Chapter 2
Center Manifolds

This chapter is devoted to center manifold theory. We present a general result on the existence of local center manifolds for infinite-dimensional systems in Section 2.2 and then discuss several particular cases and extensions, as, for instance, to parameter-dependent systems and systems possessing different symmetries in Section 2.3. We give a series of examples showing how these results apply to various situations in Section 2.2.4 and in Section 2.4. A brief description of the tools and results from the theory of linear operators needed in this chapter is given in Appendix A.

2.1 Notations

Consider two (complex or real) Banach spaces $\mathscr{X}$ and $\mathscr{Z}$. Throughout this chapter we shall use the following notations:

- $B_\varepsilon(\mathscr{X})$ is the closed ball $\{u \in \mathscr{X}\,;\, \|u\|_{\mathscr{X}} \leq \varepsilon\}$.
- $\mathscr{C}^k(\mathscr{Z},\mathscr{X})$ is the Banach space of k-times continuously differentiable functions $F : \mathscr{Z} \to \mathscr{X}$ equipped with the sup norm on all derivatives up to order k,

$$\|F\|_{\mathscr{C}^k} = \max_{j=0,\dots,k}\left(\sup_{y\in\mathscr{Z}}\left(\|D^jF(y)\|_{\mathscr{L}(\mathscr{Z}^j,\mathscr{X})}\right)\right);$$

  here, and in the following, D denotes the differentiation operator.
- For a positive constant $\eta > 0$, we define the space of exponentially growing functions

$$\mathscr{C}_\eta(\mathbb{R},\mathscr{X}) = \{u \in \mathscr{C}^0(\mathbb{R},\mathscr{X})\,;\, \|u\|_{\mathscr{C}_\eta} = \sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\|u(t)\|_{\mathscr{X}}\right) < \infty\},$$

  which is a Banach space when equipped with the norm $\|\cdot\|_{\mathscr{C}_\eta}$; we also consider the Banach space

M. Haragus, G. Iooss, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*, Universitext,
DOI 10.1007/978-0-85729-112-7_2, © EDP Sciences 2011

$$\mathscr{F}_\eta(\mathbb{R},\mathscr{X}) = \{u \in \mathscr{C}^0(\mathbb{R},\mathscr{X})\ ;\ \|u\|_{\mathscr{F}_\eta} = \sup_{t\in\mathbb{R}} \left(e^{\eta t}\|u(t)\|_{\mathscr{X}}\right) < \infty\},$$

equipped with the norm $\|\cdot\|_{\mathscr{F}_\eta}$, of functions which may grow exponentially at $-\infty$ and which tend towards 0 exponentially at $+\infty$. Notice that $\mathscr{F}_\eta(\mathbb{R},\mathscr{X}) \subset \mathscr{C}_\eta(\mathbb{R},\mathscr{X})$ with continuous embedding.

- $\mathscr{L}(\mathscr{Z},\mathscr{X})$ is the Banach space of linear bounded operators $\mathbf{L}:\mathscr{Z}\to\mathscr{X}$, equipped with the operator norm

$$\|\mathbf{L}\|_{\mathscr{L}(\mathscr{Z},\mathscr{X})} = \sup_{\|u\|_{\mathscr{Z}}=1} \left(\|\mathbf{L}u\|_{\mathscr{X}}\right).$$

If $\mathscr{Z}=\mathscr{X}$, we write $\mathscr{L}(\mathscr{X}) = \mathscr{L}(\mathscr{X},\mathscr{X})$.

- For a linear operator $\mathbf{L}:\mathscr{Z}\to\mathscr{X}$, we denote by $\operatorname{im}\mathbf{L}$ its *range*,

$$\operatorname{im}\mathbf{L} = \{\mathbf{L}u \in \mathscr{X}\ ;\ u\in\mathscr{Z}\} \subset \mathscr{X},$$

and by $\ker\mathbf{L}$ its *kernel*,

$$\ker\mathbf{L} = \{u\in\mathscr{Z}\ ;\ \mathbf{L}u = 0\} \subset \mathscr{Z}.$$

- Assume that $\mathscr{Z}\hookrightarrow\mathscr{X}$ with continuous embedding. For a linear operator $\mathbf{L}\in\mathscr{L}(\mathscr{Z},\mathscr{X})$ we denote by $\rho(\mathbf{L})$, or simply ρ, if there is no risk of confusion, the *resolvent set* of $\mathbf{L}$,

$$\rho = \{\lambda\in\mathbb{C}\ ;\ \lambda\mathbb{I}-\mathbf{L}:\mathscr{Z}\to\mathscr{X} \text{ is bijective}\}.$$

The complement of the resolvent set is the *spectrum* $\sigma(\mathbf{L})$, or simply σ,

$$\sigma = \mathbb{C}\setminus\{\rho\}.$$

Notice that when the operator $\mathbf{L}$ is real, the resolvent set and the spectrum of $\mathbf{L}$ are both symmetric with respect to the real axis in the complex plane.

2.2 Local Center Manifolds

In this section we present the main result on the existence of local center manifolds. We discuss the hypotheses in Section 2.2.1, and then in Section 2.2.3, and state the main theorem in Section 2.2.2. The proof of the theorem is given in Appendix B.1.

2.2.1 Hypotheses

Let $\mathscr{X}$, $\mathscr{Z}$, $\mathscr{Y}$ be (real or complex) Banach spaces such that

$$\mathscr{Z} \hookrightarrow \mathscr{Y} \hookrightarrow \mathscr{X},$$

with continuous embeddings. We consider a differential equation in $\mathscr{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u), \tag{2.1}$$

in which we assume that the linear part $\mathbf{L}$ and the nonlinear part $\mathbf{R}$ are such that the following holds.

Hypothesis 2.1 *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (2.1) have the following properties:*

(i) $\mathbf{L} \in \mathscr{L}(\mathscr{Z}, \mathscr{X})$;
(ii) for some $k \geq 2$, *there exists a neighborhood* $\mathscr{V} \subset \mathscr{Z}$ *of* 0 *such that* $\mathbf{R} \in \mathscr{C}^k(\mathscr{V}, \mathscr{Y})$ *and*

$$\mathbf{R}(0) = 0, \quad D\mathbf{R}(0) = 0.$$

Remark 2.2 *The condition* $\mathbf{R}(0) = 0$ *means that* 0 *is an equilibrium of the differential equation (2.1), and the condition* $D\mathbf{R}(0) = 0$ *then shows that* $\mathbf{L}$ *is the linearization of the vector field about* 0, *so that* $\mathbf{R}$ *represents the nonlinear terms which are* $O(\|u\|^2_{\mathscr{Z}})$. *More generally, for an equation which has a nonzero equilibrium,* u_*, *say, we recover these conditions after replacing* u *by* $u - u_*$ *and then taking for* $\mathbf{L}$ *the differential of the resulting vector field at* 0.

Definition 2.3 *A* solution *of the differential equation (2.1) is a function* $u : \mathscr{I} \to \mathscr{Z} \hookrightarrow X$ *defined on an interval* $\mathscr{I} \subset \mathbb{R}$, *with the following properties:*

(i) the map $u : \mathscr{I} \to \mathscr{Z}$ *is continuous;*
(ii) the map $u : \mathscr{I} \to \mathscr{X}$ *is continuously differentiable;*
(iii) the equality (2.1) holds in $\mathscr{X}$ *for all* $t \in \mathscr{I}$.

Besides Hypothesis 2.1, we make two further assumptions on the linear operator $\mathbf{L}$, which are essential for the center manifold theorem.

Hypothesis 2.4 (Spectral decomposition) *Consider the spectrum* σ *of the linear operator* $\mathbf{L}$, *and write*

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-,$$

in which

$$\sigma_+ = \{\lambda \in \sigma\,;\, \mathrm{Re}\,\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma\,;\, \mathrm{Re}\,\lambda = 0\}, \quad \sigma_- = \{\lambda \in \sigma\,;\, \mathrm{Re}\,\lambda < 0\}.$$

We assume that

(i) there exists a positive constant $\gamma > 0$ such that

$$\inf_{\lambda \in \sigma_+} (\mathrm{Re}\,\lambda) > \gamma, \quad \sup_{\lambda \in \sigma_-} (\mathrm{Re}\,\lambda) < -\gamma;$$

(ii) the set σ_0 *consists of a finite number of eigenvalues with finite algebraic multiplicities.*

Remark 2.5 *(i) The sets σ_+, σ_0, and σ_- are called* unstable, central, and stable spectrum, *respectively.*

(ii) The hypothesis above implies that the resolvent set ρ of $\mathbf{L}$ is not empty. This further implies that $\mathbf{L}$ is a closed operator in $\mathscr{X}$. Indeed, for some $\lambda \in \rho$, the operator $\lambda\mathbb{I} - \mathbf{L}$ is bijective, and since $\mathbb{I}$ and $\mathbf{L}$ belong to $\mathscr{L}(\mathscr{Z}, \mathscr{X})$, by the closed graph theorem the resolvent $(\lambda\mathbb{I} - \mathbf{L})^{-1}$ belongs to $\mathscr{L}(\mathscr{X}, \mathscr{Z})$. Now $\mathscr{L}(\mathscr{X}, \mathscr{Z}) \subset \mathscr{L}(\mathscr{X})$, so that $(\lambda\mathbb{I} - \mathbf{L})^{-1} \in \mathscr{L}(\mathscr{X})$ and then by the closed graph theorem $\lambda\mathbb{I} - \mathbf{L}$ is closed in $\mathscr{X}$. Consequently, $\mathbf{L}$ is closed in $\mathscr{X}$.

As a consequence of Hypothesis 2.4(ii), we can define the (spectral) projection $\mathbf{P}_0 \in \mathscr{L}(\mathscr{X})$, corresponding to σ_0, by the Dunford integral formula

$$\mathbf{P}_0 = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbb{I} - \mathbf{L})^{-1} \mathrm{d}\lambda, \tag{2.2}$$

where Γ is a simple, oriented counterclockwise, Jordan curve surrounding σ_0 and lying entirely in $\{\lambda \in \mathbb{C}\,;\, |\mathrm{Re}\,\lambda| < \gamma\}$. Then

$$\mathbf{P}_0^2 = \mathbf{P}_0, \quad \mathbf{P}_0\mathbf{L}u = \mathbf{L}\mathbf{P}_0 u \text{ for all } u \in \mathscr{Z},$$

and the range $\mathrm{im}\,\mathbf{P}_0$ is finite-dimensional, since σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities. In particular, it satisfies $\mathrm{im}\,\mathbf{P}_0 \subset \mathscr{Z}$, and

$$\mathbf{P}_0 \in \mathscr{L}(\mathscr{X}, \mathscr{Z}),$$

since the map $\lambda \mapsto (\lambda\mathbb{I} - \mathbf{L})^{-1} \in \mathscr{L}(\mathscr{X}, \mathscr{Z})$ is analytic in a neighborhood of Γ.

We define a second projection $\mathbf{P}_h : \mathscr{X} \to \mathscr{X}$ by

$$\mathbf{P}_h = \mathbb{I} - \mathbf{P}_0,$$

which then also satisfies

$$\mathbf{P}_h^2 = \mathbf{P}_h, \quad \mathbf{P}_h\mathbf{L}u = \mathbf{L}\mathbf{P}_h u \text{ for all } u \in \mathscr{Z},$$

and

$$\mathbf{P}_h \in \mathscr{L}(\mathscr{X}) \cap \mathscr{L}(\mathscr{Z}) \cap \mathscr{L}(\mathscr{Y}),$$

since $\mathbf{P}_0 \in \mathscr{L}(\mathscr{X}, \mathscr{Z})$ and the embeddings $\mathscr{Z} \hookrightarrow \mathscr{Y} \hookrightarrow \mathscr{X}$ are continuous[1].

Next, we consider the spectral subspaces associated with these two projections,

$$\mathscr{E}_0 = \mathrm{im}\,\mathbf{P}_0 = \ker \mathbf{P}_h \subset \mathscr{Z}, \quad \mathscr{X}_h = \mathrm{im}\,\mathbf{P}_h = \ker \mathbf{P}_0 \subset \mathscr{X},$$

which provide a decomposition of $\mathscr{X}$ into invariant subspaces,

$$\mathscr{X} = \mathscr{E}_0 \oplus \mathscr{X}_h.$$

[1] If there is no risk of confusion we shall sometimes use the same notation for an operator $\mathbf{L} \in \mathscr{L}(\mathscr{X})$, say, and its restrictions to $\mathscr{Z}$ and $\mathscr{Y}$, $\mathbf{L}\big|_{\mathscr{Z}} \in \mathscr{L}(\mathscr{Z})$ and $\mathbf{L}\big|_{\mathscr{Y}} \in \mathscr{L}(\mathscr{Y})$, respectively.

We also set

$$\mathscr{Z}_h = \mathbf{P}_h\mathscr{Z} \subset \mathscr{Z}, \quad \mathscr{Y}_h = \mathbf{P}_h\mathscr{Y} \subset \mathscr{Y},$$

and denote by $\mathbf{L}_0$ and $\mathbf{L}_h$ the restrictions of $\mathbf{L}$ to $\mathscr{E}_0$ and $\mathscr{Z}_h$, respectively,

$$\mathbf{L}_0 \in \mathscr{L}(\mathscr{E}_0), \quad \mathbf{L}_h \in \mathscr{L}(\mathscr{Z}_h, \mathscr{X}_h).$$

An immediate consequence of these definitions is that the spectrum of $\mathbf{L}_0$ is σ_0 and the spectrum of $\mathbf{L}_h$ is $\sigma_h = \sigma_+ \cup \sigma_-$.

Remark 2.6 *As already noticed, the space $\mathscr{E}_0$ is finite-dimensional by Hypothesis 2.4(ii). Then $\mathbf{L}_0$ acts in a finite-dimensional space, and the exponential $e^{\mathbf{L}_0 t}$ allows us to explicitly solve the linear ordinary differential equation*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + f(t) \tag{2.3}$$

via the variation of constant formula,

$$u_0(t) = e^{\mathbf{L}_0 t}u_0(0) + \int_0^t e^{\mathbf{L}_0(t-s)}f(s)\,\mathrm{d}s.$$

Our second hypothesis concerns the analogue of this linear problem for the operator $\mathbf{L}_h$.

Hypothesis 2.7 (Linear equation) *For any $\eta \in [0,\gamma]$ and any $f \in \mathscr{C}_\eta(\mathbb{R},\mathscr{Y}_h)$ the linear problem*

$$\frac{du_h}{dt} = \mathbf{L}_h u_h + f(t), \tag{2.4}$$

has a unique solution $u_h = \mathbf{K}_h f \in \mathscr{C}_\eta(\mathbb{R},\mathscr{Z}_h)$. Furthermore, the linear map $\mathbf{K}_h$ belongs to $\mathscr{L}(\mathscr{C}_\eta(\mathbb{R},\mathscr{Y}_h),\mathscr{C}_\eta(\mathbb{R},\mathscr{Z}_h))$, and there exists a continuous map $C : [0,\gamma] \to \mathbb{R}$ such that

$$\|\mathbf{K}_h\|_{\mathscr{L}(\mathscr{C}_\eta(\mathbb{R},\mathscr{Y}_h),\mathscr{C}_\eta(\mathbb{R},\mathscr{Z}_h))} \le C(\eta).$$

While Hypotheses 2.1 and 2.4 are rather easy to check, in applications it is much more difficult to check Hypothesis 2.7. In Section 2.2.3, we discuss this hypothesis in more detail and give standard results showing how to verify it for a large class of infinite- dimensional systems.

Exercise 2.8 *Prove that Hypothesis 2.7 is satisfied in finite dimensions when $\mathscr{X} = \mathbb{R}^n$.*
Hint: *For the differential equation (2.4) the initial condition $u_h(0)$ is uniquely determined by the exponential growth required for the solution, $u_h \in \mathscr{C}_\eta(\mathbb{R},\mathscr{Z}_h)$, which is given by*

$$u_h(t) = -\int_t^\infty e^{\mathbf{L}(t-s)}\mathbf{P}_+ f(s)\mathrm{d}s + \int_{-\infty}^t e^{\mathbf{L}(t-s)}\mathbf{P}_- f(s)\mathrm{d}s.$$

Here, $\mathbf{P}_\pm$ are the spectral projections associated to $\sigma_\pm$, which are in this case finite sets, just as σ_0, and the projections can therefore be defined by formulae similar to (2.2).

2.2.2 Main Result

In this section we state the center manifold theorem. This result has been proved for the first time in finite dimensions by Pliss [101] in 1964, in the case where the unstable spectrum σ_+ is empty, and by Kelley [77] in 1967, in the case where σ_+ is not empty. There are several versions of these results in infinite dimensions (e.g., see [47], σ_+ is empty, and [97, 122, 82], and the references therein, σ_+ is not empty), and there are analogous results for mappings (e.g., see [87, 94, 72]).

Theorem 2.9 (Center manifold theorem) *Assume that Hypotheses 2.1, 2.4, and 2.7 hold. Then there exists a map $\Psi \in \mathscr{C}^k(\mathscr{E}_0, \mathscr{Z}_h)$, with*

$$\Psi(0) = 0, \quad D\Psi(0) = 0, \tag{2.5}$$

and a neighborhood $\mathscr{O}$ of 0 in $\mathscr{Z}$ such that the manifold

$$\mathscr{M}_0 = \{u_0 + \Psi(u_0) \; ; \; u_0 \in \mathscr{E}_0\} \subset \mathscr{Z} \tag{2.6}$$

has the following properties:

(i) $\mathscr{M}_0$ is locally invariant, i.e., if u is a solution of (2.1) satisfying $u(0) \in \mathscr{M}_0 \cap \mathscr{O}$ and $u(t) \in \mathscr{O}$ for all $t \in [0,T]$, then $u(t) \in \mathscr{M}_0$ for all $t \in [0,T]$.
(ii) $\mathscr{M}_0$ contains the set of bounded solutions of (2.1) staying in $\mathscr{O}$ for all $t \in \mathbb{R}$, i.e., if u is a solution of (2.1) satisfying $u(t) \in \mathscr{O}$ for all $t \in \mathbb{R}$, then $u(0) \in \mathscr{M}_0$.

We give the proof of this theorem in Appendix B.1.

Remark 2.10 *The manifold $\mathscr{M}_0$ is called a* local center manifold *of (2.1), and the map Ψ is often referred to as the* reduction function. *Notice that $\mathscr{M}_0$ has the same* dimension *as $\mathscr{E}_0$, so it is finite-dimensional, and that it is tangent to $\mathscr{E}_0$ in 0, due to (2.5).*

Remark 2.11 *We give in Section 2.3.4 a specific center manifold theorem corresponding to the cases in which the unstable part σ_+ of the spectrum of $\mathbf{L}$ is empty.*

Center manifolds are fundamental for the study of dynamical systems near "critical situations," and in particular in bifurcation theory. Starting with an infinite-dimensional problem of the form (2.1), the center manifold theorem reduces the study of small solutions, staying sufficiently close to 0, to that of small solutions of a reduced system with finite dimension, equal to the dimension of $\mathscr{E}_0$. Indeed, such solutions belong to the center manifold $\mathscr{M}_0$, and are therefore of the form $u = u_0 + \Psi(u_0)$. The corollary below shows that solutions on the center manifold are described by a finite-dimensional system of ordinary differential equations, also called *reduced system*, which has the same dimension as $\mathscr{E}_0$.

Corollary 2.12 *Under the assumptions in Theorem 2.9, consider a solution u of (2.1) which belongs to $\mathscr{M}_0$ for $t \in \mathscr{I}$, for some open interval $\mathscr{I} \subset \mathbb{R}$. Then $u = u_0 + \Psi(u_0)$, and u_0 satisfies*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0)). \tag{2.7}$$

Furthermore, the reduction function Ψ satisfies the equality

$$D\Psi(u_0)\left(\mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0))\right) = \mathbf{L}_h \Psi(u_0) + \mathbf{P}_h \mathbf{R}(u_0 + \Psi(u_0)) \text{ for all } u_0 \in \mathscr{E}_0. \tag{2.8}$$

Proof By substituting $u = u_0 + \Psi(u_0)$ into (2.1) we obtain

$$\frac{du_0}{dt} + D\Psi(u_0)\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{L}_h \Psi(u_0) + \mathbf{R}(u_0 + \Psi(u_0)).$$

Projecting this equality with $\mathbf{P}_0$ we find that u_0 satisfies (2.7), and then projecting with $\mathbf{P}_h$ we obtain

$$D\Psi(u_0)\frac{du_0}{dt} = \mathbf{L}_h \Psi(u_0) + \mathbf{P}_h \mathbf{R}(u_0 + \Psi(u_0)).$$

Inserting du_0/dt from (2.7) in the equality above gives (2.8). □

Remark 2.13 *In applications it is important to compute the reduced vector field in (2.7), and more precisely its Taylor expansion. Very often it is enough to know the lowest order terms in its Taylor expansion, which can be computed directly from the formula $P_0\mathbf{R}(u_0 + \Psi(u_0))$. However, there are situations in which we need to know the terms at the next orders. This requires the computation of the Taylor expansion of the reduction function Ψ, as well, which can be done with the help of formula (2.8). We point out that one can compute the Taylor expansions of the reduced vector field and of the reduction function up to the order k, but these computations become more involved as k increases. Several examples of such computations are made in Section 2.4.*

Remark 2.14 *(i) Local center manifolds are in general* not unique *even though the Taylor expansion at the origin is unique. This is due to the occurrence in the proof of the theorem of a smooth cut-off function χ_0 on the space $\mathscr{E}_0$, which is not unique (see Appendix B.1). Uniqueness can be achieved under appropriate boundedness conditions on the nonlinearity $\mathbf{R}$: it should be Lipschitzian with sufficiently small Lipschitz constant. We refer to [122, Theorems 1 and 2] for a precise statement of this result. In addition, in this case the resulting center manifold is* global *in the sense that the properties in Theorem 2.9 hold with $\mathscr{O} = \mathscr{Z}$.*

(ii) Center manifolds are in general not analytic *even when the right hand side of the differential equation (2.1) is analytic in u. We refer to [114, 12, 112], and [94, pp. 44–45], [38, p. 126], [120, p. 123] for examples of analytic vector fields leading to nonanalytic center manifolds.*

(iii) A crucial hypothesis in the existing proofs on local center manifolds is Hypothesis 2.4(ii) on the set σ_0, which has to be finite. Without this hypothesis one would expect to construct an infinite- dimensional manifold. *However, this raises a number of difficulties, which, so far, have been overcome in only very*

particular situations [98, 100]. Such a construction would require we first build a "good" projection $\mathbf{P}_0$ associated with the infinite spectral set σ_0, allowing us to obtain a group property for $e^{\mathbf{L}_0 t}$ together with a subexponential growth as $t \to \pm\infty$, and then also to construct a smooth cut-off function χ_0 on the central space $\mathscr{E}_0 = \mathbf{P}_0 \mathscr{X}$.

2.2.3 Checking Hypothesis 2.7

We discuss in this section Hypothesis 2.7, and more precisely how to check it in applications. While this hypothesis always holds in finite dimensions (see Exercise 2.8), in infinite dimensions this is not always the case. Here, we distinguish between

- the *semilinear case*, $\mathscr{Y} \subset \mathscr{X}$ with $\mathscr{Y} \neq \mathscr{X}$, and
- the *quasilinear case*, $\mathscr{Y} = \mathscr{X}$.

First, we give some conditions on the resolvent of $\mathbf{L}$ which are sufficient for Hypothesis 2.7 to hold in the semilinear case. In contrast, in the quasilinear case Hypothesis 2.7 is in general not true. We discuss this situation in the second part of this section.

Semilinear Equations in Banach Spaces

We assume that Hypotheses 2.1, 2.4 hold, and show here that we may replace Hypothesis 2.7 by the following one. Though we do not make explicitly the assumption that $\mathscr{Y} \neq \mathscr{X}$, the hypothesis below can only be verified in this case.

Hypothesis 2.15 (Resolvent estimates) *Assume that there exist positive constants $\omega_0 > 0$, $c > 0$, and $\alpha \in [0,1)$ such that for all $\omega \in \mathbb{R}$, with $|\omega| \geq \omega_0$, we have that $i\omega$ belongs to the resolvent set of $\mathbf{L}$, and*

$$\|(i\omega\mathbb{I} - \mathbf{L})^{-1}\|_{\mathscr{L}(\mathscr{X})} \leq \frac{c}{|\omega|}, \tag{2.9}$$

$$\|(i\omega\mathbb{I} - \mathbf{L})^{-1}\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})} \leq \frac{c}{|\omega|^{1-\alpha}}. \tag{2.10}$$

Remark 2.16 (Hilbert spaces) *Though necessary to show that Hypothesis 2.7 holds, as we shall see in Theorem 2.20, the second inequality (2.10) is not needed for the center manifold Theorem 2.9 to hold when $\mathscr{X}$, $\mathscr{Z}$, and $\mathscr{Y}$ are Hilbert spaces. We make use of this fact in the examples presented in Section 2.4.*

We prove in Appendix B.2 that Hypothesis 2.15 above implies Hypothesis 2.7, so that the following holds.

Theorem 2.17 (Center manifold theorem in the semilinear case) *Assume that Hypotheses 2.1, 2.4, and 2.15 hold. Then*

(i) Hypothesis 2.7 is satisfied;
(ii) the result in Theorem 2.9 holds.

Remark 2.18 (Parabolic problems) *An important class of problems for which Hypothesis 2.15 usually holds is that of parabolic equations in Hilbert spaces. In such a situation the operator* $\mathbf{L}$ *is typically sectorial and generates an analytic semigroup. In particular, its resolvent satisfies Hypothesis 2.15, so that center manifold Theorem 2.9 applies provided Hypotheses 2.1 and 2.4 hold.*

Remark 2.19 *In Section 5.2.3 of Chapter 5 we give an example (waves in lattices) where (2.9) does not hold, while Hypothesis 2.7 is verified.*

Quasilinear Equations in Hilbert Spaces

We consider now the quasilinear case, $\mathscr{Y} = \mathscr{X}$. In this case Hypothesis 2.7 requires a maximal regularity property for the linear equation (2.4), and it turns out that such a property does not hold in general for spaces of continuous functions such as $\mathscr{C}_\eta(\mathbb{R}, \mathscr{X}_h)$.

Nevertheless, maximal regularity has been shown in Sobolev and Hölder spaces. We mention here the maximal regularity result by da Prato and Grisvard [21] in Sobolev spaces $W^{\theta,p}(\mathbb{R}, \mathscr{X})$, with $\theta \in (0,1)$ and $p \in (1,\infty]$, $\mathscr{X}$ is a Banach space, and the result by Mielke [96] in Sobolev spaces $L^p(\mathbb{R}, \mathscr{X})$, with $p \in (1,\infty)$, $\mathscr{X}$ is a Hilbert space. For both results, the resolvent estimate (2.9) turns out to be a sufficient condition for maximal regularity in these spaces. As for the Hölder spaces, Kirrmann [82] proved a maximal regularity result in $\mathscr{C}^{0,\alpha}(\mathbb{R}, \mathscr{X})$ with $\mathscr{X}$ a Banach space, but under a slightly different resolvent estimate.

Since these maximal regularity results hold in different spaces (Sobolev or Hölder spaces instead of spaces of continuous functions), the proof of the center manifold theorem given in Appendix B.1 does not work anymore, and needs to be adapted. Starting with the result in [96] for Hilbert spaces, Mielke [97] proved a center manifold theorem for quasilinear equations in Hilbert spaces. In Banach spaces, the maximal regularity result by Kirrmann allowed proof of a center manifold theorem [82], with a reduction function Ψ of class $\mathscr{C}^{k-1}$ instead of $\mathscr{C}^k$. We state below the result in Hilbert spaces, which uses our resolvent estimate (2.9), and refer to [97] for its proof and to [82] for the slightly different result in Banach spaces.

Theorem 2.20 (Center manifold theorem in the quasilinear case) *Assume that* $\mathscr{X}$, $\mathscr{Z}$, *and* $\mathscr{Y}$ *are Hilbert spaces, and that Hypotheses 2.1 and 2.4 hold. If the linear operator* $\mathbf{L}_h$ *satisfies (2.9), then the result in Theorem 2.9 holds.*

2.2.4 Examples

We show in this section how to apply the center manifold theorem in two examples. The first one is a fourth order ODE, for which $\mathscr{X} = \mathbb{R}^4$, while the second one is a parabolic PDE, for which $\mathscr{X}$ is a Banach space of continuous functions.

A Fourth Order ODE

Consider the fourth order ODE

$$u^{(4)} - u'' - au^2 = 0, \tag{2.11}$$

where a is a given real number.

Formulation as a First Order System

We start by writing the equation (2.11) in the form (2.1). We set $U = (u, u_1, u_2, u_3)$ with $u_1 = u'$, $u_2 = u'' - u$, $u_3 = u_2'$, and then the equation is equivalent with the system

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}(U), \tag{2.12}$$

in which

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{R}(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ au^2 \end{pmatrix}.$$

Here $\mathbf{L}$ is a 4×4-matrix and $\mathbf{R}$ is a smooth vector field in $\mathbb{R}^4$, so that we can choose

$$\mathscr{X} = \mathscr{Y} = \mathscr{Z} = \mathbb{R}^4.$$

Checking the Hypotheses

Clearly, Hypothesis 2.1 is satisfied for $\mathbf{L}$ and $\mathbf{R}$ as above, for any $k \geq 2$ and the neighborhood $\mathscr{V} = \mathbb{R}^4$.

Next, in order to check Hypothesis 2.4 we have to compute the spectrum of $\mathbf{L}$, i.e., the eigenvalues of $\mathbf{L}$. A direct calculation gives

$$\sigma(\mathbf{L}) = \{-1, 0, 1\},$$

with ± 1 simple eigenvalues, and 0 a geometrically simple and algebraically double eigenvalue. Consequently, Hypothesis 2.4 is also satisfied with

$$\sigma_+ = \{1\}, \quad \sigma_0 = \{0\}, \quad \sigma_- = \{-1\}.$$

Finally, according to the result in Exercise 2.8, Hypothesis 2.7 holds in this case since $\mathscr{X}$ is finite-dimensional.

Consequently, we can apply center manifold Theorem 2.9, and conclude the existence of a local center manifold of class $\mathscr{C}^k$ for any arbitrary, but fixed, $k \geq 2$. Since 0 is an algebraically double eigenvalue, the space $\mathscr{E}_0$ is two-dimensional, so that the center manifold is two-dimensional.

Reduced Equation

Our purpose is to compute the Taylor expansion, up to order 2, of the vector field in the reduced equation.

We start by computing a basis for $\mathscr{E}_0$, which is the two-dimensional generalized kernel of $\mathbf{L}$. Solving successively the eigenvalue problem $\mathbf{L}\zeta_0 = 0$ and the generalized eigenvalue problem $\mathbf{L}\zeta_1 = \zeta_0$, we find a basis $\{\zeta_0, \zeta_1\}$ for $\mathscr{E}_0$ given by

$$\zeta_0 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \zeta_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

According to the center manifold Theorem 2.9, solutions on the center manifold are of the form

$$U(t) = U_0(t) + \Psi(U_0(t)), \tag{2.13}$$

in which $\Psi(0) = 0$, $D\Psi(0) = 0$, and $U_0(t) \in \mathscr{E}_0$, so that

$$U_0(t) = A(t)\zeta_0 + B(t)\zeta_1, \tag{2.14}$$

where A and B are real-valued functions. The reduced system is an ODE for $U_0 = (A, B)$, and according to Corollary 2.12 it is given by

$$\frac{dU_0}{dt} = \mathbf{L}_0 U_0 + \mathbf{P}_0 \mathbf{R}(U_0 + \Psi(U_0)), \tag{2.15}$$

where $\mathbf{L}_0$ is the restriction of $\mathbf{L}$ to $\mathscr{E}_0$, and $\mathbf{P}_0$ is the spectral projection onto $\mathscr{E}_0$. We compute the expansion, up to order 2, of the vector field in (2.15), by calculating successively the 2×2-matrix $\mathbf{L}_0$, the spectral projector $\mathbf{P}_0$, and the expansion of $\mathbf{P}_0 \mathbf{R}(U_0 + \Psi(U_0))$.

First, since $\mathbf{L}_0$ is the restriction of $\mathbf{L}$ to the space $\mathscr{E}_0$, in the basis $\{\zeta_0, \zeta_1\}$ of $\mathscr{E}_0$ calculated above we find that the 2×2-matrix representing $\mathbf{L}_0$ is given by

$$\mathbf{L}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

since $\mathbf{L}\zeta_0 = 0$ and $\mathbf{L}\zeta_1 = \zeta_0$. Next, there are several ways of computing the spectral projection $\mathbf{P}_0$ in finite dimensions. Here, we compute $\mathbf{P}_0$ with the help of the adjoint matrix

$$\mathbf{L}^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

since this calculation also works in infinite dimensions, provided the operator $\mathbf{L}$ possesses an adjoint $\mathbf{L}^*$. Recall that the adjoint matrix $\mathbf{L}^*$ satisfies

$$\langle \mathbf{L}U, V\rangle = \langle U, \mathbf{L}^*V\rangle \text{ for all } U, V \in \mathbb{R}^4,$$

where $\langle\cdot,\cdot\rangle$ is the usual Euclidean scalar product in $\mathbb{R}^4$.

We claim that the spectral projection $\mathbf{P}_0$ is given by

$$\mathbf{P}_0 U = \langle U, \zeta_0^*\rangle\zeta_0 + \langle U, \zeta_1^*\rangle\zeta_1, \tag{2.16}$$

where $\{\zeta_0^*, \zeta_1^*\}$ is a dual basis satisfying

$$\mathbf{L}^*\zeta_0^* = \zeta_1^*, \quad \mathbf{L}^*\zeta_1^* = 0, \quad \langle \zeta_i, \zeta_j^*\rangle = \delta_{ij} \text{ for all } i, j \in \{0,1\}. \tag{2.17}$$

Indeed, since $\mathbf{P}_0$ is a linear map from $\mathbb{R}^4$ onto $\mathscr{E}_0$, there exist two vectors $\zeta_0^*, \zeta_1^* \in \mathbb{R}^4$ such that $\mathbf{P}_0 U$ is given by (2.16). Next, since $\mathbf{P}_0$ is a projection, $\mathbf{P}_0^2 = \mathbf{P}_0$, it follows that $\mathbf{P}_0\zeta_0 = \zeta_0$ and $\mathbf{P}_0\zeta_1 = \zeta_1$, which implies that the last equality in (2.17) holds for all $i, j \in \{0,1\}$. Finally, the spectral projection $\mathbf{P}_0$ commutes with $\mathbf{L}$, $\mathbf{P}_0\mathbf{L} = \mathbf{L}\mathbf{P}_0$, which implies that

$$\langle \mathbf{L}U, \zeta_0^*\rangle = \langle U, \zeta_1^*\rangle, \quad \langle \mathbf{L}U, \zeta_1^*\rangle = 0 \text{ for all } U \in \mathbb{R}^4,$$

and these equalities are equivalent with the first two equalities in (2.17). This proves the claim.

It is now straightforward to compute the vectors ζ_0^* and ζ_1^* in (2.16). We obtain that

$$\zeta_0^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \zeta_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Finally, it remains to compute the Taylor expansion up to order 2 of $\mathbf{P}_0\mathbf{R}(U_0 + \Psi(U_0))$. Notice that since the last component of the vector ζ_0^* vanishes, the scalar product $\langle \mathbf{R}(U_0 + \Psi(U_0)), \zeta_0^*\rangle = 0$, so that

$$\mathbf{P}_0\mathbf{R}(U_0 + \Psi(U_0)) = \langle \mathbf{R}(U_0 + \Psi(U_0)), \zeta_1^*\rangle\zeta_1.$$

Furthermore, since $\Psi(0) = 0$ and $D\Psi(0) = 0$, we have $\Psi(U_0) = O(\|U_0\|^2)$, which together with the fact that $\mathbf{R}$ is a quadratic map implies that

$$\mathbf{P}_0\mathbf{R}(U_0+\Psi(U_0)) = \langle \mathbf{R}(U_0), \zeta_1^* \rangle \zeta_1 + O(\|U_0\|^3).$$

The explicit formulas for $\mathbf{P}_0$, $\mathbf{R}$, and U_0 give

$$\mathbf{R}(U_0) = \mathbf{R}(A\zeta_0 + B\zeta_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ aA^2 \end{pmatrix},$$

so that

$$\mathbf{P}_0\mathbf{R}(U_0+\Psi(U_0)) = \left(aA^2 + O((|A|+|B|)^3)\right)\zeta_1.$$

Together with the explicit formula for $\mathbf{L}_0$ above, this implies that the reduced system (2.15), in the basis $\{\zeta_0, \zeta_1\}$, is

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= aA^2 + O((|A|+|B|)^3). \end{aligned}$$

Remark 2.21 *(i) In the calculation of the expansion up to order 2 of the reduced system, it was not necessary to compute the expansion of Ψ. This property is always true because $\Psi(U_0) = O(\|U_0\|^2)$ and $\mathbf{R}(U) = O(\|U\|^2)$. However, the expansion of Ψ is necessary when computing the expansion up to order 3, or higher, of the reduced system. For instance, for a computation up to order 3 one needs to compute the terms of order 2 in the expansion of Ψ. This can be done by substituting the Ansatz*

$$\Psi(A,B) = \Psi_{20}A^2 + \Psi_{11}AB + \Psi_{02}B^2 + O((|A|+|B|)^3) \tag{2.18}$$

in the identity (2.8). Then the vectors Ψ_{20}, Ψ_{11}, and Ψ_{02} are determined by identifying powers of A and B in this identity and taking into account that these vectors belong to the space $(\mathbb{I}-\mathbf{P}_0)\mathbb{R}^4$, i.e., they are orthogonal to both ζ_0^* and ζ_1^*.

(ii) An alternative way of computing the reduced system, is by directly substituting the formulas (2.13), (2.14), and (2.18) into the first order system (2.12) and calculating the Taylor expansions of both sides of the resulting system. We use this alternative approach in most of the examples in Section 2.4. It turns out that such an approach is particularly convenient when the center manifold reduction is followed by a normal form transformation (see Chapter 3, Section 3.4).

A Parabolic PDE

Consider the parabolic boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u + g\left(u, \frac{\partial u}{\partial x}\right) \tag{2.19}$$

$$u(0,t) = u(\pi,t) = 0, \tag{2.20}$$

where $u(x,t) \in \mathbb{R}$ for $(x,t) \in (0,\pi) \times \mathbb{R}$, and $g \in \mathscr{C}^k(\mathbb{R}^2, \mathbb{R})$, $k \geq 2$, satisfying

$$g(0,v) = 0 \text{ for all } v \in \mathbb{R}, \text{ and } g(u,v) = O(|u|^2 + |v|^2) \text{ as } (u,v) \to 0.$$

Formulation and Hypothesis 2.1

First we write the problem (2.19)–(2.20) in form (2.1) by setting

$$\mathbf{L}u = \frac{d^2 u}{dx^2} + u, \quad \mathbf{R}(u) = g\left(u, \frac{du}{dx}\right),$$

and choosing the Banach space

$$\mathscr{X} = C^0([0,\pi])$$

of real-valued continuous functions on $[0,\pi]$. Then $\mathbf{L}$ is a closed linear operator in $\mathscr{X}$ with domain

$$\mathscr{Z} = \{u \in C^2([0,\pi]) \; ; \; u(0) = u(\pi) = 0\},$$

taken such that $\mathbf{L}u \in \mathscr{X}$ for $u \in \mathscr{Y}$, and such that the functions in $\mathscr{Y}$ satisfy the boundary conditions (2.20). The nonlinear terms $\mathbf{R}$ satisfy $\mathbf{R}(u) \in C^1([0,\pi])$ and $(\mathbf{R}(u))(0) = (\mathbf{R}(u))(\pi) = 0$ for $u \in \mathscr{Y}$. We therefore set

$$\mathscr{Y} = \{u \in C^1([0,\pi]) \; ; \; u(0) = u(\pi) = 0\},$$

and then we have $\mathbf{R} \in C^k(\mathscr{Z}, \mathscr{Y})$. In particular, these show that $\mathbf{L}$ and $\mathbf{R}$ satisfy Hypothesis 2.1.

Spectrum and Hypothesis 2.4

Next, we investigate the spectrum of $\mathbf{L}$ and check Hypothesis 2.4. For this we have to solve the linear equation

$$\lambda u - \mathbf{L}u = f$$

for $\lambda \in \mathbb{C}$, $f \in \mathscr{X}$, and $u \in \mathscr{Z}$; that is, we have to find solutions $u \in C^2([0,\pi])$ of the linear problem

$$\lambda u - u - u'' = f \tag{2.21}$$

$$u(0) = u(\pi) = 0 \tag{2.22}$$

for $f \in C^0([0,\pi])$. The second order ODE (2.21) has a unique solution $u \in C^2([0,\pi])$ satisfying the boundary conditions (2.22), for $f \in C^0([0,\pi])$, precisely when the associated homogeneous equation

$$u'' + u - \lambda u = 0 \tag{2.23}$$

possesses no nontrivial solutions. When this is the case, then λ belongs to the resolvent set $\rho(\mathbf{L})$ of $\mathbf{L}$. A direct calculation shows that (2.23) has nontrivial solutions for $\lambda = 1 - n^2$, with n any positive integer. We conclude that the resolvent set and the spectrum of $\mathbf{L}$ are, respectively,

$$\rho(\mathbf{L}) = \mathbb{C} \setminus \sigma(\mathbf{L}), \quad \sigma(\mathbf{L}) = \{\lambda \in \mathbb{C}\,;\, \lambda = 1 - n^2,\ n \in \mathbb{N}^*\};$$

(here, and later in the text, $\mathbb{N}^* = \{n \in \mathbb{N}\,;\, n \geq 1\}$).

With the notations from Hypothesis 2.4 we now have

$$\sigma_+ = \varnothing, \quad \sigma_0 = \{0\}, \quad \sigma_- \subset (-\infty, -3],$$

so that part (i) of this hypothesis holds. Next, the kernel of $\mathbf{L}$ is one-dimensional, spanned by $\xi_0 = \sin x$, so that the eigenvalue $\lambda = 0$ has geometric multiplicity one. A generalized eigenvector v associated to the eigenvalue 0 satisfies the ODE

$$v'' + v = \sin x,$$

and the boundary conditions (2.22). Multiplying this equation by $\sin x$, integrating over $[0,\pi]$, and then integrating twice by parts on the left hand side gives

$$\int_0^\pi v''(x)\sin x\,dx + \int_0^\pi v(x)\sin x\,dx = -\int_0^\pi v(x)\sin x\,dx + \int_0^\pi v(x)\sin x\,dx = 0,$$

while the right hand side is equal to

$$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2},$$

so that there are no solutions to the ODE above. This proves that 0 is a simple eigenvalue of $\mathbf{L}$, with algebraic multiplicity one, as well, and then shows that part (ii) of Hypothesis 2.4 holds. Notice that the spectral subspace $\mathscr{E}_0$ associated to σ_0 is one-dimensional, spanned by ξ_0, so that we expect in this case to find a one-dimensional center manifold.

Checking Hypothesis 2.7

Finally, we have to check Hypothesis 2.7. For this we use the result in Theorem 2.17, so that we have to verify the estimates on the resolvent (2.9) and (2.10). Since our problem is formulated in Banach spaces we need to check both inequalities (see Remark 2.16).

Consider $\omega \neq 0$. Since $\sigma_0 = \{0\}$, we have that $i\omega$ belongs to the resolvent set of $\mathbf{L}$, so that the equation

$$(i\omega\mathbb{I} - \mathbf{L})u = f$$

has a unique solution $u \in \mathscr{Z}$ for $f \in \mathscr{X}$. This solution satisfies

$$(i\omega - 1)u - u'' = f$$
$$u(0) = u(\pi) = 0,$$

and a direct computation gives

$$u(x) = \frac{1}{\gamma \sinh(\gamma\pi)} \left(\int_0^x \sinh(\gamma\xi)\sinh(\gamma(\pi - x)) f(\xi) d\xi \right.$$
$$\left. + \int_x^\pi \sinh(\gamma x)\sinh(\gamma(\pi - \xi)) f(\xi) d\xi \right)$$

in which

$$\gamma = \sqrt{i\omega - 1}.$$

We need to show that

$$\|u\|_{C^0} \leq \frac{c}{|\omega|}\|f\|_{C^0}, \qquad \|u\|_{C^2} \leq \frac{c}{|\omega|^{1-\alpha}}\|f\|_{C^1} \tag{2.24}$$

for $|\omega| \geq \omega_0$ and constants $c > 0$ and $\alpha \in [0,1)$, which then proves that (2.9) and (2.10) hold.

We write

$$u(x) = \frac{1}{\gamma \sinh(\gamma\pi)} \left(\frac{1}{2}\int_0^x \cosh(\gamma(\pi + \xi - x)) f(\xi) d\xi \right.$$
$$\left. + \frac{1}{2}\int_x^\pi \cosh(\gamma(\pi + x - \xi)) f(\xi) d\xi - \frac{1}{2}\int_0^\pi \cosh(\gamma(x + \xi - \pi)) f(\xi) d\xi \right),$$

and $\gamma = \gamma_r + i\gamma_i$, $\gamma_r > 0$. Using the inequalities

$$|\sinh(a + ib)| \geq \sinh(a), \qquad |\cosh(a + ib)| \leq 1 + \sinh(a),$$

which hold for real numbers $a > 0$ and $b \in \mathbb{R}$, we estimate

$$|u(x)| \leq \frac{\|f\|_{C^0}}{2|\gamma|\sinh(\gamma_r\pi)} \left(\int_0^x (1 + \sinh(\gamma_r(\pi + \xi - x))) d\xi \right.$$
$$+ \int_x^\pi (1 + \sinh(\gamma_r(\pi + x - \xi))) d\xi + \int_0^{\pi - x} (1 + \sinh(\gamma_r(\pi - x - \xi))) d\xi$$
$$\left. + \int_{\pi - x}^\pi (1 + \sinh(\gamma_r(x + \xi - \pi))) d\xi \right)$$
$$= \frac{\|f\|_{C^0}}{|\gamma|\gamma_r \sinh(\gamma_r\pi)} (\gamma_r\pi + \cosh(\gamma_r\pi) - 1) \leq \frac{2\|f\|_{C^0}}{|\gamma|\gamma_r}.$$

This proves the first inequality in (2.24).

Similar calculations show that

$$\|u'\|_{C^0} \leq \frac{c}{|\omega|^{1/2}} \|f\|_{C^0},$$

and it remains to estimate $\|u''\|_{C^0}$. Now we use the fact that $f \in \mathscr{Y}$, in order to obtain the second inequality in (2.24), with $\alpha \neq 0$. (We point out that $\|u''\|_{C^0} \leq c\|f\|_{C^0}$, since $u'' = \gamma^2 u - f$, which gives the second inequality in (2.24) for $\alpha = 1$, only.) Integrating by parts in the formula for u we find, for $f \in C^1([0,\pi])$,

$$\begin{aligned} u''(x) &= \gamma^2 u(x) - f(x) \\ &= \frac{1}{\sinh(\gamma\pi)} \Big(-\sinh(\gamma(\pi - x)) f(0) - \sinh(\gamma x) f(\pi) \\ &\qquad - \int_0^x \cosh(\gamma\xi) \sinh(\gamma(\pi - x)) f'(\xi) d\xi \\ &\qquad + \int_x^\pi \sinh(\gamma x) \cosh(\gamma(\pi - \xi)) f'(\xi) d\xi \Big). \end{aligned}$$

Using the fact that $f(0) = f(\pi) = 0$ for $f \in \mathscr{Y}$, and arguing as above, we find

$$\|u''\|_{C^0} \leq \frac{c}{|\omega|^{1/2}} \|f'\|_{C^0},$$

which completes the proof of (2.24). Notice that the equalities $f(0) = f(\pi) = 0$ were essential in this last part of the proof, taking $f \in C^1([0,\pi])$, only, does not allow us to obtain the second inequality in (2.24) with $\alpha \neq 0$. However, such boundary conditions on f are not necessary when the Banach spaces $C^k([0,\pi])$ are replaced by the Sobolev spaces $H^k(0,\pi)$, for which one can prove the second inequality in (2.24), with $\alpha = 3/4$, without imposing $f(0) = f(\pi) = 0$ (see [122]).

Reduced Equation

Hypotheses 2.1, 2.4, and 2.7 being satisfied, we can now apply center manifold Theorem 2.9. This gives us a one-dimensional center manifold $\mathscr{M}_0$ as in (2.6), parameterized by $u_0 \in \mathscr{E}_0$. Notice that $\mathbf{L}_0 u_0 = 0$ in this case, so that the linear term in the reduced system (2.7) vanishes. Furthermore, since $\mathscr{E}_0$ is spanned by ξ_0, we may write

$$u_0(t) = A(t)\xi_0 \in \mathscr{E}_0, \quad A(t) \in \mathbb{R}.$$

Replacing this formula in the reduced system (2.7) we obtain a first order ODE for A,

$$\frac{dA}{dt} = f_0(A),$$

with $f_0(A) = O(A^2)$ as $A \to 0$. For concrete nonlinear terms g in (2.19), one can compute explicitly the Taylor expansion of f_0 (see Remark 2.13), and then easily determine the dynamics near 0 of the reduced equation, since it is a first order ODE. We present examples of such computations in Section 2.4.

2.3 Particular Cases and Extensions

2.3.1 Parameter-Dependent Center Manifolds

In the same frame as above, we consider a *parameter-dependent* differential equation in $\mathscr{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,\mu), \tag{3.1}$$

where $\mathbf{L}$ is a linear operator as in Section 2.2 and $\mathbf{R}$ is defined for (u,μ) in a neighborhood of $(0,0)$ in $\mathscr{Z} \times \mathbb{R}^m$. Here $\mu \in \mathbb{R}^m$ is a parameter that we assume to be small. More precisely, we keep Hypotheses 2.4, 2.7, and replace Hypothesis 2.1 by the following:

Hypothesis 3.1 *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (3.1) have the following properties:*

(i) $\mathbf{L} \in \mathscr{L}(\mathscr{Z}, \mathscr{X})$;
(ii) for some $k \geq 2$, *there exist neighborhoods* $\mathscr{V}_u \subset \mathscr{Z}$ *and* $\mathscr{V}_\mu \subset \mathbb{R}^m$ *of* 0 *such that* $\mathbf{R} \in \mathscr{C}^k(\mathscr{V}_u \times \mathscr{V}_\mu, \mathscr{Y})$ *and*

$$\mathbf{R}(0,0) = 0, \quad D_u\mathbf{R}(0,0) = 0.$$

Remark 3.2 *The equalities above on* $\mathbf{R}$ *imply that* 0 *is an equilibrium of (3.1) for* $\mu = 0$, *and that* $\mathbf{L}$ *represents the linearization of the vector field about this equilibrium at* $\mu = 0$. *Now, if* $\mathbf{L}$ *has a bounded inverse, then this equilibrium persists for small* μ. *More precisely, by arguing with the implicit function theorem, we find that there is a family of stationary solutions* $u = u(\mu)$ *of (3.1) for* μ *close to* 0, *i.e., such that*

$$\mathbf{L}u(\mu) + \mathbf{R}(u(\mu),\mu) = 0.$$

On the contrary, if $\mathbf{L}$ *does not have a bounded inverse, then this equilibrium may not persist for some values of* μ *near* 0.

The analogue of center manifold Theorem 2.9 for the parameter-dependent equation (3.1) is the following result.

Theorem 3.3 (Parameter-dependent center manifolds) *Assume that Hypotheses 3.1, 2.4, and 2.7 hold. Then there exists a map* $\Psi \in \mathscr{C}^k(\mathscr{E}_0 \times \mathbb{R}^m, \mathscr{Z}_h)$, *with*

$$\Psi(0,0) = 0, \quad D_u\Psi(0,0) = 0, \tag{3.2}$$

and a neighborhood $\mathscr{O}_u \times \mathscr{O}_\mu$ *of* $(0,0)$ *in* $\mathscr{Z} \times \mathbb{R}^m$ *such that for* $\mu \in \mathscr{O}_\mu$, *the manifold*

$$\mathscr{M}_0(\mu) = \{u_0 + \Psi(u_0,\mu) \; ; \; u_0 \in \mathscr{E}_0\} \tag{3.3}$$

has the following properties:

(i) $\mathscr{M}_0(\mu)$ *is locally invariant, i.e., if* u *is a solution of (3.1) satisfying* $u(0) \in \mathscr{M}_0(\mu) \cap \mathscr{O}_u$ *and* $u(t) \in \mathscr{O}_u$ *for all* $t \in [0,T]$*, then* $u(t) \in \mathscr{M}_0(\mu)$ *for all* $t \in [0,T]$.

(ii) $\mathscr{M}_0(\mu)$ *contains the set of bounded solutions of (3.1) staying in* $\mathscr{O}_u$ *for all* $t \in \mathbb{R}$*, i.e., if* u *is a solution of (3.1) satisfying* $u(t) \in \mathscr{O}_u$ *for all* $t \in \mathbb{R}$*, then* $u(0) \in \mathscr{M}_0(\mu)$.

Proof We consider (3.1) as a particular case of a system of the form (2.1), namely,

$$\frac{d\widetilde{u}}{dt} = \widetilde{\mathbf{L}}\widetilde{u} + \widetilde{\mathbf{R}}(\widetilde{u}), \tag{3.4}$$

by setting

$$\widetilde{u} = (u,\mu),$$

and

$$\begin{aligned} \widetilde{\mathbf{L}}\widetilde{u} &= (\mathbf{L}u + D_\mu \mathbf{R}(0,0)\mu, 0), \\ \widetilde{\mathbf{R}}(\widetilde{u}) &= (\mathbf{R}(u,\mu) - D_\mu \mathbf{R}(0,0)\mu, 0). \end{aligned}$$

We show that $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{R}}$ verify Hypotheses 2.1, 2.4, and 2.7, with Banach spaces

$$\widetilde{\mathscr{X}} = \mathscr{X} \times \mathbb{R}^m, \quad \widetilde{\mathscr{Z}} = \mathscr{Z} \times \mathbb{R}^m, \quad \widetilde{\mathscr{Y}} = \mathscr{Y} \times \mathbb{R}^m,$$

and then the result in the theorem follows from Theorem 2.9.

First, Hypothesis 2.1 is an immediate consequence of Hypothesis 3.1. Next, we show that the spectral sets $\widetilde{\sigma}_\pm$, $\widetilde{\sigma}_0$ of $\widetilde{\mathbf{L}}$ satisfy

$$\widetilde{\sigma}_\pm = \sigma_\pm, \quad \widetilde{\sigma}_0 \setminus \{0\} = \sigma_0 \setminus \{0\}, \tag{3.5}$$

where $\sigma_\pm$, σ_0 are the spectral sets of $\mathbf{L}$, and that $\widetilde{\sigma}_0$ consists of purely imaginary eigenvalues with finite algebraic multiplicities. These properties imply then that Hypothesis 2.4 holds.

Indeed, let us consider the linear equation

$$(\widetilde{\mathbf{L}} - \lambda)\widetilde{u} = \widetilde{v},$$

where $\widetilde{v} = (v, \nu) \in \mathscr{X} \times \mathbb{R}^m$. This means that

$$\begin{aligned} (\mathbf{L} - \lambda)u + D_\mu \mathbf{R}(0,0)\mu &= v, \\ -\lambda\mu &= \nu. \end{aligned}$$

Hence, if $\lambda \neq 0$ we have $\mu = -\nu/\lambda$ and

$$(\mathbf{L} - \lambda)u = v + \lambda^{-1} D_\mu \mathbf{R}(0,0)\nu.$$

Consequently, in $\mathbb{C} \setminus \{0\}$, the resolvent set of $\mathbf{L}$ is identical to the resolvent set of $\widetilde{\mathbf{L}}$. In particular, we have that (3.5) holds. Furthermore, for $\widetilde{\mathbf{L}}$ we can define the spectral projections $\widetilde{\mathbf{P}}_0$, $\widetilde{\mathbf{P}}_h$, and the corresponding spectral spaces $\widetilde{\mathscr{E}_0}$, $\widetilde{\mathscr{X}_h}$ as in Section 2.2.1.

Next, notice that $\mathscr{X}_h \times \{0\}$ is an invariant subspace for $\widetilde{\mathbf{L}}$, since

$$\widetilde{\mathbf{L}}(u_h, 0) = (\mathbf{L}_h u_h, 0) \in \mathscr{X}_h \times \{0\} \text{ for all } u_h \in \mathscr{Z}_h.$$

From this equality we further deduce that

$$\sigma(\widetilde{\mathbf{L}}\big|_{\mathscr{X}_h \times \{0\}}) = \sigma(\mathbf{L}_h) = \sigma_+ \cup \sigma_- = \widetilde{\sigma}_+ \cup \widetilde{\sigma}_-.$$

Consequently, $\mathscr{X}_h \times \{0\} \subset \widetilde{\mathscr{X}_h}$, and since

$$\operatorname{codim} \widetilde{\mathscr{X}_h} \leq \operatorname{codim} (\mathscr{X}_h \times \{0\}) = \dim \mathscr{E}_0 + m < \infty,$$

we conclude that

$$\dim \widetilde{\mathscr{E}_0} = \operatorname{codim} \widetilde{\mathscr{X}_h} < \infty.$$

In particular, this shows that $\widetilde{\sigma}_0$ consists of purely imaginary eigenvalues with finite algebraic multiplicities and proves Hypothesis 2.4.

In order to prove Hypothesis 2.7 it is enough to show that $\widetilde{\mathscr{X}_h} = \mathscr{X}_h \times \{0\}$, and then the conditions on $\widetilde{\mathbf{L}}$ in Hypothesis 2.7 follow from the analogue ones on $\mathbf{L}$. We claim that

$$\widetilde{\mathscr{E}_0} = \{(u_0 - \mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu, \mu) \; ; \; u_0 \in \mathscr{E}_0, \; \mu \in \mathbb{R}^m\} =: \mathscr{F}_0.$$

Then this implies that

$$\operatorname{codim} \widetilde{\mathscr{X}_h} = \dim \widetilde{\mathscr{E}_0} = \dim \mathscr{E}_0 + m = \operatorname{codim} (\mathscr{X}_h \times \{0\}),$$

and since $\mathscr{X}_h \times \{0\} \subset \widetilde{\mathscr{X}_h}$ we conclude that $\widetilde{\mathscr{X}_h} = \mathscr{X}_h \times \{0\}$.

It remains to prove the claim $\widetilde{\mathscr{E}_0} = \mathscr{F}_0$. First, take $\widetilde{u} = (u, \mu) \in \widetilde{\mathscr{E}_0} \subset \widetilde{\mathscr{Z}}$. We write $u = u_0 + u_h$ with $u_0 \in \mathscr{E}_0$, $u_h \in \mathscr{Z}_h$, and compute

$$\widetilde{\mathbf{L}}\widetilde{u} = (\mathbf{L}_h u_h + D_\mu \mathbf{R}_h(0,0)\mu, 0) + (\mathbf{L}_0 u_0 + D_\mu \mathbf{R}_0(0,0)\mu, 0),$$

where $\mathbf{R}_h = \mathbf{P}_h \mathbf{R}$ and $\mathbf{R}_0 = \mathbf{P}_0 \mathbf{R}$. The first term on the right hand side of the above equality belongs to $\mathscr{X}_h \times \{0\} \subset \widetilde{\mathscr{X}_h}$, whereas the second term belongs to $\mathscr{E}_0 \times \{0\} \subset \widetilde{\mathscr{E}_0}$. Then, since $\widetilde{\mathbf{L}}\widetilde{u} \in \widetilde{\mathscr{E}_0}$, the first term vanishes, so that

$$\mathbf{L}_h u_h + D_\mu \mathbf{R}_h(0,0)\mu = 0.$$

Now $\mathbf{L}_h$ has a bounded inverse because 0 does not belong to its spectrum, so that we find

$$u_h = -\mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu.$$

Summarizing, for $\widetilde{u} \in \widetilde{\mathscr{E}}_0$, we have

$$\widetilde{u} = (u,\mu) = (u_0 + u_h,\mu) = (u_0 - \mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu,\mu),$$

which proves that $\widetilde{\mathscr{E}}_0 \subset \mathscr{F}_0$.

Next, notice that

$$\widetilde{\mathbf{L}}(u_0 - \mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu,\mu) = (\mathbf{L}_0 u_0 + D_\mu \mathbf{R}_0(0,0)\mu,0) \in \mathscr{E}_0 \times \{0\} \subset \mathscr{F}_0,$$

so that $\mathscr{F}_0$ is an invariant subspace for $\widetilde{\mathbf{L}}$. Consider the bases $\{e_j; j = 1,\dots,\dim\mathscr{E}_0\}$ and $\{f_k; k = 1,\dots,m\}$ of $\mathscr{E}_0$ and $\mathbb{R}^m$, respectively. Then the set

$$\{(e_j,0),(-\mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)f_k,f_k)\ ;\ j = 1,\dots,\dim\mathscr{E}_0,\ k = 1,\dots,m\}$$

is a basis for $\mathscr{F}_0$, in which we find that the matrix of $\widetilde{\mathbf{L}}\big|_{\mathscr{F}_0}$ is of the form

$$\begin{pmatrix} M_0 & M_1 \\ 0 & 0 \end{pmatrix},$$

with M_0 the matrix of $\mathbf{L}_0$ in the basis $\{e_j; j = 1,\dots,\dim\mathscr{E}_0\}$ and M_1 a matrix of size $m \times \dim\mathscr{E}_0$. The set of eigenvalues of M_0 is precisely the set σ_0, and we then conclude that

$$\sigma(\widetilde{\mathbf{L}}\big|_{\mathscr{F}_0}) = \sigma_0 \cup \{0\} \subset \widetilde{\sigma}_0.$$

In particular, this implies that $\mathscr{F}_0 \subset \widetilde{\mathscr{E}}_0$, which completes the proof of $\widetilde{\mathscr{E}}_0 = \mathscr{F}_0$. $\square$

Remark 3.4 *The analogue of the* reduced equation *(2.7) in this situation is*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \Psi(u_0,\mu),\mu) \stackrel{def}{=} f(u_0,\mu), \tag{3.6}$$

where we observe that $f(0,0) = 0$ and $D_{u_0} f(0,0) = \mathbf{L}_0$ has the spectrum σ_0. Similarly, we have the analogue of the equality (2.8),

$$\begin{aligned} D_{u_0}\Psi(u_0,\mu) f(u_0,\mu) = {} & \mathbf{L}_h \Psi(u_0,\mu) \\ & + \mathbf{P}_h \mathbf{R}(u_0 + \Psi(u_0,\mu),\mu) \text{ for all } u_0 \in \mathscr{E}_0. \end{aligned} \tag{3.7}$$

Exercise 3.5 *Consider a system of the form (3.1) for which* 0 *is a solution for all values of μ, i.e., such that* $\mathbf{R}(0,\mu) = 0$ *for all μ in a neighborhood of* 0 *in* $\mathbb{R}^m$. *Show that*

$$\Psi(0,\mu) = 0, \quad f(0,\mu) = 0,$$

for μ sufficiently small. Furthermore, set

$$\mathbf{L}_\mu = \mathbf{L} + D_u \mathbf{R}(0,\mu) \in \mathscr{L}(\mathscr{Z},\mathscr{X}) \quad \text{and} \quad \mathbf{A}_\mu = \frac{\partial f}{\partial u_0}(0,\mu).$$

Show that eigenvalues of $\mathbf{A}_\mu$ are precisely the eigenvalues of $\mathbf{L}_\mu$, which are the continuation for small μ of the purely imaginary eigenvalues of $\mathbf{L}$ *(i.e., those of* $\mathbf{L}_0$*)).*

Hint: *Identify the terms linear in u_0 in the identity*

$$\left(\mathbb{I}+D_{u_0}\Psi(u_0,\mu)\right)f(u_0,\mu)=\mathbf{L}(u_0+\Psi(u_0,\mu))+\mathbf{R}(u_0+\Psi(u_0,\mu),\mu)\ \textit{for all}\ u_0\in\mathscr{E}_0.$$

Remark 3.6 (Case when σ_0 does not lie on the imaginary axis) *A situation arising in some applications is one in which the eigenvalues in σ_0 of the operator $\mathbf{L}$ in (3.1) do not lie on the imaginary axis but stay close to the imaginary axis. More precisely, we still have the spectral decomposition in Hypothesis 2.4, satisfying the properties (i) and (ii), but with σ_0 such that*

$$\sigma_0=\{\lambda\in\sigma\ ;\ |\mathrm{Re}\,\lambda|\leq\delta\}\tag{3.8}$$

for some $\delta\ll\gamma$ sufficiently small. This means that σ_0 consists of a finite number of eigenvalues λ_j, $j=1,\dots,r$ of $\mathbf{L}$, with real parts that are small but not necessarily 0:

$$\mathrm{Re}\,\lambda_j=\varepsilon_j,\quad |\varepsilon_j|\leq\delta,\quad j=1,\dots,r.$$

In such a situation we can apply the result in Theorem 3.3 by arguing in the following way:

Consider the bounded linear operator

$$\mathbf{A}_\nu=\sum_{j=1}^{r}\nu_j\mathbf{P}_j\ \textit{for}\ \nu=(\nu_1,\dots,\nu_r)\in\mathbb{R}^r,$$

where $\mathbf{P}_j$ denotes the spectral projection associated with the eigenvalue $\lambda_j\in\sigma_0$ of $\mathbf{L}$. When $\nu=\varepsilon$, $\varepsilon=(\varepsilon_1,\dots,\varepsilon_r)$, the operator

$$\mathbf{L}'=\mathbf{L}-\mathbf{A}_\varepsilon,\quad \varepsilon=(\varepsilon_1,\dots,\varepsilon_r),$$

satisfies Hypothesis 2.4, the effect of adding $-\mathbf{A}_\varepsilon$ to $\mathbf{L}$ being that all eigenvalues in σ_0 are shifted on the imaginary axis. Consequently, we can apply the result in Theorem 3.3 to the modified system

$$\frac{du}{dt}=\mathbf{L}'u+\mathbf{R}'(u,\mu'),$$

where $\mu'=(\mu,\nu)$ and

$$\mathbf{R}'(u,\mu')=\mathbf{A}_\nu u+\mathbf{R}(u,\mu),$$

which satisfies the hypotheses in Theorem 3.3 with the parameter $\mu'=(\mu,\nu)\in\mathbb{R}^{m+r}$. We recover the original equation by taking $\nu=\varepsilon$, and find the invariant manifolds $\mathscr{M}_0(\mu,\varepsilon)$ for this equation, provided ε is sufficiently small, such that $(0,\varepsilon)$ belongs to the neighborhood $\mathscr{O}_{\mu'}$ of $(0,0)$ in $\mathbb{R}^{m+r}$ given by Theorem 3.3. This latter property is achieved when δ in (3.8) is sufficiently small, i.e., when the eigenvalues in σ_0 are close enough to the imaginary axis.

Remark 3.7 *(i) In (3.1) the parameter μ occurs only in the term $\mathbf{R}$, which takes values in $\mathscr{Y}$. A more general study would be for cases where μ also occurs in the linear terms which take values in $\mathscr{X}$. Then one would have a family of* *operators* $\mathbf{L}_\mu$ *with domains which may also depend upon* μ. *Such a situation requires a more delicate analysis, which does not enter in our setting.*

(ii) It is possible to develop the theory for a parameter μ *lying in a (infinite-dimensional) Banach space instead of* $\mathbb{R}^m$. *Nevertheless, for such a situation one needs to go back and adapt the proof of the general result in Theorem 2.9. The proof of Theorem 3.3 given above does not extend to this situation, since it relies upon the fact that* $\mathbb{R}^m$ *is finite-dimensional (one has that* $\dim \widetilde{\mathscr{E}}_0 = \dim \mathscr{E}_0 + m$, *and this quantity is infinite when* $\mathbb{R}^m$ *is replaced by an infinite- dimensional Banach space, so that the extended system (3.4) does not satisfy Hypothesis 2.4(ii)). We refer the reader to [73] for an example of a problem with a parameter varying in a function space, and for which the continuity of the reduction function* Ψ *with respect to the parameter, is only valid in* $\mathscr{X}$, *not in* $\mathscr{Z}$.

2.3.2 Nonautonomous Center Manifolds

We present in this section an extension of the result of center manifold Theorem 2.9 to the case of nonautonomous equations of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,t). \tag{3.9}$$

We replace here Hypothesis 2.1 by the following assumptions on $\mathbf{L}$ and $\mathbf{R}$.

Hypothesis 3.8 *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (3.9) have the following properties:*

(i) $\mathbf{L} \in \mathscr{L}(\mathscr{Z},\mathscr{X})$;
(ii) for some $k \geq 2$, *there exists a neighborhood* $\mathscr{V} \subset \mathscr{Z}$ *of* 0 *such that* $\mathbf{R} \in \mathscr{C}^k(\mathscr{V} \times \mathbb{R}, \mathscr{Y})$ *and*

$$\mathbf{R}(0,t) = 0, \quad D_u\mathbf{R}(0,t) = 0.$$

In addition, we assume that for any sufficiently small ε, *there exist positive constants* $\delta_0(\varepsilon) = O(\varepsilon^2)$ *and* $\delta_1(\varepsilon) = O(\varepsilon)$ *such that*

$$\sup_{u\in B_\varepsilon(\mathscr{Z})} \|\mathbf{R}(u,t)\|_{\mathscr{Y}} = \delta_0(\varepsilon), \qquad \sup_{u\in B_\varepsilon(\mathscr{Z})} \|D_u\mathbf{R}(u,t)\|_{\mathscr{L}(\mathscr{Z},\mathscr{Y})} = \delta_1(\varepsilon). \tag{3.10}$$

The equalities in the formula (3.10) above, show that the nonlinear term $\mathbf{R}$ is bounded with respect to all $t \in \mathbb{R}$, uniformly for u in any sufficiently small closed ball $B_\varepsilon(\mathscr{Z})$. Furthermore, the dependency in t of the system (3.9) is in the nonlinear term $\mathbf{R}$, only. In this sense, the following theorem is a "perturbation" result of center manifold Theorem 2.9.

Theorem 3.9 (Nonautonomous center manifolds) *Assume that Hypotheses 3.8, 2.4, and 2.7 hold. Then, there exist a map* $\Psi \in \mathscr{C}^k(\mathscr{E}_0 \times \mathbb{R}, \mathscr{Z}_h)$ *and* $c > 0$, *with*

$$\Psi(0,t) = 0, \quad D_{u_0}\Psi(0,t) = 0,$$

and

$$\sup_{u_0\in B_\varepsilon(\mathscr{E}_0)} \|\Psi(u_0,t)\|_{\mathscr{Z}} = c\delta_0(\varepsilon), \qquad \sup_{u_0\in B_\varepsilon(\mathscr{E}_0)} \|D_u\Psi(u_0,t)\|_{\mathscr{L}(\mathscr{Z})} = c\delta_1(\varepsilon),$$

for sufficiently small ε*, and a neighborhood* $\mathscr{O}$ *of* 0 *in* $\mathscr{Z}$ *such that the manifold*

$$\mathscr{M}_0(t) = \{u_0 + \Psi(u_0,t)\ ;\ (u_0,t) \in B_\varepsilon(\mathscr{E}_0)\times\mathbb{R}\} \subset \mathscr{Z}$$

has the following properties:

(i) the set $\{(t,u(t)) \in \mathbb{R}\times\mathscr{M}_0(t)\}$ *is a local integral manifold of (3.9);*
(ii) any solution u *of (3.9) staying in* $\mathscr{O}$ *for all* $t\in\mathbb{R}$ *satisfies* $u(t)\in\mathscr{M}_0(t)$.

We give a brief proof of this result in Appendix B.3 (see also [95] for a complete proof).

Remark 3.10 *The analogue of the reduced equation (2.7) in this situation is*

$$\frac{du_0}{dt} = \mathbf{L}_0u_0 + \mathbf{P}_0\mathbf{R}(u_0+\Psi(u_0,t),t) \stackrel{def}{=} f(u_0,t), \tag{3.11}$$

whereas the analogue of the equality (2.8) is

$$\begin{aligned}\partial_t\Psi(u_0,t) + D_{u_0}\Psi(u_0,t)f(u_0,t) &= \mathbf{L}_h\Psi(u_0,t)\\ &\quad +\mathbf{P}_h\mathbf{R}(u_0+\Psi(u_0,t),t)\ \textit{for all}\ u_0\in\mathscr{E}_0.\end{aligned}$$

There are at least two particular cases of equation (3.9) that are important in applications:

(i) the case in which the map $\mathbf{R}$ is periodic with respect to t, and
(ii) the case in which $\lim_{t\to\infty}\mathbf{R}(u,t)\to\mathbf{R}_\infty(u)$ or $\lim_{t\to-\infty}\mathbf{R}(u,t)\to\mathbf{R}_{-\infty}(u)$.

In these cases the reduction function Ψ, and then also the reduced system, has similar properties. We show in Appendix B.3 that the following result holds.

Corollary 3.11 (Special cases) *Assume that the hypothesis in Theorem 3.9 holds.*

(i) If the map $\mathbf{R}$ *is periodic with respect to* t*,* $\mathbf{R}(u,t) = \mathbf{R}(u,t+\tau)$ *for some* $\tau>0$*, then one can find a reduction function* Ψ *that is periodic, with the same period, namely* $\Psi(u_0,t) = \Psi(u_0,t+\tau)$ *for any* $(u_0,t)\in B_\varepsilon(\mathscr{E}_0)\times\mathbb{R}$.
(ii) Assume that there exist a map $\mathbf{R}_\infty\in\mathscr{C}^k(\mathscr{V},\mathscr{Y})$ *and* $d_0>0$ *such that*

$$\|\mathbf{R}(u,t)-\mathbf{R}_\infty(u)\|_{\mathscr{Y}} \le ce^{-d_0t}\ \textit{for all}\ (u,t)\in\mathscr{V}\times\mathbb{R}^+.$$

Then the result in center manifold Theorem 2.9 holds for the autonomous equation

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}_\infty(u), \tag{3.12}$$

and there exists $c'>0$ *such that*

$$\|\Psi(u_0,t)-\Psi_\infty(u_0)\|_{\mathscr{Z}_h} \le c' e^{-d_0 t} \text{ for all } (u_0,t)\in B_\varepsilon(\mathscr{E}_0)\times\mathbb{R}^+,$$

where Ψ_∞ is the reduction function for the autonomous equation (3.12). A similar result holds when $\|\mathbf{R}(u,t)-\mathbf{R}_{-\infty}(u)\|_{\mathscr{Y}} \le ce^{d_0 t}$ for all $(u,t)\in\mathscr{V}\times\mathbb{R}^-$.

2.3.3 Symmetries and Reversibility

We discuss in this section three cases of equations possessing a certain symmetry. In each case we show that this symmetry is inherited by both the reduction function Ψ and the reduced system.

Equivariant Systems

We start with the case of an equation that is equivariant under the action of a linear operator. More precisely, we make the following assumptions.

Hypothesis 3.12 (Equivariant equation) *We assume that there exists a linear operator $\mathbf{T}\in\mathscr{L}(\mathscr{X})\cap\mathscr{L}(\mathscr{Z})$, which commutes with the vector field in equation (2.1),*

$$\mathbf{TL}u=\mathbf{LT}u, \quad \mathbf{TR}(u)=\mathbf{R}(\mathbf{T}u).$$

We further assume that the restriction $\mathbf{T}_0$ of $\mathbf{T}$ to the subspace $\mathscr{E}_0$ is an isometry.

Notice that the fact that the operator $\mathbf{T}$ commutes with the vector field in the equation (2.1) implies that the subspace $\mathscr{E}_0$ is invariant under the action of $\mathbf{T}$, so that the restriction $\mathbf{T}_0$ in the hypothesis above is well defined. Indeed, since $\mathbf{T}$ commutes with $\mathbf{L}$, it also commutes with its resolvent $(\lambda\mathbb{I}-\mathbf{L})^{-1}$, and from the Dunford integral formula (2.2) it follows that $\mathbf{T}$ commutes with the spectral projector $\mathbf{P}_0$. Consequently, the spectral subspace $\mathscr{E}_0$ associated with $\mathbf{P}_0$ is invariant under the action of $\mathbf{T}$.

We show in Appendix B.4 that the following result holds in this situation.

Theorem 3.13 (Center manifold theorem for equivariant equations) *Under the assumptions in Theorem 2.9, we further assume that Hypothesis 3.12 holds. Then one can find a reduction function Ψ in Theorem 2.9 which commutes with $\mathbf{T}$, i.e.,*

$$\mathbf{T}\Psi(u_0)=\Psi(\mathbf{T}_0u_0) \text{ for all } u_0\in\mathscr{E}_0,$$

and such that the vector field in the reduced equation (2.7) commutes with $\mathbf{T}_0$.

We point out that analogous results hold for the parameter-dependent equation (3.1) and in the nonautonomous case for the equation (3.9).

Reversible Systems

Next, we consider the case of reversible equations, when the vector field in (2.1) anticommutes with a symmetry $\mathbf{S}$. More precisely, we make the following assumptions.

Hypothesis 3.14 (Reversible equation) *Assume that there exists a linear symmetry* $\mathbf{S} \in \mathscr{L}(\mathscr{X}) \cap \mathscr{L}(\mathscr{Z})$, *with*

$$\mathbf{S}^2 = \mathbb{I}, \quad \mathbf{S} \neq \mathbb{I},$$

and which anticommutes with the vector field in (2.1),

$$\mathbf{S}\mathbf{L}u = -\mathbf{L}\mathbf{S}u, \quad \mathbf{S}\mathbf{R}(u) = -\mathbf{R}(\mathbf{S}u). \tag{3.13}$$

Notice that in this case, if $t \mapsto u(t)$ is a solution of (2.1), then $t \mapsto \mathbf{S}u(-t)$ is also a solution of (2.1). Moreover, the spectrum of the linear operator $\mathbf{L}$ is symmetric with respect to the origin in the complex plane. Indeed, from the first equality in (3.13) we deduce that

$$\mathbf{S}(\lambda\mathbb{I} - \mathbf{L})^{-1} = (\lambda\mathbb{I} + \mathbf{L})^{-1}\mathbf{S},$$

which shows that the resolvent set $\rho(\mathbf{L})$ as well as its complement $\sigma(\mathbf{L})$ are symmetric with respect to the origin. In particular, for real systems, besides the usual symmetry with respect to the real axis, in this case the spectrum of $\mathbf{L}$ is also symmetric with respect to the imaginary axis. We also point out that if λ is an eigenvalue of $\mathbf{L}$ with the associated eigenvector ζ, then $-\lambda$ is an eigenvalue with the associated eigenvector $\mathbf{S}\zeta$.

As in the case of equivariant equations with Hypothesis 3.12, we have that the spectral subspace $\mathscr{E}_0$ is invariant under the action of $\mathbf{S}$. Indeed, since the spectrum of the operator $\mathbf{L}$ is symmetric with respect to the origin in the complex plane, we may choose the curve Γ in the Dunford integral formula (2.2) such that it is also symmetric with respect to the origin in the complex plane. Then a direct calculation shows that the spectral projection $\mathbf{P}_0$ given by (2.2) commutes with $\mathbf{S}$, so that $\mathscr{E}_0$ is invariant under the action of $\mathbf{S}$.

By arguing as in the case of equivariant equations, we obtain here the following result.

Theorem 3.15 (Center manifold theorem for reversible equations) *Under the assumptions of Theorem 2.9, we further assume that Hypothesis 3.14 holds. Then one can find a reduction function* Ψ *in Theorem 2.9 that commutes with* $\mathbf{S}$,

$$\mathbf{S}\Psi(u_0) = \Psi(\mathbf{S}_0 u_0) \textit{ for all } u_0 \in \mathscr{E}_0,$$

where $\mathbf{S}_0$ *is the restriction of* $\mathbf{S}$ *to the subspace* $\mathscr{E}_0$ *and such that the reduced equation is reversible, i.e., the vector field in (2.7) anticommutes with* $\mathbf{S}_0$.

A similar result holds for the parameter-dependent equation (3.1), whereas in the nonautonomous case for equation (3.9) the following holds.

Corollary 3.16 (Reversible nonautonomous equations) *Under the assumptions of Theorem 3.9, we further assume that the equation (3.9) is reversible, i.e., there exists a symmetry* $\mathbf{S} \in \mathscr{L}(\mathscr{X}) \cap \mathscr{L}(\mathscr{Z})$*, with* $\mathbf{S}^2 = \mathbb{I}$ *and* $\mathbf{S} \neq \mathbb{I}$*, such that*

$$\mathbf{SL}u = -\mathbf{LS}u, \quad \mathbf{SR}(u,t) = -\mathbf{R}(\mathbf{S}u, -t).$$

Then, one can find a reduction function Ψ *in the Theorem 3.9 that satisfies*

$$\mathbf{S}\Psi(u_0,t) = \Psi(\mathbf{S}_0 u_0, -t) \text{ for all } u_0 \in \mathscr{E}_0,$$

and the reduced equation is reversible, i.e., the vector field in (3.11) satisfies

$$\mathbf{S}_0 f(u_0,t) = -f(\mathbf{S}_0 u_0, -t) \text{ for all } u_0 \in \mathscr{E}_0.$$

Continuous Symmetry

We end this section with the case where equation (2.1) is equivariant under a one-parameter group of isometries. We focus on the case of the underlying group $\mathbb{R}$, and, instead of a single equilibrium at the origin, the equation has a "line" of equilibria. This situation is encountered in the applications in Sections 5.1.2, 5.1.3, and 5.2.2 of Chapter 5. Other groups of symmetries can be treated in the same spirit, however, this may require more specific tools and further evolved algebra. We refer the reader to the book [16] for such cases. More precisely, we make here the following hypotheses.

Hypothesis 3.17 (Continuous symmetry) *Assume that there exists a continuous one-parameter group of isometries* $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}} \subset \mathscr{L}(\mathscr{Z}) \cap \mathscr{L}(\mathscr{X})$*, which commutes with the vector field in (2.1), that is, such that the following properties hold:*

(i) *the map* $\alpha \in \mathbb{R} \mapsto \mathbf{T}_\alpha \in \mathscr{L}(\mathscr{Z}) \cap \mathscr{L}(\mathscr{X})$ *is continuous;*
(ii) $\mathbf{T}_0 = \mathbb{I}$ *and* $\mathbf{T}_{\alpha+\beta} = \mathbf{T}_\alpha \mathbf{T}_\beta$ *for all* $\alpha,\ \beta \in \mathbb{R}$*;*
(iii) $\mathbf{T}_\alpha \mathbf{L}u = \mathbf{L}\mathbf{T}_\alpha u$ *and* $\mathbf{T}_\alpha \mathbf{R}(u) = \mathbf{R}(\mathbf{T}_\alpha u)$ *for all* $\alpha \in \mathbb{R}$.

Further assume that the infinitesimal generator τ *of the group* $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}} \subset \mathscr{L}(\mathscr{X})$ *belongs to* $\mathscr{L}(\mathscr{Z},\mathscr{Y})$*,*

$$\tau := \frac{d\mathbf{T}_\alpha}{d\alpha}\Big|_{\alpha=0} \in \mathscr{L}(\mathscr{Z},\mathscr{Y}).$$

Hypothesis 3.18 (Equilibria) *Assume that equation (2.1) has a nontrivial equilibrium* $u^* \in \mathscr{Z}$*,*

$$\mathbf{L}u^* + \mathbf{R}(u^*) = 0, \quad u^* \neq 0,$$

satisfying $\tau u^* \in \mathscr{Z} \setminus \{0\}$.

An immediate consequence of the hypotheses above is that equation (2.1) possesses a *line of equilibria* given by $\{\mathbf{T}_\alpha u^* \in \mathscr{Z}; \alpha \in \mathbb{R}\}$. Furthermore, since $\tau u^* \in \mathscr{Z}$, we may differentiate the identity

$$\mathbf{L}\mathbf{T}_\alpha u^* + \mathbf{R}(\mathbf{T}_\alpha u^*) = 0$$

at $\alpha = 0$ and obtain

$$\mathbf{L}\tau u^* + D\mathbf{R}(u^*)\tau u^* = 0. \tag{3.14}$$

This shows that τu^* belongs to the kernel of the linearization $\mathbf{L} + D\mathbf{R}(u^*)$ of the vector field at the equilibrium u^* (this eigenvector is often called the "Goldstone mode" by physicists).

Our purpose is to construct *a local center manifold along this line of equilibria* in $\mathscr{Z}$, taking into account the continuous symmetry of the equation. We make the Ansatz

$$u(t) = \mathbf{T}_{\alpha(t)}(u^* + v(t)), \tag{3.15}$$

replacing the unknown u by the pair (α, v), with $\alpha(t) \in \mathbb{R}$ and $v(t) \in \mathscr{Z}$ satisfying a transversality condition that we define now. For this we decompose the space $\mathscr{X}$ in the subspace spanned by τu^*, parallel to the line of equilibria, and a complementary subspace. Consider the linear form φ^* in the dual space $\mathscr{X}^*$ such that $\langle \tau u^*, \varphi^* \rangle = 1$ (e.g., see [76, p. 135]). We define the subspace $\mathscr{H} \subset \mathscr{X}$ transverse to τu^*,

$$\mathscr{H} = \{v \in \mathscr{X} \ ; \ \langle v, \varphi^* \rangle = 0\},$$

which provides us with a decomposition of $\mathscr{X}$ into two complementary closed subspaces,

$$\mathscr{X} = \{\tau u^*\} \oplus \mathscr{H}.$$

The linear operators

$$\Pi_0 u = \langle u, \varphi^* \rangle \tau u^*, \quad \Pi_{\mathscr{H}} = \mathbb{I} - \Pi_0$$

are projections onto the subspaces $\{\tau u^*\}$ and $\mathscr{H}$, respectively. Since $\tau u^* \in \mathscr{Z}$, we have that $\Pi_{\mathscr{H}} u \in \mathscr{Z}$ (resp., $\Pi_{\mathscr{H}} u \in \mathscr{Y}$) if $u \in \mathscr{Z}$ (resp., $u \in \mathscr{Y}$), so that we have similar decompositions for $\mathscr{Z}$ and $\mathscr{Y}$. We now choose v in (3.15) such that $v(t)$ belongs to $\mathscr{H}$, i.e.,

$$\Pi_0 v(t) = 0 \quad \Longleftrightarrow \quad \langle v(t), \varphi^* \rangle = 0.$$

Next, we substitute the Ansatz (3.15) into the equation (2.1) and obtain the equation

$$\tau \mathbf{T}_\alpha (u^* + v) \frac{d\alpha}{dt} + \mathbf{T}_\alpha \frac{dv}{dt} = \mathbf{L}\mathbf{T}_\alpha v + \mathbf{R}(\mathbf{T}_\alpha (u^* + v)) - \mathbf{R}(\mathbf{T}_\alpha u^*),$$

where we have used the fact that $\mathbf{T}_\alpha u^*$ is an equilibrium of (2.1). Using the equivariance property in Hypothesis 3.17(iii) we find

$$(\tau u^* + \tau v) \frac{d\alpha}{dt} + \frac{dv}{dt} = \mathbf{A} v + \tilde{\mathbf{R}}(v),$$

in which

$$\mathbf{A}v = \mathbf{L}v + D\mathbf{R}(u^*)v, \quad \tilde{\mathbf{R}}(v) = \mathbf{R}(u^* + v) - \mathbf{R}(u^*) - D\mathbf{R}(u^*)v.$$

Projecting successively with Π_0 and $\Pi_{\mathscr{H}}$, this gives the first order system for (α, v),

$$\frac{d\alpha}{dt} = (1 + \langle \tau v, \varphi^* \rangle)^{-1} \langle \mathbf{A}v + \tilde{\mathbf{R}}(v), \varphi^* \rangle \stackrel{\text{def}}{=} g(v) \tag{3.16}$$

$$\frac{dv}{dt} = \Pi_{\mathscr{H}} \mathbf{A}v + \Pi_{\mathscr{H}} \tilde{\mathbf{R}}(v) - g(v) \Pi_{\mathscr{H}} \tau v, \tag{3.17}$$

which holds for $v \in \mathscr{Z}$ sufficiently small.

The key property of the system (3.16)–(3.17) is that the vector field is independent of α, which in particular does not appear in the equation (3.17). This equation decouples, so that we can solve it separately, and once v is known we obtain α from the first equation. The differential equation (3.17) is of the form of (2.1), with the spaces $\mathscr{X}$, $\mathscr{Z}$, $\mathscr{Y}$ replaced by

$$\mathscr{X}' = \mathscr{H}, \quad \mathscr{Z}' = \Pi_{\mathscr{H}} \mathscr{Z}, \quad \mathscr{Y}' = \Pi_{\mathscr{H}} \mathscr{Y},$$

respectively, and operators $\mathbf{L}$ and $\mathbf{R}$ replaced by

$$\mathbf{L}' = \Pi_{\mathscr{H}} \mathbf{A}, \quad \mathbf{R}'(v) = \Pi_{\mathscr{H}} (\tilde{\mathbf{R}}(v) - g(v) \tau v), \tag{3.18}$$

respectively. In particular, this means that thanks to the choice of the Ansatz (3.15), the dimension of the problem is decreased by one, the space $\mathscr{X}$ being replaced by $\mathscr{H}$. In fact we suppressed the direction τu^*, which belongs to the kernel of $\mathbf{A}$ as shown by (3.14). Furthermore, once we obtain a local center manifold for equation (3.17), we have a center manifold for equation (2.1), with one additional dimension, in a neighborhood of the line of stationary solutions $\{\mathbf{T}_\alpha u^* \in \mathscr{Z}; \alpha \in \mathbb{R}\}$. More precisely, we have the following result.

Theorem 3.19 (Center manifolds in presence of continuous symmetry) *Assume that Hypothesis 2.1 holds and that the linear operator $\mathbf{L}' = \Pi_{\mathscr{H}} \mathbf{A}$ in (3.18) acting in $\mathscr{X}'$ satisfies Hypotheses 2.4 and 2.7. Then for the differential equation (3.17) the result in Theorem 2.9 holds.*

Let $\mathscr{O}'$, Ψ', and $\mathscr{E}_0'$ be respectively the neighborhood of the origin in $\mathscr{Z}'$, the reduction function, and the spectral subspace, given by Theorem 2.9 for (3.17). Consider the "tubular" neighborhood

$$\mathscr{O} = \{\mathbf{T}_\alpha (u^* + v) \; ; \; v \in \mathscr{O}', \; \alpha \in \mathbb{R}\} \subset \mathscr{Z}$$

of the line of equilibria $\{\mathbf{T}_\alpha u^* \in \mathscr{Z}; \alpha \in \mathbb{R}\}$, and the manifold

$$\mathscr{M}_0 = \{\mathbf{T}_\alpha (u^* + v_0 + \Psi(v_0)) \; ; \; v_0 \in \mathscr{E}_0', \; \alpha \in \mathbb{R}\} \subset \mathscr{Z}. \tag{3.19}$$

Then for differential equation (2.1) the following properties hold:

(i) The manifold $\mathscr{M}_0$ is locally invariant, i.e., if u is a solution of (2.1) satisfying $u(0) \in \mathscr{M}_0 \cap \mathscr{O}$ and $u(t) \in \mathscr{O}$ for all $t \in [0, T]$, then $u(t) \in \mathscr{M}_0$ for all $t \in [0, T]$.

(ii) *$\mathscr{M}_0$ contains the set of solutions of (2.1) staying in $\mathscr{O}$ for all $t \in \mathbb{R}$, i.e., if u is a solution of (2.1) satisfying $u(t) \in \mathscr{O}$ for all $t \in \mathbb{R}$, then $u(0) \in \mathscr{M}_0$.*

We point out that in this situation the center manifold $\mathscr{M}_0$ contains the solutions which stay close to the line of equilibria for all $t \in \mathbb{R}$. These solutions are of the form

$$u = \mathbf{T}_\alpha(u^* + v_0 + \Psi(v_0)),$$

with α and v_0 satisfying the reduced system

$$\frac{d\alpha}{dt} = g(v_0 + \Psi(v_0)) \tag{3.20}$$

$$\begin{aligned}\frac{dv_0}{dt} &= \Pi_{\mathscr{H}}\mathbf{A}v_0 + \mathbf{P}_0'\left(\Pi_{\mathscr{H}}\tilde{\mathbf{R}}(v_0 + \Psi(v_0))\right)\\ &\quad -\mathbf{P}_0'\left(g(v_0 + \Psi(v_0))\Pi_{\mathscr{H}}\tau(v_0 + \Psi(v_0))\right),\end{aligned} \tag{3.21}$$

in which g is defined in (3.16) and $\mathbf{P}_0'$ is the spectral projector for the linear operator $\mathbf{L}' = \Pi_{\mathscr{H}}\mathbf{A}$ defined as in Section 2.2.1. Furthermore, for such a solution we have that v_0 is a small bounded solution of the equation (3.21), whereas α given by (3.20) has bounded derivative and may grow linearly in t.

Similar results hold for the parameter-dependent equation (3.1) and for the nonautonomous equation (3.9).

2.3.4 Empty Unstable Spectrum

A particular case, which appears in some applications, e.g. in parabolic problems, occurs when the unstable spectrum σ_+ of $\mathbf{L}$ is empty. Then we complete general Hypothesis 2.7 by the following assumptions, which allow us to obtain further information about the center manifolds in this case.

Hypothesis 3.20 (Empty unstable spectrum) *Assume that $\sigma_+ = \varnothing$ and that for any $\eta \in [0, \gamma]$ the following properties hold:*

(i) *For any $f \in \mathscr{F}_\eta(\mathbb{R}, \mathscr{Y}_h)$ the linear problem*

$$\frac{du_h}{dt} = \mathbf{L}_h u_h + f$$

has a unique solution $u_h = \mathbf{K}_h f \in \mathscr{F}_\eta(\mathbb{R}, \mathscr{Z}_h)$. Furthermore, the linear map $\mathbf{K}_h$ belongs to $\mathscr{L}(\mathscr{F}_\eta(\mathbb{R}, \mathscr{Y}_h), \mathscr{F}_\eta(\mathbb{R}, \mathscr{Z}_h))$, and there exists a continuous map $C : [0, \gamma] \to \mathbb{R}$ such that

$$\|\mathbf{K}_h\|_{\mathscr{L}(\mathscr{F}_\eta(\mathbb{R},\mathscr{Y}_h),\mathscr{F}_\eta(\mathbb{R},\mathscr{Z}_h))} \leq C(\eta).$$

(ii) *The linear initial value problem*

$$\frac{du_h}{dt} = \mathbf{L}_h u_h, \quad u_h|_{t=0} = u_h(0) \in \mathscr{Z}_h,$$

has a unique solution $u_h(t) \in \mathscr{C}^0(\mathbb{R}^+, \mathscr{Z}_h)$, which satisfies

$$\|u_h(t)\|_{\mathscr{X}} \leq c_\eta e^{-\eta t} \text{ for all } t \geq 0$$

for some positive constant c_η.

As for Hypothesis 2.7, we have that these assumptions are satisfied, provided Hypothesis 2.15 holds (see Remark B.2 in Appendix B.2).

Exercise 3.21 *Prove that Hypothesis 3.20 is satisfied in finite dimensions when $\mathscr{X} = \mathbb{R}^n$ and $\sigma_+ = \varnothing$.*

Theorem 3.22 (Center manifold theorem for empty unstable spectrum) *Under the assumptions of Theorem 2.9, further assume that Hypothesis 3.20 holds. Then in addition to the properties in Theorem 2.9 the following holds.*

The local center manifold $\mathscr{M}_0$ is locally attracting, i.e., any solution of (2.1) that stays in $\mathscr{O}$ for all $t > 0$ tends exponentially towards a solution of (2.1) on $\mathscr{M}_0$. More precisely, if $u(0) \in \mathscr{O}$ and the solution $u(t; u(0))$ of (2.1) satisfies $u(t; u(0)) \in \mathscr{O}$ for all $t > 0$, then there exists $\widetilde{u} \in \mathscr{M}_0 \cap \mathscr{O}$ and $\gamma' > 0$ such that

$$u(t; u(0)) = u(t; \widetilde{u}) + O(e^{-\gamma' t}) \text{ as } t \to \infty.$$

(Here we denoted by $u(t; u(0))$ the solution of (2.1) satisfying $u|_{t=0} = u(0)$).

We prove this result in Appendix B.5. In addition, since according to the proof of Theorem 3.3, the parameter-dependent equation (3.1) can be regarded as a particular case of equation (2.1), we can extend the result above to equation (3.1).

Theorem 3.23 (Parameter-dependent center manifolds) *Assume that Hypotheses 3.1, 2.4, 2.7, and 3.20 hold. Then in addition to the properties in Theorem 3.3 the following holds.*

The local center manifold $\mathscr{M}_0(\mu)$ is locally attracting, i.e., any solution of (3.1) that stays in $\mathscr{O}_u$ for all $t > 0$ tends exponentially towards a solution of (3.1) on $\mathscr{M}_0(\mu)$. More precisely, if $u(0) \in \mathscr{O}_u$ and the solution $u(t; u(0))$ of (3.1) satisfies $u(t; u(0)) \in \mathscr{O}_u$ for all $t > 0$, then there exists $\widetilde{u} \in \mathscr{M}_0(\mu) \cap \mathscr{O}_u$ and $\gamma' > 0$ such that

$$u(t; u(0)) = u(t; \widetilde{u}) + O(e^{-\gamma' t}) \text{ as } t \to \infty.$$

(Here we denoted by $u(t; u(0))$ the solution of (3.1) satisfying $u|_{t=0} = u(0)$.)

2.4 Further Examples and Exercises

We end this chapter with some further examples in which we apply the different variants of center manifold Theorem 2.9 presented in Section 2.3. In each example we show how to check the hypotheses and discuss the reduced system. In contrast to the second example given in Section 2.2.4, here we work in Hilbert spaces, which, in particular, simplifies the checking of Hypothesis 2.7 (see Remark 2.16). In addition, these examples are such that $u = 0$ is a solution of the system for all values of the parameter(s), except for the example in Section 2.4.3, case V, and the example in Section 2.4.4. This property allows us to use the result in Exercise 3.5, and so simplify some computations.

2.4.1 A Fourth Order ODE

Consider the fourth order ODE

$$u^{(4)} - u'' - \mu u - au^2 = 0, \tag{4.1}$$

where μ is a small parameter and a a given real number. For $\mu = 0$ this is precisely equation (2.11), studied in Section 2.2.4.

Formulation as a First Order System

We start by writing equation (2.11) in the form (3.1). As in Section 2.2.4, we set $U = (u, u_1, u_2, u_3)$ with $u_1 = u'$, $u_2 = u'' - u$, $u_3 = u_2'$, and then the equation is equivalent to the system

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}(U, \mu), \tag{4.2}$$

in which

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{R}(U, \mu) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu u + au^2 \end{pmatrix}.$$

Here $\mathbf{L}$ is the same 4×4-matrix, and $\mathbf{R} : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is a smooth map, so that we choose again

$$\mathscr{X} = \mathscr{Y} = \mathscr{Z} = \mathbb{R}^4.$$

In addition, notice that the system (4.2) possesses a reversibility symmetry, i.e., $\mathbf{L}$ and $\mathbf{R}(\cdot, \mu)$ anticommute with

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This symmetry is a consequence of the fact that the equation (4.1) is invariant under the reflection $t \mapsto -t$.

Checking the Hypotheses

Clearly, Hypothesis 3.1 is satisfied for $\mathbf{L}$ and $\mathbf{R}$ as above for any $k \geq 2$, and neighborhoods $\mathscr{V}_u = \mathbb{R}^4$ and $\mathscr{V}_\mu = \mathbb{R}$. We have seen, in Section 2.2.4, that $\mathbf{L}$ satisfies Hypothesis 2.4 with

$$\sigma_+ = \{1\}, \quad \sigma_0 = \{0\}, \quad \sigma_- = \{-1\},$$

and that Hypothesis 2.7 holds because $\mathscr{X}$ is finite-dimensional. Consequently, we can apply center manifold Theorem 3.3, and conclude the existence of a local two-dimensional center manifold of class $\mathscr{C}^k$ for any arbitrary, but fixed, $k \geq 2$ for any μ sufficiently small.

In addition, since system (4.2) is reversible, Hypothesis 3.14 is also satisfied, so that according to Theorem 3.15 the reduced equation is reversible, i.e., the vector field in this equation anticommutes with the symmetry $\mathbf{S}_0$ induced by $\mathbf{S}$ on $\mathscr{E}_0$.

Reduced Equation

We compute now the Taylor expansion, up to order 2, of the vector field in the reduced equation. Clearly, for $\mu = 0$ we have the expansion found in Section 2.2.4.

Consider the basis $\{\zeta_0, \zeta_1\}$ of $\mathscr{E}_0$ computed in Section 2.2.4, and notice that $\mathbf{S}$ acts on this basis through

$$\mathbf{S}\zeta_0 = \zeta_0, \quad \mathbf{S}\zeta_1 = -\zeta_1,$$

so that

$$\mathbf{S}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, according to Theorems 3.3 and 3.15, solutions on the center manifold are of the form

$$U(t) = U_0(t) + \Psi(U_0(t), \mu), \tag{4.3}$$

in which $\Psi(0,\mu) = 0$, $D\Psi(0,0) = 0$, $\Psi(\mathbf{S}_0 U_0, \mu) = \mathbf{S}\Psi(U_0, \mu)$, and $U_0(t) \in \mathscr{E}_0$, so that

$$U_0(t) = A(t)\zeta_0 + B(t)\zeta_1, \tag{4.4}$$

where A and B are real-valued functions. Notice that $\Psi(0,\mu) = 0$, because $\mathbf{R}(0,\mu) = 0$ (see Exercise 3.5). The reduced system is an ODE for $U_0 = (A, B)$, which now depends upon μ, and according to (3.6) it is given by

$$\frac{dU_0}{dt} = \mathbf{L}_0 U_0 + \mathbf{P}_0 \mathbf{R}(U_0 + \Psi(U_0, \mu), \mu), \tag{4.5}$$

where $\mathbf{L}_0$ and $\mathbf{P}_0$ are as in Section 2.2.4. Again, since the last component of the vector ζ_0^* vanishes, we have that

$$\mathbf{P}_0\mathbf{R}(U_0+\Psi(U_0,\mu),\mu) = \langle \mathbf{R}(U_0+\Psi(U_0,\mu),\mu),\zeta_1^*\rangle\zeta_1,$$

and since now $\Psi(U_0) = O(\|U_0\|(|\mu|+\|U_0\|))$, we conclude that

$$\mathbf{P}_0\mathbf{R}(U_0+\Psi(U_0,\mu),\mu) = \langle \mathbf{R}(U_0),\zeta_1^*\rangle\zeta_1 + O(\|U_0\|(|\mu|^2+\|U_0\|^2)).$$

The explicit formulas for $\mathbf{P}_0$, $\mathbf{R}$, and U_0 give

$$\mathbf{R}(U_0) = \mathbf{R}(A\zeta_0+B\zeta_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mu A + aA^2 \end{pmatrix},$$

so that

$$\mathbf{P}_0\mathbf{R}(U_0+\Psi(U_0,\mu),\mu) = \left(-\mu A + aA^2 + O((|A|+|B|)(|\mu|^2+|A|^2+|B|^2))\right)\zeta_1.$$

We conclude that the reduced system (2.15), in the basis $\{\zeta_0,\zeta_1\}$, is

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= -\mu A + aA^2 + O((|A|+|B|)(|\mu|^2+|A|^2+|B|^2)). \end{aligned}$$

In addition, the vector field in this system anticommutes with the matrix $\mathbf{S}_0$, which implies that the right hand side in the second equation above is even in B, so that the higher order terms in the expansion are in fact of order $O((|A|+|B|^2)(|\mu|^2+|A|^2+|B|^4))$.

Remark 4.1 *(i) For the calculation of the terms that are linear in A and B in the reduced equation, we can also use the result in Exercise 3.5. According to this result, the two eigenvalues of the 2×2-matrix obtained by linearizing the vector field in the reduced equation at $(A,B) = (0,0)$ are precisely the two eigenvalues of the matrix*

$$\mathbf{L}_\mu = \mathbf{L} + D\mathbf{R}(0,\mu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu & 0 & 0 & 0 \end{pmatrix},$$

which are the continuation of the double eigenvalue 0 of $\mathbf{L}$ for small μ. A direct calculation gives the eigenvalues

$$\lambda^2 = \frac{1\pm\sqrt{1+4\mu}}{2}.$$

Hence, the two eigenvalues close to 0 satisfy $\lambda_\pm^2 = -\mu + O(|\mu|^2)$. Next, the 2×2-matrix obtained by linearizing the vector field in the reduced equation at $(A,B) = (0,0)$ is of the form

$$\begin{pmatrix} 0 & 1 \\ \alpha(\mu) & 0 \end{pmatrix},$$

since, as we have seen above, B is the only term in the first component of the vector field, and the second component is even in B. Consequently,

$$\alpha(\mu) = -\mu + O(|\mu|^2),$$

which gives the same result as above.

(ii) *For the computation of an expansion up to order 3, or higher, one needs to compute the terms of order 2 in the expansion of Ψ (see also Remark 2.21). This can be done by substituting the Ansatz*

$$\begin{aligned} \Psi(A,B) &= \Psi_{101}\mu A + \Psi_{011}\mu B + \Psi_{200}A^2 + \Psi_{110}AB + \Psi_{020}B^2 \\ &\quad + O((|A|+|B|)(|\mu|^2+|A|^2+|B|^2)) \end{aligned} \tag{4.6}$$

in the identity (3.7), and then the vectors Ψ_{ijk} are determined by identifying powers of μ, A, and B. Besides the fact that these vectors belong to the space $(\mathbb{I}-\mathbf{P}_0)\mathbb{R}^4$, so that they are orthogonal to both ζ_0^* and ζ_1^*, due to the reversibility symmetry they also satisfy

$$\begin{gathered} \mathbf{S}\Psi_{101} = \Psi_{101}, \quad \mathbf{S}\Psi_{011} = -\Psi_{011}, \quad \mathbf{S}\Psi_{200} = \Psi_{200}, \\ \mathbf{S}\Psi_{110} = -\Psi_{110}, \quad \mathbf{S}\Psi_{020} = \Psi_{020}. \end{gathered}$$

(iii) *An alternative way of computing the reduced system is to directly substitute formulas (4.3), (4.4), and (4.6) into the first order system (4.2) and calculate the Taylor expansions of both sides of the resulting system. We use this approach in examples that follow, in this section.*

(iv) *The terms in the expansion of the vector field that do not depend upon μ can be computed separately, by setting $\mu = 0$ from the beginning. The other terms, depending upon μ, can be calculated afterwards by restricting to such terms in the Taylor expansions.*

2.4.2 Burgers Model

We consider the initial boundary value problem

$$\frac{\partial\phi}{\partial t} = \frac{1}{\mathscr{R}}\frac{\partial^2\phi}{\partial x^2} + \phi - \frac{\partial(\phi^2)}{\partial x} + U\phi, \tag{4.7}$$

$$\frac{dU}{dt} = -\frac{1}{\mathscr{R}}U - \int_0^1 \phi^2(x,t)dx, \tag{4.8}$$

$$\phi(0,t) = \phi(1,t) = 0, \tag{4.9}$$

where $\phi(x,t) \in \mathbb{R}$ and $U(t) \in \mathbb{R}$ for $(x,t) \in (0,1) \times \mathbb{R}$. This model equation, introduced by J. M. Burgers [11], is a one-dimensional model used for understanding instabilities in viscous fluid flows. In this system ϕ represents a velocity fluctuation, U is the induced perturbation on the mean basic flow, and $\mathscr{R}$ is the Reynolds number, proportional to the inverse of viscosity. The product $U\phi$ represents the interaction between the mean flow and the perturbation, the derivative of ϕ^2 represents inertial terms, and the integral represents Reynolds stresses.

Formulation as a First Order Equation

We start by writing the problem (4.7)–(4.9) in the form (2.1), but now with linear part $\mathbf{L}$ depending upon the parameter $\mathscr{R}$, $\mathbf{L} = \mathbf{L}_{\mathscr{R}}$. We set

$$u = \begin{pmatrix} \phi \\ U \end{pmatrix}, \quad \mathbf{L}_{\mathscr{R}} u = \begin{pmatrix} \frac{1}{\mathscr{R}} \frac{\partial^2 \phi}{\partial x^2} + \phi \\ -\frac{1}{\mathscr{R}} U \end{pmatrix}, \quad \mathbf{R}(u) = \begin{pmatrix} -\frac{\partial(\phi^2)}{\partial x} + U\phi \\ -\int_0^1 \phi^2(x,\cdot)dx \end{pmatrix},$$

and choose the Hilbert space

$$\mathscr{X} = L^2(0,1) \times \mathbb{R}.$$

As in the example given in Section 2.2.4, we include the boundary conditions (4.9) in the domain of definition $\mathscr{Y}$ of the operator $\mathbf{L}_{\mathscr{R}}$, by taking

$$\mathscr{Z} = (H^2(0,1) \cap H_0^1(0,1)) \times \mathbb{R}.$$

Finally, we set

$$\mathscr{Y} = H_0^1(0,1) \times \mathbb{R},$$

so that $\mathbf{R}(u) \in \mathscr{Z}$ for $u \in \mathscr{Y}$. Notice that the system commutes with the symmetry $\mathbf{T}$ defined by

$$\mathbf{T} \begin{pmatrix} \phi(x) \\ U \end{pmatrix} = \begin{pmatrix} -\phi(1-x) \\ U \end{pmatrix},$$

which is an isometry in both $\mathscr{X}$ and $\mathscr{Z}$.

This formulation of the problem does not quite enter into the setting of center manifold theorems presented in the previous sections, because the linear operator depends upon the parameter $\mathscr{R}$. The next step consists in determining the spectrum of this operator in order to detect the "critical" values of the parameter $\mathscr{R}$, where its spectrum contains purely imaginary eigenvalues. These values are bifurcation points. Then we choose such a bifurcation point and apply the result in the parameter-dependent version of the center manifold theorem, Theorem 3.3, by taking $\mathbf{L}$ to be the operator $\mathbf{L}_{\mathscr{R}}$ at this bifurcation point.

Spectrum of the Linear Operator

The linear operator $\mathbf{L}_{\mathscr{R}}$ is a closed operator in $\mathscr{X}$ with domain $\mathscr{Z}$. Since the domain $\mathscr{Z}$ is compactly embedded in $\mathscr{X}$, the operator $\mathbf{L}_{\mathscr{R}}$ has compact resolvent. Consequently, its spectrum consists of isolated eigenvalues, only, which all have finite algebraic multiplicity. In order to determine the spectrum we then solve the eigenvalue problem

$$\mathbf{L}_{\mathscr{R}}u = \lambda u, \quad u \in \mathscr{Z},$$

which is equivalent to the system

$$\begin{aligned} &\phi'' + \mathscr{R}(1-\lambda)\phi = 0 \quad \phi(0) = \phi(1) = 0, \\ &\left(\lambda + \frac{1}{\mathscr{R}}\right) U = 0. \end{aligned}$$

The two equations in this system are decoupled, so that we can determine ϕ and U separately. The second equation gives the eigenvalue $\lambda_0 = -1/\mathscr{R}$, with eigenvector $(0,1)$, whereas by solving the first equation we find the sequence of eigenvalues $\lambda_k = 1 - k^2\pi^2/\mathscr{R}$, with eigenvectors $(\sin(k\pi x), 0)$ for $k \in \mathbb{N}^*$. Upon varying the parameter $\mathscr{R}$, we find that there is a sequence $(\mathscr{R}_k)_{k\in\mathbb{N}^*}$ of critical values of $\mathscr{R}$, where the part σ_0 of the spectrum of $\mathbf{L}_{\mathscr{R}}$ is not empty:

$$\mathscr{R}_k = k^2\pi^2, \quad k \in \mathbb{N}^*.$$

At each such value, $\sigma_0 = \{0\}$ and it is easy to check that the operators $\mathbf{L}_{\mathscr{R}_k}$ satisfy spectral Hypothesis 2.4. Furthermore, in each case the kernel of the operator $\mathbf{L}_{\mathscr{R}_k}$ is one-dimensional, spanned by the vector $(\sin(k\pi x), 0)$, so that 0 has geometric multiplicity one, and by arguing as in the example in Section 2.2.4 we conclude that its algebraic multiplicity is also one.

Checking Hypotheses 3.1 and 2.7

We restrict our analysis to the first bifurcation point $\mathscr{R} = \mathscr{R}_1 = \pi^2$. We set $\mu = \mathscr{R} - \mathscr{R}_1$ and write the system in the form (3.1) by taking

$$\mathbf{L} = \mathbf{L}_{\mathscr{R}_1}, \quad \mathbf{R}(u,\mu) = \mathbf{R}(u) + (\mathbf{L}_{\mathscr{R}_1+\mu} - \mathbf{L}_{\mathscr{R}_1})u.$$

Then $\mathbf{L}$ satisfies Hypothesis 3.1, whereas we now have $\mathbf{R}(u,\mu) \in \mathscr{X}$, instead of $\mathscr{Y}$, for $u \in \mathscr{Z}$, because of the term $(\mathbf{L}_{\mathscr{R}_1+\mu} - \mathbf{L}_{\mathscr{R}_1})u$, which belongs to $\mathscr{X}$ but not to $\mathscr{Y}$. Since $\mathbf{R}(u)$ is quadratic, and

$$\|\mathbf{R}(u)\|_{\mathscr{X}} \le C\|u\|_{\mathscr{Z}}^2 \text{ for all } u \in \mathscr{Z},$$

for some positive constant C, we have that $\mathbf{R} \in C^k(\mathscr{Z} \times \mathscr{V}_\mu, \mathscr{X})$ for any positive integer k, where $\mathscr{V}_\mu = \mathbb{R} \setminus \{\mathscr{R}_1\}$. Consequently, $\mathbf{R}$ satisfies Hypothesis 3.1 with $\mathscr{X}$

instead of $\mathscr{Y}$. We are in the presence of a "quasilinear" equation with this formulation.

Remark 4.2 *Alternatively, one could go back to the original system (4.7)–(4.9), and rescale the time t through $t = \mathscr{R}t'$, which then allows us to recover a formulation for which Hypothesis 3.1 holds with the space $\mathscr{Z}$ introduced above. With this second formulation we are in the presence of a "semilinear" equation. Since our problem is formulated in Hilbert spaces we can apply the center manifold theorem to both formulations, Theorem 2.20 to the first one and Theorem 2.17 to the second one. We choose here the first formulation above as a quasilinear equation. However, this won't be possible in Banach spaces, e.g., if the Sobolev spaces H^k are replaced by C^k, in which one has to choose this second formulation as a semilinear equation (see Section 2.2.3).*

It remains to check that Hypothesis 2.7 holds. For this we use now the result in Theorem 2.20 which shows that it is enough to check the estimate on the resolvent (2.9). For $f = (\psi, V) \in \mathscr{X}$, we have to show that the solution $u = (\phi, U) \in \mathscr{Y}$ of the system

$$(i\omega - 1)\phi - \frac{1}{\pi^2}\phi'' = \psi$$
$$\left(i\omega + \frac{1}{\pi^2}\right)U = V,$$

satisfies

$$\|u\|_{\mathscr{X}} = \left(\|\phi\|^2_{L^2(0,1)} + |U|^2\right)^{1/2} \le \frac{c}{|\omega|}\|f\|_{\mathscr{X}} = \frac{c}{|\omega|}\left(\|\psi\|^2_{L^2(0,1)} + |V|^2\right)^{1/2},$$

for $|\omega| \ge \omega_0$ and some positive constant c. First, from the second equation we immediately find

$$|U| = \frac{\pi^2}{\sqrt{1 + \pi^4\omega^2}}|V|, \tag{4.10}$$

whereas for the solution ϕ of the first equation we can proceed as in the example in Section 2.2.4 (explicitly compute the solution and then estimate its norm). Alternatively, we can make use of the fact that we know that this solution exists and belongs to $H^2(0,1) \cap H^1_0(0,1)$ for $\psi \in L^2(0,1)$, when $\omega \ne 0$, since any $i\omega \ne 0$ belongs to the resolvent set of $\mathbf{L}_{\mathscr{R}_1}$. Then multiplying the equation by $\bar{\phi}$, integrating over $(0,1)$, and integrating once by parts we obtain

$$(i\omega - 1)\|\phi\|^2_{L^2(0,1)} + \frac{1}{\pi^2}\|\phi'\|^2_{L^2(0,1)} = \int_0^1 \psi(x)\bar{\phi}(x)\,dx.$$

Upon taking the imaginary parts of both sides of this equality we find

$$\omega\|\phi\|^2_{L^2(0,1)} = \operatorname{Im}\int_0^1 \psi(x)\bar{\phi}(x)\,dx,$$

so that

$$|\omega| \|\phi\|^2_{L^2(0,1)} \leq \int_0^1 |\psi(x)\bar{\phi}(x)|\, dx \leq \|\psi\|_{L^2(0,1)} \|\phi\|_{L^2(0,1)}.$$

Consequently,

$$\|\phi\|_{L^2(0,1)} \leq \frac{1}{|\omega|} \|\psi\|_{L^2(0,1)},$$

which together with (4.10) gives the desired estimate and proves that Hypothesis 2.7 holds.

Center Manifold

Hypotheses 3.1, 2.4, and 2.7 being satisfied, we can now apply center manifold Theorem 3.3. Since 0 is a simple eigenvalue, the space $\mathscr{E}_0$ is one-dimensional, which gives us the family of one-dimensional center manifolds $\mathscr{M}_0(\mu)$, as in (3.3), for sufficiently small μ. As in the example in Section 2.2.4, we have that $\mathbf{L}_0 u_0 = 0$, so that the linear term in the reduced system (2.7) vanishes. Further denote by ξ_0 the eigenvector

$$\xi_0 = (\sin(\pi x), 0)$$

which spans $\mathscr{E}_0$, and write

$$u_0(t) = A(t)\xi_0 \in \mathscr{E}_0, \quad A(t) \in \mathbb{R}.$$

Replacing this formula in the reduced system (3.6) we obtain a first order ODE for A,

$$\frac{dA}{dt} = f_0(A, \mu),$$

with $f_0(A,\mu) = O(|A|(|\mu| + |A|))$, as $(A,\mu) \to (0,0)$.

Now, recall that the system commutes with the symmetry $\mathbf{T}$, so that the result in Theorem 3.13 holds, as well. Then the vector field in the reduced system commutes with the induced symmetry $\mathbf{T}_0$ on $\mathscr{E}_0$. Since $\mathbf{T}\xi_0 = -\xi_0$, this symmetry acts on A through $A \mapsto -A$. In particular, this shows that the vector field f_0 is odd in A, so that we may write

$$\frac{dA}{dt} = a\mu A + bA^3 + O(|A|(|\mu|^2 + A^4)).$$

We expect to find here a pitchfork bifurcation (see Section 1.1.2, Chapter 1). In order to analyze this bifurcation we compute the coefficients a and b.

Pitchfork Bifurcation

The coefficient a can be computed with the help of the result in Exercise 3.5, which shows that $\partial f_0/\partial A(0,\mu)$ is the eigenvalue of $\mathbf{L}_{\mathscr{R}_1+\mu}$ vanishing at $\mu = 0$. This latter eigenvalue is

$$\lambda_1 = 1 - \frac{\pi^2}{\mathscr{R}_1 + \mu} = \frac{\mu}{\pi^2} - \frac{\mu^2}{\pi^4} + O(|\mu|^3),$$

so that we find

$$a = \frac{1}{\pi^2}.$$

Next, in order to compute b we write for u on the center manifold

$$u(t) = A(t)\xi_0 + \Psi(A(t), \mu), \tag{4.11}$$

in which $u_0(t) = A(t)\xi_0$ and Ψ is the reduction function. Recall that $\mathbf{R}(u, 0) = \mathbf{R}(u)$ is quadratic, so that we may write

$$\mathbf{R}(u,0) = \mathbf{R}_2(u,u), \quad \mathbf{R}_2(u,v) = \begin{pmatrix} -\frac{\partial(\phi\psi)}{\partial x} + \frac{1}{2}U\psi + \frac{1}{2}V\phi \\ -\int_0^1 \phi(x,\cdot)\psi(x,\cdot)dx \end{pmatrix},$$

where $v = (\psi, V)$. We set $\mu = 0$ in the following calculations, and consider the expansion

$$\Psi(A,0) = A^2\Psi_2 + A^3\Psi_3 + O(A^4),$$

in which $\mathbf{T}\Psi_2 = \Psi_2$, and $\mathbf{T}\Psi_3 = -\Psi_3$, because Ψ commutes with the symmetry $\mathbf{T}$. Now we substitute u from (4.11) into

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}_2(u,u), \tag{4.12}$$

and taking into account that

$$\frac{dA}{dt} = bA^3 + O(|A|^5)$$

when $\mu = 0$, we identify the powers of A in this equality. At orders $O(A^2)$ and $O(A^3)$, we find, respectively,

$$\begin{aligned} \mathbf{L}\Psi_2 &= -\mathbf{R}_2(\xi_0, \xi_0), \\ \mathbf{L}\Psi_3 &= -2\mathbf{R}_2(\xi_0, \Psi_2) + b\xi_0. \end{aligned}$$

A necessary condition for solving these equations is that the right hand sides of both equalities lie in the range of $\mathbf{L}$, or equivalently, lie in the space orthogonal to the kernel of the adjoint of $\mathbf{L}$. A direct calculation shows that here $\mathbf{L}^* = \mathbf{L}$, i.e., $\mathbf{L}$ is self-adjoint, so that its kernel is spanned by ξ_0. Further, recall that $\Psi(A, \mu)$ belongs to $\mathscr{Z}_h$, the space defined by $\mathscr{Z}_h = (\mathbb{I} - \mathbf{P}_0)\mathscr{Z}$, where $\mathbf{P}_0$ is the spectral projection onto $\mathscr{E}_0$, associated with σ_0. It is this property which allows one to uniquely determine Ψ_2 and Ψ_3 from the equalities above. However, in this particular example we can get the desired result without explicitly computing the projection $\mathbf{P}_0$.

First,

$$\mathbf{R}_2(\xi_0, \xi_0) = \begin{pmatrix} -\pi\sin(2\pi x) \\ -\frac{1}{2} \end{pmatrix},$$

which is clearly orthogonal to ξ_0 in $\mathscr{X}$, and a direct calculation gives

$$\Psi_2 = \begin{pmatrix} -\frac{\pi}{3}\sin(2\pi x) \\ -\frac{\pi^2}{2} \end{pmatrix} + \alpha\xi_0$$

for some $\alpha \in \mathbb{R}$. Now, recall that $\mathbf{T}\Psi_2 = \Psi_2$, which together with the fact that $\mathbf{T}\xi_0 = -\xi_0$, implies that $\alpha = 0$. Next, we compute

$$2\mathbf{R}_2(\xi_0, \Psi_2) = \begin{pmatrix} \pi^2\sin(3\pi x) - \frac{5\pi^2}{6}\sin(\pi x) \\ 0 \end{pmatrix}.$$

The solvability condition for the second equation is

$$0 = \langle b\xi_0 - 2\mathbf{R}_2(\xi_0, \Psi_2), \xi_0 \rangle = \frac{1}{2}b + \frac{5\pi^2}{12},$$

so that

$$b = -\frac{5\pi^2}{6}.$$

Summarizing, the reduced equation is

$$\frac{dA}{dt} = \frac{1}{\pi^2}\mu A - \frac{5\pi^2}{6}A^3 + O(|A|(|\mu|^2 + |A|^4)),$$

in which the right hand side is odd in A. According to the result in Theorem 1.9 in Chapter 1, we have here a *supercritical pitchfork bifurcation*, in which a pair of steady solutions emerges from 0 as $\mathscr{R}$ crosses $\mathscr{R}_1$. These steady solutions are stable, whereas the trivial solution $A = 0$ is stable for $\mathscr{R} < \mathscr{R}_1$ and unstable for $\mathscr{R} > \mathscr{R}_1$ (see Figure 4.1).

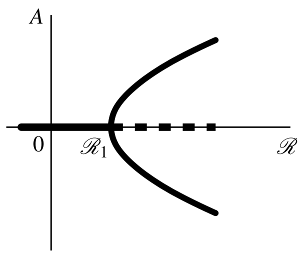

Fig. 4.1 Supercritical pitchfork bifurcation, which occurs at the first bifurcation point $\mathscr{R}_1 = \pi^2$ in the Burgers model.

Exercise 4.3 *Consider the integro-differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 - e^{-\nu u} - K\int_0^\pi u(x,t)dx,$$
$$\frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=\pi} = 0,$$

where $u(x,t) \in \mathbb{R}$ *for* $(x,t) \in (0,\pi)\times\mathbb{R}$*, and* K*,* ν *are real parameters.*

(i) Check that $u = 0$ *is a solution of this problem for all* K *and* ν*. Write the system in the form (2.1) with linear operator* $\mathbf{L} = \mathbf{L}_{K,\nu}$*, depending upon the two parameters* K *and* ν*.*

(ii) Show that the system is equivariant under the symmetry $\mathbf{T}$ *defined by*

$$\mathbf{T}u(x,t) = u(\pi - x, t).$$

(iii) Show that the spectrum of $\mathbf{L}_{K,\nu}$ *is a discrete set,* $\sigma = \{\lambda_n \in \mathbb{R}; n \in \mathbb{N}\}$*, consisting of the eigenvalues*

$$\lambda_0 = \nu - K\pi, \quad \lambda_n = \nu - n^2, \;\; n = \mathbb{N}^*,$$

with associated eigenvectors

$$\xi_n = \cos(nx), \;\; n \in \mathbb{N}.$$

Give the action of the symmetry $\mathbf{T}$ *on these eigenvectors.*

(iv) Assume $K\pi > 1$*, and set* $\nu = 1+\mu$*. Write the system in the form (3.1) and show that it possesses a center manifold of dimension 1. Show that the reduced equation takes the form*

$$\frac{dA}{dt} = \mu A + bA^3 + O(|A|(|\mu|^2 + |A|^4)), \quad b = \frac{1}{6} + \frac{1}{4(K\pi - 1)} > 0.$$

(Notice that the coefficient b *tends towards* ∞ *when* $K\pi \to 1$*. This is due to the invalidity of the study when* $K\pi$ *is close to 1, since at* $K\pi = 1$ *there are two "critical" eigenvalues,* λ_0 *and* λ_1*, instead of only one for* $K\pi > 1$*.)*

(v) Consider $K\pi$ *and* ν *close to 1, and set* $\mu = \nu - 1$ *and* $\varepsilon = \nu - K\pi$*. Write the system in the form (3.1) and show that it possesses a center manifold of dimension 2. Show that the reduced system is given by*

$$\frac{dA}{dt} = \mu A - AB + \frac{1}{6}A^3 + h.o.t.$$
$$\frac{dB}{dt} = (\mu - \varepsilon)B - \frac{1}{4}A^2 - \frac{1}{2}B^2 + h.o.t.,$$

in which the first component of the vector field is odd in A, and the second component is even in A. Here and in the remainder of this book "h.o.t." denotes higher order terms.

2.4.3 Swift–Hohenberg Equation

We consider the Swift–Hohenberg equation (SHE)

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \mu u - u^3, \tag{4.13}$$

where $u = u(x,t) \in \mathbb{R}$ for $(x,t) \in \mathbb{R}^2$, and μ is a real parameter. The Swift–Hohenberg equation arises as a model for hydrodynamical instabilities. We refer to [18] for a detailed analysis of this equation.

Notice that $u = 0$ is a solution of (4.13) and that the equation is invariant under spatial translations $x \mapsto x+\alpha$, $\alpha \in \mathbb{R}$, and the reflections $x \mapsto -x$ and $u \mapsto -u$.

Linear Stability Analysis

We first analyze the linear stability of the trivial solution $u = 0$. We look for solutions of the form

$$u(x,t) = \widehat{u}e^{ikx+\lambda t}, \tag{4.14}$$

where k is a real wavenumber and λ and $\widehat{u}$ may be complex numbers, of the linearized SHE

$$\frac{\partial u}{\partial t} = -\left(1+\frac{\partial^2}{\partial x^2}\right)^2 u + \mu u.$$

Inserting (4.14) into the linearized equation gives the linear dispersion relation

$$\lambda(\mu,k) = \mu - (1-k^2)^2. \tag{4.15}$$

The solution $u = 0$ is linearly stable (resp., unstable) with respect to the mode e^{ikx} if $\mathrm{Re}\,\lambda(\mu,k) < 0$ (resp., $\mathrm{Re}\,\lambda(\mu,k) > 0$).

The dispersion relation (4.15) shows that $\lambda(\mu,k)$ is real for all k and μ. For a fixed μ, the solution $u = 0$ is stable with respect to all modes e^{ikx} for which $\mu < (1-k^2)^2$, and unstable with respect to all modes for which $\mu > (1-k^2)^2$. The modes e^{ikx} such that $(1-k^2)^2 = \mu$ are the critical modes at the threshold from stability to instability. We plot in Figure 4.2 the curve $\lambda(\mu,k) = 0$. This shows that,

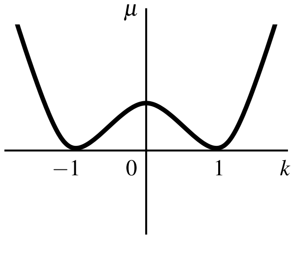

Fig. 4.2 Critical curve $\lambda(\mu,k) = 0$ for the Swift–Hohenberg equation.

upon increasing μ, the first critical modes, $k = \pm 1$, occur at $\mu = 0$. These modes correspond to 2π-periodic solutions $e^{\pm ix}$ of the linearized equation, at the threshold of linear instability. We therefore expect spatially 2π-periodic solutions to play a particular role in the dynamics of the equation, and restrict ourselves to this type of solutions in our analysis.

Center Manifolds

We write the equation in the form (2.1), with linear operator $\mathbf{L} = \mathbf{L}_\mu$ depending upon the parameter μ, by setting

$$\mathbf{L}_\mu = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 + \mu, \quad \mathbf{R}(u) = -u^3,$$

and choosing the spaces of 2π-periodic functions

$$\mathscr{X} = L^2_{\text{per}}(0, 2\pi), \quad \mathscr{Y} = \mathscr{Z} = H^4_{\text{per}}(0, 2\pi).$$

Then $\mathbf{L}_\mu$ is a closed operator in $\mathscr{X}$ with domain $\mathscr{Z}$, and $\mathbf{R}$ is a cubic map in $\mathscr{Z}$, satisfying

$$\|\mathbf{R}(u)\|_{\mathscr{Z}} \leq C\|u\|^3_{\mathscr{Z}},$$

so that $\mathbf{R} \in C^k(\mathscr{Z})$ for any positive integer k.

Next, we compute the spectrum of $\mathbf{L}_\mu$. As for the operator in the previous example, Section 2.4.2, the domain $\mathscr{Y}$ of $\mathbf{L}_\mu$ is compactly embedded in $\mathscr{X}$, so that $\mathbf{L}_\mu$ has a compact resolvent. Consequently, its spectrum consists only of isolated eigenvalues with finite multiplicities. Since we work in spaces of 2π-periodic functions, we can use Fourier analysis to solve the eigenvalue problem and conclude that

$$\sigma = \{\lambda_n = \mu - (1 - n^2)^2 \; ; \; n \in \mathbb{N}\}.$$

All these eigenvalues are real, and there is a sequence $(\mu_n = (1 - n^2)^2)_{n \in \mathbb{N}}$ of values of μ for which 0 is an eigenvalue of $\mathbf{L}_\mu$. The smallest value, $\mu_1 = 0$, is the one at which the solution $u = 0$ loses its stability when increasing μ. We apply center manifold Theorem 3.3 for values of μ close to this critical value $\mu_1 = 0$.

We proceed as in the example in Section 2.4.2 and first rewrite the equation in the form (3.1), with

$$\mathbf{L} = \mathbf{L}_0, \quad \mathbf{R}(u, \mu) = \mathbf{R}(u) + (\mathbf{L}_\mu - \mathbf{L}_0)u.$$

From the arguments above it follows that $\mathbf{L}$ and $\mathbf{R}$ satisfy Hypothesis 3.1 and that Hypothesis 2.4 holds with $\sigma_0 = \{0\}$. Furthermore, 0 is an eigenvalue with geometric multiplicity two, with associated eigenvectors $e^{\pm ix}$, and by arguing as in Section 2.2.4, one can show that its algebraic multiplicity is two as well. (Alternatively, notice that $\mathbf{L}_\mu$ is self-adjoint in $\mathscr{X}$ so that its eigenvalues are all semisimple. In particular, 0 is then a double eigenvalue of $\mathbf{L}$.) Finally, Hypothesis 2.7 can be checked as in the example in Section 2.4.2. Applying Theorem 3.3, we conclude that the equation possesses a two-dimensional center manifold for μ sufficiently small.

Symmetries

An important role in this example is played by the different symmetries of the SHE mentioned above. The invariance under spatial translations $x \mapsto x+\alpha$, $\alpha \in \mathbb{R}$, and the reflections $x \mapsto -x$ and $u \mapsto -u$ imply that the equation is equivariant with respect to the isometries defined by

$$(\mathbf{T}_\alpha u)(x) = u(x+\alpha), \;\; \alpha \in \mathbb{R}, \quad (\mathbf{T}u)(x) = u(-x), \quad (\mathbf{U}u)(x) = -u(x).$$

All these symmetries, $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}}$, $\mathbf{T}$, and $\mathbf{U}$, satisfy Hypothesis 3.12. Consequently, the result in Theorem 3.13 holds with any of these symmetries. The family $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}}$ also satisfies Hypothesis 3.17. However, we haven't in this case a nontrivial equilibrium satisfying Hypothesis 3.18, so that we cannot argue as for Theorem 3.19 in this example.

In addition, notice that

$$\mathbf{T}_\alpha = \mathbf{T}_{\alpha+2\pi}, \quad \mathbf{T}\mathbf{T}_\alpha = \mathbf{T}_{-\alpha}\mathbf{T}, \quad \mathbf{U}\mathbf{T}_\alpha = \mathbf{T}_\alpha\mathbf{U}, \;\; \alpha \in \mathbb{R}.$$

The first equality is a consequence of the fact that we restrict our analysis to 2π-periodic functions in x. In particular, the first two equalities show that (4.13) is equivariant under the representation of the group $O(2)$ by $(\mathbf{T}, (\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}})$.

Steady $O(2)$ Bifurcation

We discuss now the reduced system given by Theorems 3.3 and 3.13. Recall that the subspace $\mathscr{E}_0$ is two-dimensional, spanned by the complex conjugated eigenvector $\zeta = e^{ix}$ and $\overline{\zeta} = e^{-ix}$, so that it is convenient in this case to write

$$u_0 = A\zeta + \bar{A}\,\overline{\zeta}, \quad A(t) \in \mathbb{C},$$

for real-valued $u_0(t) \in \mathscr{E}_0$. Then we set for the real-valued solutions on the center manifold

$$u = A\zeta + \bar{A}\overline{\zeta} + \Psi(A,\bar{A},\mu), \quad A(t) \in \mathbb{C},$$

where $\Psi(A(t),\bar{A}(t),\mu) \in \mathscr{Z}_h$. The reduced equation reads

$$\frac{dA}{dt} = f(A,\bar{A},\mu), \tag{4.16}$$

together with the complex conjugated equation for $\bar{A}$. In addition, since the original equation is equivariant under the actions of $\mathbf{T}_\alpha$ and $\mathbf{T}$, by the result in Theorem 3.13, we have that the reduced vector field $(f,\overline{f})$ is equivariant under the actions of the induced symmetries. Since

$$\mathbf{T}_\alpha\zeta = e^{i\alpha}\zeta, \quad \mathbf{T}_\alpha\overline{\zeta} = e^{-i\alpha}\overline{\zeta}, \quad \mathbf{T}\zeta = \overline{\zeta}, \quad \mathbf{T}\overline{\zeta} = \zeta,$$

the action of the induced symmetries on the pair $(A,\bar{A})$ is given by the 2×2-matrices

$$\mathbf{T}_\alpha : \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad \mathbf{T} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This shows that we are in the setting of the study made in Section 1.2.4, Chapter 1, on steady bifurcations with $O(2)$ symmetry. Consequently, we have that

$$f(A,\bar{A},\mu) = Ag(|A|^2,\mu),$$

where the function g is of class C^{k-1} in $(A,\bar{A},\mu)$ and real-valued. We consider the Taylor expansion of g and write

$$\frac{dA}{dt} = aA\mu + bA|A|^2 + O(|A|(|\mu|^2 + |A|^4)).$$

In polar coordinates, for $A = re^{i\phi}$, this gives the system (2.43)–(2.44) studied in Section 1.2.4.

We now compute the coefficients a and b in order to determine the nature of this bifurcation. For this we proceed as in the previous example in Section 2.4.2. First, using the result in the Exercise 3.5, we obtain

$$\frac{\partial f}{\partial A}(0,\mu) = \lambda_1 = \mu,$$

so that

$$a = 1.$$

Next, we set $\mu = 0$ in the following calculations and consider the expansion of the reduction function Ψ,

$$\Psi(A,\bar{A},0) = \sum_{p,q} \Psi_{pq} A^p \overline{A}^q.$$

Here $\Psi_{qp} \in \mathscr{Z}_h$ are such that

$$\Psi_{qp} = \overline{\Psi_{pq}}, \quad \Psi_{00} = \Psi_{10} = \Psi_{01} = 0.$$

The first equality shows that Ψ is real-valued, whereas the last equalities come from (3.2). Furthermore, from the equivariance of the equation with respect to $\mathbf{U}$, we conclude that $\Psi(-A,-\overline{A},0) = -\Psi(A,\overline{A},0)$ for all A, and thus $\Psi_{pq} = 0$ when $p+q$ is even. Summarizing, we find the expansion

$$\Psi(A,\bar{A},0) = \Psi_{30}A^3 + \Psi_{03}\bar{A}^3 + \Psi_{21}A^2\bar{A} + \Psi_{12}A\bar{A}^2 + O(|A|^5),$$

where $\Psi_{03} = \overline{\Psi_{30}}$ and $\Psi_{12} = \overline{\Psi_{21}}$.

Now by arguing as in the calculation of the coefficient b in the example in Section 2.4.2, we obtain the equalities

$$\mathbf{L}\Psi_{30} = e^{3ix},$$
$$\mathbf{L}\Psi_{21} = 3e^{ix} + be^{ix}.$$

The solvability condition for the second equation gives

$$b = -3.$$

Summarizing, the reduced equation is

$$\frac{dA}{dt} = \mu A - 3A|A|^2 + O(|A|(|\mu|^2 + |A|^4)), \tag{4.17}$$

and the reduced vector field possesses an $O(2)$ equivariance, just as in Hypothesis 2.14. According to the result in Theorem 2.18 in Chapter 1, we have here a *steady bifurcation with $O(2)$ symmetry*, in which a family $(A_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}})$ of stable equilibria emerges from 0, as μ crosses 0. A direct calculation gives

$$A_\alpha = \sqrt{\frac{\mu}{3}}e^{i\alpha} + O(|\mu|^{3/2})$$

for $\mu > 0$, and the corresponding family of steady 2π-periodic solutions of SHE,

$$u_\alpha(x) = 2\sqrt{\frac{\mu}{3}}\cos(x+\alpha) + O(|\mu|^{3/2}). \tag{4.18}$$

We point out that $u_\alpha = \mathbf{T}_\alpha u_0$, so that the solutions in this family are obtained by spatially translating u_0.

Remark 4.4 *These steady 2π-periodic solutions of the SHE are called* roll solutions. *Actually, such solutions exist for a range of periods close to 2π, for any sufficiently small μ. One can prove the existence of all these rolls in a similar way. Looking for periodic solutions of the SHE with wavenumbers k close to 1, instead of wavenumbers $k = 1$, only, and normalizing the period to 2π in the equation, one finds an equation having an additional parameter, the wavenumber k. The normalization of the period allows us to use the same function spaces $\mathscr{X}$ and $\mathscr{Z}$, and this reduction procedure can be performed with two parameters, k close to 1 and μ small.*

Symmetry Breaking

We briefly discuss here several scenarios in which we perturb the Swift–Hohenberg equation, by adding a small term, in such a way that one, or more, of the symmetries of the SHE is broken. We are interested in the effect of the perturbation on the reduced equation (4.17).

I. First we consider the perturbed equation obtained by adding the term εu^2 in the right hand side of the SHE, with ε a small real parameter. This term breaks the equivariance of the equation with respect to the symmetry $\mathbf{U}$ but preserves the $O(2)$ equivariance with respect to $(\mathbf{T},(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}})$. The center manifold analysis remains the same, up to the equivariance in $\mathbf{U}$, which is lost, and to the appearance of the additional small parameter ε. However, this parameter does not play a role in checking the different hypotheses, its effect being that now the reduced vector field $(f,\bar{f})$ depends upon ε as well. Since the $O(2)$ equivariance is preserved, we still have the particular form

$$f(A,\bar{A},\mu,\varepsilon) = Ag(|A|^2,\mu,\varepsilon),$$

with g of class C^{k-1} and real-valued.

Notice that at $\varepsilon = 0$ we find exactly the reduced vector field obtained for the unperturbed equation. Furthermore, we have here a new symmetry, which is the invariance of the SHE under $(u,\varepsilon)\mapsto(-u,-\varepsilon)$. It is then straightforward to check that this induces the invariance of the reduced equation under the action of $(A,\varepsilon)\mapsto(-A,-\varepsilon)$. In particular, this shows that the map g above is even in ε. This fact is useful in the computation of the Taylor expansion of g.

II. Next, we add the term $\varepsilon\partial u/\partial x$ in the right hand side of the SHE, with ε a small real parameter. This situation actually reduces to the unperturbed SHE, by the change of variables $u(x,t)=\widetilde{u}(x+\varepsilon t,t)$. It is easy to see that u is a solution of the perturbed SHE if and only if $\widetilde{u}$ is a solution of the unperturbed SHE. In particular, our previous analysis gives us in this case the family of traveling wave solutions $u_\alpha(x+\varepsilon t)$, with u_α the steady 2π-periodic solution in (4.18). These traveling waves have small speeds $-\varepsilon$, are 2π-periodic in the spatial variable x, and are periodic in time with large period $2\pi/\varepsilon$.

Our interest in considering this example is to see the effect of such a term on the different symmetries of the SHE and then on the reduced system. This term breaks the symmetry $\mathbf{T}$, but preserves the symmetries $\mathbf{T}_\alpha$ and $\mathbf{U}$. In particular, instead of an $O(2)$ equivariance we have now an $SO(2)$ equivariance. However, one can argue as in Section 1.2.4 and conclude that the map f in the reduced system is of the form

$$f(A,\bar{A},\mu,\varepsilon) = Ag(|A|^2,\mu,\varepsilon),$$

with g of class C^{k-1}, and complex-valued but not necessarily real-valued anymore.

In this situation, we have the additional invariance of the SHE under $(x,\varepsilon)\mapsto(-x,-\varepsilon)$. On the center manifold, this induces the symmetry acting by $(A,\varepsilon)\mapsto(\bar{A},-\varepsilon)$, so that g satisfies

$$g(|A|^2,\mu,\varepsilon) = \overline{g(|A|^2,\mu,-\varepsilon)}.$$

Consequently, the real part g_r of g is even in ε, whereas the imaginary part g_i of g is odd in ε. This leads to the equation

$$\frac{dA}{dt} = (\mu + c\varepsilon^2 + id\varepsilon)A - 3A|A|^2 + h.o.t.,$$

which in polar coordinates $A = re^{i\phi}$ reads

$$\begin{aligned}\frac{dr}{dt} &= (\mu + c\varepsilon^2)r - 3r^3 + h.o.t.\\ \frac{d\phi}{dt} &= d\varepsilon + h.o.t.\end{aligned} \tag{4.19}$$

Here the real coefficients c and d can be computed explicitly, just as the coefficients a and b in (4.16), and we have used the fact that the reduced system at $\varepsilon = 0$ is the same as the reduced system found for the unperturbed equation. It is then straightforward to find the solutions

$$r_0(\mu, \varepsilon^2) = \left(\frac{\mu + c\varepsilon^2}{3}\right)^{1/2} + h.o.t., \quad \phi_0 = \omega t + \alpha, \;\; \omega = d\varepsilon + h.o.t.,$$

with any $\alpha \in \mathbb{R}$. These give the solutions of the perturbed SHE equation

$$u(x,t) = 2r_0(\mu, \varepsilon^2)\cos(x + \omega t + \alpha) + h.o.t..$$

The lowest order term in this solution is clearly a traveling wave, with speed $-\omega$. A careful use of the symmetries mentioned above, together with the invariance of the equation under translations in the time t, allows us to show that these solutions are indeed traveling waves.

Exercise 4.5 *Show that $c = 0$ and $d = 1$ in the reduced system (4.19).*

III. Consider now the additional term $\varepsilon u\partial u/\partial x$ on the right hand side of the SHE. This term breaks the symmetries $\mathbf{T}$ and $\mathbf{U}$, but preserves the composed symmetry $\widetilde{\mathbf{T}} = \mathbf{T} \circ \mathbf{U}$ and the family $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}}$. Consequently, we still have an $O(2)$ equivariance of the system, but now with $\widetilde{\mathbf{T}}$ instead of $\mathbf{T}$. The action of $\widetilde{\mathbf{T}}$ on the pair $(A, \bar{A})$ is given by the 2×2-matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

However this does not change the form of the reduced equation, the map f being again of the form

$$f(A, \bar{A}, \mu, \varepsilon) = Ag(|A|^2, \mu, \varepsilon).$$

In addition, we have here the symmetry $(u, \varepsilon) \mapsto (-u, -\varepsilon)$, which implies that

$$g(|A|^2, \mu, -\varepsilon) = g(|A|^2, \mu, \varepsilon).$$

IV. We introduce now an additional term $\varepsilon_1 u\partial u/\partial x + \varepsilon_2 u^2$, in which we have two small parameters ε_1 and ε_2. This term breaks the symmetries $\mathbf{T}$, $\mathbf{U}$, and also $\widetilde{\mathbf{T}} = \mathbf{T} \circ \mathbf{U}$, but preserves the symmetries $\mathbf{T}_\alpha$, $\alpha \in \mathbb{R}$. Consequently, we still have an $SO(2)$ equivariance, just as in the case **II**, which allows us to conclude that the map f in the reduced system is of the form

$$f(A,\bar{A},\mu,\varepsilon_1,\varepsilon_2) = Ag(|A|^2,\mu,\varepsilon_1,\varepsilon_2),$$

with g of class C^{k-1} and complex-valued.

In addition, we now find the new symmetries

$$(u,\varepsilon_1,\varepsilon_2) \mapsto (-u,-\varepsilon_1,-\varepsilon_2), \quad (u(x),\varepsilon_1,\varepsilon_2) \mapsto (u(-x),-\varepsilon_1,\varepsilon_2).$$

Their action on $(A,\bar{A})$ is given by

$$(A,\bar{A},\varepsilon_1,\varepsilon_2) \mapsto (-A,-\overline{A},-\varepsilon_1,-\varepsilon_2), \quad (A,\bar{A},\varepsilon_1,\varepsilon_2) \mapsto (\bar{A},A,-\varepsilon_1,\varepsilon_2).$$

We can then conclude that the map g satisfies

$$g(|A|^2,\mu,\varepsilon_1,\varepsilon_2) = g(|A|^2,\mu,-\varepsilon_1,-\varepsilon_2), \quad g(|A|^2,\mu,\varepsilon_1,\varepsilon_2) = \overline{g(|A|^2,\mu,-\varepsilon_1,\varepsilon_2)},$$

so that the reduced equation is

$$\frac{dA}{dt} = \mu A - 3A|A|^2 + (c_1\varepsilon_1^2 + id\varepsilon_1\varepsilon_2 + c_2\varepsilon_2^2)A|A|^2 + h.o.t..$$

In polar coordinates $A = re^{i\phi}$, we find the system

$$\begin{aligned}\frac{dr}{dt} &= \mu r - 3r^3 + (c_1\varepsilon_1^2 + c_2\varepsilon_2^2)r^3 + h.o.t.\\ \frac{d\phi}{dt} &= d\varepsilon_1\varepsilon_2 r^2 + h.o.t..\end{aligned}$$

By arguing as for the system (4.19) in case **II**, one can show in this case the existence of bifurcating traveling waves with speeds of order $O(\mu\varepsilon_1\varepsilon_2)$.

Exercise 4.6 *Show that $c_1 = -1/9$, $d = 4/3$, and $c_2 = 20/9$ in the reduced system.*

V. Consider now the case of an inhomogeneous additional term $\varepsilon h(x)$, on the right hand side of the SHE, where $h : \mathbb{R} \to \mathbb{R}$ is an even 2π-periodic function and ε a small parameter, again. Notice that in this case the trivial solution $u = 0$ is no longer a solution for $\varepsilon \neq 0$.

This term now breaks the translation invariance $\mathbf{T}_\alpha$, $\alpha \in \mathbb{R}$, and the reflection $\mathbf{U}$, but preserves the symmetry $\mathbf{T}$. As in the previous cases we find a two-dimensional center manifold and a reduced equation of the form

$$\frac{dA}{dt} = f(A,\bar{A},\mu,\varepsilon)$$

for $A(t) \in \mathbb{C}$. At $\varepsilon = 0$, the map f is the one obtained for the unperturbed equation,

$$f(A,\bar{A},\mu,0) = Ag(|A|^2,\mu) = \mu A - 3A|A|^2 + h.o.t.,$$

whereas for $\varepsilon \neq 0$ the equivariance with respect to $\mathbf{T}$ implies that

$$f(A,\bar{A},\mu,\varepsilon) = \overline{f(\bar{A},A,\mu,\varepsilon)}.$$

Consequently, the reduced equation is of the form

$$\frac{dA}{dt} = c\varepsilon + \mu A - 3A|A|^2 + h.o.t.,$$

where c is a real constant. Notice that the constant term on the right hand side of this equation is real, because of the property of f above, and nonzero, since $u = 0$ is no longer a solution of the perturbed equation.

Exercise 4.7 *Show that the coefficient c in the reduced system is given by*

$$c = \frac{1}{2\pi}\int_0^{2\pi} h(x)\cos x dx.$$

Remark 4.8 (Steady solutions) *Notice that the steady solutions of this system are easy to compute. They are real, $A = A_r$, with A_r satisfying*

$$c\varepsilon + A_r(\mu - 3A_r^2) + h.o.t. = 0.$$

We plot in Figure 4.3 the bifurcation diagram for the steady solutions of this reduced equation. As for the stability of these steady solutions, it can be determined from the

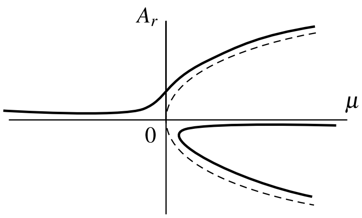

Fig. 4.3 Bifurcation diagram in the (μ,A_r)-plane for the steady solutions of the reduced system in the SHE perturbed by an inhomogeneity $\varepsilon h(x)$ in the case $c\varepsilon > 0$. The solid lines represent the branches of steady solutions for a fixed, small ε, whereas the dashed lines represent the branch of steady solutions for $\varepsilon = 0$.

eigenvalues of the linearized vector field at $A = A_r$. A direct calculation gives the two eigenvalues $\mu - 9A_r^2 + h.o.t.$ and $\mu - 3A_r^2 + h.o.t.$. In particular, in the case represented in the bifurcation diagram in Figure 4.3, the upper branch is stable (both eigenvalues are negative), while the lower branch is unstable (at least one eigenvalue is positive). We point out that this result differs from the classical result occuring in a perturbed pitchfork bifurcation. Notice that one eigenvalue is 0 at the turning point of the lower branch, but that this does not change the stability here, because of the second eigenvalue. Moreover, observe that all these steady solutions are symmetric, invariant under **T***, since they are real.*

VI. Finally, we consider the Swift–Hohenberg equation (4.13), but instead of looking for solutions that are 2π-periodic in x, we seek solutions that satisfy the boundary conditions

$$u(\pm h,t) = \frac{\partial u}{\partial x}(\pm h,t) = 0 \tag{4.20}$$

on some interval $[-h,h]$. We assume that h is large enough, so that we regard this new problem as a "small" perturbation of the equation (4.13).

Replacing the spatial periodicity of the solutions by the boundary conditions (4.20) breaks the translational invariance $\mathbf{T}_a$, but does not break the symmetries $\mathbf{T}$ and $\mathbf{U}$, and $u = 0$ remains a solution of the new problem. As a consequence, the eigenvalues of the linear operator $\mathbf{L}_\mu$ are no longer double, and for $\mu = 0$ the former 0 eigenvalue splits into two simple, negative eigenvalues, which are close to 0, of order $O(1/h^3)$ as $h \to \infty$. The other eigenvalues are all negative and at least of order $O(1/h^2)$. It is then convenient to rescale the variables in order to push the eigenvalues of order $O(1/h^2)$ at a distance of order $O(1)$ from the imaginary axis. Then the two eigenvalues of order $O(1/h^3)$ are changed into eigenvalues of order $O(1/h)$, which allows us to use a center manifold reduction, as described in Remark 3.6, when the critical spectrum σ_0 does not lie on the imaginary axis, but stays close to it. In addition to the original parameter μ, we now have a second small parameter $\varepsilon = O(1/h)$, so that this case is indeed a small perturbation of the original problem.

Taking into account the fact that 0 is always a solution, and that in this new problem only the translational symmetry is broken, by arguing as in the previous cases one finds that the reduced equation is now modified at main orders as follows:

$$\frac{dA}{dt} = (\mu + a\varepsilon)A + b\varepsilon\bar{A} - 3A|A|^2,$$

where a and b are real coefficients. Using polar coordinates $A = re^{i\phi}$, we find the system

$$\begin{aligned}\frac{dr}{dt} &= r(\mu + a\varepsilon + b\varepsilon\cos 2\phi - 3r^2)\\ \frac{d\phi}{dt} &= -b\varepsilon\sin(2\phi).\end{aligned}$$

Steady solutions are found for $\phi \in \{0, \pi/2, \pi, 3\pi/2\}$. Note that changing $\phi \mapsto \phi + \pi$ is equivalent to changing $r \mapsto -r$, so that we can restrict to the two cases $\phi = 0$ and $\phi = \pi/2$. The case $\phi = 0$ leads to *symmetric solutions*, i.e., invariant under $\mathbf{U}$, since $A = \bar{A}$, whereas the case $\phi = \pi/2$ leads to *antisymmetric solutions*, since $\bar{A} = -A$. It turns out that *symmetric solutions* bifurcate for $\mu = -(a+b)\varepsilon$ and have the amplitude given by $r_S^2 = 1/3(\mu + (a+b)\varepsilon)$. Their stability is determined by the sign of the two eigenvalues $-6r_S^2$, $-2b\varepsilon$. *Antisymmetric solutions* bifurcate for $\mu = (b-a)\varepsilon$, and have the amplitude given by $r_A^2 = 1/3(\mu + (a-b)\varepsilon)$. Their stability is determined by the sign of the two eigenvalues $-6r_A^2$, $2b\varepsilon$. In particular, it follows that the *stabilities of these two branches of solutions are opposite* (see Figure 4.4 for a typical bifurcation diagram).

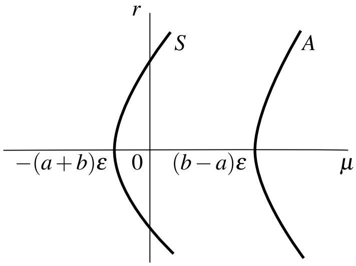

Fig. 4.4 Bifurcation diagram for the Swift–Hohenberg equation with boundary conditions (4.20), for a fixed $\varepsilon = O(1/h)$. The two curves S and A represent the branches of symmetric and antisymmetric solutions, respectively.

Remark 4.9 *This question has a major physical importance for many hydrodynamic stability problems where, for a large aspect ratio apparatus, one replaces, for mathematical convenience, the physical boundary conditions by periodic boundary conditions (large periods), as for example in Section 5.1 of Chapter 5. On the model equation SHE, a complete mathematical justification of the new amplitude equation obtained for Dirichlet–Neumann boundary conditions, as a perturbation of the periodic case, can be found in [125], while this is still a mathematically open problem for classical hydrodynamic stability problems like the ones in Section 5.1 of Chapter 5.*

2.4.4 Brusselator Model

Consider the system of PDEs

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \delta_1 \frac{\partial^2 u_1}{\partial x^2} - (\beta+1)u_1 + u_1^2 u_2 + \alpha \\ \frac{\partial u_2}{\partial t} &= \delta_2 \frac{\partial^2 u_2}{\partial x^2} + \beta u_1 - u_1^2 u_2, \end{aligned} \tag{4.21}$$

in which δ_1, δ_2, α, and β are positive constants, and $u = (u_1, u_2)$ is a function of $(x,t) \in (0,1) \times \mathbb{R}$, together with the boundary conditions

$$u_1(0,t) = u_1(1,t) = \alpha, \quad u_2(0,t) = u_2(1,t) = \frac{\beta}{\alpha}. \tag{4.22}$$

Remark 4.10 *This system is called* inhomogeneous Brusselator*, and arises in modeling an autocatalytic chemical reaction as described in Remark 2.9 of Chapter 1. In contrast to the homogeneous Brusselator considered in Section 1.2.2, in the inhomogeneous case the products are not homogeneously mixed during the reaction. In such a case, diffusion phenomena occur, so that u_1 and u_2 are now functions of a* *space variable x, besides the time t. We assume that $x \in (0,1)$, which is, of course, a simplification of the reality. The coefficients δ_1 and δ_2 in the system (4.21) represent the diffusion coefficients of the products X and Y. The Dirichlet boundary conditions (4.22) correspond in the chemical reaction to a control at the boundary for the concentrations of the products X and Y, which are maintained at the constant values given by the equilibrium solution of the homogeneous system. Notice that with such boundary conditions, the equilibrium $(u_1,u_2) = (\alpha, \beta/\alpha)$ found in Section 1.2.2 remains a solution of the PDE, but the periodic solution arising in the Hopf bifurcation for the homogeneous system is no longer a solution. It is not difficult to check that this solution does not satisfy the boundary conditions.*

First Formulation and Spectrum

We set

$$(u_1,u_2) = \left(\alpha, \frac{\beta}{\alpha}\right) + (v_1,v_2).$$

Then $v = (v_1,v_2)$ satisfies the system

$$\begin{aligned}\frac{\partial v_1}{\partial t} &= \delta_1 \frac{\partial^2 v_1}{\partial x^2} + (\beta - 1)v_1 + \alpha^2 v_2 + 2\alpha v_1 v_2 + \frac{\beta}{\alpha} v_1^2 + v_1^2 v_2\\ \frac{\partial v_2}{\partial t} &= \delta_2 \frac{\partial^2 v_2}{\partial x^2} - \beta v_1 - \alpha^2 v_2 - 2\alpha v_1 v_2 - \frac{\beta}{\alpha} v_1^2 - v_1^2 v_2,\end{aligned} \tag{4.23}$$

and the Dirichlet boundary conditions

$$v_1(0,t) = v_1(1,t) = 0, \quad v_2(0,t) = v_2(1,t) = 0. \tag{4.24}$$

In this way we have replaced the constant solution $(u_1,u_2) = (\alpha, \beta/\alpha)$ by the trivial solution $v = 0$. This system is of the form (2.1) with

$$\mathbf{L} = \begin{pmatrix} \delta_1 \frac{d^2}{dx^2} + \beta - 1 & \alpha^2 \\ -\beta & \delta_2 \frac{d^2}{dx^2} - \alpha^2 \end{pmatrix}, \quad \mathbf{R}(v) = \begin{pmatrix} 2\alpha v_1 v_2 + \frac{\beta}{\alpha} v_1^2 + v_1^2 v_2 \\ -2\alpha v_1 v_2 - \frac{\beta}{\alpha} v_1^2 - v_1^2 v_2 \end{pmatrix},$$

where both $\mathbf{L}$ and $\mathbf{R}$ depend upon parameters.

Next, we choose the spaces

$$\mathscr{X} = (L^2(0,1))^2, \quad \mathscr{Z} = \mathscr{Y} = (H^2(0,1) \cap H_0^1(0,1))^2,$$

such that the boundary conditions are included in the definition of $\mathscr{Y}$. Then $\mathbf{L}$ is a closed operator in $\mathscr{X}$, with domain $\mathscr{Z}$, and $\mathbf{R}$ a smooth map in $\mathscr{Y}$. As in the previous examples, $\mathscr{Y}$ is compactly embedded in $\mathscr{X}$, so that $\mathbf{L}$ has compact resolvent and its spectrum consists of isolated eigenvalues with finite multiplicities.

We determine now the spectrum of $\mathbf{L}$. Since the set $\{\sin(n\pi x)\,;\, n \in \mathbb{N}\}$ forms a basis of $H_0^1(0,1)$, we can look for solutions $v = (v_1,v_2)$ of the eigenvalue problem

$$\mathbf{L}v = \lambda v, \quad \lambda \in \mathbb{C},$$

of the form

$$v_1 = \sum_{n\in\mathbb{N}} v_1^{(n)} \sin(n\pi x), \quad v_2 = \sum_{n\in\mathbb{N}} v_2^{(n)} \sin(n\pi x).$$

Then λ is an eigenvalue of $\mathbf{L}$ if there exists $n \neq 0$ such that there exists a nontrivial solution $(v_1^{(n)}, v_2^{(n)})$ of the system

$$\begin{aligned}(\delta_1 n^2\pi^2 - \beta + 1 + \lambda)v_1^{(n)} - \alpha^2 v_2^{(n)} &= 0\\ \beta v_1^{(n)} + (\delta_2 n^2\pi^2 + \alpha^2 + \lambda)v_2^{(n)} &= 0.\end{aligned} \tag{4.25}$$

Consequently, the eigenvalues λ are roots of the characteristic polynomials

$$P_n(X) = X^2 + (\beta_n - \beta)X + \delta_2 n^2\pi^2(\gamma_n - \beta),$$

where

$$\beta_n = 1 + \alpha^2 + n^2\pi^2(\delta_1 + \delta_2), \quad \gamma_n = 1 + \alpha^2\frac{\delta_1}{\delta_2} + n^2\pi^2\delta_1 + \frac{\alpha^2}{n^2\pi^2\delta_2}.$$

The two roots of P_n have negative real parts provided

$$\beta < \beta_n, \quad \beta < \gamma_n.$$

Notice that the sequence $(\beta_n)_{n\geq 1}$ is increasing and that $\gamma_n \geq \left(1 + \alpha\sqrt{\delta_1/\delta_2}\right)^2$, so that for any β satisfying

$$\beta < \beta_1, \quad \beta < \left(1 + \alpha\sqrt{\frac{\delta_1}{\delta_2}}\right)^2,$$

the roots of all these polynomials have negative real parts. When

$$\beta = \beta_1 < \left(1 + \alpha\sqrt{\frac{\delta_1}{\delta_2}}\right)^2,$$

the polynomial P_1 has purely imaginary roots, whereas the other polynomials all have roots with negative real parts. We conclude that the eigenvalues of $\mathbf{L}$ have negative real parts (are all stable) if $\beta < \beta_1$, and that a pair of eigenvalues crosses the imaginary axis at $\beta = \beta_1$, provided

$$\beta_1 < \left(1+\alpha\sqrt{\frac{\delta_1}{\delta_2}}\right)^2.$$

This inequality is equivalent to

$$\alpha^2\left(\frac{\delta_1}{\delta_2}-1\right)+2\alpha\sqrt{\frac{\delta_1}{\delta_2}}-\pi^2(\delta_1+\delta_2)>0, \tag{4.26}$$

and we assume in the following that this condition holds, so that we have a Hopf bifurcation at $\beta=\beta_1$.

Center Manifolds

We focus on this Hopf bifurcation and set $\beta=\beta_1+\mu$. In order to apply the result in Theorem 3.3, we rewrite the system (4.23) in the form

$$\frac{dv}{dt}=\mathbf{L}v+\mathbf{R}(v,\mu) \tag{4.27}$$

in the space $\mathscr{X}$, with

$$\mathbf{L}=\begin{pmatrix}\delta_1\frac{d^2}{dx^2}+\beta_1-1 & \alpha^2\\ -\beta_1 & \delta_2\frac{d^2}{dx^2}-\alpha^2\end{pmatrix},$$

and

$$\mathbf{R}(v,\mu)=\mu\mathbf{R}_{01}v+\mathbf{R}_{20}(v,v)+\mathbf{R}_{30}(v,v,v)+\mu\mathbf{R}_{21}(v,v), \tag{4.28}$$

where

$$\mathbf{R}_{01}v=\begin{pmatrix}v_1\\ -v_1\end{pmatrix},\quad \mathbf{R}_{20}(u,v)=\begin{pmatrix}\alpha(u_1v_2+u_2v_1)+\frac{\beta_1}{\alpha}u_1v_1\\ -\alpha(u_1v_2+u_2v_1)-\frac{\beta_1}{\alpha}u_1v_1\end{pmatrix},$$

$$3\mathbf{R}_{30}(u,v,w)=\begin{pmatrix}u_1v_1w_2+u_1v_2w_1+u_2v_1w_1\\ -u_1v_1w_2-u_1v_2w_1-u_2v_1w_1\end{pmatrix},\quad \mathbf{R}_{21}(u,v)=\begin{pmatrix}\frac{1}{\alpha}u_1v_1\\ -\frac{1}{\alpha}u_1v_1\end{pmatrix}.$$

Then $\mathbf{L}$, which is closed in $\mathscr{X}$ with domain $\mathscr{Z}$, and $\mathbf{R}(v,\mu)$, which has a polynomial form and is continuous in $\mathscr{Z}$, satisfy Hypothesis 3.1. Next, the analysis above implies that the operator $\mathbf{L}$ satisfies Hypothesis 2.4, with

$$\sigma_0=\{\pm i\omega\},\quad \omega^2=\alpha^2+\alpha^2\pi^2(\delta_1-\delta_2)-\delta_2^2\pi^4.$$

The eigenvectors ζ and $\bar{\zeta}$ associated with the eigenvalues $i\omega$ and $-i\omega$, respectively, are given by

$$\zeta = \sin(\pi x)\begin{pmatrix}1\\ \gamma\end{pmatrix}, \quad \gamma = \frac{i\omega - \alpha^2 - \delta_2\pi^2}{\alpha^2} = \frac{-\beta_1}{i\omega + \alpha^2 + \delta_2\pi^2}.$$

Notice that $\omega^2 > 0$ thanks to the condition (4.26), since

$$\omega^2 = \pi^2\delta_2\left(\alpha^2\left(\frac{\delta_1}{\delta_2} - 1\right) + 2\alpha\sqrt{\frac{\delta_1}{\delta_2}} - \pi^2(\delta_1 + \delta_2)\right) + (\alpha - \pi^2\sqrt{\delta_1\delta_2})^2.$$

Finally, one can proceed as in the example in Section 2.4.2 and check the inequality (2.9), which implies that Hypothesis 2.7 holds as well.

Applying Theorem 3.3 we conclude that the system possesses a two-dimensional center manifold for sufficiently small μ. For the solutions on the center manifold we write

$$v = z\zeta + \overline{z\zeta} + \Psi(z, \bar{z}, \mu), \quad z \in \mathbb{C}, \tag{4.29}$$

in which $v_0(t) = z(t)\zeta + \overline{z\zeta(t)} \in \mathscr{E}_0$ and Ψ takes values in $\mathscr{Z}_h$. The reduced equation

$$\frac{dz}{dt} = f(z, \bar{z}, \mu),$$

together with the complex conjugated equation, has the linear part

$$\begin{pmatrix} \frac{\partial f}{\partial z}(0,0,0) & \frac{\partial f}{\partial \bar{z}}(0,0,0) \\ \frac{\partial \bar{f}}{\partial z}(0,0,0) & \frac{\partial \bar{f}}{\partial \bar{z}}(0,0,0) \end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix},$$

in which $\pm i\omega$ are the two eigenvalues in σ_0. In particular, it is of the form (2.5), so that we can use the results on the Hopf bifurcation in Section 1.2.1, Chapter 1, to analyze this reduced equation.

Hopf Bifurcation

According to the analysis in Section 1.2.1, there is a polynomial change of variables that transforms this reduced equation into the normal form

$$\frac{dA}{dt} = i\omega A + a\mu A + bA|A|^2 + O(|A|(|\mu| + |A|^2)^2). \tag{4.30}$$

Our goal now is to compute the coefficients a and b in this normal form. To do so it is convenient to incorporate this change of variables in the formula (4.29), and write

$$v = A\zeta + \overline{A\zeta} + \Psi(A, \bar{A}, \mu), \quad A \in \mathbb{C}, \tag{4.31}$$

in which $v_0(t) = A(t)\zeta + \overline{A(t)\zeta} \in \mathscr{E}_0$, but now Ψ takes values in $\mathscr{Z}$, instead of $\mathscr{Z}_h$.

First, according to the result in the Exercise 3.5, the coefficient a can be obtained from the eigenvalue λ of $\mathbf{L} + \mu\mathbf{R}_{01}$ which is equal to $i\omega$ when $\mu = 0$. More precisely,

a is the coefficient of the $O(\mu)$ term in the expansion in μ of this eigenvalue. Going back to the eigenvalue problem (4.25) we find that this eigenvalue λ is a root of the characteristic polynomial P_1 for $\beta = \beta_1 + \mu$, i.e.,

$$\lambda^2 - \mu\lambda + \omega^2 - \mu\pi^2\delta_2 = 0.$$

Here we have used the fact that $i\omega$ is a root of P_1 when $\beta = \beta_1$. This gives the solutions

$$\lambda_+ = i\omega + \mu\left(\frac{1}{2} - i\pi^2\frac{\delta_2}{2\omega}\right) + O(\mu^2), \quad \lambda_- = \bar{\lambda}_+,$$

so that the coefficient a in (4.30) is

$$a = \frac{1}{2} - i\pi^2\frac{\delta_2}{2\omega}. \tag{4.32}$$

To compute b we proceed as in the previous examples. We set $\mu = 0$ in the following calculations. Inserting v from (4.31) into (4.27), we find the equality

$$(\zeta + \partial_A\Psi)\frac{dA}{dt} + (\bar{\zeta} + \partial_{\bar{A}}\Psi)\frac{d\bar{A}}{dt} = \mathbf{L}(A\zeta + \overline{A\zeta} + \Psi) + \mathbf{R}(A\zeta + \overline{A\zeta} + \Psi, 0). \tag{4.33}$$

Using the expansion (4.28) of $\mathbf{R}$, the expansion of Ψ

$$\Psi(A, \bar{A}, 0) = \sum_{p,q} \Psi_{pq} A^p \overline{A}^q,$$

where $\Psi_{qp} \in \mathscr{X}$ are such that

$$\Psi_{qp} = \overline{\Psi_{pq}}, \quad \Psi_{00} = \Psi_{10} = \Psi_{01} = 0,$$

and replacing dA/dt by the right hand side of (4.30), we identify the powers of $(A, \bar{A})$ in (4.33). At orders A^2, $A\bar{A}$, and $A^2\bar{A}$, we find, respectively,

$$(2i\omega - \mathbf{L})\Psi_{20} = \mathbf{R}_{20}(\zeta, \zeta), \tag{4.34}$$

$$-\mathbf{L}\Psi_{11} = 2\mathbf{R}_{20}(\zeta, \bar{\zeta}), \tag{4.35}$$

$$(i\omega - \mathbf{L})\Psi_{21} = -b\zeta + 2\mathbf{R}_{20}(\bar{\zeta}, \Psi_{20}) + 2\mathbf{R}_{20}(\zeta, \Psi_{11}) + 3\mathbf{R}_{30}(\zeta, \zeta, \bar{\zeta}). \tag{4.36}$$

Recall that $\pm i\omega$ are the only purely imaginary eigenvalues of $\mathbf{L}$. Then from the first two equations we can compute Ψ_{20} and Ψ_{11}, since $(2i\omega - \mathbf{L})$ and $\mathbf{L}$ are invertible, and from the solvability condition for the third equation we find b. We show below how these quantities can be explicitly computed. The arguments are typical for such types of bifurcation problems arising for PDEs.

First, we obtain

$$\mathbf{R}_{20}(\zeta,\zeta) = \left(\alpha\gamma + \frac{\beta_1}{2\alpha}\right)(1-\cos(2\pi x))\begin{pmatrix}1\\-1\end{pmatrix},$$
$$2\mathbf{R}_{20}(\zeta,\bar{\zeta}) = \left(2\alpha\gamma_r + \frac{\beta_1}{\alpha}\right)(1-\cos(2\pi x))\begin{pmatrix}1\\-1\end{pmatrix},$$

where γ_r in the second formula represents the real part of γ. The equation (4.34) is a linear nonhomogeneous system of two differential equations of second order. Its solution set is a four-dimensional space, and the solution Ψ_{20} is uniquely determined by the Dirichlet boundary conditions at $x=0$ and $x=1$, which must be satisfied by the functions in $\mathscr{X}$. We introduce the following 2×2-matrices:

$$\mathscr{M}(ni\omega, \nu^2) = \begin{pmatrix} ni\omega + 1 - \beta_1 - \delta_1\nu^2 & -\alpha^2 \\ \beta_1 & ni\omega + \alpha^2 - \delta_2\nu^2 \end{pmatrix},$$

representing the action of the operator $in\omega - \mathbf{L}$ on the exponential $e^{\nu x}v$, with $v\in\mathbb{C}^2$, so that

$$(ni\omega - \mathbf{L})e^{\nu x}v = \mathscr{M}(ni\omega, \nu^2)e^{\nu x}v.$$

Then the solutions of the homogeneous equation $(2i\omega - \mathbf{L})v = 0$ are linear combinations of the four basic solutions,

$$v_{1+}e^{\nu_1 x}, \quad v_{1-}e^{-\nu_1 x}, \quad v_{2+}e^{\nu_2 x}, \quad v_{2-}e^{-\nu_2 x},$$

in which $\pm\nu_1$, $\pm\nu_2$ are the four solutions of

$$\det(\mathscr{M}(2i\omega, \nu^2)) = 0$$

and the vectors $v_{1\pm}\in\mathbb{C}^2$ and $v_{2\pm}\in\mathbb{C}^2$ belong to the kernels of $\mathscr{M}(2i\omega,\nu_1^2)$ and $\mathscr{M}(2i\omega,\nu_2^2)$, respectively. Next, notice that the operator $2i\omega - \mathbf{L}$ preserves the linear subspaces spanned by $\cos(n\pi x)$ (and also $\sin(n\pi x)$), so that we can look for a particular solution of (4.34) in the form

$$\Psi_{20}^0 = \alpha_{20} + \beta_{20}\cos(2\pi x),$$

in which $\alpha_{20}\in\mathbb{C}^2$ and $\beta_{20}\in\mathbb{C}^2$ are the unique solutions of

$$\mathscr{M}(2i\omega, 0)\alpha_{20} = \left(\alpha\gamma + \frac{\beta_1}{2\alpha}\right)\begin{pmatrix}1\\-1\end{pmatrix},$$
$$\mathscr{M}(2i\omega, -4\pi^2)\beta_{20} = \left(\alpha\gamma + \frac{\beta_1}{2\alpha}\right)\begin{pmatrix}-1\\1\end{pmatrix}. \tag{4.37}$$

Summarizing, we have that

$$\Psi_{20} = \alpha_{20} + \beta_{20}\cos(2\pi x) + \gamma_{20}e^{\nu_1 x} + \delta_{20}e^{-\nu_1 x} + \chi_{20}e^{\nu_2 x} + \kappa_{20}e^{-\nu_2 x},$$

in which α_{20} and β_{20} are uniquely determined from (4.37), the vectors γ_{20}, δ_{20}, χ_{20}, and κ_{20} satisfy

$$\mathscr{M}(2i\omega,\nu_1^2)\gamma_{20}=0,\quad \mathscr{M}(2i\omega,\nu_1^2)\delta_{20}=0,$$
$$\mathscr{M}(2i\omega,\nu_2^2)\chi_{20}=0,\quad \mathscr{M}(2i\omega,\nu_2^2)\kappa_{20}=0,$$

and are uniquely determined from the Dirichlet boundary conditions at $x=0$ and $x=1$ for Ψ_{20}:

$$\alpha_{20}+\beta_{20}+\gamma_{20}+\delta_{20}+\chi_{20}+\kappa_{20}=0,$$
$$\alpha_{20}+\beta_{20}+\gamma_{20}e^{\nu_1}+\delta_{20}e^{-\nu_1}+\chi_{20}e^{\nu_2}+\kappa_{20}e^{-\nu_2}=0.$$

In the same way, we solve equation (4.35) and find the solution

$$\Psi_{11}=\alpha_{11}+\beta_{11}\cos(2\pi x)+\gamma_{11}e^{\mu_1 x}+\delta_{11}e^{-\mu_1 x}+\chi_{11}e^{\mu_2 x}+\kappa_{11}e^{-\mu_2 x},$$

where $\pm\mu_1\in\mathbb{C}$ and $\pm\mu_2\in\mathbb{C}$ are the four solutions of

$$\det(\mathscr{M}(0,\mu^2))=0,$$

and the vectors on the right hand side are uniquely determined from

$$\mathscr{M}(0,0)\alpha_{11}=\left(2\alpha\gamma_r+\frac{\beta_1}{\alpha}\right)\begin{pmatrix}1\\-1\end{pmatrix},\ \mathscr{M}(0,-4\pi^2)\beta_{11}=\left(2\alpha\gamma_r+\frac{\beta_1}{\alpha}\right)\begin{pmatrix}-1\\1\end{pmatrix},$$

$$\mathscr{M}(0,\mu_1^2)\gamma_{11}=0,\quad \mathscr{M}(0,\mu_1^2)\delta_{11}=0,\quad \mathscr{M}(0,\mu_2^2)\chi_{11}=0,\quad \mathscr{M}(0,\mu_2^2)\kappa_{11}=0,$$

and the equalities

$$\alpha_{11}+\beta_{11}+\gamma_{11}+\delta_{11}+\chi_{11}+\kappa_{11}=0,$$
$$\alpha_{11}+\beta_{11}+\gamma_{11}e^{\mu_1}+\delta_{11}e^{-\mu_1}+\chi_{11}e^{\mu_2}+\kappa_{11}e^{-\mu_2}=0.$$

Finally, in equation (4.36) we compute

$$2\mathbf{R}_{20}(\bar\zeta,\Psi_{20})+2\mathbf{R}_{20}(\zeta,\Psi_{11})+3\mathbf{R}_{30}(\zeta,\zeta,\bar\zeta)=\begin{pmatrix}f(x)\\-f(x)\end{pmatrix},$$

where

$$f(x)=(2\gamma+\bar\gamma)\left(\frac{3}{4}\sin(\pi x)-\frac{1}{4}\sin(3\pi x)\right)$$
$$+\left(2\alpha(\psi_{20}^{(2)}+\psi_{11}^{(2)})+\left(2\alpha\bar\gamma+\frac{2\beta_1}{\alpha}\right)\psi_{20}^{(1)}+\left(2\alpha\gamma+\frac{2\beta_1}{\alpha}\right)\psi_{11}^{(1)}\right)\sin(\pi x).$$

Here we have denoted

$$\Psi_{20}=\begin{pmatrix}\psi_{20}^{(1)}\\ \psi_{20}^{(2)}\end{pmatrix},\quad \Psi_{11}=\begin{pmatrix}\psi_{11}^{(1)}\\ \psi_{11}^{(2)}\end{pmatrix}.$$

The solvability condition for the equation (4.36) is that its right hand side should be orthogonal to the kernel of the adjoint operator, $-i\omega+\mathbf{L}^*$. A direct computation shows that this kernel is one-dimensional and spanned by the vector

$$\zeta^* = \sin(\pi x)\begin{pmatrix} -\gamma \\ 1 \end{pmatrix},$$

from which we obtain that

$$b = \frac{2(1+\bar{\gamma})}{\bar{\gamma}-\gamma}\int_0^1 \sin(\pi x) f(x)dx.$$

According to the result in Theorem 2.6 in Chapter 1, on the Hopf bifurcation, a supercritical (resp., subcritical) Hopf bifurcation occurs at $\mu=0$ if the real part b_r of b is negative (resp., positive). Notice that the bifurcating periodic solution corresponds to an *oscillating chemical reaction*.

2.4.5 Elliptic PDE in a Strip

Consider the elliptic problem

$$\begin{aligned} &\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\nu v+g\left(v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)=0,\\ &v(x,0)=v(x,\pi)=0, \end{aligned}$$

where $v(x,y)\in\mathbb{R}$ for $(x,y)\in\mathbb{R}\times(0,\pi)$, ν is a real parameter, and we assume that $g\in C^k(\mathbb{R}^3,\mathbb{R})$, with $g(u,v,w)=O(|u|^2+|v|^2+|w|^2)$, as $(u,v,w)\to 0$. We further assume that g is even in its second argument.

Formulation and Symmetries

This problem enters our setting when we take as our time variable the unbounded spatial variable $x\in\mathbb{R}$, and so write the problem in the form

$$\frac{du}{dx}=\mathbf{L}_\nu u+\mathbf{R}(u). \tag{4.38}$$

This formulation of the problem is also called "spatial dynamics" formulation. The idea of spatial dynamics goes back to [80] and was used for finding bounded solutions of elliptic PDEs in cylindrical domains.

We obtain the equation (4.38) by taking $u=(u_1,u_2)=(v,\partial v/\partial x)$, and

$$\mathbf{L}_\nu=\begin{pmatrix} 0 & 1 \\ -\frac{d^2}{dy^2}-\nu & 0 \end{pmatrix}, \quad \mathbf{R}(u)=\begin{pmatrix} 0 \\ -g\left(u_1,u_2,\frac{du_1}{dy}\right) \end{pmatrix}.$$

We choose the spaces

$$\mathscr{X} = H_0^1(0,\pi) \times L^2(0,\pi), \quad \mathscr{Z} = \left(H^2(0,\pi) \cap H_0^1(0,\pi)\right) \times H_0^1(0,\pi),$$

such that $\mathbf{L}_\nu$ is a closed operator in $\mathscr{X}$ with domain $\mathscr{Z}$, which contains the boundary conditions, and

$$\mathscr{Y} = \left(H^2(0,\pi) \cap H_0^1(0,\pi)\right) \times H^1(0,\pi),$$

such that $\mathbf{R}(u) \in \mathscr{Y}$ for $u \in \mathscr{Z}$, and $\mathbf{R}$ is of class $C^k(\mathscr{Z},\mathscr{Y})$.

Notice that the elliptic equation is invariant under $(x,y) \mapsto (-x,y)$, since we assumed that g is even in its second argument. This induces a reversibility symmetry for (4.38), i.e., the vector field on the right hand side anticommutes with the symmetry $\mathbf{S}$ defined by

$$\mathbf{S}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}.$$

As in the previous examples we next look at the spectrum of $\mathbf{L}_\nu$. We have again that $\mathscr{Y}$ is compactly embedded in $\mathscr{X}$, so $\mathbf{L}_\nu$ has a compact resolvent and its spectrum consists of isolated eigenvalues with finite multiplicities. The eigenvalue problem reads

$$\begin{aligned} u_2 &= \lambda u_1 \\ -u_1'' - \nu u_1 &= \lambda u_2, \end{aligned}$$

in which u_1 satisfies the boundary conditions $u_1(0) = u_1(\pi) = 0$. Then a direct computation shows that the spectrum σ of $\mathbf{L}_\nu$ is

$$\sigma = \{\lambda_n^\pm = \pm\sqrt{n^2 - \nu}\ ;\ n \in \mathbb{N}^*\}. \tag{4.39}$$

Notice that σ is symmetric with respect to both the imaginary and the real axis in the complex plane, due to the reversibility symmetry and the fact that $\mathbf{L}_\nu$ is a real operator. When $\nu \neq p^2$, for any integer p, the eigenvalues are all simple and real except for a finite number that are purely imaginary. When $\nu = p^2$ for some nonzero integer p, we find that 0 is a double eigenvalue, $\lambda_p^\pm = 0$, and the eigenvalues $\lambda_n^\pm$ with $n < p$ are purely imaginary, whereas the eigenvalues $\lambda_n^\pm$ with $n > p$ are real. Consequently, we can use the center manifold theorem, for values of ν close to $\nu_p = p^2$, for any $p \geq 1$.

Reduced System

We focus here on values of ν close to $\nu_1 = 1$ and set $\nu = 1 + \mu$. We rewrite the equation (4.38) in the form

$$\frac{du}{dx} = \mathbf{L}u + \mathbf{R}(u,\mu),$$

with

$$\mathbf{L} = \mathbf{L}_1, \quad \mathbf{R}(u,\mu) = \mathbf{R}(u) + (\mathbf{L}_{1+\mu} - \mathbf{L}_1)u.$$

From the arguments above it is now easy to check that Hypotheses 3.1 and 2.4 hold. In Hypothesis 2.4 we have $\sigma_0 = \{0\}$ with 0 a geometrically simple and algebraically double eigenvalue. The corresponding spectral subspace $\mathscr{E}_0$ is spanned by

$$\zeta_0 = \begin{pmatrix} \sin y \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ \sin y \end{pmatrix},$$

which satisfy $\mathbf{L}\zeta_0 = 0$ and $\mathbf{L}\zeta_1 = \zeta_0$, respectively. Also notice that

$$\mathbf{S}\zeta_0 = \zeta_0, \quad \mathbf{S}\zeta_1 = -\zeta_1.$$

Further, Hypothesis 2.7 can be checked as in the example in Section 2.4.2, using the result in Theorem 2.17 and showing that the estimate (2.9) holds. In addition, the reversibility symmetry $\mathbf{S}$ satisfies Hypothesis 3.14.

We can now apply the results in Theorems 3.3 and 3.15 and obtain a family of two-dimensional center manifolds for μ sufficiently small. For solutions on the center manifold we write

$$u = A\zeta_0 + B\zeta_1 + \Psi(A,B,\mu),$$

where $A(t) \in \mathbb{R}$, $B(t) \in \mathbb{R}$, and Ψ takes values in $\mathscr{Z}_h$. This leads to a reduced equation of the form

$$\begin{aligned} \frac{dA}{dx} &= f(A,B,\mu) \\ \frac{dB}{dx} &= g(A,B,\mu), \end{aligned} \tag{4.40}$$

in which the vector field (f,g) satisfies

$$(f,g)(0,0,\mu) = (0,0), \quad D(f,g)(0,0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In addition, the vector field is reversible, it anticommutes with the symmetry $\mathbf{S}_0$ induced by $\mathbf{S}$ acting through

$$\mathbf{S}_0(A,B) = (A,-B),$$

and the reduction function Ψ commutes with $\mathbf{S}$,

$$\mathbf{S}\Psi(A,B,\mu) = \Psi(A,-B,\mu). \tag{4.41}$$

We shall further analyze this reduced system in Chapter 4, which is devoted to reversible systems.

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Local Bifurcations, Center Manifolds, and Normal Forms
in Infinite-Dimensional Dynamical Systems
Haragus, M.; Iooss, G.
2011, XI, 329 p., Softcover
ISBN: 978-0-85729-111-0