\mathbf{z} = (z_1, z_2) \in \mathbf{L}^2(\Omega, \mathrm{d}\mu)$ such that

$$\int_I |u - u_k|^2 \, \mathrm{d}x_1 \to 0, \ \int_I \left| \frac{\partial u_k}{\partial x_1} - z_1 \right|^2 \mathrm{d}x_1 \to 0, \ \int_I \left| \frac{\partial u_k}{\partial x_2} - z_2 \right|^2 \mathrm{d}x_1 \to 0.$$

Thus, due to (2.51) each element of $H^1(\varOmega, \mathrm{d}\mu)$ is uniquely defined by the respective element of the 1D Sobolev space $H^1([a,b])$ and $z_1 = \partial u/\partial x_1$. Thus, $\nabla^\mu u = (\partial u/\partial x_1, z_2)$. We now show that the component $z_2$ can be an arbitrary element of $L^2(I)$. In other words,

$$\varGamma^\mu(0) = \{(0,\alpha)\}\,, \quad \alpha \in L^2(I) = L^2(\varOmega, \mathrm{d}\mu).$$

Indeed, since $\varGamma^\mu(0)$ is closed in $\mathbf{L}^2(\varOmega, \mathrm{d}\mu)$ and $C^\infty(\overline{I})$ is dense in $L^2(I)$, it is sufficient to verify that $(0,\alpha) \in \varGamma^\mu(0)$ for $\alpha \in C^\infty(\overline{I})$. To do so, we set $u_k(x_1,x_2) = x_2\alpha(x_1)$. Then $u_k \to 0$ strongly in $L^2(\varOmega, \mathrm{d}\mu)$ as $k \to \infty$ and, moreover,

$$\left.\frac{\partial u_k}{\partial x_1}\right|_{x_2=0} = 0, \qquad \left.\frac{\partial u_k}{\partial x_2}\right|_{x_2=0} = \alpha(x_1).$$

Hence, the required conclusion is obtained. To conclude this example, we note that a Borel function $u = u(x_1,x_2)$ belongs to the space $H^1(\varOmega, \mathrm{d}\mu)$ if and only if $u \in H^1(\varOmega)$ and the restriction (trace) of $u$ to the segment $I$ is an $H^1$ function of a single variable. Note also that the trace of a function in $H^1(\varOmega)$ is defined on $I$ and is an element of the space $H^{1/2}(I)$, in general!

Let us consider several examples of the Sobolev spaces $H^1(\varOmega, \mathrm{d}\mu)$.

*Example 2.51.* **Network node**. Consider the segments $I_1, I_2, \ldots, I_N$ starting at the origin and directed along vectors $v_1, v_2, \ldots, v_N$. Suppose that $v_i/|v_i| \neq v_j/|v_j|$ for $i \neq j$, and a bounded open domain $\varOmega \subset \mathbb{R}^2$ contains this star structure. Let $\mu_1, \mu_2, \ldots, \mu_N$ be the 1D measures on the segments $I_1, I_2, \ldots, I_N$, respectively (see, for instance, (2.52)). Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be arbitrary positive numbers. We set

$$\mu = \sum_{i=1}^{N} \lambda_i \mu_i.$$

Then it is easy to verify that a Borel function $u = u(x_1,x_2)$ belongs to the space $H^1(\varOmega, \mathrm{d}\mu)$ if and only if its restriction to each segment $I_j$ is an $H^1$ function of a single variable and the values of the restricted functions at the origin coincide for all segments (recall that, by the Sobolev embedding theorem, an $H^1$ function of a single variable is continuous).

*Example 2.52.* **Junction**. Let $\varOmega = (-1,1)^3$, $G = \left[-\frac{1}{8}, \frac{1}{8}\right]^3$,

$$\varPi = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^3 \;:\; x_3 = 0, (x_1,x_2) \in \left[0, \frac{3}{8}\right] \times \left[-\frac{1}{8}, \frac{1}{8}\right] \right\},$$

$$I = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^3 \;:\; x_2 = x_3 = 0, x_1 \in \left[\frac{1}{4}, \frac{1}{2}\right] \right\},$$

$\mu_1$ is the standard Lebesgue measure on the segment $I$, $\mu_2$ is the planar Lebesgue measure on $\Pi$, and $\mathrm{d}x$ is the spatial Lebesgue measure in $\mathbb{R}^3$ restricted to the cube $G$. Introduce the measure on $\Omega$ as follows:

$$\mathrm{d}\mu = \mathrm{d}\mu_1 + \mathrm{d}\mu_2 + \mathrm{d}x.$$

Note that the trace of an $H^1(\Omega)$ function is defined on $\Pi$ and is an element of the space $H^{1/2}(\Pi)$. The trace of an $H^1(\Pi)$ function on the segment $I$ is also well defined. However, unlike the 2D case, in the case of dimension 3, the traces of an $H^1(\Omega)$ function on 1D segments are not defined. Therefore, we have to use the following obvious fact: The space $H^1(\Omega, \mathrm{d}\mu)$ is isomorphic to the direct sum of the spaces $H^1(\Omega, \chi_G \mathrm{d}x)$ and $H^1(\Omega, \mathrm{d}\mu_1 + \mathrm{d}\mu_2)$. Thus, the function $u$ belongs to $H^1(\Omega, \mathrm{d}\mu)$ if $u = \widehat{u} + \widetilde{u}$, where $\widehat{u} \in H^1(G)$, $\widetilde{u} \in H^1(\Pi) \cap H^1(I, \mathrm{d}\mu_2)$, and $\widetilde{u}|_I$ is an element of $H^1(I)$. For the other properties of the space $H^1(\Omega, \mathrm{d}\mu)$, we refer to [58, 59].

We now describe some properties of the subspace $\Gamma^\mu(0)$.

(i) If $\mathbf{g} \in \Gamma^\mu(0)$ and $a \in L^\infty(\Omega, \mathrm{d}\mu)$, then $a\mathbf{g} \in \Gamma^\mu(0)$.
To see this, it suffices to consider the case $a \in C^\infty(\overline{\Omega})$. Then if follows from the definition of $\Gamma^\mu(0)$ that

$$au_k \overset{L^2(\Omega,\mathrm{d}\mu)}{\longrightarrow} 0,\ \nabla^\mu(au_k) = \nabla^\mu a\, u_k + a\nabla u_k \overset{\mathbf{L}^2(\Omega,\mathrm{d}\mu)}{\longrightarrow} a\mathbf{g}.$$

Hence, $a\mathbf{g} \in \Gamma^\mu(0)$.

(ii) Let $\Pi$ be the orthogonal projection of $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ onto $\Gamma^\mu(0)$. Then

$$\Pi(a\mathbf{g}) = a\Pi(\mathbf{g}), \quad \forall\, \mathbf{g} \in \mathbf{L}^2(\Omega, \mathrm{d}\mu),\ a \in L^\infty(\Omega, \mathrm{d}\mu).$$

Indeed, setting $\widehat{\mathbf{g}} = \Pi(\mathbf{g})$, we obtain $\widehat{\mathbf{g}} \in \Gamma^\mu(0)$ and $\mathbf{g} - \widehat{\mathbf{g}} \perp \Gamma^\mu(0)$. Then, by property (i), we have

$$a\widehat{\mathbf{g}} \in \Gamma^\mu(0), \quad (a(\mathbf{g} - \widehat{\mathbf{g}}), h) = 0,\ \forall\, h \in \Gamma^\mu(0),$$

as required.

(iii) The set $\Gamma^\mu(0) \cap \mathbf{L}^\infty(\Omega, \mathrm{d}\mu)$ is dense in $\Gamma^\mu(0)$. In fact, if $a_k$ is the characteristic function of the set $\{x \in \Omega \ :\ |\mathbf{g}| \le k\}$, then $a_k\mathbf{g} \in \Gamma^\mu(0)$ and $a_k\mathbf{g} \to \mathbf{g}$ as $k \to \infty$.

(iv) There exists a $\mu$-measurable subspace $D(x) \subset \mathbb{R}^n$ such that

$$\Gamma^\mu(0) = \left\{\mathbf{g} \in \mathbf{L}^2(\Omega, \mathrm{d}\mu) \ :\ \mathbf{g}(x) \in D(x)\right\}. \tag{2.53}$$

Indeed, let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the natural basis of $\mathbb{R}^n$. We set $\boldsymbol{\xi}_i = \Pi\mathbf{e}_i$ and define $D(x)$ as the linear span of the vectors $\{\boldsymbol{\xi}_1(x), \dots, \boldsymbol{\xi}_n(x)\}$. We denote by $B$ the subspace of $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ defined by the right-hand side of (2.53) and show that $\Gamma^\mu(0) = B$.
Let $\mathbf{g} \in \Gamma^\mu(0) \cap \mathbf{L}^\infty(\Omega, \mathrm{d}\mu)$. Then, by property (ii), we obtain

$$\mathbf{g} = g_1\mathbf{e}_1 + \cdots + g_n\mathbf{e}_n = \Pi g = g_1\boldsymbol{\xi}_1 + \cdots + g_n\boldsymbol{\xi}_n \in B.$$

In view of property (iii) and the fact that $B$ is closed, we have the inclusion $\Gamma^\mu(0) \subseteq B$. To verify the reverse inclusion $B \subseteq \Gamma^\mu(0)$, we note that

$$a(x)\sum_{i=1}^{n} \lambda_i\boldsymbol{\xi}_i \in \Gamma^\mu(0), \quad \forall\, a \in L^\infty(\Omega, \mathrm{d}\mu),\ \forall\, \lambda_i \in \mathbb{R}^1$$

because of properties (i) and (ii) and the fact that $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n \in \Gamma^\mu(0)$. The set of such elements is dense in $B$. Let us assume that there exists a $\mathbf{b} \in B$ such that $\mathbf{b} \neq 0$ and

$$\left(a\mathbf{b}, \sum_{i=1}^{n} \lambda_i\boldsymbol{\xi}_i\right)_{\mathbf{L}^2(\Omega,\mathrm{d}\mu)} = 0.$$

Since $a$ is an arbitrary function, it follows that

$$\left(\mathbf{b}(x), \sum_{i=1}^{n} \lambda_i\boldsymbol{\xi}_i(x)\right)_{\mathbb{R}^n} = 0$$

for $\mu$-almost all $x \in \Omega$, which means that $b(x) = 0$. The proof is complete.

**Definition 2.53.** *We say that a gradient $\nabla^\mu u$ is tangential for $u \in H^1(\Omega, \mathrm{d}\mu)$ if $\nabla^\mu u \perp \Gamma^\mu(0)$ (or in the equivalent form, if $\nabla^\mu u(x) \in T(x)$ for $\mu$-almost all $x \in \Omega$, where $T(x) = (D(x))^\perp$).*

It is clear that each function in the Sobolev space $H^1(\Omega, \mathrm{d}\mu)$ has a unique tangential gradient. It is also obvious that if $\nabla^\mu u(x)$ is some gradient of $u$ and $P(x)$ is the orthogonal projection $\mathbb{R}^n \to T(x)$, then $P(x)\nabla^\mu u(x)$ is the tangential gradient. Combining these results, we come to the following conclusion (see [256]): *There exists a $\mu$-measurable subspace $T(x)$ such that the set of gradients of each function in $H^1(\Omega, \mathrm{d}\mu)$ has the representation $\nabla^\mu u(x) + \mathbf{g}(x)$, where $\nabla^\mu u(x) \in T(x)$ and $\mathbf{g}$ is an arbitrary vector in $\mathbf{L}^2(\Omega, \mathrm{d}\mu)$ such that $\mathbf{g}(x) \in T^\perp(x)$.* The subspace $T(x)$ is called the tangential space at the point $x$ and $\nabla^\mu u$ is called the tangential gradient. For more results concerning the Sobolev spaces with respect to a measure and their applications, we refer to Chechkin, Zhikov, Lukkassen and Piatnitski [58], Bouchitté, Buttazzo and Seppecher [25], and Fragalà and Mantegazza [110].

## 2.8 Boundary value problems in Sobolev spaces with measures

The asymptotic behavior of thin and reticulated structures such as, for example, shells, plates, thin films, rod structures, skeletons, and so on is widely discussed in the mathematical and engineering literatures. As a rule, the goal of an asymptotic analysis is to reduce dimension (i.e., to reduce the original problem to a problem on some structure of smaller dimension). For instance, the equation on a thin 3D plate is replaced with an equation on a 2D domain [257], the system describing a rod construction is reduced to a family of ordinary differential equations [265], and so on. The reduced system is usually simpler, especially from the point of view of numerical analysis.

Asymptotic methods for such problems are well known in the literature, but, as a rule, they are derived under strict restrictions on the geometry and smoothness of the structure (see, for instance, [14, 22, 66, 200, 209, 216]). There are numerous works in mechanics where asymptotics for different special models were obtained at the physical level of rigor.

From mathematical point of view, it is of great interest to construct a general approach to the asymptotic analysis of different classes of boundary value problems on thin and reticulated structures, that would be associated with the corresponding periodic measures, singular measures, partially singular measures, as well as measures converging to singular ones. A successful attempt to create such a theory was made in [25, 26, 254, 256, 257, 265]. In this section, following these works, we define the meaning of boundary value problems in spaces with measures and clarify the idea of Sobolev spaces with an arbitrary measure. The main motivation of our intention can be clarified by a simple example. Let $\mu$ be a positive finite Borel measure defined on a smooth bounded domain $\Omega$. We have the following variational problem:

$$\inf_{\varphi\in C_0^\infty(\Omega)} \int_\Omega \Big(A(x)\nabla\varphi(x)\cdot\nabla\varphi(x)+\varphi^2(x)-2f(x)\varphi(x)\Big)\,\mathrm{d}\mu(x),$$

where $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ is a coercive matrix of positive elements and $f \in C(\overline{\Omega})$ is a given function. Our aim is to describe a minimizer to this problem as an element of an appropriate Sobolev space with respect to the measure $\mu$ and associate it with a solution to the corresponding Euler equation.

Throughout this section, we assume that $\Omega$ is an open domain in $\mathbb{R}^n$ and $\mu$ is a periodic Borel (e.g., probability) measure of $\mathcal{M}_0^p(\Omega)$ such that $\int_\square \mathrm{d}\mu = 1$, where $\square = [0,1)^n$ is the cell (or the torus) of periodicity for $\mu$. Let $H^1_{per}(\square, \mathrm{d}\mu) = W^{1,2}_{per}(\square, \mathrm{d}\mu)$ be the periodic Sobolev space with respect to the measure $\mu$. Since $\mu$ can be identified with the corresponding periodic measure in $\mathbb{R}^n$, we will also make use the Sobolev space $H^1(\mathbb{R}^n, \mathrm{d}\mu)$.

Let $A = [a_{ij}(x)]_{i,j=1,\dots,n}$ be a continuous function with values in the space of symmetric $n\times n$ matrices satisfying the uniform ellipticity condition

$$\alpha|\xi|^2_{\mathbb{R}^n} \le (A(x)\xi,\xi)_{\mathbb{R}^n} \le \alpha^{-1}|\xi|^2_{\mathbb{R}^n}, \quad \alpha>0,\ \xi\in\mathbb{R}^n \tag{2.54}$$

for all $x \in \mathbb{R}^n$. Let $f$ be a given element of $L^2(\mathbb{R}^n, \mathrm{d}\mu)$ and let $\lambda > 0$.

To begin with, we define the notion of divergence with respect to the measure.

**Definition 2.54.** *Suppose that* $g \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ *and* $\mathbf{v} \in \big[L^2(\mathbb{R}^n, \mathrm{d}\mu)\big]^n$. *We say that* $g(x) = \operatorname{div}^\mu \mathbf{v}(x)$ *if*

$$\int_{\mathbb{R}^n} g(x)\varphi(x)\,\mathrm{d}\mu(x) = -\int_{\mathbb{R}^n} \mathbf{v}(x)\cdot\nabla\varphi(x)\,\mathrm{d}\mu(x), \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

It is easy to see that, in this definition, instead of smooth functions $\varphi$ one can take functions $\varphi \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$.

**Definition 2.55.** *We say that a pair* $(u, \nabla^\mu u)$, *where* $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *and* $\nabla^\mu u$ *is a* $\mu$*-gradient of* $u$, *satisfies the equation*

$$-\operatorname{div}^\mu\Big(A(x)\nabla^\mu u\Big) + \lambda u = f \tag{2.55}$$

*in* $L^2(\mathbb{R}^n, \mathrm{d}\mu)$ *if for any* $v \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *and any gradient* $\nabla^\mu v$ *of* $v$, *we have*

$$\int_{\mathbb{R}^n} A(x)\nabla^\mu u\cdot\nabla^\mu v\,\mathrm{d}\mu + \lambda\int_{\mathbb{R}^n} uv\,\mathrm{d}\mu = \int_{\mathbb{R}^n} fv\,\mathrm{d}\mu. \tag{2.56}$$

**Definition 2.56.** *A function* $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *is called a solution to (2.55) if the integral identity (2.56) holds for some of the gradients of* $u$ *and for any* $v \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ *and any gradient* $\nabla^\mu v$ *of* $v$.

Note that in Definitions 2.55 and 2.56, instead of functions $v \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$, one can take the test functions $v$ in $C_0^\infty(\mathbb{R}^n)$.

*Remark 2.57.* In the special case when the matrix $A(x)$ in (2.55) is identity, relation (2.56) takes the form

$$\int_{\mathbb{R}^n} \nabla^\mu u\cdot\nabla^\mu v\,\mathrm{d}\mu + \lambda\int_{\mathbb{R}^n} uv\,\mathrm{d}\mu = \int_{\mathbb{R}^n} fv\,\mathrm{d}\mu.$$

Then the corresponding expression $\operatorname{div}^\mu\nabla^\mu u$ is usually denoted by $\Delta_\mu u$ and called the $\mu$-Laplacian of $u$.

The main result of this section can be formulated as follows:

**Lemma 2.58.** *Let* $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ *be a symmetric matrix satisfying conditions (2.54). Then for every* $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$, *(2.55) has a unique solution* $(u, \nabla^\mu u)$, $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$. *Moreover, the choice of a* $\mu$*-gradient of* $u$ *is uniquely determined by the condition of orthogonality of the vector* $A(x)\nabla^\mu u$ *and the subspace* $\Gamma_\mu(0)$ *of the gradients of* $0$.

*Proof.* Since the matrix $A = A(x)$ is positive definite, the left-hand side of (2.56) is the inner product in $W^2(\mathbb{R}^n, \mathrm{d}\mu)$ (see the definition of the space $W^2(\mathbb{R}^n, \mathrm{d}\mu)$ in Sects. 2.7 and 6.4), whereas the right-hand side of (2.56) is a continuous linear functional on $W^2(\mathbb{R}^n, \mathrm{d}\mu)$. By the Riesz representation theorem, there exists a pair $(u, \nabla^\mu u) \in W^2(\mathbb{R}^n, \mathrm{d}\mu)$ satisfying relation (2.56). Taking for a test function in (2.56) the pair $(0, \mathbf{z})$, where $\mathbf{z} \in \Gamma_\mu(0)$, we see that the vector $A(x)\nabla^\mu u$ is orthogonal to $\Gamma^\mu(0)$. Using the Riesz theorem again, we conclude that there is a unique element $\mathbf{z}$ of the set of $\mu$-gradients of $u$ satisfying the orthogonality condition. Thus, the solution $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ to (2.55) is unique. $\square$

*Remark 2.59.* By Lemma 2.58, the gradient $\nabla^\mu u$ chosen by a solution $u$ to the problem (2.55) satisfies the condition $\nabla^\mu u \perp \Gamma_\mu(0)$. Hence, in view of Definition 2.53, $\nabla^\mu u$ is the tangential gradient for a function $u \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$. Moreover, in the case when the measure $\mu$ is defined as in Example 2.51, the orthogonality $A(x)\nabla^\mu u|_{x=0} \perp \Gamma_\mu(0)$ is equivalent to the classical Kirchoff condition.

Let us apply Lemma 2.58 to establish some additional properties of the solutions to the elliptic equation (2.55).

**Lemma 2.60.** *Under the assumptions of Lemma 2.58, the set of solutions to (2.55), for all $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$, is dense in the space $L^2(\mathbb{R}^n, \mathrm{d}\mu)$.*

*Proof.* Denote by $\mathcal{D}$ the set of solutions to the (2.55) when $f$ runs over the entire space $L^2(\mathbb{R}^n, \mathrm{d}\mu)$. Assume that there is a nontrivial element $g \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ which is orthogonal to $\mathcal{D}$. For a fixed $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$, we denote by $u^f \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ the corresponding solution to (2.55). Then $u^g \in H^1(\mathbb{R}^n, \mathrm{d}\mu)$ is a solution to the equation

$$-\mathrm{div}^\mu\Big(A(x)\nabla^\mu u\Big) + \lambda u = g.$$

Taking $u^g$ as a test function in (2.56) and $u^f$ for a test function in the last equation and taking the difference of the obtained integral identities, we come to the relation

$$\int_{\mathbb{R}^n} f u^g \,\mathrm{d}\mu = 0, \quad \forall\, f \in L^2(\mathbb{R}^n, \mathrm{d}\mu).$$

Hence, $u^g = 0$ and $g = 0$, which contradicts the assumption that $g$ is nontrivial. The proof is complete. $\square$

Further, we note that (2.55) can be written in the operator form $\mathcal{A}\,u + \lambda u = f$, where $\mathcal{A}$ is a self-adjoint symmetric operator. Indeed, let $u^f$ be any element of $\mathcal{D}$ (i.e., $u^f$ is a solution to (2.55) for some $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$). We set

$$\mathcal{A}\,u^f = f - \lambda u^f.$$

Then, in view of Lemma 2.58, the operator $(\mathcal{A} + \lambda I)^{-1}$ sends a function $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ to the corresponding unique solution $u^f$ of (2.55). Since this operator is nonnegative, bounded, and symmetric, we come to the required conclusion.

The following assertion can be proved in the same way as in the case of classical variational problems in the space $H^1(\mathbb{R}^n)$ (see, for instance, [169]).

**Proposition 2.61.** *Let $f \in L^2(\mathbb{R}^n, \mathrm{d}\mu)$ and let $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ be a symmetric matrix satisfying conditions (2.54). Then for every $\lambda > 0$, the variational problem*

$$\inf_{\varphi\in H^1(\mathbb{R}^n,\mathrm{d}\mu)} \left\{ \int_{\mathbb{R}^n} \Big( A(x)\nabla^\mu u \cdot \nabla^\mu u + \lambda u^2 \Big)\,\mathrm{d}\mu - \int_{\mathbb{R}^n} 2fu\,\mathrm{d}\mu \right\} \tag{2.57}$$

*has a unique minimum point in $H^1(\mathbb{R}^n, \mathrm{d}\mu)$ and it is a solution to (2.55).*

Thus, (2.55) or the equivalent equation $\mathcal{A}\,u + \lambda u = f$ is an Euler equation for the variation problem (2.57).

The developed technique allows us to study not only problems in the entire space $\mathbb{R}^n$ but also various boundary value problems. We illustrate this by an example of a Dirichlet problem (we consider other types of boundary problems in the second part of this book).

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\mu$ be a positive finite Borel measure on $\Omega$.

**Definition 2.62.** *We say that a function $u \in L^2(\Omega, \mathrm{d}\mu)$ belongs to the space $H^1_0(\Omega, \mathrm{d}\mu)$ and $\mathbf{z} \in \mathbf{L}^2(\Omega, \mathrm{d}\mu) = \big[L^2(\Omega, \mathrm{d}\mu)\big]^n$ is a $\mu$-gradient of $u$ if there exists a sequence $\{u_k \in C_0^\infty(\Omega)\}_{k=1}^\infty$ such that*

$$\begin{aligned} u_k \to u \quad &\mathit{in} \quad L^2(\Omega, \mathrm{d}\mu) \quad \mathit{as} \quad k \to \infty, \\ \nabla u_k \to \mathbf{z} \quad &\mathit{in} \quad \mathbf{L}^2(\Omega, \mathrm{d}\mu) \quad \mathit{as} \quad k \to \infty. \end{aligned}$$

Consider the Dirichlet problem

$$-\mathrm{div}^\mu\Big(A(x)\nabla^\mu u\Big) + \lambda u = f \quad \text{in} \quad L^2(\Omega, \mathrm{d}\mu), \tag{2.58}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.59}$$

As before, we assume that the matrix $A \in C(\overline{\Omega}, \mathbb{R}^{n\times n})$ is symmetric and satisfies the uniform ellipticity condition in $\Omega$ like (2.54).

**Definition 2.63.** *We say that $u \in H^1_0(\Omega, \mathrm{d}\mu)$ is a solution to the Dirichlet problem (2.58)–(2.59) if the integral identity*

$$\int_\Omega A(x)\nabla^\mu u \cdot \nabla^\mu v\,\mathrm{d}\mu + \lambda \int_\Omega uv\,\mathrm{d}\mu = \int_\Omega fv\,\mathrm{d}\mu$$

*holds true for any $v \in H^1_0(\Omega, \mathrm{d}\mu)$.*

The existence and uniqueness of a solution to the problem (2.58)–(2.59) can be established in the same way as in the proof of Lemma 2.58. We recall that a gradient of the solution that satisfies the above integral identity is chosen in a unique way.

## 2.9 On weak compactness of a class of bounded sets in Banach spaces

Let $\mathbb{X}$ be a Banach space and $\mathbb{X}^*$ its dual. Let us recall that the weak-$*$ topology of $\mathbb{X}^*$ is the locally convex topology $\sigma(\mathbb{X}^*, \mathbb{X})$ in which the linear functionals

$$X^* \ni f \mapsto \langle f, u\rangle_{\mathbb{X}^*,\mathbb{X}}\,, \quad u \in \mathbb{X}$$

are continuous.

Due to the Banach–Alaoglu theorem, it is well known that any bounded weakly-$*$ closed subset $B \subset \mathbb{X}^*$ is compact with respect to the weak-$*$ topology of $\mathbb{X}^*$. However, it is hard to verify the property of weak-$*$ closure for sets which differ from a ball, even for relatively simple sets $B \subset \mathbb{X}^*$. Therefore, the main goal of this section is to study the compactness property for other types of subsets in $\mathbb{X}^*$ (see, for instance, [151, 152]).

Let $\mathbb{X}$ and $\{\mathbb{Z}_j\}_{j=1}^k$ be Banach spaces and let $\mathbb{X}^*$ and $\left\{\mathbb{Z}_j^*\right\}_{j=1}^k$ be their duals, respectively. Let

$$\{\Lambda_j : (D(\Lambda_j) \subset \mathbb{Z}_j) \to \mathbb{X}\}_{j=1}^k$$

be linear mappings defined on the domains $D(\Lambda_j)$ which are dense subsets of the corresponding spaces $\mathbb{Z}_j$. Let

$$\left\{\Lambda_j^* : \left(D(\Lambda_j^*) \subset \mathbb{X}^*\right) \to \mathbb{Z}_j^*\right\}_{j=1}^k$$

be their dual mappings, respectively. We set

$$\mathcal{D}^* = \bigcap_{j=1}^k D(\Lambda_j^*)$$

and endow this set with the graph norm

$$\|y^*\|_* = \|y^*\|_{\mathbb{X}^*} + \sum_{j=1}^k \left\|\Lambda_j^* y^*\right\|_{\mathbb{Z}_j^*}\,. \tag{2.60}$$

By $\mathbb{Y}^*$ we denote the normed space $\mathcal{D}^*$ equipped with the norm $\|\cdot\|_*$. Let $\mathbb{X}_\sigma^*$ and $\mathbb{Z}_{j,\sigma}^*$ be the spaces $\mathbb{X}^*$ and $\mathbb{Z}_j^*$, respectively, endowed with the topologies $\sigma(\mathbb{X}^*, \mathbb{X})$ and $\sigma(\mathbb{Z}_j^*, \mathbb{Z}_j)$. The graphs $\mathrm{gr}\,\Lambda_j^*$ of operators $\Lambda_j^*$ are closed in $\mathbb{X}_\sigma^* \times \mathbb{Z}_{j,\sigma}^*$ for every $j = 1, 2, \ldots, k$ (see [241]). Hence, $\mathbb{Y}^*$ is a Banach space. By $\mathbb{X}_w$ we denote the Banach space $X$ endowed with the $\sigma(\mathbb{X}, \mathbb{X}^*)$ topology. Let us consider the following class of subsets in $\mathbb{Y}^*$:

$$\mathcal{K}^* = \left\{\xi \in \mathbb{Y}^* \ : \ \|y^*\|_{\mathbb{X}^*} \le l_0,\ \|\Lambda_j^* y^*\|_{\mathbb{Z}_j^*} \le l_j,\ j = 1, 2, \ldots, k\right\}, \tag{2.61}$$

where $l_0, l_1, \ldots, l_k$ are positive numbers.

Having used the notation $\mathcal{D} = \mathbb{X} \times \prod_{j=1}^{k} D(\Lambda_j)$, we note that the set $\mathcal{D}$ can be associated with the family of linear continuous functionals $G_\phi(\cdot)$ on $\mathbb{Y}^*$. Indeed, let $\phi = (\phi_0, \phi_1, \ldots, \phi_k)$ be a fixed element of $\mathcal{D}$. Then the functional $G_\phi(\cdot)$ can be defined as follows:

$$G_\phi(y^*) = \langle y^*, \phi \rangle := \langle y^*, \phi_0 \rangle_{\mathbb{X}^*,\mathbb{X}} + \sum_{j=1}^{k} \left\langle \Lambda_j^* y^*, \phi_j \right\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j} .$$

*Remark 2.64.* Let $\sigma(\mathbb{Y}^*, \mathcal{D})$ be the weakest topology on $\mathbb{Y}^*$ with respect to which all functionals $G_\phi(\cdot)$ are continuous. Then the pair $(\mathbb{Y}^*, \sigma(\mathbb{Y}^*, \mathcal{D}))$ is a locally convex topological space.

**Definition 2.65.** *We say that a sequence $\{y_n^*\}_{n=1}^{\infty} \subset \mathbb{Y}^*$ converges $\mathcal{D}$-weakly to an element $y^* \in \mathbb{Y}^*$, written $y_n^* \xrightharpoonup{\mathcal{D}} y^*$, provided*

$$\lim_{n\to\infty} \langle y_n^*, \phi_0 \rangle_{\mathbb{X}^*,\mathbb{X}} = \langle y^*, \phi_0 \rangle_{\mathbb{X}^*,\mathbb{X}} ,$$

$$\lim_{n\to\infty} \left\langle \Lambda_j^* y_n^*, \phi_j \right\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j} = \left\langle \Lambda_j^* y^*, \phi_j \right\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j}$$

*for all $\phi = (\phi_0, \phi_1, \ldots, \phi_k) \in \mathcal{D}$.*

Note that this concept can be easy extended to the case of $\mathcal{D}$-weakly convergent nets (or generalized sequences) $\{y_\alpha^*\}_{\alpha\in A} \subset \mathbb{Y}^*$, where $A$ is a directed set of indices.

The following result deals with the compactness property of the set $\mathcal{K}^*$.

**Theorem 2.66.** *Let $\mathbb{X}$ and $\{\mathbb{Z}_j\}_{j=1}^{k}$ be Banach spaces and let $\mathbb{X}^*$, $\left\{\mathbb{Z}_j^*\right\}_{j=1}^{k}$ be their duals, respectively. Let*

$$\{\Lambda_j : (D(\Lambda_j) \subset \mathbb{Z}_j) \to \mathbb{X}\}_{j=1}^{k}$$

*be a given family of linear mappings with dense domains $D(\Lambda_j)$ in $\mathbb{Z}_j$ for each $j = 1, \ldots, k$. Then the set $\mathcal{K}^*$, defined by (2.61), is $\mathcal{D}$-weakly compact in $\mathbb{Y}^*$ for every collection of positive numbers $l_0, l_1, \ldots, l_k$.*

*Proof.* Let $l_0, l_1, \ldots, l_k$ be given positive numbers. Let $\{y_\alpha^*\}_{\alpha\in A}$ be an arbitrary net in $\mathcal{K}^*$. To prove this theorem, we have to show that there is a subnet $\{x_\beta^*\}_{\beta\in B}$ of the original net $\{y_\alpha^*\}_{\alpha\in A}$, which $\mathcal{D}$-weakly converges in $\mathbb{Y}^*$ to some element of $\mathcal{K}^*$. Taking into account the structure of the set $\mathcal{K}^*$, we see that $\{y_\alpha^*\}_{\alpha\in A}$ belongs to the close ball $B_{\mathbb{X}^*}(0, l_0)$ centered at the origin with radius $l_0$. Hence, by the Banach–Alaoglu theorem (see Yosida [251] or Kantorovich and Akilov [128]), there exists a subnet of $\{y_\alpha^*\}_{\alpha\in A}$, denoted by $\{x_\beta^*\}_{\beta\in B_0}$, such that $x_\beta^* \to y^*$ in $\mathbb{X}_\sigma^*$ and $\|y^*\|_{\mathbb{X}^*} \le l_0$. Setting now $j = 1$, we consider the net $\{\Lambda_1^* x_\beta^*\}_{\beta\in B_0}$. Since the elements $\Lambda_1^* x_\beta^*$ belong to the closed ball $B_{\mathbb{Z}_1^*}(0, l_1)$, it follows that there exists a subnet $\left\{z_\gamma^*\right\}_{\gamma\in B_1}$ such that $\Lambda_1^* z_\gamma^* \to \mu_1$ in $Z_{1,\sigma}^*$ and $\|\mu_1\|_{\mathbb{Z}_1^*} \le l_1$. Repeating this iteration process under $j = 2,\dots,k$, finally, we can extract a subnet $\{d_\nu^*\}_{\nu\in B_k}$ of the origin net $\{y_\alpha^*\}_{\alpha\in A}$ such that

$$\begin{gathered}\langle d_\nu^*, \phi_0\rangle_{\mathbb{X}^*,\mathbb{X}} \overset{\nu\in B_k}{\longrightarrow} \langle y^*, \phi_0\rangle_{\mathbb{X}^*,\mathbb{X}}\,,\\ \left\langle \Lambda_j^* d_\nu^*, \phi_j\right\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j} \overset{\nu\in B_k}{\longrightarrow} \langle \mu_j, \phi_j\rangle_{\mathbb{Z}_j^*,\mathbb{Z}_j}\,,\\ \|y^*\|_{\mathbb{X}^*} \le l_0, \quad \|\mu_j\|_{\mathbb{Z}_j^*} \le l_j,\ j = 1,2,\dots,k,\end{gathered}$$

for each $\phi = (\phi_0, \phi_1, \dots, \phi_k) \in \mathcal{D}$.

In view of the properties of the domains $D(\Lambda_j)$, the operators $\Lambda_j^* : \left(D(\Lambda_j^*) \subset \mathbb{X}^*\right) \to \mathbb{Z}_j^*$ are closed in $\mathbb{X}_\sigma^* \times \mathbb{Z}_{j,\sigma}^*$. Hence, $\mu_j = \Lambda_j^* y^*$ for every $j = 1,2,\dots,k$. As a result, we have $d_\nu^* \overset{\mathcal{D}}{\rightharpoonup} y^*$, where $\|y^*\|_{\mathbb{X}^*} \le l_0$ and $\|\Lambda_j^* y^*\|_{\mathbb{Z}_j^*} \le l_j$ for $j = 1,\dots,k$. Thus, $y^* \in \mathcal{K}^*$, which concludes the proof. □

To illustrate the possible applications of this theorem, we give the following example.

*Example 2.67.* Let $\Omega$ be open bounded subset of $\mathbb{R}^n$ with a Lipschitz boundary. Let $\mathbb{X} = \mathbb{Z}_j = L^1(\Omega)$ for all $j = 1,\dots,k$. We define the collection of linear mappings $\{\Lambda_j : (D(\Lambda_j) \subset \mathbb{Z}_j) \to \mathbb{X}\}_{j=1}^k$ as follows:

$$\Lambda_j y := -\partial y/\partial x_j, \quad j = 1,2,\dots,k,$$

where $D(\Lambda_j)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|y\|_j = \|y\|_{L^1(\Omega)} + \|\partial y/\partial x_j\|_{L^1(\Omega)}.$$

Thus, if $y \in C(\Omega) \cap D(\Lambda_j)$, then

$$y \in L^1(\Omega), \ \ \operatorname{supp} y(x) \ \text{ is a compact in } \ \Omega \ \text{ and } \ \partial y/\partial x_j \in L^1(\Omega),$$

where the partial derivatives $\partial y/\partial x_j$ we mean in the weak sense (see (2.6)).

It is easy to see that the following chain of embeddings holds:

$$C_0^1(\Omega) \subset W_0^{1,1}(\Omega) \subset D(\Lambda_j) \subset L^1(\Omega), \quad \forall\, j = 1,\dots,k.$$

Since $C_0^1(\Omega)$ is dense in $L^1(\Omega)$ with respect to the topology induced by the norm $\|\cdot\|_{L^1(\Omega)}$, it follows that the weak closure of the set $C_0^1(\Omega)$ coincides with the entire space $L^1(\Omega)$. Therefore, the domains $D(\Lambda_j)$ are weakly dense in $L^1(\Omega)$ for all $j = 1,\dots,k$. In view of this, the dual operators

$$\left\{\Lambda_j^* : \left(D(\Lambda_j^*) \subset L^\infty(\Omega)\right) \to L^\infty(\Omega)\right\}_{j=1}^k$$

are well defined and closed. Moreover, in this case, we have $\Lambda_j^* = \partial/\partial x_j$. As a result, we obtain

$$\mathcal{D}^* := \bigcap_{j=1}^{k} D(\Lambda_j^*) = W^{1,\infty}(\Omega),$$

where

$$W^{1,\infty}(\Omega) = \left\{ y(x) \ : \ y \in L^\infty(\Omega), \ \frac{\partial y}{\partial x_j} \in L^\infty(\Omega), \ j = 1, \dots, k \right\}$$

is the Banach space equipped with the norm

$$\|y\|_{W^{1,\infty}(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |y(x)| + \sum_{j=1}^{k} \operatorname*{ess\,sup}_{x \in \Omega} \left| \frac{\partial y(x)}{\partial x_j} \right|.$$

Hence $\mathbb{Y}^* = W^{1,\infty}(\Omega)$ and, therefore, due to the Theorem 2.66, the bounded set

$$\mathcal{K}^* = \left\{ y \in W^{1,\infty}(\Omega) \ : \ \|y\|_{L^\infty(\Omega)} \leq l_0, \ \left\| \frac{\partial y}{\partial x_j} \right\|_{L^\infty(\Omega)} \leq l_j, \ j = 1, \dots, k \right\}$$

is $\mathcal{D}$-weakly compact in $W^{1,\infty}(\Omega)$ for every positive numbers $l_0, l_1, \dots, l_k$. Following Definition 2.65 and using the fact that $L^1(\Omega)$ is a separable Banach space, it means that for any net $\{y_\alpha^*\}_{\alpha \in A} \subset \mathcal{K}^*$, there exists a sequence $\{x_i^*\}_{i \in \mathbb{N}}$ (which is a subnet of $\{y_\alpha^*\}_{\alpha \in A}$) such that

$$\lim_{i \to \infty} \langle x_i^*, \phi \rangle_{\mathbb{Y}^*, \mathbb{Y}} = \langle y, \phi \rangle_{\mathbb{Y}^*, \mathbb{Y}}, \quad \forall \, \phi = (\phi_0, \dots, \phi_k) \in L^1(\Omega) \times \prod_{j=1}^{k} D(\Lambda_j),$$

where $y \in \mathcal{K}^*$ and

$$\langle y, \phi \rangle_{\mathbb{Y}^*, \mathbb{Y}} = \int_\Omega y(x) \phi_0(x) \, \mathrm{d}x + \sum_{j=1}^{k} \int_\Omega \phi_j(x) \frac{\partial y(x)}{\partial x_j} \, \mathrm{d}x.$$

Thus, the set $\mathcal{K}^*$ is sequentially $\mathcal{D}$-weakly compact.

http://www.springer.com/978-0-8176-8148-7

Optimal Control Problems for Partial Differential
Equations on Reticulated Domains
Approximation and Asymptotic Analysis
Kogut, P.I.; Leugering, G.R.
2011, XVI, 636 p. 26 illus., Hardcover
ISBN: 978-0-8176-8148-7
A product of Birkhäuser Basel