Chapter 2
Stabilization of Abstract Parabolic Systems

We discuss here a few stabilization techniques for nonlinear parabolic-like equations in Hilbert spaces. The abstract theory of stabilization presented below captures most of the techniques to be developed for the specific problems which are treated in the next chapters. As a matter of fact, most of the stabilization results for Navier–Stokes equations can be formulated and proven for control systems in Hilbert spaces governed by so-called *abstract parabolic* systems to be defined below.

2.1 Nonlinear Parabolic-like Systems

Consider a real Hilbert space H with the scalar product $(\cdot,\cdot)$ and norm $|\cdot|_H$ and $F: D(F) \subset H \to H$ a nonlinear operator on H with domain $D(F)$. In almost all the situations considered in the following, F is of the form

$$Fy = Ay + F_0(y), \quad \forall y \in D(A), \tag{2.1}$$

where A is a closed and densely defined linear operator on H with domain $D(A)$ and $F_0 : D(F_0) \subset H \to H$ is a nonlinear operator.

We assume that

(i) $-A$ generates a C_0-analytic semigroup on H.
(ii) F_0 is Gâteaux differentiable on $D(A)$, that is,

$$F_0'(y^*)(z) = \lim_{\lambda \to 0} F_0(y^* + \lambda z) - F_0(y^*) \tag{2.2}$$

exists in H for all $y^*, z \in D(A)$, $F_0(0) = 0$, and for some $\alpha \in (0,1)$

$$|F_0'(y^*)z|_H \le \alpha |Az|_H + C|z|, \quad \forall z \in D(A). \tag{2.3}$$

It is easily seen that, for each $y \in D(A)$, the operator

$$\mathscr{A} = A + F_0'(y^*), \qquad D(\mathscr{A}) = D(A) \tag{2.4}$$

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is closed, densely defined and $-\mathscr{A}$ generates a C_0-semigroup on H. (See Theorem 1.12.) The linear operator $\mathscr{A}$ can be viewed as the linearization of F in y^*.

A nonlinear operator of the form $\frac{d}{dt}+F$ with F satisfying Conditions (i), (ii) is called *abstract parabolic* operator.

The standard example is $H=L^2(\mathscr{O})$ and $F: L^2(\mathscr{O}) \to L^2(\mathscr{O})$ defined by

$$F(y)(x) = -\Delta y(x) + \beta(y(x)) + g(\nabla y(x)), \quad \text{a.e. } x \in \mathscr{O}, \tag{2.5}$$

where $\mathscr{O}$ is an open, bounded domain of R^d with smooth boundary $\partial\mathscr{O}$, $1 \le d \le 3$, and $\beta \in C^1(R)$, $g \in C^1(R^d)$, $\beta' \in L^\infty(R)$, $g' \in L^\infty(R^d)$, $\beta(0)=0$, $g(0)=0$. In this case, we have for all $y \in D(A) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$,

$$F_0'(y)z = \beta'(y)\, z + g'(\nabla y)z, \quad \text{a.e. } x \in \mathscr{O}, \forall z \in L^2(\mathscr{O}),$$

and Conditions (i), (ii) are obviously satisfied. More general, C^1-functions β, g with polynomial growth satisfy (ii) if the growth of β and the dimension d of the space are correlated via Sobolev imbedding theorem but the details are omitted.

Consider the Cauchy problem

$$\begin{aligned} &\frac{dy}{dt} + F(y) = 0, \quad t \ge 0, \\ &y(0) = y_0. \end{aligned} \tag{2.6}$$

Under appropriate conditions on F, this problem is well-posed, but we do not discuss this here since Cauchy problems of this type were presented in Sect. 1.4. We simply assume that (2.6) generates a semigroup (semiflow)

$$y(t) = y(t, y_0), \quad \forall t \ge 0. \tag{2.7}$$

An equilibrium (steady-state) solution y_e to system (2.6) is a solution to the stationary equation

$$F(y_e) = 0. \tag{2.8}$$

The equilibrium solution y_e is said to be *stable* or, more precisely, *asymptotically stable* if

$$\lim_{t\to\infty} y(t, y_0) = y_e,$$

for all y_0 in a neighborhood $\mathscr{V}$ of y_e. As is well-known, the stability of y_e can be reduced to the stability of the solution $y=0$ to the system

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A}y + G(y) = 0, \quad t \ge 0, \\ &y(0) = y_0 - y_e, \end{aligned} \tag{2.9}$$

where $\mathscr{A}$ is defined by (2.4) with $y^* = y_e$

$$G(y) = F_0(y+y_e) - F_0(y_e) - F_0'(y_e)(y). \tag{2.10}$$

If the equilibrium solution y_e to system (2.6) is not stable (that is, asymptotically stable), the standard way to stabilize it is to associate with (2.6) a *control system*

$$\begin{aligned}&\frac{dy}{dt}+F(y)=Bu, \quad t\geq 0,\\&y(0)=y_0,\end{aligned} \tag{2.11}$$

where the controller function $u:[0,\infty)\to U$ takes values in another space U (which is assumed Hilbert everywhere in the following) and B is a linear, closed operator from U to H.

The stabilization problem is to find a controller $u\in L^2(0,\infty;U)$ such that the corresponding solution $y=y(t,y_0,u)$ to system (2.11) has the property that

$$\lim_{t\to\infty} y(t,y_0,u)=y_e,$$

for y_0 in a neighborhood of y_e. In such a case, system (2.11) is said to be *stabilizable*. If the *stabilizable controller* u is in feedback form, that is,

$$u(t)=K(y(t)), \quad \forall t\geq 0, \tag{2.12}$$

where K is a given operator form H to U, then system (2.11) is said to be *feedback stabilizable*.

The *stabilization problem* for such a system is to find a feedback controller of the form (2.12) which stabilizes the equilibrium solution y_e, that is, in a neighborhood $\mathscr{V}(y_e)$ of y_e we have that

$$\lim_{t\to\infty} y_K(t,y_0)=y_e, \quad \forall y_0\in\mathscr{V}(y_e), \tag{2.13}$$

where $y=y_K$ is the solution to the *closed-loop system*

$$\begin{aligned}&\frac{dy}{dt}+F(y)-BKy=0, \quad \forall t\geq 0,\\&y(0)=y_0.\end{aligned} \tag{2.14}$$

We do not discuss here the existence of solutions y to (2.14) which follows under specific assumptions on F and K by the general results presented in Sect. 1.4. It should be emphasized, however, that the true controller in (2.11) is the "acting" controller $v(t)=Bu(t)$ which is the realization of the input controller u under the operator B. Larger is the space $R(B)$, more probably is the stabilization effect but a large space $R(B)$ means also a large space of controllers u, which implies of course an expensive stabilization procedure. The true objective of the stabilization theory is to obtain stabilization via a "minimal" class of input controllers u. Roughly speaking, this means that the space $\{v=Bu\}$ of "acting" controllers should be a proper subspace of H or, as the case will be in the examples presented below, it has zero element intersection with the space H. (This happens, for instance, if B is unbounded.)

Two classes of controllers (or, more precisely, of control systems of the form (2.11)), are largely used in stabilization theory of *parameter distributed systems*, that is, of infinite-dimensional systems represented by partial differential equations: *internal controllers* and *boundary controllers*.

1° Internal control systems are control systems of the form (2.11), where B is a linear continuous operator form U to H (that is, $B \in L(U, H)$). A typical example is

$$(Bu)(x) = \mathbf{1}_{\mathscr{O}_0}(x)u(x), \quad \forall x \in \mathscr{O}, \tag{2.15}$$

where $\mathbf{1}_{\mathscr{O}_0}$ is the characteristic function of a subdomain $\mathscr{O}_0 \subset \mathscr{O}$, that is,

$$\mathbf{1}_{\mathscr{O}_0}(x) = 1, \quad \forall x \in \mathscr{O}_0; \qquad \mathbf{1}_{\mathscr{O}_0}(x) = 0, \quad \forall x \in \mathscr{O}_0^c = \mathscr{O} \setminus \mathscr{O}_0. \tag{2.16}$$

This means that the corresponding control system with F given by (2.5) is

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \beta(y) + g(\nabla y) = \mathbf{1}_{\mathscr{O}_0} u && \text{in } (0,\infty) \times \mathscr{O}, \\ &y(0,x) = y_0(x), && x \in \mathscr{O}, \\ &y(t,x) = 0, && \forall t \geq 0,\ x \in \partial\mathscr{O}. \end{aligned} \tag{2.17}$$

In this case, the acting controller $v = \mathbf{1}_{\mathscr{O}} u$ is active on the subset $\mathscr{O}_0$ of $\mathscr{O}$ only. In terms of automatic control theory, this means that the control actuation is on the subset $\mathscr{O}_0$. So, the objective of the stabilization problem in this case is to construct a controller u (in feedback form) such that $y(t, y_0) \to y_e$ in $L^2(\mathscr{O})$ as $t \to \infty$. Of course, on this line other types of internal stabilizable controllers are relevant, but that presented above is most important.

2° Boundary control systems. An abstract boundary control problem is that in which $B \in L(U, X')$ where X is a Hilbert space such that $X \subset H$ algebraically and topologically and X' is the dual of X in the duality induced by H, that is, with H as pivot space. In other words, $X \subset H \subset X'$ algebraically and topologically. The precise description of this functional setting will be given in Sect. 2.2.

A typical example of such a control system, if one invokes once again the parabolic operator (2.5) is,

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \beta(y) + g(\nabla y) = 0 && \text{in } (0,\infty) \times \mathscr{O}, \\ &y(0,x) = y_0(x), && x \in \mathscr{O}, \\ &\frac{\partial y}{\partial n} = u && \text{on } (0,\infty) \times \partial\mathscr{O}, \end{aligned} \tag{2.18}$$

where the flux $u \in L^2_{\mathrm{loc}}(0,\infty; L^2(\partial\mathscr{O}))$, is a boundary controller. This is a control system with flux actuation on the boundary $\partial\mathscr{O}$. Such a system can be written as (2.11), where $H = L^2(\mathscr{O})$, $U = L^2(\partial\mathscr{O})$ and $B \in L(U, (H^1(\mathscr{O}))')$ given by

$$(Bu, \psi) = \int_{\partial\mathscr{O}} u(\xi)\psi(\xi)d\xi, \quad \forall \psi \in H^1(\mathscr{O}). \tag{2.19}$$

(Here, $(H^1(\mathscr{O}))'$ is the dual of $H^1(\mathscr{O}) \subset L^2(\mathscr{O})$ in the pairing $(\cdot,\cdot)$ induced by scalar product $(\cdot,\cdot)$ of $L^2(\mathscr{O})$.)

A more delicate problem arises in the case of Dirichlet boundary control system

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \beta(y) + g(\nabla y) = 0 && \text{in } (0,\infty) \times \mathscr{O}, \\ &y(0,x) = y_0(x), && x \in \mathscr{O}, \\ &y = u, && \text{on } (0,\infty) \times \partial\mathscr{O}, \end{aligned} \tag{2.20}$$

where $u \in L^2_{\text{loc}}(0,\infty; L^2(\partial\mathscr{O}))$.

In order to represent (2.20) into form (2.11), we consider first the so-called *Dirichlet map* $\widetilde{y} = Du$ which is defined as the solution to the Dirichlet problem

$$\begin{aligned} -\Delta\widetilde{y} &= 0 \quad \text{in } \mathscr{O}, \\ \bar{y} &= u \quad \text{on } \partial\mathscr{O}. \end{aligned} \tag{2.21}$$

It turns out (see, e.g., [60]) that $D \in L(L^2(\partial\mathscr{O}), H^{\frac{1}{2}}(\mathscr{O}))$. Then, subtracting (2.20) and (2.21), we obtain that

$$\begin{aligned} &\frac{\partial}{\partial t}(y - Du) - \Delta(y - Du) + \beta(y) + g(\nabla y) = -\frac{\partial}{\partial t} Du, \quad t > 0, \text{ in } (0,\infty) \times \mathscr{O}, \\ &y - Du = 0, \quad \text{on } \partial\mathscr{O}, \\ &(y - Du)(0) = y_0 - Du(0), \quad \text{in } \mathscr{O}. \end{aligned}$$

Substituting $y - Du = z$, we reduce the latter to the differential equation in $H = L^2(\mathscr{O})$,

$$\begin{aligned} &\frac{d}{dt} z + A_0 z + \beta(z + Du) + g(\nabla z + \nabla Du) = -\frac{d}{dt} Du, \quad t \geq 0, \\ &z(0) = y_0 - Du(0), \end{aligned}$$

where $A_0 = -\Delta$, $D(A_0) = H^1_0(\mathscr{O}) \cap H^2(\mathscr{O})$.

Equivalently,

$$\begin{aligned} z(t) = e^{-A_0 t}(y_0 - Du(0)) &- \int_0^t e^{-A_0(t-s)}(\beta(z(s) + Du(s)) + g(\nabla z + \nabla Du))ds \\ &- \int_0^t e^{-A_0(t-s)} \frac{d}{ds} Du(s) ds, \quad t \geq 0. \end{aligned}$$

Integrating by parts, we obtain that

$$\begin{aligned} y(t) = e^{-A_0 t} y_0 + \int_0^t A_0 e^{-A_0(t-s)} Du(s) ds - \int_0^t e^{-A_0(t-s)}(\beta(y(s)) + g(\nabla y(s)))ds, \\ \forall t \geq 0, \end{aligned}$$

that is,

$$\frac{dy}{dt} + A_0 y + \beta(y + g(\nabla y)) = A_0 Du, \quad t \geq 0,$$
$$y(0) = y_0, \tag{2.22}$$

where the operator $Bu = A_0 Du$, $\forall u \in L^2(\partial\mathscr{O})$ is defined from $U = L^2(\partial\mathscr{O})$ to $(D(A_0))'$ by (see (1.9))

$$Bu(\psi) = (Du, A_0\psi), \quad \forall \psi \in D(A_0) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O}). \tag{2.23}$$

Clearly, $B \in L(U, (D(A_0))')$. Hence, we are in the general situation presented above where $U = L^2(\partial\mathscr{O})$, $H = L^2(\mathscr{O})$ and $X = D(A_0) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$, $X' = (D(A_0))'$.

The general feature of boundary control systems of the form (2.18) and (2.22) is that the control operator B is unbounded from U to H and so the "acting" controller $v = Bu$ takes values in a larger space $X' \supset H$. In the case of Dirichlet boundary control system (2.20) (equivalently (2.22)) the space X' is a distribution space, while in the case of Neumann boundary control system (2.18) it is an abstract space $(H^1(\mathscr{O}))'$. It is useful to notice that in both cases (and this is a general property of abstract boundary control systems) the space $\{v = Bu;\ u \in U\}$ of "acting" controllers has zero element intersection with H and so it is a "meager" control set.

The first step to stabilization of steady-state solution y_e to system (2.6) is of course the stabilization of the linearized system

$$\frac{dy}{dt} + \mathscr{A} y = 0, \quad t \geq 0,$$
$$y(0) = y_0, \tag{2.24}$$

where $\mathscr{A}$ is given by (2.4) with $y^* = y_e$.

We discuss this problem separately for the internal and the boundary stabilization case.

2.2 Internal Stabilization of Linearized System

Consider the controlled system

$$\frac{dy}{dt} + \mathscr{A} y = Bu, \quad y \geq 0,$$
$$y(0) = y_0, \tag{2.25}$$

where, in agreement with Hypotheses (i) and (ii), we assume that

(H1) *$-\mathscr{A}$ generates a C_0-analytic semigroup and the resolvent $(\lambda I - \mathscr{A})^{-1}$ of $\mathscr{A}$ is compact in H.*

As regards the operator $B : U \to H$, we assume that

(H2) $B \in L(U, H)$.

Hypothesis (H1) implies, via Fredholm–Riesz theory (see Theorem 1.1), that the operator $\mathscr{A}$ has a countable set of eigenvalues λ_j and corresponding eigenvectors φ_j, that is,

$$\mathscr{A}\varphi_j = \lambda_j \varphi_j, \quad j = 1, \dots. \tag{2.26}$$

We recall (see Sect. 1.1) that, for each λ_j, there is a finite number m_j of linear independent eigenvectors $\{\varphi_j^i\}_{i=1}^{m_j}$ and m_j is called the multiplicity of λ_j. It should be emphasized that some of the eigenvectors φ_j^i, $i = 1, \dots, m_j$, might be generalized eigenvectors (that is, $(\mathscr{A} - \lambda_j)^k \varphi_j^k = 0$, $1 < k \le m_j$). The algebraic multiplicity m_j of λ_j is the number of generalized eigenvectors while the geometric multiplicity $\widetilde{m}_j$ is the number of proper vectors φ_j^k (that is, $\mathscr{A}\varphi_j^k = \lambda_j \varphi_j^k$, $1 \le k \le \widetilde{m}_j$). In general, we have $1 < \widetilde{m}_j \le m_j$, for all j. An eigenvalue λ_j is called *semisimple* if all the eigenvectors are proper (that is, $\widetilde{m}_j = m_j$).

The spectrum $\sigma(\mathscr{A}) = \{\lambda_j\}_{j=1}^{\infty}$ is said to be *semisimple* if all the eigenvalues λ_j are semisimple. An eigenvalue λ_j is said to be simple if $m_j = 1$.

From now on, each eigenvalue λ_j will be repeated according to its multiplicity m_j in order to have a correspondence $\lambda_j \to \varphi_j$, $j = 1, \dots$.

Taking into account that some of the eigenvalues λ_j might be complex, it will be convenient in the sequel to view $\mathscr{A}$ as a linear operator (again denoted $\mathscr{A}$) in the complexified space $\widetilde{H} = H + iH$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product of $\widetilde{H}$ and by $|\cdot|_{\widetilde{H}}$ its norm. We denote again by $\sigma(\mathscr{A})$ the spectrum of $\mathscr{A}$ and notice that each finite part of the spectrum, let say $\{\lambda_j\}_{j=1}^N$, can be separated from the rest of spectrum by a rectifiable contour Γ_N in the complex space $\mathbb{C}$. If we denote by $\mathscr{X}_N$ the linear space generated by eigenvectors $\{\varphi_j\}_{j=1}^N$, that is,

$$\mathscr{X}_N = \text{lin span}\{\varphi_j\}_{j=1}^N,$$

then the operator $P_N = \widetilde{H} \to \mathscr{X}_N$ defined by

$$P_N = \frac{1}{2\pi i} \int_{\Gamma_N} (\lambda I - \mathscr{A})^{-1} d\lambda \tag{2.27}$$

is the algebraic projection of $\widetilde{H}$ onto $\mathscr{X}_N$ (that is, $\mathscr{X} = P_N \widetilde{H}$). (See, Theorem 1.3.) Moreover, the operator

$$\mathscr{A}_N = P_N \mathscr{A} \tag{2.28}$$

maps the space $\mathscr{X}_N$ into itself and $\sigma(\mathscr{A}_N) = \{\lambda_j\}_{j=1}^N$. In fact, $\mathscr{A}_N : \mathscr{X}_N \to \mathscr{X}_N$ is finite-dimensional and can be represented by an $N \times N$ matrix.

If $\mathscr{A}^*$ is the dual operator of $\mathscr{A}$, then its eigenvalues are precisely $\overline{\lambda}_j$, $j = 1, \dots,$ and the corresponding eigenvectors

$$\mathscr{A}^* \varphi_j^* = \overline{\lambda}_j \varphi_j^*, \quad j = 1, \dots$$

have the same properties as φ_j. In particular, the multiplicity of φ_j^* coincides with the multiplicity m_j of φ_j and the dual operator P_N^* of P_N is given by

$$P_N^* = \frac{1}{2\pi} \int_{\overline{\Gamma}_N} (\lambda I - \mathscr{A}^*)^{-1} d\lambda,$$

while $\mathscr{X}_N^* = \text{lin span}\{\varphi_j^*\}_{j=1}^N = P_N^* \widetilde{H}$.

We have also the following proposition.

Proposition 2.1 *Assume that the spectrum $\sigma(\mathscr{A})$ is semisimple. Then there is a biorthogonal system $\{\varphi\}_{j=1}^\infty$, $\{\varphi_j^*\}_{j=1}^\infty$ of eigenfunctions, that is,*

$$\langle \varphi_j, \varphi_i^* \rangle = \delta_{ij}, \quad i, j = 1, \dots, \tag{2.29}$$

$$\mathscr{A}\varphi_j = \lambda_j \varphi_j, \qquad \mathscr{A}^* \varphi_j^* = \overline{\lambda}_j \varphi_j^*. \tag{2.30}$$

Here and everywhere in the following, $\langle \cdot, \cdot \rangle$ is the scalar product in the complexified space $\widetilde{H} = H + iH$.

The proof of Proposition 2.1 follows immediately if taking into account that for $\lambda_j \neq \lambda_i$ we have by (2.30) that

$$\lambda_j \langle \varphi_j, \varphi_i^* \rangle = \langle \varphi_j, \mathscr{A}^* \varphi_i^* \rangle = \lambda_i \langle \varphi_j, \varphi_i^* \rangle.$$

If $\lambda_j = \lambda_i$, then (2.29) follows via the Schmidt orthogonalization procedure.

Let $\gamma > 0$ and let $N = \inf\{j;\ \operatorname{Re} \lambda_j \geq \gamma\}$. By Assumption (H1) it follows that $N < \infty$. Let

$$M = \max\{m_j;\ j = 1, \dots, N\}. \tag{2.31}$$

First, we study the stabilization of System (2.25) under Hypotheses (H1), (H2) and

(H3) *The eigenvalues $\{\lambda_j,\ j = 1, \dots, N\}$ are semisimple.*

Let $\mathscr{B}$ be the $N \times M$ matrix

$$\mathscr{B} = \| \langle B\varphi_j^*, \varphi_i^* \rangle \|_{i=1}^{N}{}_{j=1}^{M} \tag{2.32}$$

and let D_k, $k = 1, \dots, \ell$, be the matrices

$$\begin{aligned} D_1 &= \left\| \langle \varphi_i^*, B^* \varphi_j^* \rangle \right\|_{i=1\, j=1}^{m_1, M}, \\ D_2 &= \left\| \langle \varphi_i^*, B^* \varphi_j^* \rangle \right\|_{i=m_1+1, j=1}^{m_1+m_2,\, M}, \dots, \\ D_\ell &= \left\| \langle \varphi_i^*, B^* \varphi_j^* \rangle \right\|_{i=m_{\ell-1}+1, j=1}^{m_{\ell-1}+m_\ell, M}. \end{aligned} \tag{2.33}$$

Theorem 2.1 *Assume that Hypotheses* (H1)$\sim$(H3) *hold. Assume also that*

$$\operatorname{rank} D_k = m_k, \quad \forall k = 1, 2, \dots, \ell. \tag{2.34}$$

Then there is a controller $u = u(t)$ of the form

$$u(t) = \sum_{j=1}^{M} v_j(t)\varphi_j^*, \quad v_j \in L^2(0,\infty), \tag{2.35}$$

which stabilizes exponentially the complexified system (2.25) with exponent decay $-\gamma$. Moreover, the controller $v = \{v_j\}_{j=1}^M$ can be chosen in the feedback form

$$v_j(t) = -\langle B\varphi_j^*, R_0 y^*(t)\rangle, \quad j = 1, \dots, M,\ t \geq 0, \tag{2.36}$$

where $R_0 \in L(\widetilde{H}, \widetilde{H})$, $R_0 = R_0^*$, $R_0 \geq 0$ is the solution to the algebraic Riccati equation (2.48).

It should be said that M is the minimal dimension of the stabilizable controller u. As a matter of fact, we can replace M in Theorem 2.1 by any number $M \leq \widetilde{M} \leq N$ for which (2.34) holds. In particular, if

$$\det \| \langle B\varphi_j^*, \varphi_i^* \rangle \|_{i=1}^{N}{}_{j=1}^{N} \neq 0,$$

then one might take $\widetilde{M} = N$. However, depending on the multiplicity of eigenvalues λ_j, this number $\widetilde{M}$ might be $< N$. For instance, we have

Corollary 2.1 *Assume that the eigenvalues $\{\lambda_j\}_{j=1}^N$ are simple and $\langle B\varphi_j^*, \varphi_1^*\rangle \neq 0$, $\forall j = 1, \dots, N$. Then the stabilizable controller u can be chosen of the form*

$$u(t) = v(t)\varphi_1^*, \quad \forall t \geq 0,$$

where $v(t) = -\langle B\varphi_1^*, R_0 y^*(t)\rangle$.

In Theorem 2.1, $\{\varphi_j^*\}$ is the dual system of eigenvectors satisfying (2.29) and (2.30).

Proof of Theorem 2.1 We represent the solution y to System (2.25) as $y = y_u + y_s$, where $y_u = P_N y$, $y_s = (I - P_N)y$. Recalling Notation (2.28), we may rewrite System (2.25) with controller (2.35) as

$$\begin{aligned} \frac{dy_u}{dt} + \mathscr{A}_u y_u &= \sum_{j=1}^{M} v_j(t) P_N B\varphi_j^*, \quad t \geq 0, \\ y_u(0) &= P_N y_0, \end{aligned} \tag{2.37}$$

$$\begin{aligned} \frac{dy_s}{dt} + \mathscr{A}_s y_s &= \sum_{j=1}^{M} v_j(t)(I - P_N) B\varphi_j^*, \quad t \geq 0, \\ y_s(0) &= (I - P_N) y_0, \end{aligned} \tag{2.38}$$

where $\mathscr{A}_u = P_N \mathscr{A}$, $\mathscr{A}_s = (I - P_N)\mathscr{A}$.

Recalling that spaces $X_u = P_N \widetilde{H}$ and $X_s = (I - P_N)\widetilde{H}$ are invariant to $\mathscr{A}$, we have that $\sigma(\mathscr{A}_u) = \{\lambda_j\}_{j=1}^N$, $\sigma(\mathscr{A}_s) = \{\lambda_j\}_{j=N+1}^\infty$. Moreover, (2.37) is finite-dimensional while (2.38) is an infinite-dimensional system. We note however that $-\mathscr{A}_s$ generates a C_0-analytic semigroup in X_s and together with $\sigma(\mathscr{A}_s) \subset \{\lambda;\ \mathrm{Re}\,\lambda > \gamma\}$, this implies that (see Theorem 1.14)

$$\|e^{-\mathscr{A}_s t}\|_{L(\widetilde{H},\widetilde{H})} \le Ce^{-\gamma t}, \quad \forall t \ge 0. \tag{2.39}$$

Now, coming back to System (2.37) and representing the solution y_u as

$$y_u(t) = \sum_{j=1}^N y_j(t)\varphi_j,$$

by (2.29) we may rewrite it as

$$\begin{aligned} & y_i'(t) + (\Lambda y(t))_i = \sum_{j=1}^M v_j(t)\langle B\varphi_j^*, \varphi_i^*\rangle, \quad i = 1, \dots, N, \\ & y_i(0) = \langle y_0, \varphi_i^*\rangle, \end{aligned} \tag{2.40}$$

where Λ is the matrix $\|\langle \mathscr{A}\varphi_j, \varphi_i^*\rangle\|_{i,j=1}^N$. Equivalently,

$$\begin{aligned} & y'(t) + \Lambda y(t) = \mathscr{B}v(t), \quad t \ge 0, \\ & y(0) = \{y_i(0)\}_{i=1}^N \end{aligned} \tag{2.41}$$

where $v(t) = \{v_j(t)\}_{j=1}^M$, $y(t) = \{y_i(t)\}_{i=1}^N$ and $\mathscr{B}$ is the matrix (2.32).

We note that, by virtue of Assumption (H3), Λ is a diagonal matrix of the form

$$\Lambda = \left\| \begin{matrix} J_1 & & 0 \\ & J_2 & \\ 0 & & \ddots & \\ & & & J_\ell \end{matrix} \right\|$$

where

$$J_j = \left\| \begin{matrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{matrix} \right\|$$

has the dimension $m_j \times m_j$, $m_1 + m_2 + \dots + m_\ell = N$.

Lemma 2.1 is the main step of the proof.

Lemma 2.1 *System* (2.40) *is exactly null controllable, that is, there is*

$$v(t) = \{v_j(t)\}_{j=1}^M \subset L^2(0, T; \mathbb{C}^M)$$

such that $y_i(T) = 0$ *for* $i = 1, \dots, N$.

Proof It is well-known that the finite-dimensional system (2.40) is exactly controllable if and only if

$$\mathscr{B}^* e^{-\Lambda t} x = 0, \quad \forall t \geq 0, \tag{2.42}$$

implies $x = 0$. (This is a variant of the Kalman controllability criterion.)

Taking into account that $\mathscr{B}^* = \| \langle \varphi_j^*, B^* \varphi_i^* \rangle \|_{i,j=1}^{N,M}$ and

$$e^{-\Lambda t} x = \begin{Vmatrix} e^{-\lambda_1 t} x_1 \\ \vdots \\ e^{-\lambda_1 t} x_{m_1} \\ e^{-\lambda_2 t} x_{m_1+1} \\ \vdots \\ e^{-\lambda_\ell t} x_N \end{Vmatrix}, \quad x = \begin{Vmatrix} x_1 \\ \vdots \\ x_N \end{Vmatrix},$$

(2.42) reduces to

$$e^{-\lambda_1 t} \sum_{i=1}^{m_1} b_{ij} x_i + e^{-\lambda_2 t} \sum_{i=m_1+1}^{m_2} b_{ij} x_i + \cdots + e^{-\lambda_\ell t} \sum_{i=m_{\ell-1}+1}^{m_\ell} b_{ij} x_i = 0,$$
$$\forall t \geq 0, \ j = 1, 2, \ldots, M,$$

where $b_{ij} = \langle \varphi_i^*, B \varphi_j^* \rangle$.

This yields

$$\sum_{i=1}^{m_1} b_{ij} x_i = 0, \quad \sum_{i=m_1+1}^{m_2} b_{ij} x_i = 0, \quad \ldots, \quad \sum_{i=m_{\ell-1}+1}^{b} b_{ij} x_i = 0, \quad j = 1, \ldots, M$$

and by Assumption (2.34) of the theorem the latter implies $x \equiv 0$, as claimed. □

Hence, there is a system $\{v_i\}_{i=1}^M \subset L^2(0, T; \mathbb{C}^M)$ such that the corresponding solution $y_u \in C([0, T]; \mathbb{C}^N)$ to (2.37) satisfies

$$y_u(0) = P_N y_0, \qquad y_u(T) = 0, \tag{2.43}$$

where T is arbitrary but fixed. Without loss of generality, we may assume that $v_j(t) = 0$, $\forall t \geq T$. If we plug this controller in (2.38), it follows by (2.39) that

$$\begin{aligned} |y_s(t)|_{\widetilde{H}} &\leq C e^{-\gamma t} |(I - P_N) y_0|_{\widetilde{H}} + C \int_0^T e^{-\gamma(t-s)} \left(\sum_{j=1}^M |v_j(s)| \right) ds \\ &\leq C_1 e^{-\gamma t} |y_0|_{\widetilde{H}}, \quad \forall t \geq 0. \end{aligned} \tag{2.44}$$

(The latter is the consequence of the fact that the controller $v = \{v_j\}_{j=1}^N$ can be chosen in such a way that $\int_0^T |v(t)|^2 dt \le c|P_N y_0|_{\widetilde{H}}$.)

It is useful to notice for later use that starting from the controller $\{v_i\}_{i=1}^M = v$, which steers $P_N y$ into origin, we may construct via the algebraic Riccati equation associated with the stabilizable finite-dimensional system (2.37) a feedback controller $v^*(t) = R y_u^*(t)$, which exponentially stabilizes (2.37) and $v^* \in C^1([0,\infty), \mathbb{C}^M)$

$$|y_u^*(t)| + |v^*(t)| + |(v^*)'(t)| \le C e^{-\gamma t} |P_N y_0|_{\widetilde{H}}.$$

(For internal stabilization, this choice of v^* is not relevant, but it is however so in boundary stabilization.) Then, by (2.43) and (2.44) we see that there is a controller u of the form (2.35) which stabilizes the asymptotically system (2.25) with exponential rate $-\gamma$.

In order to find a stabilizable feedback controller $u = K(y)$ for (2.25), we proceed in a standard way (see, e.g., [32, 60]). Namely, we associate with (2.25) the infinite horizon optimal control problem

Minimize

$$\int_0^\infty (|y(t)|_{\widetilde{H}}^2 + |v(t)|_{\mathbb{C}^M}^2) dt \tag{2.45}$$

subject to

$$\frac{dy}{dt} + \mathscr{A} y - \gamma y = \sum_{i=1}^M v_j B\varphi_j^* \stackrel{\text{def}}{=} Dv, \quad t \ge 0. \tag{2.46}$$

By the first part of the proof, System (2.46) is stabilizable and so (2.45) has a unique solution $\{y^* = v^*\}$. Moreover, this optimal controller $v^* = \{v_j^*\}$ is given in the feedback form

$$v^*(t) = -D^* R_0 y^*(t), \quad \forall t \ge 0, \tag{2.47}$$

where D^* is the dual operator of D, that is, $D^* p = \{\langle B\varphi_j^*, p\rangle\}_{j=1}^M$, $\forall p \in \widetilde{H}$, and $R_0 \in L(\widetilde{H}, \widetilde{H})$ is the self-adjoint positive solution to the algebraic Riccati equation

$$\langle \mathscr{A} y - \gamma y, R_0 y\rangle + \frac{1}{2}\, |D^* R_0 y|_{\widetilde{H}}^2 = \frac{1}{2}\, |y|_H^2, \quad \forall y \in D(\mathscr{A}). \tag{2.48}$$

In fact, R_0 is given by

$$\langle R_0 y_0, y_0\rangle = \int_0^\infty (|y^*(t)|_{\widetilde{H}}^2 + |v^*(t)|_{\mathbb{C}^M}^2) dt, \quad \forall y_0 \in \widetilde{H}. \tag{2.49}$$

Substituting (2.47) into (2.35), we obtain the desired result. $\square$

2.2.1 The Case of Not Semisimple Eigenvalues

It turns out that in the case where some of the eigenvalues λ_j, $j = 1, \dots, N$, are not semisimple, that is, the corresponding eigenvectors $\{\varphi_j^i\}_{i=1}^{m_j}$ are generalized,

$$(\mathscr{A} - \lambda_j)^i \varphi_j^i = 0, \quad i = 1, \dots, m_j, \ j = 1, \dots, N,$$

Theorem 2.1 still remains true but the argument becomes very technical in absence of a biorthogonal system of the eigenfunctions $\{\varphi_j\}_{j=1}^N$, $\{\varphi_j^*\}_{j=1}^N$ (see Sect. 3.3 for the treatment of this case for Navier–Stokes systems).

In order to avoid a tedious argument, we establish here a weaker form (as regards the dimension of the controller) of Theorem 2.1 in this general case.

Theorem 2.2 *Assume that Hypotheses* (H1) *and* (H2) *hold and that*

$$\det \|\langle B\varphi_j^*, P_N^* \varphi_i \rangle\|_{i,j=1}^N \neq 0. \tag{2.50}$$

Then there is a controller u of the form

$$u(t) = \sum_{j=1}^N v_j(t) \varphi_j^*, \quad t \geq 0, \ v_j \in L^2(0, \infty), \tag{2.51}$$

which stabilizes the exponentially system (2.25) *with exponent $-\gamma$. Moreover, the stabilizing controller $v = \{v_j\}_{j=1}^N$ can be chosen in the feedback form* (2.36).

Proof As in the previous case, it suffices to show that the finite-dimensional control system (2.37) where $M = N$ is exactly controllable on some interval $[0, T]$. If we represent y_u as

$$y_u = \sum_{i=1}^N y_i \varphi_i,$$

we obtain as above that

$$\sum_{i=1}^N y_i'(t) \langle \varphi_i, \varphi_j^* \rangle + \sum_{i=1}^N \langle \mathscr{A}_u \varphi_i, \varphi_j^* \rangle y_i(t) = \sum_{i=1}^N \langle B\varphi_i^*, P_N^* \varphi_j^* \rangle v_i(t), \quad j = 1, \dots, N.$$

If we set

$$\Lambda = \|\langle \mathscr{A}_u \varphi_i, \varphi_j^* \rangle\|_{i,j=1}^N, \qquad L = \|\langle \varphi_i, \varphi_j^* \rangle\|_{i,j=1}^N$$

and

$$\widetilde{\mathscr{B}} = \|\langle B\varphi_i^*, P_N^* \varphi_j \rangle\|_{i,j=1}^N,$$

we obtain that $y(t)=\{y_i(t)\}_{i=1}^N$ and $v(t)=\{v_i(t)\}_{i=1}^N$ satisfy the system

$$\begin{aligned} &\frac{dy}{dt}+L^{-1}\Lambda y=L^{-1}\widetilde{\mathscr{B}}v, \quad t\ge 0,\\ &y(0)=P_N y_0. \end{aligned} \tag{2.52}$$

(We note that since the systems $\{\varphi_j\}_{j=1}^N$ and $\{\varphi_j^*\}_{j=1}^N$ are linearly independent, the Gram matrix L is not singular.)

Since by Assumption (2.50) the matrix $\widetilde{\mathscr{B}}$ and, consequently, $L^{-1}\widetilde{\mathscr{B}}$ are nonsingular, we conclude that System (2.52) is exactly null controllable on each interval $(0,T]$ and from now on the proof is exactly the same as that of Theorem 2.1. □

Remark 2.1 The difference between Theorems 2.1 and 2.2 is that the latter provides stabilization but with a larger dimension of the controller u.

It should be said that, in specific situations, Condition (2.50) as well as Assumption (2.34) of Theorem 2.1 regarding the non zero minors D_k, $m\le k\le M$, of the matrix (2.33) are checked via unique continuation results for eigenfunctions or solutions to homogeneous partial differential equations of elliptic type. To be more specific, let us come back to the parabolic system (2.17). Then the corresponding linearized system is

$$\begin{aligned} &\frac{\partial y}{\partial t}-\Delta y+\beta'(y_e)y+g'(\nabla y_e)\cdot\nabla y=u\mathbf{1}_{\mathscr{O}_0} && \text{in } (0,\infty)\times\mathscr{O},\\ &y=0 && \text{on } (0,\infty)\times\partial\mathscr{O},\\ &y(0,x)=y_0(x) && \text{in } \mathscr{O}. \end{aligned} \tag{2.53}$$

In this case, as seen earlier, $Bu=u\mathbf{1}_{\mathscr{O}_0}$, $\mathscr{A}y=-\Delta y+\beta'(y_e)y+g'(\nabla y_e)\cdot\nabla y$, $D(\mathscr{A})=H_0^1(\mathscr{O})\cap H^2(\mathscr{O})$, and the dual eigenfunctions φ_j^* are solutions to the elliptic equation

$$\begin{aligned} &-\Delta\varphi_j^*+\beta'(y_e)\varphi_j^*-\operatorname{div}(g'(\nabla y_e)\varphi_j^*)=\overline{\lambda}_j\varphi_j^* && \text{in } \mathscr{O},\\ &\varphi_j^*=0 && \text{on } \partial\mathscr{O}. \end{aligned} \tag{2.54}$$

(Or, eventually, $(\mathscr{A}^*-\overline{\lambda}_j)^k\varphi_j^*=0$, $k=1,2,\dots,m_j$.) We have

$$\mathscr{B}=\left\|\int_{\mathscr{O}_0}\varphi_j^*\overline{\varphi}_i^*\,dx\right\|_{i,j=1}^{N,M}$$

and D_k are of the form

$$D_k=\left\|\int_{\mathscr{O}_0}\varphi_j^*\overline{\varphi}_i^*\,dx\right\|_{i=m_{k-1}+1,j=1}^{m_{k-1}+m_k,M}.$$

Then Condition (2.34) reduces to: for each p and $p+k \le M$, the system $\{\varphi_j^*\}_{j=p}^{p+k}$ is linearly independent on $\mathscr{O}_0 \subset \mathscr{O}$. But the latter condition is automatically satisfied by the solutions φ_j^* to (2.54) because, by the unique continuation property of solutions to elliptic equations, each φ_j^* which is zero on $\mathscr{O}_0$ is zero everywhere. By a simple induction argument, this implies that if the system $\{\varphi_j^*\}_{j=p}^{k+p}$ is linearly dependent on $\mathscr{O}_0$, then it is linearly dependent on $\mathscr{O}$, which is of course absurd. (See also Sect. 3.8.)

Hence Theorem 2.1 applies in the present case to stabilize (2.53). More about this subject will be said in the next chapter. If the eigenvalues λ_j are not semisimple, one must invoke Theorem 2.2 with Condition (2.50) which is also automatically satisfied.

Theorems 2.1 and 2.2 are proven in the complexified space $\widetilde{H} = H + iH$. However, if we set

$$\psi_j^1 = \operatorname{Re}\varphi_j^*, \quad \psi_j^2 = \operatorname{Im}\varphi_j^*, \quad j = 1, \dots, N,$$

then it follows that there is a controller of the form

$$\widetilde{u}(t) = \sum_{j=1}^{M} (v_j^1(t)\psi_j^1 + v_j^2(t)\psi_j^2),$$

which stabilizes the real system (2.25) and which can be written as in the proof of Theorem 2.1 in the feedback form

$$v_j^i(t) = -\langle B\psi_j^1, Ry^*(t)\rangle, \quad i = 1, 2.$$

It should be noticed that if $M = N$, then the dimension of the controller remains the same because a complex eigenvalue λ_j arises always in the system together with its conjugate $\overline{\lambda}_j$. Thus the dimension M^* of the real controller $\widetilde{u}$ is dependent of the maximum multiplicity m_j of complex eigenvalues λ_j, $\lambda = 1, \dots, \ell$. More precisely, we have

$$M^* = 2M \quad \text{if } M \text{ is equal to } \max_j\{m_j;\ \lambda_j \text{ complex}\}. \tag{2.55}$$

Therefore, we have

1° $M^* = N$ if $M = N$.
2° $M^* = M$ if one of the eigenvalues λ_j of maximum multiplicity is real. (In particular, if all the eigenvalues are real.)
3° $M^* = 2$ if all the eigenvalues are simple but complex-valued.
4° $M^* = 1$ if all the eigenvalues are simple and real.

We have, therefore, the following stabilization result for the real system (2.25).

Corollary 2.2 *Assume that $U = H$ and that Hypotheses* (H1)∼(H3) *and* (2.34) *hold. Then there is a real-valued controller u of the form*

$$u = \sum_{j=1}^{M^*} v_j(t)\psi_j, \quad v_j \in L^2(0,\infty),$$

where M^* is defined by (2.55) *and ψ_j is either* $\operatorname{Re}\varphi_j^*$ *or* $\operatorname{Im}\varphi_j^*$, *which stabilizes exponentially System* (2.25) *with decaying rate $-\gamma$.*

Similarly, under the hypotheses of Theorem 2.2 we have the following corollary.

Corollary 2.3 *Assume that $U = H$ and that Hypotheses* (H1)∼(H2) *and* (2.50) *hold. Then there is a real-valued controller u of the form*

$$u(t) = \sum_{j=1}^{N} v_j(t)\psi_j, \quad v_j \in L^2(0,\infty),$$

which stabilizes System (2.25). *Here ψ_j is either* $\operatorname{Re}\varphi_j^*$ *or* $\operatorname{Im}\varphi_j^*$.

2.2.2 Direct Proportional Stabilization of Unstable Modes

The previous method of stabilization of the linear system (2.25) might be called *spectral controllability-based* approach. Its advantage is that it provides a linear stabilizing feedback controller with a minimal dimension M which depends on spectral properties of unstable eigenvalues. On the other hand, the construction of this feedback controller in the form (2.47) involves an infinite-dimensional Riccati equation (see (2.48)). Below, we describe a simpler design of stabilizing feedback controller which is conceptually different from the previous one.

We assume that the operator $\mathscr{A}$ satisfies Assumptions (H1)–(H3) and let N be such that $\operatorname{Re}\lambda_j \le \gamma$, $j = 1,\dots,N$, where $\gamma > 0$ is arbitrary but fixed. If $\{\varphi_j\}_{j=1}^N$ and $\{\varphi_j^*\}_{j=1}^N$ are the corresponding eigenvectors of $\mathscr{A}$ and $\mathscr{A}^*$, respectively, we consider the feedback controller

$$u(t) = -\eta \sum_{i=1}^{N} \langle y(t), \varphi_i^* \rangle \phi_i \tag{2.56}$$

where $\{\phi_j\}$ is a system of functions such that

$$\langle \phi_i, B^*\varphi_j^* \rangle = \delta_{ij}, \quad i,j = 1,\dots,N. \tag{2.57}$$

Such a system can be found of the form

$$\phi_i = \sum_{k=1}^{N} \alpha_{ki}\varphi_k^*, \quad i = 1,\dots,N,$$

where $\alpha_{ki} \in \mathbb{C}$ are chosen from the system

$$\sum_{k=1}^{N} \alpha_{ki} \langle \varphi_k^*, B^*\varphi_j^* \rangle = \delta_{ij}, \quad i,j = 1,\dots,N.$$

Assuming that

$$\det \|\langle \varphi_k^*, B^*\varphi_j^*\rangle\|_{k,j=1}^N \neq 0, \tag{2.58}$$

then clearly there is a system $\{\alpha_{ki}\}$ such that $\{\phi_i\}$ satisfy Condition (2.57).

Assume also that

$$\eta \geq \gamma - \operatorname{Re}\lambda_j, \quad j = 1, \dots, N. \tag{2.59}$$

Theorem 2.3 *Under Assumptions* (H1), (H2), (H3) *and* (2.58), (2.59), *the solution* y *to the closed-loop system*

$$\begin{aligned} &y' + \mathscr{A}y + \eta \sum_{i=1}^N \langle y, \varphi_j^*\rangle B\phi_j = 0, \quad t \geq 0, \\ &y(0) = y_0, \end{aligned} \tag{2.60}$$

satisfies $|y(t)| \leq Ce^{-\gamma t}|y_0|,\ t \geq 0$.

Proof As in the proof of Theorem 2.1, we rewrite System (2.60) as (see (2.37), (2.38))

$$\begin{aligned} &\frac{dy_u}{dt} + \mathscr{A}_u y_u = -\eta P_N \sum_{i=1}^N \langle y, \varphi_i^*\rangle B\phi_i, \\ &y_u(0) = P_N y_0, \\ &\frac{dy_s}{dt} + \mathscr{A}_s y_s = -\eta(I - P_N) \sum_{i=1}^N \langle y, \varphi_i^*\rangle B\phi_i, \\ &y_s(0) = P_N y_0. \end{aligned}$$

Setting

$$y_u = \sum_{j=1}^N y_j\varphi_j$$

and taking into account (2.29), (2.57), we obtain that

$$y_i' + \lambda_i y_i = -\eta y_i, \quad i = 1, \dots, N,$$

and, therefore,

$$|y_u(t)| \leq e^{-(\operatorname{Re}\lambda_i + \eta)t}|y_0| \leq e^{-\gamma t}|y_0|.$$

Then, substituting y into the right-hand side of the system in y_s, and taking into account that

$$|e^{-\mathscr{A}_s t}y_0| \leq Ce^{-\gamma t}|y_0|, \quad \forall t \geq 0,$$

we conclude the proof. □

It should be said that in the above construction Assumption (H3) can be dispensed with. Indeed, we may replace the system $\{\varphi_j\}_{j=1}^N$ by an orthonormal system $\{\widetilde{\varphi}_j\}_{j=1}^N$ obtained by Schmidt's algorithm and choose the controller u of the form

$$u(t) = -\eta \sum_{i=1}^{N} \langle y(t), \widetilde{\varphi}_i \rangle \widetilde{\phi}_i, \tag{2.61}$$

where $\{\widetilde{\phi}_i\}_{i=1}^N$ are chosen such that

$$\langle \widetilde{\phi}_i, B^* P_N^* \widetilde{\varphi}_j \rangle = \delta_{ij} \quad \text{for } i, j = 1, \ldots, N.$$

The latter is possible if one assumes

$$\det \| \langle P_N^* \widetilde{\varphi}_i, B^* \widetilde{\phi}_j \rangle \| \neq 0. \tag{2.62}$$

We obtain, therefore, the following theorem.

Theorem 2.4 *Under Assumptions* (H1), (H2) *and* (2.58), (2.59), *the closed-loop*

$$\begin{aligned} &y' + \mathscr{A} y + \eta \sum_{i=1}^{N} \langle y, \widetilde{\varphi}_i \rangle B \widetilde{\phi}_i = 0, \\ &y(0) = y_0, \end{aligned}$$

is exponentially stable.

Coming back to Example (2.53), we note that Condition (2.62) is obviously satisfied in this case by the unique continuation property of eigenfunctions φ_j.

Remark 2.2 It should be noticed that though the structure of the controller (2.56) is very simple, it is not however robust. More precisely, it might be very sensitive to structural perturbations of the system which modify the spectrum and, implicitly, the basic system (2.57) from which the controller (2.61) is derived.

2.3 Boundary Stabilization of Linearized System

We consider System (2.25) under the following assumptions on the operator B.

(H4) $B \in L(U, (D(\mathscr{A}^*))')$.

Here, $D(\mathscr{A}^*)$ is the domain of adjoint operator $\mathscr{A}^*$ endowed with the graph norm and $(D(\mathscr{A}^*))'$ is its dual in the pairing $\langle \cdot, \cdot \rangle$ with the pivot space $\widetilde{H}$.

More precisely, $(D(\mathscr{A}^*))'$ is the completion of the space $D(\mathscr{A}^*)$ in the norm $\|y\|_{(D(\mathscr{A}^*))'} = |(\lambda I - \mathscr{A}^*)^{-1}y|$, $y \in \widetilde{H}$, where $\lambda \in \rho(\mathscr{A}^*)$ is arbitrary but fixed. (See (1.7)–(1.9).) Then, for each $y_0 \in \widetilde{H}$, $u \in L^2(0, T; U)$ and $T > 0$, the function

$$y(t) = e^{-\mathscr{A}t}y_0 + \int_0^t e^{-\mathscr{A}(t-s)}Bu(s)ds, \quad t \geq 0, \tag{2.63}$$

belongs to $C([0, T]; (D(\mathscr{A}^*))')$ and it is a generalized (mild) solution to System (2.25) under Assumptions (H1) and (H4). In general, y does not belong to $C([0, T]; \widetilde{H})$, but this happens, however, under additional assumptions on B (see [32, 60]).

It should be emphasized that in this formulation the space $(D(\mathscr{A}^*))'$ becomes the basic space of the system. The operator $\mathscr{A} : H \to (D(\mathscr{A}^*))'$ (or, more exactly, its extension to $(D(A^*))'$, $\widetilde{\mathscr{A}}$), defined by $\langle \widetilde{\mathscr{A}}y, \psi\rangle = \langle y, \mathscr{A}^*\psi\rangle$, $\forall \psi \in D(\mathscr{A}^*)$ generates a C_0-analytic semigroup on $(D(\mathscr{A}^*))'$, again denoted by $e^{-\mathscr{A}t}$. Moreover, the spectrum of this extension coincides with that of the original operator.

We denote by $\mathscr{D}$ the matrix

$$\mathscr{D} = \|\langle B^*\varphi_j^*, B^*\varphi_i\rangle_U\|_{i,j=1}^{N\ M} \tag{2.64}$$

and

$$\mathscr{D}_k = \|\langle B^*\varphi_i^*, B^*\varphi_j\rangle_U\|_{i=m_{k-1}+1,j=1}^{m_{k-1}+m_k,M}, \quad k = 1, \dots, \ell.$$

Here, $\{\varphi_j^*\}_{j=1}^N$ are, as in the previous case, the eigenvectors of $\mathscr{A}^*$ corresponding to the eigenvalues $\{\overline{\lambda}_j,\ 1 \leq j \leq N\}$, $\langle \cdot, \cdot\rangle_U$ is the scalar product of U and $B^*:D(\mathscr{A}^*) \to U$ the dual operator.

Theorem 2.5 *Assume that Hypothesis* (H1), (H3), *and* (H4) *hold and also that*

$$\operatorname{rank} \mathscr{D}_k = m_k, \quad k = 1, \dots, \ell. \tag{2.65}$$

Then there is a controller

$$u(t) = \sum_{j=1}^M v_j(t)B^*\varphi_j^* \tag{2.66}$$

which stabilizes exponentially System (2.25) *in* $(D(\mathscr{A}^*))'$. *Moreover,* $v_j(t)$ *can be chosen in feedback form*

$$v_j(t) = R_j(y(t)), \quad j = 1, \dots, N. \tag{2.67}$$

Proof We proceed as in Theorem 2.1. Namely, we write System (2.25) as

$$\frac{dy_u}{dt} + \mathscr{A}_u y_u = \sum_{j=1}^M v_j(t)P_N BB^*\varphi_j^*, \tag{2.68}$$

$$\frac{dy_s}{dt} + \mathscr{A}_s y_s = \sum_{j=1}^{M} v_j(t)(I - P_N)BB^*\varphi_j^*, \tag{2.69}$$

where $\mathscr{A}_u = P_N\mathscr{A}$ and $\mathscr{A}_s$ is the extension of $(I - P_N)\mathscr{A}$ to all of $\widetilde{H}$ and with values in $(D(\mathscr{A}^*))'$.

As in the previous case, System (2.68) can be put in the form (see (2.40))

$$\begin{aligned} &y' + \Lambda y = \mathscr{D}v, \quad t \in (0, T), \\ &y(0) = P_N y_0 \end{aligned} \tag{2.70}$$

and, by Assumption (2.65), Lemma 2.1 remains true in the present case and we may conclude, as in the proof of Theorem 2.2, that there is a stabilizing controller u of the form (2.66). (We note that Estimate (2.39) remains valid here in $(D(\mathscr{A}^*))'$ for the extended semigroup $e^{-\mathscr{A}_s t}$.) Hence

$$\|y(t)\|_{(D(\mathscr{A}^*))'} \le Ce^{-\gamma t}|y_0|_{\widetilde{H}}, \quad \forall t \ge 0. \qquad \square$$

Remark 2.3 It should be said that, under additional assumptions, one has strong stabilization of (2.25) in the space $\widetilde{H}$. Indeed, we have by (2.63) and (2.69) that

$$y_s(t) = e^{-\mathscr{A}_s t} P_N y_0 + \int_0^t e^{-\mathscr{A}(t-s)} \sum_{j=1}^{M} v_j(s)(I - P_N)BB^*\varphi_j ds.$$

Since v_j can be taken in such a way that $|v_j'(t)| \le Ce^{-\delta t}$, $\forall t > 0$, $j = 1, \dots, M$, then, if $\mathscr{A}^{-1}B \in L(\widetilde{H}, \widetilde{H})$, we see that

$$|y_s(t)|_{\widetilde{H}} \le Ce^{-\gamma t}|y_0|_{\widetilde{H}}.$$

The construction of the feedback controller is similar to that from the proof of Theorem 2.1, so it will be omitted.

We must remark that the stabilization effect of Controller (2.66) is in a weaker topology than that of Controller (2.35) designed in Theorem 2.1. This is due to the singularity of the operator B and, in particular, of the weaker regularity property of the function $t \to e^{-\mathscr{A}t}Bu$. However, as we see later in some specific situations and, in particular, to that of boundary control systems governed by the Stokes–Oseen operator which will be treated in Sect. 3.4, this stabilization result can be strengthen to the strong topology of H. We come back to Theorem 2.5 and to the boundary control problem (2.20) or, more precisely, to its linearization

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \beta'(y_e)y + g'(\nabla y_e) \cdot \nabla y = 0 & \text{in } (0, \infty) \times \mathscr{O}, \\ y = u & \text{on } (0, \infty) \times \partial\mathscr{O}, \\ y(0, x) = y_0(x), & x \in \mathscr{O}. \end{cases} \tag{2.71}$$

As seen earlier, (2.71) can be written as (2.25) in the space $H = L^2(\mathscr{O})$, where $U = L^2(\partial\mathscr{O})$, $Bu = A_0 Du$, $A_0 = -\Delta$ with $D(A_0) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$ and $D : L^2(\partial\mathscr{O}) \to H^{\frac{1}{2}}(\mathscr{O})$ is the Dirichlet map defined by (2.21). Then, we have

$$B^* p = -\frac{\partial p}{\partial n}, \quad \forall p \in D(\mathscr{A}^*) = D(A_0)$$

and so the stabilizing controller u in (2.71) with actuation on the boundary is of the form

$$u(t,x) = -\sum_{j=1}^{M} v_j(t)\, \frac{\partial \varphi_j^*}{\partial n}\,(x), \quad t \ge 0,\ x \in \partial\mathscr{O}. \tag{2.72}$$

As regards Condition (2.65), in this case it reduces to

$$\det \left\| \int_{\partial\mathscr{O}} \frac{\partial \varphi_j^*}{\partial n}\, \frac{\partial \overline{\varphi}_i^*}{\partial n}\, dx \right\|_{i=m_{k-1},\, j=1}^{i=m_{k-1}+m_k, M} \neq 0. \tag{2.73}$$

Condition (2.73) is equivalent to the linear independence of the system $\{\frac{\partial \varphi_j^*}{\partial n}\}_p^{p+k}$ in $L^2(\partial\mathscr{O})$, a condition which automatically holds for solutions φ_j^* of (2.54). In fact, it is a consequence of the fact that if a solution to (2.54) has zero normal derivative on all of $\partial\mathscr{O}$ or on some part of it with nonempty interior it is everywhere zero. (The uniqueness of the Cauchy problem.)

It should be mentioned that, in this case, Remark 2.3 applies and so, Controller (2.72) stabilizes (2.71) in $L^2(\mathscr{O})$ topology.

As in the previous section, one might construct in this case a stabilizing feedback of the form (2.56) or (2.61) for System (2.25). Namely,

$$u(t) = -\eta \sum_{j=1}^{N} \langle y, \widetilde{\varphi}_j \rangle B^* \widetilde{\phi}_j \tag{2.74}$$

where $\{\widetilde{\varphi}_j\}$ and $\{\widetilde{\phi}_j\}$ are chosen as in Theorem 2.4.

We have, therefore, as in the previous case, the following theorem.

Theorem 2.6 *Under Assumptions* (H1), (H3), (H4) *and*

$$\det \| \langle B^* \phi_j, P_N^* B^* \widetilde{\varphi}_i \rangle \|_{i,j=1}^{N} \neq 0,$$

if $\eta > 0$ is sufficiently large, the feedback controller (2.74) *stabilizes exponentially System* (2.25) *in* $(D(\mathscr{A}^*))'$.

2.4 Stabilization by Noise of the Linearized Systems

Here, we study the stabilization by noise of the linear system (2.25), where the operator $\mathscr{A}$ satisfies Assumptions (H1), (H3).

Roughly speaking, the noise stabilization of (2.25) means to design a stochastic controller of the form $\sum_{i=1}^{N} \psi_i \dot{\beta}_i$, where $\dot{\beta}_i$ are white noises, which stabilizes the system in probability.

In other words, the solution X to the system

$$\begin{aligned} &\dot{X}(t) + \mathscr{A} X(t) = \sum_{j=1}^{N} B\psi_j(t)\dot{\beta}_j(t), \quad t > 0, \\ &X(0) = x \end{aligned} \tag{2.75}$$

is asymptotically convergent to zero in probability as $t \to \infty$.

Here, $\{\beta_j\}_{j=1}$ is an independent system of complex Brownian motions in a probability space $\{\Omega, \mathbb{P}, \mathscr{F}, \mathscr{F}_t\}_{t>0}$ and $\{\psi_j\} \subset L^\infty(0, \infty; U)$.

Equation (2.75) should be taken of course in Ito's sense, that is (see Sect. 4.5),

$$\begin{aligned} &dX(t) + \mathscr{A} X(t)dt = \sum_{j=1}^{N} B\psi_j(t)d\beta_j(t), \quad t > 0, \\ &X(0) = x, \end{aligned} \tag{2.76}$$

or, equivalently,

$$X(t) = e^{-\mathscr{A}t}x + \int_0^t \sum_{j=1}^{N} e^{-\mathscr{A}(t-s)} B\psi_j(s)d\beta_j(s). \tag{2.77}$$

Here, $B \in L(U, H)$.

We see below that, under quite general assumptions, such a stabilizable feedback controller exists and has a simple form.

Let $\{\varphi_j\}_{j=1}^N$, $\{\varphi_j^*\}_{j=1}^N$ be the eigenvectors system satisfying (2.29), (2.30) and we also assume that

$$\det \| \langle B\varphi_i^*, \varphi_j^* \rangle \|_{i,j=1}^N \neq 0. \tag{2.78}$$

Consider the system $\{\phi_j\}_{j=1}^N \subset H$ defined by

$$\phi_j = \sum_{i=1}^{N} \alpha_{ij}\varphi_i^*, \quad j = 1, \dots, N, \tag{2.79}$$

where α_{ij} are chosen in such a way that

$$\sum_{i=1}^{N} \alpha_{ik} \langle B\varphi_i^*, \varphi_j^* \rangle = \delta_{jk}, \quad j, k = 1, \dots, N. \tag{2.80}$$

By Condition (2.78) it is clear that (2.80) has solution and, by (2.79) and (2.80), we see that

$$\langle B\phi_j, \varphi_i^* \rangle = \delta_{ij}, \quad i, j = 1, \dots, N. \tag{2.81}$$

We consider the stochastic feedback controller

$$u(t) = \eta \sum_{i=1}^{N} \langle X(t), \varphi_i^* \rangle \phi_i \dot{\beta}_i$$

and we show that it stabilizes in probability the control system (2.25). Namely, one has the following theorem.

Theorem 2.7 *For each $x \in \widetilde{H}$ and $\eta^2 > 2(\gamma - \operatorname{Re}\lambda_j)$, $\forall j = 1, \dots, N$, the equation*

$$\begin{aligned} &dX(t) + \mathscr{A}X(t)dt = \eta \sum_{i=1}^{N} \langle X(t), \varphi_i^* \rangle B\phi_i d\beta_i \quad \text{in } (0,\infty),\ \mathbb{P}\text{-a.s.}, \\ &X(0) = x, \end{aligned} \tag{2.82}$$

has a unique solution $X \in C_W([0,T]; L^2(\Omega, \widetilde{H}))$, $\forall T > 0$, such that

$$\mathbb{P}\left[\lim_{t\to\infty} e^{\gamma t} |X(t)|_{\widetilde{H}} = 0\right] = 1. \tag{2.83}$$

Here, $C_W([0,T]; L^2(\Omega, \widetilde{H}))$ is the space of all adapted square-mean $\widetilde{H}$-valued continuous processes on $[0,T]$ and, as mention earlier, (2.82) is understood in the following "mild" sense

$$X(t) = e^{-\mathscr{A}t}x + \eta \sum_{i=1}^{N} \int_0^t \langle X(s), \varphi_i^* \rangle e^{-\mathscr{A}(t-s)}(B\phi_i)(s)d\beta_i(s), \quad t \geq 0. \tag{2.84}$$

(See Sect. 4.5.)

Proof of Theorem 2.7 The idea, already used before, is to decompose (2.82) in a finite-dimensional system and an infinite-dimensional exponentially stable system. To this end, we set $X_u = P_N X$, $X_s = (I - P_N)X$ and we rewrite (2.82) as

$$\begin{aligned} &dX_u(t) + \mathscr{A}_u X_u(t)dt = \eta P_N \sum_{i=1}^{N} \langle X_u(t), \varphi_i^* \rangle B\phi_i d\beta_i(t), \quad \mathbb{P}\text{-a.s.},\ t \geq 0, \\ &X_u(0) = P_N x, \end{aligned} \tag{2.85}$$

$$\begin{aligned} &dX_s(t) + \mathscr{A}_s X_s(t)dt = \eta(I - P_N) \sum_{i=1}^{N} \langle X_u(t), \varphi_i^* \rangle B\phi_i d\beta_i(t), \quad \mathbb{P}\text{-a.s.},\ t \geq 0, \\ &X_s(0) = (I - P_N)x. \end{aligned} \tag{2.86}$$

Then, we may represent X_u as $\sum_{i=1}^{N} y_i(t)\varphi_i$ and so reduce (2.85) via the biorthogonal relations (2.29) and (2.81) to the finite-dimensional stochastic system

$$\begin{aligned} &dy_j + \lambda_j y_j dt = \eta y_j d\beta_j, \quad j = 1, \dots, N,\ t \geq 0,\ \mathbb{P}\text{-a.s.}, \\ &y_j(0) = y_j^0, \end{aligned} \tag{2.87}$$

where $y_j^0 = \langle P_N x, \varphi_j^* \rangle$.

It is well-known that the solution y_j to the stochastic differential equation (2.87) is given by

$$y_j(t) = e^{-\lambda_j t - \frac{\eta^2}{2}\, t + \eta\beta_j(t)}\, y_j^0 = 1, \ldots, N, \tag{2.88}$$

and, therefore, there is $\varepsilon > 0$ such that

$$|y_j(t)|e^{(\varepsilon+\gamma)t} \le e^{\eta\beta(t)}|y_j^0|, \quad \mathbb{P}\text{-a.s.}$$

Taking into account that, for each $\lambda > 0$ and $r > 0$, we have (see Lemma 4.6 in Sect. 4.5)

$$\mathbb{P}\left(\sup_{t\ge 0} e^{\beta_j(t)-\lambda t} \ge r\right) = r^{-2\lambda}. \tag{2.89}$$

We infer, therefore, by (2.88) that for each $r > 0$ there is $\Omega_r \subset \Omega$ such that

$$|y_j(t)|e^{\gamma t} \le C r^{\eta}|y_j^0| \quad \text{in } \Omega_r,$$

where C is independent of r and $\mathbb{P}(\Omega_r) \ge 1 - r^{-2\varepsilon}$. This implies that

$$\lim_{t\to\infty} |y_j(t)|e^{\gamma t} = 0, \quad \mathbb{P}\text{-a.s.}$$

and also

$$\int_0^\infty |y_j(t)|^2 e^{2\gamma t} dt < \infty, \quad \mathbb{P}\text{-a.s.}$$

We have therefore that

$$\lim_{t\to\infty} e^{2\gamma t}|X_u(t)|^2_{\widetilde{H}} = 0, \quad \mathbb{P}\text{-a.s.}, \tag{2.90}$$

$$\int_0^\infty e^{2\gamma t}|X_u(t)|^2_{\widetilde{H}} dt < \infty, \quad \mathbb{P}\text{-a.s.} \tag{2.91}$$

Next, we come back to the infinite-dimensional system (2.86). Since, as seen earlier, the operator $-\mathscr{A}_s$ generates a γ-exponentially stable C_0-semigroup on $\widetilde{H}$, by the Lyapunov theorem there is $Q \in L(\widetilde{H}, \widetilde{H})$, $Q = Q^* \ge 0$ such that

$$\mathrm{Re}\langle Qx, \mathscr{A}_s x - \gamma x\rangle = \frac{1}{2}\, |x|^2_{\widetilde{H}}, \quad \forall x \in D(\mathscr{A}_s).$$

(We note that, though Q is not positively definite in the sense that

$$\inf\{\langle Qx, x\rangle;\ |x| = 1\} > 0,$$

we have, nevertheless, that $\langle Qx, x\rangle > 0$ for all $x \ne 0$.)

Applying Ito's formula in (2.86) (see Theorem 4.8) to the function

$$\varphi(x) = \frac{1}{2}\,\langle Qx, x\rangle,$$

we obtain that

$$\begin{aligned}
&\frac{1}{2}\,d\langle QX_s(t), X_s(t)\rangle + \frac{1}{2}\,|X_s(t)|^2_{\widetilde{H}}dt + \gamma\,\langle QX_s(t), X_s(t)\rangle dt\\
&\quad= \frac{1}{2}\,\eta^2\sum_{i=1}^{N}(QY_i(t), Y_i(t))_H dt + \eta\sum_{i=1}^{N}((\mathrm{Re}(QX_s(t)), \mathrm{Re}\,Y_i(t))_H\\
&\qquad+ (\mathrm{Im}(QX_s(t)), \mathrm{Im}\,Y_i(t))_H)d\beta_i(t),
\end{aligned}$$

where Y_i are stochastic processes defined by

$$Y_i(t) = \langle X_u(t), \varphi_i^*\rangle(I - P_N)B\phi_i, \quad i = 1, \dots, N.$$

This yields

$$\begin{aligned}
&e^{2\gamma t}\langle QX_s(t), X_s(t)\rangle + \int_0^t e^{2\gamma s}|X_s(s)|^2_{\widetilde{H}}ds\\
&\quad= \langle Q(I - P_N)x, (I - P_N)x\rangle\\
&\qquad+ \eta^2\sum_{i=1}^{N}\int_0^t e^{2\gamma s}\langle QY_i(s), Y_i(s)\rangle ds\\
&\qquad+ 2\eta\sum_{i=1}^{N}\int_0^t e^{2\gamma s}((\mathrm{Re}(QX_s(s)), \mathrm{Re}\,Y_i(s))_H\\
&\qquad+ (\mathrm{Im}(QX_s(s)), \mathrm{Im}\,Y_i(s)))_H)d\beta_i(s), \quad t \ge 0,\ \mathbb{P}\text{-a.s.}
\end{aligned} \tag{2.92}$$

Now, we apply Lemma 4.5 in Sect. 4.5 to the stochastic processes Z, I, I_1, M defined below

$$\begin{aligned}
&Z(t) = e^{2\gamma t}\langle QX_s(t), X_s(t)\rangle,\\
&I(t) = \int_0^t e^{2\gamma s}|X_s(s)|^2_{\widetilde{H}}ds, \qquad I_1(t) = \eta^2\sum_{i=1}^{N}\int_0^t e^{2\gamma s}\langle QY_i, Y_i\rangle ds,\\
&M(t) = 2\eta\sum_{i=1}^{N}\int_0^t e^{2\gamma s}((\mathrm{Re}(QX_s(s)), \mathrm{Re}, Y_i(s))_H\\
&\qquad\times (\mathrm{Im}(QX_s(s)), \mathrm{Im}\,Y_i(s)))_H)d\beta_i(s),\\
&\quad \mathbb{P}\text{-a.s.},\ t \ge 0.
\end{aligned}$$

Because, by the first step of the proof (see (2.91)), $I_1(\infty) < \infty$, we conclude that

$$\lim_{t\to\infty} e^{2\gamma t}\langle QX_s(t), X_s(t)\rangle = 0, \quad \mathbb{P}\text{-a.s.},$$

and, since Q is positive definite in the sense that $\langle Qx, x\rangle = (Qx, x)_H > 0$ for all $x \in \widetilde{H}$, we have that

$$\lim_{t\to\infty} e^{\gamma t}|X_s(t)|_{\widetilde{H}} = 0, \quad \mathbb{P}\text{-a.s.}$$

Recalling that $X = X_u + X_s$ and again invoking (2.91), the latter implies (2.83), thereby completing the proof of Theorem 2.7. □

Now, we illustrate Theorem 2.7 on Example (2.53). (Other more sophisticated examples will be discussed in Sect. 4.1.) In this case, $Bu = \mathbf{1}_{\mathscr{O}_0}u$ where $\mathbf{1}_{\mathscr{O}_0}$ is the characteristic func tion on some open subdomain $\mathscr{O}_0 \subset \mathscr{O}$. Then, Condition (2.78) reduces to

$$\det\left\| \int_{\mathscr{O}_0} \varphi_i^* \overline{\varphi}_j^* \right\|_{i,j=1}^N \neq 0,$$

which clearly holds because as noticed earlier the eigenvalue system $\{\varphi_i^*\}_{i=1}^N$ is linearly independent on $\mathscr{O}_0$. Then the stochastic feedback controller is, in this case, of the form

$$\eta \sum_{i=1}^N \left(\int_{\mathscr{O}} X(t,\zeta)\varphi_i^*(\zeta)d\zeta\, \mathbf{1}_{\mathscr{O}_0}(\xi)\phi_i(\xi) \right) \dot{\beta}_i(t) \tag{2.93}$$

and by Theorem 2.7 it stabilizes exponentially in probability equation (2.53), that is,

$$\begin{aligned}
&dX - \Delta X\,dt + \beta'(y_e)X\,dt + g'(\nabla y_e)\nabla X\,dt \\
&\quad = \eta\sum_{i=1}^N \int_{\mathscr{O}} X(t,\zeta)\varphi_i^*(\zeta)d\zeta\,\mathbf{1}_{\mathscr{O}_0}(\xi)\phi_i(\xi)d\beta_i(t), \quad t \geq 0,\ \xi \in \mathscr{O}, \\
&X = 0 \quad \text{on } (0,\infty)\times\partial\mathscr{O}, \\
&X(0,\xi) = x(\xi), \quad \xi \in \mathscr{O}.
\end{aligned}$$

Moreover, the feedback controller (2.93) has the support in $\mathscr{O}_0$.

2.4.1 The Boundary Stabilization by Noise

We consider here System (2.25), where $\mathscr{A}$ satisfies Assumptions (H1), (H3), and B satisfies (H4). We consider the stochastic differential equation

$$\begin{aligned}
&dX + \mathscr{A}X\,dt = \eta\sum_{i=1}^N BB^*\phi_i\langle X, \varphi_i^*\rangle d\beta_i(t), \quad t \geq 0, \\
&X(0) = x.
\end{aligned} \tag{2.94}$$

Here, ϕ_i are defined by (2.79) where α_{ij} are chosen as in (2.80), that is,

$$\sum_{i=1}^{N} \alpha_{ij} \langle B^* \varphi_i^*, B^* \varphi_k^* \rangle = \delta_{jk}, \quad j, k = 1, \dots, N. \tag{2.95}$$

We assume that

$$\det \| \langle B^* \varphi_i^*, B^* \varphi_k^* \rangle \|_{i,k=1}^{N} \neq 0 \tag{2.96}$$

and so α_{ij} are well-defined. By (2.95) we have, therefore, that

$$\langle B B^* \phi_i, \varphi_j^* \rangle = \delta_{ij}, \quad i, j = 1, \dots, N. \tag{2.97}$$

(As in the previous case, we refer to Sect. 4.5 for the existence of a solution $X \in C_W([0,T]; L^2(\widetilde{\mathscr{O}}, H))$ to (2.94).)

We have the following theorem.

Theorem 2.8 *For $|\eta|$ large enough, we have*

$$\mathbb{P}\left[\lim_{t\to\infty} e^{\gamma t} \|X(t)\|_{(D(\mathscr{A}^*))'} = 0\right] = 1.$$

Proof We argue as into the proof of Theorem 2.7. Namely, we decompose System (2.94) in two parts

$$dX_u + \mathscr{A}_u X_u dt = \eta P_N \sum_{i=1}^{N} B B^* \phi_i \langle X_u, \varphi_i^* \rangle d\beta_i \tag{2.98}$$

and

$$dX_s + \mathscr{A}_s X_s dt = \eta (I - P_N) \sum_{i=1}^{N} B B^* \phi_i \langle X_u, \varphi_i^* \rangle d\beta_i$$

and treat the finite-dimensional stochastic system (2.98) exact as in the previous case. After that, the proof continues exactly as in the proof of Theorem 2.7. The details are omitted. □

In the case of the boundary control system (2.71), (2.94) has the form

$$\begin{aligned} & dX + \widetilde{\mathscr{A}} X\, dt = -\eta \sum_{i=1}^{N} B \left(\frac{\partial \phi_i}{\partial n}\right) \int_{\mathscr{O}} X \overline{\varphi}_i^* d\xi d\beta_i, \\ & X(0) = x, \end{aligned} \tag{2.99}$$

where $\widetilde{\mathscr{A}} : L^2(\mathscr{O}) \to (D(\mathscr{A}^*))'$ is the extension of $\mathscr{A}$ on all of $\widetilde{H} = L^2(\mathscr{O})$ and $B = A_0 D$.

In terms of boundary control system, this equation can be, equivalently, written as

$$\begin{aligned} &dX - \Delta X\,dt + \beta'(y_e)X\,dt + g'(\nabla y_e)\cdot\nabla X\,dt = 0 && \text{in } (0,\infty)\times\mathscr{O},\\ &X(0,\xi) = x(\xi), && \xi\in\mathscr{O},\\ &X(t,\xi) = -\eta\sum_{i=1}^N \frac{\partial\phi_i}{\partial n}\left(\int_{\mathscr{O}} X\overline{\varphi}_i^*\,d\xi\right)\dot\beta_i(t) && \text{on } (0,\infty)\times\partial\mathscr{O}. \end{aligned}$$

Then, by Theorem 2.8, for $|\eta|$ large enough, the stochastic boundary controller

$$u(t) = -\eta\sum_{i=1}^N \frac{\partial\phi_i}{\partial n}\left(\int_{\mathscr{O}} X(t,\xi) - y_e(\xi)\varphi_i^*\,d\xi\right)\dot\beta_i(t) \quad \text{on } (0,\infty)\times\partial\mathscr{O}$$

stabilizes exponentially in probability the equilibrium state $X = y_e(t)$ of System (2.20). (As a matter of fact, the above stochastic feedback controller stabilizes the linearization of (2.20).)

Remark 2.4 We notice that the noise controller arising in Theorem 2.7 has a similar structure as the deterministic feedback controller (2.56) and, apparently, the latter is simpler. However, as remarked earlier, the noise controller is robust, which is not the case with (2.56). We come back later on to this discussion.

2.5 Internal Stabilization of Nonlinear Parabolic-like Systems

We come back to the nonlinear system (2.11) or, more precisely, to

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A}y + G(y) = 0, \quad t\ge 0,\\ &y(0) = y_0, \end{aligned} \tag{2.100}$$

where $\mathscr{A} : D(\mathscr{A})\subset H\to H$ is of the form (2.4), that is

$$\mathscr{A} = A + F_0'(y_e), \tag{2.101}$$

and G is given by (2.10). We assume everywhere in this section that

(j) *A is a self-adjoint positive definite linear operator with domain $D(A)$, that is,*

$$(Ay, y) \ge \delta|y|^2, \quad \forall y\in H \text{ for some } \delta > 0.$$

Moreover, assume that the space $V = D(A^{\frac12})$ with the norm $\|y\|_V = |A^{\frac12}y|^2$ is compactly imbedded in H.

(jj) *$F_0'(y_e) : H\to H$ is linear, closed, densely defined and*

$$|F_0'(y_e)y| \le C|A^{\frac12}y|, \quad \forall y\in D(A^{\frac12}).$$

As regards the operator $G : D(G) \subset H \to H$, it is made precise later on. Here, H is a real Hilbert space with the scalar product denoted by $(\cdot, \cdot)$ and the norm $|\cdot|$.

Under Assumptions (j) and (jj), it is easily seen that $\mathscr{A}$ defined by (2.101) satisfies Assumption (H1) and also Assumptions (i), (ii) from Sect. 2.1.

We keep the notation of Sect. 2.2 for the spectrum and eigenvectors $\{\varphi_j\}$ of $\mathscr{A}$. Also, N is the number of eigenvalues λ_j with $\operatorname{Re}\lambda_j \le \gamma$.

Our goal here is to construct a stabilizable feedback controller u for System (2.11), that is, a map

$$u = -K(y - y_e), \tag{2.102}$$

such that the solution $y = y(t)$ to the system

$$\begin{aligned} &\frac{dy}{dt} + F(y) = -BK(y - y_e), \quad t \ge 0, \\ &y(0) = \widetilde{y}_0, \end{aligned} \tag{2.103}$$

has the property that

$$|y(t) - y_e| \le C|\widetilde{y}_0 - y_e|e^{-\gamma t}, \quad \forall t \ge 0, \tag{2.104}$$

for all $\widetilde{y}_0$ in a neighborhood of y_e. (Here, $B \in L(H, H)$.) This means that the feedback controller (2.102) stabilizes exponentially the equilibrium solution y_e and the corresponding system (2.103) is the *closed-loop system* associated with feedback law (2.102).

If we translate y into $y - y_e$, this reduces to the stabilization of null solution to (2.100), where G is given by (2.10) and the corresponding closed-loop system is

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A}y + Gy = -BKy, \quad t \ge 0, \\ &y(0) = y_0 = \widetilde{y}_0 - y_e. \end{aligned} \tag{2.105}$$

Here, we prove that, under the above assumptions, there is a feedback controller $u = -Ky$ which stabilizes System (2.105) or, more precisely, its zero solution.

In fact, the feedback controller $u = -Ky$ will be a stabilizable feedback controller for the linearized equation (2.25) associated with (2.105). We have shown in Sect. 2.2 that such a feedback controller can be obtained from an infinite horizon linear quadratic problem associated with the control system

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A}y - \gamma y = \sum_{j=1}^{M^*} v_j B\psi_j, \quad t \ge 0, \\ &y(0) = y_0. \end{aligned} \tag{2.106}$$

Here and everywhere in the following, M^* is determined by (2.55) under the assumptions of Theorem 2.1 and $M^* = N$ under that of Theorem 2.2 (see Corollaries 2.2 and 2.3). The system $\{\psi_j\}_{j=1}^{M^*}$ is that made precise in Corollary 3.1, respectively Corollary 3.2. We need, however, a sharper feedback controller and this can be obtained in a similar way analyzing more closely under present assumptions on $\mathscr{A}$ the solution R to the corresponding Riccati equation (2.48). It is clear that the properties of R will depend also of the structure of the linear quadratic cost functional we associate to the control system (2.106). Here, it is of the form

$$J_\alpha(y,v) = \frac{1}{2}\int_0^\infty (|A^\alpha y(t)|^2 + |v(t)|^2_{M^*})dt, \tag{2.107}$$

where $|v|^2_{M^*} = \sum_{j=1}^{M^*} |v_j|^2$ and $A^\alpha,\ 0 \le \alpha \le 1$, is the fractional power of order α of the operator A. In examples to partial differential equations A is an operator of elliptic type and $|y|_\alpha = |A^\alpha y|$ is the Sobolev norm of order 2α of y. In particular, $|A^0 y| = |y|$ is the L^2-norm. We may view J_α as a cost functional with $D(A^\alpha)$-gain.

We consider here two situations.

1° $\alpha = \frac{3}{4}$ (high-gain Riccati-based feedback)
2° $\alpha = 0$ (low-gain Riccati-based feedback)

In both cases, we construct a Riccati-based linear feedback operator $u = -Ky$ which inserted into the nonlinear system

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A}y + Gy = Bu, \quad t \ge 0, \\ &y(0) = y_0, \end{aligned} \tag{2.108}$$

stabilizes exponentially the zero solution in a neighborhood of origin.

2.5.1 High-gain Riccati-based Stabilizable Feedback

Let

$$\Phi_\alpha(y_0) = \inf\{J_\alpha(y,v);\ (y,v) \text{ subject to (2.106)}\}. \tag{2.109}$$

We have the following proposition.

Proposition 2.2 *Let $\alpha = \frac{3}{4}$. Then there is a linear self-adjoint operator $R : D(R) \subset H \to H$ such that*

$$\frac{1}{2}(Ry_0, y_0) = \Phi_\alpha(y_0), \quad \forall y_0 \in D(A^{\frac{1}{4}}), \tag{2.110}$$

$$a_1|A^{\frac{1}{4}}y_0|^2 \le (Ry_0, y_0) \le a_2|A^{\frac{1}{4}}y_0|^2, \quad \forall y_0 \in D(A^{\frac{1}{4}}), \tag{2.111}$$

$$R \in L(D(A^{\frac{3}{4}}), D(A^{\frac{1}{4}})) \cap L(D(A^{\frac{1}{2}}), H) \cap L(D(A^{\frac{1}{4}}), (D(A^{\frac{1}{4}}))'), \tag{2.112}$$

$$(\mathscr{A}y_0 - \gamma y_0, Ry_0) + \frac{1}{2}\sum_{i=1}^{M^*}(B\psi_j, Ry_0)^2 = \frac{1}{2}|A^{\frac{3}{4}}y_0|^2, \quad \forall y_0 \in D(A), \tag{2.113}$$

where $a_i > 0,\ i = 1, 2$. *Moreover, the corresponding feedback controller*

$$u = -\sum_{i=1}^{M^*} (B\psi_j, Ry)\psi_j$$

stabilizes exponentially the linearized system (2.25), *that is*,

$$\begin{aligned} &\int_0^\infty e^{2\gamma t} |A^{\frac{3}{4}} y(t)|^2 dt \le C\|y_0\|_W^2, \quad \forall y_0 \in W, \\ &\|y(t)\|_W \le Ce^{-\gamma t}\|y_0\|_W, \qquad\qquad \forall y_0 \in W, \end{aligned} \tag{2.114}$$

where $W = D(A^{\frac{1}{4}})$.

Proof By Corollaries 2.2 and 2.3, we know that $\Phi_\alpha(y_0) < \infty$ for each $y_0 \in H$ and, therefore, there is a pair $(y^*, v^*) \in L^2(0, \infty; D(A^\alpha)) \cap L^2(0, \infty; R^{M^*})$ satisfying System (2.106) and such that $J_\alpha(y^*, v^*) = \Phi_\alpha(y_0)$.

By (2.106), that is,

$$\begin{aligned} &\frac{d}{dt} y^* + Ay^* + F_0'(y_e)y^* - \gamma y^* = Dv^*(t), \quad t \ge 0, \\ &y^*(0) = y_0, \end{aligned} \tag{2.115}$$

where $Dv = \sum_{j=1}^{M^*} v_j B\psi_j,\ v = \{v_j\}_{j=1}^{M^*}$, we obtain by multiplication with $A^{\frac{1}{2}} y^*$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|A^{\frac{1}{4}} y^*(t)|^2 + |A^{\frac{3}{4}} y^*(t)|^2 + (F_0'(y_e)y^*(t), A^{\frac{1}{2}} y^*(t)) - \gamma(y^*(t), A^{\frac{1}{2}} y^*(t)) \\ &\quad = (Dv^*(t), A^{\frac{1}{2}} y^*(t)), \quad \text{a.e., } t > 0. \end{aligned}$$

Taking into account (j) and (jj), we obtain that

$$a_1|A^{\frac{1}{4}} y_0|^2 \le \Phi_{\frac{3}{4}}(y_0) \le a_2|A^{\frac{1}{4}} y_0|^2 \quad \text{for } y_0 \in D(A^{\frac{1}{4}}),\ \text{where } a_1, a_2 > 0.$$

If we denote by R the Gâteaux derivative of the function $\Phi_{\frac{3}{4}}$ on $D(A^{\frac{1}{4}})$, we have

$$R \in L(D(A^{\frac{1}{4}}), (D(A^{\frac{1}{4}}))') \quad \text{and} \quad \frac{1}{2}(Ry_0, y_0) = \Phi_{\frac{3}{4}}(y_0), \quad \forall y_0 \in D(A^{\frac{1}{4}}).$$

Then, (2.110) and (2.111) follows.

By the dynamic programming principle, we have that, for all $0 < T < \infty$, (y^*, v^*) is also optimal in the problem

$$\text{Min}\left\{\frac{1}{2}\int_t^T (|A^{\frac{3}{4}} y(s)|^2 + |v(s)|_{M^*}^2) ds + \Phi_{\frac{3}{4}}(y(T))\right.$$

$$\left.\text{subject to (2.115), } y(t) = y^*(t)\right\} = \Phi_{\frac{3}{4}}(y(t)).$$

Then, by the maximum principle we have (D^* is the adjoint of D)

$$v^*(t) = D^* p_T(t), \quad \text{a.e., } t \in (0, T), \tag{2.116}$$

where p_T is the solution to the dual system

$$\begin{aligned} &p_T' - Ap_T - (F_0'(y_e))^* p_T + \gamma p_T = A^{\frac{3}{2}} y^* \quad \text{on } (0, T), \\ &p_T(T) = -Ry^*(T). \end{aligned} \tag{2.117}$$

Moreover, we have also

$$p_T(t) = -Ry^*(t), \quad \forall t \in [0, T]. \tag{2.118}$$

Now, if $y_0 \in D(A^{\frac{1}{2}})$, then as easily follows by (2.115) we have $Ay^* \in L^2(0, \infty; H)$ and so $A^{\frac{3}{2}} y^* \in L^2(0, \infty; (D(A^{\frac{1}{2}}))')$.

Next, one multiplies (2.117) by $A^{-\frac{1}{2}} p_T$ and integrate on (t, T). We obtain that

$$\begin{aligned} &\frac{1}{2}|A^{-\frac{1}{4}} p_T(t)|^2 + \int_t^T |A^{\frac{1}{4}} p_T(s)|^2 ds \\ &\quad \le \frac{1}{2}|A^{-\frac{1}{4}} p_T(T)|^2 + C\int_t^T (|Ay^*(s)|^2 + |p_T(s)|^2) ds, \quad \forall t \in (0, T), \end{aligned}$$

because $|((F_0'(y_e))^* p, A^{-\frac{1}{2}} p)| \le C|p|$.

Now, invoking the interpolating inequality $|p| \le |A^{-\frac{1}{4}} p|^{\frac{1}{2}} |A^{\frac{1}{4}} p|^{\frac{1}{2}}$, we obtain via Gronwall's lemma that

$$A^{-\frac{1}{4}} |p_T(t)|^2 + \int_t^T |A^{\frac{1}{4}} p_T(s)|^2 ds \le C, \quad \forall t \in [0, T].$$

If we multiply (2.117) by $(T - t)p_T(t)$ and integrate on (t, T), we get

$$(T - t)|p_T(t)|^2 + \int_t^T (T - s)\|p_T(s)\|^2 ds \le \int_t^T |p_T(s)|^2 ds + C \le C_1.$$

Hence, $p_T(t) \in H$ for all $t \in [0, T)$ and so $p_T(0) = -Ry_0 \in H$. Hence, $R(D(A^{\frac{1}{2}})) \subset H$ and $R \in L(D(A^{\frac{1}{2}}), H)$.

Now, if $y_0 \in D(A^{\frac{3}{4}})$, it follows (by (2.115)) that $A^{\frac{5}{4}} y^* \in L^2(0, T; H)$ and so, by (2.117), we get as above that $|A^{\frac{1}{4}} p_T(t)| \in C[0, T - \delta; H)$ and, therefore, $p_T(0) = -Ry_0 \in D(A^{\frac{1}{4}})$. Hence, $R \in L(D(A^{\frac{3}{4}}), D(A^{\frac{1}{4}}))$, as claimed.

Now, to find the Riccati equation (2.113), we start with the equation

$$\Phi_{\frac{3}{4}}(y^*(t)) = \frac{1}{2}\int_t^\infty \left| |A^{\frac{3}{4}} y^*(s)|^2 + |v^*(s)|^2_{M^*} \right| ds, \quad \forall t \ge 0, \tag{2.119}$$

and recall that (see (2.116) and (2.118)),

$$v^*(t) = -D^* R y^*(t), \quad \forall t \geq 0, \tag{2.120}$$

where D^* is the adjoint of D, that is,

$$D^* p = \{(B\psi_j, p)\}_{j=1}^{M^*}, \quad \forall p \in H.$$

Taking into account that

$$\begin{aligned} \frac{d}{dt}\Phi_{\frac{3}{4}}(y^*(t)) &= \left(Ry^*(t), \frac{dy^*}{dt}(t)\right) \\ &= -(Ry^*(t), \mathscr{A} y^*(t) - \gamma y^*(t) + DD^* R y^*(t)), \quad \forall t \geq 0, \end{aligned}$$

we obtain by (2.119) that

$$\frac{1}{2}(|A^{\frac{3}{2}} R y^*(t)|^2 + |D^* R y^*(t)|^2) = (\mathscr{A} y^*(t) - \gamma y^*(t), Ry^*(t)) + |D^* R y^*(t)|^2, \\ \forall t \geq 0,$$

and for $t = 0$ we get (2.113), as claimed.

As regards (2.114), it follows immediately by (2.115). □

2.5.2 Low-gain Riccati-based Stabilizable Feedback

Proposition 2.3 *Let $\alpha = 0$. Then there is a linear self-adjoint positively semidefinite operator $R_0 \in L(H, H)$ such that $R_0 \in L(H, D(A))$ and*

$$\frac{1}{2}(Ry_0, y_0) = \Phi_0(y_0), \quad \forall y_0 \in H, \tag{2.121}$$

$$(\mathscr{A} y_0 - \gamma y_0, R_0 y_0) + \frac{1}{2}\sum_{i=1}^{M^*}(B\psi_i, R_0 y_0)^2 = \frac{1}{2}|y_0|^2, \quad \forall y_0 \in D(A). \tag{2.122}$$

Moreover, the feedback law

$$u = -\sum_{j=1}^{M^*}(B\psi_i, R_0 y)\psi_i$$

stabilizes exponentially with decaying rate $-\gamma$ System (2.25), that is,

$$|y(t)| \leq Ce^{-\gamma t}|y_0|, \quad \forall y_0 \in H.$$

The proof is standard and similar to that of Proposition 2.2.

We notice that also in this case we have (see (2.118))

$$\widetilde{p}_T(t) = -R_0 y^*(t), \quad \forall t \in [0, T],$$

where $\widetilde{p}_T$ is the solution to (2.117) with the right-hand side y^*. This implies an additional regularity for R_0, namely that $R_0 y_0 = -p(0) \in D(A)$ and, therefore, $R_0 \in L(H, D(A))$.

2.5.3 *Internal Stabilization of Nonlinear System via High-gain Riccati-based Feedback*

We assume here, besides (j) and (jj), that the following hypothesis holds.

(jjj) *G is locally Lipschitz from $V = D(A^{\frac{1}{2}})$ to $V' = (D(A^{\frac{1}{2}}))'$ and*

$$|(Gy - Gz, y - z)| \le \|y - z\|^2 + C_\varepsilon |y - z|^2 \tag{2.123}$$

for all $\|y\| + \|z\| \le \frac{1}{\varepsilon}$ and $\varepsilon > 0$. Moreover,

$$|Gy| \le \eta(\|y\|_W)|A^{\frac{3}{4}} y|^{\frac{3}{2}}, \quad \forall y \in D(A^{\frac{3}{4}}), \tag{2.124}$$

where $\eta : R \to R^+$ is continuous, increasing and $\eta(0) = 0$.

Here, $\|y\| = |A^{\frac{1}{2}} y|^2$ and $W = D(A^{\frac{1}{4}})$ with the norm $\| \cdot \|_W = | \cdot |_{D(A^{\frac{1}{4}})}$.

Theorem 2.9 *Under Assumptions* (j), (jj) *and* (jjj) *there is a neighborhood $\mathscr{U}_\rho = \{y \in W;\ \|y\|_W < \rho\}$ of the origin such that for all $y_0 \in \mathscr{U}_\rho$ the Cauchy problem*

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A} y + G(y) = -\sum_{j=1}^{M^*} (B\psi_j, Ry) B\psi_j, \quad \forall t \ge 0, \\ &y(0) = y_0, \end{aligned} \tag{2.125}$$

has a unique solution

$$y \in C([0, \infty); H) \cap L^2(0, \infty; D(A^{\frac{3}{4}})). \tag{2.126}$$

Moreover,

$$\begin{aligned} &\int_0^\infty e^{2\gamma t} |A^{\frac{3}{4}} y(t)|^2 dt \le C \|y_0\|_W^2, \\ &\|y(t)\|_W \le C e^{-\gamma t} \|y_0\|_W, \quad \forall t \ge 0,\ y_0 \in \mathscr{U}_\rho. \end{aligned} \tag{2.127}$$

Here $R \in L(W, W')$ is provided by Proposition 2.2.

Theorem 2.9 amounts to saying that

$$Ky = \sum_{i=1}^{M^*} (B\psi_j, Ry)\psi_j \tag{2.128}$$

is an exponentially stabilizable feedback for System (2.100) (see (2.105)).

We get therefore the following stabilization result for the equilibrium solution y_e to System (2.6), that is,

$$\begin{aligned} &\frac{dy}{dt} + Ay + F_0(y) = 0, \quad t \geq 0, \\ &y(0) = y_0. \end{aligned} \tag{2.129}$$

Corollary 2.4 *Assume that A, F_0 and $G(y) \equiv F_0(y) - F_0(y_e)$ satisfy Assumptions* (j)–(jjj). *Then the feedback controller $u = -K(y - y_e)$ stabilizes exponentially System* (2.129) *in a neighborhood of y_e. More precisely, there is $\rho > 0$ such that the closed-loop system*

$$\begin{aligned} &\frac{dy}{dt} + Ay + F_0 y = -BK(y - y_e), \quad t \geq 0, \\ &y(0) = y_0 \end{aligned} \tag{2.130}$$

has a unique solution $y \in C([0, \infty; H) \cap L^2_{\mathrm{loc}}(0, \infty; D(A^{\frac{3}{4}}))$ which satisfies

$$\int_0^\infty e^{2\gamma t} |A^{\frac{3}{4}}(y(t) - y_e)|^2 dt < C\|y_0 - y_e\|_W^2, \tag{2.131}$$

$$\|y(t) - y_e\|_W \leq Ce^{-\gamma t}\|y_0 - y_e\|_W, \tag{2.132}$$

for all $t \geq 0$ and $\|y_0 - y_e\|_W < \rho$.

Proof of Theorem 2.9 First, we prove that the Cauchy problem (2.125) is well-posed for $y_0 \in \mathscr{U}_\rho$, where ρ is sufficiently small. To this end, we consider the truncation G_ε of the operator G, that is,

$$G_\varepsilon(y) = \begin{cases} G(y) & \text{for } \|y\| \leq \frac{1}{\varepsilon}, \\ G(\frac{y}{\varepsilon\|y\|}) & \text{for } \|y\| > \frac{1}{\varepsilon}. \end{cases} \tag{2.133}$$

Clearly, G_ε is Lipschitz from V to V' and by (2.123) we see also that

$$(G_\varepsilon y - G_\varepsilon z, y - z) \leq \|y - z\|^2 + C_\varepsilon |y - z|^2, \quad \forall y, z \in V.$$

Then, recalling (2.112), by Theorem 1.15 we conclude that for each $y_0 \in H$ there is a unique solution $y_\varepsilon \in C([0, \infty; H)) \cap L^2_{\mathrm{loc}}(0, \infty; D(A))$, $\frac{dy_\varepsilon}{dt} \in W^1_{\mathrm{loc}}(0, \infty; H)$ to the equation

$$\begin{aligned}&\frac{dy_\varepsilon}{dt}+\mathscr{A}y_\varepsilon+G_\varepsilon(y_\varepsilon)+BKy_\varepsilon=0,\quad \text{a.e., } t>0,\\&y_\varepsilon(0)=y_0.\end{aligned}\tag{2.134}$$

If we multiply (2.134) by Ry_ε (scalarly in H) and recall that by (2.113)

$$(\mathscr{A}y-\gamma y, Ry)+\frac{1}{2}\,(BKy,Ry)=\frac{1}{2}\,|A^{\frac{3}{4}}|^2,\quad \forall y\in D(A),$$

we obtain by (2.112), (2.124) and (2.134) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(Ry_\varepsilon(t),y_\varepsilon(t))+\gamma(Ry_\varepsilon(t),y_\varepsilon(t))+\frac{1}{2}(BKy_\varepsilon(t),Ry_\varepsilon(t))+\frac{1}{2}\,|A^{\frac{3}{4}}y_\varepsilon(t)|^2\\&\quad\le |G_\varepsilon(y_\varepsilon(t))\,|Ry_\varepsilon(t)|\le C|A^{\frac{3}{4}}y(t)|^{\frac{3}{2}}\|y_\varepsilon(t)\|\eta(\|y_\varepsilon(t)\|_W),\quad \text{a.e., } t>0.\end{aligned}$$

Taking into account the interpolation inequality

$$\|y\|\le|A^{\frac{3}{4}}y|^{\frac{1}{2}}\|y\|_W^{\frac{1}{2}},\quad \forall y\in D(A^{\frac{3}{4}}),$$

we have, for $\|y_0\|_W<\rho$,

$$\frac{d}{dt}\,(Ry_\varepsilon(t),y_\varepsilon(t))+2\gamma(Ry_\varepsilon(t),y_\varepsilon(t))+(BKy_\varepsilon(t),Ry_\varepsilon(t))+\frac{1}{2}\,|A^{\frac{3}{4}}y_\varepsilon(t)|^2\le 0$$

a.e., on $(0,T^*(y_0))$, where $T^*(y_0)=\sup\{t>0;\ \|y_\varepsilon(t)\|_W\le\rho\}$ and $\rho>0$ is chosen by the condition

$$C\eta(\rho)\le\frac{1}{2}\cdot$$

This yields $T^*(y_0)=\infty$ and

$$(Ry_\varepsilon(t),y_\varepsilon(t))\le e^{-2\gamma t}(Ry_0,y_0)\le Ce^{-2\gamma t}\|y_0\|_W^2,\quad \forall t\ge 0,\tag{2.135}$$

$$\int_0^\infty e^{2\gamma t}|A^{\frac{3}{4}}y_\varepsilon(t)|^2dt\le C\|y_0\|_W^2.\tag{2.136}$$

Taking into account that $G_\varepsilon(y)=G(y)$ for $\|y\|\le\frac{1}{\varepsilon}$, it follows by (2.133), (2.135) that for each $y_0\in V\cap\mathscr{W}_\rho$ there is a solution (obviously unique by virtue of Assumption (2.123)) $y=y_\varepsilon$ to (2.125) satisfying Estimate (2.127).

The stabilizable feedback law (2.128) has the unpleasant feature that the operator R is computed from a high $D(A^{\frac{3}{4}})$-gain Riccati equation (2.113) which involves some computational problem. An alternative is to use the feedback law given in Proposition 2.3. □

2.5.4 Internal Stabilization of Nonlinear System via Low-gain Riccati-based Feedback

We study here the effect of the linear feedback

$$u(t) = -\sum_{i=1}^{M^*} (B\psi_j, R_0 y)\psi_j = -Ly(t) \tag{2.137}$$

in the system

$$\begin{aligned} &\frac{dy}{dt} + \mathscr{A} y + Gy = Bu, \quad t \geq 0, \\ &y(0) = y_0, \end{aligned} \tag{2.138}$$

where R_0 is the solution to Riccati equation (2.122) given by Proposition 2.3.

Denote by $\Gamma : D(\Gamma) \subset H \to H$ the operator

$$\Gamma y = \mathscr{A} y + BLy, \qquad D(\Gamma) = D(A). \tag{2.139}$$

By (j) it is easily seen that $-\Gamma$ generates a C_0-analytic semigroup $e^{-\Gamma t}$ on H and, by Proposition 2.3, $e^{-\Gamma t}$ is exponentially stable, that is,

$$|e^{-\Gamma t} y_0| \leq Ce^{-\gamma t}|y_0|, \quad \forall t \geq 0, \ y_0 \in H.$$

Further estimates on $z(t) = e^{-\Gamma t} y_0$ are given below. If we multiply the equation

$$\frac{dz}{dt} + \mathscr{A} z + BLz = 0, \quad t \geq 0,$$

by $A^{\frac{1}{2}} z$, we get

$$\frac{d}{dt} \|z(t)\|_W^2 + |A^{\frac{3}{4}} z(t)|^2 \leq C_1 |z(t)|^2, \quad \text{a.e., } t > 0$$

and, therefore,

$$\begin{aligned} &\frac{d}{dt} (\|z(t)\|_W^2 e^{2\gamma t}) + e^{2\gamma t} |A^{\frac{3}{4}} z(t)|^2 \\ &\leq C_2 e^{2\gamma t} (|z(t)|^2 + \|z(t)\|_W^2) \\ &\leq C_3 e^{2\gamma t} |z(t)|^2 + \frac{1}{2} e^{2\gamma t} |A^{\frac{3}{4}} z(t)|^2. \end{aligned}$$

Finally,

$$\|z(t)\|_W^2 e^{2\gamma t} + \int_0^t e^{2\gamma s} |A^{\frac{3}{4}} z(s)|^2 ds \leq C_4 \|y_0\|_W^2, \quad \forall t \geq 0. \tag{2.140}$$

Now, we rewrite (2.138) with the controller u given by (2.137) as

$$y(t) = e^{-\Gamma t} y_0 - \int_0^t e^{-\Gamma(t-s)} G y(s) ds, \quad t \geq 0. \tag{2.141}$$

We assume here the following hypothesis on G.

(jv) $\|Gy - Gz\|_W \leq C|y - z|_{\frac{3}{4}}(|y|_{\frac{3}{4}} + |z|_{\frac{3}{4}}), \ \forall y, z \in D(A^{\frac{3}{4}})$.

Theorem 2.10 *Under Assumptions* (j), (jj) *and* (jv) *for each* $y_0 \in \mathscr{U}_\rho$ *and* ρ *sufficiently small there is a unique solution to* (2.141)

$$y \in C([0, \infty); W) \cap L^2(0, \infty; D(A^{\frac{3}{4}})). \tag{2.142}$$

Moreover, one has

$$\|y(t)\|_W \leq Ce^{-\gamma t}\|y_0\|_W, \quad \forall t \geq 0, \ y_0 \in \mathscr{U}_\rho. \tag{2.143}$$

Proof The proof will be sketched only. We are going to apply the contraction principle to the operator defined by the right-hand side $\Lambda(y)$ of (2.141),

$$y \to \Lambda(y) : L^2(0, \infty; D(A^{\frac{3}{4}})) \to L^2(0, \infty; D(A^{\frac{3}{4}}))$$

defined on the set

$$\mathscr{K}_r = \left\{ y \in L^2(0, \infty; D(A^{\frac{3}{4}})); \int_0^\infty |A^{\frac{3}{4}} y(t)|^2 dt \leq r \right\}$$

where r will be suitable chosen. By (2.140) we have, for $y \in \mathscr{K}_r$ and $y_0 \in \mathscr{U}_\rho$, via Young inequality and Hypothesis (jv) that

$$\begin{aligned} \|\Lambda(t)\|^2_{L^2(0,\infty;D(A^{\frac{3}{4}}))} &\leq C(\|y_0\|^2_W + \|G(y)\|^2_{L^1(0,\infty;W)}) \\ &\leq C\|y_0\|^2_W + C_1\|y\|^4_{L^2(0,\infty;D(A^{\frac{3}{4}}))}) \leq C\rho^2 + C_1 r^2. \end{aligned} \tag{2.144}$$

Here, we have used the obvious estimates

$$|A^{\frac{3}{4}} e^{-\mathscr{A}t} y_0| \leq |A^{\frac{1}{2}} e^{-\mathscr{A}t} A^{\frac{1}{4}} y_0|, \quad \forall t > 0, \ y_0 \in W,$$

and the fact that, as easily follows by Hypothesis (j) and (jj), we have

$$\|A^{\frac{1}{2}} e^{-\mathscr{A}t} z_0\|_{L^2(0,\infty;H)} \leq C|z_0|, \quad \forall z_0 \in H.$$

By (2.144) we see that, for $0 < r \leq \mu(\rho)$ sufficiently small, we have

$$C\rho^2 + C_1 r^4 \leq r^2$$

and so, the operator Λ leaves invariant the set $\mathscr{K}_r$.

On the other hand, we see in a similar way by (jv) that

$$\begin{aligned}
&\|\Lambda(y_1)-\Lambda(y_2)\|^2_{L^2(0,\infty;D(A^{\frac{3}{4}})}\\
&\quad\le C_1\|G(y_1)-G(y_2)\|^2_{L^1(0,\infty;W)})\\
&\quad\le C_1\left(\int_0^\infty |A^{\frac{3}{4}}(y_1-y_2)|(|A^{\frac{3}{4}}y_1|+|A^{\frac{3}{4}}y_2|)dt\right)^2\\
&\quad\le C_2\int_0^\infty |A^{\frac{3}{4}}(y_1-y_2)|^2dt\int_0^\infty (|A^{\frac{3}{4}}y_1|^2+|A^{\frac{3}{4}}y_2|^2)dt\\
&\quad\le C_2r^2\|y_1-y_2\|^2_{L^2(0,\infty;D(A^{\frac{3}{4}}))},\qquad \forall y_1,y_2\in\mathscr{K}_r.
\end{aligned}$$

Hence, choosing r sufficiently small ($r<\frac{1}{\sqrt{C_2}}$), we have that Λ is a contraction on $\mathscr{K}_r$ and, therefore, (2.141) has a unique solu tion y satisfying (2.142).

In order to prove (2.143), we write (2.141) as

$$\frac{dy}{dt}+\mathscr{A}y+BLy+Gy=0,\qquad y(0)=y_0,$$

and repeat the previous estimates (2.140).

We get as above that, for $y_0\in\mathscr{U}_\rho$,

$$\|e^{\gamma t}y(t)\|_W\le C\|y_0\|_W,\quad \forall t\ge 0.\qquad\square$$

Theorems 2.9 and 2.10 can be applied to Example (2.17) if one assumes that β and g are C^2-functions with polynomial growth and $y_e\in L^\infty(\mathscr{O})$.

One might prove also that the linear feedback controller provided by Theorem 2.4, that is,

$$u(t)=-\eta\sum_{j=1}^N\langle y,\widetilde{\varphi}_j\rangle\widetilde{\phi}_j$$

inserted into (2.138) stabilizes exponentially the system in a neighborhood $\mathscr{U}_\rho$ of the origin. The proof is identical with that of Theorem 2.9 or 2.10, but once again the details are omitted.

2.5.5 High-gain Feedback Controller Versus Low-gain Controller and Robustness

Roughly speaking, Theorems 2.9 and 2.10 provide the same type of stability for the control system associated with (2.100). One might suspect, however, that the radius of stability of $\mathscr{U}_\rho$ established via Lyapunov function (Ry,y) is more exact and bigger in the first case, but this does not seem to be the principal advantage of the first approach. As a matter of fact, in both situations the stabilizing feedback controller is obtained from a linear quadratic control problem (so-called LQG design method from automatic control theory) but with different quadratic cost criteria and here arises the major difference between them because, as we show in Chap. 5, the high-gain feedback controller used in Theorem 2.1 is more robust than (2.137). The robustness of a control feedback is a central problem in automatic control and roughly speaking it is the property of the system to remain insensitive to disturbances or model imperfections. If we consider System (2.100), where $\mathscr{A}$ and G are imperfectly known but remain in a certain "neighborhood" $\mathscr{V}(\mathscr{A}^*, G^*)$ of a given state-system $\frac{dy}{dt} + \mathscr{A}^* y + G^*(y) = 0$, we say that the stabilizing feedback controller is robust in this class if $u = -Ky$ is a stabilizing feedback for all $(\mathscr{A}, G) \in \mathscr{V}(\mathscr{A}^*, G^*)$. It is well-known that a feedback controller obtained from LQG is always robust in a certain sense if all the output variables are measurable but it is also clear that the robustness performance is dependent (at least in infinite-dimensional setting) of the cost functional. One principal tool to evaluate and improve the robustness in this case is the H^∞-theory we shall speak about in Chap. 5. For the nonlinear system it is more difficult to evaluate or compare the robustness performances but one can see that the given feedback controller is more robust than another if its invariant stability class $\mathscr{V}(\mathscr{A}^*, G^*)$ is larger (measured in the same topology) than another. From this point of view, we show below that the high-gain feedback controller designed here is more robust than the low-gain feedback controller designed in Theorem 2.10.

Theorem 2.11 *Under the assumptions of Theorem 2.9, the feedback controller $u = -Ky$ given by (2.128) is still stabilizable with the rate γ for all the systems of the form*

$$\frac{dy}{dt} + \widetilde{\mathscr{A}} y + \widetilde{G} y = -BKy, \quad t \ge 0, \tag{2.145}$$

where $(\widetilde{\mathscr{A}}, \widetilde{G})$ satisfy Assumptions (j), (jj) *and* (jjj) *and*

$$|\widetilde{\mathscr{A}} y - \mathscr{A} y| \le C\varepsilon |Ay|, \quad |\widetilde{G} y - Gy| \le \varepsilon |Gy|, \quad \forall y \in \mathscr{U}_\rho, \tag{2.146}$$

and $\varepsilon > 0$ is sufficiently small.

Proof If we multiply (2.145) by Ry and use (2.113), we obtain by (2.146) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(Ry(t), y(t)) + \frac{1}{2}\,|A^{\frac{3}{4}} y(t)|^2 + \gamma(Ry(t), y(t)) + \frac{1}{2}\,(BKy(t), Ry(t))\\&\quad \le C\varepsilon |(Ay, Ry)| + (1+\varepsilon)|Gy|\,|Ry|, \quad \text{a.e., } t \in (0, T^*),\end{aligned}$$

where $T^* = \sup\{t;\ y(t) \in \mathscr{W}_\rho\}$.

On the other hand, as seen in Proposition 2.2, we have

$$|(Ay, Ry)| \le C|A^{\frac{3}{4}} y|^2\, \|y\|_W, \quad \forall y \in D(A)$$

and this yields, for all ε sufficiently small,

$$\frac{1}{2}\frac{d}{dt}(Ry(t), y(t)) + \left(\frac{1}{2} - \varepsilon\right)|A^{\frac{3}{4}}y(t)|^2 + \gamma(Ry(t), y(t)) \\ + \frac{1}{2}(BKy(t), Ry(t)) \leq C\eta(\rho)|A^{\frac{3}{4}}y(t)|^2$$

and this implies, as seen earlier,

$$a_1\|y(t)\|_W^2 \leq (Ry(t), y(t)) \leq e^{-\gamma t}(Ry_0, y_0), \quad \forall t \geq 0.$$

Then, arguing as above, we find that

$$\|y(t)\|_W \leq Ce^{-\gamma t}\|y_0\|_W, \quad \forall t \geq 0,$$

for all y_0, with $\|y_0\|_W \leq \rho$ suitable chosen. This completes the proof. □

Theorem 2.11 amounts to saying that the feedback controller found by the high-gain Riccati equation keeps unaltered its stabilizing property for small but sharp deviations of the system. For instance, in case of the parabolic system (2.17), it turns out that it still operates with the same stabilizing rate γ on perturbed parabolic systems of the form

$$\frac{\partial y}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^{\varepsilon}(x)\frac{\partial y}{\partial x_i}\right) + \beta_\varepsilon(y) + g_\varepsilon(\nabla y) = -BKu \quad \text{in } (0,T)\times\mathscr{O},$$
$$y = 0 \quad \text{on } \partial\mathscr{O}, \qquad y(0,x) = y_0(x) \quad \text{in } \mathscr{O},$$

where $|a_{ij}^{\varepsilon} - a_{ij}| \leq C\varepsilon$, for $i, j = 1, \ldots, n$ and

$$\|\beta_\varepsilon - \beta\|_{C^2(R)} + \|g_\varepsilon - g\|_{C^2(R)} \leq C\varepsilon.$$

In other words, it remains stabilizable to small structural perturbation of the system.

In particular, it follows by Theorem 2.11 that, if

$$\frac{dy}{dt} + \mathscr{A}_h y + G_h y = Bu, \quad t \geq 0 \tag{2.147}$$

is a finite element approximation of (2.100), then, if $|(\mathscr{A}_h - \mathscr{A})y|$, $|(G_h - G)y| \to 0$ as $h \to 0$ uniformly on $D(A)$ respectively, on $D(A^{\frac{3}{4}})$ (and this usually happens), then the stabilizing high-gain feedback law $u = -K_h y$ for (2.147) is, for h small, still stabilizable for System (2.100). This fact allows to stabilize the state-system (2.100) using approximating feedback laws provided by the finite-dimensional Riccati equation (2.113), that is,

$$(\mathscr{A}_h y - \gamma y, R_h y) + \frac{1}{2}\sum_{i=1}^{M^*}(B\psi_i, R_h y)^2 = \frac{1}{2}|A_h^{\frac{3}{4}}y|^2.$$

(We notice that by the stability of the spectrum $\sigma(\mathscr{A})$ (see [59]), the spectral index M^* is invariant to small perturbations of $\mathscr{A}$.)

Now, analyzing the stability performances of low-gain Riccati-based feedback (2.137), it is easily seen that, in general, it is not robust to structural sharp perturbations of the form mentioned above or, more precisely, its robustness region is smaller than that of high-gain Riccati-based feedback discussed above.

For instance, it is not stabilizable for the linear system

$$\frac{dy}{dt} + \mathscr{A}_\varepsilon y = Bu, \quad t \geq 0,$$

where $\mathscr{A}_\varepsilon = \mathscr{A} - \varepsilon A$. Indeed, in this case we have by (2.122) (we take $\gamma = 0$),

$$\frac{1}{2}\frac{d}{dt}(R_0 y, y) + \frac{1}{2}|y|^2 + \frac{1}{2}(BKy, R_0 y) = \varepsilon(Ay, R_0 y)$$

and, obviously, this does not imply $\lim_{t\to 0}(R_0 y(t), y(t)) = 0$, as desired.

2.6 Stabilization of Time-periodic Flows

2.6.1 The Functional Setting

We consider here the controlled evolution system

$$\frac{dy}{dt}(t) + Ay(t) + B(t, y(t)) = Du(t), \quad t \in R, \tag{2.148}$$

in a Hilbert space H with the norm $|\cdot|$ and scalar product denoted $(\cdot,\cdot)$.

The following assumptions will be in effect throughout this section.

(k) A is a linear, self adjoint positive definite operator in H with domain $D(A)$. A^{-1} is completely continuous.

For $0 < \alpha < 1$ we denote, as usually, by A^α the fractional power of order α of A and by $|x|_\alpha = |A^\alpha x|$ the norm of $D(A^\alpha)$.

(kk) $B : R \times D(A^\alpha) \to H$, *where* $\frac{1}{4} \leq \alpha < 1$, *satisfies the conditions*

$$B(t+T, y) = B(t, y), \quad \forall (t, y) \in R \times D(A^\alpha); \tag{2.149}$$

$$|B(t,0) - B(s,0)| \leq C_1|t-s|, \quad \forall s, t \in R; \tag{2.150}$$

$$|B_y(t,y) - B_y(s,z)|_{L(D(A^\alpha),H)} \leq C_2(|y|_{\frac{1}{4}} + |z|_{\frac{1}{4}})(|t-s| + |y-z|_\alpha),$$
$$\forall y, z \in D(A^\alpha);\ t, s \in R; \tag{2.151}$$

$$|B_y(t,y)|_{L(D(A^{\frac{1}{2}}),H)} \leq C(1 + |Ay|), \quad \forall y \in D(A). \tag{2.152}$$

Here, $B_y(t,\cdot) \in L(D(A^\alpha), H)$ is the (*Fréchet*) *derivative of* $B(t,\cdot)$. We note that by (2.151) it follows that

$$|B(t,y) - B(t,z)| \le C_3(|y|_{\frac{1}{4}} + |z|_{\frac{1}{4}})|y-z|_\alpha,\ \ y,z \in D(A^\alpha), \quad t \in R. \tag{2.153}$$

(kkk) $D \in L(U,H)$ *where* U *is a Hilbert space with the norm* $|\cdot|_U$ *and the scalar product* $\langle\cdot,\cdot\rangle_U$.

Now, let $y_\pi \in C^1(R, D(A))$ be a T-periodic solution to (2.148), that is,

$$\begin{aligned} &\frac{d}{dt}\, y_\pi(t) + Ay_\pi(t) + B(t, y_\pi(t)) = 0, \quad && t \in R,\\ &y_\pi(t) = y_\pi(t+T), && \forall t \in R. \end{aligned} \tag{2.154}$$

Let $\mathscr{A}(t) \equiv A + B_y(t, y_\pi(t))$. By Assumptions (k) and (kk), and the fact that $y_\pi \in C^1(R; D(A))$, we see that the resolvent $R(\lambda; \mathscr{A}(t)) = (\lambda I + \mathscr{A}(t))^{-1}$, $t \in R^+$, exists for all complex $\lambda \in \Sigma$, where $\Sigma = \{\lambda;\ |\arg(\lambda - a)| \le \phi\}$ for some $a > 0$ and $\phi > \frac{\pi}{2}$. Moreover, there is a positive constant C such that $\|R(\lambda; \mathscr{A}(t))\| \le \frac{C}{|\lambda - a|}$ for all $\lambda \in \Sigma$, $t \in R^+$, and there exists a constant $C_1 > 0$ such that

$$\|(\mathscr{A}(t) - \mathscr{A}(s))(aI - \mathscr{A})^{-1}(\tau)\| \le C_1|t-s|, \quad \text{for all } s,t,\tau \in R^+.$$

Then (see, e.g., [54, 66]), there is a unique evolution operator $S(t,s)$, $0 \le s \le t < \infty$, such that

$$\begin{aligned} \frac{d}{dt} S(t,s)x + \mathscr{A}(t)S(t,s)x = 0, \quad 0 \le s < t < \infty,\\ S(s,s)x = x \in H. \end{aligned}$$

Moreover, $S(t,s)$ is strongly continuous in (t,s) with values in $L(D(A^\beta), D(A^\beta))$ for any $0 \le \beta < 1$ and (see [54], p. 191)

$$\begin{aligned} &|S(t,s)x|_\beta \le C(t-s)^{\gamma-\beta}|x|_\gamma, \quad 0 \le \gamma \le \beta < 1,\ t > s,\\ &\left|\frac{d}{dt}\, S(t,s)x\right|_\beta \le C(t-s)^{\gamma-\beta-1}|x|_\gamma. \end{aligned} \tag{2.155}$$

If we set $y(t) = S(t,0)y_0$, then $y(t) \in C(R^+; H)$ is the solution to the system

$$y'(t) + \mathscr{A}(t)y(t) = 0, \quad y(0) = y_0.$$

Now, we let $U(t) = S(T+t,t)$, $t \in R^+$, be the periodic map (Poincaré map) and recall that (see, e.g., [54], p. 198), $U(T+t) = U(t)$ and the spectrum $\sigma(U(t))$ is independent of t. Since A^{-1} is completely continuous, $U(t)$ is completely continuous as well. Moreover, $\sigma(U(t)) \setminus \{0\}$ consists entirely of eigenvalues $\{\lambda_j\}_{j=1}^\infty$, $|\lambda_j| \to 0$ as $j \to \infty$. Each eigenvalue λ_j is repeated accor ding to its algebraic multiplicity m_j. Let $U^*(t)$ and D^* be the adjoint of $U(t)$ and D, respectively. Then

$\sigma(U^*(t)) \setminus \{0\} = \{\overline{\lambda}_j\}_{j=1}^{\infty}$. We denote by X_m^* the space spanned by $\{\psi_i^*\}_{i=1}^m$, where ψ_i^*, $i = 1, \dots, m$, are eigenvectors of $U^*(T)$ corresponding to eigenvalues $\{\overline{\lambda}_j\}_{j=1}^m$. We assume that the following hypothesis holds.

(A1) $\ker\{D^*|_{X_m^*}\} = \{0\}, \ \forall m.$

In particular, Assumption (A1) implies the following unique continuation property: *If* $U^*(T)\varphi^* = \lambda\varphi^*$, *where* $\lambda \in \sigma(U^*(T)) \setminus \{0\}$, *and* $D^*\varphi^* = 0$, *then* $\varphi^* = 0$.

Assumption (A1) is a consequence of the following one.

(A1)$'$ *If z satisfies for $\psi \in X_m^*$ the equation*

$$\begin{aligned} &z' - \mathscr{A}^*(t)z = 0 \quad in \ (0, T), \\ &z(0) = \lambda z(T) + \psi, \end{aligned} \tag{2.156}$$

and $D^*z(T) = 0$, then $z \equiv 0$.

Indeed, if ψ_1^*, ψ_2^* are linearly independent eigenvectors, $U^*(T)\psi_i^* = \lambda_i \psi_i^*$, $i = 1, 2$ and $D^*\psi_1^* = \mu D^*\psi_2^*$, then $z(t) = S^*(T, t)\psi_1^* - \mu S^*(T, t)\psi_2^*$ satisfies (2.156) for $\psi = \psi_1^* - \mu\psi_2^* \in X_m^*$ and $D^*z(T) = 0$. Hence $z \equiv 0$ (that is, $\psi_2^* = C\psi_1^*$). The case of m eigenfunctions $\{\psi_i^*\}_{i=1}^m$ follows by induction from the previous one. A more delicate situation is that when system $\{\psi_i^*\}_{i=1}^m$ contains generalized eigenvectors ψ^*, that is, $(U^*(T) - \lambda_j I)^q \psi^* = 0$ for some $1 < q < m_j$, but we omit the proof.

In the classical Floquet theory, the eigenvalues λ of $U(t)$ are the characteristic multipliers of the linear system and $\gamma = -(\frac{1}{T}) \log \lambda$ are the Floquet exponents. One knows that, if there is a characteristic multiplier with modulus greater than one, then the periodic solution y_π is unstable.

The main result of Sect. 2.6, Theorem 2.12 below, amounts to saying that under Assumptions (k)–(kkk), and (A1), (A1)$'$ there is a feedback controller u which stabilizes exponentially the solution y_π. Moreover, the controller u has a finite-dimensional structure $u(t) = \sum_{i=1}^N u_i(t)w_i$, where $\{w_i\}$ is a given system in U and N is the number of characteristic multipliers (repeated according to their algebraic multiplicity) with modulus greater than or equal to one.

2.6.2 Stabilization of the Linearized Time-periodic System

Let $\mathscr{A}(t) = A + B_y(t, y_\pi(t))$ with the domain $D(\mathscr{A}(t)) = D(A)$. We consider the linear system

$$\begin{cases} y'(t) + \mathscr{A}(t)y(t) = Du(t), & t \in R^+, \\ y(0) = x, \end{cases} \tag{2.157}$$

where $x \in H$. System (2.157) is just the linearization of (2.148) in $y = y_\pi(t)$.

Unless stated explicitly, by solution$y$ to (2.157) we mean “mild” solution, that is,

$$y(t) = S(t, 0)x + \int_0^t S(t, s)Du(s)ds, \quad t \geq 0, \tag{2.158}$$

where $S(t,s)$ is the evolution generated by $\mathscr{A}(t)$. We notice, however, that such a solution is a strong solution. More precisely, y is absolutely continuous on every compact interval (δ, T) and satisfies, a.e., (2.157). It suffices to check this for $x = 0$ because, as noticed earlier, $\left|\frac{d}{dt} S(t,0)x\right| \le Ct^{-1}$, for all $t > 0$. By (2.155) and (2.158), we have

$$|y|_{L^2(0,T;D(A^\alpha))} \le C|Du|_{L^2(0,T;H)}.$$

Then, by Assumption (2.151) we have that

$$|B_y(t, y_\pi(t))y|_{L^2(0,T;H)} \le C_1|u|_{L^2(0,T;H)}$$

and since $-A$ generates an analytic C_0-semigroup, we see that (see Sect. 1.3)

$$\frac{d}{dt}\,y,\ Ay \in L^2(\delta, T; H), \quad \forall \delta > 0.$$

Lemma 2.2 *There is a controller u of the form*

$$u(t) = \sum_{i=1}^{N} u_i(t)w_i, \quad t \ge 0, \tag{2.159}$$

where $\{w_i\}_{i=1}^N \subset U$ is a linearly independent system and $u_i \in L^2(R^+)$, $i = 1,\dots,N$, are such that

$$u_i(t) = 0 \quad \textit{for } t \ge T,\ i = 1,\dots,N, \tag{2.160}$$

$$\int_0^T \sum_{i=1}^{N} |u_i(t)|^2 dt \le C|x|^2, \tag{2.161}$$

$$|y(t)| \le Ce^{-\delta t}|x|, \quad \forall t \ge 0,\ \textit{where } \delta > 0. \tag{2.162}$$

Here, $y \in C(R^+; H)$ is the solution to (2.157).

Proof As in the previous cases, we can replace H by its complexified space, again denoted by $\widetilde{H}$. Similarly, we replace U by $\widetilde{U} = U + iU$. As noticed earlier, the periodic map $U(t) = S(T+t,t)$ has the property that

$$\sigma(U(t)) \setminus \{0\} = \{\lambda\}_{j=1}^{\infty}, \quad \lambda_j \to 0, \forall t \in R.$$

Let $\eta > 0$ be arbitrarily small but fixed. Then, outside the disk

$$\Sigma = \{\lambda \in \mathbb{C};\ |\lambda| < 1 - \eta\},$$

there remains a finite number of eigenvalues $\{\lambda_j\}_{j=1}^N$ only. (Recall the eigenvalues λ_j are repeated according to their algebraic multiplicity m_j and so $N = m_1 + m_2 +$

$\cdots + m_k$, where k is the number of distinct eigenvalues in Σ.) Then, for each $t \in R^+$, the space $\widetilde{H}$ can be decomposed as

$$\widetilde{H} = H_1(t) \oplus H_2(t), \quad \forall t \geq 0,$$

where $H_1(t) = P_1(t)H$, $H_2(t) = (I - P_1(t))H$, and $P_1(t)$ is defined by

$$P_1(t) = \frac{1}{2\pi i} \int_\Gamma (\lambda I - U(t))^{-1} d\lambda, \quad t \in R^+,$$

where Γ is a contour surrounding $\{\lambda_j\}_{j=1}^N$ but not other eigenvalues.

It is clear that $P_1(t) = P_1(t+T)$, for all $t \in R^+$, and

$$\sigma(U(t)|_{H_1(t)}) = \{\lambda_j\}_{j=1}^N, \qquad \sigma(U(t)|_{H_2(t)}) = \{\lambda_j\}_{j=N+1}^\infty.$$

It follows that (see, e.g., [54], p. 198)

$$\begin{aligned} &\dim H_1(t) = N \quad \text{and} \quad H_1(t+T) = H_1(t), \quad \forall t \geq 0; \\ &S(t,s) : H_i(s) \to H_i(t), \quad i = 1,2,\ 0 \leq s \leq t < \infty, \\ &\quad \text{that is,} \quad S(t,s)P_i(s) = P_i(t)S(t,s), \quad i = 1,2; \end{aligned} \tag{2.163}$$

$$|S(t,s)x| \leq Ce^{-\delta(t-s)}|x|, \quad \forall x \in H_2(s),\ t \geq s, \tag{2.164}$$

where C and δ are positive constants independent of t, s and x.

Let $U^*(t)$, $S^*(t,s)$ be the adjoints of $U(t)$ and $S(t,s)$ respectively, and let

$$H_1^*(t) = P_1^*(t)H, \quad H_2^*(t) = (I - P_1^*(t))H, \quad t \geq 0,$$

where P_1^* is the adjoint of P_1, that is,

$$P_1^*(t) = \frac{1}{2\pi i} \int_{\Gamma^*} (\lambda I - U^*(t))^{-1} d\lambda.$$

Here, Γ^* is a contour surrounding $\{\overline{\lambda}_j\}_{j=1}^N$ (the eigenvalues of $U^*(t)$). We have

$$\dim H_1^*(t) = N, \quad \forall t \in R^+,$$

and

$$\begin{aligned} &S(t+T, s+T) = S(t,s), \quad 0 < s \leq t < \infty, \\ &S^*(t+T, s+T) = S^*(t,s), \\ &S^*(t,s) : H_i^*(t) \to H_i^*(s), \quad 0 \leq s \leq t < \infty,\ i = 1,2. \end{aligned}$$

Now, let

$$u = \sum_{i=1}^N u_i w_i,$$

where $w_i \in U$ is specified later and let $u_i \in L^2(R^+)$, $i = 1, \dots, N$. We represent the mild solution $y = y^u$ to (2.25), where $u = \sum_{i=1}^N u_i w_i$, that is,

$$y(t) = S(t,0)x + \int_0^t S(t,s)\left(\sum_{i=1}^N u_i(s) D w_i\right) ds, \tag{2.165}$$

as $y(t) = y_1(t) + y_2(t)$, and $y_i(t) \in H_i(t)$, $i = 1, 2$, are given by

$$y_1(t) = S(t,0)x^1 + \int_0^t S(t,s) P_1(s)\left(\sum_{i=1}^N u_i(s) D w_i\right) ds, \tag{2.166}$$

$$y_2(t) = S(t,0)x^2 + \int_0^t S(t,s)(I - P_1(s))\left(\sum_{i=1}^N u_i(s) D w_i\right) ds. \tag{2.167}$$

Here, $x^1 = P_1(0)x$ and $x^2 = (I - P_1(0))x$.

By (2.164), it follows that

$$|y_2(t)| \le C\left(e^{-\delta t}|x^2| + \int_0^t e^{-\delta(t-s)} \left|\sum_{i=1}^N u_i(s) D w_i\right|_U ds\right), \quad t \ge 0. \tag{2.168}$$

Next, we are going to show that there are $u_i(s)$, $i = 1, \dots, N$, such that $y_1(T) = 0$ (that is, (2.166) is exactly null controllable). To this end, we note first that for each $\xi \in H_1^*(T)$, $S^*(T,s)\xi = q(s) \in H_1^*(s)$, $0 \le s \le T$, is the so lution to the backward adjoint equation

$$q_t - \mathscr{A}^*(t)q = 0, \quad t \in (0,T), \quad q(T) = \xi, \tag{2.169}$$

where $\mathscr{A}^*(t)$ is the adjoint of $\mathscr{A}(t)$. To this purpose, we recall that the exact null controllability of (2.166) is equivalent to the following observability inequality

$$|S^*(T,0)\xi|^2 \le C \int_0^T \sum_{i=1}^N \left|\langle w_i, D^* S^*(T,s)\xi\rangle_U\right|^2 ds, \quad \forall \xi \in H_1^*(T). \tag{2.170}$$

As the space $H_1^*(T)$ is finite-dimensional, Inequality (2.170) is equivalent to the following one:

if $\langle w_i, D^* S^*(T,s)\xi\rangle_U = 0$, $\forall i = 1, \dots, N$, for all $s \in [0,T]$, then $\xi = 0$.

Inasmuch as $\xi \in H_1^*(T)$, we may write it as

$$\xi = \sum_{j=1}^N C_j \psi_j^*,$$

where $\{\psi_j^*,\ j = 1, \dots, N\}$ is a basis of $H_1^*(T)$ formed by eigenvectors of $U^*(T)$ corresponding to $\overline{\lambda}_j$, $j = 1, \dots, N$. By Assumption (A1), $\{D^*\psi_j^*\}_{j=1}^N$ is a linearly independent system in U. Thus, we may find a system $\{w_i\}_{i=1}^N \subset U$ such that

$$\det \| \langle w_i, D^* \psi_j^* \rangle_U \| \neq 0.$$

For instance, one might choose $\{w_i\}_{i=1}^N$ as the solution to the algebraic system

$$\langle w_i, D^* \psi_j^* \rangle_U = \delta_{ij}, \quad i, j = 1, \dots, N. \tag{2.171}$$

Thus, if $\langle w_i, D^* S^*(T, s)\xi \rangle = 0$, $i = 1, \dots, N$, $s \in [0, T]$, then $\xi = 0$. Hence, (2.166) is exactly null controllable. Thus, there exist $\{w_i\}_{i=1}^N \subset U$ and $u_i \in L^2(R^+)$, $i = 1, \dots, N$, such that $y_1(t), u_i(t) = 0$ for $t \geq T$, and $\int_0^T |u_i(t)|^2 dt \leq C|x^1|^2$. Then, by (2.168) we have

$$|y_1(t)| \leq C_{\gamma_0} e^{-\gamma_0 t} |x|, \quad \forall t \geq 0,$$

for any $\gamma_0 > 0$. This implies that, for some $\delta > 0$,

$$|y(t)| \leq C e^{-\delta t} |x| \quad \text{for } t \geq 0,$$

as claimed. □

Now, if we represent $\psi_j^* = (\psi_j^1)^* + i(\psi_j^2)^*$, where $(\psi_j^1)^*, (\psi_j^2)^* \in H$, we may assume that $w_i \in U$, $i = 1, \dots, N$, and that the controller u is real-valued.

It is clear that Lemma 2.2 remains true on any interval $[s, T + s]$. However, the dependence of s of constants C arising in the above estimates is crucial for latter development and must be analyzed. Thus, we are lead to consider the system

$$\begin{aligned} & y'(t) + \mathscr{A}(t)y(t) = Du(t), \quad t \geq s, \\ & y(s) = x. \end{aligned} \tag{2.172}$$

Lemma 2.3 *For each $s \in [0, \infty)$, there is a controller $u_s(t) = \sum_{i=1}^N u_i^s(t) w_i$, where $\{w_i\}_{i=1}^N \subset H$, was given by Lemma 2.2, such that $u_s(t) = 0$ for $t \geq s + T$ and the solution y_s to (2.172) satisfies*

$$|y_s(t)| \leq C e^{-\delta(t-s)} |x|, \qquad \int_0^T |u_s(t)|^2 dt \leq C|x|^2, \tag{2.173}$$

for some positive constants C and δ independent of s and x.

Proof We show first that there exists a controller $u_s(t)$ of the form $\sum_{i=1}^N u_i^s(t) w_i$, where $\{w_i\}_{i=1}^N$ are as in Lemma 2.2, such that (2.173) holds. (As before, we work in the complexified space H.) After that, we prove that $C(s)$ and $\delta(s)$ are independent of s.

As seen in the proof of Lemma 2.2, the existence of such a $u_s(t)$ is equivalent to the following observability inequality

$$|S^*(T + s, s)\xi|^2 \leq C \int_s^{T+S} \left| \sum_{i=1}^N \langle w_i, D^* S^*(T - s, \sigma)\xi \rangle_U \right|^2 d\sigma,$$

for all $\xi \in H_1^*(T+s)$, which is equivalent to the following unique continuation property:

if $\langle w_i, D^*S^*(T+s,\sigma)\xi\rangle_U$, $i=1,\dots,N$, $\sigma \in [s, T+s]$, then $\xi = 0$.

Assume that

$$\langle w_i, D^*S^*(T+s,T)\xi\rangle_U = 0 \quad \text{for all } i=1,\dots,N.$$

Since $S^*(T+s,T): H_1^*(T+s) \to H_1^*(T)$ is one to one (see [54], p. 198), we have

$$S^*(T+s,T)\xi \in H_1^*(T) \quad \text{for } \xi \in H_1^*(T+s).$$

Hence, we may write

$$S^*(T+s,T)\xi = \sum_{j=1}^N \eta_j \psi_j^*,$$

where $\{\psi_j^*\}_{j=1}^N$ is the basis of $H_1^*(T)$ formed by the eigenvectors of $U^*(T)$ corresponding to $\{\overline{\lambda}_j\}_{j=1}^N$. Then, by the same argument as the used in the proof of Lemma 2.2, we obtain that $\xi = 0$, as desired.

Now, we turn to prove the independence of C and δ in (2.173) as functions of s.

By the substitution $t \to t+s$, we rewrite (2.172) as

$$y_s'(t) + \mathscr{A}_s(t)y_s(t) = Du_s(t), \quad t \geq 0,\ y_s(0) = x, \tag{2.174}$$

where $\mathscr{A}_s(t) = \mathscr{A}(t+s)$. We denote by $S_s(t,\sigma)$ the evolution generated by $\mathscr{A}_s(t)$. It is clear that $S_s(t,\sigma) = S(t+s,\sigma)$. We have, of course, $\mathscr{A}_T(t) = \mathscr{A}(t)$ and $S_T(t,\sigma) = S(t,\sigma)$. By the previous discussion, the solution y_s to (2.174) may be written as $y_s(t) = y_s^1(t) + y_s^2(t)$, where

$$y_s^1(t) = S_s(t,0)P_1(s)x + \int_0^t S_s(t,\eta)P_1(\eta+s)Du_s(\eta+s)d\eta.$$

By periodicity, it suffices to assume that $s \in [0,T]$. Let $u_s^* \in L^2(0,T)$ be such that $(y_s^*)^1(T) = 0$, where y_s^* is given as above with $u_s = u_s^*$. It turns out that u_s^* can be determined by

$$u_s^*(t) = \lim_{\varepsilon \to 0} u_s^\varepsilon(t) \quad \text{strongly in } L^2(0,T), \tag{2.175}$$

where

$$u_s^\varepsilon = \arg\min \left\{ \int_0^T |u(t)|_U^2 dt + \frac{1}{\varepsilon}\,|y_s(T)|^2;\ u \in L^2(0,T;U), \right.$$
$$\left. y_s^1(t) = S_s(t,0)P_1(s)x + \int_0^t S_s(t,\eta)P_1(\eta+s)Du(\eta)d\eta \right\}. \tag{2.176}$$

Let $(\widetilde{y}_T, \widetilde{u}_T)$ be such that

$$(\widetilde{y}_T(t))' + \mathscr{A}(t)\widetilde{y}_T(t) = P_1(t)D\widetilde{u}_T(t), \quad t \in [0,T], \\ \widetilde{y}_T(0) = P_1(T)x, \qquad \widetilde{y}_T(T) = 0. \tag{2.177}$$

By (2.176), we have that

$$\int_0^T |u_s^\varepsilon(t)|_U^2 dt + \frac{1}{\varepsilon}\,|y_s(T)|^2 \le \int_0^T |u_T^*(t)|^2 dt, \tag{2.178}$$

and by (2.175), we conclude that

$$\int_0^T |u_s^*(t)|_U^2 dt \le \int_0^T |u_T^*(t)|_U^2 dt. \tag{2.179}$$

On the other hand, by (2.161) it follows that u_T^* can be chosen in such a way that

$$\int_0^T |u_T^*(t)|_U^2 dt \le C|x|^2, \tag{2.180}$$

where C is a positive constant independent of x. Thus, by (2.179), we infer that

$$\int_0^T |u_s^*(t)|_U^2 dt \le C|x|^2, \quad \forall x \in [0,T],\ s \in [0,T], \tag{2.181}$$

where C is independent of s.

Then, arguing as in the proof of Lemma 2.2, we obtain Estimate (2.173) independent of s, as claimed. □

Remark 2.5 As seen in the proof of Lemma 2.2, the dimension N of basis $\{w_j\}_{j=1}^N$ arising in construction of stabilizing controller is equal to the number of Floquet exponents for $\mathscr{A}(t)$ with nonnegative real parts.

2.6.3 The Stabilizing Riccati Equation

Throughout this sequel, we assume that Assumptions (k), (kk) hold with $\frac{1}{4} \le \alpha \le \frac{5}{8}\cdot$

Consider the infinite horizon optimal control problem

$$\varphi(s,x) = \mathrm{Min}\left\{\frac{1}{2}\int_s^\infty \left(|A^{\frac{3}{4}}y(t)|^2 + \sum_{i=1}^N |u_i(t)|^2\right) dt\right\} \tag{2.182}$$

subject to $u_i \in L^2(s,\infty),\ i = 1,\dots,N$, and

$$y'(t) + Ay(t) + A_0(t)y(t) = \sum_{i=1}^N u_i(t)Dw_i, \quad t \ge 0, \\ y(s) = x, \tag{2.183}$$

where $\{w_i\}_{i=1}^N \subset U$ are as in Lemma 2.2 and $A_0(t) = B_y(t, y_\pi(t))$. We set $W = D(A^{\frac{1}{4}})$ with the norm $|\cdot|_W = |\cdot|_{\frac{1}{4}}$ and $\widetilde{D} \in L(R^N, H)$ given by $\widetilde{D}u = \sum_{i=1}^N u_i Dw_i$, $u = \{u_i\}_{i=1}^N$. By $|u|$ we denote here the Euclidean norm of u. We also denote $\widetilde{D}^*$ the dual of $\widetilde{D}$ (that is, $\widetilde{D}^* y = \{(Dw_i, y)\}_{i=1}^N$). We note that, since A is self-adjoint, the mild solution y to (2.183) is strong solution and $y \in W^{1,2}(\delta, T; H)$ for all $0 < \delta < T$.

Lemma 2.4 *For each $s \geq 0$ there is a symmetric and positive operator $R(s) \in L(W, W')$ such that*

$$\varphi(s, x) = \frac{1}{2}\,(R(s)x, x), \quad \forall x \in W,\ s \geq 0.$$

There exist positive constants $\gamma_1, \gamma_2, \gamma_3$ independent of s, such that

$$\gamma_1 |x|_W^2 \leq (R(s)x, x) \leq \gamma_2 |x|_W^2, \quad \forall x \in W,\ s \geq 0, \tag{2.184}$$

and

$$|R(s)x| \leq \gamma_3 |A^{\frac{1}{2}} x|, \quad \forall x \in D(A^{\frac{1}{2}}),\ s \geq 0. \tag{2.185}$$

Moreover, $R(s)$ satisfies the Riccati equation

$$\begin{cases} (R'(s)x, x) - 2(R(s)x, (A + A_0(s))x) - \sum_{i=1}^M (R(s)x, Dw_i)^2 + |A^{\frac{3}{4}} x|^2 = 0, \\ \quad \forall x \in D(A),\ s \geq 0, \\ R(t + T) = R(t), \quad \forall t \in (0, \infty). \end{cases} \tag{2.186}$$

Here, $R'(s) \in L(D(A), (D(A))')$ is the weak derivative of $R(s)$, that is,

$$(R'(t)x, y) = \frac{d}{dt}\,(R(t)x, y), \quad \forall x, y \in D(A).$$

We denote by the same symbol $(\cdot, \cdot)$ the scalar product of H and the pairing between W and its dual space W', $W \subset H \subset W'$.

Proof For any $s \geq 0$, it follows from Lemma 2.3 that there exist $u_i \in L^2(0, \infty)$ with $u_i(t) = 0$ for $t \geq s + T$, $i = 1, \ldots, N$, such that

$$|y(t)| \leq C e^{-\delta t} |x| \quad \text{for } t \geq s, \tag{2.187}$$

and

$$\int_s^{T+s} \sum_{i=1}^N |u_i(t)|^2 dt \leq C|x|^2, \quad \forall s > 0, \tag{2.188}$$

for some positive constants C and δ independent of s. Here, y is the solution to (2.183). Moreover, it is readily seen that for each x the function $\varphi(s, x)$ is T-periodic.

Now, we fix $x \in W$. Multiplying (2.183) by $A^{\frac{1}{2}}y$, we get

$$\frac{d}{dt}|A^{\frac{1}{4}}y(t)|^2 + 2|A^{\frac{3}{4}}y(t)|^2 \leq 2|A_0(t)y(t)|\,|A^{\frac{1}{2}}y(t)| + |\widetilde{D}u|\,|A^{\frac{1}{2}}y(t)|, \quad \text{a.e., } t > s. \tag{2.189}$$

Since $y_\pi \in C^1(R; D(A))$, by Assumption (kk) we obtain, via the interpolation inequality, that

$$|A_0(t)y(t)||A^{\frac{1}{2}}y(t)| \leq C|y(t)|^2_{\frac{1}{2}} \leq C|y(t)|^{\frac{4}{3}}_{\frac{3}{4}}|y(t)|^{\frac{2}{3}} \leq \frac{1}{4}\,|y(t)|^2_{\frac{3}{4}} + C|y(t)|^2, \quad \text{a.e., } t > 0. \tag{2.190}$$

(Here and throughout the proof of this lemma, we denote by C several positive constants independent of s, t and x.)

Integrating (2.189) over (s, ∞) and using (2.187), (2.188) and (2.190), we obtain that

$$\int_s^\infty |A^{\frac{3}{4}}y(t)|^2 dt \leq C\|x\|^2_W,$$

which implies that $\varphi(s, x) \leq C\|x\|^2_W$ for some $C > 0$ independent of s and x.

On the other hand, it is readily seen that, for each $x \in W$, Problem (2.182) has a unique pair (u^*, y^*), $u^* = \{u_i^*\}_{i=1}^N \in (L^2(s, \infty))^N$. Multiplying (2.183), where $(y, u) = (y^*, u^*)$, by $A^{\frac{1}{2}}y^*$ and integrating on (s, ∞), we obtain that

$$\frac{1}{2}|x|^2_W \leq \int_s^\infty \left(|A^{\frac{3}{4}}y^*(t)|^2 + \left|(A_0(t)y^*(t), A^{\frac{1}{2}}y^*(t))\right| + \left|(\widetilde{D}u^*(t), A^{\frac{1}{2}}y^*(t))\right|\right) dt$$

$$\leq C\int_s^\infty (|A^{\frac{3}{4}}y^*|^2 + |u^*|^2)dt = C\varphi(s, x).$$

Hence, there is a constant $C > 0$ independent of s such that

$$C\|x\|^2_W \leq \varphi(s, x).$$

In other words, $D(\varphi(s, \cdot)) = W$, for all $s \geq 0$, where $D(\varphi(s, \cdot))$ is the domain of $\varphi(s, \cdot)$. This implies that, for each $s \geq 0$, there is a linear positive and symmetric operator $R(s) : H \to H$ with the domain $D(R(s)) \subset W$ such that

$$\varphi(s, x) = \frac{1}{2}\,(R(s)x, x), \quad \forall x \in D(R(s)).$$

Moreover, $R(s)$ extends to all of W and $R(s) \in L(W, W')$.

We now turn to prove (2.184) and (2.185). To this end, we consider the optimization problem

$$\varphi_n(s, x) = \text{Min}\left\{\frac{1}{2}\int_s^n \left(|A^{\frac{3}{4}}y(t)|^2 + |u(t)|^2\right) dt\right\}, \tag{2.191}$$

subject to

$$y'(t) + Ay(t) + A_0(t)y(t) = \widetilde{D}u(t), \quad t \in (s,n), \ y(s) = x, \tag{2.192}$$

where $u = \{u_i\}_{i=1}^N \in L^2(s,n)^N$.

By the previous discussion, for each n there is a linear symmetric operator $R_n(s) \in L(W, W')$, $R_n(s) : D(R_n(s)) \subset H \to H$, such that

$$\frac{1}{2}\,(R_n(s)x, x) = \varphi_n(s,x), \quad \forall x \in W, \ s \ge 0.$$

It is readily seen that, for $n \to \infty$,

$$(R_n(s)x, x) \to (R(s)x, x), \quad \forall x \in W, \ s \ge 0,$$

and, therefore,

$$R_n(s)x \to R(s)x \quad \text{weakly in } W', \forall x \in W, \ s \ge 0.$$

Hence, it suffices to prove Estimates (2.185) and the right-hand side part of (2.184) for R_n only.

Let $x \in D(A^{\frac{1}{2}})$ and let (y^n, u^n) be optimal for Problem (2.191). Then, by the maximum principle, we see that

$$u^n(t) = \widetilde{D}^* q^n(t) = \{\langle Dw_i, q^n(t)\rangle\}_{i=1}^N, \tag{2.193}$$

for all $t \in [x, n]$, where q^n is the solution to the Hamiltonian system

$$\begin{aligned} &y_t^n(t) + Ay^n(t) + A_0(t)y^n(t) = \widetilde{D}\widetilde{D}^* q^n(t), \quad s < t < n, \\ &q_t^n(t) - Aq^n(t) - A_0^* q^n(t) = A^{\frac{3}{2}} y^n(t), \quad s < t < n, \\ &y^n(s) = x, \qquad q^n(n) = 0. \end{aligned} \tag{2.194}$$

Moreover, one has

$$R_n(s)x = -q^n(s), \quad s \in [0, n]. \tag{2.195}$$

On the other hand, if (y_s^*, u_s^*) is an optimal pair for Problem (2.182), then we have

$$\begin{aligned} \int_s^n \left(|A^{\frac{3}{4}} y^n(t)|^2 + |u^n(t)|^2\right) dt &\le \int_s^n \left(|A^{\frac{3}{4}} y_s^*(t)|^2 + |u_s^*(t)|^2\right) dt \\ &\le 2\varphi(s,x) \le C|x|_W^2. \end{aligned} \tag{2.196}$$

Now, we multiply the first equation of (2.194) by Ay^n to get

$$\begin{aligned} \frac{1}{2}\,\frac{d}{dt}\,|y^n(t)|_{\frac{1}{2}}^2 + |Ay^n(t)|^2 &\le |(A_0(t)y^n(t), Ay^n(t))| \\ &\quad + \left|(\widetilde{D}\widetilde{D}^* q^n(t), Ay^n(t))\right|. \end{aligned} \tag{2.197}$$

We have

$$\begin{aligned}\left|(A_0(t)y^n(t), Ay^n(t))\right| &\le C\left|y^n(t)\right|_{\frac{1}{2}}|Ay^n(t)|\\ &\le \frac{1}{4}|Ay^n(t)|^2 + C|y^n(t)|^2_{\frac{1}{2}},\end{aligned} \tag{2.198}$$

and

$$\begin{aligned}\left|(\widetilde{D}\widetilde{D}^* q^n(t), Ay^n(t))\right| &\le \left|\widetilde{D}u^n(t)\right|\ |Ay^n(t)|\\ &\le |Ay^n(t)|^2 + C|\widetilde{D}u(t)|^2.\end{aligned} \tag{2.199}$$

Combining (2.197)–(2.199), we obtain that

$$\begin{aligned}\frac{d}{dt}|y^n(t)|^2_{\frac{1}{2}} + |Ay^n(t)|^2 &\le C\left(|\widetilde{D}u^n(t)|^2 + |y^n(t)|^2_{\frac{1}{2}}\right)\\ &\le \frac{1}{2}|Ay^n(t)|^2 + C|\widetilde{D}u^n(t)|^2\end{aligned}$$

and, integrating above on (s,n) and using (2.196), we get the estimate

$$\int_s^n |Ay^n(t)|^2 dt \le C|x|^2_{\frac{1}{2}}. \tag{2.200}$$

Multiplying the second equation of (2.194) by q^n, integrating over (s,n) and using (2.200), we see that

$$|q^n(t)| \le C|x|_{\frac{1}{2}}, \quad \forall t \in (s,n),$$

for some $C > 0$ independent of n and s, which together with (2.195) implies

$$|R_n(s)x| \le C|x|_{\frac{1}{2}}, \quad \forall s > 0,\ x \in D(A^{\frac{1}{2}}),$$

as claimed.

Finally, by a standard argument involving (2.194) and (2.195), we obtain that

$$\begin{cases}(R_n'(s)x,x) - 2(R_n(s)x,(A+A_0(s))x) - |\widetilde{D}^*R_n(s)x|^2 + |A^{\frac{3}{4}}x|^2 = 0,\\ \quad \forall x \in D(A),\ s \ge 0,\ n \in \mathbb{N},\\ R_n(n) = 0.\end{cases}$$

By passing to the limit for $n \to \infty$ in the latter equality, we obtain (2.186), as desired. □

2.6.4 Stabilization of Nonlinear System (2.148)

In (2.148) we insert the (feedback) controller $u(t) = \sum_{i=1}^{N} u_i(t)w_i$, where

$$u_i(t) = -\langle w_i, D^*R(t)(y(t) - y_\pi(t))\rangle_U, \quad i = 1, \dots, N. \tag{2.201}$$

Here, $\{w_i\}_{i=1}^N$ are as in Lemma 2.2 and $R(t) \in L(W, W')$ is given by Lemma 2.4. Consider the corresponding closed-loop system

$$\begin{cases} y'(t) + Ay(t) + B(t, y(t)) + \sum_{i=1}^{N} \langle w_i, D^*R(t)(y(t) - y_\pi(t))\rangle_U Dw_i = 0, \\ \quad t \ge 0, \\ y(0) = y_0. \end{cases} \tag{2.202}$$

Lemma 2.5 *Let* $y_0 \in D(A^\alpha)$. *Then there is* $0 < T_0 = T_0(|y_0|_\alpha)$ *such that* (2.202) *has a unique mild solution* $y \in C([0, T]; D(A^\beta))$, $\beta = \max(\alpha, \frac{1}{2})$ *on the interval* $[0, T_0)$. *Moreover, y is absolutely continuous on each compact interval of* $(0, T_0)$, *satisfies, a.e., on* $(0, T_0)$ (2.202) *and*

$$y \in W^{1,2}(\delta, T_0; H), \quad Ay, B(t, y) \in L^2(\delta, T_0; H), \quad \forall 0 < \delta < T_0. \tag{2.203}$$

Proof Since the proof is standard, we only sketch it. We write (2.202) as the integral equation

$$\begin{aligned} y(t) &= e^{-At}y_0 \\ &\quad - \int_0^t e^{-A(t-s)} \left(B(s; y(s)) + \sum_{i=1}^{N} \langle w_i, D^*R(s)(y(s) - y_\pi(s))\rangle_U Dw_i \right) ds \end{aligned} \tag{2.204}$$

and apply the Banach fixed-point theorem in the space

$$X = \{y \in C([0, T_0]; D(A^\beta));\ |y(t)|_\beta \le \mu,\ t \in [0, T_0]\},$$

where $\beta = \alpha$ if $\alpha \ge \frac{1}{2}$ and $\beta = \frac{1}{2}$ if $\alpha < \frac{1}{2} \cdot$ By (2.153) and (2.182), it follows that the operator Γ, defined by the right-hand side of (2.204), maps X into itself if $|y_0|_\beta \le \frac{M}{2}$ and $0 < T_0 < \delta(\mu)$ is sufficiently small. Moreover, Γ is a contraction on X. This means that (2.204) has a unique solution $y \in C([0, T_0]; D(A^\beta))$, as desired. Since $|B(s, y)| \le C(1 + |y|_\alpha |y|_{\frac{1}{4}})$ and $|R(s)y| \le C|y|_{\frac{1}{2}}$, we conclude that $y \in W^{1,2}(\delta, T_0; H)$, that is,

$$\frac{dy}{dt},\ Ay \in L^2(\delta, T_0; H), \quad \forall 0 < \delta < T_0,$$

and, therefore, y is a strong solution to (2.202). This completes the proof. □

Theorem 2.12 *Assume that Hypotheses* (k)−(kkk) *and* (A1) *hold with* $\alpha = \frac{5}{8}$. *Then there is* $\rho > 0$ *such that, for* $y_0 \in D(A^\alpha)$, $\|y_0 - y_\pi(0)\|_W < \rho$, (2.201) *has a unique strong solution* $y \in C([0,\infty); H)$, *such that* $\frac{dy}{dt} \in L^2_{\rm loc}(0,\infty; H)$, $Ay \in L^2_{\rm loc}(0,\infty; H)$, $B(t,y) \in L^2_{\rm loc}(0,\infty; H)$ *and*

$$\int_0^\infty |A^{\frac{3}{4}}(y(t) - y_\pi(t)|^2 dt \le C\|y_0 - y_\pi\|_W^2, \tag{2.205}$$

$$\|y(t) - y_\pi(t)\|_W \le Ce^{-\delta t}\|y_0 - y_\pi(0)\|_W, \quad \forall t \ge 0, \tag{2.206}$$

for some δ, $C > 0$.

Proof Let $z(t) = y(t) - y_\pi(t)$, then we have

$$\begin{cases} z'(t) + Az(t) + B(t, z(t) + y_\pi(t)) - B(t, y_\pi(t)) \\ \quad + \sum_{i=1}^N \langle w_i, D^*R(t)z(t)\rangle_U Dw_i = 0, \quad t > 0, \\ z(0) = z_0 \equiv y_0 - y_\pi(0). \end{cases} \tag{2.207}$$

Let $F(t,z(t)) = B(t, z(t) + y_\pi(t)) - B(t, y_\pi(t)) - A_0(t)z(t)$. It follows from (2.151) that

$$|F(t,z)| \le C(1 + |z|_{\frac{1}{4}})|z|_\alpha^2, \quad \forall t \ge 0,\ \forall z \in D(A^\alpha), \tag{2.208}$$

where $C > 0$ is independent of t. Now, we rewrite (2.207) as

$$\begin{aligned} &z'(t) + Az(t) + A_0(t)z(t) + F(t,z(t)) + \sum_{i=1}^M \langle w_i, D^*R(t)z(t)\rangle_U Dw_i = 0, \\ &z(0) = z_0 \equiv y_0 - y_\pi(0). \end{aligned} \tag{2.209}$$

Multiplying (2.209) by Rz, we obtain

$$\begin{aligned} &(z'(t), R(t)z(t)) + (Az(t), R(t)z(t)) + (A_0(t)z(t), R(t)z(t)) \\ &\quad + \sum_{i=1}^N \langle w_i, D^*R(t)z(t)\rangle_U^2 = -(F(t,z(t), R(t)z(t))). \end{aligned} \tag{2.210}$$

Note that

$$\frac{d}{dt}(R(t)z(t), z(t)) = (R'(t)z(t), z(t)) + 2(R(t)z(t), z'(t)), \quad \text{a.e., } t > 0.$$

Then, by (2.186), (2.210), we see that

$$\begin{aligned} &\frac{d}{dt}(R(t)z(t), z(t)) + |A^{\frac{3}{4}}z(t)|^2 + \sum_{i=1}^N \langle w_i, D^*R(t)z(t)\rangle_U^2 \\ &\quad = -2(F(t,z(t)), R(t)z(t)), \end{aligned} \tag{2.211}$$

for all t in the interval of existence $(0, T_0)$ of $z(t)$.

By (2.150), (2.151), (2.152), (2.184), (2.185) and (2.208), we get via interpolation that, for $\alpha = \frac{5}{8}$,

$$\begin{aligned}
2|(F(t,z), R(t)z)| &\le 2|F(t,z)|\,|R(t)z| \le C|z|_{\alpha}^{2}(1+|z|_{\frac{1}{4}})|z|_{\frac{1}{2}} \\
&\le C|z|_{\frac{3}{4}}^{4\alpha-\frac{1}{2}}(1+|z|_{\frac{1}{4}})|z|_{\frac{1}{4}}^{\frac{7}{2}-4\alpha} = C|z|_{\frac{3}{4}}^{2}|z|_{\frac{1}{4}}(1+|z|_{\frac{1}{4}}) \\
&\le C\gamma_1^{-\frac{1}{2}}(R(t)z,z)^{\frac{1}{2}}(1+C\gamma_1^{-\frac{1}{2}}(R(t)z,z)^{\frac{1}{2}})|z|_{\frac{3}{4}}^{2}, \\
&\quad \forall t>0,\ z \in D(A^{\frac{3}{4}}).
\end{aligned} \tag{2.212}$$

We set

$$\mathscr{U}_t = \left\{ z \in W;\ (R(t)z,z)^{\frac{1}{2}}(1+C\gamma_1^{-\frac{1}{2}}(R(t)z,z)^{\frac{1}{2}}) < \frac{\gamma_1^{\frac{1}{2}}}{2C} \right\}.$$

Equivalently,

$$\mathscr{U}_t = \{z \in W;\ (R(t)z,z) \le \eta^2(\gamma_1)\},$$

where $\eta(\gamma_1)$ is the real positive solution to equa tion $2C\lambda(1+C\gamma_1^{-\frac{1}{2}}\lambda) = \gamma_1^{\frac{1}{2}}$.

We see that, for all $t \ge 0$, we have for θ_1, θ_2 appropriately chosen,

$$\{z \in Q;\ \|z\|_W < \theta_1\} \subset \mathscr{U}_t \subset \{z \in Q;\ \|z\|_W < \theta_2\}. \tag{2.213}$$

Choose $\|z_0\|_W < \theta_1$ and consider the maximal interval $(0, T_1)$ with the property that $z(t) \in \mathscr{U}_t$, $\forall t \in (0, T_1)$. By (2.212) and (2.184), we see that $T_1 = +\infty$, that is, the solution $z(t)$ exists globally and $z(t) \in \mathscr{U}_t$, $\forall t \ge 0$. Moreover, it follows that

$$\|z(t)\|_W \le Ce^{-\delta t}\|z_0\|_W, \quad \forall t \ge 0$$

and

$$\int_0^\infty |A^{\frac{3}{4}}z(t)|^2 dt \le C\|z_0\|_W^2.$$

This completes the proof of Theorem 2.12. □

We shall briefly discuss some semilinear time-periodic parabolic equations which can be treated as special cases of Theorem 2.12. Throughout in the sequel, $\mathscr{O}$ is a bounded, open domain of R^d with smooth boundary $\partial\mathscr{O}$.

Example 2.1 Consider the controlled semilinear parabolic equation:

$$\begin{cases} y_t(x,t) - \Delta y(x,t) + f_1(x,t,y(x,t)) + f_2(x,t)\cdot\nabla y(x,t) \\ \quad = m(x)u(x,t), \quad x \in \mathscr{O},\ t \in R^+, \\ y(x,t) = 0, \quad \forall (x,t) \in \partial\mathscr{O} \times R^+. \end{cases} \tag{2.214}$$

Here $m = \mathbf{1}_{\mathscr{O}_0}$ is as above the characteristic function of an open domain $\mathscr{O}_0 \subset \mathscr{O} \subset R^3$ and $f_1 : \mathscr{O} \times R \times R \to R$, $f_2 : \mathscr{O} \times R \to R^3$ are given continuous functions which are T-periodic in t. More precisely, we assume that

(ℓ) f_i, $i = 1, 2$, *are analytic in* (t, y) *and*

$$|(f_1)_y(x,t,y) - (f_1)_y(x,s,z)| \le C_1(|y-z| + |t-s|)(|y|^p + |z|^p), \quad (2.215)$$

where $0 \le p \le \frac{5}{8}$.

Indeed, taking into account that, by the Sobolev imbedding theorem, $D(A^\alpha) \subset L^q(\mathscr{O})$ for $q < 2d/(d-4\alpha)$, $d \ge 2$, we see that Assumption (kk) holds with $\alpha = \frac{5}{8}$, $A = -\Delta$, $D(A) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$ and

$$B(t,y)(x) = f_1(x,t,y(x)) + f_2(x,t) \cdot \nabla y(x),$$
$$Du = mu, \quad \forall u \in U = L^2(\mathscr{O}).$$

Let y_π be a T-periodic solution to (2.214), that is,

$$\begin{aligned} &(y_\pi)_t - \Delta y_\pi + f_1(x,t,y_\pi) + f_2(x,t) \cdot \nabla y_\pi = 0 \quad \text{in } \mathscr{O} \times R, \\ &y_\pi = 0 \quad \text{on } \partial\mathscr{O} \times R, \\ &y_\pi(x,t+T) = y_\pi(x,t), \quad \forall (x,t \in \mathscr{O} \times R. \end{aligned} \quad (2.216)$$

Assuming that $t \to y_\pi(t)$ is analytic as function with values in $H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$, it follows that Assumption (A1)$'$ is satisfied. Indeed, if z and ψ_i, $i = 1, \dots, N$, are solutions to the equation

$$\begin{aligned} &z_t + \Delta z + (f_1)_y z + \operatorname{div}(f_2 z) = 0, && \text{in } (0,T) \times \mathscr{O}, \\ &z = 0 && \text{on } (0,T) \times \partial\mathscr{O}, \end{aligned} \quad (2.217)$$

and satisfy, for some $\lambda, \lambda_i^* \in \mathbb{C}$,

$$\begin{aligned} &z(0,x) = \lambda z(T,x) + \sum_{i=1}^N \psi_i(0,x), \\ &\psi_i(0,x) = \lambda_i^* \psi_i(T,x), \quad i = 1, \dots, N, \end{aligned} \quad (2.218)$$

one must show that $z(T,x) = 0$ on $\mathscr{O}_0$ implies $z \equiv 0$.

We set $\psi = \sum_{i=1}^N \psi_i$ and $\zeta = z - \psi$. By periodicity, we extend ζ as solution to (2.217) on $(-T, 0) \times \mathscr{O}$. By Assumption ($\ell$), it follows that ζ is analytic in t on some interval $(-\delta, \delta)$ and so

$$\zeta(t,x) = \sum_{k=0}^\infty \frac{1}{k!} \zeta_x^{(k)}(0,x) t^k, \quad \forall x \in \mathscr{O}, \ -\delta < t < \delta.$$

Since $\zeta(0,x) = 0$ on $\mathscr{O}_0$, we infer that $\zeta(t,x) = 0$ on $(-\delta,\delta) \times \mathscr{O}_0$ and so, by the unique continuation property of solutions to linear parabolic equations, we have that $\zeta \equiv 0$. Then, by (2.218), we see that $z(T, x) = 0$, $\forall x \in \mathscr{O}$, and by (2.217) we conclude that $z \equiv 0$, as claimed.

Then we may apply Theorem 2.12 and conclude as follows.

Corollary 2.5 *There is a system of functions* $\{w_j\}_{j=1}^N \subset L^2(\mathscr{O})$ *such that the feedback controller*

$$u(x,t) = -\sum_{i=1}^{N} w_i(x) \int_{\mathscr{O}} R(t)(y(x,t) - y_\pi(x,t))dx$$

exponentially stabilizes the periodic solution y_π *in a neighborhood*

$$\mathscr{U} = \{y_0 \in W;\ \|y_0 - y_\pi(0)\|_W < \rho\}.$$

Here $R(t) : D(A^{\frac{1}{4}}) \to D((A^{\frac{1}{4}})')$ is the periodic solution to Riccati equation (2.186).

We recall that (see, e.g., [60], p. 186)

$$W = D(A^{\frac{1}{4}}) = H_{00}^{\frac{1}{2}}(\mathscr{O}) = \{y \in H^{\frac{1}{2}}(\mathscr{O});\ f(x)(\mathrm{dist}(x, \partial\mathscr{O}))^{-\frac{1}{2}} \in L^2(\mathscr{O})\}.$$

Example 2.2 The reaction-diffusion controlled system ("Belousov–Zhabotinski" system)

$$\begin{cases} y_t - \Delta y - y(1 - y - az) - bz = m(x)u + f_1(t) \quad \text{in } \mathscr{O} \times R, \\ z_t - \Delta z + cyz + dz = m(x)v + f_2(t) \quad \text{on } \mathscr{O} \times R, \\ \frac{\partial y}{\partial n} = 0, \quad \frac{\partial z}{\partial n} = 0 \quad \text{on } \partial\mathscr{O} \times R, \end{cases} \tag{2.219}$$

where a, b, c and d are positive constants, f_1, f_2 and T periodic functions, and $m = \mathbf{1}_{\mathscr{O}_0}$ is relevant in the theory of chemical reactions.

If (y_π, z_π) is a nontrivial T-periodic solution to the ordinary differential system

$$\begin{cases} y_\pi' - y_\pi(1 - y_\pi - az_\pi) - bz_\pi = f_1(t), \quad t \in R, \\ z_\pi' + cy_\pi z_\pi + dz_\pi = f_2(t), \\ y_\pi(0) = y_\pi(T), \qquad z_\pi(0) = z_\pi(T), \end{cases} \tag{2.220}$$

then (2.219) has $y \equiv y_\pi$, $z = z_\pi$ as a periodic solution with period T.

We consider the matrix $C(t)$ associated with the linearization of System (2.220) around $\{y_\pi, z_\pi\}$ and recall that this solution is asymptotically stable if the Floquet exponents associated with the monodromy matrix

$$\Phi = Y(T), \qquad Y'(t) = C(t)Y(t), \qquad Y(0) = I,$$

are in the left complex half plane, otherwise it might be asymptotically unstable. However, by Theorem 2.12 this periodic solution to (2.219) is stabilizable by internal controllers with support in $\mathscr{O}_0$.

Indeed, we may apply here Theorem 2.12, where $H = (L^2(\mathscr{O}))^2$ and

$$A(y,z) = \begin{pmatrix} -\Delta y \\ -\Delta z \end{pmatrix},$$

$$D(A) = \left\{ (y,z) \in (H^2(\mathscr{O}))^2;\ \frac{\partial y}{\partial n} = 0,\ \frac{\partial z}{\partial n} = 0 \text{ on } \partial\mathscr{O} \right\},$$

$$B(y,z) = \begin{pmatrix} -y(1-y-az) - bz \\ cyz + dz \end{pmatrix},$$

$$D\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} mu \\ mv \end{pmatrix}.$$

Since Assumptions (k)∼(kkk) are obviously satisfied, we check (A1)$'$ only.

Let $(\bar{y}, \bar{z})$ and $(\bar{y}_1, \bar{z}_1)$ satisfy the system

$$\bar{z}_t + \Delta \bar{y} + (1 - y_\pi - a z_\pi)\bar{y} - (c y_\pi + d)\bar{z} = 0 \quad \text{in } \mathscr{O} \times (0,T),$$

$$\bar{z}_t + \Delta \bar{z} - c z_\pi \bar{y} + (b - a y_\pi)\bar{z} = 0,$$

$$\frac{\partial}{\partial n}\ \bar{y} = 0, \qquad \frac{\partial \bar{z}}{\partial n} = 0 \quad \text{on } \partial\mathscr{O} \times (0,T),$$

$$\bar{y}(x,0) = \lambda \bar{y}(x,T) + \bar{y}_1(x,0), \qquad \bar{z}(x,0) = \lambda \bar{z}(x,T) + \bar{z}_1(x), \qquad \forall x \in \mathscr{O}.$$

Assuming that (y_π, z_π) are analytic, then, arguing as above, if $\bar{y}(x,T) = \bar{z}(x,T) \equiv 0$, it follows via the unique continuation property of solutions to linear parabolic systems that $\bar{y} \equiv 0$, $\bar{z} \equiv 0$ on $\mathscr{O} \times (0,T)$, as desired.

Then, by Theorem 2.12 there is a feedback controller

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{N} w_i(x) \int_{\mathscr{O}_0} (R_{11}(t)(y(x,t) - y_\pi(t)) \\ &\quad + R_{12}(t)(z(x,t) - z_\pi(x,t)))w_i(x)dx, \\ v(x,t) &= \sum_{i=1}^{N} w_i(x) \int_{\mathscr{O}_0} (R_{12}(t)(y(x,t) - y_\pi(t)) \\ &\quad + R_{22}(t)(z(x,t) - z_\pi(x,t)))w_i(x)dx, \end{aligned}$$

which exponentially stabilizes the periodic solution (y_π, z_π).

Here,

$$R(t) = \begin{Vmatrix} R_{11}(t) & R_{12}(t) \\ R_{12}(t) & R_{22}(t) \end{Vmatrix}$$

is the T-periodic solution to the corresponding Riccati equation (2.186).

2.7 Comments to Chap. 2

Most of the results in this chapter are new and appear for the first time in this general form. However, in some particular cases these results were established earlier. For instance, Theorems 2.1 and 2.2 were previously established in the special case of Navier–Stokes equations in [26]. (See also [12].) The results of Sect. 2.4 and, in particular, Theorem 2.7 were first established for the linearized Navier–Stokes equations in [14], but the treatment extended *mutatis mutandis* to the present general case. The results of Sect. 2.5 are new in this general framework, but are straightforward extensions to similar results established firstly for Navier–Stokes equations [26, 27] or for nonlinear parabolic equations [28]. The results of Sect. 2.6 are taken from [29]. As regards the stabilization by noise, which is new in this context, it is developed in Chap. 4 for systems governed by Navier–Stokes equations.

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