Preface

In this text, we present an introduction to the topological dynamics of two classical flows associated with surfaces of curvature -1, namely the geodesic and horocycle flows. Since the end of the nineteenth century, many texts have been written on this subject.

Why have we undertaken this project?

In the course of several talks that we have given on this topic, notably during some summer workshops which were organized by the University of Savoie, we have often regretted not being able to recommend a book to those who wanted to find out about this area for themselves. Due to their determination and their enthusiasm for the subject, we decided to go beyond the stage of regret and write up our notes.

In the past thirty years, some very strong connections have been established between dynamical systems and number theory. The intersection of these two fields relies on a change in point of view which, in dimension 2, essentially consists of considering a real number to be a point on the *boundary at infinity* of the Poincaré half-plane and of associating this point to a geodesic, on the modular surface, pointing in its direction (Sect. VII.3). This case study is still the source of inspiration for a large number of specialists. It is sometimes so present in our minds that it is absent from our texts. One of our motivations has been to put this idea back in the spotlight.

Who does this book address?

The reader is expected to have some knowledge of differential geometry and topological dynamics. Our goal has been to produce a text which is readable by a motivated student. Experts in other areas who are interested in this subject have also been considered.

We have attempted to keep the reader from being overwhelmed by proofs which are either too detailed or too succinct by punctuating the text with exercises.

In what spirit was the text written?

This text has been written primarily with the idea of highlighting, in a relatively elementary framework, the existence of gateways between some mathematical fields, and the advantages of using them.

We have chosen not to address the historical aspects of this field and to reserve most of the references until the end of each chapter in the Comments section. Some of our proofs have been borrowed from the literature. In some cases, we have worked to simplify those proofs.

Since the degree of difficulty (or of simplicity) of each chapter is relatively similar, the unfolding of this text may not appear to reach a climax. The applications are the true focus of its progression.

What does the text cover?

We begin with a chapter introducing the geometry of the hyperbolic plane and the dynamics of *Fuchsian groups*, inspired by S. Katok's book "Fuchsian Groups" [41], and A. Katok's and V. Climenhaga's "Lectures on Surfaces" [39]. The action of a Fuchsian group on the Poincaré half plane $\mathbb{H}$ is properly discontinuous. If it is not finite, its orbits accumulate on the boundary at infinity $\mathbb{H}(\infty)$ of $\mathbb{H}$ in a constellation of points called the *limit set* of the group. We focus on several ways in which these points are approximated, which requires us to define *conical* and *parabolic* points, and to introduce the notion of *geometrically finite* groups (Sect. I.4).

In Chap. II, we study some examples of these groups with special attention given to *Schottky groups* and the *modular group*. In each of these cases, we construct a method of encoding their limit sets into sequences and establish a one-to-one correspondence between certain geometric properties of limit points and other combinatorial properties of sequences. For example, the coding introduced for the modular group allows us to interpret the continued fraction expansion of the real numbers in terms of hyperbolic geometry. This coding also allows us to identify a relationship between the *golden ratio* and the length of the shortest compact geodesic on the *modular surface* (Sect. II.4).

In Chap. III, we study the topological dynamics of the geodesic flow $g_{\mathbb{R}}$ on the quotient T^1S of the unitary tangent bundle of $\mathbb{H}$ by a Fuchsian group Γ. The main idea here is to connect the dynamics of this flow to that of the action of Γ on $\mathbb{H}(\infty)$. We show that if Γ is not elementary, the set of periodic elements with respect to $g_{\mathbb{R}}$ is dense in the *non-wandering set* $\Omega_g(T^1S)$ of this flow, and that there exist some trajectories which are dense in $\Omega_g(T^1S)$ (Sects. III.3 and III.4). Also, when Γ is geometrically finite, we construct a compact set which intersects every trajectory in $\Omega_g(T^1S)$ (Sect. III.2).

In Chap. IV, we restrict ourselves to the case in which the Fuchsian group is a Schottky group. Using the coding of its limit set constructed in Chap. II, we develop a symbolic approach which allows us to study the topology of trajectories of the geodesic flow on $\Omega_g(T^1S)$ and to appreciate its complexity. For example, we construct some trajectories in $\Omega_g(T^1S)$ which are neither compact nor dense, and we obtain, in the general case of non-elementary Fuchsian groups, the existence of non-periodic, minimal compact sets which are invariant with respect to the geodesic flow (Sect. IV.3).

Chapter V is devoted to the study of the horocycle flow $h_{\mathbb{R}}$ on T^1S. The method that we use relies on a correspondence between horocycles of $\mathbb{H}$ and non-zero vectors in $\mathbb{R}^2$, modulo $\pm\,\mathrm{Id}$. This vectorial point of view allows us to link the action of $h_{\mathbb{R}}$ on T^1S to that of a linear group on a vector space, and to determine, for example, the existence of trajectories which are dense in the non-wandering set $\Omega_h(T^1S)$ of this flow (Sects. V.2 and V.3). When the group Γ is geometrically finite, the dynamics of the horocycle flow, unlike that of the geodesic flow, is simple since a trajectory in $\Omega_h(T^1S)$ is either dense or periodic (Sect. V.4). Despite this very different behavior, these two flows are intimately related in the sense that the flow $h_{\mathbb{R}}$ reflects the collective behavior of asymptotic trajectories of the flow $g_{\mathbb{R}}$.

The last two chapters are dedicated to some applications of the study of these flows, one in the area of linear actions, the other in that of Diophantine approximations.

In Chap. VI, we focus on the *Lorentz* space $\mathbb{R}^3$ equipped with a bilinear form of signature $(2,1)$. We connect the topology of orbits of a discrete group G of orthogonal transformations of this form to that of the trajectories of the geodesic and horocycle flows on the quotient of the unitary tangent bundle of $\mathbb{H}$ over a Fuchsian group. Translating the results proved about the horocycle flow into this vectorial context, one obtains, for example, a complete description of the orbits of G located in the *light cone*, when this group is of finite type (Sect. VI.3).

In Chap. VII, we translate the Diophantine approximation of a real number by rational ones into the terms of hyperbolic geometry. Relying on the dynamics of the geodesic flow on the modular surface, we rediscover among other things that a real number is *badly approximated* if and only if the coefficients involved in its continued fraction expansion are bounded (Sect. VII.3).

We have chosen to make geometric simplicity a priority and to avoid discussing the metric aspects of these flows. As such, we have limited the scope of some statements and have occasionally hidden some important ideas in these arguments. So that the reader does not come away with the impression that we have said everything there is to say about these subjects, at the end of each chapter we have added some comments in which we recast our treatment of the material into a general Riemannian context and introduce the reader to the vast field of ergodic geometry. We conclude these comments with some open problems as a reminder that this general area has its share of unknown answers and that it has a place in contemporary research.

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