Chapter 2
Infinite Products and Elementary Functions

The objective in this chapter is to lay out a working background for dealing with infinite products and their possible applications. The reader will be familiarized with a specific topic that is not often included in traditional texts on related courses of mathematical analysis, namely the infinite product representation of elementary functions.

It is known [9] that the theory of some special functions is, to a certain extent, linked to infinite products. In this regard, one might recall, for example, the elliptic integrals, gamma function, Riemann's zeta function, and others. But note that special functions are not targeted in this book at all. Our scope is limited exclusively to the use of infinite products for the representation of elementary functions.

We will recall and discuss those infinite product representations of elementary functions that are available in the current literature. Note that they have been derived by different methods, but the number of them is limited. In Sect. 2.1, Euler's classical derivation procedure will be analyzed. His elegant elaborations in this field were directed toward the derivation of infinite product representations for trigonometric as well as the hyperbolic sine and cosine functions. The work of Euler on infinite products was inspirational [26] for many generations of mathematicians. It will be frequently referred to in this brief volume as well.

Some alternative derivation techniques proposed for infinite product representations of trigonometric functions will be reviewed in detail in Sect. 2.2. The closing Sect. 2.3 brings to the reader's attention a variety of possible techniques for the derivation of infinite product forms of other elementary functions. We will instruct the reader on how to obtain the infinite product representations of elementary functions that are available in standard texts and handbooks.

2.1 Euler's Classical Representations

Both infinite series and infinite products could potentially be helpful in the area of approximation of functions. Infinite series represent a traditional instrument in contemporary mathematics. One of its classical implementations is the representation of

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functions, which is applicable to different areas of mathematical analysis. Approximation of functions and numerical differentiation and integration can be pointed out as some, but not the only, such areas. Although infinite products have also been known and developed for centuries [26], and can potentially be used in solving a variety of mathematical problems, the range of their known implementations is not as broad as that of infinite series.

The focus in the present volume is on just one of many possible implementations of infinite products, namely the representation of elementary functions. Pioneering results in this field were obtained over two hundred fifty years ago. They are associated with the name of one of the most prominent mathematicians of all time, Leonhard Euler. According to historians [26], his mind had been preoccupied with this topic for quite a long span of time. And it took him nearly ten years to ultimately derive the following now classical representation for the trigonometric sine function:

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right). \tag{2.1}$$

We will analyze in this section the derivation procedure proposed by Euler and also review, in further sections, some other procedures proposed later for the derivation of the representation in (2.1). Euler also showed that his procedure appears effective for the trigonometric cosine function and derived the following infinite product representation:

$$\cos x = \prod_{k=1}^{\infty} \left(1 - \frac{4x^2}{(2k-1)^2\pi^2}\right). \tag{2.2}$$

It is evident from the classical relations

$$\sin iz = i \sinh z \quad \text{and} \quad \cos iz = \cosh z$$

between the trigonometric and hyperbolic functions, which represent the analytic continuation of the trigonometric functions into the complex plane, that the infinite product representations

$$\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right) \tag{2.3}$$

and

$$\cosh x = \prod_{k=1}^{\infty} \left(1 + \frac{4x^2}{(2k-1)^2\pi^2}\right) \tag{2.4}$$

for the hyperbolic sine and cosine functions directly follow from (2.1) and (2.2), respectively.

As we will show later, Euler's direct approach can be successfully applied to the derivation of the representations in (2.3) and (2.4).

To let the reader enjoy the elegance of the approach, we will consider first the case of the representation in (2.1) and follow it in some detail. In doing so, we write down the trigonometric sine function, using Euler's formula, in the exponential form

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

and replace the exponential functions with their limit expressions reducing the above to

$$\begin{aligned}\sin x &= \frac{1}{2i} \lim_{n\to\infty} \left[\left(1 + \frac{ix}{n}\right)^n - \left(1 - \frac{ix}{n}\right)^n \right] \\ &= -\frac{i}{2} \lim_{n\to\infty} \left[\left(1 + \frac{ix}{n}\right)^n - \left(1 - \frac{ix}{n}\right)^n \right]. \end{aligned} \tag{2.5}$$

We then apply Newton's binomial formula to both polynomials in the brackets. This yields

$$\left(1 + \frac{ix}{n}\right)^n = 1 + n\frac{ix}{n} + \frac{n(n-1)}{2!}\left(\frac{ix}{n}\right)^2 + \cdots = \sum_{k=0}^{n} \binom{k}{n} \left(\frac{ix}{n}\right)^k \tag{2.6}$$

and

$$\left(1 - \frac{ix}{n}\right)^n = 1 - n\frac{ix}{n} + \frac{n(n-1)}{2!}\left(\frac{ix}{n}\right)^2 - \cdots = \sum_{k=0}^{n} (-1)^k \binom{k}{n} \left(\frac{ix}{n}\right)^k. \tag{2.7}$$

Once these expressions are substituted into (2.5), all the real terms in the brackets (the terms in even powers of x) cancel out. As soon as the common factor of $2ix$ is factored out in the remaining odd-power terms of x, the right-hand side of (2.5) reduces to a compact form, and we have

$$\sin x = x \lim_{n\to\infty} \sum_{k=0}^{(n-1)/2} (-1)^k \binom{2k+1}{n} \frac{x^{2k}}{n^{2k+1}}. \tag{2.8}$$

Of all the stages in Euler's procedure, which, as a whole, represents a real work of art, the next stage is perhaps the most critical and decisive. Factoring the polynomial in (2.8) into the trigonometric form

$$\sin x = x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left[1 - \frac{(1 + \cos 2k\pi/n)\, x^2}{(1 - \cos 2k\pi/n)\, n^2} \right],$$

after trivial trigonometric transformations, we obtain

$$\begin{aligned}\sin x &= x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 - \frac{x^2 \cos^2 k\pi/n}{n^2 \sin^2 k\pi/n} \right) \\ &= x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 - \frac{x^2}{n^2 \tan^2 k\pi/n} \right). \end{aligned}$$

To take the limit, the second additive term in the parentheses of the above finite product is multiplied and divided by the factor $k^2\pi^2$. This yields

$$\sin x = x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 - \frac{x^2 k^2 \pi^2}{n^2 k^2 \pi^2 \tan^2 k\pi/n}\right)$$
$$= x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left[1 - \frac{x^2}{k^2\pi^2}\left(\frac{k\pi/n}{\tan k\pi/n}\right)^2\right],$$

which can be written, on account of the standard limit

$$\lim_{\vartheta\to 0} \frac{\vartheta}{\tan\vartheta} = 1,$$

as the classical Euler representation

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right).$$

An interesting observation can be drawn from a comparison of the above infinite product form with the classical Maclaurin series expansion

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

of the sine function. These two forms share a common feature and are different at the same time. As to the common feature, both, the partial products of Euler's infinite product representation and partial sums of the Maclaurin expansion are odd-degree polynomials. But what makes the two forms different is that the partial products of Euler's representation are somewhat more relevant to the sine function. That is, they share same zeros $x_k = k\pi$ with the original sine function, whereas the Maclaurin expansion does not. It is evident that this property of the infinite product representation could be essential in applications.

To examine the convergence pattern of Euler's representation and compare it to that of Maclaurin's series, the reader is invited to take a close look at Figs. 2.1 and 2.2. Sequences of the Euler partial products and Maclaurin partial sums are depicted, illustrating the difference between the two formulations.

As to the derivation of the infinite product representation of the trigonometric cosine function, which was shown in (2.2), we diligently follow the procedure just described for the sine function. That is, after using Euler's formula

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

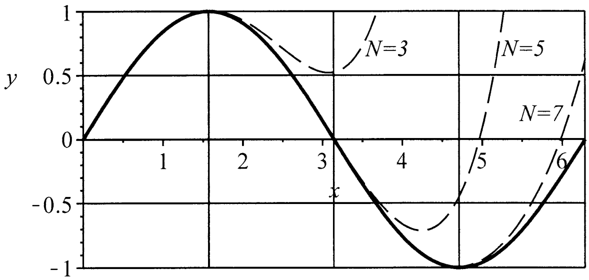

Fig. 2.1 Convergence of the series expansion for $\sin x$

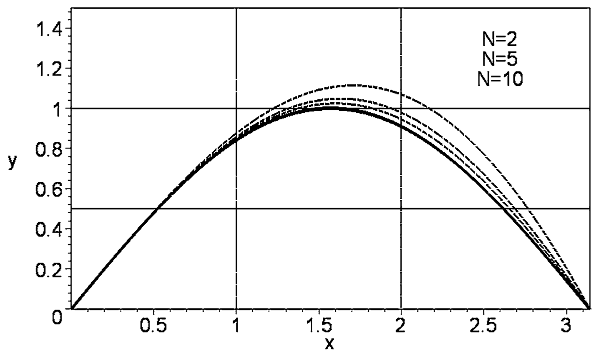

Fig. 2.2 Convergence of the product expansion for $\sin x$

and expressing the exponential functions in the limit form

$$\cos x = \frac{1}{2} \lim_{n\to\infty} \left[\left(1 + \frac{ix}{n}\right)^n + \left(1 - \frac{ix}{n}\right)^n \right],$$

we substitute the Newtonian polynomials from (2.6) and (2.7) into the right-hand side of the above relation. It can readily be seen that, in contrast to the case of the sine function, all the odd-power terms cancel out; and we subsequently arrive at the following even-degree polynomial-containing representation

$$\cos x = \lim_{n\to\infty} \sum_{k=0}^{(n-1)/2} (-1)^k \binom{2k}{n} \frac{x^{2k}}{n^{2k}}$$

for the cosine function. The polynomial under the limit sign can be factored in a similar way as in (2.8). In this case, we obtain

$$\cos x = \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left\{ 1 - \frac{[1 + \cos(2k-1)\pi/n]\, x^2}{[1 - \cos(2k-1)\pi/n]\, n^2} \right\},$$

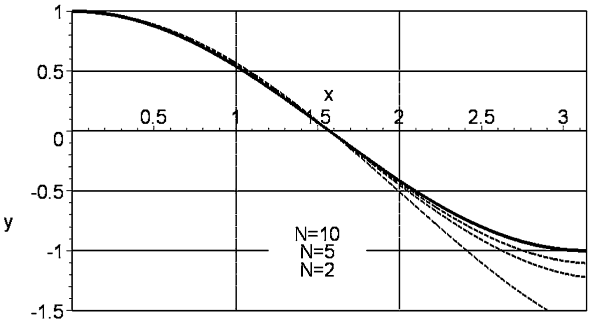

Fig. 2.3 Convergence of the product expansion for $\cos x$

which, after a trivial trigonometric transformation, becomes

$$\cos x = \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 - \frac{x^2 \cos^2(2k-1)\pi/2n}{n^2 \sin^2(2k-1)\pi/2n}\right)$$
$$= \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 - \frac{x^2}{n^2 \tan^2(2k-1)\pi/2n}\right).$$

Similarly to the case of the sine function, we take the limit in the above relation, which requires some additional algebra. That is, the second additive term in the parentheses of the finite product is multiplied and divided by $(2k-1)^2\pi^2/4n^2$. This yields

$$\cos x = \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 - \frac{4x^2(2k-1)^2\pi^2}{4n^2(2k-1)^2\pi^2 \tan^2(2k-1)\pi/2n}\right),$$

which immediately transforms into

$$\cos x = \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left[1 - \frac{4x^2}{(2k-1)^2\pi^2}\left(\frac{(2k-1)\pi/2n}{\tan(2k-1)\pi/2n}\right)^2\right].$$

The latter, in turn, reads ultimately as the classical Euler expansion for the cosine shown in (2.2):

$$\cos x = \prod_{k=1}^{\infty} \left(1 - \frac{4x^2}{(2k-1)^2\pi^2}\right).$$

Note that, similarly to the case of the sine function, the above infinite product representation also shares the zeros $x_k = (2k-1)\pi/2$ with the original cosine function.

The convergence pattern of the above infinite product representation can be observed in Fig. 2.3.

We turn now to the case of the hyperbolic sine function whose expansion is presented in (2.3). Its derivation can be conducted in a manner similar to that for the trigonometric sine. Indeed, representing the hyperbolic sine function with Euler's formula

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

one customarily expresses both the exponential functions in the limit form. This results in

$$\sinh x = \frac{1}{2} \lim_{n\to\infty} \left[\left(1 + \frac{x}{n}\right)^n - \left(1 - \frac{x}{n}\right)^n \right]. \tag{2.9}$$

Once the Newton binomial formula is used for both polynomials in the brackets, one obtains

$$\left(1 + \frac{x}{n}\right)^n = 1 + n\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \ldots = \sum_{k=0}^{n} \binom{k}{n} \left(\frac{x}{n}\right)^k$$

and

$$\left(1 - \frac{x}{n}\right)^n = 1 - n\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} - \ldots = \sum_{k=0}^{n} (-1)^k \binom{k}{n} \left(\frac{x}{n}\right)^k.$$

As in the derivation of the trigonometric sine function, all the even-power terms in x in (2.9) cancel out, while the remaining odd-power terms possess a common factor of $2x$. Once the latter is factored out, the expression in (2.9) simplifies to the compact form

$$\sinh x = x \lim_{n\to\infty} \sum_{k=0}^{(n-1)/2} \binom{2k+1}{n} \frac{x^{2k}}{n^{2k+1}},$$

which factors as

$$\sinh x = x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left[1 + \frac{x^2(1 + \cos 2k\pi/n)}{n^2(1 - \cos 2k\pi/n)}\right].$$

Elementary trigonometric transformations yield

$$\begin{aligned}\sinh x &= x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 + \frac{x^2 \cos^2 k\pi/n}{n^2 \sin^2 k\pi/n}\right) \\ &= x \lim_{n\to\infty} \prod_{k=1}^{(n-1)/2} \left(1 + \frac{x^2}{n^2 \tan^2 k\pi/n}\right).\end{aligned}$$

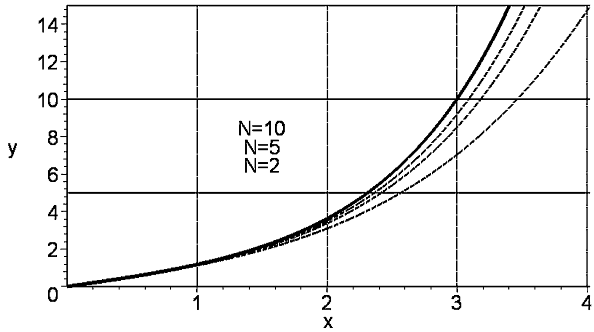

Fig. 2.4 Convergence of the product expansion for $\sinh x$

Taking the limit as in the case of the trigonometric sine function, one ultimately transforms the above relation into the classical Euler form in (2.3):

$$\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right).$$

The convergence pattern of the above product representation can be observed in Fig. 2.4.

As to the derivation procedure for the case of the hyperbolic cosine function, we will not go through its specifics, because it can be accomplished in exactly the same way as that for the trigonometric cosine. To better understand the peculiarities of the procedure, the reader is, however, urgently recommended to carefully pass through its details.

It is worth noting that since Euler there have been proposed various procedures for the derivation of the infinite product representations of the trigonometric and hyperbolic functions. In the next section, we plan to review some of those procedures.

Over a dozen infinite product representations of elementary functions are available in current handbooks (see, for example, [9]). The present volume reviews them in detail and describes, in addition, an interesting approach to the problem based on the construction of Green's functions for the two-dimensional Laplace equation. This results, in particular, in infinite product representations [28] alternative to those in (2.1) and (2.2) for the trigonometric sine and cosine functions. A number of otherwise unavailable infinite product representations will also be derived for some other trigonometric and hyperbolic functions.

2.2 Alternative Derivations

In all fairness, Euler's derivation of the infinite product representations of the trigonometric (hyperbolic) sine and cosine functions, which were reviewed in Sect. 2.1, must be referred to as *classical*. This assertion is unreservedly justified by the chronology. Indeed, Euler was the first to propose his derivation.

The reader will later be exposed to an unusual approach to the representation of elementary functions by infinite products, which was proposed by the author. This approach had resulted [28] in novel representations for many elementary functions. But before going any further into the details of that approach, let us revisit the classical Euler representation of the trigonometric sine function, and proceed through some of its other derivations that are well known and can readily be found in the classical literature [5] on the subject.

The first of those derivations can be handled with DeMoivre's formula [5] for a complex number in trigonometric form. It will be written down here for its odd $(2n+1)$ exponent:

$$(\cos w + i \sin w)^{2n+1} = \cos(2n+1)w + i \sin(2n+1)w. \tag{2.10}$$

On the other hand, using the binomial formula, the left-hand side of the above can be expanded as

$$\begin{aligned}(\cos w + i \sin w)^{2n+1} = {} & \cos^{2n+1} w + i(2n+1)\cos^{2n} w \sin w \\ & - \binom{2n+1}{2} \cos^{2n-1} w \sin^2 w \\ & - i\binom{2n+1}{3} \cos^{2n-2} w \sin^3 w \\ & + \cdots + (-1)^n \sin^{2n+1} w.\end{aligned} \tag{2.11}$$

Equating the imaginary parts of the left-hand sides in (2.10) and (2.11), we obtain

$$\begin{aligned}\sin(2n+1)w &= (2n+1)\cos^{2n} w \sin w - \binom{2n+1}{3} \cos^{2n-2} w \sin^3 w \\ &\quad + \cdots + (-1)^n \sin^{2n+1} w \\ &= \sin w \Bigg[(2n+1)\cos^{2n} w \\ &\quad - \binom{2n+1}{3} \cos^{2n-2} w \sin^2 w + \cdots + (-1)^n \sin^{2n} w \Bigg].\end{aligned} \tag{2.12}$$

Since the second factor (the one in the brackets) contains only even exponents of the sine and cosine functions, it can be represented as a polynomial $P_n(\sin^2 w)$, where the degree of $\sin^2 x$ never exceeds n. On the other hand, for any fixed value of n, the left-hand side of (2.12) takes on the value zero at the n points $w_k = k\pi/(2n+1)$, $k = 1, 2, 3, \ldots, n$, on the open segment $(0, \pi/2)$. This implies that the zeros of $P_n(s)$ are the values $s_k = \sin w_k$, allowing the polynomial to be expressed as

$$P_n(s) = \beta \prod_{k=1}^{n} \left(1 - \frac{s}{\sin^2 w_k}\right), \tag{2.13}$$

where the factor β is yet to be determined. In going through its determination, we can rewrite the relation in (2.12), in light of (2.13), in the following compact form

$$\frac{\sin(2n+1)w}{\sin w} = \beta \prod_{k=1}^{n} \left[1 - \left(\frac{\sin w}{\sin w_k} \right)^2 \right] \tag{2.14}$$

in terms of w_k and take the limit as w approaches zero:

$$\lim_{w \to 0} \frac{\sin(2n+1)w}{\sin w} = \beta \lim_{w \to 0} \prod_{k=1}^{n} \left[1 - \left(\frac{\sin w}{\sin w_k} \right)^2 \right].$$

The limit on the left-hand side of the above is $2n+1$, while the limit on the right-hand side is equal to 1. This suggests for the factor β the value $2n+1$, and the relation in (2.14) transforms into

$$\sin(2n+1)w = (2n+1)\sin w \prod_{k=1}^{n} \left\{ 1 - \left[\frac{\sin w}{\sin(k\pi/(2n+1))} \right]^2 \right\}. \tag{2.15}$$

Substituting $x = (2n+1)w$, we rewrite (2.15) as

$$\sin x = (2n+1)\sin \frac{x}{2n+1} \prod_{k=1}^{n} \left\{ 1 - \left[\frac{\sin(x/(2n+1))}{\sin(k\pi/(2n+1))} \right]^2 \right\}. \tag{2.16}$$

Since

$$\lim_{n \to \infty} \left[(2n+1)\sin \frac{x}{2n+1} \right] = x,$$

while

$$\lim_{n \to \infty} \frac{\sin(x/(2n+1))}{\sin(k\pi/(2n+1))} = \frac{x}{k\pi},$$

the relation in (2.16) transforms, as n approaches infinity, into the classical Euler representation in (2.1):

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2} \right).$$

Clearly, the derivation procedure just reviewed is based on a totally different idea compared to that used by Euler. Recall another alternative derivation of the Euler representation of the sine function, which can be carried out using the Laurent series expansion [5]

$$\cot z - \frac{1}{z} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi} \right) \tag{2.17}$$

for the cotangent function of a complex variable. Note that the summation in (2.17) assumes that the $k = 0$ term is omitted.

Evidently, the opening terms of the above series have isolated singular points (poles) in any bounded region D of the complex plane. If, however, a few initial terms of the series in (2.17) are truncated, then the series becomes absolutely and uniformly convergent in a bounded region. This assertion can be justified by considering the general term

$$\frac{1}{z-k\pi}+\frac{1}{k\pi}=\frac{z}{k\pi(z-k\pi)}$$

of the series, for which the following estimate holds:

$$\left|\frac{z}{k\pi(z-k\pi)}\right|=\left|\frac{z}{k^2\pi(z/k-\pi)}\right|\le\frac{T}{\pi|T/k-\pi|}\cdot\frac{1}{k^2},$$

where T represents the upper bound of the modulus of the variable z, that is, $|z|<T$.

It can be shown that the first factor on the right-hand side of the above inequality has the finite limit T/π^2 as k approaches infinity. Thus, the series in (2.17) converges (at the rate of $1/k^2$) absolutely and uniformly in any bounded region. In other words, both the left-hand side and the right-hand side in (2.17) are regular functions at $z=0$. This makes it possible for the series in (2.17) to be integrated term by term. Taking advantage of this fact, we integrate both sides in (2.17) along a path joining the origin $z=0$ to a point $z\in D$. This yields

$$\log\frac{\sin z}{z}\Bigg|_{z=0}^{z=z}=\sum_{k=-\infty}^{\infty}\left[\log(z-k\pi)+\frac{z}{k\pi}\right]_{z=0}^{z=z},$$

and after choosing the branch of the logarithm that vanishes at the origin, we obtain

$$\begin{aligned}\log\frac{\sin z}{z}&=\sum_{k=-\infty}^{\infty}\left[\log\left(1-\frac{z}{k\pi}\right)+\frac{z}{k\pi}\right]\\&=\sum_{k=-\infty}^{\infty}\log\left(\left(1-\frac{z}{k\pi}\right)\exp\frac{z}{k\pi}\right)\\&=\log\left(\prod_{k=-\infty}^{\infty}\left(1-\frac{z}{k\pi}\right)\exp\frac{z}{k\pi}\right).\end{aligned}\tag{2.18}$$

Exponentiating (2.18), we rewrite it as

$$\sin z=z\prod_{k=-\infty}^{\infty}\left(1-\frac{z}{k\pi}\right)\exp\frac{z}{k\pi}.\tag{2.19}$$

Recall that the factor $k=0$ is omitted in the above infinite product. Coupling then the kth factor

$$\left(1-\frac{z}{k\pi}\right)\exp\frac{z}{k\pi}$$

and the $-k$th factor

$$\left(1+\frac{z}{k\pi}\right)\exp\left(-\frac{z}{k\pi}\right)$$

in (2.19), we ultimately obtain the classical Euler representation of (2.1):

$$\sin z = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right).$$

So, two different derivations for the expansion in (2.1) have been reviewed in this section. They are alternative to the classical Euler procedure discussed in Sect. 2.1. This issue will be revisited again in Chap. 6, where yet another alternative derivation procedure for infinite product representations of elementary functions will be presented. It was recently proposed by the author and reported in [27, 28], and is based on a novel approach.

The objective in the next section is to introduce the reader to a limited number of infinite product representations of elementary functions that can be found in the current literature.

2.3 Other Elementary Functions

The classical Euler representations of the trigonometric and hyperbolic sine and cosine functions, whose derivation has been reproduced in this volume, could be employed in obtaining infinite product expansions for some other elementary functions. However, only a limited number of those expansions are available in the literature. All of them are listed in handbooks on the subject (see, for example, [6, 9]).

In this section, we are going to revisit the expressions for elementary functions in terms of infinite products available in literature and advise the reader on methods that could be applied for their derivation. In doing so, we begin with the representation

$$\cos x-\cos y = 2\left(1-\frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left[1-\frac{x^2}{(2k\pi+y)^2}\right]\left[1-\frac{x^2}{(2k\pi-y)^2}\right] \tag{2.20}$$

listed in [9] as #1.432(1). In order to derive it, the difference of cosines on the left-hand side of (2.20) can be converted to the product form

$$\cos x-\cos y = 2\sin\frac{y+x}{2}\sin\frac{y-x}{2}$$

and multiplied and divided then by the factor $\sin^2\frac{y}{2}$, yielding

$$\cos x-\cos y = 2\,\frac{\sin^2\frac{y}{2}}{\sin^2\frac{y}{2}}\sin\frac{y+x}{2}\sin\frac{y-x}{2}\,.$$

Leaving the $\sin^2 \frac{y}{2}$ factor in the numerator in its current form while expressing the other three sine factors with the aid of the classical Euler infinite product representation in (2.1), one obtains

$$\cos x - \cos y = \sin^2 \frac{y}{2} \frac{y^2 - x^2}{2} \prod_{k=1}^{\infty} \left(1 - \frac{(y+x)^2}{4k^2\pi^2}\right) \prod_{k=1}^{\infty} \left(1 - \frac{(y-x)^2}{4k^2\pi^2}\right)$$
$$\times \left[\frac{y}{2} \prod_{k=1}^{\infty} \left(1 - \frac{y^2}{4k^2\pi^2}\right)\right]^{-2},$$

which can be rewritten in a more compact form. To proceed with this, we combine all the three infinite products into a single product form. This yields

$$\cos x - \cos y = 2\,\frac{y^2 - x^2}{y^2} \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \frac{[1 - \frac{(y+x)^2}{4k^2\pi^2}][1 - \frac{(y-x)^2}{4k^2\pi^2}]}{(1 - \frac{y^2}{4k^2\pi^2})^2},$$

or, after performing elementary algebra on the expression under the product sign, we have

$$\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \frac{[4k^2\pi^2 - (x+y)^2][4k^2\pi^2 - (x-y)^2]}{(4k^2\pi^2 - y^2)^2}.$$

Upon factoring the differences of squares under the product sign, the above relation transforms into

$$\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \frac{[2k\pi + (x+y)][2k\pi - (x+y)]}{(2k\pi + y)^2}$$
$$\times \frac{[2k\pi + (x-y)][2k\pi - (x-y)]}{(2k\pi - y)^2}.$$

At this point, we regroup the numerator factors under the product sign. That is, we combine the first and fourth factors, as well as the second and third factors. This yields

$$\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \frac{[(2k\pi + y) + x][(2k\pi + y) - x]}{(2k\pi + y)^2}$$
$$\times \frac{[(2k\pi - y) - x][(2k\pi - y) + x]}{(2k\pi - y)^2},$$

reducing the above relation to the form

$$\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \frac{(2k\pi + y)^2 - x^2}{(2k\pi + y)^2} \frac{(2k\pi - y)^2 - x^2}{(2k\pi - y)^2}$$
$$= 2\left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left[1 - \frac{x^2}{(2k\pi + y)^2}\right]\left[1 - \frac{x^2}{(2k\pi - y)^2}\right].$$

This completes the derivation of the representation in (2.20).

A derivation procedure similar to that just described for the expansion in (2.20) can be employed for obtaining another infinite product expression of an elementary function. This is the representation

$$\cosh x - \cos y = 2\left(1+\frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left[1+\frac{x^2}{(2k\pi+y)^2}\right]\left[1+\frac{x^2}{(2k\pi-y)^2}\right], \tag{2.21}$$

which is also available in the existing literature (see #1.432(2) in [9]).

To put the derivation procedure for the relation in (2.21) on the effective track just used in the case of the representation in (2.20), we express the hyperbolic cosine function in terms of the trigonometric cosine,

$$\cosh x = \cos ix,$$

and simply trace out the procedure described earlier in detail for the case of (2.20):

$$\begin{aligned}
\cosh x - \cos y &= \cos ix - \cos y = 2\sin\frac{y+ix}{2}\sin\frac{y-ix}{2}\\
&= 2\frac{\sin^2\frac{y}{2}}{\sin^2\frac{y}{2}}\ \sin\frac{y+ix}{2}\sin\frac{y-ix}{2}\\
&= 2\sin^2\frac{y}{2}\cdot\frac{y+ix}{2}\prod_{k=1}^{\infty}\left[1-\frac{(y+ix)^2}{4k^2\pi^2}\right]\\
&\quad\times\frac{y-ix}{2}\prod_{k=1}^{\infty}\left[1-\frac{(y-ix)^2}{4k^2\pi^2}\right]\left[\frac{y^2}{4}\prod_{k=1}^{\infty}\left(1-\frac{y^2}{4k^2\pi^2}\right)^2\right]^{-1}.
\end{aligned}$$

Upon grouping all the infinite product factors, the above reads

$$2\,\frac{y^2+x^2}{y^2}\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\frac{[1-\frac{(y+ix)^2}{4k^2\pi^2}][1-\frac{(y-ix)^2}{4k^2\pi^2}]}{(1-\frac{y^2}{4k^2\pi^2})^2},$$

and transforms then as

$$\begin{aligned}
&2\left(1+\frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\frac{[4k^2\pi^2-(ix+y)^2][4k^2\pi^2-(ix-y)^2]}{(4k^2\pi^2-y^2)^2}\\
&= 2\left(1+\frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\frac{[2k\pi-(ix+y)][2k\pi+(ix+y)]}{(2k\pi+y)^2}\\
&\quad\times\frac{[2k\pi-(ix-y)][2k\pi+(ix-y)]}{(2k\pi-y)^2}
\end{aligned}$$

$$= \left(1+\frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\frac{(2k\pi+y)^2+x^2}{(2k\pi+y)^2}\frac{(2k\pi-y)^2+x^2}{(2k\pi-y)^2}$$

$$= 2\left(1+\frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left[1+\frac{x^2}{(2k\pi+y)^2}\right]\left[1+\frac{x^2}{(2k\pi-y)^2}\right].$$

We turn now to another infinite product representation of an elementary function that is available in the literature,

$$\cos\frac{\pi x}{4}-\sin\frac{\pi x}{4}=\prod_{k=1}^{\infty}\left[1+\frac{(-1)^k x}{2k-1}\right], \tag{2.22}$$

listed in [9], for example, as #1.433. This infinite product converges at the slow rate of $1/k$. We can offer two alternative expansions of the function

$$\cos\frac{\pi x}{4}-\sin\frac{\pi x}{4}$$

whose convergence rate is notably faster compared to that of (2.22). To derive the first such expansion, we convert the difference of trigonometric functions in (2.22) to a single cosine function. This can be done by multiplying and dividing it by a factor of $\sqrt{2}/2$:

$$\cos\frac{\pi x}{4}-\sin\frac{\pi x}{4}=\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\frac{\pi x}{4}-\frac{\sqrt{2}}{2}\sin\frac{\pi x}{4}\right)$$

$$=\sqrt{2}\left(\cos\frac{\pi}{4}\cos\frac{\pi x}{4}-\sin\frac{\pi}{4}\sin\frac{\pi x}{4}\right)=\sqrt{2}\cos\frac{\pi(1+x)}{4}.$$

Upon expressing the above cosine function by the classical Euler infinite product form in (2.2), the first alternative version of the expansion in (2.22) appears as

$$\cos\frac{\pi x}{4}-\sin\frac{\pi x}{4}=\sqrt{2}\cos\frac{\pi(1+x)}{4}=\sqrt{2}\prod_{k=1}^{\infty}\left[1-\frac{(1+x)^2}{4(2k-1)^2}\right]. \tag{2.23}$$

If in contrast to the derivation just completed, the left-hand side of (2.22) is similarly expressed as a single sine function

$$\cos\frac{\pi x}{4}-\sin\frac{\pi x}{4}=\sqrt{2}\sin\frac{\pi(1-x)}{4},$$

then one arrives, with the aid of the classical Euler infinite product form for the sine function in (2.1), at another alternative representation to that in (2.22),

$$\cos\frac{\pi x}{4}-\sin\frac{\pi x}{4}=\frac{\pi\sqrt{2}}{4}(1-x)\prod_{k=1}^{\infty}\left[1-\frac{(1-x)^2}{16k^2}\right]. \tag{2.24}$$

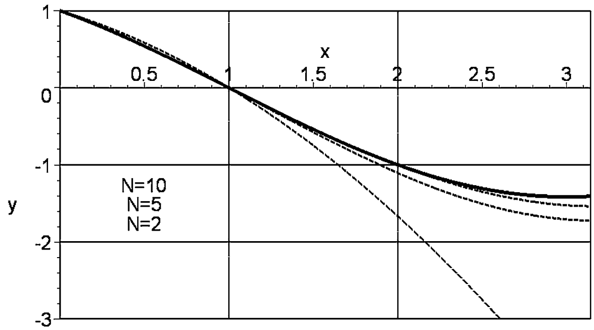

Fig. 2.5 Convergence of the representation in (2.22)

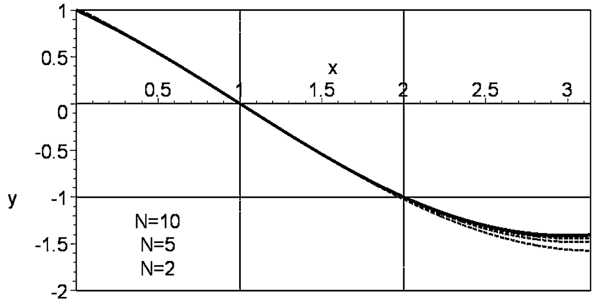

Fig. 2.6 Convergence of the representation in (2.22)

It is evident that the versions in (2.23) and (2.24) are more efficient computationally than that in (2.22). Indeed, they converge at the rate $1/k^2$, in contrast to the rate $1/k$ for the expansion in (2.22).

As to the representations in (2.23) and (2.24), it can be shown that the relative convergence of the latter must be slightly faster. This assertion directly follows from observation of the denominators in their fractional components. Indeed, the inequality

$$4(2k-1)^2 = 16k^2 - 16k + 4 < 16k^2$$

holds for any integer k, since $16k - 4 > 0$.

Relative convergence of the representations in (2.22) and (2.24) can be observed in Figs. 2.5 and 2.6, where their second, fifth, and tenth partial products are plotted on the interval $[0, \pi]$.

Derivation of the next infinite product representation of an elementary function, which is available in [9] (see #1.434),

$$\cos^2 x = \frac{1}{4}(\pi + 2x)^2 \prod_{k=1}^{\infty} \left[1 - \frac{(\pi + 2x)^2}{4k^2\pi^2} \right], \tag{2.25}$$

is as straightforward as it gets. Indeed, once the cosine function is converted to the sine form

$$\cos^2 x = \sin^2\left(\frac{\pi}{2} + x\right),$$

the implementation of the classical representation for the sine function in (2.1) completes the job.

At this point, we turn to another representation,

$$\frac{\sin \pi(x+a)}{\sin \pi a} = \frac{x+a}{a} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k-a}\right)\left(1 + \frac{x}{k+a}\right), \tag{2.26}$$

which is presented in [9] as #1.435. If the sine functions in the numerator and denominator are expressed in terms of the classical Euler form, then (2.26) reads

$$\frac{\sin \pi(x+a)}{\sin \pi a} = \frac{\pi(x+a) \prod_{k=1}^{\infty} [1 - \frac{\pi^2(x+a)^2}{k^2\pi^2}]}{\pi a \prod_{k=1}^{\infty} [1 - \frac{\pi^2 a^2}{k^2\pi^2}]}.$$

And upon performing a chain of straightforward transformations, the above representation converts ultimately into (2.26),

$$\begin{aligned}
\frac{\sin \pi(x+a)}{\sin \pi a} &= \frac{x+a}{a} \prod_{k=1}^{\infty} \frac{1 - \frac{(x+a)^2}{k^2}}{1 - \frac{a^2}{k^2}} \\
&= \frac{x+a}{a} \prod_{k=1}^{\infty} \frac{(1 - \frac{x+a}{k})(1 + \frac{x+a}{k})}{(1 - \frac{a}{k})(1 + \frac{a}{k})} \\
&= \frac{x+a}{a} \prod_{k=1}^{\infty} \frac{(k-a) - x}{(k-a)} \cdot \frac{(k+a) + x}{(k+a)} \\
&= \frac{x+a}{a} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k-a}\right)\left(1 + \frac{x}{k+a}\right).
\end{aligned}$$

For another infinite product representation of an elementary function available in the literature, we turn to

$$1 - \frac{\sin^2 \pi x}{\sin^2 \pi a} = \prod_{k=-\infty}^{\infty} \left[1 - \frac{x^2}{(k-a)^2}\right], \tag{2.27}$$

which is listed as #1.436 in [9].

To proceed with the derivation in this case, we convert the infinite product in (2.27) to an equivalent form. In doing so, we isolate the term with $k = 0$ (which is equal to $(1 - x^2/a^2)$) of the product, and group the kth and the $-k$th terms by pairs. This transforms the relation in (2.27) into

$$1-\frac{\sin^2\pi x}{\sin^2\pi a}=\left(1-\frac{x^2}{a^2}\right)\prod_{k=1}^{\infty}\left[1-\frac{x^2}{(k-a)^2}\right]\left[1-\frac{x^2}{(k+a)^2}\right]. \tag{2.28}$$

To verify the above identity, transform its left-hand side as

$$1-\frac{\sin^2\pi x}{\sin^2\pi a}=\frac{\sin^2\pi a-\sin^2\pi x}{\sin^2\pi a}$$

and decompose the numerator as a difference of squares:

$$\frac{\sin^2\pi a-\sin^2\pi x}{\sin^2\pi a}=\frac{(\sin\pi a-\sin\pi x)(\sin\pi a+\sin\pi x)}{\sin^2\pi a}. \tag{2.29}$$

At the next step, convert the difference and the sum of the sine functions in (2.29) to the product forms

$$\sin\pi a-\sin\pi x=2\,\sin\frac{\pi(a-x)}{2}\cos\frac{\pi(a+x)}{2}$$

and

$$\sin\pi a+\sin\pi x=2\,\sin\frac{\pi(a+x)}{2}\cos\frac{\pi(a-x)}{2}.$$

With this, we regroup the numerator in (2.29) as

$$2\sin\frac{\pi(a+x)}{2}\cos\frac{\pi(a+x)}{2}\cdot 2\sin\frac{\pi(a-x)}{2}\cos\frac{\pi(a-x)}{2},$$

where the first double product represents the sine function $\sin\pi(a+x)$, while the second double product is $\sin\pi(a-x)$. This finally transforms the left-hand side in (2.28) into

$$\frac{\sin\pi(a+x)\sin\pi(a-x)}{\sin^2\pi a}.$$

At this point, replacing all the sine functions with their classical Euler infinite product form, we rewrite the above as

$$\frac{(a+x)(a-x)}{a^2}\prod_{k=1}^{\infty}\frac{[1-\frac{(a+x)^2}{k^2}][1-\frac{(a-x)^2}{k^2}]}{(1-\frac{a^2}{k^2})^2},$$

which transforms into

$$\frac{a^2-x^2}{a^2}\prod_{k=1}^{\infty}\frac{[k^2-(a+x)^2][k^2-(a-x)^2]}{(k-a)^2(k+a)^2}. \tag{2.30}$$

The numerator under the infinite product sign can be decomposed as

$$(k-a-x)(k+a+x)(k-a+x)(k+a-x).$$

So, grouping the first factor with the third, and the second with the fourth, one converts the numerator in (2.30) into

$$\left[(k-a)^2-x^2\right]\left[(k+a)^2-x^2\right],$$

which transforms (2.30) to

$$\left(1-\frac{x^2}{a^2}\right)\prod_{k=1}^{\infty}\frac{(k-a)^2-x^2}{(k-a)^2}\,\frac{(k+a)^2-x^2}{(k+a)^2}$$

and finally to

$$\left(1-\frac{x^2}{a^2}\right)\prod_{k=1}^{\infty}\left[1-\frac{x^2}{(k-a)^2}\right]\left[1-\frac{x^2}{(k+a)^2}\right].$$

This completes the derivation of the representation in (2.27).

The next infinite product representation of an elementary function that will be reviewed here, is also taken from [9]. It is #1.437:

$$\frac{\sin 3x}{\sin x}=-\prod_{k=-\infty}^{\infty}\left[1-\left(\frac{2x}{x+k\pi}\right)^2\right]. \tag{2.31}$$

To verify this identity, we decompose first the difference of squares in the product as

$$-\prod_{k=-\infty}^{\infty}\left[1-\left(\frac{2x}{x+k\pi}\right)^2\right]=-\prod_{k=-\infty}^{\infty}\left(1-\frac{2x}{x+k\pi}\right)\left(1+\frac{2x}{x+k\pi}\right)$$

and then convert the above infinite product to an equivalent form. Namely, by splitting off the term with $k=0$, which is evidently equal to -3, and pairing the kth and the $-k$th terms, the above product transforms into

$$3\prod_{k=1}^{\infty}\left(1-\frac{2x}{x+k\pi}\right)\left(1-\frac{2x}{x-k\pi}\right)\left(1+\frac{2x}{x+k\pi}\right)\left(1+\frac{2x}{x-k\pi}\right)$$

and

$$3\prod_{k=1}^{\infty}\left(\frac{k\pi-x}{x+k\pi}\right)\left(\frac{-k\pi-x}{x-k\pi}\right)\left(\frac{3x+k\pi}{x+k\pi}\right)\left(\frac{3x-k\pi}{x-k\pi}\right).$$

Clearly, the first two factors under the product sign cancel, leaving the right-hand side of (2.31) as

$$3\prod_{k=1}^{\infty}\left(\frac{3x+k\pi}{x+k\pi}\right)\left(\frac{3x-k\pi}{x-k\pi}\right). \tag{2.32}$$

As to the left-hand side in (2.31), we reduce both the sine functions in it to the infinite product form

$$\frac{\sin 3x}{\sin x}=3\prod_{k=1}^{\infty}\frac{1-\frac{(3x)^2}{k^2\pi^2}}{1-\frac{x^2}{k^2\pi^2}}=3\prod_{k=1}^{\infty}\frac{9x^2-k^2\pi^2}{x^2-k^2\pi^2},$$

which is identical to the expression in (2.32). Thus, the identity in (2.31) is ultimately verified.

We turn next to an infinite product representation of another elementary function,

$$\frac{\cosh x-\cos\alpha}{1-\cos\alpha}=\prod_{k=-\infty}^{\infty}\left[1+\left(\frac{x}{2k\pi+a}\right)^2\right], \tag{2.33}$$

which is listed in [9] as #1.438. To verify this identity, we transform its left-hand side as

$$\frac{\cosh x-\cos\alpha}{1-\cos\alpha}=\frac{\cos ix-\cos\alpha}{1-\cos\alpha}=\frac{\sin\frac{a+ix}{2}\sin\frac{a-ix}{2}}{\sin^2\frac{a}{2}}.$$

We then express the sine functions by the classical Euler infinite product form, and perform some obvious elementary transformations. This yields

$$\frac{\frac{a+ix}{2}\frac{a-ix}{2}}{\frac{a^2}{4}}\prod_{k=1}^{\infty}\frac{[1-\frac{(a+ix)^2}{4k^2\pi^2}][1-\frac{(a-ix)^2}{4k^2\pi^2}]}{(1-\frac{a^2}{4k^2\pi^2})^2},$$

or

$$\frac{a^2+x^2}{a^2}\prod_{k=1}^{\infty}\frac{[4k^2\pi^2-(a+ix)^2][4k^2\pi^2-(a-ix)^2]}{(2k\pi+a)^2(2k\pi-a)^2},$$

which transforms as

$$\left(1+\frac{x^2}{a^2}\right)\prod_{k=1}^{\infty}\frac{(2k\pi-a-ix)(2k\pi+a+ix)}{(2k\pi+a)^2}\frac{(2k\pi-a+ix)(2k\pi+a-ix)}{(2k\pi-a)^2}.$$

Combining the first factor with the third, and the second with the fourth in the numerator, one converts the above into

$$\left(1+\frac{x^2}{a^2}\right)\prod_{k=1}^{\infty}\frac{(2k\pi-a)^2+x^2}{(2k\pi-a)^2}\frac{(2k\pi+a)^2+x^2}{(2k\pi+a)^2},$$

which can be represented as

$$\left(1+\frac{x^2}{a^2}\right)\prod_{k=1}^{\infty}\left[1+\frac{x^2}{(2k\pi-a)^2}\right]\left[1+\frac{x^2}{(2k\pi+a)^2}\right]. \tag{2.34}$$

It can be shown that the above infinite product (where the multiplication is assumed from one to infinity) transforms to that in (2.33), where we "sum" from negative infinity to positive infinity. To justify this assertion, we formally break down the product in (2.34) into two pieces,

$$\left(1+\frac{x^2}{a^2}\right)\prod_{m=1}^{\infty}\left[1+\frac{x^2}{(2m\pi-a)^2}\right]\cdot\prod_{k=1}^{\infty}\left[1+\frac{x^2}{(2k\pi+a)^2}\right],$$

and change the multiplication index in the first of the products via $m=-k$. This converts the above expression to

$$\left(1+\frac{x^2}{a^2}\right)\prod_{k=-1}^{-\infty}\left[1+\frac{x^2}{(2k\pi+a)^2}\right]\cdot\prod_{k=1}^{\infty}\left[1+\frac{x^2}{(2k\pi+a)^2}\right],$$

which is just the right-hand side of the relation in (2.33). Thus, the identity in (2.33) is verified.

We have finished our review of infinite product expansions of elementary functions that can be directly derived with the aid of the classical Euler representations for the trigonometric sine and cosine functions.

A few expansions, whose derivation will be conducted in the remaining part of this section, illustrate a variety of other possible approaches to the problem. Let us recall an alternative to the Euler's (2.1) infinite product expansion of the trigonometric sine function. That is,

$$\sin x = x\prod_{k=1}^{\infty}\cos\frac{x}{2^k}, \tag{2.35}$$

which also has been known for centuries and is listed, in particular, in [9] as #1.439. A formal comment is appropriate as to the convergence of the infinite product in (2.35). It converges to nonzero values of the sine function for any value of the variable x that does not make the argument of the cosine equal to $\pi/2+n\pi$, whereas it diverges to zero at such values of x, matching zero values of the sine function.

The derivation strategy that we are going to pursue in the case of (2.35) is based on the definition of the value of an infinite product. The strategy has two stages. First, a compact expression must be derived for the Kth partial product $P_K(x)$,

$$P_K=\prod_{k=1}^{K}\cos\frac{x}{2^k},$$

of the infinite product in (2.35). Then the limit of $P_K(x)$ is obtained as K approaches infinity.

To obtain a compact form of the partial product $P_K(x)$ for (2.35), we rewrite its first factor $\cos\frac{x}{2}$ as

$$\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{\sin x}{2\sin\frac{x}{2}}.$$

Similarly, the second factor $\cos\frac{x}{2^2}$ and the third factor $\cos\frac{x}{2^3}$ in $P_K(x)$ turn out to be

$$\cos\frac{x}{2^2} = \frac{2\sin\frac{x}{2^2}\cos\frac{x}{2^2}}{2\sin\frac{x}{2^2}} = \frac{\sin\frac{x}{2}}{2\sin\frac{x}{2^2}}$$

and

$$\cos\frac{x}{2^3} = \frac{2\sin\frac{x}{2^3}\cos\frac{x}{2^3}}{2\sin\frac{x}{2^3}} = \frac{\sin\frac{x}{2^2}}{2\sin\frac{x}{2^3}}.$$

Proceeding like this with the next-to-the-last factor $\cos\frac{x}{2^{K-1}}$ and the last factor $\cos\frac{x}{2^K}$ in $P_K(x)$, we express them as

$$\cos\frac{x}{2^{K-1}} = \frac{2\sin\frac{x}{2^{K-1}}\cos\frac{x}{2^{K-1}}}{2\sin\frac{x}{2^{K-1}}} = \frac{\sin\frac{x}{2^{K-2}}}{2\sin\frac{x}{2^{K-1}}}$$

and

$$\cos\frac{x}{2^K} = \frac{2\sin\frac{x}{2^K}\cos\frac{x}{2^K}}{2\sin\frac{x}{2^K}} = \frac{\sin\frac{x}{2^{K-1}}}{2\sin\frac{x}{2^K}}.$$

Once all the factors are put together, we have a series of cancellations, and the partial product $P_K(x)$ eventually reduces to the form

$$P_K(x) = \frac{\sin x}{2^K\sin\frac{x}{2^K}}. \tag{2.36}$$

Upon multiplying the numerator and denominator in (2.36) by x and regrouping the factors

$$P_K(x) = \frac{x\sin x}{x2^K\sin\frac{x}{2^K}} = \frac{\frac{x}{2^K}}{\sin\frac{x}{2^K}}\frac{\sin x}{x},$$

the partial product of the representation in (2.35) is prepared for taking the limit. Thus, we finally obtain

$$\lim_{K\to\infty} P_K(x) = \prod_{k=1}^{\infty}\cos\frac{x}{2^k} = \lim_{K\to\infty}\frac{\frac{x}{2^K}}{\sin\frac{x}{2^K}}\frac{\sin x}{x} = \frac{\sin x}{x},$$

which completes the derivation of the representation in (2.35).

Recall another infinite product representation,

$$\sinh x = x \prod_{k=1}^{\infty} \cosh \frac{x}{2^k}, \tag{2.37}$$

which is available in [20]. It is evident that its derivation can also be conducted in exactly same way as for the one in (2.35).

In what follows, the strategy just illustrated will be applied to the derivation of an infinite product representation for another elementary function, that is,

$$\frac{1}{1-x} = \prod_{k=0}^{\infty} \left(1 + x^{2^k}\right), \quad |x| < 1. \tag{2.38}$$

It can also be found in [20]. To proceed with the derivation, we transform the general term in (2.38) as

$$1 + x^{2^k} = \frac{1 - x^{2^{k+1}}}{1 - x^{2^k}}$$

and write down the Kth partial product $P_K(x)$ of the representation in (2.38) explicitly as

$$\begin{aligned} P_K(x) &= \prod_{k=0}^{K} \left(1 + x^{2^k}\right) \\ &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^8}{1-x^4} \cdot \ldots \cdot \frac{1-x^{2^{K-1}}}{1-x^{2^{K-2}}} \cdot \frac{1-x^{2^K}}{1-x^{2^{K-1}}}. \end{aligned}$$

It is evident that nearly all the terms in the above product cancel. Indeed, the only terms left are the denominator $1-x$ of the first factor and the numerator $1-x^{2^K}$ of the last factor. This reduces the partial product $P_K(x)$ to the compact form

$$\prod_{k=0}^{K} \left(1 + x^{2^k}\right) = \frac{1 - x^{2^K}}{1-x},$$

whose limit, as K approaches infinity, is

$$\lim_{K\to\infty} \prod_{k=0}^{K} \left(1 + x^{2^k}\right) = \lim_{K\to\infty} \frac{1 - x^{2^K}}{1-x} = \frac{1}{1-x}$$

for values of x such that $|x| < 1$.

From a comparison of the infinite product representation in (2.38) with the Maclaurin series $\sum_{k=0}^{\infty} x^k$ of the function $1/(1-x)$, it follows that the two are equivalent to each other, with the relation

$$P_K = S_{2^K}$$

between the partial product of (2.38) and the partial sum of the series. This observation means that the infinite product in (2.38) converges, at least formally, at a much faster rate.

To complete the review of methods customarily used for the infinite product representation of elementary functions, let us recall an approach to the square root function $\sqrt{1+x}$, which is described in [20], for example. The function is first transformed as

$$\sqrt{1+x} = \frac{2(x+1)}{x+2}\sqrt{(1+x)\frac{(x+2)^2}{4(x+1)^2}}, \tag{2.39}$$

and the radicand on the right-hand side is then simplified as

$$(1+x)\frac{(x+2)^2}{4(x+1)^2} = \frac{(x+2)^2}{4(x+1)},$$

resulting in

$$\begin{aligned}\sqrt{1+x} &= \frac{2(x+1)}{x+2}\sqrt{\frac{(x+2)^2}{4(x+1)}} \\ &= \frac{2(x+1)}{x+2}\sqrt{\frac{x^2+4x+4}{4x+4}} = \frac{2(x+1)}{x+2}\sqrt{1+\frac{x^2}{4x+4}}.\end{aligned} \tag{2.40}$$

This suggests for the radical factor

$$\sqrt{1+\frac{x^2}{4(x+1)}}$$

of the right-hand side in (2.40) the same transformation that has just been applied to the function $\sqrt{1+x}$ in (2.39). This yields

$$\sqrt{1+x} = \frac{2(x+1)}{x+2}\cdot\frac{2(\frac{x^2}{4(x+1)}+1)}{\frac{x^2}{4(x+1)}+2}\sqrt{1+\frac{(\frac{x^2}{4(x+1)})^2}{4(\frac{x^2}{4(x+1)}+1)}}.$$

Proceeding further with this algorithm, one arrives at the infinite product representation

$$\sqrt{1+x} = \prod_{k=0}^{\infty}\frac{2(A_k+1)}{A_k+2} \tag{2.41}$$

for the square root function, where the parameter A_k can be obtained from the recurrence

$$A_0 = x \quad \text{and} \quad A_{k+1} = \frac{A_k^2}{4(A_k+1)}, \quad k = 0, 1, 2, \ldots.$$

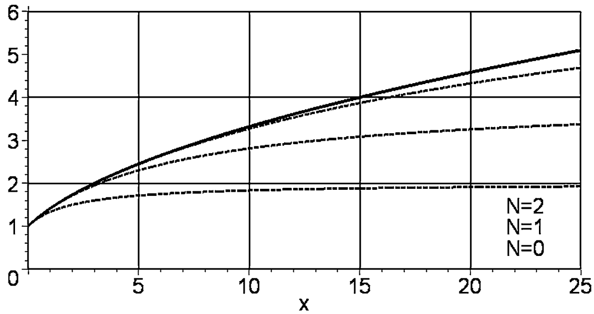

Fig. 2.7 Convergence of the expansion in (2.41)

It appears that the convergence rate of the expansion in (2.41) is extremely fast. This assertion is illustrated with Fig. 2.7, where the partial products P_0, P_1, and P_2 of the representation are depicted.

This completes the review that we intended to provide the reader of infinite product representations of elementary functions available in the current literature.

In the next chapter, the reader's attention will be directed to a totally different subject. Namely, we will begin a review of a collection of methods that are traditionally used for the construction of Green's functions for the two-dimensional Laplace equation.

The purpose for such a sharp turn is twofold. First, we aim at giving a more comprehensive, in comparison with other relevant sources, review of the available procedures for the construction of Green's functions for a variety of boundary-value problems for the Laplace equation. Second, one of those procedures represents a significant issue for Chap. 6, where an innovative approach will be discussed for the expression of elementary functions in terms of infinite products.

2.4 Chapter Exercises

2.1 Use Euler's approach and derive the infinite product representation in (2.4) for the hyperbolic cosine function.

2.2 Verify the infinite product representation in (2.24).

2.3 Derive the infinite product representation in (2.37) for the hyperbolic sine function.

2.4 Verify the infinite product representation

$$\cos x - \sin x = \prod_{n=1}^{\infty}\left[1 + \frac{(-1)^n 4x}{(2n-1)\pi}\right].$$

2.5 Derive an infinite product representation for the function

$$a\sin x + b\cos x,$$

where a and b are real factors.

2.6 Derive an infinite product representation for the function

$$\sin x + \sin y.$$

2.7 Derive an infinite product representation for the function

$$\cos x + \cos y.$$

2.8 Verify the infinite product representation

$$\tan x + \cot x = \frac{1}{x}\prod_{k=1}^{\infty}\left(1 + \frac{4x^2}{k^2\pi^2 - 4x^2}\right).$$

2.9 Derive an infinite product representation for the function

$$\cot x + \cot y.$$

2.10 Verify the infinite product representation

$$\cosh x - \cosh y = \frac{x^2 - y^2}{2}\prod_{k=1}^{\infty}\left[1 + \frac{x^2 + y^2}{2k^2\pi^2} + \frac{(x^2 - y^2)^2}{16k^4\pi^4}\right].$$

2.11 Derive an infinite product representation for the function

$$\coth x + \coth y.$$

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