# II

# Examples of Fuchsian groups

In this chapter, we study concrete examples of Fuchsian groups and illustrate the results of the previous chapter.

The first family of groups that we will consider consists of geometrically finite free groups, called *Schottky* groups. Its construction is based on the dynamics of isometries.

The second family comes from number theory. It consists of three non-uniform lattices: the *modular* group $\mathrm{PSL}(2, \mathbb{Z})$, its congruence modulo 2 subgroup and its commutator subgroup.

We will study each of these groups according the same general outline:

- description of a fundamental domain;
- shape of the associated topological surface;
- properties of its isometries;
- study of its limit set;
- characterization of its parabolic points.

We will also construct a coding of the limit sets of Schottky groups and of the modular group. We will use this coding in Chap. IV to study the dynamics of the geodesic flow, and in Chap. VII to translate the behavior of geodesic rays on the modular surface into the terms of Diophantine approximations.

## 1 Schottky groups

The Poincaré disk model $\mathbb{D}$ is the ambient space in this discussion. We will fix a point 0 in this set, which is not necessary the origin of the disk.

Recall that, if $g$ is a positive isometry of $\mathbb{D}$ which does not fix 0, then the set $D_0(g) = D(g)$ represents the closed half-plane in $\mathbb{D}$ bounded by the perpendicular bisector of the hyperbolic segment $[0, g(0)]_h$, containing $g(0)$. The sets $D(g)$ and $D(g^{-1})$ are disjoint (resp. tangent) if and only if $g$ is hyperbolic (resp. parabolic) (Property I.2.7). Moreover we have:

$$g(D(g^{-1})) = \mathbb{D} - \mathring{D}(g).$$

F. Dal'Bo, *Geodesic and Horocyclic Trajectories*, Universitext,
DOI 10.1007/978-0-85729-073-1_2, 

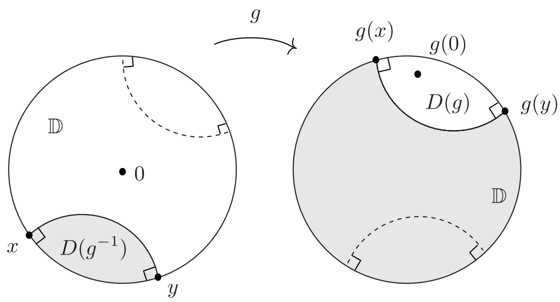


**Fig. II.1.** $g$ hyperbolic

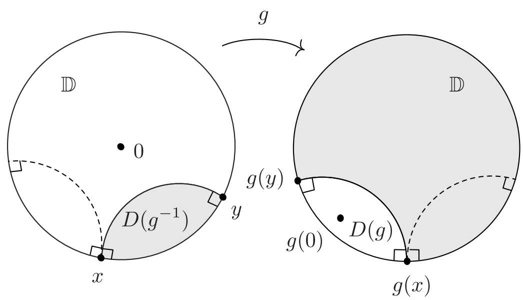


**Fig. II.2.** $g$ parabolic

**Definition 1.1.** Let $p$ be an integer $\geqslant 2$. A Schottky group of rank $p$ is a subgroup of $G$ which has a collection of non-elliptic, non-trivial generators $\{g_1, \ldots, g_p\}$ satisfying the following condition: there exists a point 0 in $\mathbb{D}$ such that the closures in $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{D}(\infty)$ of the sets $D_0(g_i^{\pm 1}) = D(g_i^{\pm 1})$, for $i = 1, \ldots, p$, satisfy

$$(\overline{D(g_i) \cup D(g_i^{-1})}) \cap (\overline{D(g_j) \cup D(g_j^{-1})}) = \varnothing,$$

for all $i \neq j$ in $\{1, \ldots, p\}$.

Let $S(g_1, \ldots, g_p)$ denote such a group whose collection of generators is $\{g_1, \ldots, g_p\}$.

In the rest of this discussion, in order to avoid notational clutter, we will restrict ourselves to the case where $p = 2$.

Figure II.3 represents the four possible configurations associated with Schottky groups of rank 2.

Schottky groups are not especially difficult to find. The following lemma shows that their construction only requires two non-elliptic isometries.

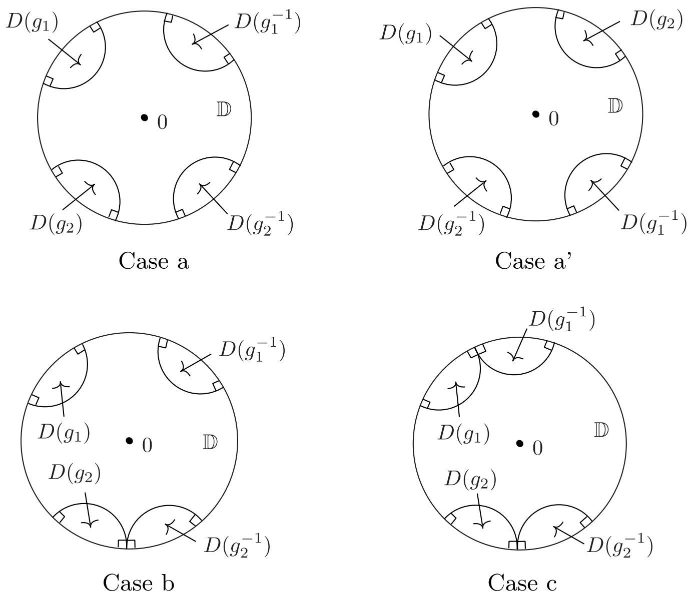


**Fig. II.3.**

**Lemma 1.2.** *Let $g$ and $g'$ be two non-elliptic isometries in $G$ which have no common fixed points. Then there exists $N > 0$ such that $g^N$ and $g'^N$ generate a Schottky group $S(g^N, g'^N)$.*

*Proof.* Fix a point 0 in $\mathbb{D}$. The sequence $(g^n(0))_{n \geqslant 1}$ converges to a point $x$ which is fixed by $g$. Thus the sequence of perpendicular bisectors of $[0, g^n(0)]_h$ similarly converges to $x$. Since $g$ and $g'$ do not have any fixed points in common, for large enough $n$ the closed sets $\overline{D(g^n) \cup D(g^{-n})}$ and $\overline{D(g'^n) \cup D(g'^{-n})}$ are disjoint. $\square$

Since a non-elementary Fuchsian group contains infinitely many non-elliptic isometries which have no common fixed points, we deduce from Lemma 1.2 the following result

**Corollary 1.3.** *A non-elementary Fuchsian group contains infinitely many Schottky groups.*

### 1.1 Dynamics of Schottky groups

Fix a Schottky group $S(g_1, g_2)$ of rank 2. The *alphabet* of this group is by definition the set $\mathcal{A} = \{g_1^{\pm 1}, g_2^{\pm 1}\}$. A product of $n$ letters $s_1 \cdots s_n$ in $\mathcal{A}$ is said to be a *reduced word* of $S(g_1, g_2)$ if $n = 1$, or if $n > 1$ and $s_i \neq s_{i+1}^{-1}$ for all $1 \leqslant i \leqslant n-1$. The integer $n$ is called the *length* of $s_1 \cdots s_n$. We associate to a reduced word $s_1 \cdots s_n$ the set $D(s_1)$ if $n = 1$, and if $n > 1$ the set $D(s_1, \ldots, s_n)$ defined by:

$$D(s_1, \ldots, s_n) = s_1 \cdots s_{n-1} D(s_n).$$

**Property 1.4.** *Let $s_1 \cdots s_n$ be a reduced word. The following properties hold*:

(i) $s_1 \cdots s_n(\mathbb{D} - \mathring{D}(s_n^{-1})) \subset D(s_1)$.
(ii) *If $n \geqslant 2$, $D(s_1, \ldots, s_n) \subset D(s_1, \ldots, s_{n-1})$.*
(iii) *If $s_1 \cdots s_n$ and $s'_1 \cdots s'_n$ are two distinct reduced words, then the half-planes $D(s_1, \ldots, s_n)$ and $D(s'_1, \ldots, s'_n)$ are either tangent or disjoint.*

*Proof.*

(i) We prove (i) by induction on $n \geqslant 1$. When $n = 1$, part (i) is a consequence of the following relation:

$$\forall\, s \in \mathcal{A}, \quad s(\mathbb{D} - \mathring{D}(s^{-1})) = D(s).$$

Suppose that (i) is true up to $n = N$, for some $N \geqslant 1$. Consider the reduced word $s_1 \cdots s_N s_{N+1}$. By induction hypothesis, one has

$$s_2 \cdots s_{N+1}(\mathbb{D} - \mathring{D}(s_{N+1}^{-1})) \subset D(s_2).$$

Since $s_2 \neq s_1^{-1}$, the set $D(s_2)$ is contained in $\mathbb{D} - \mathring{D}(s_1^{-1})$. Furthermore, the set $s_1(\mathbb{D} - \mathring{D}(s_1^{-1}))$ is equal to $D(s_1)$, thus

$$s_1 \cdots s_{N+1}(\mathbb{D} - \mathring{D}(s_{N+1}^{-1})) \subset D(s_1).$$

(ii) Since $s_n \neq s_{n-1}^{-1}$, one has $D(s_n) \subset \mathbb{D} - \mathring{D}(s_{n-1}^{-1})$. Thus the set $s_{n-1}D(s_n)$ is contained in $D(s_{n-1})$, which implies (ii).

(iii) Let $k \geqslant 1$ be the smallest integer $\leqslant n$ such that $s'_k \neq s_k$. Proving part (iii) reduces to proving that $D(s'_k, \ldots, s'_n)$ and $D(s_k, \ldots, s_n)$ are tangent or disjoint.
By (i) and (ii), the set $D(s'_k, \ldots, s'_n) = s'_k \cdots s'_{n-1} D(s'_n)$ is a subset of $D(s'_k)$. Likewise $D(s_k, \ldots, s_n)$ is a subset of $D(s_k)$. Since $s_k \neq s'_k$, the sets $D(s_k)$ and $D(s'_k)$ are tangent or disjoint, hence the half-planes $D(s'_k, \ldots, s'_n)$ and $D(s_k, \ldots, s_n)$ are as well. □

It is of interest to note that the only hypothesis which played a role in proving these properties was the following: if $a$ and $b$ are contained in $\mathcal{A}$ with $a \neq b^{-1}$, then $D(a)$ is contained in $\mathbb{D} - \mathring{D}(b^{-1})$. As a result, these properties remain valid for *generalized Schottky* groups of rank $p$. This means that the group admits a collection of non-elliptic generators $\{g_1, \ldots, g_p\}$ satisfying the following weaker condition:

$$(D(g_i) \cup D(g_i^{-1})) \cap (D(g_j) \cup D(g_j^{-1})) = \varnothing,$$

for all $i \neq j$ in $\{1, \ldots, p\}$.

We will meet such groups again in Sect. 3.

**Exercise 1.5.** Prove that in each of the cases a, a', b, c represented in Fig. II.3, the configuration of half-planes $D(g_1, g_2)$ is as follows in Fig. II.4.

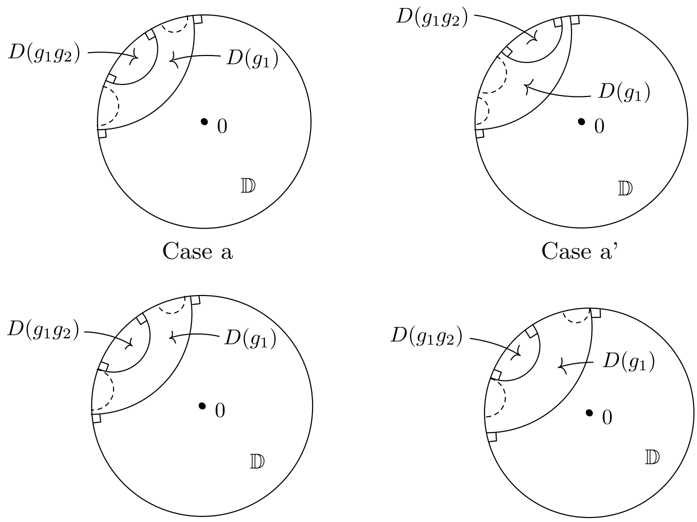


**Fig. II.4.**

Let $g$ in $G$ and $z \in \mathbb{D}$ such that $g(z) \neq z$. Following the notation used in Chap. I, we denote by $\mathbb{D}_z(g)$ the closed half-plane containing $z$, bounded by the perpendicular bisector of the segment $[z, g(z)]_h$. We have: $\mathbb{D}_z(g) = \mathbb{D} - \mathring{D}_z(g)$. Recall that the Dirichlet domain $\mathcal{D}_z(\Gamma)$ of a Fuchsian group $\Gamma$ centered at $z$ is the intersection of all sets $\mathbb{D}_z(\gamma)$, with $\gamma$ in $\Gamma - \{\mathrm{Id}\}$.

**Proposition 1.6.** *The group $S(g_1, g_2)$ is free with respect to $g_1, g_2$, and is discrete. Furthermore, the set $\bigcap_{\substack{i=1,2\\ \varepsilon=\pm 1}} \mathbb{D} - \mathring{D}(g_i^\varepsilon)$ is the Dirichlet domain of this group centered at the point* $0$.

*Proof.* Let $s_1, \ldots, s_n$ be a reduced word. From Property 1.4(i), the point $s_1 \cdots s_n(0)$ is contained is $D(s_1)$. The sets $D(s_1)$ and $\mathring{\mathcal{D}}(S(g_1, g_2))$ are disjoint. In particular $s_1 \cdots s_n(0) \neq 0$, which shows that $S(g_1, g_2)$ is free.

Let us prove that $S(g_1, g_2)$ is discrete. Consider a sequence $(\gamma_n)_{n \geqslant 1}$ in $S(g_1, g_2) - \{\mathrm{Id}\}$. Each $\gamma_n$ can be written as a reduced word $s_{n,1} \cdots s_{n,\ell_n}$. Passing to a subsequence $(\gamma_{n_k})_{k \geqslant 1}$, one may assume that $s_{n_k,1} = s_1$ for all $k \geqslant 1$. The point $\gamma_{n_k}(0)$ is contained in $D(s_1)$, thus there exists $c > 0$ such that $d(\gamma_{n_k}(0), 0) > c$ for all $k \geqslant 1$. This shows that $(\gamma_n)_{n \geqslant 1}$ cannot converge to the identity and thus $S(g_1, g_2)$ is discrete.

Since $S(g_1, g_2)$ is free and discrete, none of its elements is elliptic. Therefore, the Dirichlet domain $\mathcal{D}_0(S(g_1, g_2))$ centered at 0 is well defined.

It only remains to show that $F = \bigcap_{\substack{i=1,2\\ \varepsilon=\pm1}} \mathbb{D} - \mathring{D}(g_i^\varepsilon)$ and $\mathcal{D}_0(S(g_1, g_2))$ are equal. The set $\mathcal{D}_0(S(g_1, g_2))$ is certainly a subset of $F$, since $\mathbb{D}_0(g_i) = \mathbb{D} - \mathring{D}(g_i)$. If it is a proper subset, there exist $z$ in $\mathcal{D}_0(S(g_1, g_2))$ and $\gamma$ in $S(g_1, g_2) - \{\mathrm{Id}\}$ such that $\gamma(z)$ is contained in $\mathring{F}$. Writing $\gamma$ as a reduced word $s_1 \cdots s_n$, Property 1.4(i) implies that the point $\gamma(z)$ is an element of $D(s_1)$. This is impossible since $\gamma(z)$ is contained in $\mathring{\mathcal{D}}(S(g_1, g_2))$. □

Since the group $S(g_1, g_2)$ is discrete and admits a Dirichlet domain having finitely many edges, one can state the following result:

**Corollary 1.7.** *The group $S(g_1, g_2)$ is a geometrically finite Fuchsian group.*

Notice that the proof of Proposition 1.6 is essentially an application of Property 1.4(i). As such, this proposition and its corollary are also valid for generalized Schottky groups.

From the dynamic point of view, Schottky groups $S(g_1, g_2)$, where $g_1$ and $g_2$ are hyperbolic, are—in some sense—the simplest non-elementary Fuchsian groups.

The following exercise shows that the construction of these Schottky groups can be extended to an infinite collection of generators.

**Exercise 1.8.** Let $(g_i)_{i \geqslant 1}$ be an infinite sequence of non-elliptic isometries in $G$ satisfying the following condition for all $i \neq j$:

$$(D(g_i) \cup D(g_i^{-1})) \cap (D(g_j) \cup D(g_j^{-1})) = \varnothing.$$

Prove that the group generated by this sequence is a Fuchsian free group which is not geometrically finite.

We now focus on the nature of the isometries of the $S(g_1, g_2)$.

**Property 1.9.** *Let $S(g_1, g_2)$ be a Schottky group.*

(i) *If $g_1$ and $g_2$ are both hyperbolic, then every element of $S(g_1, g_2) - \{\mathrm{Id}\}$ is hyperbolic.*
(ii) *If not, the non-hyperbolic isometries in $S(g_1, g_2) - \{\mathrm{Id}\}$ are conjugate to powers of parabolic generators in $S(g_1, g_2)$.*

*Proof.* Since the group $S(g_1, g_2)$ is free, it does not contain elliptic isometries. Let $x$ be a point in $\mathbb{D}(\infty)$ fixed by a non-trivial parabolic isometry of $S(g_1, g_2)$. Applying Corollary I.4.10, we obtain $\gamma$ in $\Gamma$ such that $y = \gamma(x)$ is in $\mathcal{D}_0(S(g_1, g_2))(\infty)$. Since $y$ is in $L(S(g_1, g_2))$, this point is an endpoint of an edge of $\mathcal{D}_0(S(g_1, g_2))$. Using the dynamics of the parabolic isometries, we have that any open arc in $\mathbb{D}(\infty)$ with extremity $y$ meets $L(S(g_1, g_2))$. It follows that $y$ is the common endpoint of two edges, and hence that it is fixed by a parabolic generator $a \in \mathcal{A}$ (see Fig. II.3). The group $S(g_1, g_2)$ being discrete, any isometry in $S(g_1, g_2)$ fixing $y$ belongs to the cyclic group generated by $a$. This implies that $x$ is fixed by an isometry of the form $\gamma a^k \gamma^{-1}$, for some $k \neq 0$. □

These properties cannot be extended to generalized Schottky groups. It will be shown in the next section that some such groups can contain parabolic isometries which are not conjugate to powers of generators.

Using Proposition I.2.17, we obtain that the surfaces $S(g_1, g_2)\backslash\mathbb{D}$ associated to each of the four cases in Fig. II.3 are of the form shown in Fig. II.5.

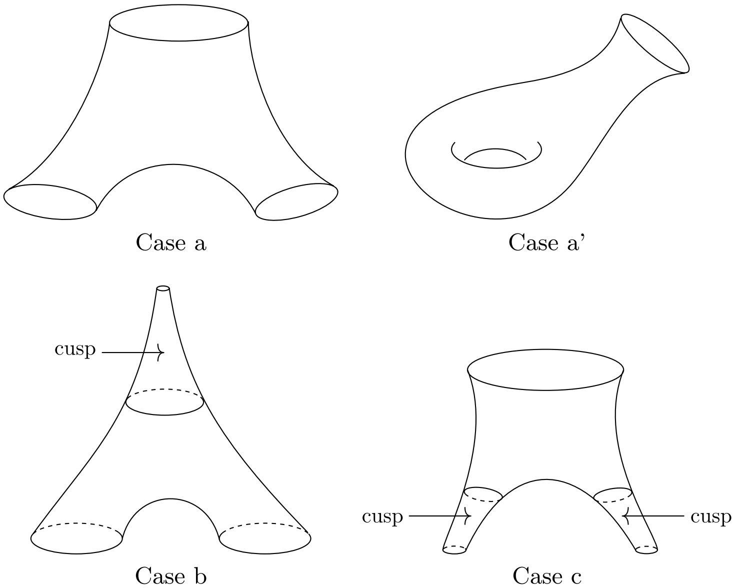


**Fig. II.5.**

## 1.2 Limit set

Clearly, the limit set of a Schottky group $S(g_1, g_2)$ is a proper subset of $\mathbb{D}(\infty)$ since it does not meet the non-empty, open, circular arcs included in $\mathcal{D}_0(S(g_1, g_2))(\infty)$.

What is the topological structure of $L(S(g_1, g_2))$? To begin answering this question, we prove the following lemma:

**Lemma 1.10.** *Given a sequence $(s_i)_{i \geqslant 1}$ in $\mathcal{A}$ satisfying $s_{i+1} \neq s_i^{-1}$, the sequence of Euclidean diameters of the sets $D(s_1, \ldots, s_n)$ converges to zero.*

*Proof.* By Property 1.4(ii), the half-planes $D(s_1, \ldots, s_n)$ are nested. If the sequence of Euclidean radii do not converge to 0, there is a compact set $K$ in $\mathbb{D}$ such that for all geodesics of the form $s_1 \cdots s_{n-1}(\mathcal{C}_n)$, where $\mathcal{C}_n$ is the boundary of $D(s_n)$, we have $K \cap s_1 \cdots s_{n-1}(\mathcal{C}_n) \neq \varnothing$. The geodesics $\mathcal{C}_n$ are edges of the Dirichlet domain $\mathcal{D}_0(S(g_1, g_2))$, thus the image of this domain under the maps $s_1 \cdots s_{n-1}$ intersects $K$. Yet this is impossible since Property I.2.15 states that this domain is locally finite. □

**Proposition 1.11.**

$$L(S(g_1, g_2)) = \bigcap_{n=1}^{+\infty} \bigcup_{\substack{\text{reduced words}\\ \text{of length } n}} \overline{D(s_1, \ldots, s_n)}.$$

*Proof.* Let $y$ be an element of $L(S(g_1, g_2))$, and consider a sequence $(\gamma_n)_{n \geqslant 1}$ in $S(g_1, g_2)$ such that $\lim_{n \to +\infty} \gamma_n(0) = y$. Write $\gamma_n$ as a reduced word $\gamma_n = s_{n,1} \cdots s_{n,\ell_n}$, where $s_{n,i} \in \mathcal{A}$ and $s_{n,i} \neq s_{n,i+1}^{-1}$. Since $\mathcal{A}$ is finite, one can assume (by passing to a subsequence) that there exist $(s_i)_{i \geqslant 1}$ with $s_{i+1} \neq s_i^{-1}$, and a sequence of positive integers $(\ell_n)_{n \geqslant 1}$ which is strictly increasing such that $\gamma_n = s_1 \cdots s_{\ell_n}$.

The point $s_{\ell_n}(0)$ is an element of $D(s_{\ell_n})$, therefore $\gamma_n(0)$ is in $D(s_1, \ldots, s_{\ell_n})$, for any $n \geqslant 1$. Since the sets $D(s_1, \ldots, s_n)$ are nested and their diameter go to 0 (Lemma 1.10), we have

$$\{y\} = \bigcap_{n=1}^{+\infty} \overline{D(s_1, \ldots, s_{\ell_n})}.$$

This shows that $L(S(g_1, g_2))$ is a subset of

$$\bigcap_{n=1}^{+\infty} \bigcup_{\substack{\text{reduced words}\\ \text{of length } n}} D(s_1, \ldots, s_n).$$

The reverse inclusion is a consequence of Property 1.4 and Lemma 1.10. □

Using this proposition, we obtain a construction of $L(S(g_1, g_2))$ by an iterative procedure analogous to the construction of *Cantor sets*. More precisely, consider the case in which $g_1$ and $g_2$ are hyperbolic and denote by $I_1, I_2, I_3, I_4$ the connected components of the following set:

$$\mathbb{D}(\infty) - \bigcup_{a \in \mathcal{A}} D(a)(\infty).$$

Step 1: Remove these four arcs from the set $\mathbb{D}(\infty)$. One obtains

$$\mathbb{D}(\infty) - \bigcup_{1 \leqslant i \leqslant 4} I_i = \bigcup_{a \in \mathcal{A}} D(a)(\infty).$$

Step 2: For each $a$, remove the four arcs $a(I_1), a(I_2), a(I_3), a(I_4)$ from the set $D(a)(\infty)$. One obtains

$$\forall\, a \in \mathcal{A}, \quad \mathbb{D}(a)(\infty) - \bigcup_{1 \leqslant i \leqslant 4} (I_i) = \bigcup_{1 \leqslant i \leqslant 4} \bigcup_{\substack{b \in \mathcal{A} \\ b \neq a^{-1}}} D(a,b)(\infty).$$

More generally, one does the following for $n \geqslant 2$:
Step $n$: Remove from each set of the form $D(s_1, \ldots, s_{n-1})(\infty)$, where $s_1 \cdots s_{n-1}$ is a reduced word, the four arcs $s_1 \cdots s_{n-1}(I_1)$, $s_1 \cdots s_{n-1}(I_2)$, $s_1 \cdots s_{n-1}(I_3)$, $s_1 \cdots s_{n-1}(I_4)$. One obtains

$$D(s_1, \ldots, s_{n-1})(\infty) - \bigcup_{1 \leqslant i \leqslant 4} s_1 \cdots s_{n-1}(I_i) = \bigcup_{\substack{s \in \mathcal{A} \\ s \neq s_{n-1}^{-1}}} D(s_1, \ldots, s_{n-1}, s)(\infty).$$

It follows from Proposition 1.11, that the set $L(S(g_1, g_2))$ is the intersection over the integers $n \geqslant 1$, of $4 \times 3^{n-1}$ arcs obtained at Step $n$ of this procedure (Fig. II.6).

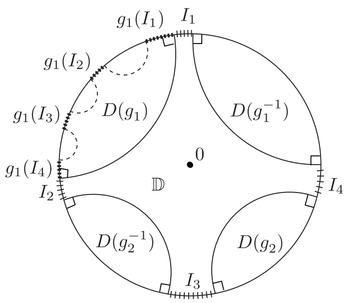


**Fig. II.6.**

If some of the generators is parabolic, then the procedure above must be modified by grouping together arcs of the form $D(s_1, \ldots, s_n)(\infty)$ and $D(s'_1, \ldots, s'_n)(\infty)$ having an endpoint in common.

**Exercise 1.12.** Prove that the set $L(S(g_1, g_2))$ is a totally discontinuous set (i.e., its connected components are points), without isolated points.
(Hint: [13].)

Recall that the Nielsen region $N(S(g_1, g_2))$ of $S(g_1, g_2)$ is the convex hull of the set of points in $\mathbb{D}$ belonging to geodesics whose endpoints are in the limit set $L(S(g_1, g_2))$ (Sect. I.4).

Figure II.7 shows the form of the intersection of $N(S(g_1, g_2))$ with the Dirichlet domain $\mathcal{D}_0(S(g_1, g_2))$ in cases a, a', b, c associated with Fig. II.3.

If neither of its generators are parabolic, the group $S(g_1, g_2)$ is geometrically finite and does not contain any parabolic isometries, and thus this group

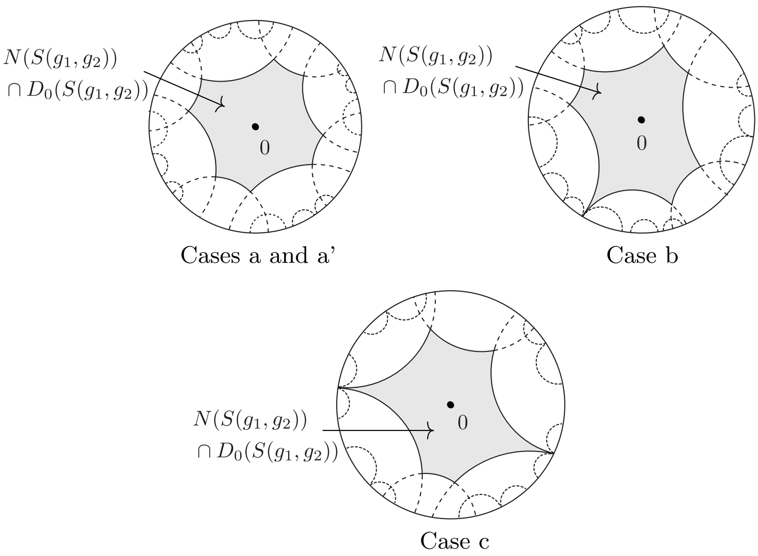


**Fig. II.7.**

is convex-cocompact (Corollary I.4.17). This property is shown by Fig. II.7 (cases a and a') since the set $N(S(g_1, g_2)) \cap \mathcal{D}_0(S(g_1, g_2))$ is compact (Definition I.4.5). This is not the case if at least one of the generators is parabolic.

Thus one can state the following property.

**Property 1.13.** *The group $S(g_1, g_2)$ is convex-cocompact if and only if $g_1$ and $g_2$ are hyperbolic.*

## 2 Encoding the limit set of a Schottky group

At this stage, we enter the world of symbolic dynamics which will be explored further in Chap. IV.

The purpose of this section is simply to construct a dictionary between $L(S(g_1, g_2))$ and a specific set of sequences of elements of $\mathcal{A}$. Using this dictionary, we establish a correspondence between some properties of these sequences and some geometric properties of points in $L(S(g_1, g_2))$.

Since the group $S(g_1, g_2)$ is free, there is a bijection between the set of finite reduced sequences $s_1, \ldots, s_n$, with $s_i \neq s_{i+1}^{-1}$, and $S(g_1, g_2) - \{\mathrm{Id}\}$. Let us extend this bijection to the set of infinite reduced sequences $\Sigma^+$ defined by

$$\Sigma^+ = \{(s_i)_{i \geqslant 1} \mid s_i \in \mathcal{A},\, s_{i+1} \neq s_i^{-1}\}.$$

Let $(s_i)_{i \geqslant 1}$ be such a sequence. Define

$$\gamma_n = s_1 \cdots s_n.$$

The point $\gamma_n(0)$ is in the half-plane $D(s_1,\ldots,s_n)$. As a consequence of Property 1.4(ii), the half-planes $(D(s_1,\ldots,s_n))_{n\geqslant 1}$ are nested and, by Lemma 1.10, their Euclidean diameter tends to 0. Hence the sequence $(\gamma_n(0))_{n\geqslant 1}$ converges to a point in $L(S(g_1,g_2))$.

Let $f:\Sigma^+\to L(S(g_1,g_2))$ denote the function which sends $s=(s_i)_{i\geqslant 1}$ to the point $x$ defined by

$$x(s)=\lim_{n\to+\infty}\gamma_n(0).$$

**Exercise 2.1.** Prove that the function $f$ is surjective.
(Hint: Use Property 1.11.)

Is this function injective? To answer this question, the cases with and without parabolic generators must be considered separately.

Consider the subset $\Sigma_c^+$ of $\Sigma^+$ consisting of sequences $(s_n)_{n\geqslant 1}\in\Sigma^+$ for which, if the term $s_n$ is parabolic, there exists $m>n$ such that $s_m\neq s_n$.

Recall that, since $S(g_1,g_2)$ is geometrically finite, its limit set can be decomposed into a disjoint union of the set of its parabolic points $L_p((S(g_1,g_2))$ and its conical points $L_c(S(g_1,g_2))$ (Theorem I.4.13).

**Proposition 2.2.** *If $g_1$ and $g_2$ are hyperbolic, then the function $f:\Sigma^+\to L(S(g_1,g_2))$ is a bijection. Otherwise, this function is surjective but not injective and its restriction to $\Sigma_c^+$ is a bijection onto $L_c(S(g_1,g_2))$.*

*Proof.*
*Case 1: $g_1$ and $g_2$ hyperbolic.* In this case $\Sigma_c^+=\Sigma^+$ and for all $a$ in $\mathcal{A}$ and $b$ in $\mathcal{A}-\{a\}$, the closure of the sets $D(a)$ and $D(b)$ are disjoint. Let $s=(s_i)_{i\geqslant 1}$ and $s'=(s_i')_{i\geqslant 1}$ be in $\Sigma^+$. Suppose that there exists $i\geqslant 1$ such that $s_i\neq s_i'$. Let $k$ denote the smallest of these integers and define $\gamma=s_1\cdots s_{k-1}$ if $k>1$ and $\gamma=\mathrm{Id}$ otherwise. For all $n\geqslant k$, the points $\gamma^{-1}s_1\cdots s_n(0)$ and $\gamma^{-1}s_1'\cdots s_n'(0)$ are contained in $D(s_k)$ and $D(s_k')$ respectively. These two sets are disjoint, thus $\lim_{n\to+\infty}s_1\cdots s_n(0)\neq\lim_{n\to+\infty}s_1'\cdots s_n'(0)$.

This shows that $f$ is injective.
*Case 2: $g_1$ or $g_2$ is parabolic.* Suppose that $g_1$ is parabolic. In this case, the sequences $(g_1^n(0))_{n\geqslant 1}$ and $(g_1^{-n}(0))_{n\geqslant 1}$ converge to the same point, thus $f$ is not injective.

Let us show that $f(\Sigma_c^+)=L_c(S(g_1,g_2))$. Let $s=(s_i)_{i\geqslant 1}$ be in $(\Sigma^+-\Sigma_c^+)$ and let $k$ denote the smallest integer for which $s_k$ is parabolic and $s_i=s_k$ for all $i\geqslant k$. Define $\gamma=s_1\cdots s_{k-1}$ if $k>1$ and $\gamma=\mathrm{Id}$ otherwise. The point $\gamma^{-1}f(s)$ is parabolic since it is fixed by $s_k$, thus $f(s)$ is parabolic. As a result, the set $f(\Sigma^+-\Sigma_c^+)$ is a subset of $L_p(S(g_1,g_2))$.

Let $y$ be in $L_p(S(g_1,g_2))$. By Property 1.9(ii), there exists $\gamma$ in $S(g_1,g_2)$ such that $\gamma(y)$ is fixed by a parabolic generator $g$. Let $s=(s_i)_{i\geqslant 1}$ be a sequence in $\Sigma^+$ such that $f(s)=y$. Since $\gamma$ can be written as a finite reduced word, there exists $s'=(s_i')_{i\geqslant 1}$ in $\Sigma^+$, $k\geqslant 0$ and $k'\geqslant 0$ such that $\gamma(y)=f(s')$, and for all $i\geqslant 1$, $s_{k'+i}'=s_{k+i}$. The point $\gamma(y)$ is in $D(s_1')(\infty)$. On the other hand,

$\gamma(y)$ is the point of tangency between $D(g)$ and $D(g^{-1})$, hence $s'_1 \in \{g, g^{-1}\}$. If we replace $\gamma(y)$ with $s_1^{-1}(y)$, following the same argument we obtain $s'_1 = s'_2$.

Continuing this process iteratively, we find that the sequence $s'$ must be constant. Hence for some $k \geqslant 0$, the sequence $(s_{k+i})_{i\geqslant 1}$ is constant. This implies that $s$ is not contained in $\Sigma_c^+$. Hence $f^{-1}(L_p(S(g_1, g_2))) = (\Sigma - \Sigma_c^+)$.

In conclusion, $f(\Sigma - \Sigma_c^+) = L_p(S(g_1, g_2))$ and hence $f(\Sigma_c^+) = L_c(S(g_1, g_2))$.

Finally we must show that the function $f$ restricted to $\Sigma_c^+$ is injective. Let $s = (s_i)_{i\geqslant 1}$ and $s' = (s'_i)_{i\geqslant 1}$ be elements of $\Sigma_c^+$. Suppose that $s$ and $s'$ are distinct and denote by $k$ the smallest integer $i \geqslant 1$ such that $s_i \neq s'_i$. Define $\gamma = s_1 \cdots s_{k-1}$ if $k \neq 1$ and $\gamma = \mathrm{Id}$ otherwise.

If $s_k^{-1} \neq s'_k$, or if one of these two letters are hyperbolic, then the set $D(s_k)(\infty) \cap D(s'_k)(\infty)$ is empty, thus $\gamma^{-1} f(s) \neq \gamma^{-1} f(s')$.

If $s_k^{-1} = s'_k$ and $s_k$ is parabolic, consider the smallest $i > k$ such that $s_i \neq s_k$ and define $g = \gamma s_k \cdots s_{i-1}$. Then

$$g^{-1}(f(s)) = \lim_{n\to+\infty} s_i \cdots s_{i+n}(0),$$
$$g^{-1}(f(s')) = \lim_{n\to+\infty} s_{i-1}^{-1} \cdots s_k^{-1} s_k^{-1} s'_{k+1} \cdots s'_{k+n}(0).$$

Since $s'_{k+1} \neq s_k$, the point $g^{-1}(f(s'))$ is in $D(s_{i-1}^{-1})(\infty)$. Furthermore, $g^{-1}(f(s))$ is in $D(s_i)(\infty)$, and $D(s_i)(\infty) \cap D(s_{i-1}^{-1})(\infty) = \varnothing$, since $s_i$ is not contained in $\{s_{i-1}, s_{i-1}^{-1}\}$. Therefore $g^{-1}(f(s')) \neq g^{-1}(f(s))$ and hence $f(s) \neq f(s')$. $\square$

It follows that the fixed points of parabolic isometries in $L(S(g_1, g_2))$ are encoded (non-uniquely) by the sequences $(s_i)_{i\geqslant 1}$ in $\Sigma^+$ which are constant for large $i$, and whose repeated term is a parabolic generator.

Let us now analyze the encoding of all the fixed points of the isometries in $S(g_1, g_2)$. Consider the *shift function* $T : \Sigma^+ \to \Sigma^+$ defined by

$$T((s_i)_{i\geqslant 1}) = (s_{i+1})_{i\geqslant 1}.$$

A sequence $s$ is *periodic* if there exists $k \geqslant 1$ such that $T^k s = s$. In this case, one writes

$$s = (\overline{s_1, \ldots, s_k}).$$

More generally, if there exists $n \geqslant 1$ such that $T^n s$ is periodic, then the sequence $s$ is said to be *almost periodic*.

Note that if $s$ is a sequence in $(\Sigma^+ - \Sigma_c^+)$, there exists $k \geqslant 0$ such that $T^k(s) = (\overline{s_{k+1}})$ with $s_{k+1}$ parabolic. Hence, such a sequence is almost periodic and $f(s)$ is the fixed point of $\gamma s_{k+1} \gamma^{-1}$, where $\gamma = s_1 \cdots s_k$. The following property generalizes this connection between almost periodic sequences and fixed points of isometries of $S(g_1, g_2)$.

**Property 2.3.** *A point $y$ in $L(S(g_1, g_2))$ is fixed by a non-trivial isometry in $S(g_1, g_2)$ if and only if there exists an almost periodic sequence $s$ in $\Sigma^+$ such that $f(s) = y$.*

*Proof.* Let $y$ be in $L(S(g_1, g_2))$. Suppose that there exists a non-trivial $\gamma$ in $S(g_1, g_2)$ such that $y = \lim_{n\to+\infty} \gamma^n(0)$. Write $\gamma$ as a reduced word $\gamma = s_1 \cdots s_n$. If $s_1 \neq s_n^{-1}$, then the periodic sequence $s = (\overline{s_1, \ldots, s_n})$ is contained in $\Sigma^+$ and $y = f(s)$.

Otherwise, consider the largest $1 \leqslant k < n$ such that $s_k = s_{n-k+1}^{-1}$, and define $g = s_1 \cdots s_k$. The point $g^{-1}(y)$ is fixed by the reduced word $s_{k+1} \cdots s_{n-k}$. Since $s_{k+1} \neq s_{n-k}^{-1}$, one has $g^{-1}(y) = f(\overline{s_{k+1}, \ldots, s_{n-k}})$. Consider the almost periodic sequence $s'$ defined by $s'_i = s_i$ for all $1 \leqslant i \leqslant k$ and $T^k(s') = (\overline{s_{k+1}, \ldots, s_{n-k}})$. Since $s_{n-k} \neq s_{k+1}^{-1}$, this sequence is in $\Sigma^+$ and $f(s') = y$.

Conversely, consider an almost periodic sequence $s$ in $\Sigma^+$. Let $k \geqslant 0$ be such that $T^k(s)$ is the periodic sequence $s' = (\overline{s_{k+1} \cdots s_n})$. Then

$$f(s') = \lim_{p\to+\infty} (s_{k+1} \cdots s_n)^p(0),$$

hence $f(s')$ is fixed by $\gamma = s_{k+1} \cdots s_n$. If $k = 0$, then $f(s') = f(s)$; otherwise $f(s) = g(f(s'))$ with $g = s_1 \cdots s_k$. Thus $f(s)$ is fixed by $g\gamma g^{-1}$. □

Since $\Gamma$ is geometrically finite, we have $L(S(g_1, g_2)) = L_p(S(g_1, g_2)) \cup L_c(S(g_1, g_2))$. By definition of conical points, if $x$ is a point in the set $L(S(g_1, g_2)) - L_p(S(g_1, g_2))$, then there exists a sequence $(\gamma_n)_{n\geqslant 1}$ in $S(g_1, g_2)$ such that $(\gamma_n(0))_{n\geqslant 1}$ converges to $x$, remaining at a bounded distance from the geodesic ray $[0, x)$.

How to construct such a sequence $(\gamma_n)_{n\geqslant 1}$? The answer is found in the coding.

Since $x$ is not parabolic, there exists a unique sequence $s$ in $\Sigma_c^+$ satisfying $f(s) = x$. Consider a new sequence $s' = (s'_i)_{i\geqslant 1}$ constructed from $s$ by grouping together consecutive terms corresponding to the same parabolic generator, defined by:

- $s'_1 = s_1$ if $s_1$ is hyperbolic and $n = 1$,
- $s'_1 = s_1^n$ if $s_1$ is parabolic, with $n \geqslant 1$ satisfying $s_1 = s_2 = \cdots = s_n$ and $s_{n+1} \neq s_1$. Such an $n$ exists by the definition of $\Sigma_c^+$.

Repeat this procedure, beginning with the sequence $(s_{n+i})_{i\geqslant 1}$, to find $s'_2$. Step by step, this procedure produces a sequence $(s'_i)_{i\geqslant 1}$ satisfying the following properties:

(i) $s'_i = a_i^{ni}$ with $a_i \in \mathcal{A}$. If $a_i$ is hyperbolic, then $n_i = 1$ and $a_{i+1} \neq a_i^{-1}$; if $a_i$ is parabolic, then $n_i \in \mathbb{N}^*$ and $a_{i+1} \neq a_i^{\pm 1}$.
(ii) For all $i \geqslant 1$, the arcs $D(a_i)(\infty)$ and $D(a_{i+1})(\infty)$ are disjoint.
(iii) $\lim_{n\to+\infty} s'_1 \cdots s'_n(0) = x$.

Note that, if $g_1$ and $g_2$ are hyperbolic, then $s = s'$.

**Property 2.4.** *Let $x$ in $L_c(S(g_1, g_2))$. There exists $\varepsilon > 0$ such that the sequence $(s'_1 \cdots s'_n(0))_{n\geqslant 1}$ is contained in an $\varepsilon$-neighborhood of the geodesic ray $[0, x)$.*

*Proof.* Write $\gamma_n = s'_1 \cdots s'_n$. Fix a point $y \neq x$ in $\mathcal{D}_0(S(g_1, g_2))(\infty)$.

The point $\gamma_n^{-1}(x)$ is in $D(s'_{n+1})(\infty)$. Since $y$ does not belong to the interior of the arcs $D(a)(\infty)$ for all $a$ in $\mathcal{A}$, it follows from Property 1.4(i) that the point $\gamma_n^{-1}(y)$ is in $D(s'_n)(\infty)$. The construction of the sequence $(s'_i)_{i\geqslant 1}$ requires the arcs $D(s'_i)(\infty)$ and $D(s'_{i+1})(\infty)$ to be disjoint. Thus the Euclidean distance between $\gamma_n^{-1}(x)$ and $\gamma_n^{-1}(y)$ is bounded below by a positive constant which does not depend on $n$. This property implies that there exists a compact subset of $\mathbb{D}$ whose image under $\gamma_n$ intersects the geodesic $(yx)$. Take $z$ in the geodesic $(yx)$. Since $\lim_{n\to+\infty} \gamma_n(0) = x$, the sequence $(\gamma_n(0))_{n\geqslant 1}$ converges to $x$ and remains within a bounded distance from the geodesic ray $[z, x)$. It follows that there exists $\varepsilon > 0$ such that the sequence $(\gamma_n(0))_{n\geqslant 1}$ is in the $\varepsilon$-neighborhood of the geodesic ray $[0, x)$. □

## 3 The modular group and two subgroups

Let us now return to the Poincaré half-plane. In this section, we study the action on $\mathbb{H}$ of the *modular group* $\mathrm{PSL}(2,\mathbb{Z})$ composed of Möbius transformations $h$ of the form

$$h(z) = \frac{az+b}{cz+d} \quad \text{with } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1,$$

and of two of its subgroups.

### 3.1 The modular group

By definition, the modular group is Fuchsian. We are going to describe one of its Dirichlet domains.

**Exercise 3.1.** Prove that only the trivial isometry in $\mathrm{PSL}(2,\mathbb{Z})$ fixes the point $2i$.

Define the isometries $\mathcal{T}_1(z) = z+1$ and $s(z) = -1/z$. These two isometries will allow us to construct the Dirichlet domain of the modular group.

**Property 3.2.**

$$\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z})) = \{z \in \mathbb{H} \mid |z| \geqslant 1 \textit{ and } -1/2 \leqslant \operatorname{Re} z \leqslant 1/2\}.$$

*Proof.* Set $E = \{z \in \mathbb{H} \mid |z| \geqslant 1 \text{ and } -1/2 \leqslant \operatorname{Re} z \leqslant 1/2\}$. By definition of the Dirichlet domain (see Sect. I.2.3), $\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z}))$ is contained in the set $\mathbb{H}_{2i}(\mathcal{T}_1) \cap \mathbb{H}_{2i}(\mathcal{T}_1^{-1}) \cap \mathbb{H}_{2i}(s)$. On the other hand,

$$\begin{aligned} \mathbb{H}_{2i}(\mathcal{T}_1) &= \{z \in \mathbb{H} \mid \operatorname{Re} z \leqslant 1/2\}, \\ \mathbb{H}_{2i}(\mathcal{T}_1^{-1}) &= \{z \in \mathbb{H} \mid \operatorname{Re} z \geqslant -1/2\} \quad \text{and} \\ \mathbb{H}_{2i}(s) &= \{z \in \mathbb{H} \mid |z| \geqslant 1\}, \end{aligned}$$

hence $\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z})$ is contained in $E$ (Fig. II.8).

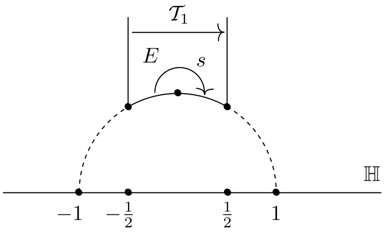


**Fig. II.8.**

Let $z$ be in $\mathring{E}$. Suppose that there exists $\gamma(z) = (az+b)/(cz+d)$ in $\mathrm{PSL}(2,\mathbb{Z}) - \{\mathrm{Id}\}$ such that $\gamma(z)$ is in $E$. Then $c \neq 0$ necessarily since $|\mathrm{Re}(z+b)| > 1/2$ for all $b \in \mathbb{Z}^*$. One has $\mathrm{Im}(\gamma z) = \mathrm{Im}\, z/|cz+d|^2$. Also, since $z$ is contained in $\mathring{E}$,

$$|cz+d|^2 > (|c|-|d|)^2 + |c||d|.$$

Therefore, $c \neq 0$ implies $\mathrm{Im}\, z > \mathrm{Im}(\gamma(z))$.

If $\gamma(z)$ is contained in $\mathring{E}$, the same reasoning using $\gamma^{-1}$ allows us to conclude that $\mathrm{Im}(\gamma(z)) > \mathrm{Im}\, z$, a contradiction.

In conclusion, for all $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z}) - \{\mathrm{Id}\}$, one has $\gamma\mathring{E} \cap \mathring{E} = \varnothing$. This implies that $\mathring{E}$ is contained in $\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z}))$. Thus $\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z})) = E$. $\square$

This proposition immediately produces the following result.

**Corollary 3.3.** *The group* $\mathrm{PSL}(2,\mathbb{Z})$ *is a non-uniform lattice.*

**Exercise 3.4.** Verify that the *modular surface* $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ has the form of Fig. II.9.
(Hint: use Proposition I.2.17.)

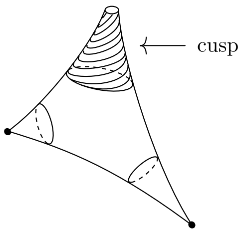


**Fig. II.9.**

In Sect. 4, we will use another fundamental domain constructed from $\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z}))$, as defined in the following exercise.

**Exercise 3.5.** Prove that the set

$$\begin{aligned}\Delta = \mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z})) \cap \{z \in \mathbb{H} \mid \operatorname{Re} z \geqslant 0\} \\ \cap\, \mathcal{T}_1(\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z})) \cap \{z \in \mathbb{H} \mid \operatorname{Re} z \leqslant 0\}),\end{aligned}$$

is a fundamental domain of the modular group (Fig. II.10).

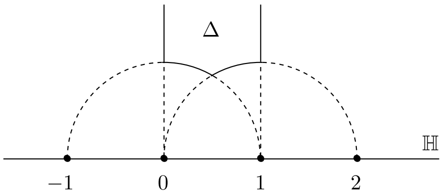


**Fig. II.10.**

The group $\mathrm{PSL}(2,\mathbb{Z})$, unlike Schottky groups, contains elliptic elements, as $s(z) = -1/z$ and $r(z) = (z-1)/z$.

**Proposition 3.6.** *An elliptic element of* $\mathrm{PSL}(2,\mathbb{Z})$ *is conjugate in* $\mathrm{PSL}(2,\mathbb{Z})$ *to some power of either $s$ or $r$.*

**Exercise 3.7.** Prove Proposition 3.6.
(Hint: use the fact that the trace of such element is $-1$, $0$, or $1$.)

Which isometries in the modular group are parabolic? As a consequence of Corollary I.4.10, if $p$ is such an isometry, the orbit of its fixed point intersects $\mathcal{D}_{2i}(\mathrm{PSL}(2,\mathbb{Z}))(\infty)$. This set is reduced to the point $\infty$, which is parabolic since it is fixed by $\mathcal{T}_1$. The stabilizer of this point in $\mathrm{PSL}(2,\mathbb{Z})$ is generated by $\mathcal{T}_1$, hence $p$ is conjugate in $\mathrm{PSL}(2,\mathbb{Z})$ to a power of $\mathcal{T}_1$.

From the preceding discussion, it follows that all parabolic points are of the form $\gamma(\infty)$, with $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$. If $\gamma$ does not fix the point $\infty$, the transformation $\gamma$ can be written as $\gamma(z) = (az+b)/(cz+d)$ with $c \neq 0$, thus $\gamma(\infty)$ is the rational number $a/c$.

Conversely, let $p/q$ be in $\mathbb{Q}$, with $\gcd(p,q) = 1$. Consider $p', q'$ in $\mathbb{Z}$ such that $pq' - qp' = 1$. Define $g(z) = (pz+p')/(qz+q')$. This isometry belongs to $\mathrm{PSL}(2,\mathbb{Z})$ and $g(\infty) = p/q$ thus $p/q$ is fixed by $g\mathcal{T}_1 g^{-1}$.

We have proved the following facts.

**Property 3.8.**

(i) *The parabolic isometries of the modular group are conjugate in* $\mathrm{PSL}(2,\mathbb{Z})$ *to powers of* $\mathcal{T}_1$.
(ii) $L_p(\mathrm{PSL}(2,\mathbb{Z})) = \mathbb{Q} \cup \{\infty\}$.

Therefore, the rational numbers correspond to the parabolic points of $\mathrm{PSL}(2,\mathbb{Z})$, distinct from $\infty$. Furthermore, since this group is a lattice, its limit set is $\mathbb{H}(\infty)$ and is the disjoint union of the set of parabolic points and conical points. It follows that the irrational numbers correspond to conical points.

This geometric characterization of a rational is the key to the last section of Chap. VII. It allows us to relate the theory of Diophantine approximation to the behavior of geodesics on the modular surface.

### 3.2 Congruence modulo 2 subgroup and the commutator subgroup

One interesting aspect of these two subgroups is that they share the same fundamental domain. However, this domain is Dirichlet only in the first case.

*The congruence modulo* 2 *subgroup.* Let $P$ be the group homomorphism of $\mathrm{PSL}(2,\mathbb{Z})$ into $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$ sending any Möbius transformation $h(z) = (az+b)/(cz+d)$ with integer coefficients to the matrix

$$P(h) = \begin{pmatrix} a^\bullet & b^\bullet \\ c^\bullet & d^\bullet \end{pmatrix},$$

where $n^\bullet$ denotes the class of $n$ in $\mathbb{Z}/2\mathbb{Z}$. The group $\Gamma(2) = P^{-1}\left(\begin{smallmatrix} 1^\bullet & 0^\bullet \\ 0^\bullet & 1^\bullet \end{smallmatrix}\right)$ is called the congruence modulo 2 subgroup [41, Chap. V.5]. This is a normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ of index 6.

Let $r$ be the Möbius transformation which sends $z$ to $r(z) = (z-1)/z$. Then

$$(*) \qquad \Gamma(2)\backslash \mathrm{PSL}(2,\mathbb{Z}) = \{\overline{\mathrm{Id}}, \overline{r}, \overline{r^2}, \overline{\mathcal{T}_1^{-1}}, \overline{\mathcal{T}_1^{-1}r}, \overline{\mathcal{T}_1^{-1}r^2}\}.$$

Consider the set $\Delta'$ (Fig. II.11) defined by

$$\Delta' = \Delta \cup r\Delta \cup r^2\Delta \cup \mathcal{T}_1^{-1}(\Delta) \cup \mathcal{T}_1^{-1}r(\Delta) \cup \mathcal{T}_1^{-1}r^2(\Delta).$$

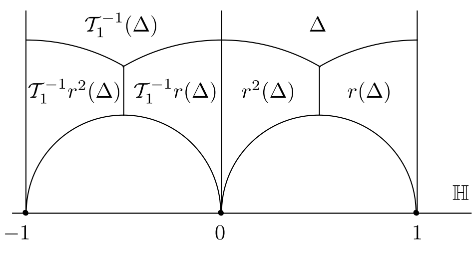


**Fig. II.11.**

**Exercise 3.9.** Prove that $\Delta'$ is a fundamental domain of $\Gamma(2)$.
(Hint: use Exercise 3.5 and the relation (*).)

The isometry $\mathcal{T}_{-1}(z) = z/(z+1)$ will useful in the following discussion.

**Property 3.10.** *The domain $\Delta'$ is the Dirichlet domain of $\Gamma(2)$ at the point $i$.*

*Proof.* None of the elements of $\Gamma(2) - \{\mathrm{Id}\}$ fixes $i$. Furthermore, the isometries $\mathcal{T}_1^{\pm 2}, \mathcal{T}_{-1}^{\pm 2}$ belong to $\Gamma(2)$, so one has

$$\begin{aligned}
\mathbb{H}_i(\mathcal{T}_1^2) &= \{z \in \mathbb{H} \mid \operatorname{Re} z \leqslant 1\},\\
\mathbb{H}_i(\mathcal{T}_1^{-2}) &= \{z \in \mathbb{H} \mid \operatorname{Re} z \geqslant -1\},\\
\mathbb{H}_i(\mathcal{T}_{-1}^2) &= \{z \in \mathbb{H} \mid |z + 1/2| \geqslant 1/2\}, \quad \text{and}\\
\mathbb{H}_i(\mathcal{T}_{-1}^{-2}) &= \{z \in \mathbb{H} \mid |z - 1/2| \geqslant 1/2\}.
\end{aligned}$$

Therefore, $\Delta' = \mathbb{H}_i(\mathcal{T}_1^2) \cap \mathbb{H}_i(\mathcal{T}_1^{-2}) \cap \mathbb{H}_i(\mathcal{T}_{-1}^2) \cap \mathbb{H}_i(\mathcal{T}_{-1}^{-2})$. It follows that $\mathcal{D}_i(\Gamma(2))$ is contained in $\Delta'$. However, since $\mathcal{D}_i(\Gamma(2))$ and $\Delta'$ are two fundamental domains of $\Gamma(2)$, one has $\Delta' = \mathcal{D}_i(\Gamma(2))$. $\square$

**Corollary 3.11.** *The group $\Gamma(2)$ is a non-uniform lattice.*

**Exercise 3.12.** Verify that the surface $\Gamma(2)\backslash\mathbb{H}$ is of the form given by Fig. II.12.
(Hint: use Proposition I.2.17.)

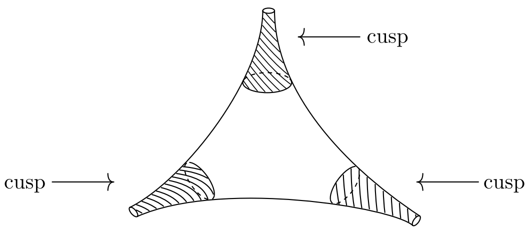


**Fig. II.12.**

The group $\Gamma(2)$ is a generalized Schottky group (see Sect. 1.1 for the definition). Recall that the Möbius transformation $\psi(z) = i\,\frac{z-i}{z+i}$ (see Sect. 1.5) sends $\mathbb{H}$ into the Poincaré disk. We have $0 = \psi(i)$. Set:

$$\mathcal{A} = \{\psi\mathcal{T}_1^2\psi^{-1}, \psi\mathcal{T}_1^{-2}\psi^{-1}, \psi\mathcal{T}_1^2\psi^{-1}, \psi\mathcal{T}_1^{-2}\psi^{-1}\}.$$

For all $a$ in $\mathcal{A}$, the half-planes $D(a)$ bounded by the perpendicular bisectors of the hyperbolic segments $[0, a(0)]_h$ are either tangent or disjoint. From Proposition 1.6, the group generated by $\mathcal{T}_1^2$ and $\mathcal{T}_{-1}^2$ is therefore free (and discrete) and it has $\Delta'$ as its fundamental domain. Since this group is a subset of $\Gamma(2)$ and has the same fundamental domain, they are equal.

Thus we have the following property:

**Property 3.13.** *The group $\Gamma(2)$ is generated by $\mathcal{T}_1^2$ and $\mathcal{T}_{-1}^2$, and is free relative to these generators.*

We now consider the parabolic isometries in $\Gamma(2)$.

**Property 3.14.** *The parabolic isometries of $\Gamma(2)$ are conjugate to powers of $\mathcal{T}_1^2$, $\mathcal{T}_{-1}^2$ or $\mathcal{T}_{-1}^{-2}\mathcal{T}_1^2$ in $\Gamma(2)$.*

*Proof.* Notice first that the point $\infty$ is fixed by $\mathcal{T}_1^2$, the point $0$ is fixed by $\mathcal{T}_{-1}^2$, the point $-1$ is fixed by $\mathcal{T}_{-1}^{-2}\mathcal{T}_1^2$, and the point $1$ is fixed by $\mathcal{T}_{-1}^2\mathcal{T}_1^{-2}$. These four points are parabolic. Furthermore, $-1$ and $1$ are in the same orbit since $\mathcal{T}_1^2(-1) = 1$, and the sets $\Gamma(2)(0), \Gamma(2)(\infty), \Gamma(2)(1)$ are three disjoint orbits.

Let $\gamma$ be a parabolic isometry in $\Gamma(2)$. From Corollary I.4.10, its fixed point is contained in one of the three orbits described above. Also since each of $\mathcal{T}_1^2, \mathcal{T}_{-1}^{-2}\mathcal{T}_1^2, \mathcal{T}_{-1}^2$ generates the stabilizer in $\Gamma(2)$ of its fixed point, $\gamma$ is conjugate to a power of one of these three isometries. $\square$

Note that, unlike Schottky groups, the parabolic isometries of a generalized Schottky group admitting a collection of non-elliptic generators $\{g_1, \ldots, g_p\}$ satisfying the condition $(D(g_i) \cup D(g_i^{-1})) \cap (D(g_j) \cup D(g_j^{-1})) = \varnothing$, for all $i \neq j$ in $\{1, \ldots, p\}$ are not always conjugate to powers of the parabolic generators $g_i$.

**Exercise 3.15.** Prove that the set $L_p(\Gamma(2))$ is equal to $\mathbb{Q} \cup \{\infty\}$.
(Hint: use Property 3.8 and the fact that $\Gamma(2)$ is normal in $\mathrm{PSL}(2, \mathbb{Z})$, (see also [41, Chap. V, Example F]).)

*The commutator subgroup.* We now introduce another subgroup of the modular group defined this time by a given collection of generators. Let $\alpha_1$ and $\alpha_2$ be two Möbius transformations defined as

$$\alpha_1(z) = \frac{z+1}{z+2}, \qquad \alpha_2(z) = \frac{z-1}{-z+2},$$

and consider the half-planes

$$B(\alpha_1) = \{z \in \mathbb{H} \mid |z - 1/2| \leqslant 1/2\}, \qquad B(\alpha_1^{-1}) = \{z \in \mathbb{H} \mid \operatorname{Re} z \leqslant -1\},$$
$$B(\alpha_2) = \{z \in \mathbb{H} \mid |z + 1/2| \leqslant 1/2\}, \qquad B(\alpha_2^{-1}) = \{z \in \mathbb{H} \mid \operatorname{Re} z \geqslant 1\}.$$

For all $i = 1, 2$ and $\varepsilon = \pm 1$, one has

$$\alpha_i^{\varepsilon}(\mathbb{H} - \mathring{B}(\alpha_i^{-\varepsilon})) = B(\alpha_i^{\varepsilon}).$$

Note that we have (Fig. II.13) the following relation:

$$\Delta' = \bigcap_{\substack{\varepsilon = \pm 1 \\ i = 1,2}} \mathbb{H} - \mathring{B}(\alpha_i^{-\varepsilon}).$$

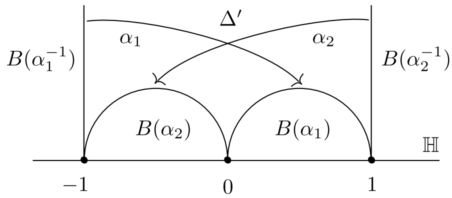


**Fig. II.13.**

**Exercise 3.16.**

(i) Prove that if the perpendicular bisector of a hyperbolic segment $[z_1, z_2]_h$ is a vertical half-line, then $\operatorname{Im} z_1 = \operatorname{Im} z_2$.
(Hint: use Exercise I.1.8.)

(ii) Conclude that there is no point $z$ in $\mathbb{H}$ such that the half-planes $B(\alpha_1^{-1})$ and $B(\alpha_2^{-1})$ are bounded by the perpendicular bisectors of the segments $[z, \alpha_1^{-1}(z)]_h$ and $[z, \alpha_2^{-1}(z)]_h$ respectively.

Let $\Gamma$ be the group generated by $\alpha_1$ and $\alpha_2$. The group $\Gamma$ being a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, it is Fuchsian. Observe that $\Gamma$ is not a generalized Schottky group with respect to $\alpha_1$ and $\alpha_2$, since there is no $z \in \mathbb{D}$ such that $B(\alpha_i^{\varepsilon}) = D_z(\alpha_i^{\varepsilon})$. Even so, the arguments presented in the first part of the proof of Proposition 1.6 being purely dynamic, they still apply.

**Property 3.17.** *The group $\Gamma$ generated by $\alpha_1$ and $\alpha_2$ is free relative to these generators. Moreover $\Gamma$ is the commutator subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$ (*i.e., it is generated by the elements $[g,h] = ghg^{-1}h^{-1}$, where $g$ and $h$ are in* $\mathrm{PSL}(2,\mathbb{Z})$).

**Exercise 3.18.** Prove Property 3.17.
(Hint: For the first part of Property 3.17, see proof of Proposition 1.6. For the second part, use the identities $[s, \mathcal{T}_1^{-1}] = \alpha_1$ and $[s, \mathcal{T}_1] = \alpha_2$.)

**Exercise 3.19.** Prove that $\Gamma$ is a normal subgroup of index 6 in $\mathrm{PSL}(2,\mathbb{Z})$. Moreover, give the structure of the two finite groups $\mathrm{PSL}(2,\mathbb{Z})/\Gamma(2)$ and $\mathrm{PSL}(2,\mathbb{Z})/\Gamma$.

Let us prove that the set $\Delta'$ is a fundamental domain of $\Gamma$. Notice that, since $\Gamma$ is not a generalized Schottky group, the second part of Proposition 1.6 does not apply directly.

**Property 3.20.** *The set $\Delta'$ is a fundamental domain of $\Gamma$.*

*Proof.* Define $\mathcal{A} = \{\alpha_1^{\pm 1}, \alpha_2^{\pm 1}\}$. Let us show that, for all $z$ in $\mathbb{H}$, there exists $\gamma$ in $\Gamma$ such that $\gamma(z)$ belongs to $\Delta'$.

If $z$ is not in $\Delta'$, there exists $a_1$ in $\mathcal{A}$ such that $z$ belongs to $B(a_1)$. Define $z_1 = a_1^{-1}(z)$. If $z_1$ is contained in $\Delta'$, one may define $z_n = z_1$ for all $n \geqslant 1$. Otherwise there exists $a_2$ in $\mathcal{A} - \{a_1^{-1}\}$ such that $z_1$ belongs to $B(a_2)$ and one may define $z_2 = a_2^{-1}(z_1)$. Following this process, we construct the sequence $(z_n)_{n\geqslant 1}$.

If $(z_n)_{n\geqslant 1}$ is constant after some $N$, then $a_N^{-1}\cdots a_1^{-1}(z)$ is contained in $\Delta'$. Otherwise, define $\gamma_n = a_1 \cdots a_n$. By construction, $\gamma_n^{-1}(z)$ is contained in $B(a_{n+1})$. Consider a subsequence $(\gamma_{n_p})_{p\geqslant 1}$ such that $a_{n_p+1} = a$. The point $z$ belongs to $\gamma_{n_p}(B(a))$ for all $p$. For all $p \geqslant 2$, the half-plane $\gamma_{n_p}(B(a))$ is contained in $\gamma_{n_{p-1}}(B(a))$ (see the argument in Property 1.4(ii)). The point $z$ lies in each of these half-planes, hence there is a compact $K$ in $\mathbb{H}$ such that the image by $a_1 \cdots a_{n_p}$ of the geodesic bounding $B(a)$ meets $K$, for all $p \geqslant 1$. This geodesic is a subset of $\Delta'$ and $\Delta' = \Delta \cup r\Delta \cup r^2\Delta \cup \mathcal{T}_1^{-1}\Delta \cup \mathcal{T}_1^{-1}r\Delta \cup \mathcal{T}_1^{-1}r^2\Delta$. Therefore, there exist infinitely many elements $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$ such that $\gamma\Delta$ intersects $K$. This contradicts the fact that $\Delta$ is a locally finite fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ (since it is a Dirichlet domain). Thus the sequence $(z_n)_{n\geqslant 1}$ is necessarily constant and there exists $\gamma$ in $\Gamma$ such that $\gamma(z) \in \Delta'$.

Furthermore, for any non-trivial $\gamma$ in $\Gamma$, the open set $\gamma(\mathring{\Delta}')$ is a subset of some open half-plane $\mathring{B}(a)$ with $a \in \mathcal{A}$ (see the argument from Property 1.4(i)), thus its intersection with $\mathring{\Delta}'$ is empty. □

**Exercise 3.21.** Verify that the surface $\Gamma\backslash\mathbb{H}$ is of the form given by Fig. II.14. (Hint: use the fact that since $\Delta'$ is locally finite, and that the function from $\Delta'$ modulo $\Gamma$ to in $\Gamma\backslash\mathbb{H}$, sending $\Gamma z \cap \Delta'$ to $\Gamma z$, is a homeomorphism (see Proposition I.2.17).)

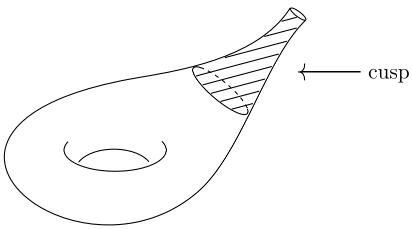


**Fig. II.14.**

Since the domain $\Delta'$ is not a Dirichlet domain, we cannot use the results of Chap. I to conclude that $\Gamma$ is a non-uniform lattice. This property is in fact true, but requires proof. Our chosen proof is not especially direct, but has the advantage of illustrating some interesting properties of the group and of using Criterion I.4.17: $\Gamma$ is a non-uniform lattice if and only if $L(\Gamma) = \mathbb{H}(\infty)$ and $L(\Gamma) = L_p(\Gamma) \cup L_h(\Gamma)$, with $L_p(\Gamma) \neq \varnothing$.

Let us consider the non-trivial translation

$$[\alpha_2^{-1}, \alpha_1^{-1}] = \alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1.$$

**Exercise 3.22.** Prove that $[\alpha_2^{-1}, \alpha_1^{-1}]$ generates the stabilizer of the point $\infty$ in $\Gamma$.

**Property 3.23.** *The parabolic isometries of $\Gamma$ are conjugate to powers of $[\alpha_2^{-1}, \alpha_1^{-1}]$ in $\Gamma$.*

*Proof.* Fix $z$ in $\mathring{\Delta}'$. Consider a parabolic isometry $\gamma$ in $\Gamma$ and write it in the form of a reduced word $s_1 \cdots s_n$, where $\mathcal{A} = \{\alpha_1^{\pm 1}, \alpha_2^{\pm 1}\}$. One may assume that $s_1 \neq s_n^{-1}$. In this case, $\lim_{k\to+\infty} \gamma^k(i)$ is contained in $B(s_1)$ and $\lim_{k\to+\infty} \gamma^{-k}(i)$ is contained in $B(s_n^{-1})$. These two limits are equal to the unique fixed point $x$ of $\gamma$, thus $x$ is an element of $\{-1, 0, 1, \infty\}$. If $x = \infty$, then $\gamma$ belongs to the group generated by $[\alpha_2^{-1}, \alpha_1^{-1}]$. Otherwise, since $1 = \alpha_1(\infty)$, $-1 = \alpha_2(\infty)$, and $0 = \alpha_2^{-1}\alpha_1(\infty)$, the element $\gamma$ is conjugate to a power of $[\alpha_2^{-1}, \alpha_1^{-1}]$. $\square$

**Exercise 3.24.** Let $H$ be a non-elementary Fuchsian group and $N$ be a normal subgroup. Prove that $L(H) = L(N)$.
(Hint: use the minimality of the limit set.)

**Property 3.25.** *The group $\Gamma$ is a non-uniform lattice and $L_p(\Gamma) = \mathbb{Q} \cup \{\infty\}$.*

*Proof.* The group $\Gamma$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$, thus one has $L(\Gamma) = \mathbb{H}(\infty)$. Let us examine $L_p(\Gamma)$. We know that the point $\infty$ is contained in this set. Furthermore, for all $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$, the Möbius transformation $\gamma[\alpha_2^{-1}, \alpha_1^{-1}]\gamma^{-1}$ is a parabolic isometry in $\Gamma$ which fixes $\gamma(\infty)$. Thus $L_p(\Gamma) = \mathbb{Q} \cup \{\infty\}$ (Property 3.8).

Consider now an irrational number $x$. This point is horocyclic with respect to $\mathrm{PSL}(2,\mathbb{Z})$, hence there exists a sequence $(\gamma_n)_{n\geqslant 1}$ in the modular group satisfying

$$\lim_{n\to+\infty} B_x(z, \gamma_n(z)) = +\infty.$$

Since $\Gamma$ has finite index in $\mathrm{PSL}(2,\mathbb{Z})$, passing to a subsequence one has $\gamma_n = g_n\gamma$, where $g_n$ is in $\Gamma$ and $\lim_{n\to+\infty} B_x(z, g_n(z)) = +\infty$. Thus $x$ is a horocyclic point with respect to $\Gamma$.

In conclusion,

$$L(\Gamma) = \mathbb{H}(\infty), \quad L_p(\Gamma) = \mathbb{Q} \cup \{\infty\} \quad \text{and} \quad L(\Gamma) = L_p(\Gamma) \cup L_h(\Gamma). \quad \square$$

## 4 Expansions of continued fractions

We have shown in the preceding section that $\mathbb{Q} \cup \{\infty\}$ is the set of parabolic points associated to the modular group. In this section, we continue to weave the relationships between number theory and hyperbolic geometry, by creating a geometric context for representations of irrational numbers $x$ as continued fractions.

We begin by recalling the algorithmic definition of this representation. We define $x_0 = x$ and $n_0 = E(x_0)$, where $E(x)$ designates the integer part of $x$. For all $i \geqslant 1$, define $x_i$ and $n_i$ by the following recurrence relation:

$$x_i = 1/(x_{i-1} - n_{i-1}) \quad \text{and} \quad n_i = E(x_i).$$

Let $[n_0; n_1, \ldots, n_k]$ denote the rational number defined by

$$[n_0; n_1, \ldots, n_k] = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ddots + \cfrac{1}{n_{k-1} + \cfrac{1}{n_k}}}}$$

The sequence of rational numbers $([n_0; n_1, \ldots, n_k])_{k \geqslant 1}$ converges to $x$ (see [42]). The *continued fraction expansion* of $x$ is by definition the expression of $x$ as $[n_0; n_1, \ldots]$.

### 4.1 Geometric interpretation of continued fraction expansions

Consider the hyperbolic triangle $T$ having infinite vertices at the points $\infty, 1, 0$. This triangle is related to the fundamental domain $\Delta$ of $\mathrm{PSL}(2, \mathbb{Z})$ introduced in Exercise 3.5 and defined to be

$$\Delta = \{z \in \mathbb{H} \mid 0 \leqslant \operatorname{Re} z \leqslant 1,\ |z| \geqslant 1 \text{ and } |z-1| \geqslant 1\}.$$

More precisely, if $r$ denotes the transformation defined by $r(z) = (z-1)/z$, one has (see Fig. II.15)

$$T = \Delta \cup r\Delta \cup r^2\Delta.$$

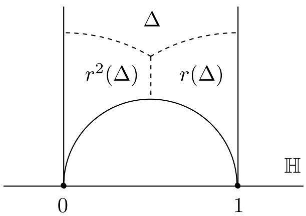


**Fig. II.15.**

It follows that

$$\bigcup_{\gamma \in \mathrm{PSL}(2,\mathbb{Z})} \gamma T = \mathbb{H} \quad \text{and if} \quad \gamma \mathring{T} \cap \mathring{T} \neq \varnothing, \quad \text{then } \gamma \in \{\mathrm{Id}, r, r^2\}.$$

This tiling of $\mathbb{H}$ by images of $T$ is called the *Farey tiling*. Let $L$ denote the unoriented geodesic whose endpoints are 0 and $\infty$. The images of $L$ under $\mathrm{PSL}(2,\mathbb{Z})$ are called *Farey lines*.

Recall that

$$\mathcal{T}_1(z) = z+1 \quad \text{and} \quad \mathcal{T}_{-1}(z) = z/(z+1).$$

The endpoints of $\mathcal{T}_1(L)$ are the points 1, $\infty$. Those of $\mathcal{T}_{-1}(L)$ are 0, 1. Thus the edges of $T$ and of $\gamma T$ for $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$, are Farey lines (Fig. II.16).

**Property 4.1.** *Let $L^+$ be the geodesic $L$ oriented from 0 to $\infty$. For any oriented Farey lines $(xy)$, there exists an unique $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$, such that $(xy) = \gamma(L^+)$.*

*Proof.* By definition, given an oriented Farey lines $(xy)$, there exists $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$ such that $\gamma(L)$ is the unoriented Farey line whose endpoints are $x$ and $y$. If $\gamma(0) = x$ and $\gamma(\infty) = y$, then $\gamma(L^+) = (xy)$. Otherwise, $\gamma s(L^+) = (xy)$, where $s(z) = -1/z$.

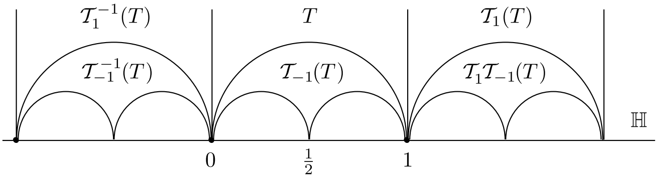


**Fig. II.16.**

It follows that there exists $g$ in $\mathrm{PSL}(2,\mathbb{Z})$ satisfying $g(L^+) = (xy)$. This element is unique since $\mathrm{PSL}(2,\mathbb{Z})$ does not contain any non-trivial isometry fixing the points 0 and $\infty$. $\square$

**Definition 4.2.** Let $n > 1$. A collection of $n$ oriented Farey lines $L_1^+, \dots, L_n^+$ are called *consecutive*, if there exists $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$ and $\varepsilon$ in $\{\pm 1\}$ such that for all $1 \leqslant i \leqslant n$

$$L_i^+ = \gamma \mathcal{T}_\varepsilon^i L^+.$$

Set $L_i^+ = (x_i y_i)$. Suppose that $L_1^+, \dots, L_n^+$ are consecutive. If $\varepsilon = 1$, then $y_i = \gamma(\infty)$ for all $1 \leqslant i \leqslant n$; likewise if $\varepsilon = -1$, then $x_i = \gamma(0)$ for all $1 \leqslant i \leqslant n$.

Figures II.17, II.18, II.19 show several cases in which three oriented Farey lines are consecutive.

If $x$ is an element of $\mathbb{H}(\infty)$, the irrationality of $x$ is characterized by the number of Farey lines which intersect the geodesic ray $[i, x)$.

**Proposition 4.3.** *Let $x$ be in $\mathbb{H}(\infty)$. The ray $[i, x)$ intersects finitely many Farey lines if and only if $x$ is in $\mathbb{Q} \cup \{\infty\}$.*

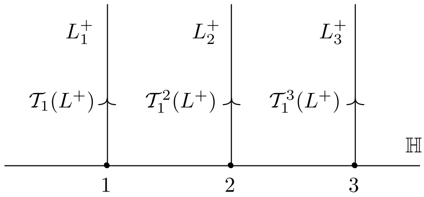


**Fig. II.17.**

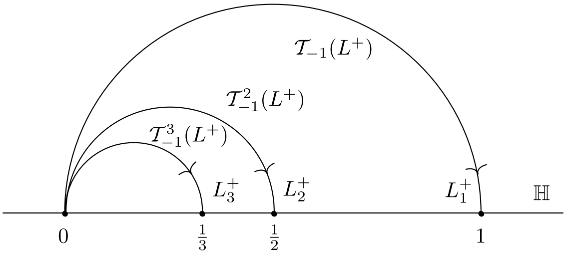


**Fig. II.18.**

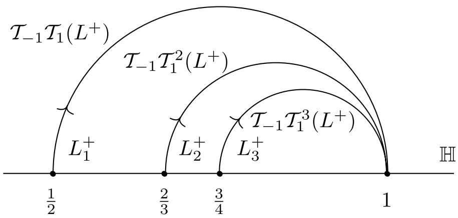


**Fig. II.19.**

*Proof.* Suppose that $x$ belongs to $\mathbb{Q} \cup \{\infty\}$. In this case, after Property 3.8(ii), there exists $\gamma$ in $\mathrm{PSL}(2, \mathbb{Z})$ such that $\gamma(x) = \infty$. Since the ray $\gamma^{-1}([i, x))$ is a vertical half-line passing through $\gamma^{-1}(i)$, there exists $n$ in $\mathbb{Z}$ and $z$ in $[i, x)$ such that the ray $\mathcal{T}_1^n \gamma^{-1}([z, x))$ is in $T$. The domain $\Delta$ is locally finite (since it is constructed from a finite number of subsets of a Dirichlet domain (Property 3.2)), thus $\mathcal{T}_1^n \gamma^{-1}([i, z]_h)$ intersects only a finite number of images of $T$ under $\mathrm{PSL}(2, \mathbb{Z})$.

It follows that $\mathcal{T}_1^n \gamma^{-1}([i, x))$—and thus $[i, x)$—only intersects finitely many Farey lines.

Suppose now that $[i, x)$ only intersects a finite number of Farey lines. Then there exists $z$ in $[i, x)$ and $\gamma$ in $\mathrm{PSL}(2, \mathbb{Z})$ such that $[z, x)$ is contained in $\gamma(T)$. In other words, $[\gamma^{-1}(z), \gamma^{-1}(x))$ is a geodesic ray contained in $T$. Hence, $\gamma^{-1}(x)$ is in $\{0, 1, \infty\}$ and thus $x$ is in $\mathbb{Q} \cup \{\infty\}$. $\square$

From now on, we focus on positive irrational numbers. Let $x$ be such a real number and $r : [0, +\infty) \to [i, x)$ be the arclength parametrization of $[i, x)$.

According to the previous proposition, $[i,x)$ crosses infinitely many Farey lines. Let $(L_n)_{n\geqslant 1}$ denote the sequence of Farey lines crossed by $(r(t))_{t>0}$ in order by increasing $t$. For each $n$, let $L_n^+ = (x_n y_n)$ be the orientation on $L_n$ defined by: at the point of intersection $r(t_n)$ between $[i,x)$ and $L_n$, the oriented angle from $[r(t_n),x)$ to $[r(t_n),y_n)$ belongs to $(0,\pi)$. If $L_n^+$ is not a vertical half-line, then $L_n^+=(x_n y_n)$ with $x_n<y_n$, if $L_n^+$ is vertical, then $y_n=\infty$ (Fig. II.20). Denote by $\gamma_n$ the unique element of $\mathrm{PSL}(2,\mathbb{Z})$ such that (Property 4.1)

$$\gamma_n(L^+) = L_n^+.$$

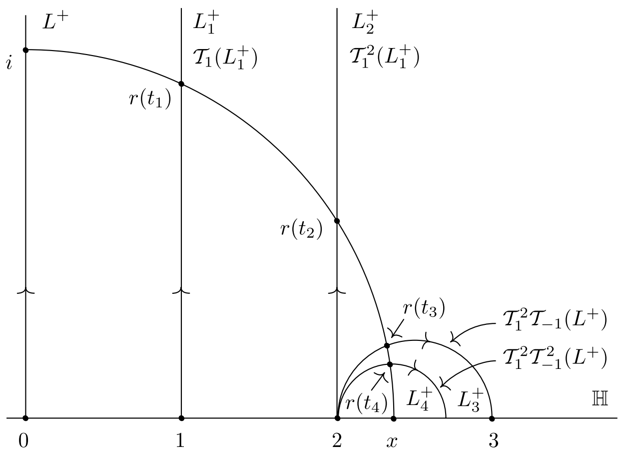


**Fig. II.20.**

Consider the geodesic ray $\gamma_n^{-1}([i,x))$. This ray crosses $L^+$ at the point $\gamma_n^{-1}(r(t_n))$. By construction, at their point of intersection $\gamma_n^{-1}(r(t_n))$, the angle oriented from $([\gamma_n^{-1}(r(t_n)),\gamma_n^{-1}(x))$ to $[\gamma_n^{-1}(r(t_n)),\infty)$ belongs to $(0,\pi)$. Thus $\gamma_n^{-1}([i,x))$ meets the Farey lines $\mathcal{T}_{-1}(L^+)=(01)$ or $\mathcal{T}_1(L^+)=(1\infty)$ and hence, $\gamma_n^{-1}(L_{n+1}^+)$ is equal to $\mathcal{T}_{-1}(L^+)$ or $\mathcal{T}_1(L^+)$. It follows that $\gamma_{n+1}=\gamma_n\mathcal{T}_{\varepsilon_n}$, where $\varepsilon_n=\pm 1$.

Set

$$n_0 = \begin{cases} \max\{n\geqslant 1 \mid \forall\, k\in[1,n],\ \varepsilon_k=1\} & \text{if } \varepsilon_1=1,\\ 0 & \text{if } \varepsilon_1=-1,\end{cases}$$

$$n_p = \max\{n>n_{p-1} \mid \forall\, k\in(n_{p-1},n],\ \varepsilon_k=(-1)^p\},\quad \text{if } p\geqslant 1.$$

Note that $n_k$ is a positive integer for all $k\geqslant 1$.

**Exercise 4.4.** Prove that, for all $k\geqslant 1$, the positive integer $n_k$ represents the largest integer $p\geqslant 1$ such that $L^+_{n_{p-1}+1},\dots,L^+_{n_{p-1}+p}$ are oriented consecutive Farey lines.

By construction of the integers $n_k$, we have for all $k \geqslant 1$:

$$\gamma_{n_k} = \mathcal{T}_1^{n_0} \cdots \mathcal{T}_{(-1)^k}^{n_k}.$$

Moreover, notice that, for $k \geqslant 1$, the intervals of $\mathbb{R}$ with extremities $\gamma_{n_k}(0)$ and $\gamma_{n_k}(\infty)$ are nested. Set $g_k = \gamma_{n_k}$. The next exercise relates the rational number $[n_0; n_1, \ldots, n_k]$, which was defined at the beginning of this section, to the point $g_k(0)$.

**Exercise 4.5.** Let $k \geqslant 2$. Prove that if $k$ is even, then $g_k(0) = [n_0; n_1, \ldots, n_k]$ and $g_k(\infty) = [n_0; n_1 \cdots n_{k-1}]$. Also if $k$ is odd, then $g_k(0) = [n_0; n_1, \ldots, n_{k-1}]$ and $g_k(\infty) = [n_0; n_1 \cdots n_k]$.
(Hint: use the relations $\mathcal{T}_1^n(z) = z + n$ and $\mathcal{T}_{-1}^n(z) = 1/(n + 1/z)$.)

The following proposition shows that $[n_0; n_1, \ldots]$ is the continued fraction expansion of $x$.

**Proposition 4.6.** *The sequence of rational numbers* $([n_0; n_1, \ldots, n_k])_{k \geqslant 1}$ *converges to* $x$. *In addition, if there exists a sequence* $(n'_k)_{k \geqslant 1}$ *satisfying* $n'_0 \in \mathbb{N}$, $n'_k \in \mathbb{N}^*$ *for all* $k \geqslant 1$ *and* $\lim_{k \to +\infty} [n'_0; n'_1, \ldots, n'_k] = x$, *then* $n_k = n'_k$ *for all* $k \geqslant 0$.

*Proof.* The geodesic $g_k(L)$ intersects the geodesic ray $[i, x)$. By construction, the point $x$ belongs to the interval of $\mathbb{R}$ having endpoints $g_k(0)$, $g_k(\infty)$, and these intervals are nested. For all $k \geqslant 1$, the rational numbers $g_k(0)$ and $g_k(\infty)$ belong to the interval $[n_0, n_0 + 1]$ (Exercise 4.5), thus $0 < |g_k(0) - g_k(\infty)| \leqslant 1$. Let us show that $\lim_{k \to +\infty} |g_k(0) - g_k(\infty)| = 0$. Suppose that there exists $d > 0$ and a subsequence $(g_{k_p})_{p \geqslant 1}$ such that $|g_{k_p}(0) - g_{k_p}(\infty)| > d$. In this case, the geodesic $g_{k_p}(L)$ intersects the Euclidean segment $I$ in $\mathbb{H}$ whose endpoints are $n_0 + id/2$ and $n_0 + 1 + id/2$. Since $L$ is an edge of $T$, and since $T$ is the finite union of images of $\varDelta$, there exist infinitely many isometries $\gamma$ in $\mathrm{PSL}(2, \mathbb{Z})$ such that $\gamma\varDelta$ intersects the compact set $I$. This contradicts the fact that $\varDelta$ is locally finite (since it is a finite union of subsets of a Dirichlet domain). One concludes from this property that the sequence $([n_0; n_1, \ldots, n_k])_{k \geqslant 1}$ converges to $x$.

Let us now show uniqueness. Suppose that $([n'_0; n'_1, \ldots, n'_k])_{k \geqslant 1}$ converges to $x$. Then

$$\lim_{p \to +\infty} \mathcal{T}_1^{n'_0} \cdots \mathcal{T}_1^{n'_{2p}}(0) = \lim_{p \to +\infty} \mathcal{T}_1^{n_0} \cdots \mathcal{T}_1^{n_{2p}}(0).$$

The rational number $\mathcal{T}_1^{n'_0} \cdots \mathcal{T}_1^{n'_{2p}}(0)$ is in $(n'_0, n'_0 + 1)$ and $\mathcal{T}_1^{n_0} \cdots \mathcal{T}_1^{n_{2p}}(0)$ is in $(n_0, n_0 + 1)$. Thus $n'_0 = n_0$. It follows that,

$$\lim_{p \to +\infty} \mathcal{T}_{-1}^{n'_1} \cdots \mathcal{T}_1^{n'_{2p}}(0) = \lim_{p \to +\infty} \mathcal{T}_{-1}^{n_1} \cdots \mathcal{T}_1^{n_{2p}}(0).$$

Applying the same reasoning to the sequences

$$\left(\frac{1}{[0; n'_1, \ldots, n'_{2p}]}\right)_{p \geqslant 1} = ([n'_1; n'_2, \ldots, n'_{2p}])_{p \geqslant 1} \quad \text{and}$$
$$\left(\frac{1}{[0; n_1, \ldots, n_{2p}]_{p \geqslant 1}}\right) = ([n_1; n_2, \ldots, n_{2p}])_{p \geqslant 1},$$

one obtains $n_1 = n'_1$. Iteratively, one has $n_k = n'_k$ for all $k \geqslant 0$. □

In summary, to find the continued fraction expansion of a positive irrational number $x$ in terms of hyperbolic geometry, it suffices to identify $x$ with a point in $\mathbb{H}(\infty)$, to associate to the ray $[i, x)$ the infinite sequence of oriented Farey lines $(L_i^+ = (x_i y_i))_{i \geqslant 1}$ in the order in which this ray crosses them per the procedure described above, and to count the maximal number of consecutive oriented Farey lines. Then

- $n_0 = E(x)$, and $n_1$ is defined by: $x_{n_0+1} = \cdots = x_{n_1} = n_0$, $x_{n_1+1} \neq n_0$;
- for all $k \geqslant 1$:
  - if $k$ is even, then $n_k$ is defined by $y_{n_{k-1}} = y_{n_{k-1}+1} = \cdots = y_{n_k}$ and $y_{n_k+1} \neq y_{n_k}$,
  - if $k$ is odd, then $n_k$ is defined by $x_{n_{k-1}} = x_{n_{k-1}+1} = \cdots = x_{n_k}$ and $x_{n_k+1} \neq x_{n_k}$.

**Examples 4.7.** Verify that in the situation of Fig. II.21, $n_0 = 2$, $n_1 = 2$, and $n_2 \geqslant 2$.

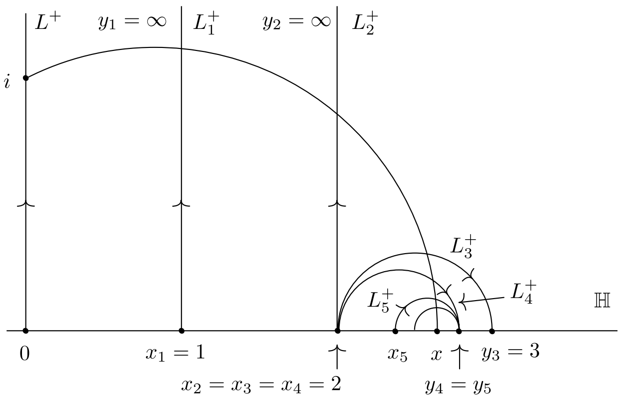


**Fig. II.21.**

Let $S^+$ denote the set of integer sequences defined by

$$S^+ = \{(n_i)_{i \geqslant 0} \mid n_0 \in \mathbb{N},\, n_i \in \mathbb{N}^* \text{ for } i \geqslant 1\}.$$

Consider the function $F : \mathbb{R}^+ - \mathbb{Q}^+ \to S^+$, which sends $x$ to the sequence $(n_i)_{i \geqslant 0}$ corresponding to the continued fraction expansion of $x$.

**Property 4.8.** *The map $F : \mathbb{R}^+ - \mathbb{Q}^+ \to S^+$ is bijective.*

*Proof.* According to Proposition 4.6, it is enough to show that $F$ is surjective. For any sequence $(n_k)_{k\geqslant 0}$ in $S^+$, set $g_k = \mathcal{T}_1^{n_0} \circ \cdots \circ \mathcal{T}_{(-1)^k}^{n_k}$. Reusing the argument from the proof of Proposition 4.6, one obtains $\lim_{k\to+\infty} |g_k(0) - g_k(\infty)| = 0$. Moreover the intervals with extremities $g_k(0)$ and $g_k(\infty)$ are nested. More precisely, if $k$ is even, then $g_{k+1}(\infty) = g_k(\infty)$, and $g_{k+1}(0)$ is in the segment having endpoints $g_k(\infty), g_k(0)$. If $k$ is odd, then $g_{k+1}(0) = g_k(0)$ and $g_{k+1}(\infty)$ is in the segment having endpoints $g_k(\infty), g_k(0)$. Thus the sequences $(g_k(0))_{k\geqslant 1}$ and $(g_k(\infty))_{k\geqslant 1}$ converge to the same positive real number $x$. Exercise 4.5 implies that $x = \lim_{k\to+\infty}[n_0; n_1, n_2, \ldots, n_k]$. The real $x$ is irrational since $[i, x)$ intersects each $g_k(L)$ (Property 4.3). $\square$

We have restricted ourselves to positive irrational number. However, if $y$ is a negative irrational, one may identify it with a sequence of integers $(m_i)_{i\geqslant 0}$, such that $m_0 = E(y) \in \mathbb{Z}$ and $(m_i)_{i\geqslant 1} = F(y - m_0)$.

In this way, one obtains a bijection between the set of irrational numbers with the set of sequences of integers whose terms are positive—with the possible exception of the first.

Let us come back to the geometry. Since the modular group is a non-uniform lattice, its limit set is the disjoint union of parabolic points and conical points. We know that conical points correspond to irrational numbers. In the same spirit as Property 2.4 for Schottky groups, let us prove that the continued fraction expansion of an irrational number $x$ is related to a sequence $(\gamma_k(i))_{k\geqslant 0}$ with $\gamma_k$ in $\mathrm{PSL}(2,\mathbb{Z})$, remaining at a bounded distance from the geodesic ray $[i, x)$. It suffices to prove this relation when $x$ is positive.

**Property 4.9.** *Let $x$ be a positive irrational number. Set $F(x) = (n_i)_{i\geqslant 0}$ and $\gamma_k = \mathcal{T}_1^{n_0} \cdots \mathcal{T}_{-1}^{n_{2k+1}}$ for all $k \geqslant 0$. The sequence $(\gamma_k(i))_{k\geqslant 0}$ remains at a uniformly bounded distance from the geodesic ray $[i, x)$.*

*Proof.* Let $s(z) = -1/z$. Note that the point $i$ belongs to the geodesic $(s(x)\, x)$. Since $-1/x < 0$ and $\mathrm{Re}\,\gamma_k(i) > 0$, proving Property 4.9 amounts to proving that the sequence $(\gamma_k(i))_{k\geqslant 0}$ remains at a uniformly bounded distance from the geodesic $(s(x)x)$, and hence that the sequence of couples of points $((\gamma_k^{-1}(s(x)), \gamma_k^{-1}(x)))_{k\geqslant 0}$ is contained in a compact subset of $\mathbb{H}(\infty) \times \mathbb{H}(\infty)$ minus its diagonal (Proposition I.3.14).

The continued fraction expansion of $\gamma_k^{-1}(x)$ is $[n_{2k+2}; n_{2k+3}, \ldots]$. Since $n_{2k+2}$ is non-zero, the real $\gamma_k^{-1}(x)$ is greater than 1. Furthermore, since $s\mathcal{T}_1^{-1}s = \mathcal{T}_{-1}$ and $s\mathcal{T}_{-1}^{-1}s = \mathcal{T}_1$, one has

$$s\gamma_k^{-1}(s(x)) = \mathcal{T}_1^{n_{2k+1}}\mathcal{T}_{-1}^{n_{2k}} \cdots \mathcal{T}_{-1}^{n_0}(x).$$

Hence, the continued fraction expansion of the real $s\gamma_k^{-1}s(x)$ is

$$\begin{cases} [n_{2k+1}; n_{2k}, \ldots, n_0, n_0, n_1, n_2, \ldots] & \text{if } n_0 \neq 0, \\ [n_{2k+1}; n_{2k}, \ldots, n_1, n_1, n_2, \ldots] & \text{otherwise.} \end{cases}$$

Since $n_{2k+1} \neq 0$, in both cases one obtains that the real $\gamma_k^{-1}(s(x))$ belongs to $(-1, 0)$. It follows that the geodesics $((\gamma_k^{-1}(x')\gamma_k^{-1}(x)))_{k\geqslant 0}$ remain at a bounded distance from $i$. □

## 4.2 Application to the hyperbolic isometries of the modular group

We focus here on positive irrational numbers such that the sequence $F(x) = (n_i)_{i\geqslant 0}$ is almost periodic (i.e., for some $k \geqslant 0$, the sequence $(n_{k+i})_{i\geqslant 0}$ is periodic). As with the coding of the conical points of a Schottky group, we are going to show that almost periodic sequences encode the fixed points of hyperbolic isometries of the modular group.

**Property 4.10.**

(i) *A positive irrational number $x$ is fixed by a hyperbolic isometry in* $\mathrm{PSL}(2,\mathbb{Z})$ *if and only if the sequence $F(x)$ is almost periodic.*
(ii) *An isometry in* $\mathrm{PSL}(2,\mathbb{Z})$ *is hyperbolic if and only if it is conjugate in* $\mathrm{PSL}(2,\mathbb{Z})$ *to an isometry of the form $\mathcal{T}_1^{m_1}\mathcal{T}_{-1}^{m_2}\cdots\mathcal{T}_{-1}^{m_k}$ with $m_i > 0$ and $k$ even.*

*Proof.*

(i) Let $x$ be a positive irrational number. Suppose that the sequence $F(x) = (n_i)_{i\geqslant 0}$ is periodic, in which case $n_0$ is non-zero. Let $T$ denote the period of this sequence, and define $k = T - 1$ if $T$ is even, and $k = 2T - 1$ otherwise. Then $x = \lim_{p\to+\infty}(\mathcal{T}_1^{n_0}\cdots\mathcal{T}_{-1}^{n_k})^p(0)$. This shows that $x$ is fixed by $\mathcal{T}_1^{n_0}\cdots\mathcal{T}_{-1}^{n_k}$, which is hyperbolic since $x$ is irrational.
If $F(x)$ is almost periodic, after $q$ initial terms for some $q \geqslant 1$, it suffices to apply the preceding reasoning to the point $(\mathcal{T}_1^{n_0}\cdots\mathcal{T}_{-1}^{n_q})^{-1}(x)$ if $q$ is odd, and to the point $(\mathcal{T}_1^{n_0}\cdots\mathcal{T}_{-1}^{n_{q-1}})^{-1}(x)$ if $q$ is even.
Consider now a hyperbolic isometry $\gamma$ in $\mathrm{PSL}(2,\mathbb{Z})$. Let $F(\gamma^+) = (n_i)_{i\geqslant 0}$ and set $g_k = \mathcal{T}_1^{n_0}\cdots\mathcal{T}_{(-1)^k}^{n_k}$. Recall that the geodesic ray $[i, x)$ meets all oriented Farey lines $L_k^+ = (g_k(0)g_k(\infty))$. According to Exercise 4.5, the sequences $(g_k(0))_{k\geqslant 0}$ and $(g_k(\infty))_{k\geqslant 0}$ converge to $\gamma^+$. Thus for large enough $k$, the Euclidean segment having endpoints $g_k(0), g_k(\infty)$ does not contain $\gamma^-$, and hence there exists $k' > k$ such that $\gamma L_k^+ = L_{k'}^+$. Applying Property 4.1, one obtains that $\gamma g_k = g_{k'}$. It follows that $\gamma = g_k\mathcal{T}_{(-1)^{(k+1)}}^{n_{k+1}}\cdots\mathcal{T}_{(-1)^{k'}}^{n'_k}g_k^{-1}$. If $k$ and $k'$ are both odd or even, then the sequence $F(g_k^{-1}(\gamma^+))$ is periodic. Otherwise, $F(\mathcal{T}_{(-1)^{(k+1)}}^{-n_{k+1}}g_k^{-1}(\gamma^+))$ is periodic. In both cases, the sequence $F(\gamma^+)$ is almost periodic.
(ii) Let $\gamma$ be a hyperbolic isometry in $\mathrm{PSL}(2,\mathbb{Z})$. After conjugating $\gamma$ by a translation, one may suppose that $\gamma^+ > 0$. According to the end of the proof of part (i), $\gamma$ is conjugate to $\mathcal{T}_1^{n_{k+1}}\cdots\mathcal{T}_{(-1)^{k'}}^{n'_k}$ if $k$ and $k'$ are both

odd or even, and to $\mathcal{T}_1^{n_{k+2}} \cdots \mathcal{T}_{(-1)^{k'}}^{n'_k}$, otherwise. Hence it is conjugate to an isometry of the form $\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_p}$, with $m_i > 0$ and $p$ even.
Conversely, an isometry of the form $\mathcal{T}_1^{m_1}\mathcal{T}_{-1}^{m_2} \cdots \mathcal{T}_{-1}^{m_k}$ with $m_i > 0$ and $k$ even, is hyperbolic since it fixes $\lim_{p\to+\infty}(\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_k})^p(0)$ and $\lim_{p\to-\infty}(\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_k})^p(0)$, which are distinct real numbers. □

This property allows us to establish a relationship between the fixed points of hyperbolic isometries of the modular group and the *quadratic real numbers*, which are solutions to equations like

$$ax^2 + bx + c = 0 \quad \text{with } a \in \mathbb{N}^* \text{ and } b, c \in \mathbb{Z}.$$

**Proposition 4.11.** *Let $x$ be an irrational number. Then the following are equivalent*:

(i) *$x$ is fixed by a hyperbolic isometry in* $\mathrm{PSL}(2,\mathbb{Z})$;
(ii) *$x$ is quadratic.*

We give a proof of this well-known result (see for example [42]) using the transformations $\mathcal{T}_1$ and $\mathcal{T}_{-1}$.

*Proof.* The implication from (i) to (ii) requires only two facts. The first one is that a fixed point $x$ of a Möbius transformation $\gamma(z) = (az+b)/(cz+d)$ satisfies the Diophantine equation

$$Ax^2 + Bx - C = 0,$$

with $A = c, B = d - a$ and $C = -b$. The second one is that the integer $c$ is non-zero since $\gamma$ is hyperbolic.

Let us now prove that (ii) implies (i). Let $\alpha$ and $\beta$ be two distinct roots of an equation of the form

$$Ax^2 + Bx - C = 0,$$

with $A \neq 0$ and $B$, $C$ in $\mathbb{Z}$.

We want to show that these two real numbers are fixed by some hyperbolic isometry of the modular group. After replacing them by $g(\alpha)$ and $g(\beta)$, where $g$ is in $\mathrm{PSL}(2,\mathbb{Z})$, one may assume that $\alpha > 0$ and $\beta < 0$. Hence $A > 0$ and $C > 0$. Set $F(\alpha) = (n_i)_{i\geqslant 0}$. For all even integer $k > 0$, define the real numbers

$$x_k = (\mathcal{T}_1^{n_0} \cdots \mathcal{T}_{-1}^{n_{k-1}})^{-1}(\alpha) \quad \text{and} \quad y_k = (\mathcal{T}_1^{n_0} \cdots \mathcal{T}_{-1}^{n_{k-1}})^{-1}(\beta),$$

and set $x_0 = \alpha$, $y_0 = \beta$.

We have $x_k = \lim_{p\to+\infty}[n_k; n_{k+1}, \ldots, n_{k+p}]$, hence $x_k$ is positive. Furthermore, an induction argument shows that $y_k$ is negative and that the two real numbers $x_k$ and $y_k$ are solutions of an equation of the form

$$A_k x^2 + B_k x - C_k = 0,$$

where $A_k, B_k, C_k \in \mathbb{Z}$, $A_k > 0$, $C_k > 0$ and $B_k^2 + 4A_kC_k = B^2 + 4AC$. Thus the coefficients $A_k, B_k, C_k$ belong to a finite set. It follows that there exist two even integers $k_2 > k_1 \geqslant 0$ such that $A_{k_1} = A_{k_2}, B_{k_1} = B_{k_2}, C_{k_1} = C_{k_2}$. This implies that $x_{k_1} = x_{k_2}$, and hence

$$\mathcal{T}_1^{n_{k_1}} \cdots \mathcal{T}_{-1}^{n_{k_2-1}}(x_{k_2}) = x_{k_2}.$$

We obtain that the real number $x_{k_2}$ is the fixed point of the hyperbolic isometry $g' = \mathcal{T}_1^{n_{k_1}} \cdots \mathcal{T}_{-1}^{n_{k_2-1}}$. Since $\alpha = \mathcal{T}_1^{n_0} \cdots \mathcal{T}_{-1}^{n_{k_2-1}}(x_{k_2})$, this real is fixed by a conjugate of $g'$. The same reasoning applied to $y_{k_2}$ implies that $\beta$ is similarly fixed by the same hyperbolic isometry. □

We conclude this section by focusing on the displacements of isometries in $\mathrm{PSL}(2,\mathbb{Z})$. Recall that the displacement $\ell(\gamma)$ of an isometry $\gamma$ is defined (see I.2.2) by

$$\ell(\gamma) = \inf_{z \in \mathbb{H}} d(z, \gamma(z)).$$

**Exercise 4.12.** Let $\gamma(z) = (az+b)/(cz+d)$ be a hyperbolic isometry in $G$. Denote $\lambda$ the eigenvalue of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ whose absolute value is $> 1$. Prove the equality

$$\ell(\gamma) = 2\ln|\lambda|.$$

The following property relates the fixed point of a hyperbolic isometry in $\mathrm{PSL}(2,\mathbb{Z})$ to its displacement. Let $\sigma : (\mathbb{R} - \mathbb{Q}) \cap (1, +\infty) \to (\mathbb{R} - \mathbb{Q}) \cap (1, +\infty)$ defined by

$$\sigma(x) = \frac{1}{x - n_0},$$

where $n_0$ is the first term of the sequence $F(x)$. Notice that the sequence $F(\sigma(x))$ is the shifted sequence $(n_{i+1})_{i \geqslant 0}$.

**Property 4.13.** *If $\gamma$ is an isometry of the form $\gamma = \mathcal{T}_1^{m_1}\mathcal{T}_{-1}^{m_2} \cdots \mathcal{T}_{-1}^{m_k}$, with $m_i$ in $\mathbb{N}^*$ and $k$ even, then*

$$\ell(\gamma) = 2\ln(\gamma^+ \times \sigma(\gamma^+) \times \cdots \times \sigma^{k-1}(\gamma^+)).$$

*Proof.* Set $\gamma(z) = (az+b)/(cz+d)$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\lambda = e^{\ell(\gamma)/2}$. We have

$$M \begin{pmatrix} \gamma^+ \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} \gamma^+ \\ 1 \end{pmatrix}.$$

Consider the matrices $D_n = \left(\begin{smallmatrix} 0 & 1 \\ 1 & n \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. These matrices satisfy the following relations:

$$R^2 = \mathrm{Id}, \quad D_n R = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \quad \text{and} \quad R D_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Using these relations, one obtains

$$\lambda \begin{pmatrix} 1 \\ \gamma^+ \end{pmatrix} = D_{m_1} \cdots D_{m_k} \begin{pmatrix} 1 \\ \gamma^+ \end{pmatrix}.$$

If $x \neq 0$, then $D_m \left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right) = x \left(\begin{smallmatrix} 1 \\ m+1/x \end{smallmatrix}\right)$. Therefore, $D_{m_k} \left(\begin{smallmatrix} 1 \\ \gamma^+ \end{smallmatrix}\right) = \gamma^+ \left(\begin{smallmatrix} 1 \\ m_k+1/\gamma^+ \end{smallmatrix}\right)$. However, $m_k + 1/\gamma^+ = \sigma^{k-1}(\gamma^+)$, hence

$$\lambda \begin{pmatrix} 1 \\ \gamma^+ \end{pmatrix} = \gamma^+ D_{m_1} \cdots D_{m_{k-1}} \begin{pmatrix} 1 \\ \sigma^{k-1}(\gamma^+) \end{pmatrix}.$$

Repeating this process, one obtains

$$\lambda \begin{pmatrix} 1 \\ \gamma^+ \end{pmatrix} = \gamma^+ \sigma^{k-1}(\gamma^+) \cdots \sigma^2(\gamma^+) D_{m_1} \begin{pmatrix} 1 \\ \sigma(\gamma^+) \end{pmatrix}.$$

Furthermore, $D_{m_1} \left(\begin{smallmatrix} 1 \\ \sigma(\gamma^+) \end{smallmatrix}\right) = \sigma(\gamma^+) \left(\begin{smallmatrix} 1 \\ \gamma^+ \end{smallmatrix}\right)$, thus $\lambda = \gamma^+ \prod_{i=1}^{k-1} \sigma^i(\gamma^+)$. □

Using this property, one obtains an interpretation—in terms of hyperbolic geometry—of the *golden ratio*

$$\mathcal{N} = \frac{1+\sqrt{5}}{2}.$$

**Corollary 4.14.** *If* $\gamma$ *is a hyperbolic isometry in* $\mathrm{PSL}(2, \mathbb{Z})$, *then*

$$\ell(\gamma) \geqslant 2 \ln(\mathcal{T}_1 \mathcal{T}_{-1})^+ = 4 \ln \mathcal{N}.$$

*Proof.* Suppose $\gamma$ is a hyperbolic isometry in $\mathrm{PSL}(2, \mathbb{Z})$. According to Property 4.10(ii), we can suppose that this isometry is of the form $\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_k}$, with $m_i > 0$. Let us prove that $\ell(\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_k}) > 4 \ln \mathcal{N}$, if some $m_i \neq 1$. Let $x$ be the attractive fixed point of this isometry, the sequence $F(x)$ is the periodic sequence $m_1, \ldots, m_k, m_1, \ldots, m_k, m_1, \ldots$. For $1 \leqslant i \leqslant k$, notice that the real $\sigma^i(x)$ is of the form $m_{i+1} + 1/(m_{i+2} + x_i)$, where $0 < x_i < 1$. Therefore, if one of the $m_i$ is equal to 2, there exist $j, l$ with $0 \leqslant j, l \leqslant k-1$ and $j \neq l$ such that $\sigma^j(x) = 2 + 1/y$, where $y > 1$, and $\sigma^l(x) = m_{l+1} + 1/(2 + x_l)$, with $0 < x_l < 1$. From these remarks and Property 4.13, one obtains

$$\ell(\gamma) > 2 \ln(2 \times (1 + 1/3)) = 2 \ln 8/3 > 4 \ln \mathcal{N}.$$

If one of the $m_i$ is $\geqslant 3$, then $\ell(\gamma) \geqslant 2 \ln 3 > 4 \ln \mathcal{N}$.

Suppose now that all of the $m_i$ are 1, then $\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_k}$ is a power of $\mathcal{T}_1 \mathcal{T}_{-1}$. The attractor $x$ of $\mathcal{T}_1 \mathcal{T}_{-1}$ satisfies $x = 1 + 1/x$, hence $x = \mathcal{N}$ and, after Property 4.13, one has $\ell(\mathcal{T}_1 \mathcal{T}_{-1}) = 2 \ln \mathcal{N}^2$.

In conclusion, $\ell(\mathcal{T}_1^{m_1} \cdots \mathcal{T}_{-1}^{m_k}) > 4 \ln \mathcal{N}$, except if $k = 2$ and $m_1 = m_2 = 1$. □

Corollary 4.14 has a geometric interpretation on the modular surface $S = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$. We will see in Chap. III that the set of all $\ell(\gamma)$, where $\gamma$ is a hyperbolic isometry of $\mathrm{PSL}(2,\mathbb{Z})$, is the set of the lengths of all compact geodesics in the surface $S$ (see Sect. III.3 and Appendix B for the notion of a geodesic on $S$). In this context, Corollary 4.14 says that the real $4\ln\mathcal{N}$ corresponds to the length of the shortest compact geodesic on the modular surface.

## 5 Comments

The construction of Schottky groups and the coding of their limit sets that have been introduced in this chapter can be generalized to pinched Hadamard manifolds $X$ (see the Comments in Chap. I) whose group of positive isometries contains at least two non-elliptic elements $g_1, g_2$ having no common fixed points. Under this condition, for large enough $n_0$, there exist two disjoint compact subsets $K_1$ and $K_2$ of $X(\infty)$ satisfying the following relation for all $i = 1, 2$ and $|n| \geqslant n_0$:

$$g_i^n(X(\infty) - K_i) \subset K_i.$$

An application of the *Ping-Pong Lemma* [36] shows that the group generated by $g_1^{n_0}, g_2^{n_0}$ is a free Kleinian group. Such a group is geometrically finite [21]. Without any other hypotheses on $X$, these groups—which are again called *Schottky groups*—and their variants are in general the only accessible non-elementary Kleinian groups.

On the other hand, if one restricts to the case in which $X$ is a symmetric space of rank 1, one can construct Kleinian groups (in general, lattices) using number theory.

In the particular case of the Poincaré half-plane, the arithmetic groups are known [41, Chap. 5]. The modular groups and $\Gamma(2)$ belong to this rich family. Let us mention (see [45]) a rather unexpected example of a lattice contained in the group of Möbius transformations having rational coefficients which is not commensurable to the modular group but for which the set of parabolic points is still $\mathbb{Q} \cup \{\infty\}$.

The geometric construction of continued fraction expansions that was introduced in this chapter is essentially taken from two articles [57, 20]. In addition to these two references, the text of C. Series in [8] also studies the limit set of Schottky groups and the modular group, but goes further toward the construction of a coding of the limit set of an arbitrary geometrically finite Fuchsian group.

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