Preface

Many mathematicians are aware of some of the dramatic interactions between ergodic theory and other parts of the subject, notably Ramsey theory, infinite combinatorics, and Diophantine number theory. These notes are intended to provide a gentle route to a tiny sample of these results. The intended readership is expected to be mathematically sophisticated, with some background in measure theory and functional analysis, or to have the resilience to learn some of this material along the way from other sources.

In this volume we develop the beginnings of ergodic theory and dynamical systems. While the selection of topics has been made with the applications to number theory in mind, we also develop other material to aid motivation and to give a more rounded impression of ergodic theory. Different points of view on ergodic theory, with different kinds of examples, may be found in the monographs of Cornfeld, Fomin and Sinaĭ [60], Petersen [282], or Walters [374]. Ergodic theory is one facet of dynamical systems; for a broad perspective on dynamical systems see the books of Katok and Hasselblatt [182] or Brin and Stuck [44]. An overview of some of the more advanced topics we hope to pursue in a subsequent volume may be found in the lecture notes of Einsiedler and Lindenstrauss [80] in the Clay proceedings of the Pisa Summer school.

Fourier analysis of square-integrable functions on the circle is used extensively. The more general theory of Fourier analysis on compact groups is not essential, but is used in some examples and results. The ergodic theory of commuting automorphisms of compact groups is touched on using a few examples, but is not treated systematically. It is highly developed elsewhere: an extensive treatment may be found in the monograph by Schmidt [332]. Standard background material on measure theory, functional analysis and topological groups is collected in the appendices for convenience.

Among the many *lacunae*, some stand out: Entropy theory; the isomorphism theory of Ornstein, a convenient source being Rudolph [324]; the more advanced spectral theory of measure-preserving systems, a convenient source being Nadkarni [264]; finally Pesin theory and smooth ergodic theory, a source being Barreira and Pesin [19]. Of these omissions, entropy theory is perhaps the most fundamental for applications in number theory, and this was the reason for not including it here. There is simply too much to say about entropy to fit into this volume, so we will treat this important topic, both in general terms and in more detail in the algebraic context needed for number theory, in a subsequent volume. The notion is mentioned in one or two places in this volume, but is never used directly.

No Lie theory is assumed, and for that reason some arguments here may seem laborious in character and limited in scope. Our hope is that seeing the language of Lie theory emerge from explicit matrix manipulations allows a relatively painless route into the ergodic theory of homogeneous spaces. This will be carried further in a subsequent volume, where some of the deeper applications will be given.

Notation and Conventions

The symbols $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}$ denote the natural numbers, non-negative integers and integers; $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the rational numbers, real numbers and complex numbers; $\mathbb{S}^1$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the multiplicative and additive circle respectively. The elements of $\mathbb{T}$ are thought of as the elements of $[0, 1)$ under addition modulo 1. The real and imaginary parts of a complex number are denoted $x = \Re(x+\mathrm{i}y)$ and $y = \Im(x+\mathrm{i}y)$. The order of growth of real- or complex-valued functions f, g defined on $\mathbb{N}$ or $\mathbb{R}$ with $g(x) \neq 0$ for large x is compared using Landau's notation:

$$f \sim g \text{ if } \left|\frac{f(x)}{g(x)}\right| \longrightarrow 1 \text{ as } x \to \infty;$$

$$f = \mathrm{o}(g) \text{ if } \left|\frac{f(x)}{g(x)}\right| \longrightarrow 0 \text{ as } x \to \infty.$$

For functions f, g defined on $\mathbb{N}$ or $\mathbb{R}$, and taking values in a normed space, we write $f = \mathrm{O}(g)$ if there is a constant $A > 0$ with $\|f(x)\| \leqslant A\|g(x)\|$ for all x. In particular, $f = \mathrm{O}(1)$ means that f is bounded. Where the dependence of the implied constant A on some set of parameters $\mathscr{A}$ is important, we write $f = \mathrm{O}_{\mathscr{A}}(g)$. The relation $f = \mathrm{O}(g)$ will also be written $f \ll g$, particularly when it is being used to express the fact that two functions are commensurate, $f \ll g \ll f$. A sequence $a_1, a_2, \dots$ will be denoted (a_n). Unadorned norms $\|x\|$ will only be used when x lives in a Hilbert space (usually L^2) and always refer to the Hilbert space norm. For a topological space X, $C(X)$, $C_{\mathbb{C}}(X)$, $C_c(X)$ denote the space of real-valued, complex-valued, compactly supported continuous functions on X respectively, with the supremum norm. For sets A, B, denote the set difference by

$$A \smallsetminus B = \{x \mid x \in A, x \notin B\}.$$

Additional specific notation is collected in an index of notation on page 467.

Statements and equations are numbered consecutively within chapters, and exercises are numbered in sections. Theorems without numbers in the main body of the text will not be proved; appendices contain background material in the form of numbered theorems that will not be proved here.

Several of the issues addressed in this book revolve around *measure rigidity*, in which there is a natural measure that other measures are compared with. These natural measures will usually be Haar measure on a compact or locally compact group, or measures constructed from Haar measures, and these will usually be denoted m.

We have not tried to be exhaustive in tracing the history of the ideas used here, but have tried to indicate some of the rich history of mathematical developments that have contributed to ergodic theory. Certain references to earlier and to related material is generally collected in endnotes at the end of each chapter; the presence of these references should not be viewed in any way as authoritative. Statements in these notes are informed throughout by a desire to remain rooted in the familiar territory of ergodic theory. The standing assumption is that, unless explicitly noted otherwise, metric spaces are complete and separable, compact groups are metrizable, discrete groups are countable, countable groups are discrete, and measure spaces are assumed to be Borel probability spaces (this assumption is only relevant starting with Sect. 5.3; see Definition 5.13 for the details). A convenient summary of the measure-theoretic background may be found in the work of Royden [320] or of Parthasarathy [280].

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