Chapter 2
Complex Numbers

2.1 Introduction

Complex numbers have been described as the 'king' of numbers, probably because they resolve all sorts of mathematical problems where ordinary real numbers fail. For example, the rather innocent looking equation

$$1 + x^2 = 0$$

has no real solution, which seems amazing when one considers the equation's simplicity. But one does not need a long equation to show that the algebra of real numbers is unable to cope with objects such as

$$x = \sqrt{-1}.$$

However, this did not prevent mathematicians from finding a way around such an inconvenience, and fortuitously the solution turned out to be an incredible idea that is used everywhere from electrical engineering to cosmology. The simple idea of declaring the existence of a quantity i, such that $i^2 = -1$, permits us to express the solution to the above equation as

$$x = \pm i.$$

All very well, you might say, but what is i? What is mathematics? One could also ask, and spend an eternity searching for an answer! i is simply a mathematical object whose square is -1. Let us continue with this strange object and see how it leads us into the world of rotations.

2.2 Complex Numbers

A complex number has two parts: a *real* part and an *imaginary* part. The real part is just an ordinary number that may be zero, positive or negative, and the imaginary part is another real number multiplied by i. For example, $2 + 3i$ is a complex number

J. Vince, *Rotation Transforms for Computer Graphics*,
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where 2 is the real part and $3i$ is the imaginary part. The following are all complex numbers:

$$2, \quad 2+2i, \quad 1-3i, \quad -4i, \quad 17i.$$

Note the convention to place the real part first, followed by i. However, if i is associated with a trigonometric function such as *sin* or *cos*, it is usual to place i in front of the function: $i\sin\theta$ or $i\cos\theta$, to avoid any confusion that it is part of the function's angle.

All that we have to remember is that whenever we manipulate complex numbers, the occurrence of i^2 is replaced by -1.

2.2.1 Axioms

The axioms defining the behaviour of complex numbers are identical to those associated with real numbers. For example, given two complex numbers z_1 and z_2 they obey the following rules:
Addition:

$$\begin{aligned} \text{Commutative} \quad & z_1 + z_2 = z_2 + z_1 \\ \text{Associative} \quad & (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)\,. \end{aligned}$$

Multiplication:

$$\begin{aligned} \text{Commutative} \quad & z_1 z_2 = z_2 z_1 \\ \text{Associative} \quad & (z_1 z_2)\, z_3 = z_1\,(z_2 z_3) \\ \text{Distributive} \quad & z_1\,(z_2 + z_3) = z_1 z_2 + z_1 z_3 \\ & (z_1 + z_2)\, z_3 = z_1 z_3 + z_2 z_3. \end{aligned}$$

2.3 The Modulus

The *modulus* of a complex number $a+bi$ is defined as $\sqrt{a^2+b^2}$. For example, the modulus of $3+4i$ is 5. In general, the modulus of a complex number z is written $|z|$:

$$\begin{aligned} z &= a + bi \\ |z| &= \sqrt{a^2 + b^2}. \end{aligned}$$

We'll see why this is so when we cover the polar representation of a complex number.

2.4 Addition and Subtraction

Given two complex numbers:

$$\begin{aligned} z_1 &= a + bi \\ z_2 &= c + di \\ z_1 \pm z_2 &= (a \pm c) + (b \pm d)\, i \end{aligned}$$

where the real and imaginary parts are added or subtracted, respectively. For example:

$$\begin{aligned} z_1 &= 5 + 3i \\ z_2 &= 3 + 2i \\ z_1 + z_2 &= 8 + 5i \\ z_1 - z_2 &= 2 + i. \end{aligned}$$

2.5 Multiplication by a Scalar

A scalar is just an ordinary number, and may be used to multiply a complex number using normal algebraic rules. For example, the complex number $a + bi$ is multiplied by the scalar λ as follows:

$$\lambda\,(a + bi) = \lambda a + \lambda bi$$

and a specific example:

$$2\,(3 + 5i) = 6 + 10i.$$

2.6 Product of Two Complex Numbers

The product of two complex numbers is evaluated by creating all the terms algebraically, and collecting up the real and imaginary terms:

$$\begin{aligned} z_1 &= a + bi \\ z_2 &= c + di \\ z_1 z_2 &= (a + bi)\,(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)\, i \end{aligned}$$

which is another complex number. For example:

$$\begin{aligned}
z_1 &= 3+4i \\
z_2 &= 5-2i \\
z_1 z_2 &= (3+4i)(5-2i) \\
&= 15-6i+20i-8i^2 \\
&= 15+14i+8 \\
&= 23+14i.
\end{aligned}$$

Remember that the addition, subtraction and multiplication of complex numbers obey the normal axioms of algebra. Also, the multiplication of two complex numbers, and their addition always results in a complex number, that is, the two operations are closed.

2.7 The Complex Conjugate

A special case exists when we multiply two complex numbers together where the only difference between them is the sign of the imaginary part:

$$\begin{aligned}
(a+bi)(a-bi) &= a^2 - abi + abi - b^2 i^2 \\
&= a^2 + b^2.
\end{aligned}$$

As this real value is such an interesting result, $a-bi$ is called the *complex conjugate* of $a+bi$. In general, the complex conjugate of

$$z = a+bi$$

is written either with a bar $\bar{z}$ symbol or an asterisk z^* as

$$z^* = a-bi$$

and implies that

$$zz^* = a^2 + b^2 = |z|^2.$$

2.8 Division of Two Complex Numbers

The complex conjugate provides us with a mechanism to divide one complex number by another. For instance, consider the quotient

$$\frac{a+bi}{c+di}.$$

This can be resolved by multiplying the numerator and denominator by the complex conjugate $c-di$ to create a real denominator:

$$
\begin{aligned}
\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\
&= \frac{ac-adi+bci-bdi^2}{c^2+d^2} \\
&= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i.
\end{aligned}
$$

Another special case is when $a=1$ and $b=0$:

$$
\frac{1}{c+di}=(c+di)^{-1}=\left(\frac{c}{c^2+d^2}\right)-\left(\frac{d}{c^2+d^2}\right)i
$$

which is the *inverse* of a complex number.

Let's evaluate the quotient:

$$
\frac{4+3i}{3+4i}.
$$

Multiplying top and bottom by the complex conjugate $3-4i$ we have

$$
\begin{aligned}
\frac{4+3i}{3+4i} &= \frac{(4+3i)(3-4i)}{(3+4i)(3-4i)} \\
&= \frac{12-16i+9i-12i^2}{25}=\frac{24}{25}-\frac{7}{25}i.
\end{aligned}
$$

2.9 The Inverse

Although we have already discovered the inverse of a complex number, let's employ another strategy by declaring

$$
z_1=\frac{1}{z}
$$

where z is a complex number.

Next, we divide both sides by the complex conjugate of z to create

$$
\frac{z_1}{z^*}=\frac{1}{zz^*}.
$$

But we have previously shown that $zz^*=|z|^2$, therefore,

$$
\frac{z_1}{z^*}=\frac{1}{|z|^2}
$$

and rearranging, we have

$$
z_1=\frac{z^*}{|z|^2}.
$$

In general

$$
\frac{1}{z}=z^{-1}=\frac{z^*}{|z|^2}.
$$

As an illustration let's find the inverse of $3+4i$

$$\begin{aligned}\frac{1}{3+4i} &= (3+4i)^{-1}\\ &= \frac{3-4i}{25}\\ &= \frac{3}{25}-\frac{4}{25}i.\end{aligned}$$

Let's test this result by multiplying z by its inverse:

$$(3+4i)\left(\frac{3}{25}-\frac{4}{25}i\right)=\frac{9}{25}-\frac{12}{25}i+\frac{12}{25}i+\frac{16}{25}=1$$

which confirms the correctness of the inverse.

2.10 The Complex Plane

Leonhard Euler (1707–1783) (whose name rhymes with *boiler*) played a significant role in putting complex numbers on the map. His ideas on rotations are also used in computer graphics to locate objects and virtual cameras in space, as we shall see later on.

Consider the scenario depicted in Fig. 2.1. Any number on the number line is related to the same number with the opposite sign via a rotation of 180°. For example, when 2 is rotated 180° about zero, it becomes -2, and when -3 is rotated 180° about zero it becomes 3.

But as we know that $i^2=-1$ we can write

$$-n=i^2n.$$

If we now regard i^2 as a rotation through 180°, then i could be a rotation through 90°!

Figure 2.2 shows how complex numbers can be interpreted as 2D coordinates using the *complex plane* where the real part is the horizontal coordinate and the

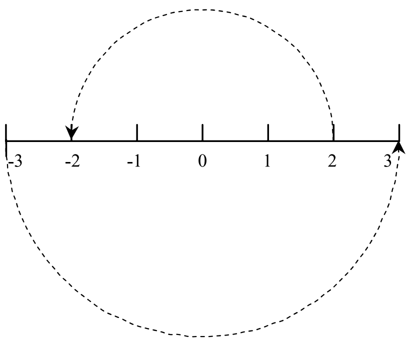


Fig. 2.1 Rotating numbers through 180° reverses their sign

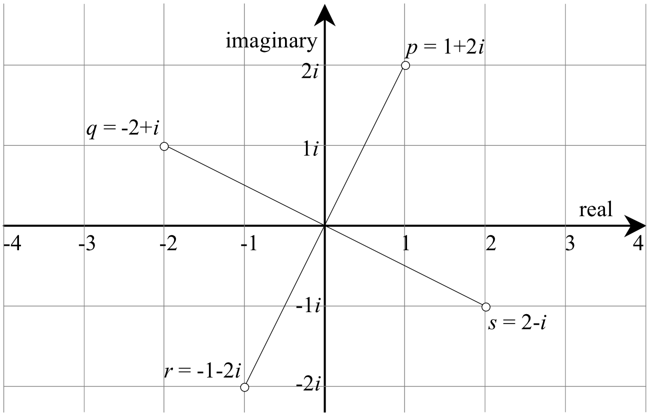


Fig. 2.2 The graphical representation of complex numbers

imaginary part is the vertical coordinate. The figure also shows four complex numbers:

$$p = 1 + 2i, \quad q = -2 + i, \quad r = -1 - 2i, \quad s = 2 - i$$

which happen to be 90° apart. For example, the complex number p in Fig. 2.2 is rotated 90° to q by multiplying it by i:

$$\begin{aligned} i\,(1 + 2i) &= i + 2i^2 \\ &= -2 + i. \end{aligned}$$

The point q is rotated another 90° to r by multiplying it by i:

$$\begin{aligned} i\,(-2 + i) &= -2i + i^2 \\ &= -1 - 2i. \end{aligned}$$

The point r is rotated another 90° to s by multiplying it by i:

$$\begin{aligned} i\,(-1 - 2i) &= -i - 2i^2 \\ &= 2 - i. \end{aligned}$$

Finally, the point s is rotated 90° back to p by multiplying it by i:

$$\begin{aligned} i\,(2 - i) &= 2i - i^2 \\ &= 1 + 2i. \end{aligned}$$

2.11 Polar Representation

The complex plane provides a simple mechanism to represent complex numbers graphically. This in turn makes it possible to use a *polar representation* as shown

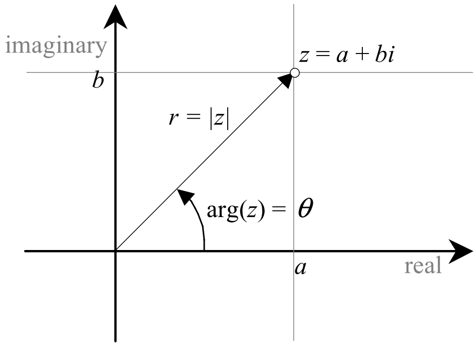


Fig. 2.3 Polar representation of a complex number

in Fig. 2.3 where we see the complex number $z = a + bi$ representing the oriented line r. The length of r is obviously $\sqrt{a^2 + b^2}$, which is why the modulus of a complex number has the same definition. We can see from Fig. 2.3 that the horizontal component of z is $r\cos\theta$ and the vertical component is $r\sin\theta$, which permits us to write

$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + ri\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$

Note that i has been placed in front of the sin function.

The angle θ between r and the real axis is called the *argument* and written $\arg(z)$, and in this case

$$\arg(z) = \theta.$$

One of Euler's discoveries concerns the relationship between the series for exponential e, sin and cos:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which enables us to write

$$z = re^{i\theta}.$$

We are now in a position to revisit the product and quotient of two complex numbers using polar representation. For example:

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ w &= s(\cos\phi + i\sin\phi) \\ zw &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs\big(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi\big) \\ &= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\big) \end{aligned}$$

and as

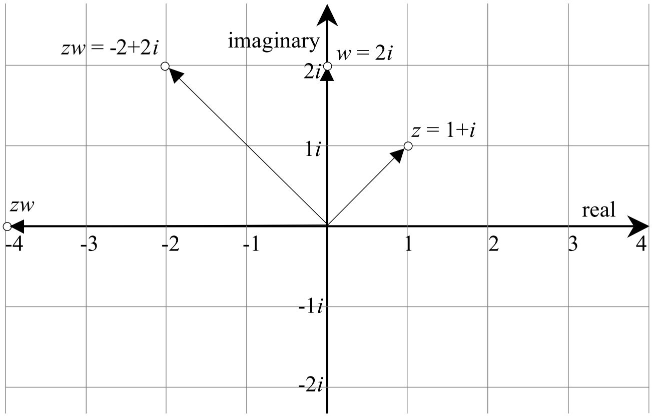


Fig. 2.4 The product of two complex numbers

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$zw = rs\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big).$$

So the product of two complex numbers creates a third one with modulus

$$|zw| = rs$$

and argument

$$\arg(zw) = \arg(z) + \arg(w) = \theta + \phi.$$

Let's illustrate this with an example. Figure 2.4 shows two complex numbers

$$z = 1 + i, \quad w = 2i$$

therefore,

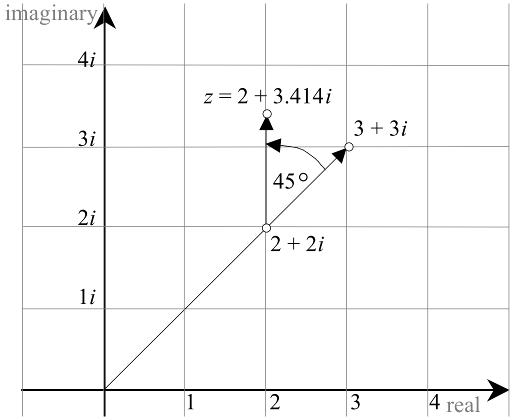
$$|z| = \sqrt{2}, \quad \arg(z) = 45^\circ$$
$$|w| = 2, \quad \arg(w) = 90^\circ$$
$$|zw| = 2\sqrt{2}$$
$$\arg(zw) = 135^\circ$$

which is another complex number $-2 + 2i$.

2.12 Rotors

The above observations imply that multiplying a complex number by another, whose modulus is unity, causes no scaling. For example, multiplying $3 + 4i$ by $1 + 0i$ creates the same complex number, unscaled and unrotated. However, multiplying $3 + 4i$ by $0 + i$ rotates it by 90° without any scaling.

Fig. 2.5 Rotating a complex number about another complex number



So to rotate $2+2i$ by 45° we must multiply it by

$$\cos 45^\circ + i\sin 45^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\begin{aligned}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)(2+2i) &= \sqrt{2} + \sqrt{2}i + \sqrt{2}i + \sqrt{2}i^2\\ &= 2\sqrt{2}i.\end{aligned}$$

So now we have a *rotor* to rotate a complex number through any angle. In general, the rotor to rotate a complex number $a+bi$ through an angle θ is

$$\mathbf{R}_\theta = \cos\theta + i\sin\theta.$$

Now let's consider the problem of rotating $3+3i$, 45° about $2+2i$ as shown in Fig. 2.5. From the figure, the result is $z \approx 2+3.414i$, but let's calculate it by subtracting $2+2i$ from $3+3i$ to shift the operation to the origin, then multiply the result by $\sqrt{2}/2 + \sqrt{2}/2i$, and then add back $2+2i$:

$$\begin{aligned}z &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\big((3+3i)-(2+2i)\big) + 2 + 2i\\ &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)(1+i) + 2 + 2i\\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} + 2 + 2i\\ &= 2 + (2+\sqrt{2})i\\ &\approx 2 + 3.414i\end{aligned}$$

which is correct. Therefore, to rotate any point (x, y) through an angle θ we convert it into a complex number $x+yi$ and multiply by the rotor $\cos\theta + i\sin\theta$:

$$
\begin{aligned}
x' + y'i &= (\cos\theta + i\sin\theta)(x + yi) \\
&= (x\cos\theta - y\sin\theta) + (x\sin\theta + y\cos\theta)\,i
\end{aligned}
$$

where (x', y') is the rotated point.

But as we shall see in Chap. 4, this is the transform for rotating a point (x, y) about the origin:

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
$$

Before moving on let's consider the effect the complex conjugate of a rotor has on rotational direction, and we can do this by multiplying $x + yi$ by the rotor $\cos\theta - i\sin\theta$:

$$
\begin{aligned}
x' + y'i &= (\cos\theta - i\sin\theta)(x + yi) \\
&= x\cos\theta + y\sin\theta - (x\sin\theta + y\cos\theta)\,i
\end{aligned}
$$

which in matrix form is

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

which is a rotation of $-\theta$.

Therefore, we define a rotor $\mathbf{R}_\theta$ and its conjugate $\mathbf{R}_\theta^\dagger$ as

$$
\begin{aligned}
\mathbf{R}_\theta &= \cos\theta + i\sin\theta \\
\mathbf{R}_\theta^\dagger &= \cos\theta - i\sin\theta
\end{aligned}
$$

where $\mathbf{R}_\theta$ rotates $+\theta$, and $\mathbf{R}_\theta^\dagger$ rotates $-\theta$. The dagger symbol '†' is chosen as it is used for rotors in multivectors, which are covered later.

2.13 Summary

There is no doubt that complex numbers are amazing objects and arise simply by introducing the symbol i which squares to -1. It is unfortunate that the names 'complex' and 'imaginary' are used to describe them as they are neither complex nor imaginary, but very simple. We will come across them again in later chapters and see how they provide a way of rotating 3D points.

In this chapter we have seen that complex numbers can be added, subtracted, multiplied and divided, and they can even be raised to a power. We have also come across new terms such as: *complex conjugate*, *modulus* and *argument*. We have also discovered the *rotor* which permits us to rotate 2D points.

In the mid-19th century, mathematicians started to look for the 3D equivalent of complex numbers, and after many years of work, Sir William Rowan Hamilton invented *quaternions* which are the subject of a later chapter.

2.13.1 Summary of Complex Operations

Complex number

$$z = a + bi \quad \text{where } i^2 = -1.$$

Addition and subtraction

$$\begin{aligned} z_1 &= a + bi \\ z_2 &= c + di \\ z_1 \pm z_2 &= (a \pm c) + (b \pm d)\, i. \end{aligned}$$

Scalar product

$$\lambda z = \lambda a + \lambda b i.$$

Modulus

$$|z| = \sqrt{a^2 + b^2}.$$

Product

$$z_1 z_2 = (ac - bd) + (ad + bc)\, i.$$

Complex conjugate

$$z^* = a - bi.$$

Division

$$\frac{z_1}{z_2} = \left(\frac{ac + bd}{c^2 + d^2} \right) + \left(\frac{bc - ad}{c^2 + d^2} \right) i.$$

Inverse

$$z^{-1} = \frac{z^*}{|z|^2}.$$

Polar form

$$\begin{aligned} z &= r\,(\cos\theta + i \sin\theta) \\ r &= |z| \\ \theta &= \arg(z) \\ z &= r e^{i\theta}. \end{aligned}$$

Rotors

$$\begin{aligned} \mathbf{R}_\theta &= \cos\theta + i \sin\theta \\ \mathbf{R}_\theta^\dagger &= \cos\theta - i \sin\theta. \end{aligned}$$

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