Infinite Sequences 2

Infinite sequences and their limits are basic concepts in mathematical analysis with widely spread applications in numerical mathematics, theory of equations and many other parts of mathematics. The original concept can be traced back to the 17th century. Development of functional analysis and basics of modern mathematics at the beginning of the 20th century showed its strength and significance.

In this chapter you will learn how and where infinite sequences originated, how they can be described and defined, and what kind of questions and problems can be posed and solved. By solving the easy problems you will learn how to approach more involved ones and finally you will be ready to solve the most difficult tasks formulated in this section.

The solution of almost all of the problems here can be made easier when using software packages comparable to S. Wolfram's MATHEMATICA®, or Maple® of Waterloo Maple Inc., although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of the corresponding commands. In this chapter the following commands (and related ones) may help:
In MATHEMATICA®:

```
Limit, ListPlot, AppendTo, Do, Reduce
ContinuedFractions, EllipticK, Floor, Ceiling,
Integrate, ArithmeticGeometricMean, Fibonacci,...
```

Additional help can be found in packages

```
DiscreteMath`*.*`.
```

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In Maple®:
`evalf, seq, floor, ceil, Reduce, assign, plot, int, listplot, numtheory[cfrac],combinat[fibonacci], EllipticK, GaussAGM`

Additional help can be found after entering `?index` and pressing `Enter`

In most of the examples numerical experiments may give a starting point of reasoning or help to verify initial conjectures. Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at
www.mathworld.wolfram.com
or at http://en.wikipedia.org/wiki/Portal:Mathematics or at
http://eom.springer.de.

Suggestions

– A good start for beginners are examples I 01, I 02, I 06, I 11, I 17, I 25, I 26 and further along the downward arrows.
– Those research inclined will find interesting stimuli in I 15, I 16, I 22, I 24, I 32, I 43, I 52, I 63 and may use the upward arrows to find help.
– Teachers could use I 08, I 10, I 27, I 29, I 33, I 44 to motivate subsequent work.
– All readers are strongly encouraged to modify, generalize or simplify the given problems, to find alternative settings , formulate and solve their own examples and find the context of these problems to the given ones.

Problems

An infinite sequence can be described by a list of several of its first terms, like $a_1, a_2, a_3, \ldots$. Such a description is not sufficient since it does not define the sequence uniquely. Except for a few special cases it is not used in mathematics; you may find it in riddles in newspapers and journals. The simplest way to define a sequence is a formulation of a rule creating its terms, for instance the sequence a_n given by the rule $a_n = n^3 - n^2$ gives $0, 4, 18, 48, 100, \ldots$. If not stated otherwise it is understood that n are natural numbers, i.e. $n = 1, 2, 3, \ldots$. Such definition of sequences is called explicit. The rule defining a sequence can also give its n-th term as a function of some previous terms, like in $a_n = qa_{n-1}$, $n > 1$. Such definition can be called implicit or recursive. The definition is unique if some initial terms are given, e.g. $a_1 = 1$ in this example. The notation a_n can also be replaced by $a(n)$, since more complicated indexes may cause difficulties in printing. This notation also accords with the notation of functions.

I 01 • **[M]** ↓ **I 11** ↓ **I 13** ↓ **I 25** ↓ **I 34** ↓ **I 43**

Plot the first few terms of the sequences below and guess their limits, if they exist. Can you prove that your guess is correct?

(i) $x_n = \dfrac{n+1}{n-1}$, (ii) $x_n = n\sin\left(\dfrac{1}{n}\right)$,

(iii) $x_n = \left(1 + \dfrac{(-1)^n}{n}\right)^n$,

(iv) $x_n = (\sqrt{n+1} - \sqrt{n-1})\sqrt{n}$,

(v) $x_n = n^{\frac{1}{n}}$, (vi) $x_n = n \bmod 5$.

I 02 • **[M]** ↓ **I 35** ↓ **I 36**

Let x_n be a sequence defined recursively by the equations below. Express its n-th term as a function of n, determine whether the sequence converges and if so find the limit of the sequence.

(i) $x_{n+1} = x_n + d, \quad x_0 \in R,\ d \in R,$

(ii) $x_{n+1} = x_n + (-1)^n, \quad x_0 \in R,$

(iii) $x_{n+1} = \lambda x_n, \quad \lambda \in R,\ x_0 \in R,$

(iv) $x_{n+1} = (n+1)x_n, \quad x_0 = 1,$

(v) $x_{n+1} = \sqrt{x_n}, \quad x_0 \geq 0,$

(vi) $x_{n+1} = \alpha^{n+1} x_n, \quad x_0 = 1,\ \alpha \in R,$

(vii) $x_{n+1} = \exp(x_n), \quad x_0 = 0.$

I 03 • ↓ **I 05** ↓ **I 52**

Find, if it exists, $\lim_{n\to\infty} \frac{1}{n}\exp(i\pi n/3)$ and $\lim_{n\to\infty} \exp(i\pi n/3)$ with $i^2 = -1$.

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I 04 • ↓ **I 05** ↓ **I 06** ↓ **I 07**

Let $a(n) = 1 + (-1)^n$, $b(n) = 1 + (-1)^{n+1}$. Neither of these has a limit, although both are bounded. Form the sequences $c(n) = a(n)b(n)$, $d(n) = a(n) + b(n)$ and find, if possible, $\lim_{n\to\infty} c(n)$ and $\lim_{n\to\infty} d(n)$. Can you modify sequences $a(n), b(n)$ so that they become divergent and unbounded, and still the deduction on $c(n)$ and $d(n)$ holds true?

Hint: Recall the basic theorems.

I 05 • ↑ **I 03**

The limit of the sequence $\{\cos \pi n\}$ does not exist. Take any $\lambda > 0$ and consider the sequence $c_n = \frac{1}{n^\lambda + \cos \pi n}$. Verify that $\lim_{n\to\infty} c(n) = 0$.

Hint: Recall the basic theorems.

I 06 • ↓ **I 07**

Assume that c_n is a bounded sequence and $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$. What conclusions can be proved for the sequences

(i) $x_n = a_n \cos \pi n$,
(ii) $y_n = b_n \sin \pi n$,
(iii) $z_n = a_n c_n$,
(iv) $w_n = b_n c_n$?

Hint: Recall the basic theorems.

I 07 • ↑ **I 06**

Let $f = \{f_n\}$, $g = \{g_n\}$ be given sequences such that $\lim_{n\to\infty} f_n g_n = 0$.

(i) What conclusions can be proved for the sequences f and g if they are both convergent?
(ii) What if only one of them is convergent?
(iii) Can it happen that none of them is convergent?

Hint: Recall the basic theorems.

I 08 • • ↓ **PI 08** ↓ **I 09**

Consider a sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = A$ and the sequence $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Show that $\lim_{n\to\infty} y_n = A$.

I 09 • ↑ **I 08**

Consider a sequence $\{x_n\}$ of positive numbers with $\lim_{n\to\infty} x_n = A$ and the sequences

$$z_n = \sqrt[n]{x_1 x_2 \dots x_n}, \qquad w_n = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Find $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$, if $A > 0$.

Hint: Consider $\log z_n$ and $\frac{1}{w_n}$.

I 10 •• ↓ **I 64**

Consider the unit circle and its circumscribed and inscribed regular polygons with n sides. Denote by $A(n)$ and $a(n)$, $n > 2$, the area of the circumscribed and inscribed polygons, respectively.

(i) Prove that $\lim_{n\to\infty} A(n) = \lim_{n\to\infty} a(n) = \pi$.
(ii) Prove that these quantities satisfy the equations

$$a(2n) = \sqrt{a(n)A(n)}, \qquad A(2n) = \frac{2a(2n)A(n)}{a(2n) + A(n)}$$

and show how to calculate $a(2^m)$ and $A(2^m)$.
(iii) Find $a(6)$ and $A(6)$ without referring to trigonometric functions and evaluate, using the arithmetic and harmonic means above, the first few terms of the sequences $a(3 \cdot 2^k)$ and $A(3 \cdot 2^k)$. Can you find the value of k such that the corresponding term of one of these sequences approximates π by a number in the interval $(223/71, 220/70)$?

This approach to finding the area of a disc is traditionally attributed to Archimedes (287–212 BC). It appeared in his treatise 'On the measurement of the circle'. He used the recurrence relation starting with a regular hexagon and doubling the number of sides in each step. The result of his computation was an approximation of π by a number between $310/71$ and $310/70$. It is, however, unlikely that Archimedes was the discoverer of this value of π, since it implicitly appeared earlier in the quadrature of the circle attributed to Dinostratus.

I 11 • **[M]** $\uparrow$ **I 01** $\downarrow$ **I 12**

Some of the sequences in I 01 converge to the same value. Denote by $N(\varepsilon)$ the smallest value of n for which the inequality $|a(n) - 1| < \varepsilon$ is satisfied. Find this value for a fixed ε in the sequences mentioned in I 01. Is there any difference in the behavior of $N(\varepsilon)$ for various sequences?

Hint: Use the `ListPlot` in MATHEMATICA® or `plots[pointplot]` or `plot` command in Maple®. Give estimates of N for various values of ε.

I 12 • **[M]** $\uparrow$ **I 01** $\uparrow$ **I 11** $\downarrow$ **I 13** $\downarrow$ **I 14** $\downarrow$ **I 15** $\downarrow$ **I 17**

Perhaps you feel that the sequences in I 01 approach their limits with different speed? Would it be possible to measure this 'speed' by comparison with some standard at least in case of finite limits? Can you formalize a concept of 'speed of convergence'? How can this concept be redefined for sequences with infinite limits?

Hint: Use a 'standard' scale of sequences (e.g. n^p, $p \in R$) for comparison of $|a_n - A|$ (see $\downarrow$ ord).

I 13 • **[M]** $\uparrow$ **I 01**

Find the order of convergence for the convergent sequences in examples I 01 and I 10. What is the largest possible value of p for

(i) $\dfrac{n}{\sqrt{n^2+n}} - 1 = O(n^{-p})$, (ii) $\dfrac{n}{\sqrt{n^2+1}} - 1 = O(n^{-p})$?

I 14 • ↑ I 12 ↑ I 13 ↓ I 15

With given sequences $f(n) = O(n^{-p})$, $g(n) = O(n^{-q})$, $p, q > 0$, find the 'O-estimates' for the following sequences:

(i) $af(n)$, $a \neq 0$,
(ii) $f(n) + g(n)$,
(iii) $f(n)g(n)$.

I 15 •• ↑ I 12 ↑ I 14 ↓ I 16

Assume that the sequence satisfies $x(n) = O(n^{-p})$, $n \to \infty$. Find the order of convergence for the following subsequences

(i) $y(n) = x(2n)$,
(ii) $z(n) = x(n^2)$,
(iii) $w(n) = x(\mu(n))$,

where μ is an increasing function $\mu : \mathbf{N} \to \mathbf{N}$.

I 16 ••• ↓ PI 16 ↑ I 12 ↑ I 14 ↑ I 15

Let x_n be a convergent sequence, $\lim_{n\to\infty} x_n = A$.

(i) Show that for any $p > 0$ there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that the order of its convergence is n^{-p}, i.e. $|y_n - A| = O(n^{-p})$, $n \to \infty$.
(ii) Show that for a given sequence $\{g_n\}$ of positive numbers converging to zero there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $|y_n - A| = O(g_n)$, $n \to \infty$.

I 17 • ↓ I 27 ↓ I 61 ↓ I 62

Consider a sequence x_n satisfying

$$|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|,$$

where $0 < \lambda < 1$.

(i) Is such sequence convergent?
(ii) What is its order of convergence?

Hint: Find that $|x_{n+1} - x_n| < \lambda^n |x_1 - x_0|$ and use the Bolzano–Cauchy theorem.

I 20 • •

Find the limit of the sequence $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

Hint: Compare a_n with integral sums of $\int_1^2 \frac{1}{x} dx$ over an equidistant partition of the interval $[1, 2]$.
In another approach take into account that the n-th term of the sequence equals the area below the graph of a piecewise constant function with discontinuities at integer numbers from the interval $(n, 2n)$ and compare it with the integral $\int_n^{2n} \frac{1}{x} dx$.

I 21 •• ↑ I 08 ↑ I 09

Let N be a fixed integer and let $\{a_n\}$ be a sequence of positive real numbers.

(i) Find the limit of the sequence

$$y_n = \sqrt[n]{a_1^n + a_2^n + \cdots + a_N^n}.$$

(ii) Reconsider this limit for

$$z_n = \sqrt[n]{a_1^n + a_2^n + \cdots + a_n^n}.$$

Hint: In (i) factor out $\max a_i$

I 22 ●● ● [M] ↓ **PI 22** ↑ **I 10** ↓ **I 23**

Investigate the sequences

$$x(n) = \int_0^{\pi/2} \sin^n x dx \quad \text{and} \quad y(n) = \int_0^{\pi/2} \cos^n x dx.$$

As one of your results prove that

$$\lim_{n\to\infty} \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{\pi}{2}.$$

The formula bears the name of Wallis. John Wallis (1616–1703), one of the leading English mathematicians before Newton, made his most important contribution in analysis of infinitesimals.

I 23 ●● ↓ **PI 23** ↑ **I 21** ↓ **I 24**

Show that the sequence $y(n) = (\int_0^{\pi} |\sin^n x| dx)^{1/n}$ is bounded. Find its limit.

I 24 ● ●● ↓ **PI 24** ↑ **I 21** ↑ **I 23**

Let f be a continuous function on the closed and bounded interval (a, b). Investigate the sequence

$$x(n) = \left(\int_a^b |f(x)|^n dx\right)^{1/n}$$

and show that

$$\lim_{n\to\infty} x(n) = \max(|f(x)|, x\varepsilon(a,b)).$$

I 25 ●● ↓ **PI 25** ↑ **I 01**

Prove that the limit of the sequence $x_n = (1 + 1/n)^n$ exists.

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I 26 • **[M]**

Let a function f be continuous on the closed interval $[a, b]$ and let $f(x) \in [a, b]$ for all $x \in [a, b]$. Choose any $x_0 \in (a, b)$ and construct a broken line passing successively the points $(x_0, f(x_0))$, $(f(x_0), f(x_0))$, $(f(x_0), f(f(x_0)))$, $(f(f(x_0)), f(f(x_0)))$, $(f(f(x_0)), f(f(f(x_0))))$, ... Compare this broken line with the plot of the function f and the function $g(x) = x$.

Hint: Design some experiments for various functions, e.g. $f(x) = \exp(-x)$, $f(x) = \sin x$, and for various 'starting points' x_0.

I 27 •• ↑ **I 17**

Consider the sequence $x(n+1) = \frac{1}{2}(x(n) + \frac{a}{x(n)})$, $a > 0$. Prove that its limit exists for every $x_0 \neq 0$, find it and estimate the speed of convergence towards this limit.

Hint: Assume first that $x(0) > 0$. Find the minimal value of the right-hand side and use it to prove that $x(n)$ is decreasing and $x(n) > \sqrt{a}$ for $n > 0$. What about $x_0 < 0$? Also, prove that $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$, implying the estimate of the speed of convergence.

I 28 •• ↑ **I 01** ↑ **I 26**

Investigate the limit of the sequence defined by the following recurrence

$$x_{n+1} = \frac{1}{2}(1 + x_n^3)$$

with a given real value x_0.

Hint: Do there exist real values of x_0 such that x_n is a constant sequence? There are three of them and they subdivide the real axis into 4 intervals. With x_0 in one of them, decide whether the sequence x_n is increasing or decreasing by analyzing $x_{n+1} - x_n$.

I 29 •• ↓ **PI 29** ↓ **I 30** ↓ **I 33**

Prove the following identity

$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \frac{1+\sqrt{5}}{2}.$$

Hint: Find the recurrence formulae for both infinite sequences. Verify that both sequences are convergent using the theorem on bounded and monotone sequences. Find the limits of the two sequences.

The value $\lambda = (1+\sqrt{5})/2$ is called the golden section ratio. This notion goes back at least to the Pythagorean school and appears e.g. in Euclid's Elements. It was a part of the theory of architecture and art as well. The main reason for this could be the 'self-propagating' nature of λ: If a point C_1 divides a segment $\overline{AB}$ so that $|AC_1| : |C_1B| = \lambda : 1$, and a point C_2 divides a segment $\overline{AC}$ so that $|AC_2| = |C_1B|$ then the point C_2 divides the segment $\overline{AC_1}$ again in the ratio $\lambda : 1$. This is equivalent to the condition $\frac{a}{x} = \frac{x}{a-x} = \lambda$, where $a = |AB|$ and $x = |AC_1|$.

Can you construct the golden section of a given segment AB with a ruler and compasses only?

I 30 • **F 47** ↑ **I 27** ↑ **I 29** ↓ **I 31** ↓ **I 32** ↓ **I 62** ↓ **I 63**

Check whether the considerations in I 29 remain true for all $a > 0$ in the following identity:

$$\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}} = 1 + \cfrac{a}{1+\cfrac{a}{1+\cfrac{a}{1+\cdots}}} = \frac{1+\sqrt{1+4a}}{2}.$$

The second identity together with the corresponding recurrence relation can be a basis to develop an algorithm of evaluation of $\sqrt{b}$ (at least for $b > 1$). Can you design such an algorithm and compare its effectiveness with that of I 27?

I 31 ●● ↓ **PI 31** ↑ **I 29** ↑ **I 30** ↓ **I 32**

Analyze the sequence

$$a_n = \sqrt{1+\sqrt{2+\sqrt{3+\cdots\sqrt{n}}}}.$$

Hint: Try monotonicity and boundedness of a_n.

I 32 ●●● ↓ **PI 32** ↑ **I 29** ↑ **I 30** ↑ **I 31**

Let $\{b_n\}$ be a given sequence of positive real numbers. Define the sequence x_n by

$$x_n = \sqrt{b_1+\sqrt{b_2+\sqrt{b_3+\cdots\sqrt{b_n}}}}$$

and find conditions for its convergence.

Since 1202, when the Italian merchant and mathematician Leonardo of Pisa (ca. 1180–1250), better known as Fibonacci, introduced the following sequence, it found so many interesting applications in number theory, combinatorics, computer science and elsewhere, that it became probably the most studied infinite sequence ever. Originally Fibonacci posed the following problem: Suppose that some kind of rabbits live forever and that every month each pair bears a new pair which becomes productive from the second month on. If we start with one newborn pair, how many pairs of rabbits will there be in the n-th month?

I 33 ●● ↓ **PI 33** ↑ **I 29**

Let F_n denote the number of pairs of rabbits in the n-th month. The sequence F_n is called the Fibonacci sequence. Find the recurrence formula for F_n.
From the vast number of interesting properties of this sequence, consider and prove the following:

(i) Two consecutive Fibonacci numbers are relatively prime.
(ii) Find $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ assuming that it exists. Compare the result with I 29.
(iii) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.
(iv) $F_{n+m} = F_m F_{n+1} + F_{m-1}F_n$. Deduce that F_{nk} is a multiple of F_k.
(v) Assume that $F_n = \lambda^n$ and find possible values of λ from the equation above.
(vi) Find an explicit expression for F_n with $F_1 = F_2 = 1$, using the values λ from (v).

Hint: In (ii) divide the recurrence formula by F_{n+1}. In (iii) and (iv) use induction. In (vi) put $F_n = a\lambda^n + b\lambda^{-1}$ for some a, b.

I 34 ●● ↑ **I 01**

Take any fixed positive integer p and find, if it exists,

$$\lim_{n\to\infty} \left\lfloor \frac{n}{p} \right\rfloor \sin \frac{1}{\lceil \frac{n}{p} \rceil}.$$

Hint: Prove and take into account that

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1.$$

Start with small values of p. A plot of the functions Floor, Ceiling (in MATHEMATICA®) or floor, ceil (in Maple®) (with n replaced by x, $1 < x < 20$) may suggest the way of reasoning.

I 35 ● ↑ **I 02** ↓ **I 36**

Let $a \neq 0$ and z_0 be complex numbers. Consider the sequence given by $z_{n+1} = az_n$. Find conditions under which $\lim_{n\to\infty} z_n$ exists.

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Hint: Try to find an explicit expression for z_n.

I 36 ●● **↑ I 02 ↑ I 35**

Let A be a real square matrix of order m and x_0 a vector of dimension m. Consider the sequence $x_{n+1} = Ax_n$. Find sufficient conditions for the existence of $\lim_{n\to\infty} x_n$.

Hint: You need some facts from linear algebra and matrix theory.

I 43 ●● **↑ I 01 ↑ I 02 ↑ I 34 ↑ I 35**

Let $A = \{a(1), a(2), \ldots, a(n)\}$ be a set of numbers arranged in the increasing way. Let $a \in A$ be given. To find its position in A we compare a with $a(\lfloor n/2 \rfloor)$ which divides A into its upper and lower part. We continue with one of these parts, depending where the number a fitted. Hence, the maximal number $B(n)$ of comparisons satisfies a certain recurrence formula.

(i) Can you find it?
(ii) Calculate some of first terms of $B(n)$ and prove that $B(n) = \lfloor \log_2 n \rfloor + 1$, where $\log_2$ denotes the logarithm to the base 2.
(iii) With the O-notation show that $B(n) = O(\log n)$.

Hint: Consider the binary representation of n and compare $\lfloor \log_2 n \rfloor$ with the number of digits in this representation.

The sequence $B(n)$ arises when estimating the computational complexity (e.g. the number of operations) of the so called 'divide and conquer algorithms' based on the following paradigm: Divide the problem into two subproblems of (approximately) equal size, solve them recursively, then use the solution to solve the original problem. An example of such problem is the binary search, where the position of an element in a sorted list is found by 'halving' the list successively. This paradigm is the basis of so-called 'fast' algorithms, e.g. the Fast Fourier Transform (FFT).

I 44 • ↑ I 35 ↑ I 36

In mathematical biology various models of the 'predator-prey interaction' are analyzed. A simple model is as follows: In an isolated area the number K of owls (in thousands) and the number L of mice (in millions) is considered stabilized. Due to some external factors this equilibrium has been destroyed. Denoting

$$x(n) = \text{the number of owls after } n \text{ years} - K,$$

$$y(n) = \text{the number of mice after } n \text{ years} - L,$$

the destruction of equilibrium can be characterized by setting $x(0)$ and $y(0)$ to be nonzero.
Recall that $\Delta x(n) = x(n+1) - x(n)$ and suppose that there exist positive constants $\alpha, \beta, \gamma, \delta$ all of them < 1, such that

$$\Delta x(n) = -\alpha x(n) + \beta y(n), \qquad \Delta y(n) = -\gamma x(n) - \delta y(n).$$

The first equation describes the fact that the decrease of $x(n)$ and increase of $y(n)$ might give more food to the remaining owls. The second equation reflects that the decrease of $x(n)$ means less danger for the mice and less mice gives the remaining mice less competition for food.
Prove that for $\alpha = \gamma = 0.1$, $\beta = 0.2$, $\delta = 0.4$ the initial deviation $x(0)$, $y(0)$ will be made small after some period of time, i.e. that $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} y(n) = 0$, which means that the original equilibrium will be restored. Can you find such values of the parameters that the original equilibrium will never be restored?

Hint: Write the equations in the form

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = A \begin{pmatrix} x(n) \\ y(n) \end{pmatrix},$$

where $A = \left(\begin{smallmatrix} 1-\alpha & \beta \\ -\gamma & 1-\delta \end{smallmatrix}\right)$.

I 51 • ↑ I 01 ↑ I 03

Find the cluster points of the following sequences:

$$a_n = \sin(n\pi/2), \qquad b_n = \sin(n^2\pi/3),$$

$$c_n = 3\cos(n\pi/2) + (-1)^n.$$

Hint: For b_n consider $n = 3k$, $3k+1$, $3k+2$.

I 52 ••• ↓ PI 52 ↑ I 03

Prove that any point of the unit circle is a cluster point of the sequence $k_n = \exp(in)$, where $i^2 = -1$.

I 53 • ↑ I 03 ↑ I 52

Prove that any point of the interval $[-1, 1]$ is a cluster point of the sequence $k_n = \sin n$.

I 54 • ↑ I 53

Given a finite set of reals $a_1, a_2, \ldots, a_n$. Find a sequence x_n such that all cluster points of $\{x\}$ are exactly all the given a_i's. If the set consists of one single point a_1, is it possible to find a non-convergent sequence x such that a_1 is the only cluster point of x?

I 55 ••

Prove that for any sequence $\{a_n\}$ of positive real numbers with $\liminf a_n > 0$ there is

$$\frac{1}{\limsup a_n} = \liminf \frac{1}{a_n} \quad \text{and} \quad \frac{1}{\liminf a_n} = \limsup \frac{1}{a_n}.$$

Hint: Start with the case $0 < \liminf a_n \leq \limsup a_n < \infty$ and show that for any set $M \subset (0, \infty)$, $\sup M = \inf\{\frac{1}{x} : x \in M\}$. Then use the definition of lim inf and lim sup.

I 61 • ↑ I 08 ↑ I 10 ↑ I 33 ↓ I 62 ↓ I 63 ↓ I 64 ↓ I 65

Put $x_1 = a$, $x_2 = b$, where $a, b > 0$ and consider the sequences x_n defined by

(i) $x_n = (x_{n-1} + x_{n-2})/2$,
(ii) $x_n = \sqrt{x_{n-1}x_{n-2}}$,
(iii) $x_n = \frac{2}{\frac{1}{x_{n-1}} + \frac{1}{x_{n-2}}}$.

Find their limits if they exist.

Hint: In (ii) consider $\log x_n$, in (iii) consider $1/x_n$.

I 62 •• ↑ I 17 ↑ I 34 ↑ I 61

Consider sequences $\{x_n\}$, $\{y_n\}$ defined by

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

with $x_0 = a$, $y_0 = b$, $a, b > 0$. Find their limits if they exist.

Hint: Show that $y_n \leq x_n$ for $n \geq 1$ and that $x_{n+1} - y_{n+1} \leq (x_n - y_n)/2$.

I 63 ••• [M] **M 30 M 33 E 01 ↑ I 17 ↑ I 62**

Consider sequences $\{x_n\}$, $\{y_n\}$ defined by

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n y_n},$$

with $x_0 = a$, $y_0 = b$, $a, b > 0$. Prove the existence and equality of their limits. For a chosen a, b, evaluate the limit with an error $<10^{-4}$.

Hint: Show that for $n \geq 1$ there is $y_n \leq x_n$, x_n is decreasing and y_n is increasing. You may compare your result with results of the command `N[ArithmeticGeometricMean[a,b]]` or `GaussAGM` in MATHEMATICA® and Maple®, respectively.

This limit is called the arithmetic-geometric mean of a, b. Its value can be found using the Gauss theorem on elliptic functions (see Theorem H). It became an important tool in calculating the values of elliptic and some related special functions.

I 64 ••• [M] **↑ I 10 ↑ I 62 ↑ I 63**

Consider sequences $\{x_n\}$, $\{y_n\}$ defined by

$$x_{n+1} = \sqrt{x_n y_n}, \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

with $x(0) = a$, $y(0) = b$, $a, b > 0$. Prove that both these limits exist and that they are equal. Find this limit in terms of the arithmetic-geometric mean and for given a, b, evaluate the limit with error $<10^{-4}$.

Hint: Consider $\xi_n = \frac{1}{y_n}$, $\eta_n = \frac{1}{x_n}$ and use I 63.

I 65 ●●● **[M]** ↑ **I 62** ↑ **I 63** ↑ **I 64**

Consider sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined as follows

$$x_{n+1} = \frac{x_n + y_n + z_n}{3}, \qquad y_{n+1} = \sqrt[3]{x_n y_n, z_n}$$

$$z_{n+1} = \frac{3x_n y_n, z_n}{x_n y_n + x_n z_n + y_n z_n}$$

where x_0, y_0, z_0 are given positive numbers. Do the limits of these sequences exist? Are they mutually equal? If so, find this value (the AHG Mean of a, b, c) for given a, b, c with an error $< 10^{-4}$.

Hint: Use the inequality for arithmetic, harmonic and geometric means (Theorem F) to show that $z_n \le y_n \le x_n$ for $n > 0$.

I 66 ●● ↑ **I 62** ↑ **I 63** ↑ **I 64**

Denoting the limits of I 62, I 63, I 64 by AH, AG, HG, respectively the arithmetic-harmonic mean, arithmetic-geometric mean, harmonic-geometric mean, show that

$$HG \le AH \le AG$$

and equality holds true for $a = b$ only.

Hint: Use the inequality for arithmetic, harmonic and geometric means (Theorem F). To show e.g. that $AH \le AG$ prove that $x_n \le X_n$, $y_n \le Y_n$, for all $n \ge 0$, where x_n, y_n denote the sequences of I 62 and X_n, Y_n denote the sequences of I 63.

2

Supplementary Material

Definitions

A sequence $\{a_n\}$ of real or complex numbers is called convergent if there exists a number $A \in \mathbf{R}$ (or $\mathbf{C}$) such that for every $\varepsilon > 0$ there is an integer N such that $|a_n - A| < \varepsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = A$. If the sequence is not convergent, we call it divergent.

A sequence $\{a_n\}$ of real numbers is called increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$. Similarly, a sequence is called decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$.
A sequence $\{a_n\}$ is bounded (below, above, respectively) if there is a number $L (m, M$, respectively) such that $|a_n| \leq L$ ($a_n \geq m, a_n \leq M$, respectively) for all n.

Let $\{f_n\}$, $\{g_n\}$ be two sequences. We say that $f_n = O(g_n)$ for $n \to \infty$ if there exists a constant C (independent of n) such that $|f_n| \leq C|g_n|$ for all sufficiently large indexes n.
Let $\{a_n\}$ be a convergent sequence, $\lim_{n\to\infty} a_n = A$. We say that a sequence $\{a_n\}$ has the order of convergence of $\{g_n\}$ to A if $|a_n - A| = O(g_n), n \to \infty$.
If $\{a_n\}$ diverges to $\pm\infty$ we say that the sequence $\{a_n\}$ has the order of divergence of $\{g_n\}$ if $\frac{1}{a_n} = O(\frac{1}{g_n}), n \to \infty$.

For a sequence a_n we define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_m : m \geq n\}$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_m : m \geq n\}.$$

These numbers are called upper limit and lower limit, respectively.

For a sequence a_n of complex numbers the value A is called its cluster point if for any $\varepsilon > 0$ the inequality $|a_n - A| < \varepsilon$ is satisfied for an infinite number of indexes. (The smallest cluster point of a sequence of reals is its lower limit, the largest is its upper limit.)

Let $a = (a_1, a_2, \ldots, a_n)$ be an n-tuple of positive numbers. Their harmonic mean $H_n(a)$, geometric mean $G_n(a)$, and arithmetic mean $A_n(a)$ is defined are follows:

$$H_n(a) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}},$$

$$G_n(a) = \sqrt[n]{a_1 a_2 \ldots a_n}, \qquad A_n(a) = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The integral $\int_0^{\pi/2} \sqrt{1 - m \sin^2 x} dx = E(m)$ is called the complete elliptic integral of the second kind, $m \in [0, 1]$.
$\int_0^{\pi/2} \frac{1}{\sqrt{1-m \sin^2 x}} dx = K(m)$ is called the complete elliptic integral of the first kind.

Let A be a square $n \times n$ matrix. Any solution λ of the equation $\det(A - \lambda I) = 0$ is called the eigenvalue of matrix A, where I denotes the unit matrix.

Theorems

A

Let $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$ be finite. Then

$$\lim_{n\to\infty} (a_n + b_n) = A + B, \qquad \lim_{n\to\infty} (a_n b_n) = AB.$$

If moreover $B \neq 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}.$$

B

Every increasing bounded from above sequence (decreasing bounded from below sequence) has a finite limit.

C

(Bolzano–Cauchy theorem) A sequence $\{a_n\}$ is convergent iff for every $\varepsilon > 0$ there is an integer N such that $|a_n - a_m| < \varepsilon$ for any $n, m \geq N$.

D

(Squeeze theorem) Assume that $a_n \leq c_n \leq b_n$ for all n sufficiently large. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = A$, then also $\lim_{n\to\infty} c_n = A$.

E

(Fixed point theorem) Let f be a real-valued function defined on the interval (a, b) such that its range belongs to (a, b). If there exists a number α, $0 \leq \alpha < 1$, such that for all $(x_1, x_2) \in (a, b)$ there is

$$|f(x_1) - f(x_2)| \leq \alpha |x_1 - x_2|,$$

then there exists exactly one value x such that $f(x) = x$.

F

For any n-tuple of positive numbers $a = (a_1, a_2, \dots, a_n)$ there is

$$\min(a) \le H_n(a) \le G_n(a) \le A_n(a) \le \max(a)$$

with equality iff $a_1 = a_2 = \cdots = a_n$ (see I 19).

G

Let A be a square $m \times m$ matrix. If the moduli of all eigenvalues of A are less than 1, then $\lim_{n\to\infty} A^n = 0$, where 0 is the zero matrix.

H

Let $a, b > 0$, put $q = \max(a, b),\ p = \min(a, b)$. Then

$$\int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}} dx = \frac{1}{q} K\left(1 - \frac{p^2}{q^2}\right).$$

Moreover, Gauss proved that

$$\int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}} dx = \int_0^{\pi/2} \frac{1}{\sqrt{r^2\cos^2 x + s^2 \sin^2 x}} dx$$

when $r = \frac{a+b}{2}$ and $s = \sqrt{ab}$.

Plans of Solution

PI 08

1. It is sufficient to consider $A = 0$.
2. For any $n, p \in \mathbf{N}$ there is

$$|y_{n+p}| \le \left|\frac{x_1 + x_2 + \cdots + x_n}{n+p}\right| + \frac{p}{n+p} \max(|x_{n+1}|, |x_{n+2}|, \dots, |x_{n+p}|).$$

3. Deduce that $\lim_{n\to\infty} y_n = 0$.

PI 22

1. Show that both $x(n)$ and $y(n)$ are decreasing and bounded. Find that $x(n) = y(n)$ for all $n \geq 0$.
2. Using integration by parts show that

$$x(n-1) \geq x(n) = \frac{n-1}{n} x(n-2) \geq \frac{n-1}{n} x(n-1).$$

3. Deduce that $\lim_{n\to\infty} \frac{x(2n+1)}{x(2n)} = 1$.
4. Show that

$$\frac{x(2n+1)}{x(2n)} = \frac{1}{2n+1} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \frac{2}{\pi}$$

and thus deduce that

$$\lim_{n\to\infty} \frac{1}{2n+1} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 = \frac{\pi}{2}.$$

PI 23

1.Show that the sequence y_n is dominated by the convergent sequence $\pi^{1/n}$.
2. For $0 < \varepsilon < \frac{\pi}{2}$ put $A_\varepsilon = [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Show that

$$y(n) \geq \left(\int_{A_\varepsilon} \sin^n x dx \right)^{\frac{1}{n}}.$$

PI 24

1. The function f attains its maximum at some point $x_0 \in [a, b]$.
2. Show that $x_n \leq |f(x_0)|(b-a)^{1/n}$.
3. Show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in [x_0 - \delta, x_0 + \delta]$ we have

$$|f(x)| \geq |f(x_0)| - \varepsilon.$$

4. Prove that $x_n \geq (|f(x_0)| - \varepsilon)(2\delta)^{\frac{1}{n}}$.

PI 25

1. Show that x_n is increasing: use the Bernoulli inequality

$$(1+x)^n \geq 1 + nx, \quad x > -1, \ n \in \mathbf{N}$$

in proving that $\frac{x_{n+1}}{x_n} \geq 1$.
2. Show that $y_n = (1 + \frac{1}{n})^{n+1}$ is decreasing by proving that $\frac{y_n}{y_{n+1}} \geq 1$ and that $x_n \leq y_n$.
3. Show that $\lim_{n\to\infty} x_n$ exists. It is commonly denoted by e and it is the base of natural logarithms.

PI 26

Decipher the following command written in MATHEMATICA® language.

```
BrokenLine[f_.a_,x0_,b_,n_]:=Module[{},
li=NestList[f,x0,n];
br=Line[Flatten[Table[li[[k]],li[[k+1]],
li[[k+1]],li[[k+1]],k,1,n-1],1]];
Show[Plot[x,f[x],x,a,b],Graphics[br]]]
```

For Maple® see the Maple® worksheet.

PI 29

1. Denote $u_n = \sqrt{1+\sqrt{1+\cdots\sqrt{1}}}$, where the square root is repeated n times, $u_0 = 0$. Show that u_n is increasing and bounded (by induction). Find the relation between u_{n+1} and u_n, and then $\lim_{n\to\infty} u_n$.
2. Define v_n by

$$v_n = 1 + \frac{1}{1+\cdots+\frac{1}{1}}, \qquad v_0 = 1$$

(with n appearances of the fraction line). Plot the first few values of v_n. Find the recurrence formula for v_n.
3. Use induction to show that $1 \le v_n \le 2$ and decide whether v_n is monotonic.
4. Show that the subsequence v_{2n} is increasing and the subsequence v_{2n-1} is decreasing. Find their limits.
5. Alternatively, show that

$$\left|v_n - \frac{1+\sqrt{5}}{2}\right| \le \left(\frac{2}{1+\sqrt{5}}\right)^n \left|v_0 - \frac{1+\sqrt{5}}{2}\right|.$$

PI 31

1. Show that a_n is increasing.
2. Use n-times the identity $a - b = (a^2 - b^2)/(a+b)$ to show that

$$a_{n+1} - a_n \le \frac{\sqrt{n+1}}{2^n\sqrt{n!}} = \frac{n+1}{2^n\sqrt{(n+1)!}}.$$

3. Show that

$$a_n \le \sum_{k=0}^{\infty} \frac{k+1}{2^k\sqrt{k!}}.$$

4. Deduce that $\lim_{n\to\infty} a_n$ exists. Numerical experiments may show how this limit differs from the sum above.

PI 32
1. The sequence x_n is increasing. Hence its limit, possibly infinite, exists.
2. Use the fact that if there exists a subsequence n_k of indexes such that

$$\frac{\log b_{n_k}}{2^{n_k}} \to \infty$$

then $\lim_{n\to\infty} x_n = \infty$ to show that $x_n \geq (b_n)^{2^{-n}}$.
3. Assume that there exists a $q > 0$ such that $b_n \leq q^{2^n}$.
Use the result of I 29 to show that $x_n \leq q(1+\sqrt{5})/2$.
4. Conclude that a finite limit of $\{x_n\}$ exists if and only if

$$\limsup_{n\to\infty} \frac{\log b_n}{2^n} < \infty.$$

PI 33
In Fibonacci's problem setting the number of pairs in the first and second month is 1. Hence $F_1 = F_2 = 1$. In the $(n+2)$-th month the number of pairs equals to the sum of the number of pairs in the previous month and the number of pairs of newborn rabbits, i.e. $F_{n+2} = F_{n+1} + F_n$.

(i) If $\gcd(F_{m+1}, F_m) = s > 1$ then all F_n should be divisible by s, which is evidently wrong.

PI 52
1. For any given integer m divide the interval $(0, 2\pi)$ into m subintervals of equal length $2\pi/m$.
2. Considering numbers $k \mod 2\pi$, $k = 1, 2, \ldots, m+1$, show that two of them belong to the same subinterval of length $2\pi/m$.
3. Since π is not rational, all integer multiples of the difference of these two numbers (still $\mod 2\pi$) form a set, which has the following property: The distance from any point of the interval $(0, 2\pi)$ to some point of this set is at most $1/m$.
4. Since m is an arbitrary integer we obtain that any point of the interval $(0, 2\pi)$ is a cluster point of the sequence $a_n = n \mod 2\pi$.

Further References

Infinite sequences are dealt with in any standard textbook of Calculus.

Elliptic functions are treated in a number of monographs, e.g.

Whittaker E.T, Watson G.N., A Course of Modern Analysis, Cambridge University Press, 1927,

Hurwitz A., Courant R., Vorlesungen uber Allgemeine Funktionentheorie und Elliptische Funktionen, Springer, Berlin, 1964, and many others.

Implicitly given sequences are often solutions of difference equations. Again, a number of monographs are devoted to this topic. One of the earliest and one of the latest, where additional references can be found, are the books:

Poincaré H., Sur les equation lineaires aux differentielles ordinaires at aux differences finies, Am. J. Math. 7 (1885), 203–258.

Kelley W.G., Peterson A.C., Difference Equations, An Introduction with Applications, Academic Press, New York, 2001.

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