II

Currents and complex structures

In this chapter we introduce two new ideas, one coming from differential geometry (currents) and the other coming from analysis (complex analytic manifolds and their associated complex structures), which we will use frequently throughout the rest of this book. We start by defining currents, which for differential forms play the role that distributions play for functions, and then consider the regularisation problem for currents defined on a $\mathcal{C}^\infty$ differential manifold. Solving this problem, which is easy in $\mathbb{R}^n$ by means of convolution, obliges us to introduce kernels with similar properties to the convolution kernel. We will also study the Kronecker index of two currents, which generalises the pairing of a current and a differential form. This index enables us to prove a fairly general Stokes' formula which will be used in Chapters III and IV. We then introduce the notion of a complex analytic manifold and describe the natural complex structures which appear on the tangent space of such manifolds, which leads us to define (p,q) differential forms, the $\overline{\partial}$ operator, the Dolbeault complex and the associated cohomology groups. The holomorphic extension phenomena studied in Chapter V are linked to the vanishing of certain of these cohomology groups, and some vanishing theorems will be proved in Chapter VII. We end this chapter with the definition of the complex tangent space to the boundary of a domain in a complex analytic manifold which appears later in the definitions of CR functions (Chapter IV) and pseudoconvexity (Chapter VII).

1 Currents

By X we will always denote a $\mathcal{C}^\infty$ n-dimensional oriented differentiable manifold. For any p such that $0 \leqslant p \leqslant n$ we denote by $\mathcal{D}^p(X)$ the vector space of $\mathcal{C}^\infty$ degree p compactly supported differentiable forms on X. We will define a locally convex topology on $\mathcal{D}^p(X)$ and study its dual.

A. The topology on $\mathcal{D}^p(X)$

Assume first that X is an open set in $\mathbb{R}^n$ and denote by $\mathcal{E}(X)$ the vector space of $\mathcal{C}^\infty$ functions on X.

C. Laurent-Thiébaut, *Holomorphic Function Theory in Several Variables: An Introduction*, Universitext, DOI 10.1007/978-0-85729-030-4_2, © Springer-Verlag London Limited 2011

If K is a compact set in X and α is an element of $\mathbb{N}^n$ then for any $f \in \mathcal{E}(X)$ we set

$$p_{K,\alpha}(f) = \sup_{x \in K} |D^\alpha f(x)|.$$

The functions $p_{K,\alpha}$ are semi-norms on $\mathcal{E}(X)$. Consider the topology on $\mathcal{E}(X)$ defined by these semi-norms. The sets

$$V_{K,m,\varepsilon} = \{f \in \mathcal{E}(X) \mid \forall\, \alpha,\ |\alpha| \leqslant m,\ p_{K,\alpha}(f) < \varepsilon\}$$

form a fundamental system of neighbourhoods of zero for this topology. As X is an open set in $\mathbb{R}^n$ there is an exhaustion of X by compact sets $(K_p)_{p\in\mathbb{N}}$ and the family $(V_{K_{p,m,1/n}})_{p,m\in\mathbb{N},n\in\mathbb{N}^*}$ is a fundamental basis of the topology in a neighbourhood of 0. This topology is therefore metrisable. It is easy to check that a sequence of elements in $\mathcal{E}(X)$ converges to 0 in the above topology if and only if both the sequence and all its derivatives converge uniformly to 0 on any compact set in X. The vector space $\mathcal{E}(X)$ equipped with this topology is a Fréchet space (i.e. a locally convex, complete, metrisable topological vector space).

If φ is a $\mathcal{C}^\infty$ differential form of degree p on X then φ can be written in the following form

$$\varphi = \sum_{\substack{|I|=p \\ i_1<i_2<\cdots<i_p}} \varphi_I dx_I,$$

where $\varphi_I \in \mathcal{E}(X)$ and for any $I = (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$, we set $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. For any compact set K in X and $\alpha \in \mathbb{N}^n$ we set

$$\widetilde{p}_{K,\alpha}(\varphi) = \sup\{p_{K,\alpha}(\varphi_I) \mid I = (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p,\ i_1 < i_2 < \cdots < i_p\}.$$

The set of semi-norms $\widetilde{p}_{K,\alpha}$, where K is a compact set in X and α is an element of $\mathbb{N}^n$ defines a Fréchet space topology on the vector space $\mathcal{E}^p(X)$ of $\mathcal{C}^\infty$ degree p differential forms on X.

If Y is another open set in $\mathbb{R}^n$ and f is a $\mathcal{C}^\infty$ diffeomorphism from X to Y then the map

$$\begin{aligned} f^* : \mathcal{E}^p(Y) &\longrightarrow \mathcal{E}^p(X) \\ \varphi &\longmapsto f^*\varphi \end{aligned}$$

is a linear homeomorphism.

We now consider the case where X is a manifold. Let $\mathcal{A}$ be an atlas on X. We define a topology on $\mathcal{E}^p(X)$ using the semi-norms $\widetilde{p}_{U,K,\alpha}$ which are defined by

$$\widetilde{p}_{U,K,\alpha}(\varphi) = \widetilde{p}_{K,\alpha}\big((h^{-1})^*\varphi\big|_U\big) \quad \text{for any } \varphi \in \mathcal{E}(X)$$

for any set U which is the domain of a chart $(U,h) \in \mathcal{A}$, any compact set K in U and any $\alpha \in \mathbb{N}^n$. This topology is independent of the choice of $\mathcal{A}$: it is the coarsest topology such that the maps $(k^{-1})^* : \mathcal{E}(X) \to \mathcal{E}(k(V))$ are continuous for any chart (V,k) on X. As X is countable at infinity we can assume that the atlas $\mathcal{A}$ contains a countable number of charts and the topology defined above is therefore metrisable. It is easy to check that $\mathcal{E}^p(X)$ is a Fréchet space with this topology.

If K is a compact set in X then we denote by $\mathcal{D}^p_K(X)$ the subspace of $\mathcal{E}^p(X)$ consisting of $\mathcal{C}^\infty$ degree p differential forms supported on K. This is a closed subset in $\mathcal{E}^p(X)$ and it follows that if we equip it with the restriction of the above topology on $\mathcal{E}^p(X)$ we get a Fréchet space. We then equip $\mathcal{D}^p(X) = \bigcup_K \mathcal{D}^p_K(X)$ with the finest locally convex vector space topology for which all the inclusions

$$\mathcal{D}^p_K(X) \hookrightarrow \mathcal{D}^p(X), \quad K \text{ a compact set in } X$$

are continuous.

Remarks. A sequence of differential forms $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{D}^p(X)$ converges to $\varphi \in \mathcal{D}^p(X)$ in the above topology if and only if

1) the forms φ_j are all supported on some fixed compact set K in X.
2) $(\varphi_j)_{j\in\mathbb{N}}$ converges to φ in $\mathcal{D}^p_K(X)$.

B. Currents

Definition 1.1. A *p-dimensional current on X* is a continuous linear form on $\mathcal{D}^p(X)$. We denote by $\mathcal{D}'_p(X)$ the set of p-dimensional currents on X. It is a $\mathbb{C}$-vector space, the topological dual of $\mathcal{D}^p(X)$.

Consider $T \in \mathcal{D}'_p(X)$: this is a linear form on $\mathcal{D}^p(X)$ and hence, for any pair of forms $\varphi_1, \varphi_2 \in \mathcal{D}^p(X)$,

$$T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$$

and for any $\lambda \in \mathbb{C}$ and $\varphi \in \mathcal{D}^p(X)$

$$T(\lambda\varphi) = \lambda T(\varphi).$$

The current T is also continuous on $\mathcal{D}^p(X)$. In other words, $T\big|_{(\mathcal{D}^p_K)(X)}$ is continuous for any compact set K in X. This is equivalent to the following statement: For any sequence $(\varphi_j)_{j\in\mathbb{N}}$ of elements in $\mathcal{D}^p(X)$ which tends to 0, the sequence $T(\varphi_j)$ also tends to 0 in $\mathbb{C}$.

Throughout the following we will write $\langle T, \varphi\rangle$ for $T(\varphi)$.

Examples of currents.

i) The Dirac delta function δ_x, $x \in X$, defined by $\delta_x(\varphi) = \varphi(x)$ for any $\varphi \in \mathcal{D}^0(X)$ is a 0-dimensional current.

ii) If ω is a locally integrable differential form of degree q on X then we define an $(n-q)$-dimensional current T_ω by

$$\langle T_\omega, \varphi\rangle = \int_X \omega \wedge \varphi \quad \text{for any } \varphi \in \mathcal{D}^{n-q}(X).$$

This definition is only possible on an oriented manifold X.

iii) If Y is a $\mathcal{C}^\infty$ closed and oriented p-dimensional submanifold of X then we define a p-dimensional current $[Y]$ on X by

$$\langle [Y], \varphi\rangle = \int_Y \varphi = \int_Y i^*\varphi \quad \text{for any } \varphi \in \mathcal{D}^p(X),$$

where i is the inclusion $Y \hookrightarrow X$. The current $[Y]$ is called the *integration current on* Y.

If K is a compact set in X and $k \in \mathbb{N}$ is an integer then we denote by $(\mathcal{C}^k_K)^p(X)$ the space of $\mathcal{C}^k$ degree p differential forms supported on K. We equip this space with the topology defined by the semi-norms $\widetilde{p}_{K,\alpha}$, $|\alpha| \leqslant k$. We define $(\mathcal{C}^k_c)^p(X)$ to be the union of the spaces $(\mathcal{C}^k_K)^p(X)$ for all compact sets K in X: it is the space of $\mathcal{C}^k$ degree p compactly supported differential forms on X. We equip this space with the finest locally convex vector space topology for which all the inclusions

$$(\mathcal{C}^k_K)^p(X) \hookrightarrow (\mathcal{C}^k_c)^p(X)$$

are continuous. Consider the inclusion $\mathcal{D}^p(X) \hookrightarrow (\mathcal{C}^k_c)^p(X)$: it is a continuous map with dense image. We denote the topological dual of $(\mathcal{C}^k_c)^p(X)$ by $(\mathcal{C}^k_c)'_p(X)$: it is a subspace of $\mathcal{D}'_p(X)$ and its elements are called *currents of order k and dimension p on X*.

Example. If Y is a $\mathcal{C}^1$ closed oriented submanifold of X then the integration current on Y is a current of order 0.

C. Support of a current

In this section we will see that it is possible to obtain global information on a current by gluing local information.

If Ω is an open set in X and $T \in \mathcal{D}'_p(X)$ then we can define $T\big|_\Omega$ (or, more correctly, $T\big|_{\mathcal{D}^p(\Omega)}$) by $\langle T\big|_\Omega, \varphi\rangle = \langle T, \widetilde{\varphi}\rangle$ for any $\varphi \in \mathcal{D}^p(\Omega)$, where $\widetilde{\varphi} \in \mathcal{D}^p(X)$ is defined by $\widetilde{\varphi} = \varphi$ on Ω and $\widetilde{\varphi} = 0$ on $X \smallsetminus \Omega$.

Proposition 1.2. *Let $(\Omega_i)_{i\in I}$ be a open cover of X, and for every $i \in I$ let T_i be an element of $\mathcal{D}'_p(\Omega_i)$. Assume that $T_i\big|_{\Omega_i\cap\Omega_j} = T_j\big|_{\Omega_i\cap\Omega_j}$ for any pair (i,j). There is then a unique current $T \in \mathcal{D}'_p(X)$ such that $T\big|_{\Omega_i} = T_i$ for all $i \in I$.*

Proof. Consider a locally finite partition of unity $(\alpha_i)_{i\in I}$ subordinate to the open cover $(\Omega_i)_{i\in I}$. If $\varphi \in \mathcal{D}^p(X)$ then $\varphi = \sum_{i\in I} \alpha_i\varphi$ and the right-hand sum has only a finite number of non-zero terms. If T exists then it must have the property that, for any $\varphi \in \mathcal{D}^p(X)$,

$$\langle T, \varphi\rangle = \sum_{i\in I} \langle T, \alpha_i\varphi\rangle = \sum_{i\in I} \langle T_i, \alpha_i\varphi\rangle \tag{1.1}$$

since α_i is supported on Ω_i.

Conversely, this formula defines a continuous linear form on $\mathcal{D}^p(X)$. Indeed, let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence of elements in $\mathcal{D}^p(X)$ converging to 0. There is then a compact set K such that $\operatorname{supp}\varphi_j \subset K$ for all $j \in \mathbb{N}$ and for any $i \in I$ the sequence $(\alpha_i\varphi_j)_{j\in\mathbb{N}}$ converges to 0 in $(\mathcal{C}^\infty_{K\cap \operatorname{supp}\alpha_i})^p(X)$. It follows that $T_i(\alpha_i\varphi_j)$ converges to 0 and as only a finite number of the forms $\alpha_i\varphi_j$ are non-zero on K for any j, $T(\varphi_j) = \sum T_i(\alpha_i\varphi_j)$ tends to zero as j tends to infinity. Formula (1.1) therefore defines a p-dimensional current T on X and we now check that $T\big|_{\Omega_i} = T_i$. Consider $\varphi \in \mathcal{D}(\Omega_i)$. Then:

$$\langle T_i, \varphi\rangle = \sum_{k\in I} \langle T_i, \alpha_k\varphi\rangle$$

but $\operatorname{supp}\alpha_k\varphi \subset \Omega_k \cap \Omega_i$ and it follows that $\langle T_i, \alpha_k\varphi\rangle = \langle T_k, \alpha_k\varphi\rangle$ which implies that $\langle T_i, \varphi\rangle = \sum_{k\in I}\langle T_k, \alpha_k\varphi\rangle = \langle T, \varphi\rangle$. □

Corollary 1.3. *If T is a p-dimensional current on X then there is a largest possible open set Ω in X such that $T\big|_\Omega = 0$.*

Definition 1.4. If $T \in \mathcal{D}'_p(X)$ then the *support* of T is the complement of the largest open set on which T is identically zero.

Example. If Y is a $\mathcal{C}^\infty$ closed oriented submanifold of X then the support of the current $[Y]$ is Y.

Remark. Note that if $T \in \mathcal{D}'_p(X)$ is a current with compact support then the expression $\langle T, \psi\rangle$ is meaningful for any $\mathcal{C}^\infty$ differential form ψ on X of degree p. Indeed, let χ be a compactly supported $\mathcal{C}^\infty$ function on X such that χ is identically 1 on a neighbourhood of the support of T. We then set $\langle T, \psi\rangle = \langle T, \chi\Psi\rangle$.

Local expressions of currents. Let (U, h) be a chart of X and let $(x_1, \ldots, x_n)$ be the associated local coordinates. Consider the expression

$$T = \sum_{|I|=p} T_I dx_I, \tag{1.2}$$

where T_I is an n-dimensional current on U and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for any $I = (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$. This defines an $(n-p)$-dimensional current on U in the following way: if φ in $\mathcal{D}^{n-p}(U)$ can be written in the form $\varphi = \sum_{|J|=n-p} \varphi_J dx_J$ then we set

$$\langle T, \varphi \rangle = \sum_{|I|=p} \varepsilon(I, \complement I) \langle T_I, \varphi_{\complement I} dx_1 \wedge \cdots \wedge dx_n \rangle,$$

where $I = (i_1, \ldots, i_p)$ and $\complement I = (j_1, \ldots, j_{n-p})$ have the property that $\{i_1, \ldots, i_p, j_1, \ldots, j_{n-p}\} = \{1, \ldots, n\}$ and $\varepsilon(I, \complement I)$ is the sign of the permutation sending $(1, \ldots, n)$ to $(i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$.

Conversely, if T is any current on X then the current $T\big|_U$ can be written in the form (1.2). Indeed, if we set

$$\langle T_I, \varphi dx_1 \wedge \cdots \wedge dx_n \rangle = \varepsilon(I, \complement I) \langle T, \varphi dx_{\complement I} \rangle$$

for any $I = (i_1, \ldots, i_p)$ and any $\mathcal{C}^\infty$ function with compact support on U φ, then this formula defines a set of n-dimensional currents T_I and the current T can be written in the form (1.2) using these currents T_I.

Definition 1.5. If T is a p-dimensional current on a differentiable manifold of dimension n then the number $(n-p)$ is called the *degree* of the current T. We denote the set of degree q currents on X by $\mathcal{D}'^q(X)$.

We have just proved that a degree q current on X can be locally written as a degree q differential form whose coefficients are degree 0 currents.

Example. Degree n currents in $\mathbb{R}^n$ are simply distributions and can be naturally identified with degree 0 currents.

D. Operations on currents

We now show how to extend the classical operations on differential forms to currents and define some new operations.

Wedge product with a $\mathcal{C}^\infty$ differential form. Consider a current $T \in \mathcal{D}'^p(X)$ and a differential form $\alpha \in \mathcal{E}^q(X)$ such that $0 \leqslant p + q \leqslant n$. We define the wedge product $T \wedge \alpha$ by

$$\langle T \wedge \alpha, \varphi \rangle = \langle T, \alpha \wedge \varphi \rangle \quad \text{for any } \varphi \in \mathcal{D}^{n-p-q}(X).$$

This is a degree $p+q$ current on X. If $T = T_\omega$ is the current defined by a $\mathcal{C}^\infty$ differential form of degree p then

$$T_\omega \wedge \alpha = T_{\omega \wedge \alpha} = (-1)^{pq} T_{\alpha \wedge \omega}.$$

For any $T \in \mathcal{D}'^p(X)$ and $\alpha \in \mathcal{D}^q(X)$ we set

$$\alpha \wedge T = (-1)^{pq} T \wedge \alpha.$$

If T is a current of order k then we can define its wedge product with a $\mathcal{C}^k$ differential form in a similar way.

Boundary and differential of a current. If $T \in \mathcal{D}'^p(X)$, then we define the *boundary* bT of the current T by

$$\langle bT, \varphi \rangle = \langle T, d\varphi \rangle, \quad \text{for any } \varphi \in \mathcal{D}^{n-p-1}(X).$$

This is a $p+1$ degree current on X. The *differential* dT of the current $T \in \mathcal{D}'^p(X)$ is then defined by the formula

$$dT = (-1)^{p-1} bT.$$

Examples.

1) Let $D \Subset X$ be a open set with $\mathcal{C}^1$ boundary which is relatively compact in X. We denote by $[D]$ the degree 0 current defined by

$$\langle [D], \varphi \rangle = \int_D \varphi \quad \text{for any } \varphi \in \mathcal{D}^n(X).$$

Stokes' theorem then says that $b[D] = [bD]$.

2) Let ω be an element of $\mathcal{E}^p(X)$ and let us calculate dT_ω. For any $\varphi \in \mathcal{D}^{n-p-1}(X)$,

$$\langle dT_\omega, \varphi \rangle = (-1)^{p-1} \langle bT_\omega, \varphi \rangle = (-1)^{p-1} \langle T_\omega, d\varphi \rangle = (-1)^{p-1} \int_X \omega \wedge d\varphi;$$

but now

$$d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^p \omega \wedge d\varphi,$$

so that

$$\langle dT_\omega, \varphi \rangle = \int_X d\omega \wedge \varphi - \int_X d(\omega \wedge \varphi).$$

Since $\omega \wedge \varphi$ is a compactly supported form, Stokes' formula now says that $\int_X d(\omega \wedge \varphi) = 0$ and hence $dT_\omega = T_{d\omega}$.

Remark. If $T \in \mathcal{D}'^p(X)$ then $d(dT) = 0$.

Direct image of a current under a proper map. Let X and Y be two oriented $\mathcal{C}^\infty$ differentiable manifolds and let f be a $\mathcal{C}^\infty$ map from X to Y. We say that f is proper if and only if for any compact set K in Y $f^{-1}(K)$ is a compact set in X. If $T \in \mathcal{D}'_p(X)$ then the *direct image* of T under the proper map f is the current f_*T defined by

$$\langle f_*T, \varphi \rangle = \langle T, f^*\varphi \rangle, \quad \text{for any } \varphi \in \mathcal{D}'^p(Y).$$

(This definition is meaningful because $\operatorname{supp} f^*\varphi \subset f^{-1}(\operatorname{supp} \varphi)$ is compact for any proper f). The current f_*T is a p-dimensional current on Y. It follows from the definition of the operator f_* and Proposition 5.4 ii) of Appendix A that if T is contained in $\mathcal{D}'_p(X)$ then

$$f_* dT = d f_* T.$$

Inverse image of a current under projection. Let Y and Z be two oriented $\mathcal{C}^\infty$ differentiable manifolds and let f be the projection from $X = Y \times Z$ to Z. If $\varphi \in \mathcal{D}^\bullet(Y \times Z)$ then we define the integral $\int_Y \varphi$ to be the unique differential form on Z such that, for any $\psi \in \mathcal{D}^\bullet(Z)$,

$$\Big\langle \int_Y \varphi, \psi \Big\rangle_Z = \langle \varphi, f^*\psi \rangle_{Y\times Z} = \int_{Y\times Z} \varphi \wedge f^*\psi.$$

If $T = T_\varphi$ is the current defined by a compactly supported $\mathcal{C}^\infty$ differential form φ on X then the current f_*T_φ is the current defined by the $\mathcal{C}^\infty$ compactly supported differential form $\psi(z) = \int_Y \varphi$ on Z. (This follows from the definition of f_* and Fubini's theorem). Consider $T \in \mathcal{D}'^p(Z)$: we define the *inverse image* of the current T under the projection f by

$$\langle f^*T, \varphi \rangle = \langle T, f_*T_\varphi \rangle, \quad \text{for any } \varphi \in \mathcal{D}^{\dim X - p}(X).$$

This is a degree p current on X. It is clear that if $T = T_\omega$ is the current defined by a $\mathcal{C}^\infty$ differential form on Y then $f^*T_\omega = T_{f^*\omega}$, where $f^*\omega$ is the inverse image of the differential form ω.

2 Regularisation

Let X be an n-dimensional oriented $\mathcal{C}^\infty$ differentiable manifold. We denote by $\mathcal{D}'^\bullet(X) = \bigoplus_{p=0}^n \mathcal{D}'^p(X)$ the vector space of currents on X – this is the topological dual of the vector space $\mathcal{D}^\bullet(X)$ of compactly supported $\mathcal{C}^\infty$ differential forms on X. Traditionally, we consider two topologies on $\mathcal{D}'^\bullet(X)$:

1) The weak topology, or the topology of simple convergence on $\mathcal{D}^\bullet(X)$. More precisely, a family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+} \subset \mathcal{D}'^\bullet(X)$ converges weakly to $T \in \mathcal{D}'^\bullet(X)$ as ε tends to 0 if for every $\varphi \in \mathcal{D}^\bullet(X)$

$$\lim_{\varepsilon\to 0} \langle T_\varepsilon, \varphi \rangle = \langle T, \varphi \rangle.$$

2) The strong topology, or the topology of uniform convergence on bounded sets in $\mathcal{D}'(X)$. We recall that a subset B in $\mathcal{D}^\bullet(X)$ is bounded if the elements φ in B are all supported in some given compact set K and if for any $\alpha \in \mathbb{N}^n$ and any chart domain U of an atlas $\mathcal{A}$, $\sup_{\varphi\in B}\{\widetilde{p}_{U,K,\alpha}(\varphi)\} < +\infty$, where the $\widetilde{p}_{U,K,\alpha}$ are the semi-norms defined in § 1.A.

It is easy to see that the strong topology is finer than the weak topology.

The aim of this section is to prove that $\mathcal{E}^\bullet(X) = \bigoplus_{p=0}^n \mathcal{E}^p(X)$ is dense in $\mathcal{D}'^\bullet(X)$ with respect to either the weak or the strong topology and give a method for constructing families of $\mathcal{C}^\infty$ differential forms converging to a given current in either topology.

A current of degree 0 on X is called a *distribution* on X.

A. Regularising distributions on $\mathbb{R}^n$

We consider a positive $\mathcal{C}^\infty$ function θ which is compactly supported in a neighbourhood of 0 in $\mathbb{R}^n$ and which has the property that $\int_{\mathbb{R}^n} \theta(x)dx = 1$. For example, we can take the function defined by

$$\theta(x) = \begin{cases} c\, e^{-1/(1-\|x\|^2)} & \text{for } \|x\| \leqslant 1 \\ 0 & \text{for } \|x\| \geqslant 1, \end{cases}$$

where the constant c is chosen in such a way that $\int_{\mathbb{R}^n} \theta(x)dx = 1$. We set $\theta_\varepsilon(x) = \frac{1}{\varepsilon^n}\theta(x/\varepsilon)$ for any $\varepsilon > 0$ and we set $K_\varepsilon(x,y) = \theta_\varepsilon(x-y)$ for any $x, y \in \mathbb{R}^n$. If u is a continuous function on $\mathbb{R}^n$ then we define the regularisations u_ε of u by

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} K_\varepsilon(x,y)u(y)dy.$$

The following classical proposition and corollary will be proved in a more general form in Section B.

Proposition 2.1. *If $u \in \mathcal{D}^0(\mathbb{R}^n)$ is a compactly supported $\mathcal{C}^\infty$ function on $\mathbb{R}^n$, then the family $(u_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations of u converges to u in $\mathcal{D}^0(\mathbb{R}^n)$ as ε tends to 0. Moreover, the convergences of these series to u is uniform with respect to u over any bounded sets in $\mathcal{D}^0(\mathbb{R}^n)$.*

Definition 2.2. Let $T \in \mathcal{D}'^0(\mathbb{R}^n)$ be a distribution on $\mathbb{R}^n$. We then define *the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations* of T in the following way: for any $\varphi \in \mathcal{D}^0(\mathbb{R}^n)$ we set

$$\langle T_\varepsilon, \varphi dx_1 \wedge \cdots \wedge dx_n\rangle = \langle T, \varphi_\varepsilon dx_1 \wedge \cdots \wedge dx_n\rangle,$$

where $\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} K_\varepsilon(x,y)\varphi(y)dy$.

Corollary 2.3. *The family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations of the distribution T is a family of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ which converges both weakly and strongly to T when ε tends to 0.*

The interested reader may consult [Sc, Chap. 6] for more information on regularisation in $\mathbb{R}^n$.

Let us consider the main properties of the function K_ε defined on $\mathbb{R}^n \times \mathbb{R}^n$:

1) K_ε is a $\mathcal{C}^\infty$ function,
2) K_ε is supported in a strip containing the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ whose width is of order ε,
3) $\displaystyle\int_{\mathbb{R}^n} K_\varepsilon(x,y)dy = \int_{\mathbb{R}^n} K_\varepsilon(x,y)dx = 1$,
4) $\displaystyle\int_{\mathbb{R}^n} \Big(\frac{\partial^\alpha}{\partial x_\alpha} + (-1)^{|\alpha|+1}\frac{\partial^\alpha}{\partial y_\alpha}\Big)K_\varepsilon(x,y)dy = 0$ for any $\alpha \in \mathbb{N}^n$.

These properties are central to the proofs of Proposition 2.1 and Corollary 2.3.

B. Regularising distributions on manifolds

Let X be an n-dimensional oriented $\mathcal{C}^\infty$ differentiable manifold.

As a manifold does not have a group law in general, we can no longer use convolution to regularise distributions as in Section A. The idea is to use kernels $(K_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ which are functions defined on $X\times X$ with properties similar to the four properties mentioned in Section A.

Definition 2.4. Let π_1 and π_2 be the two projections from $X\times X$ to X. We say that a subset A in $X\times X$ is *proper* if for any compact set K in X the sets $\pi_1(\pi_2^{-1}(K)\cap A)$ and $\pi_2(\pi_1^{-1}(K)\cap A)$ are relatively compact in X.

We consider a family of nested neighbourhoods of the diagonal $\Delta\subset X\times X$ which we denote by $(U_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ and which we construct in the following way. Consider a locally finite cover $\mathcal{U}$ of Δ by open sets $\widetilde{U}=(U\times U)$ such that U is the domain of a chart (U,h) on X. We then define U_ε by

$$U_\varepsilon=\bigcup_{\widetilde{U}\in\mathcal{U}}\big\{(x,y)\in\widetilde{U}\mid \|h(x)-h(y)\|<\varepsilon\big\}.$$

Let ω be a $\mathcal{C}^\infty$ degree n nowhere vanishing differential form on X defining the orientation of X.

Definition 2.5. A *family of regularising kernels* on $X\times X$ is a family $(K_\varepsilon(x,y))_{\varepsilon\in\mathbb{R}^+}$ of positive $\mathcal{C}^\infty$ functions on $X\times X$ which has the following two properties. Firstly, for any $\varepsilon>0$ the support of K_ε must be proper, contained in U_ε and contain the diagonal $\Delta\subset X\times X$. Secondly, as ε tends to 0 in $\mathbb{R}^+$ the family of functions $(x\mapsto\int_X K_\varepsilon(x,y)\omega(y))_{\varepsilon\in\mathbb{R}^+}$ must converge uniformly on any compact set in X to the constant function 1.

Definition 2.6. Let f be a continuous function on X. The *family of regularisations of* f is the family of functions $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ defined by

$$f_\varepsilon(x)=\int_X K_\varepsilon(x,y)f(y)\omega(y)\quad\text{for any } x\in X.$$

Definition 2.7. Let $T\in\mathcal{D}'^0(X)$ be a distribution on the manifold X. We define the *family* $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ *of regularisations of* T in the following way: for any φ in $\mathcal{D}^0(V)$ we set

$$\langle T_\varepsilon,\varphi\omega\rangle=\langle T,\varphi_\varepsilon\omega\rangle,\quad\text{where }\varphi_\varepsilon(x)=\int_X K_\varepsilon(x,y)\varphi(y)\omega(y).$$

Definition 2.7 is meaningful because K_ε is $\mathcal{C}^\infty$ on $X\times X$ so φ_ε is $\mathcal{C}^\infty$ on X and the support of φ_ε is contained in the set $\pi_1(\pi_2^{-1}(\operatorname{supp}\varphi)\cap\operatorname{supp}K_\varepsilon)$ which is compact because K_ε is assumed to have proper support.

Proposition 2.8. *Let $T \in \mathcal{D}'^0(X)$ be a distribution on X. The regularisations T_ε of T are then $\mathcal{C}^\infty$ functions on X and for every $y \in X$*

$$T_\varepsilon(y) = \langle T, K_\varepsilon(x,y)\omega(x)\rangle.$$

Proof. Consider $\varphi \in \mathcal{D}^0(X)$. By definition of regularisations,

$$\langle T_\varepsilon, \varphi\omega\rangle = \langle T, \varphi_\varepsilon\omega\rangle = \Big\langle T, \Big(\int_X K_\varepsilon(x,y)\varphi(y)\omega(y)\Big)\omega(x)\Big\rangle.$$

The function $x \mapsto K_\varepsilon(x,y)$ is a $\mathcal{C}^\infty$ compactly supported function on X for any given y and its dependence on y is also $\mathcal{C}^\infty$. Moreover, as ω is a $\mathcal{C}^\infty$ differential form on X, $\langle T, K_\varepsilon(x,y)\omega(x)\rangle$ is a well-defined $\mathcal{C}^\infty$ function on X. Using the density of the vector space generated by functions of the form $u(x)v(y), u, v \in \mathcal{C}^\infty(X)$ in $\mathcal{D}^0(X \times X)$ we get

$$\Big\langle T, \Big(\int_X K_\varepsilon(x,y)\varphi(y)\omega(y)\Big)\omega(x)\Big\rangle = \big\langle\langle T, K_\varepsilon(x,y)\omega(x)\rangle, \varphi(y)\omega(y)\big\rangle$$

and hence it follows by definition of T_ε that $T_\varepsilon(y) = \langle T, K_\varepsilon(x,y)\omega(x)\rangle$. $\square$

Let us now study the convergence of the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ in the weak and strong topologies on $\mathcal{D}'(X)$. Let $\psi \in \mathcal{D}^n(X)$ be a $\mathcal{C}^\infty$ compactly supported differential form of degree n on X. Since we have assumed that ω does not vanish on X, there is a $\varphi \in \mathcal{D}^0(X)$ such that $\psi = \varphi\omega$. Then,

$$\langle T - T_\varepsilon, \psi\rangle = \langle T - T_\varepsilon, \varphi\omega\rangle = \langle T, (\varphi - \varphi_\varepsilon)\omega\rangle.$$

To prove that the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges weakly to T as ε tends to 0 it will be enough to show that $(\varphi_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ tends to φ in $\mathcal{D}^0(X)$. To prove that the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges strongly to T as ε tends to 0, it will be enough to prove that the convergence of the family $(\varphi_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ to φ is uniform with respect to φ over any bounded subset of $\mathcal{D}^0(X)$.

Definition 2.9. A (linear) finite order differential operator with $\mathcal{C}^\infty$ coefficients is a linear map $P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ such that for any local chart (U,h) on X there is a differential operator $P_{(U,h)}$ with $\mathcal{C}^\infty$ coefficients on the open set $h(U)$ in $\mathbb{R}^n$ such that, for any function $f \in \mathcal{C}^\infty(X)$,

$$(Pf) \circ h^{-1} = P_{(U,h)}(f \circ h^{-1}) \quad \text{in } h(U).$$

We denote by P^* the formal adjoint of P with respect to the scalar product on $\mathcal{D}^0(X)$ defined by $(f/g) = \int_X f(y)g(y)\omega(y)$.

Throughout the following the term “differential operator” will always denote a finite order differential operator with $\mathcal{C}^\infty$ coefficients.

Remark 2.10. Let $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ and f be functions in $\mathcal{D}^0(X)$ which are all supported on some fixed compact subset of X. The family $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges to f in $\mathcal{D}^0(X)$ as ε tends to 0 if and only if for any differential operator $P = P(x, D)$,

$$\lim_{\varepsilon\to 0}\sup_{x\in X}\bigl|P(x, D_x)f(x) - P(x, D_x)f_\varepsilon(x)\bigr| = 0.$$

Proposition 2.11. *Consider $f \in \mathcal{D}^0(X)$ and set*

$$f_\varepsilon(x) = \int_X K_\varepsilon(x, y)f(y)\omega(y).$$

The following then hold.

1) *The family $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges to f in $\mathcal{D}^0(X)$ as ε tends to 0 if and only if for any differential operator P on X*

$$(*)\qquad \lim_{\varepsilon\to 0}\sup_{x\in X}\left|\int_X \bigl((P(x, D_x) - P^*(y, D_y))K_\varepsilon(x, y)\bigr)f(y)\omega(y)\right| = 0.$$

2) *The sequence $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges to f and this convergence is uniform with respect to f on any bounded subset of $\mathcal{D}^0(X)$ if and only if the following holds:*

$$(**)\qquad \begin{cases} \textit{For any differential operator } P \textit{ on } X \\ \sup_{x\in X}\left|\int_X \bigl((P(x, D_x) - P^*(y, D_y))K_\varepsilon(x, y)\bigr)f(y)\omega(y)\right| \to 0 \\ \textit{as } \varepsilon \textit{ tends to } 0. \textit{ This convergence is uniform with respect to} \\ f \textit{ in any bounded set in } \mathcal{D}^0(X). \end{cases}$$

Proof.

a) *Necessity.* Assume either that $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges to f in $\mathcal{D}^0(V)$ as ε tends to 0 or that $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges to f as ε tends to 0 and this convergence is uniform with respect to f on any bounded subset of $\mathcal{D}^0(X)$. Then,

$$\begin{aligned}
&\int_X \bigl((P(x, D_x) - P^*(y, D_y))K_\varepsilon(x, y)\bigr)f(y)\omega(y) \\
&\qquad = \int_X (P(x, D_x)K_\varepsilon(x, y))f(y)\omega(y) - \int_X K_\varepsilon(x, y)(P(y, D_y)f(y))\omega(y) \\
&\qquad = P(x, D_x)f_\varepsilon(x) - (P(y, D_y)f(y))_\varepsilon(x) \\
&\qquad \text{(differentiating the first term under the integral sign in each chart)} \\
&\qquad = P(x, D_x)(f_\varepsilon(x) - f(x)) + P(x, D_x)f(x) - (P(y, D_y)f(y))_\varepsilon(x).
\end{aligned}$$

Since $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converges to f in $\mathcal{D}^0(X)$ and P is continuous on $\mathcal{D}^0(X)$,

$$\lim_{\varepsilon\to 0}\sup_{x\in X}|P(x, D_x)(f_\varepsilon(x) - f(x))| = 0.$$

If the convergence of the sequence $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ to f is uniform with respect to f on any bounded subset of $\mathcal{D}^0(X)$, then the above limit is also uniform with respect to f on any bounded subset of $\mathcal{D}^0(X)$.

To complete the proof it will be enough to prove the following lemma which we will then apply to the function $P(x, D_x)f$.

Lemma 2.12. *Let f be a compactly supported continuous function on X. We set $f_\varepsilon(x) = \int_X K_\varepsilon(x,y)f(y)\omega(y)$. The functions $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ then converge uniformly to f on X as ε tends to 0. If moreover $\mathcal{B}$ is an equicontinuous subset of $\mathcal{C}(X)$ consisting of functions which are all supported on some fixed compact set and $\mathcal{B}$ has the property that $\sup\{\|f\|_\infty \mid f\in\mathcal{B}\}$ is finite then the families $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ converge to f and this convergence is uniform with respect to f on $\mathcal{B}$.*

Proof. By definition of f_ε,

$$\begin{aligned} f_\varepsilon(x) - f(x) &= \int_X K_\varepsilon(x,y)f(y)\omega(y) - f(x) \\ &= \int_X K_\varepsilon(x,y)(f(y)-f(x))\omega(y) + f(x)\Big(\int_X K_\varepsilon(x,y)\omega(y) - 1\Big). \end{aligned}$$

$$\begin{aligned} \|f_\varepsilon - f\|_\infty \leqslant \Big\| \int_X K_\varepsilon(x,y)(f(y)-f(x))\omega(y)\Big\|_\infty \\ + \|f\|_\infty \Big\| \Big[\int_X K_\varepsilon(x,y)\omega(y) - 1\Big]\big|_{\operatorname{supp} f}\Big\|_\infty. \end{aligned}$$

As the function f is continuous and compactly supported it is uniformly continuous and hence

$$(\forall\,\alpha>0)(\exists\,\varepsilon_0>0)(\forall\,\varepsilon<\varepsilon_0)((x,y)\in U_\varepsilon \Longrightarrow |f(x)-f(y)|<\alpha).$$

Note that ε_0 is independent of f for $f\in\mathcal{B}$ since the elements of $\mathcal{B}$ are uniformly equicontinuous. It follows that if $\varepsilon<\varepsilon_0$ then

$$\Big\| \int_X K_\varepsilon(x,y)(f(x)-f(y))\omega(y)\Big\|_\infty \leqslant m_{L,\varepsilon}\alpha,$$

where $m_{L,\varepsilon} = \sup_{x\in L}\big(\int_X K_\varepsilon(x,y)\omega(y)\big)$ whenever L is a compact set containing $\operatorname{supp} f \cup \pi_1(\pi_2^{-1}(\operatorname{supp} f))$. By our assumptions on K_ε, $m_{L,\varepsilon}$ can be bounded independently of ε and if L' is a compact set in X containing the support of f then there is a $\varepsilon_0'>0$ such that, for any $\varepsilon<\varepsilon_0'$,

$$\Big\| \int_X K_\varepsilon(x,y)\omega(y) - 1\Big\|_{\infty,L'} < \alpha.$$

We note that if $f\in\mathcal{B}$ then the compact sets L and L' can be chosen independently of f. It follows that if $\varepsilon<\min(\varepsilon_0,\varepsilon_0')$ then

$$\|f-f_\varepsilon\|_\infty \leqslant M\alpha + \|f\|_\infty\alpha.$$

This proves that $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ tends to f as ε tends to 0 and this convergence is uniform with respect to f on $\mathcal{B}$ since $\sup\{\|f\|_\infty \mid f\in\mathcal{B}\}$ is finite. $\square$

End of the proof of Proposition 2.11.

b) *Sufficiency.* To prove that the family $(f_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ tends to f in $\mathcal{D}^0(X)$ as ε tends to 0 it will be enough to prove that, for any differential operator P on X,

$$\lim_{\varepsilon\to 0}\sup_{x\in X}|P(x,D_x)f_\varepsilon(x)-P(x,D_x)f(x)|=0$$

and

$$\begin{aligned}P(x,D_x)f(x)-P(x,D_x)f_\varepsilon(x)&=P(x,D_x)f(x)-(P(y,D_y)f(y)_\varepsilon(x)\\&\quad+(P(y,D_y)f(y))_\varepsilon(x)-P(x,D_x)f_\varepsilon(x)\end{aligned}$$

but now by Lemma 2.12 $\sup_{x\in X}|P(x,D_x)f(x)-(P(y,D_y)f(y))_\varepsilon(x)|$ tends to 0 as ε tends to 0 and

$$\begin{aligned}&\sup_{x\in X}|(P(y,D_y)f(y))_\varepsilon(x)-P(x,D_x)f_\varepsilon(x)|\\&\qquad=\sup_{x\in X}\left|\int_X((P(x,D_x)-P^*(y,D_y))K_\varepsilon(x,y))f(y)\omega(y)\right|\end{aligned}$$

tends to 0 as ε tends to 0 by assumption. The theorem follows. If $(**)$ holds then it is easy to prove that the convergence is uniform with respect to f on any closed set by repeating the proof given above mutatis mutandis. □

Consider a local chart (U,h) on X and let $\xi=(\xi_1,\dots,\xi_n)$ be the associated local coordinates. Let P be a differential operator on X. There is then an operator $P_{(U,h)}$ on $h(U)$ such that, for any $f\in\mathcal{D}^0(V)$,

$$P(f)\circ h^{-1}=P_{(U,h)}(f\circ h^{-1})=\sum_\alpha a_\alpha(\xi)D_\xi^\alpha(f\circ h^{-1}),$$

where $\alpha\in\mathbb{N}^n$, the functions a_α are $\mathcal{C}^\infty$ on $h(U)$ all except a finite number of which are zero, and $D_\xi^\alpha=\partial^{|\alpha|}/\partial\xi_1^{\alpha_1}\cdots\partial\xi_1^{\alpha_n}$.

Proposition 2.13. *Let $(K_\varepsilon(x,y))_{\varepsilon\in\mathbb{R}^+}$ be a family of regularising kernels on $X\times X$. The following are then equivalent.*

$$(*)\quad\begin{cases}\textit{For any differential operator } P \textit{ on } X \textit{ and any function } f\in\mathcal{D}^0(X)\\ \lim\limits_{\varepsilon\to 0}\sup\limits_{x\in X}\left|\int_X\big((P(x,D_x)-P^*(y,D_y))K_\varepsilon(x,y)\big)f(y)\omega(y)\right|=0.\end{cases}$$

$$(*')\quad\begin{cases}\textit{For any function } f\in\mathcal{D}^0(X) \textit{ which is supported in the domain}\\ \textit{of some chart and any multi-index } \alpha\in\mathbb{N}^n\\ \lim\limits_{\varepsilon\to 0}\sup\limits_{x\in X}\left|\int_X\big((D_x^\alpha+(-1)^{|\alpha|+1}D_y^\alpha)K_\varepsilon(x,y)\big)f(y)\omega(y)\right|=0.\end{cases}$$

Proof. It is obvious that $(*)$ implies $(*')$. Conversely, let $(U_i)_{i\in I}$ be a locally finite cover of X by chart domains and let $(\chi_i)_{i\in I}$ be a partition of unity subordinate to this cover. Consider an element $f \in \mathcal{D}^0(X)$ and let P be a differential operator on V; as the support of f meets only a finite number of the open sets U_i, $(U_{i_k})_{k=1,\dots,\ell}$,

$$\int_X \big((P(x,D_x) - P^*(y,D_y))K_\varepsilon(x,y)\big)f(y)\omega(y)$$
$$= \sum_{k=1}^{\ell}\int_X \big((P(x,D_x) - P^*(y,D_y))K_\varepsilon(x,y)\big)\chi_{i_k}(y)f(y)\omega(y).$$

Since $\chi_{i_k}f$ is supported on the domain of the chart U_{i_k} it will therefore be enough to prove $(*)$ for any function $g \in \mathcal{D}^0(X)$ which is supported in the domain of a chart of X. Moreover, by linearity, it will be enough to show that $(*)$ holds for any chart (U,h), function $g \in \mathcal{D}^0(X)$ supported on U and differential operator P on X such that $P_{(U,h)} = a(\xi)D_\xi^\alpha$. In the following calculations we identify U and the open set $h(U)$ in $\mathbb{R}^n$ in order to simplify the notation. The differential operator P can then be written in the form $P(x,D_x) = a(x)D_x^\alpha$ and

$$\int_X \big((P(x,D_x) - P^*(y,D_y)\big)K_\varepsilon(x,y))g(y)dy$$

is the sum of the three following terms

$$\text{(I)} = a(x)\int_X [(D_x^\alpha + (-1)^{|\alpha|+1}D_y^\alpha)K_\varepsilon(x,y)]g(y)dy$$
$$\text{(II)} = (-1)^{|\alpha|}a(x)\int_X (D_y^\alpha K_\varepsilon(x,y))g(y)dy$$
$$\text{(III)} = -\int_X K_\varepsilon(x,y)(P(y,D_y)g(y))dy \quad \text{(by definition of } P^*\text{)}.$$

By $(*')$, (I) converges uniformly to 0 when ε converges to 0 since the continuous function a is bounded on the support of the integral in (I). (The support of this integral is compact because the support of K_ε is proper.)

Integrating by parts we see that (II) $= a(x)(D_y^\alpha g(y))_\varepsilon(x)$ and by Lemma 2.12 this quantity converges uniformly to $a(x)D_x^\alpha g(x)$ on X.

And finally, (III) $= -(a(y)D_y^\alpha g(y))_\varepsilon(x)$ which converges uniformly to $-a(x)D_x^\alpha g(x)$ by Lemma 9.2.9. This completes the proof of the proposition. $\square$

Proposition 2.14. *Let $(K_\varepsilon(x,y))_{\varepsilon\in\mathbb{R}^+}$ be a family of regularising kernels on $X \times X$. The following are then equivalent.*

$$(**)\qquad \begin{cases} \text{For any differential operator } P \text{ on } X \\ \sup_{x\in X}\left|\int_X \big((P(x,D_x) - P^*(y,D_y))K_\varepsilon(x,y)\big)f(y)\omega(y)\right| \longrightarrow 0 \\ \text{as } \varepsilon \text{ tends to } 0. \text{ This convergence is uniform with respect to } f \\ \text{on any bounded set in } \mathcal{D}^0(X). \end{cases}$$

$$(**')\qquad \begin{cases} \text{For any chart domain and any multi-index } \alpha \\ \sup_{x\in X}\left|\int_X \big((D_x^\alpha + (-1)^{|\alpha|+1}D_y^\alpha)K_\varepsilon(x,y)\big)f(y)\omega(y)\right| \longrightarrow 0 \\ \text{as } \varepsilon \text{ tends to } 0. \text{ This convergence is uniform with respect to } f \\ \text{on any bounded set in } \mathcal{D}^0(X) \text{ whose functions are supported} \\ \text{on the domain of a local chart.} \end{cases}$$

Proof. Repeat the proof of Proposition 2.13, noting that it is still possible to use a partition of unity because the functions in a bounded set are all supported on some fixed compact set, and use the results of Lemma 2.12 on the uniformity of the convergence. □

We have therefore proved the following result.

Theorem 2.15. *Let $(K_\varepsilon(x,y))_{\varepsilon\in\mathbb{R}^+}$ be a family of regularising kernels on $X \times X$.*

- *If the equivalent conditions $(*)$ and $(*')$ hold then the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations of T converges weakly to T in $\mathcal{D}'^0(X)$.*
- *If the equivalent conditions $(**)$ and $(**')$ hold then the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations of T converges strongly to T in $\mathcal{D}'^0(X)$.*

We will finish this section by constructing regularising operators whose kernels are regularising kernels as in Definition 2.5 which satisfy conditions $(*)$ and $(**)$. This construction is due to de Rham ([Rh], § 15).

Consider a locally finite countable cover of X by chart domains $(U_i)_{i\in\mathbb{N}}$ which are homeomorphic to $\mathbb{R}^n$. Let h_i be a homeomorphism from U_i to $\mathbb{R}^n$. We can then find an open cover of X by open sets $V_i \Subset U_i$ and $\mathcal{C}^\infty$ functions f_i with compact support contained in U_i such that $f_i \equiv 1$ on $\overline{V}_i$ (cf. Appendix A, Lemmas 2.1 and 2.2). If T is a distribution on X then set $R_{i,\varepsilon}T = \overline{R}_{i,\varepsilon}f_iT + (1-f_i)T$, where $\overline{R}_{i,\varepsilon} = h_i^* r_\varepsilon h_{i_*}$ and r_ε is the convolution on $\mathbb{R}^n$ by the function θ_ε of Section 2.A. In a neighbourhood of any compact set of X the sequence of operators $R^i(\varepsilon) = R_{i,\varepsilon} \circ \cdots \circ R_{1,\varepsilon}$ is stationary. We set $R_\varepsilon = \lim_{i\to\infty} R^i(\varepsilon)$.

The regularising operators constructed by this method are called the *de Rham regularising operators*.

As an example, consider a manifold with an atlas consisting of two charts. Then,

$$R_\varepsilon T = R_{1,\varepsilon}R_{2,\varepsilon}T = \overline{R}_{1,\varepsilon}f_1\overline{R}_{2,\varepsilon}f_2T + \overline{R}_{1,\varepsilon}f_1(1-f_2)T \\ + (1-f_1)\overline{R}_{2,\varepsilon}f_2T + (1-f_1)(1-f_2)T,$$

where the last term in the sum vanishes. The kernel associated to the operator R_ε can be written as

$$K_\varepsilon(x,z) = \int_X K_{1,\varepsilon}(x,y)f_1(y)K_{2,\varepsilon}(y,z)f_2(z)dy \\ + K_{1,\varepsilon}(x,z)f_1(z)(1-f_2(z)) + (1-f_1(x))K_{2,\varepsilon}(x,z)f_2(z),$$

where K_{i,ε_i} is the kernel associated to the image under h_i of convolution with θ_{ε_i}.

The following theorem was proved by de Rham (Theorem 12, [Rh], § 15).

Theorem 2.16. *Let $(R_\varepsilon)_{\varepsilon>0}$ be a family of de Rham regularising operators and let $T \in \mathcal{D}'^0(X)$ be a distribution on X. Then:*

1) *$R_\varepsilon T$ is a $\mathcal{C}^\infty$ function on X,*
2) *The support of $R_\varepsilon T$ is contained in any given neighbourhood of the support of T for small enough ε,*
3) *$R_\varepsilon T$ converges both weakly and strongly to T as ε tends to 0.*

We leave it to the reader to check that the kernels K_ε, $\varepsilon \in \mathbb{R}^+$, associated to the operators R_ε form a regularising family of operators satisfying conditions $(*)$ and $(**)$.

C. Regularising currents

To regularise currents on X we simply replace the kernels in Section B by $\mathcal{C}^\infty$ differential forms on $X \times X$ supported on a fundamental system of neighbourhoods $(U_\varepsilon)_{\varepsilon>0}$ of the diagonal $\Delta \subset X \times X$. Let $(\psi_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ be such a family. If $\varphi \in \mathcal{D}^p(X)$ is a compactly supported $\mathcal{C}^\infty$ differential form on X and π_1 and π_2 are the two projections $X \times X \to X$ then we set

$$\varphi_\varepsilon = (-1)^{np}(\pi_1)_*(\psi_\varepsilon \wedge \pi_2^*\varphi).$$

If, moreover, the support of φ is contained in a chart domain U and ε is chosen small enough that $\pi_2^{-1}(U) \cap U_\varepsilon$ is contained in a chart domain of $X \times X$ then

$$\varphi(x) = \sum_{|I|=p} \varphi_I(x)dx_I \quad \text{for any } x \in U$$

$$\psi_\varepsilon(x,y) = \sum_{I,J} K_{\varepsilon,I,J}(x,y)dx_I \wedge dy_J \quad \text{on } \pi_2^{-1}(U) \cap U_\varepsilon,$$

and hence

$$\varphi_\varepsilon(x) = \sum_{\substack{|I|=p, I\cap J=\varnothing \\ I\cup J=\{1,\ldots,n\}}} \sigma(\tau)\Big(\int_U K_{\varepsilon,I,J}(x,y)\varphi_I(y)dy\Big)dx_I,$$

where $\sigma(\tau)$ is the signature of the permutation $(I,J) \mapsto (1,\ldots,n)$ and dy is the differential form $dy_1 \wedge \cdots \wedge dy_n$.

Reasoning as in Section B we get the following theorem.

Theorem 2.17. *Let $(\psi_\varepsilon)_{\varepsilon>0}$ be a family of $\mathcal{C}^\infty$ differential forms on $X \times X$ with proper support in a fundamental system of neighbourhoods of the diagonal $\Delta \subset X \times X$ such that, for every chart on $X \times X$,*

$$\psi_\varepsilon(x,y) = \sum_{\substack{I\cup J=\{1,\ldots,n\} \\ I\cap J=\varnothing}} K_{\varepsilon,I,J}(x,y)dx_I \wedge dy_J,$$

where the functions $K_{\varepsilon,I,J}$ have the two following properties: firstly, condition $(**')$ of Proposition 2.13 holds and secondly, the functions $(x \mapsto \int_X K_{\varepsilon,I,J}(x,y)dy)$ converge uniformly on any compact subset of the chart of definition to the constant function 1 as ε tends to 0 in $\mathbb{R}^+$. If $T \in \mathcal{D}'^p(X)$ is a degree p current on the manifold X then consider the family $(T_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations of T defined by $\langle T_\varepsilon, \varphi\rangle = \langle T, \varphi_\varepsilon\rangle$ for any $\varphi \in \mathcal{D}^{n-p}(X)$, where φ_ε is the regularisation of φ by ψ_ε. This family then converges to T in both the weak and strong topologies on $\mathcal{D}'^p(X)$ as ε tends to 0 in $\mathbb{R}^+$.

Proof. By definition of the strong topology on $\mathcal{D}'^p(X)$ it will be enough to prove that the family $(\varphi_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ of regularisations of φ converges to φ and this convergence is uniform with respect to φ over any bounded set in $\mathcal{D}^{n-p}(X)$. Let $(U_i)_{i\in I}$ be a cover of X by chart domains in X and let $(\chi_i)_{i\in I}$ be a partition of unity subordinate to this cover. If $\mathcal{B}$ is a bounded set in $\mathcal{D}^{n-p}(X)$ and $\varphi \in \mathcal{B}$ then we set $\varphi_i = \chi_i\varphi$. As the functions φ are all supported on the same compact subset of X we can write $\varphi = \sum_{i\in I'} \varphi_i$, where I' is a finite subset of I which is independent of $\varphi \in \mathcal{B}$. We obtain the desired result by linearity on applying Proposition 2.11 to each of the functions φ_i. □

Definition 2.18. If $(\psi_\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a family of double differential forms satisfying the hypotheses of Theorem 2.17 then the operators R_ε mapping $\mathcal{D}'^\bullet(X)$ to itself defined by $\langle R_\varepsilon T, \varphi\rangle = \langle T, \varphi_\varepsilon\rangle$ for any $T \in \mathcal{D}'^\bullet(X)$ and $\varphi \in \mathcal{D}^\bullet(X)$ are called *regularising operators*. Here, the functions φ_ε are the regularisations of φ obtained using the kernels ψ_ε.

3 Kronecker index of two currents

Throughout the following, X is an n-dimensional $\mathcal{C}^\infty$ differentiable manifold.

Definition 3.1. If T and S are two currents on X such that $d^\circ T + d^\circ S = n$, we will say that the *Kronecker index of* T *and* S, $\mathcal{K}(T, S)$, is defined in de Rham's sense if for any choice of families of regularising operators $(R_\varepsilon)_{\varepsilon>0}$ and $(R'_{\varepsilon'})_{\varepsilon'>0}$ commuting with the operator d the quantity $\langle R_\varepsilon T \wedge R'_{\varepsilon'} S, 1\rangle$ has a limit as ε and ε' tend to 0 which is independent of the choice of $(R_\varepsilon)_{\varepsilon>0}$ and $(R'_{\varepsilon'})_{\varepsilon'>0}$. If this is the case then we denote this limit by $\mathcal{K}(T, S)$.

If $X = \mathbb{C}^n$ then we can choose our regularising operators to be the operators associated to the convolution kernels defined in Section 2.A. If X is a manifold then the de Rham regularising operators commute with the operator d (cf. [Rh], § 15, Prop. 1).

Remarks.

1) The map $(T, S) \mapsto \mathcal{K}(T, S)$ is bilinear.
2) If one of the two currents T or S is a $\mathcal{C}^\infty$ differential form and either the support of T or the support of S is compact then the Kronecker index $\mathcal{K}(T, S)$ of T and S exists and is equal to $\langle T \wedge S, 1\rangle$. (This follows from the convergence properties of regularisations in $\mathcal{D}^\bullet(X)$ and $\mathcal{D}'^\bullet(X)$ for the strong topology.)
3) If the supports of T and S do not meet and T or S has compact support then $\mathcal{K}(T, S)$ is well defined and is equal to 0. Indeed, if ε and ε' are small enough then $R_\varepsilon T$ and $R'_{\varepsilon'} S$ will have disjoint support and it follows that $R_\varepsilon T \wedge R'_{\varepsilon'} S = 0$.

We will now give some sufficient conditions for the Kronecker index of two currents to exist.

Definition 3.2. The *singular support* of a current is the complement of the set of points which have a neighbourhood in which the current is defined by a $\mathcal{C}^\infty$ differential form. We denote by $SS(T)$ the singular support of the current T.

If $T \in \mathcal{D}'^\bullet(X)$ is a current on X and U is a neighbourhood of the singular support of T then we can write $T = T' + T''$, where T' is a current supported on U and T'' is a $\mathcal{C}^\infty$ differential form on X – simply set $T' = \rho T$ and $T'' = (1 - \rho)T$, where ρ is a positive $\mathcal{C}^\infty$ function supported on U and equal to 1 in a neighbourhood of the singular support of T.

Proposition 3.3. *If T and S are two currents on X such that $d^\circ T + d^\circ S = n$, at least one of which is of compact support and whose singular supports do not meet, then the Kronecker index $\mathcal{K}(T, S)$ of T and S is well defined.*

Proof. Under the hypotheses of the proposition there are decompositions $T = T' + T''$ and $S = S' + S''$ such that T'' and S'' are $\mathcal{C}^\infty$ differential forms and the supports of T' and S' do not meet. We can then apply 1), 2) and 3) of the above remark to get $\mathcal{K}(T, S) = \mathcal{K}(T', S'') + \mathcal{K}(T'', S') + \mathcal{K}(T'', S'')$. □

Proposition 3.4. *Let T and S be two currents on X such that $d^\circ T + d^\circ S = n-1$, at least one of which has compact support. If $\mathcal{K}(bT, S)$ or $\mathcal{K}(T, dS)$ exists then then other also exists and they are equal, i.e.*

$$\mathcal{K}(bT, S) = \mathcal{K}(T, dS).$$

Proof. Since the regularising operators R_ε and $R'_{\varepsilon'}$ commute with d and therefore with b,

$$\langle R_\varepsilon bT, R'_{\varepsilon'} S\rangle = \langle bR_\varepsilon T, R'_{\varepsilon'} S\rangle = \langle R_\varepsilon T, dR'_{\varepsilon'} S\rangle = \langle R_\varepsilon T, R'_\varepsilon dS\rangle$$

and the result follows. □

Application. Proposition 3.4 enables us to extend Stokes' theorem to a domain $D \Subset X$ with $\mathcal{C}^1$ boundary and a differential form $\omega \in \mathcal{C}_{n-1}(\overline{D})$ such that $d\omega$, calculated as a current, is continuous on $\overline{D}$. Indeed, setting $T = [D]$ and $S = \omega$ we see that $\mathcal{K}(bT, S)$ exists and is equal to $\int_{bD} \omega$ and it follows that $\int_{bD} \omega = \int_D d\omega$.

Theorem 3.5. *Let T and S be two currents on X such that $d^\circ T + d^\circ S = n$, at least one of which has compact support. The Kronecker index $\mathcal{K}(T, S)$ of the currents T and S exists if*

$$SS(T) \cap SS(bS) = \varnothing \quad \textit{and} \quad SS(bT) \cap SS(S) = \varnothing.$$

Remark. We deduce from Theorem 3.5 that the Kronecker index of T and S exists whenever T is closed with compact support and S is closed.

To prove Theorem 3.5, we need a parametrix of the operator d which does not increase the singular support. The parametrix presented below is due to J.B. Poly.

Proposition 3.6. *If T is a current on X then there are operators A and R such that*

1) $T - RT = dAT + AdT$
2) *A does not increase the singular support and R is regularising.*

Proof. We start with the case $X = \mathbb{R}^n$. Denote by δ the operator on $\mathcal{D}'(\mathbb{R}^n)$ defined as follows: if the current $T \in \mathcal{D}'(\mathbb{R}^n)$, considered as a differential form with distribution coefficients can be written as $T = \sum'_I T_I dx_I$, where $\sum'_I$ denotes $\sum_{\substack{I=(i_1,\dots,i_k)\\ i_1<\cdots<i_k}}$ then we set

$$\delta T = -\sum_J{}' \sum_i \frac{\partial}{\partial x_i} T_{iJ} dx_J.$$

Note that $d\delta + \delta d = -\Delta$, where Δ is the usual Laplacian $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$. Let E be the elementary solution to the Laplacian Δ:

$$E = \begin{cases} \dfrac{1}{(n-2)s_n r^{n-2}} & \text{if } n \geqslant 3, \\ \frac{1}{2\pi}\log r & \text{if } n = 2, \\ r/2 & \text{if } n = 1, \end{cases}$$

where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$ and s_n is the area of the unit sphere in $\mathbb{R}^n$.

For any compactly supported current S on $\mathbb{R}^n$ we set $GS = -E * S$ and $KS = \delta GS$. If S (considered as a differential form with distribution coefficients) can be written as $S = \sum_I' S_I dx_I$ then the convolution product $GS = E * S$ can be written as $GS = -\sum_I' E * S_I dx_I$ whence we get the following expression for KS

$$KS = \sum_J{}' \sum_i \frac{\partial E}{\partial x_i} * S_{iJ} dx_J.$$

The operator $K : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ thus defined has the following properties.

a) $S = dKS + KdS$ since $S = -\Delta GS = d\delta GS + \delta dGS = d\delta GS + \delta GdS$.
b) K does not increase the singular support. Indeed, Δ is elliptic and it follows that GS is $\mathcal{C}^\infty$ outside of the singular support of S.

Let us return to the case where X is a manifold. Since X is a countable union of compact sets, there is a countable locally finite cover of X by chart domains $W_i \Subset X$. By Lemmas 2.3 and 2.1 of Appendix A, we can therefore find a cover of X by open sets V_i such that $V_i \Subset W_i$ and $\mathcal{C}^\infty$ functions η_i with compact support in W_i such that $\eta_i = 1$ in a neighbourhood of V_i. For any current T on X, we set

$$A_i T = \eta_i K(\eta_i T)$$

(where by abuse of notation we consider W_i as an open set in $\mathbb{R}^n$), and

$$\begin{aligned} R_i T &= T - dA_i T - A_i dT \\ &= T - (\eta_i)^2 T + \eta_i K(d\eta_i \wedge T) - d\eta_i \wedge K(\eta_i T). \end{aligned}$$

(To be completely rigorous we should consider the chart (W_i, h_i) in X and define the operator A_i by $A_i T = \eta_i (h_i^{-1})_* \big(K(h_i)_*(\eta_i T)\big)$. The reader can easily check that the identification of W_i and its image under h_i in $\mathbb{R}^n$ does not change any of the properties of A_i but greatly simplifies the notation.)

The operators A_i and R_i have the following properties

a) $T = dA_i T + A_i dT + R_i T$ by construction, from which it follows that $dR_i T = R_i dT$.
b) A_i and R_i do not increase the singular support. Moreover, $R_i T$ is $\mathcal{C}^\infty$ on V_i – indeed $R_i T$ and $K(d\eta_i \wedge T)$ are equal on V_i, and since K does not increase the singular support, $K(d\eta_i \wedge T)$ is $\mathcal{C}^\infty$ on V_i because $d\eta_i \wedge T$ vanishes on V_i.

We set $A^k = A_k R_{k-1} \cdots R_1$ and $R^k = R_k R_{k-1} \cdots R_1$. Since the cover (W_i) is locally finite, it is easy to show that

$$RT = \lim_{k \to \infty} R^k T \quad \text{and} \quad AT = \lim_{k \to \infty} A^k T$$

exist since $R^k T$ is stationary on any open set $U \Subset X$ and $A^k T$ vanishes as soon as k is large enough. The operators A and R have the desired properties:

a) We have

$$\begin{aligned} R^{k-1}T - R^k T &= (1 - R_k) R^{k-1} T \\ &= (dA_k + A_k d) R^{k-1} T \\ &= dA^k T + A^k dT \end{aligned}$$

since R_i and d commute, and summing it follows that

$$T - RT = dAT + AdT.$$

b) A does not increase the singular support since A^k does not increase the singular support. The operator R is regularising since $R^k T$ is $\mathcal{C}^\infty$ on V_i whenever $k \geqslant i$.

Proof of Theorem 3.5. By Proposition 3.6, T and S have the following decompositions:

$$\begin{aligned} T &= RT + dAT + AdT \\ S &= RS + dAS + AdS. \end{aligned}$$

We set

$$\begin{aligned} &T_1 = RT, \quad T_2 = dAT, \quad T_3 = AdT \\ &S_1 = RS, \quad S_2 = dAS, \quad S_3 = AdS. \end{aligned}$$

By linearity, $\mathcal{K}(T, S)$ exists if each of the $\mathcal{K}(T_i, S_k)$ for $i, k = 1, 2, 3$ exists. We note that T_i is a compactly supported current for $i = 1, 2, 3$. The current T_1 is a $\mathcal{C}^\infty$ form because R is regularising. It follows that $\mathcal{K}(T_1, S_k)$ is defined for $k = 1, 2, 3$. As the current S_1 is a $\mathcal{C}^\infty$ form, $\mathcal{K}(T_i, S_1)$ exists for $i = 1, 2, 3$. For $i = k = 2$, we apply Proposition 3.4 and we get $\mathcal{K}(T_2, S_2) = 0$. The cases $i = 2$ and $k = 3$, $i = 3$ and $k = 2$ and $i = k = 3$ follow from Proposition 3.3 because the operators d and A do not increase the singular support. □

Corollary 3.7 (Stokes' formula for the Kronecker index). *Let T and S be two currents on X such that $d^0 T + d^0 S = n - 1$, at least one of which has compact support. If*

$$SS(bT) \cap SS(bS) = \varnothing$$

then the Kronecker indices $\mathcal{K}(bT, S)$ and $\mathcal{K}(T, bS)$ exist and

$$\mathcal{K}(bT, S) = (-1)^{d^0 S - 1} \mathcal{K}(T, bS).$$

Proof. This follows immediately from Theorem 3.5 and Proposition 3.4. □

Example of an application: the Cauchy–Green formula in $\mathbb{C}$. Let D be a bounded open set in $\mathbb{C}$ with $\mathcal{C}^1$ boundary containing the origin. Let ψ be a $\mathcal{C}^\infty$ function on $\mathbb{C}$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ we define degree 1 differential forms dz and $d\overline{z}$ by $dz = dx + idy$ and $d\overline{z} = dx - idy$.

For any $\mathcal{C}^\infty$ function ψ on $\mathbb{C}$ we set $T = \psi[D]$ where $[D]$ is the integration current on D. Then $SS(T) = bD$.

If $S = \frac{1}{2i\pi} dz/z$ then $dS = [0]$, where $[0]$ is the integration current on the point manifold 0. Indeed, if $\varphi \in \mathcal{D}^0(\mathbb{C})$ then, by definition of d,

$$\langle dS, \varphi \rangle = \langle S, d\varphi \rangle = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{\partial \varphi}{\partial \overline{z}}(z) \frac{dz \wedge d\overline{z}}{z}$$

since S is defined by the locally integrable differential form $\frac{1}{2i\pi} dz/z$. For any $\varepsilon > 0$, we set $B_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ and then

$$\langle dS, \varphi \rangle = \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{\mathbb{C} \setminus B_\varepsilon} \frac{\partial \varphi}{\partial \overline{z}}(z) \frac{dz \wedge d\overline{z}}{z} = - \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{\mathbb{C} \setminus B_\varepsilon} d\Big(\frac{\varphi(z)}{z} dz\Big).$$

Applying Stokes' theorem, we get

$$\int_{\mathbb{C} \setminus B_\varepsilon} d\Big(\frac{\varphi(z)}{z} dz\Big) = - \int_{\partial B_\varepsilon} \frac{\varphi(z)}{z} dz$$

since φ has compact support. Moreover,

$$\int_{\partial B_\varepsilon} \frac{\varphi(z)}{z} dz = \int_{\partial B_\varepsilon} \frac{\varphi(z) - \varphi(0)}{z} dz + \varphi(0) \int_{\partial B_\varepsilon} \frac{dz}{z}.$$

As the function φ is $\mathcal{C}^\infty$ and in particular $\mathcal{C}^1$ the first integral on the right-hand side tends to 0 as ε tends to 0. Moreover, we know by Cauchy's formula that $\frac{1}{2i\pi} \int_{\partial B_\varepsilon} dz/z = 1$ for any $\varepsilon > 0$. It follows that

$$\langle dS, \varphi \rangle = \varphi(0) = \langle [0], \varphi \rangle.$$

Moreover, $bT = -d\psi \wedge [D] + \psi \wedge [bD]$ and hence $SS(bT) = SS(T) = bD$: as we also have $SS(bS) = \{0\}$ it follows that $SS(bT) \cap SS(bS) = \varnothing$ since $0 \in D$.

We can therefore apply Corollary 3.7 and we get

$$\begin{aligned}
\mathcal{K}(bT, S) &= \mathcal{K}\Big(- d\psi \wedge [D] + \psi \wedge [bD], \frac{1}{2i\pi} \frac{dz}{z}\Big) \\
&= - \frac{1}{2i\pi} \int_D d\psi \wedge \frac{dz}{z} + \frac{1}{2i\pi} \int_{bD} \psi(z) \frac{dz}{z} \\
&= \frac{1}{2i\pi} \Big(\int_{bD} \psi(z) \frac{dz}{z} + \int_D \frac{\partial \psi}{\partial \overline{z}}(z) \frac{dz \wedge d\overline{z}}{z} \Big). \\
\mathcal{K}(T, dS) &= \mathcal{K}(\psi[D], [0]) = \langle \psi, [0] \rangle = \psi(0).
\end{aligned}$$

Finally, it follows that

$$\psi(0) = \frac{1}{2i\pi} \Big(\int_{bD} \frac{\psi(z)}{z} dz + \int_D \frac{\partial \psi}{\partial \overline{z}}(z) \frac{dz \wedge d\overline{z}}{z} \Big).$$

Geometric interpretation of the Kronecker index. We state the following result without proof: the interested reader will find more details and better sufficient conditions for the existence of the Kronecker index of two currents in [Rh, § 20] and [L-T1].

If Y and Z are two p- and $(n-p)$-dimensional closed oriented submanifolds of X which meet transversally such that either Y or Z is a compact submanifold of X, then the integration currents $[Y]$ and $[Z]$ on Y and Z are closed and therefore satisfy the hypotheses of Theorem 3.5. Then

$$\mathcal{K}([Y],[Z]) = \langle [Y \cap Z], 1 \rangle.$$

Here $Y \cap Z$ contains a finite number of points and $\langle [Y \cap Z], 1 \rangle$ is equal to the number of points of $Y \cap Z$ where the orientations of Y and Z coincide minus the number of points where they differ. More generally, if Y and Z are two closed oriented submanifolds of X of dimensions p and q respectively which meet transversally in such a way that $Y \cap Z$ is a submanifold of X and φ is a $\mathcal{C}^\infty$ differential form of degree $p+q-n$ with compact support on X, then

$$\mathcal{K}([Z],[Y] \wedge \varphi) = \langle [Z \cap Y], \varphi \rangle.$$

4 Complex analytic manifolds

When studying holomorphic functions, it is natural to try to introduce objects which play the role in the holomorphic setting which is played by differentiable manifolds in the differential setting, that is, objects which locally inherit the analytic properties of open sets in $\mathbb{C}^n$.

Definition 4.1. Let X be a topological space. A *complex atlas* on X is a set of charts (U, φ) such that the domains U form an open cover of X and the maps φ are homeomorphisms from U to an open set in $\mathbb{C}^n$ satisfying the holomorphic compatibility condition: if $U \cap U' \neq \varnothing$ then the map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \longrightarrow \varphi'(U \cap U')$$

is a biholomorphic map between two open sets in $\mathbb{C}^n$. We say that two complex atlases are compatible if their union is also a complex atlas. Compatibility is an equivalence relation.

Definition 4.2. A *complex analytic manifold* is a Hausdorff topological space which is a countable union of compact sets equipped with an equivalence class of complex atlases.

For any complex analytic manifold X, any point of $x \in X$ and any chart (U, φ) in a neighbourhood of x the map φ is a homeomorphism from U to an open set in some $\mathbb{C}^n$ and the number n is called the *complex dimension* of X at x. (Of course, exactly as for differentiable manifolds, the number n

is independent of the choice of chart (U,φ) in a neighbourhood of x.) We say that X is a *complex analytic manifold of dimension* n if for any $x \in X$ the complex dimension of X at x is n. If X and Y are two complex analytic manifolds then a map $f : X \to Y$ is said to be *holomorphic* if it is continuous and for any pair of charts (U,φ) and (V,Ψ) of X and Y such that $f(U) \subset V$, the map $\psi \circ (f\big|_U) \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic. When $Y = \mathbb{C}$ such a map will be called a holomorphic function. Of course, exactly as for $\mathcal{C}^q$ maps, holomorphy can be checked on all the charts of some given atlas only. We denote the vector space of complex-valued holomorphic functions on X by $\mathcal{O}(X)$.

Definition 4.3. Let (U,φ) be a chart of a complex analytic manifold. The function φ is then a holomorphic map from U to $\mathbb{C}^n$ and we can write $\varphi(x) = (z_1(x),\ldots,z_n(x))$ where the $z_j : U \to \mathbb{C}$ $(j = 1,\ldots,n)$ are holomorphic functions on U. The functions $(z_1,\ldots,z_n)$ are then called the *holomorphic coordinates* of X on U defined by the chart (U,φ).

Remark. It is clear that any n-dimensional complex analytic manifold X has a natural $2n$-dimensional $\mathcal{C}^\infty$ differentiable manifold structure. The tangent and cotangent spaces T_xX and T_x^*X of X at x are therefore well defined. In particular, we have a space $\mathbb{C}T_x^*X$ of complex-valued differential 1-forms at $x \in X$ which is the dual of the complexified space $\mathbb{C}T_xX$ of T_xX (cf. Appendix A, §4). Let us look at what happens in a chart (U,φ) in a neighbourhood of x whose local coordinates are $(z_1,\ldots,z_n)$ where $z_j = x_j + iy_j$, $j = 1,\ldots,n$. The family $\{(dx_1)_x,(dy_1)_x,\ldots,(dx_n)_x(dy_n)_x\}$ is then a basis for $\mathbb{C}T_x^*X$ and the corresponding dual basis in $\mathbb{C}T_xX$ is $\{(\partial/\partial x_1)_x,(\partial/\partial y_1)_x,\ldots,(\partial/\partial x_n)_x,(\partial/\partial y_n)_x\}$. It is often more convenient to consider the basis

$$\{(dz_1)_x,(d\overline{z}_1)_x,\ldots,(dz_n)_x,(d\overline{z}_n)_x\}$$

in $\mathbb{C}T_x^*X$ and the associated dual basis in $\mathbb{C}T_xX$ which we denote by

$$\Big\{\Big(\frac{\partial}{\partial z_1}\Big)_x,\Big(\frac{\partial}{\partial \overline{z}_1}\Big)_x,\ldots,\Big(\frac{\partial}{\partial z_n}\Big)_x,\Big(\frac{\partial}{\partial \overline{z}_n}\Big)_x\Big\}.$$

By definition,

$$(dz_j)_x\Big(\Big(\frac{\partial}{\partial z_k}\Big)_x\Big) = \delta_{jk}, \qquad (d\overline{z}_j)_x\Big(\Big(\frac{\partial}{\partial z_k}\Big)_x\Big) = 0,$$
$$(dz_j)_x\Big(\Big(\frac{\partial}{\partial \overline{z}_k}\Big)_x\Big) = 0 \quad \text{and} \quad (d\overline{z}_j)_x\Big(\Big(\frac{\partial}{\partial \overline{z}_k}\Big)_x\Big) = \delta_{jk}.$$

If f is a complex-valued $\mathcal{C}^1$ function on a neighbourhood of x in X then its differential df_x defines an element of T_x^*X which by definition of a dual basis can be written as

$$(df)_x = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)(dx_j)_x + \frac{\partial f}{\partial y_j}(x)(dy_j)_x$$

or alternatively

$$(df)_x = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(x)(dz_j)_x + \frac{\partial f}{\partial \overline{z}_j}(x)(d\overline{z}_j)_x$$

depending on our choice of basis. An easy calculation shows that

$$\Big(\frac{\partial}{\partial z_j}\Big)_x = \frac{1}{2}\Big(\Big(\frac{\partial}{\partial x_j}\Big)_x - i\Big(\frac{\partial}{\partial y_j}\Big)_x\Big) \quad \text{and} \quad \Big(\frac{\partial}{\partial \overline{z}_j}\Big)_x = \frac{1}{2}\Big(\Big(\frac{\partial}{\partial x_j}\Big)_x + i\Big(\frac{\partial}{\partial y_i}\Big)_x\Big)$$

which agrees with the definition given in Chapter I.

We further note that the construction of the complexifications of T_x^*X and T_xX – in other words the construction of complex-valued forms – does not involve the complex structure on X and can be carried out for any differentiable manifold. We have only used the complex structure on X when writing certain expressions in local coordinates.

We end this section by proving that any complex analytic manifold is orientable. Let X be a complex analytic manifold of dimension n. By Proposition 6.2 of Appendix A, X is orientable if it has a $\mathcal{C}^\infty$ atlas, $(U_i, h_i)_{i\in I}$, such that, for any $i, j \in I$,

$$d_{ij}(x) = \det\big[J(h_i \circ h_j^{-1})(h_j(x))\big] > 0 \quad \text{for any } x \in U_i \cap U_j.$$

Consider a complex atlas $(U_j, \varphi_j)_{j\in J}$ on X. If $\varphi_j = h_j + ik_j$ then the set $(U_j, (h_j, k_j))_{j\in J}$ is a $\mathcal{C}^\infty$ atlas on X such that, for any $x \in U_i \cap U_j$,

$$d_{ij}(x) = \det \begin{vmatrix} \overline{J(\varphi_i \circ \varphi_j^{-1})}(x) & 0 \\ 0 & J(\varphi_i \circ \varphi_j^{-1})(x) \end{vmatrix} = |J(\varphi_i \circ \varphi_j^{-1})|^2(x) > 0.$$

Any complex analytic manifold X is therefore orientable. Throughout the rest of this book all complex analytic manifolds will be equipped with the following orientation: given a complex atlas $(U_i, \varphi_i)_{i\in I}$ on X, we consider the holomorphic coordinates $(z_1, \ldots, z_n)$ associated to the chart (U_i, φ_i) and we choose the orientation associated to the $2n$-differential form

$$d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n$$

which is simply the orientation defined by

$$dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n,$$

where $z_j = x_j + iy_j$.

5 Complex structures

Let X be a complex analytic manifold of dimension n. We will show that at any point $x \in X$ the real vector space T_x^*X has a natural $\mathbb{C}$-vector space structure.

We start by considering the case where $X = \mathbb{C}^n$. $\mathbb{C}^n$ can be naturally identified with $\mathbb{R}^{2n}$ as a $\mathcal{C}^\infty$ manifold, so $T_x\mathbb{C}^n = T_x\mathbb{R}^{2n} = \mathbb{R}^{2n} = \mathbb{C}^n$, where the last equality is just the natural identification of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. Identifying $T_x\mathbb{C}^n$ with $\mathbb{C}^n$ imposes a $\mathbb{C}$ vector space structure on $T_x\mathbb{C}^n$. Let us examine this identification more carefully. We denote the multiplication by i map by $J : T_x\mathbb{C}^n \to T_x\mathbb{C}^n$; it is an $\mathbb{R}$-linear map such that $J^2 = -\operatorname{Id}$. In the standard basis

$$\Big\{\Big(\frac{\partial}{\partial x_1}\Big)_x, \Big(\frac{\partial}{\partial y_1}\Big)_x, \dots, \Big(\frac{\partial}{\partial x_n}\Big)_x, \Big(\frac{\partial}{\partial y_n}\Big)_x\Big\},$$

for $T_x\mathbb{C}^n = T_x\mathbb{R}^{2n}$, we have $J((\frac{\partial}{\partial x_j})_x) = (\frac{\partial}{\partial y_j})_x$ and $J((\frac{\partial}{\partial y_j})_x) = -(\frac{\partial}{\partial x_j})_x$, $1 \leqslant j \leqslant n$. It follows that

$$(a+ib)\nu = a\nu + bJ(\nu).$$

for any $\nu \in T_x\mathbb{C}^n$ and any complex number $a + ib \in \mathbb{C}$

The homogeneous Cauchy–Riemann equations, which encode the $\mathbb{C}$-linearity at x of $(df)_x$ for any holomorphic function germ f, can be written as

$$(df)_x(J\nu) = i(df)_x(\nu) \quad \text{for any } \nu \in T_x\mathbb{C}^n.$$

Remark. This relationship between the complex structure on $T_x\mathbb{C}^n$ and holomorphic functions implies that J is independent of the choice of coordinates on $\mathbb{C}^n$. We can therefore define J on the tangent space of a complex analytic manifold.

Theorem 5.1. *Let X be a complex analytic manifold. For any $x \in X$, there is a unique $\mathbb{R}$-linear map $J = J_x : T_xX \to T_xX$ such that, for any holomorphic function germ at x,*

$$(df)_x(J\nu) = i(df)_x(\nu) \quad \textit{for every } \nu \in T_xX.$$

Moreover, $J^2 = -\operatorname{Id}$ and setting $(a+ib)\nu = a\nu + bJ(\nu)$ for any $a + ib \in \mathbb{C}$ and $\nu \in T_xX$ yields a complex vector space structure on T_xX.

Proof. We start by proving that if J exists then it is unique.

Let $(z_1, \dots, z_n)$ be a system of holomorphic coordinates in a neighbourhood of $x \in X$ and let $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ be the underlying real coordinates. Then, for any $j = 1, \dots, n$, $(dz_j)_x = (dx_j)_x + i(dy_j)_x$ or alternatively $(dx_j)_x = \operatorname{Re}(dz_j)_x$ and $(dy_j)_x = \operatorname{Im}(dz_j)_x$.

We now calculate $J(\frac{\partial}{\partial x_k})_x$ for $k = 1, \dots, n$. We have

$$J\Big(\frac{\partial}{\partial x_k}\Big)_x = \sum_{j=1}^n \Big[dx_j\Big(J\Big(\frac{\partial}{\partial x_k}\Big)\Big)\Big]\frac{\partial}{\partial x_j} + \Big[dy_j\Big(J\Big(\frac{\partial}{\partial x_k}\Big)\Big)\Big]\frac{\partial}{\partial y_j}.$$

As z_j is holomorphic, $dz_j(\frac{\partial}{\partial x_k})_x = \delta_{jk}$, and

$$dx_j\Big(J\Big(\frac{\partial}{\partial x_k}\Big)_x\Big) = \operatorname{Re} dz_j\Big(J\Big(\frac{\partial}{\partial x_k}\Big)_x\Big) = \operatorname{Re} i dz_j\Big(\frac{\partial}{\partial x_k}\Big)_x = \operatorname{Re} i\delta_{jk} = 0,$$
$$dy_j\Big(J\Big(\frac{\partial}{\partial x_k}\Big)_x\Big) = \operatorname{Im} dz_j\Big(J\Big(\frac{\partial}{\partial x_k}\Big)_x\Big) = \operatorname{Im} i dz_j\Big(\frac{\partial}{\partial x_k}\Big)_x = \delta_{jk}.$$

It follows that $J(\frac{\partial}{\partial x_k})_x = (\frac{\partial}{\partial y_k})_x$ and likewise $J(\frac{\partial}{\partial y_k})_x = -(\frac{\partial}{\partial x_k})_x$ and hence J is unique.

We now define the map J by the above equations with respect to some chosen set of holomorphic coordinates $(z_1, \dots, z_n)$. We saw above for $\mathbb{C}^n$ that J then has the desired properties and the fact that any map having these properties is unique implies that the map J thus defined is independent of our choice of holomorphic coordinates. $\square$

We now consider the link between the complex structures on $T_x X$ and $\mathbb{C}T_x X$. The map $J : T_x X \to T_x X$ is $\mathbb{R}$-linear and $J^2 = -\mathrm{Id}$. This map therefore has no real eigenvalues and if we want to diagonalise it we have to consider the natural extension of J to a $\mathbb{C}$-linear map, also denoted J, from $\mathbb{C}T_x X$ to itself. This extension then has two eigenvalues, i and $-i$. We denote the eigenspace associated to i by $T_x^{1,0}X$ and the eigenspace associated to $-i$ by $T_x^{0,1}X$: we then have

$$\mathbb{C}T_x X = T_x^{1,0}X \oplus T_x^{0,1}X.$$

If $(z_1, \dots, z_n)$ is a system of holomorphic coordinates in a neighbourhood of x then the vectors $((\frac{\partial}{\partial z_1})_x, \dots, (\frac{\partial}{\partial z_n})_x)$ form a basis for $T_x^{1,0}X$ and the vectors $((\frac{\partial}{\partial \bar{z}_1})_x, \dots, (\frac{\partial}{\partial \bar{z}_n})_x)$ form a basis of $T_x^{0,1}X$.

We note that $T_x X$ with the $\mathbb{C}$-vector space structure defined by J is naturally isomorphic to $T_x^{1,0}X$ via the map sending ν to $\frac{1}{2}(\nu - iJ(\nu))$. This map sends the family $((\frac{\partial}{\partial x_1})_x, \dots, (\frac{\partial}{\partial x_n})_x)$ – which is a basis for $T_x X$ as a $\mathbb{C}$-vector space – to the basis $((\frac{\partial}{\partial z_1})_x, \dots, (\frac{\partial}{\partial z_n})_x)$ of $T_x^{1,0}X$ and is therefore an isomorphism.

We define $T^{1,0}(X)$ to be the disjoint union of the spaces $T_x^{1,0}(X)$ for all $x \in X$ and we denote the natural projection from $T^{1,0}(X)$ to X by p.

Definition 5.2. Let X be a complex analytic manifold and let A be an open set in X. A *field of holomorphic vectors* on A is a map $V : A \to T^{1,0}(X)$ such that $p \circ V = \mathrm{Id}$.

6 Differential forms of type (p, q)

Let X be a complex analytic manifold of dimension n and let x be a point in X. The fact that $T_x X$ has a complex vector space structure leads us to give special consideration to those elements in $\mathbb{C}T_x X$ which are $\mathbb{C}$-linear with respect to this structure.

We define the space of differential 1-forms of type $(1, 0)$ at x by

$$\Lambda^{1,0}(T_x^* X) = \{\omega \in \mathbb{C}T_x^* X \mid \omega(J\nu) = i\omega(\nu), \forall\, \nu \in \mathbb{C}T_x X\}.$$

Example. The differentials $(df)_x$ of germs of holomorphic functions at x are of type $(1, 0)$ by definition of J.

If $(z_1, \ldots, z_n)$ are local holomorphic coordinates defined in a neighbourhood of x then the family $((dz_1)_x, \ldots, (dz_n)_x)$ forms a basis for $\Lambda^{1,0}(T_x^* X)$. The conjugate space $\Lambda^{0,1}T_x^*X = \overline{\Lambda^{1,0}T_x^*X}$, which has a basis given in these coordinates by $((d\overline{z}_1)_x, \ldots, (d\overline{z}_n)_x)$, is the space of forms of type $(0,1)$ at x. There is a direct sum decomposition

$$\mathbb{C}T_x^*X = \Lambda^{1,0}T_x^*X \oplus \Lambda^{0,1}T_x^*X. \tag{6.1}$$

We now consider forms of higher degree. If ω is a complex-valued differential form of degree r then it is a linear combination of elements of the form $\omega_1 \wedge \cdots \wedge \omega_r$ where $\omega_j \in \mathbb{C}T_x^*X$. By (3.1), each ω_j, $1 \leqslant j \leqslant r$, can be written in the form $\omega_j' + \omega_j''$, where $\omega_j' \in \Lambda^{1,0}T_x^*X$ and $\omega_j'' \in \Lambda^{0,1}T_x^*X$. It follows that ω is a linear combination of elements of the form $\eta_1 \wedge \cdots \wedge \eta_r$ where each η_j is either of type $(0,1)$ or of type $(1,0)$.

We say that ω is a differential form of type (p,q) or bidegree (p,q) at x if ω is a linear combination of elements of the form $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \overline{\omega}_{j_1} \wedge \cdots \wedge \overline{\omega}_{j_q}$ where all the ω_ν are 1-forms of type $(1,0)$ at x.

We denote by $\mathcal{C}^k_{p,q}(X)$ the subspace of $\mathcal{C}^k_{p+q}(X)$ consisting of $(p+q)$-differential forms which are of type (p,q) at every point. We then have a direct sum decomposition

$$\mathcal{C}^k_r(X) = \bigoplus_{p+q=r} \mathcal{C}^k_{p,q}(X).$$

Note that $\mathcal{C}^k_{p,q}(X) = \{0\}$ if p or $q > n = \dim_{\mathbb{C}} X$.

If $(z_1, \ldots, z_n)$ are holomorphic coordinates on a chart domain $U \subset X$ then $dz_j \in \mathcal{C}^\infty_{1,0}(U)$ for any $j = 1, \ldots, n$ and any (p,q)-form $\omega \in \mathcal{C}^k_{p,q}(U)$ can then be written uniquely in the form

$$\omega = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} dz_I \wedge d\overline{z}_J,$$

where the a_{IJ} are $\mathcal{C}^k$ functions on U and the sum is taken over strictly increasing multi-indices $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$.

7 The $\overline{\partial}$ operator and Dolbeault cohomology

The decomposition of differential 1-forms on a complex analytic manifold into type $(0,1)$ and type $(1,0)$ forms induces a natural decomposition of the exterior differential operator d into a holomorphic differential operator and an antiholomorphic differential operator.

Let X be a complex analytic manifold of dimension n. If f is a $\mathcal{C}^1$ function on X then, for any $x \in X$,

$$(df)_x = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(x)(dz_j)_x + \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}(x)(d\overline{z}_j)_x.$$

We can therefore write $df = \partial f + \overline{\partial} f$ where ∂f is a differential form of type $(1,0)$ on X and $\overline{\partial} f$ is a differential form of type $(0,1)$ on X. We note that the condition $f \in \mathcal{O}(X)$ is then equivalent to $\overline{\partial} f = 0$.

The decomposition $d = \partial + \overline{\partial}$ can be extended to forms of any degree in the following way. Suppose that $\omega \in \mathcal{C}^1_{p,q}(X)$ is given in some local system of holomorphic coordinates $(z_1, \ldots, z_n)$ in a neighbourhood of $x \in X$, by $\omega = \sum_{\substack{|I|=p \\ |J|=q}} a_{IJ} dz_I \wedge d\overline{z}_J$. By definition of d,

$$dw = \sum_{\substack{|I|=p \\ |J|=q}} d(a_{IJ}) \wedge dz_I \wedge d\overline{z}_J = \sum_{\substack{|I|=p \\ |J|=q}} (\partial a_{IJ} + \overline{\partial} a_{IJ}) \wedge dz_I \wedge d\overline{z}_J .$$

We then set

$$\partial \omega = \sum_{\substack{|I|=p \\ |J|=q}} \partial(a_{IJ}) dz_I \wedge d\overline{z}_J = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^{n} \frac{\partial}{\partial z_k} a_{IJ} dz_k \wedge dz_I \wedge d\overline{z}_J$$

and

$$\overline{\partial} \omega = \sum_{\substack{|I|=p \\ |J|=q}} \overline{\partial}(a_{IJ}) dz_I \wedge d\overline{z}_J = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^{n} \frac{\partial a_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J .$$

We have therefore defined operators ∂ and $\overline{\partial}$ on $\mathcal{C}^1_{p,q}(X)$ such that $\partial(\mathcal{C}^1_{p,q}(X))$ is contained in $\mathcal{C}_{p+1,q}(X)$ and $\overline{\partial}(\mathcal{C}^1_{p,q}(X))$ is contained in $\mathcal{C}_{p,q+1}(X)$

Proposition 7.1. *The operators ∂ and $\overline{\partial}$ have the following properties:*

a) $d = \partial + \overline{\partial}$ *on* $\mathcal{C}^1_{\bullet}(X)$,
b) $\partial \circ \partial = 0$, $\overline{\partial} \circ \overline{\partial} = 0$ *and* $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$ *on* $\mathcal{C}^2_{\bullet,\bullet}(X)$,
c) ∂ *and* $\overline{\partial}$ *commute with pullback.*

Proof. Property a) follows immediately from the definitions of ∂ and $\overline{\partial}$.

Consider an element $\omega \in \mathcal{C}^2_{p,q}(X)$. As $d \circ d = 0$ and $d = \partial + \overline{\partial}$,

$$0 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial})\omega = (\partial \circ \partial)\omega + (\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)\omega + (\overline{\partial} \circ \overline{\partial})\omega .$$

But $(\partial \circ \partial)\omega$ is now of type $(p+2,q)$, $(\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)\omega$ is of type $(p+1,q+1)$ and $(\overline{\partial} \circ \overline{\partial})\omega$ is of type $(p,q+2)$. It follows that each of the terms vanishes since their sum vanishes.

Let $F : X \to Y$ be a holomorphic map: note that if $(\zeta_1, \ldots, \zeta_n)$ are holomorphic coordinates in a neighbourhood of a point $y \in Y$ then $F^*\zeta_j = \zeta_j \circ F$ is a holomorphic function in a neighbourhood of $x = F^{-1}(y)$ and hence $F^*(d\zeta_j) = d(\zeta_j \circ F)$ is a $(1,0)$ form and $F^*(d\overline{\zeta}_j) = d(\overline{\zeta_j \circ F})$ is a $(0,1)$ form. Using local coordinates, this implies that $F^*(\mathcal{C}^k_{p,q}(X)) \subset \mathcal{C}^k_{p,q}(X)$ for all

$p, q \geqslant 0$ and $k \geqslant 0$. Since Proposition 8.4 of Appendix A says that $dF^* = F^*d$, we can write

$$\partial(F^*\omega) + \overline{\partial}(F^*\omega) = d(F^*\omega) = F^*(d\omega) = F^*(\partial\omega) + F^*(\overline{\partial}\omega)$$

for any $\omega \in \mathcal{C}_{p,q}(Y)$, where the last equality holds because F^* is linear. Comparing bidegrees, we see that $\partial(F^*\omega) = F^*(\partial\omega)$ and $\overline{\partial}(F^*\omega) = F^*(\overline{\partial}\omega)$. □

The operator $\overline{\partial}$ defined above is called the *Cauchy–Riemann operator.*

We denote the space of (p, q)-forms of class $\mathcal{C}^\infty$ on X by $\mathcal{E}^{p,q}(X)$ for any $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n$. These spaces are equipped with a Fréchet space topology which can be characterised as follows: a sequence $(\omega_j)_{j\in\mathbb{N}}$ of elements in $\mathcal{E}^{p,q}(X)$ converges to 0 if and only if for any chart domain U in X on which ω_j can be written as $\omega_j = \sum_{\substack{|I|=p \\ |J|=q}} \omega^j_{IJ} dz_I \wedge d\overline{z}_J$ the sequences $(\omega^j_{IJ})_{j\in\mathbb{N}}$ converge to zero uniformly on any compact subset of U and so do all their derivatives.

The operator $\overline{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X)$ is then a continuous linear operator and its kernel, denoted by $Z^{p,q}(X)$, is therefore closed.

Definition 7.2. We define the *Dolbeault cohomology groups* to be the spaces

$$H^{p,q}(X) = Z^{p,q}(X)/\overline{\partial}\mathcal{E}^{p,q-1}(X).$$

These spaces are naturally equipped with the quotient topology which is not generally Hausdorff because the space $\overline{\partial}\mathcal{E}^{p,q-1}(X)$ is not always closed. If $\overline{\partial}\mathcal{E}^{p,q-1}(X)$ is closed then $H^{p,q}(X)$ is a Fréchet space.

These groups encode the obstruction to solving the Cauchy–Riemann equations $\overline{\partial}u = f$ for any $f \in Z^{p,q}(X)$.

We end by defining Dolbeault cohomology groups with support conditions. We denote by c the family of compact subsets of X. For any compact subset K in a manifold M, Φ denotes the family of closed sets in $X = M \setminus K$ whose closure is compact in M and Ψ denotes the family of closed sets in M which do not meet K: for simplicity we let Θ be one of these three families. The space $\mathcal{E}^{p,q}_\Theta(X)$ is then the space of $\mathcal{C}^\infty$ (p, q)-forms on X whose support is contained in the family Θ. If $\theta \in \Theta$, we denote by $\mathcal{E}^{p,q}_\theta(X)$ the subspace of $\mathcal{E}^{p,q}(X)$ consisting of (p, q)-forms supported on θ. Then $\mathcal{E}^{p,q}_\Theta(X) = \bigcup_{\theta\in\Theta} \mathcal{E}^{p,q}_\theta(X)$. We note that if Θ is one of the three families c, Φ or Ψ then X has an exhaustion $(\theta_i)_{i\in\mathbb{N}}$ by elements of Θ (i.e. $X = \bigcup_{i\in\mathbb{N}} \theta_i, \theta_i \subset \overset{\circ}{\theta}_{i+1}$). The spaces $\mathcal{E}^{p,q}_\theta(X)$ are closed in $\mathcal{E}^{p,q}(X)$; they are therefore Fréchet spaces and the topology on $\mathcal{E}^{p,q}_\Theta(X)$ is the finest topology for which the inclusions $\mathcal{E}^{p,q}_{\theta_i}(X) \hookrightarrow \mathcal{E}^{p,q}_\Theta(X)$ are all continuous. The operator $\overline{\partial}$ is then a continuous linear operator from $\mathcal{E}^{p,q}_\Theta(X)$ to $\mathcal{E}^{p,q+1}_\Theta(X)$. We set $Z^{p,q}_\Theta(X) = Z^{p,q}(X) \cap \mathcal{E}^{p,q}_\Theta(X)$.

Definition 7.3. The *Dolbeault cohomology groups with support in* Θ are the spaces

$$H^{p,q}_\Theta(X) = Z^{p,q}_\Theta(X)/\overline{\partial}\mathcal{E}^{p,q-1}_\Theta(X).$$

We equip these groups with the quotient topology which is not generally Hausdorff. They encode the obstruction to solving the Cauchy–Riemann equation in the class of forms with support in the family Θ.

8 Complex tangent space to the boundary of a domain

When we come to define CR functions (Chapter IV) and pseudoconvex domains (Chapter VI) we will need the properties of the tangent space to the boundary of a domain with smooth boundary in a complex analytic manifold. The aim of this section is to study the analytic properties of this space: in particular, we will consider its interaction with the complex structure of the surrounding manifold.

We initially only assume that X is a $\mathcal{C}^\infty$ differentiable manifold.

Definition 8.1. Let D be an open set in X. For any $1 \leqslant k \leqslant \infty$ we say that D has $\mathcal{C}^k$ boundary in a neighbourhood of $p \in \partial D$ if there is an open neighbourhood U of p in X and a real-valued $\mathcal{C}^k$ function $r \in \mathcal{C}^k(U)$ such that

$$\begin{cases} U \cap D = \{x \in U \mid r(x) < 0\} \\ dr(x) \neq 0, \quad x \in U. \end{cases} \tag{8.1}$$

We say that ∂D is $\mathcal{C}^k$ if it is $\mathcal{C}^k$ in a neighbourhood of every point. A function $r \in \mathcal{C}^k(U)$ such that (8.1) holds is called a *defining function* for D at p. If U is a neighbourhood of ∂D then r is called a global defining function.

Lemma 8.2. *Let r_1 and r_2 be two defining functions for D which are $\mathcal{C}^k$ on a neighbourhood U of $p \in \partial D$. There is then a strictly positive function $h \in \mathcal{C}^{k-1}(U)$ such that*

$$\begin{cases} r_1 = hr_2 & \text{on } U \\ dr_1(x) = h(x)dr_2(x) & \text{for all } x \in U \cap \partial D. \end{cases} \tag{8.2}$$

Proof. Note that h is unique if it exists because it is continuous on U and equal to r_1/r_2 on $U \smallsetminus \partial D$.

Without loss of generality we can assume that U is contained in a chart domain of X. Consider a point $q \in U \cap \partial D$ and choose coordinates on U such that $q = 0$ and $U \cap \partial D = \{x \in \mathbb{R}^n \mid x_n = 0\}$. We can assume that $r_2(x) = x_n$. For any $x' = (x_1, \ldots, x_{n-1})$ close enough to 0, we have $r_1(x', 0) = 0$ and hence

$$r_1(x', x_n) = r_1(x', x_n) - r_1(x', 0) = x_n \int_0^1 \frac{\partial r_1}{\partial x_n}(x', tx_n)dt.$$

We set $h(x) = \int_0^1 \frac{\partial r_1}{\partial x_n}(x', tx_n)dt$; $h(x)$ is then a $\mathcal{C}^{k-1}$ function on U such that $r_1 = hr_2$ on U.

If $k \geqslant 2$ then $dr_1(x) = r_2(x)dh(x) + h(x)dr_2(x) = h(x)dr_2(x)$ for any $x \in U \cap \partial D$.

If $k = 1$, $r_1(x) = h(x)r_2(x) = \big(h(x) - h(x',0)\big)r_2(x) + h(x',0)r_2(x)$ and hence $r_1(x) = h(x',0)dr_2(x',0) + o(x_n)$ as x_n tends to 0 since h is continuous on U and $r_2(x',0) = 0$. It therefore follows that $dr_1(x) = h(x)dr_2(x)$ for any $x \in U \cap \partial D$.

It remains to prove that h is strictly positive on U. As $h = r_1/r_2$ on $U \smallsetminus D$, h is strictly positive on $U \smallsetminus D$ because r_1 and r_2 are defining functions. As $dr_1(x) \neq 0$ on U and $dr_1(x) = h(x)dr_2(x)$ on $U \cap \partial D$, h does not vanish on $U \cap \partial D$. As h is continuous on U it is strictly positive on U. $\square$

If D is a domain with $\mathcal{C}^k$ boundary in a neighbourhood of $p \in \partial D$ then ∂D is a $\mathcal{C}^k$ differentiable manifold in a neighbourhood of p. We can therefore consider the tangent space $T_p(\partial D)$ to ∂D at p.

Proposition 8.3. *If r is a defining function for D at p then*

$$T_p(\partial D) = \{\xi \in T_p(X) \mid dr(p)(\xi) = 0\}. \tag{8.3}$$

If $(x_1, \dots, x_n)$ are local coordinates on X in a neighbourhood of p then $\xi \in T_p(\partial D)$ if and only if

$$\xi = \sum_{j=1}^{n} \xi_j \Big(\frac{\partial}{\partial x_j}\Big)_p \quad \textit{where} \quad \sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0.$$

Proof. Let U be a neighbourhood of p such that

$$\partial D \cap U = \{x \in U \mid r(x) < 0\} \quad \text{and} \quad dr(x) \neq 0 \text{ for any } x \in U.$$

We denote the inclusion of $\partial D \cap U$ in X by i. This inclusion induces an injective map $di : T_p(\partial D) \hookrightarrow T_p(X)$ such that if α is a curve in ∂D passing through p representing the vector $\nu \in T_p(\partial D)$ then $\xi = di(\nu)$ is the class of $i \circ \alpha$. Since the image of α is contained in ∂D we have $r \circ i \circ \alpha \equiv 0$ and hence

$$dr(p)(\xi) = \frac{d}{dt}(r \circ i \circ \alpha)(0) = 0$$

which proves that on identifying $T_p(\partial D)$ and $di(T_p(\partial D))$

$$T_p(\partial D) \subset \{\xi \in T_p(X) \mid dr(p)(\xi) = 0\}. \tag{8.4}$$

As both sides of (8.4) are vector spaces of dimension $(n-1)$, the inclusion of (8.4) is in fact an equality. $\square$

Remark. Equation (8.3) shows that we can identify $T_p(\partial D)$ with the set of directional derivatives at p which vanish on r.

Assume now that X is a complex analytic manifold of dimension n and D is an open set in X with $\mathcal{C}^k$ boundary in a neighbourhood of $p \in \partial D$. The complex structure on X induces an extra structure on $T_p(\partial D)$.

As we have seen above, we can identify $T_p(\partial D)$ with a real subspace of real dimension $(2n-1)$ in $T_p(X)$. If J is the complex structure on $T_p(X)$ we can consider $JT_p(\partial D)$, which is also a real subspace of real dimension $(2n-1)$ in $T_p(X)$, and hence

$$T_p^{\mathbb{C}}(\partial D) = T_p(\partial D) \cap JT_p(\partial D)$$

is a real subspace of real dimension $(2n-2)$ in $T_p(X)$ which is stable under J. This space is therefore a complex subspace of $T_p(X)$ of complex dimension $(n-1)$. We note that $T_p^{\mathbb{C}}(\partial D) \neq \{0\}$ if and only if $n \geqslant 2$. The space $T_p^{\mathbb{C}}(\partial D)$ is called the *complex tangent space* to ∂D at p. If we identify $T_p(X)$ with the complex structure J with $T_p^{1,0}(X)$ then $T_p^{\mathbb{C}}(\partial D)$ becomes a subspace of $T_p^{1,0}(X)$.

Proposition 8.4. *If r is a defining function for D at $p \in \partial D$ then*

$$T_p^{\mathbb{C}}(\partial D) = \{t \in T_p^{1,0}(X) \mid \partial r(p)(t) = 0\}.$$

If $(z_1, \ldots, z_n)$ are holomorphic local coordinates for X in a neighbourhood of p then $t \in T_p^{\mathbb{C}}(\partial D)$ if and only if

$$t = \sum_{j=1}^{n} t_j \Big(\frac{\partial}{\partial z_j}\Big)_p, \quad \textit{where} \; \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) t_j = 0.$$

Proof. As the function r is real-valued

$$dr(p) = \partial r(p) + \overline{\partial} r(p) = 2 \operatorname{Re} \partial r(p).$$

By definition of $T_p^{\mathbb{C}}(\partial D) = T_p(\partial D) \cap JT_p(\partial D)$ and (8.3),

$$T_p^{\mathbb{C}}(\partial D) = \{t \in T_p^{1,0}(X) \mid dr(p)(t) = dr(p)(Jt) = 0\}.$$

But now $\partial r(p)(Jt) = i\partial r(p)(t)$ since $\partial r(p)$ is a $(1,0)$ differential form at p and hence

$$\operatorname{Re} \big(\partial r(p)(Jt)\big) = -\operatorname{Im} \partial r(p)(t).$$

It follows that

$$\begin{aligned} T_p^{\mathbb{C}}(\partial D) &= \{t \in T_p^{1,0}(X) \mid \operatorname{Re} \partial r(p)(t) = \operatorname{Im} \partial r(p)(t) = 0\} \\ &= \{t \in T_p^{1,0}(X) \mid \partial r(p)(t) = 0\}. \end{aligned}$$ □

Let $\mathbb{C}T_p(\partial D)$ be the complexification of $T_p(\partial D)$. $T_p^{\mathbb{C}}(\partial D)$ is then an $(n-1)$-dimensional subspace of $\mathbb{C}T_p(\partial D)$.

Definition 8.5. The vector space $T_p^{0,1}(\partial D) = \overline{T_p^{\mathbb{C}}(\partial D)}$, the conjugate of $T_p^{\mathbb{C}}(\partial D)$ in $\mathbb{C}T_p(\partial D)$, is called the *space of tangential Cauchy–Riemann operators* at $p \in \partial D$.

Note that if r is a defining function for ∂D at p and $(z_1, \ldots, z_n)$ are local holomorphic coordinates on X in a neighbourhood of p then a vector $\tau \in T_p^{0,1}(\partial D)$ if and only if

$$\tau = \sum_{j=1}^{n} \tau_j \Big(\frac{\partial}{\partial \overline{z}_j}\Big)_p \quad \text{where} \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\overline{\tau}_j = 0.$$

Example. If $n = 2$ then $T_p^{0,1}(\partial D)$ is a 1-dimensional $\mathbb{C}$-vector space generated by

$$L_p = \frac{\partial r}{\partial \overline{z}_2}(p)\frac{\partial}{\partial \overline{z}_1} - \frac{\partial r}{\partial \overline{z}_1}(p)\frac{\partial}{\partial \overline{z}_2}.$$

Comments

The theory of currents is developed in Schwarz's book [Sc] and de Rham's book [Rh]. De Rham's book [Rh] also contains a discussion of regularisations on manifolds and the Kronecker index and more information on the Kronecker index can be found in [L-T1]. Whilst writing Sections 4 to 8 of this chapter the author relied on the sections dealing with similar material in Section 2 of Chapter III and Section 2 of Chapter II of M. Range's book [Ra]. The interested reader may consult R. Narasimhan's book [Na2] for more details.

http://www.springer.com/978-0-85729-029-8

Holomorphic Function Theory in Several Variables
An Introduction
Laurent-Thiébaut, C.
2011, XIII, 252 p., Softcover
ISBN: 978-0-85729-029-8