

Chapter 2

The Dolbeault-Dirac Operator

In this chapter we set the stage, by introducing complex and almost structures, the Dolbeault complex and Hermitian structures. The holomorphic Lefschetz number, defined as the alternating sum of the trace of the automorphism acting on the cohomology of the sheaf of holomorphic sections, will be expressed in terms of a selfadjoint operator, which is built out of the Dolbeault operator and its adjoint; the Dolbeault-Dirac operator in the title of this chapter. This material is very well-known but, also in order to fix the notations, we have taken our time for the description of these structures. Just for convenience, we will assume that all objects are smooth (infinitely differentiable).

2.1 The Dolbeault Complex

Let M be a manifold of even dimension $2n$, provided with an almost complex structure J . That is, for each $x \in M$, J_x is a real linear transformation in $T_x M$ such that $J_x^2 = -1$. A real linear mapping A from $T_x M$ to a complex vector space V is called complex linear and complex antilinear with respect

to the complex structure J_x in $T_x M$, if

$$A(J_x(v)) = i A(v), \quad v \in T_x M \quad (2.1)$$

and

$$A(J_x(v)) = -i A(v), \quad v \in T_x M, \quad (2.2)$$

respectively.

The space of complex linear and complex antilinear forms ($V = \mathbb{C}$) on $T_x M$ is denoted by $T_x^* M^{(1,0)}$ and $T_x^* M^{(0,1)}$, respectively. With this notation, the space of complex linear and antilinear mappings from $T_x M$ to V becomes equal to $T_x^* M^{(1,0)} \otimes V$ and $T_x^* M^{(0,1)} \otimes V$, respectively. One has the complementary projections

$$\pi^{(1,0)} : \xi \mapsto \xi^{(1,0)} := \frac{1}{2} (\xi - i \xi \circ J_x), \quad (2.3)$$

and

$$\pi^{(0,1)} : \xi \mapsto \xi^{(0,1)} := \frac{1}{2} (\xi + i \xi \circ J_x), \quad (2.4)$$

from $T_x^* M \otimes \mathbb{C}$ onto $T_x^* M^{(1,0)}$ along $T_x^* M^{(0,1)}$, and from $T_x^* M \otimes \mathbb{C}$ onto $T_x^* M^{(0,1)}$ along $T_x^* M^{(1,0)}$, respectively.

A complex-valued function f on M is called complex-differentiable or complex-analytic, or holomorphic, if, for every $x \in M$, df_x is complex linear. If $\bar{\partial} = \pi^{(0,1)} \circ d$ denotes the operator d followed by the projection (2.4), then this condition is equivalent to the differential equation $\bar{\partial}f = 0$. One also writes $\partial = \pi^{(1,0)} \circ d$, so that $d = \partial + \bar{\partial}$ on functions, and $\partial f = df$ if and only if f is holomorphic.

Let $p, q, r \in \mathbb{Z}_{\geq 0}$, with $p + q = r$. A complex-valued antisymmetric r -linear form on $T_x M$ is called of type (p, q) , if it is equal to a finite sum of forms $\alpha \wedge \beta$, where $\alpha \in \Lambda^p T_x^* M^{(1,0)}$ and $\beta \in \Lambda^q T_x^* M^{(0,1)}$. The space of forms of type (p, q) is denoted by $T_x^* M^{(p,q)}$. The point is that

$$\Lambda^r T_x^* M \otimes \mathbb{C} = \bigoplus_{p,q, p+q=r} T_x^* M^{(p,q)}, \quad (2.5)$$

so we have the projection $\pi_{p,q}$ from $\Lambda^r T_x^* M \otimes \mathbf{C}$ onto $T_x^* M^{(p,q)}$ along the sum of the other components. An L -valued version is obtained by tensoring $T_x^* M^{(p,q)}$ with L_x . A (p, q) -form ω_x on $T_x M$, which depends smoothly on $x \in M$, is called a (p, q) -form on M . The space of (p, q) -forms on M is denoted by $\Omega^{(p,q)}(M)$.

In particular we will be interested in the case $p = 0$, for which we will use the following abbreviation throughout:

$$E_x^q := T_x^* M^{(0,q)} = \Lambda^q T_x^* M^{(0,1)}, \quad E_x^0 = \mathbf{C}. \quad (2.6)$$

Note that $E_x^q = 0$ if $q > n$, and $\dim_{\mathbf{C}} E_x^q = \binom{n}{q}$ if $0 \leq q \leq n$. We will write

$$E_x := \bigoplus_{q=0}^n E_x^q, \quad (2.7)$$

$$E_x^+ = E_x^{\text{even}} = \bigoplus_{\text{even } q} E_x^q, \quad (2.8)$$

$$E_x^- = E_x^{\text{odd}} = \bigoplus_{\text{odd } q} E_x^q. \quad (2.9)$$

With the exterior product of forms and the splitting in E_x^+ and E_x^- , E_x is a supercommutative superalgebra over $\mathbf{C}$. (See [9, Section 1.3] for the definition of such algebras.)

The $E_x, x \in M$, form a complex vector bundle E over M with subbundles $E^+ = \bigcup_{x \in M} E_x^+$ and $E^- = \bigcup_{x \in M} E_x^-$. The space of sections of E , E^+ and E^- is equal to the direct sum of the spaces $\Omega^{(0,q)}$, where q runs over all the even and the odd integers $0 \leq q \leq n$, respectively.

In Chapter 5 we will introduce the spin-c Dirac operator, which will be used in the general case of an almost complex structure. In order to motivate its definition and to understand its relation to complex analysis, we assume in the remainder of this chapter that M is a complex analytic manifold. This means that around every $x \in M$ there is a system of local coordinates in

which J is equal to the standard complex structure of $\mathbf{C}^n$. This is equivalent to the condition that, at every $x \in M$, there exist n holomorphic functions z_j in a neighborhood of x in M , such that the dz_j at x are linearly independent over $\mathbf{C}$.

In such coordinates z_j , each (p, q) -form is of the form

$$\omega = \sum_{J, K} \omega_{J, K} dz_J \wedge d\bar{z}_K, \quad (2.10)$$

where J and K runs over the set of strictly increasing sequences $J = (j_i)_{i=1}^p$ and $K = (k_i)_{i=1}^q$, respectively, each $\omega_{J, K}$ is a complex-valued function, and

$$dz_J = dz_{j_1} \wedge dz_{j_2} \wedge \dots \wedge dz_{j_p}, \quad (2.11)$$

$$d\bar{z}_K = d\bar{z}_{k_1} \wedge d\bar{z}_{k_2} \wedge \dots \wedge d\bar{z}_{k_q}. \quad (2.12)$$

From this we see that $d\omega$ is the sum of a $(p+1, q)$ -form and a $(p, q+1)$ -form. Or, one again has $d = \partial + \bar{\partial}$, if one writes

$$\partial = \pi^{(p+1, q)} \circ d \quad (2.13)$$

and

$$\bar{\partial} = \pi^{(p, q+1)} \circ d \quad (2.14)$$

on (p, q) -forms.

This implies that for each $(0, 1)$ -form ω , $\pi^{(2, 0)} d\omega = 0$. For a general almost complex structure J , it need no longer be true that $d = \partial + \bar{\partial}$. For each $x \in M$, one has the antisymmetric bilinear mapping $[J, J]_x$ from $T_x M \times T_x M$ to $T_x M$, which is defined by

$$[J, J](v, w) = [Jv, Jw] - J[Jv, w] - J[v, Jw] - [v, w], \quad (2.15)$$

for any vector fields v and w in M . Using the formula

$$(d\omega)(v, w) = v\omega(w) - w\omega(v) - \omega([v, w]), \quad (2.16)$$

one gets for each $(0, 1)$ -form ω that

$$d\omega(v - iJv, w - iJw) = \omega([J, J](v, w)).$$

So the condition that $\pi^{(2,0)} d\omega = 0$ for every $(0, 1)$ -form ω is equivalent to the condition that $[J, J] = 0$. The *theorem of Newlander and Nirenberg* now says that an almost complex manifold (M, J) is complex analytic if and only if $[J, J] = 0$. This theorem is already valid if the first order derivatives of J are Hölder-continuous. Cf. Newlander and Nirenberg [62], Hörmander [41], Malgrange [55].

We continue the discussion of complex analytic manifolds. Identifying the types in $0 = d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$ on $\Omega^{(p,q)}$ -forms, one sees that $\partial^2 = 0$, $\partial\bar{\partial} + \bar{\partial}\partial = 0$, and $\bar{\partial}^2 = 0$. In particular, the operator $\bar{\partial}$ defines a complex

$$0 \rightarrow \Omega^{(p,0)} \xrightarrow{\bar{\partial}} \Omega^{(p,1)} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{(p,n)} \rightarrow 0, \quad (2.17)$$

called the *Dolbeault complex*. On the sheaves of locally defined forms, this sequence is exact, and one gets the theorem of Dolbeault that

$$\ker \bar{\partial}^{(p,q)} / \text{range } \bar{\partial}^{(p,q-1)} \simeq H^q(M, \mathcal{O}(\Omega^{(p,0)})), \quad (2.18)$$

where the right hand side denotes the q -th cohomology group of the sheaf $\mathcal{O}(\Omega^{(p,0)})$ of holomorphic $(p, 0)$ -forms over M . See for instance Griffiths and Harris [33, p. 45]. If M is compact, then the ellipticity of the complex yields that the spaces in the left hand side are finite-dimensional. We will mainly be interested in the case that $p = 0$.

A holomorphic vector bundle L over M is defined as a complex vector bundle over M for which the retriualizations are given by elements of $\text{GL}(l, \mathbb{C})$ which depend holomorphically on the base point. (Here $l = \dim_{\mathbb{C}} L_x$.) All the above remains valid for L -valued (p, q) -forms, that is, the sections of the vector bundle

$$T_x^* M^{(p,q)} \otimes L_x, \quad x \in M.$$

In particular, we have the “twisted Dolbeault complex” defined by

$$\bar{\partial}^{(0,q)} : E^q \otimes L \rightarrow E^{q+1} \otimes L,$$

and the corresponding cohomology groups

$$\ker \bar{\partial}^{(0,q)} / \text{range } \bar{\partial}^{(0,q-1)} \simeq H^q(M, \mathcal{O}(L)), \quad (2.19)$$

the q -th cohomology group of the sheaf $\mathcal{O}(L)$ of holomorphic sections of L over M .

If M is a compact complex analytic manifold, then an important quantity is the *Riemann-Roch number*

$$\text{RR}(M, L) := \sum_{q=0}^n (-1)^q \dim_{\mathbf{C}} H^q(M, \mathcal{O}(L)). \quad (2.20)$$

More generally, if γ is a complex analytic automorphism of L , then γ acts on $\mathcal{O}(L)$, and one can define its *holomorphic Lefschetz number*

$$\chi(\gamma) = \chi_{M,L}(\gamma) := \sum_{q=0}^n (-1)^q \text{trace}_{\mathbf{C}} \gamma|_{H^q(M, \mathcal{O}(L))}. \quad (2.21)$$

This is a generalization because $\chi_{M,L}(1) = \text{RR}(M, L)$.

If L is a holomorphic complex line bundle over M for which $K^* \otimes L$ is positive, then Kodaira’s vanishing theorem says that $H^q(M, \mathcal{O}(L)) = 0$ for every $q > 0$, cf. (6.34). If the latter is the case, the holomorphic Lefschetz number is equal to the trace of the action of γ on the space $H^0(M, \mathcal{O}(L))$ of all holomorphic sections of L over M , and the Riemann-Roch number is equal to the dimension of that space.

2.2 The Dolbeault-Dirac Operator

In order to define adjoints, we now introduce Hermitian structures h and h^L in the tangent bundle TM of M and the fibers of L , respectively. For each $x \in M$, h_x is a complex-valued bilinear form on $T_x M$, such that

$$h_x(v, v) > 0 \text{ if } v \in T_x M, v \neq 0, \quad (2.22)$$

and

$$h_x(J_x(v), w) = i h_x(v, w) = -h_x(v, J_x(w)), \quad v, w \in T_x M. \quad (2.23)$$

Similarly with h_x and $T_x M$ replaced by h_x^L and L_x , respectively.

It follows that the real part $\beta = \operatorname{Re} h$ of h is a Riemannian structure in M , and J_x is antisymmetric with respect to β_x . Furthermore, the imaginary part $\sigma = \operatorname{Im} h$ is a nowhere degenerate two-form in M , and J_x is infinitesimally symplectic for σ_x . Finally,

$$\beta(v, w) = \sigma(Jv, w) \quad (2.24)$$

shows that choosing two of the three structures J, β, σ determines the third.

In order to get a Hermitian structure in E , we begin by observing that

$$h_x : v \mapsto (w \mapsto h_x(v, w)) : T_x M \rightarrow T_x^* M^{(0,1)} \quad (2.25)$$

is a complex linear isomorphism from $T_x M$ onto $T_x^* M^{(0,1)}$. Using this isomorphism, we transplant the Hermitian structure of TM to a Hermitian structure $h^{(0,1)}$ in $T^* M^{(0,1)}$. That is, $h^{(0,1)}$ is determined by the condition that, if $e_j, 1 \leq j \leq n$, is a unitary local frame in TM for h , then $\epsilon_j := h e_j$ forms a unitary local frame in $T^* M^{(0,1)}$ for $h^{(0,1)}$. It is dual to the frame e_j , in the sense that $\langle e_j, \epsilon_k \rangle = \delta_{jk}$.

The Hermitian structure $h^{(0,q)}$ on $E^q = T^* M^{(0,q)}$ can now be defined by the condition that the ϵ_K form a unitary local frame in E^q , if for each strictly increasing sequence $K = (k_i)_{i=1}^q$ we write

$$\epsilon_K := \epsilon_{k_1} \wedge \epsilon_{k_2} \wedge \dots \wedge \epsilon_{k_q}. \quad (2.26)$$

The Hermitian structure h_E on the direct sum E of the E^q is defined by requiring the summands to be mutually orthogonal. And the Hermitian structure $h_{E \otimes L}$ on $E \otimes L$ by the condition that if e_j and l_k are unitary local frames in E and L , respectively, then the $e_j \otimes l_k$ form a unitary local frame in $E \otimes L$.

The Hermitian L^2 -inner product of two sections u and v of $E \otimes L$ is now defined as

$$(u, v) = \int_M h_{E \otimes L}(u, v) \, dx, \quad (2.27)$$

where dx denotes the standard volume form in M , such that

$$dx(e_1, J e_1, e_2, J e_2, \dots, e_n, J e_n) = 1. \quad (2.28)$$

(Declaring dx to be positive determines the orientation of M .)

If $D : \Gamma(M, E \otimes L) \rightarrow \Gamma(M, E \otimes L)$ is a differential operator, then the (formal) *adjoint* D^* of D is defined by

$$(D(u), v) = (u, D^*(v)), \quad u, v \in \Gamma(M, E \otimes L), \quad (2.29)$$

where one of the u, v is compactly supported. The adjective “formal” is added, if one wants to stress that no attention is being paid to questions of L^2 -closure.

In complex local coordinates, and a local holomorphic trivialization of L , we get for a constant $\mathbf{C}^l$ -valued $(0, q)$ -form ω and real-valued linear form ξ on $\mathbf{C}^n \simeq \mathbf{R}^{2n}$:

$$e^{-i\langle z, \xi \rangle} \bar{\partial} \left(e^{i\langle z, \xi \rangle} \omega \right) = i \xi^{(0,1)} \wedge \omega.$$

Because of its frequent occurrence, we will use the abbreviation

$$e(\alpha) : \omega \mapsto \alpha \wedge \omega : E_x^q \otimes L_x \rightarrow E_x^{q+1} \otimes L_x \quad (2.30)$$

for the linear mapping of *taking the exterior product with* $\alpha \in T_x^* M^{(0,1)}$ *from the left*. Using the invariance properties of principal symbols, we get that the principal symbol of the operator $\bar{\partial}$ at $\xi \in T_x^* M$ is equal to

$$\sigma_{\bar{\partial}}(\xi) = i e \left(\xi^{(0,1)} \right). \quad (2.31)$$

The principal symbol at $\xi \in T_x^* M$ of the adjoint operator

$$\bar{\partial}^* : \Gamma(M, E^{q+1} \otimes L) \rightarrow \Gamma(M, E^q \otimes L) \quad (2.32)$$

is equal to the adjoint of (2.31) with respect to the given Hermitian structures. In order to compute this, we observe that for every r and every pair of strictly increasing sequences $J = (j_i)_{i=1}^q$, $K = (k_i)_{i=1}^{q+1}$, we have $h_x(\epsilon_r \wedge \epsilon_J, \epsilon_K) = 0$, unless $|K| = \{r\} \cup |J|$, and in this case the inner product is equal to $(-1)^s$, where s is equal to the number of i such that $j_i < r$. Since the same answer is obtained for $h_x(\epsilon_J, i(e_r) \epsilon_K)$, we conclude that the adjoint of $e(\epsilon_r)$, the operator of taking exterior product with ϵ_r from the left, is equal to $i(e_r)$, the operator of *taking inner product with e_r* . Writing $\xi_r = \langle e_r, \xi \rangle$, $\eta_r = \langle J e_r, \xi \rangle$, we get

$$\xi^{(0,1)} = \sum_{r=1}^n \frac{1}{2} (\xi_r + i \eta_r) \epsilon_r. \quad (2.33)$$

Moving over every ϵ_r -term, and using that h_x is complex linear in the first variable and complex antilinear in the second variable, we get

$$\sigma_{\bar{\partial}^*}(\xi) \omega = -\frac{i}{2} \sum_{r=1}^n (\xi_r - i \eta_r) i(e_r) \omega.$$

Since the antilinearity of ω yields that

$$i(J e_r) \omega = -i i(e_r) \omega,$$

the result is ¹

$$\sigma_{\bar{\partial}^*}(\xi) = -\frac{i}{2} i(\beta^{-1} \xi) : E_x^{q+1} \otimes L_x \rightarrow E_x^q \otimes L_x. \quad (2.34)$$

We now consider the operator $D = 2(\bar{\partial} + \bar{\partial}^*)$, acting on $\Gamma(M, E \otimes L)$, with principal symbol equal to $2(\sigma_{\bar{\partial}} + \sigma_{\bar{\partial}^*})$. Here the factor two has been inserted in order to cancel the halves in (2.31), cf. (2.33), and in (2.34).

¹In [9, p. 138], [32, p. 184] and [33, p. 80] the more customary convention is used of taking the Hermitian inner product in E_q , such that $h_x(\epsilon_K, \epsilon_K) = 2^q$ for $\epsilon_K \in E^q$. As a consequence, (2.34) is equal to 1/2 times the formula for the symbol of $\bar{\partial}^*$, obtained in [32, p. 184] and [9, p. 138].

For any one-form α and vector v we have $e(\alpha)^2 = 0$, $i(v)^2 = 0$, and

$$i(v)(\alpha \wedge \omega) = \langle v, \alpha \rangle \omega - \alpha \wedge (i(v)\omega),$$

and therefore

$$(e(\alpha) + i(v))^2 = \langle v, \alpha \rangle.$$

It follows now from (2.31) and (2.34), that

$$\sigma_{D^2}(\xi) = \sigma_D(\xi)^2 = \langle \beta_x^{-1} \xi, \xi \rangle. \quad (2.35)$$

It is quite remarkable that the square of the principal symbol is a scalar, because the principal symbol itself is a linear transformation in $E_x \otimes L_x$, which is far from diagonal. It actually has no diagonal terms at all because it maps $E_x^q \otimes L_x$ to the sum of $E_x^{q+1} \otimes L_x$ and $E_x^{q-1} \otimes L_x$. In [9], an operator D which satisfies (2.35) is called a *generalized Dirac operator*. Since the generalized Dirac operator $2(\bar{\partial} + \bar{\partial}^*)$ is defined in terms of the operator $\bar{\partial}$, which appears in the Dolbeault complex (2.17), I got into the habit of calling it the *Dolbeault-Dirac operator*.

From (2.35) we see that if $\xi \in T_x M$ (real) and $\xi \neq 0$, then $\sigma_D(\xi)^2$ is invertible, and hence $\sigma_D(\xi)$ is invertible as well. That is, the operator D is elliptic. If M is compact, then this implies that the kernel of D is finite-dimensional and the range of D is equal to the orthogonal complement of the kernel of $D^* = D$. However $\bar{\partial}^2 = 0$ implies that $(\bar{\partial}^*)^2 = 0$ and hence

$$\frac{1}{4}D^2 = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

or

$$\frac{1}{4}(Du, Du) = (\bar{\partial}u, \bar{\partial}u) + (\bar{\partial}^*u, \bar{\partial}^*u).$$

So $Du = 0$ is equivalent to $\bar{\partial}u = 0$ and $\bar{\partial}^*u = 0$. Since the latter equation means that u is in the orthogonal complement of the range of $\bar{\partial}$, we get from the Dolbeault theorem (2.18) that

$$\ker D|_{\Gamma(M, E^q \otimes L)} \simeq H^q(M, \mathcal{O}(L)). \quad (2.36)$$

This allows us to translate (2.20) and (2.21) in terms of the kernel of $D = 2(\bar{\partial} + \bar{\partial}^*)$. Note that D maps sections of $E^q \otimes L$ to sections of $(E^{q+1} \oplus E^{q-1}) \otimes L$, so we have the operators

$$D^+ = D|_{\Gamma(M, E^+ \otimes L)} : \Gamma(M, E^+ \otimes L) \rightarrow \Gamma(M, E^- \otimes L) \quad (2.37)$$

and

$$D^- = D|_{\Gamma(M, E^- \otimes L)} : \Gamma(M, E^- \otimes L) \rightarrow \Gamma(M, E^+ \otimes L). \quad (2.38)$$

Collecting the terms with the same sign in (2.20), we get

$$\begin{aligned} \text{RR}(M, L) &= \dim_{\mathbf{C}} \ker D^+ - \dim_{\mathbf{C}} \ker D^- \\ &= \dim_{\mathbf{C}} \ker D^+ - \dim_{\mathbf{C}} \ker (D^+)^* = \text{index } D^+, \end{aligned} \quad (2.39)$$

the *index* of the operator D^+ . And doing the same in (2.21):

$$\chi(t) = \text{trace}_{\mathbf{C}} \gamma|_{\ker D^+} - \text{trace}_{\mathbf{C}} \gamma|_{\ker D^-}. \quad (2.40)$$



<http://www.springer.com/978-0-8176-8246-0>

The Heat Kernel Lefschetz Fixed Point Formula for the
Spin-c Dirac Operator

Duistermaat, J.J.

2011, VIII, 247 p., Softcover

ISBN: 978-0-8176-8246-0

A product of Birkhäuser Basel