

Chapter 2

Classification of Harish-Chandra Modules

In this chapter, we will prove a theorem of O. Mathieu [Mat2] saying that any simple $\mathbb{Z}$ -graded Vir-module of finite type is either a highest weight module, a lowest weight module, or a simple subquotient of the module of type $V_{a,b}$ introduced in Chapter 1. (See Theorem 2.1, for detail.) This was a conjecture of V. G. Kac [Kac3].

First, we will classify irreducible modules in the case of positive characteristic, and will prove the results in the characteristic zero case by the semi-continuity principle.

In Section 2.1, we will recall some basic notion, and will state the main results in a precise form. The rest of the sections are devoted to the proof of the main results. In Section 2.2, we will recall basic facts about the ‘partial Lie algebras’ and their ‘modules’ with detailed proof. In Section 2.3, we will prove some facts about $\mathbb{Z}$ -graded Lie algebras, and will prove that the dimensions of any simple $\mathbb{Z}$ -graded Vir-module without highest nor lowest degree are uniformly bounded. In Section 2.4, we will study representations of Lie p -algebras $W(m)$, quotients of the Witt algebra in characteristic $p \neq 2, 3$. Finally, in Section 2.5, after recalling some facts about Dedekind rings, we will prove the main theorem.

Through this chapter, an associative algebra is not necessarily unital. When an algebra has to be unital, we always indicate it.

2.1 Main Result

2.1.1 Notations and Conventions

Let $\mathbb{K}$ be a field. For an abelian group G , we say that a Lie algebra $\mathfrak{g} = \bigoplus_{\pi \in G} \mathfrak{g}_{\pi}$ over $\mathbb{K}$ is **G -graded** if it satisfies

$$[\mathfrak{g}_{\pi}, \mathfrak{g}_{\pi'}] \subset \mathfrak{g}_{\pi+\pi'} \quad (\forall \pi, \pi' \in G).$$

Remark that, in this chapter, the condition $\dim \mathfrak{g}_\pi < \infty$ is not necessarily assumed. Hence, a G -graded Lie algebra with $G = Q$ does not mean a Q -graded Lie algebra in Definition 1.6. A module $M = \bigoplus_{\pi \in G} M_\pi$ over a G -graded Lie algebra $\mathfrak{g}$ is called **G -graded** if $\mathfrak{g}_\pi \cdot M_{\pi'} \subset M_{\pi+\pi'}$ for any $\pi, \pi' \in G$. $\mathfrak{g}$ (resp. M) is said to be **finite** if $\dim \mathfrak{g}_\pi < \infty$ (resp. $\dim M_\pi < \infty$) for any $\pi \in G$.

Definition 2.1 Let $\mathfrak{g} = \bigoplus_{\pi \in G} \mathfrak{g}_\pi$ be a G -graded Lie algebra, and let $M = \bigoplus_{\pi \in G} M_\pi$ be a G -graded $\mathfrak{g}$ -module.

1. M is called a **simple G -graded $\mathfrak{g}$ -module** if M has no non-trivial G -graded submodule.
2. M is called a **G -graded simple $\mathfrak{g}$ -module** if M has no non-trivial submodule.

For simplicity, we often omit G in the terminology.

In this chapter, we mainly deal with the cases $G = \mathbb{Z}$ and $G = \mathbb{Z}/N\mathbb{Z}$. Here, we introduce some notations for $\mathbb{Z}$ -graded Lie algebras. For a $\mathbb{Z}$ -graded $\mathbb{K}$ -vector space $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and an integer a , we set

$$M^{\geq a} := \bigoplus_{n \geq a} M_n, \quad M^{\leq a} := \bigoplus_{n \leq a} M_n.$$

For simplicity, we denote $M^{\geq 1}$, $M^{\leq 1}$, $M^{\geq 0}$ and $M^{\leq 0}$ by M^+ , M^- , $M^\geq$ and $M^\leq$ respectively. A $\mathbb{Z}$ -graded Lie algebra $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ has a triangular decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^+$.

2.1.2 Definitions

Let $\mathbb{K}$ be a field of characteristic $p \neq 2, 3$. Similarly to the characteristic zero case, the Virasoro algebra over $\mathbb{K}$ is, by definition,

$$\text{Vir}_{\mathbb{K}} := \bigoplus_{n \in \mathbb{Z}} \mathbb{K}L_n \oplus \mathbb{K}C$$

as vector space satisfying

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C, \\ [C, \text{Vir}_{\mathbb{K}}] &= \{0\}. \end{aligned}$$

We set $\mathfrak{h} := \mathbb{K}L_0 \oplus \mathbb{K}C$.

We recall the definition of Harish-Chandra modules over $\text{Vir}_{\mathbb{K}}$. In this definition, we suppose that the characteristic of $\mathbb{K}$ is zero.

Definition 2.2 *Let M be an absolutely simple module over $\text{Vir}_{\mathbb{K}}$. M is called a Harish-Chandra module over $\text{Vir}_{\mathbb{K}}$ if M is $\mathfrak{h}$ -diagonalisable and any weight subspaces are finite dimensional.*

To state the classification theorem of Harish-Chandra modules over $\text{Vir}_{\mathbb{K}}$, we recall the intermediate series of the Virasoro algebra. In the sequel, we regard $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{K}$. For $n \in \mathbb{Z}$, we often regard n as an element of $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{K}$ via the canonical map $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$. For $a, b \in \mathbb{K}$, let

$$V_{a,b} = \bigoplus_{n \in \mathbb{Z}} \mathbb{K}v_n$$

be a $\mathbb{Z}$ -graded $\text{Vir}_{\mathbb{K}}$ -module defined by

$$\begin{aligned} L_s.v_n &= (as + b - n)v_{n+s}, \\ C.v_n &= 0. \end{aligned}$$

- Proposition 2.1** *1. If $a \neq 0, -1$ or $b \notin \mathbb{Z}/p\mathbb{Z}$, then $V_{a,b}$ is simple graded.
 2. If $a = 0$ and $b \in \mathbb{Z}/p\mathbb{Z}$, then there exists a submodule V of $V_{a,b}$ such that the quotient module $V_{a,b}/V$ is simple graded.
 3. If $a = -1$ and $b \in \mathbb{Z}/p\mathbb{Z}$, then there exists a simple graded submodule V of $V_{a,b}$.*

Proof. Suppose that $V_{a,b}$ is not simple graded, i.e., there exists a non-trivial proper graded submodule $M = \bigoplus_{n \in \mathbb{Z}} M_n$ of $V_{a,b}$, where $M_n \subset \mathbb{K}v_n$. Since each graded subspace of $V_{a,b}$ is one-dimensional, there exists $u \in \mathbb{Z}$ such that

$$M_u \neq \{0\} \text{ and } \{ M_{u+1} = \{0\} \text{ or } M_{u-1} = \{0\} \}.$$

In this proof, we only consider the case $M_{u-1} = \{0\}$, since the other case can be similarly treated.

Notice that

$$L_{-1}.v_u = (-a + b - u)v_{u-1} \in M_{u-1} = \{0\}.$$

Hence, $-a + b - u = 0$, and thus

$$L_s.v_u = (s + 1)av_{u+s}. \quad (2.1)$$

We consider the following cases.

$a = 0$ Since $-a + b - u = 0$ in $\mathbb{K}$, $b = u \in \mathbb{Z}/p\mathbb{Z}$. In this case, we have

$$L_{s'}.v_n = (u - n)v_{n+s'}. \quad (2.2)$$

Hence,

$$V := \bigoplus_{\substack{n \in \mathbb{Z} \\ n=b \text{ in } \mathbb{Z}/p\mathbb{Z}}} \mathbb{K}v_n$$

is a direct sum of trivial $\text{Vir}_{\mathbb{K}}$ -modules. By (2.2), we see that $V_{a,b}/V$ is simple graded.

$a \neq 0$ By (2.1), if $s \not\equiv -1 \pmod{p}$, then $v_{u+s} \in M$. Let us check whether $v_{u+s} \in M$ or not for $s \equiv -1 \pmod{p}$. Since $-a + b - u = 0$ in $\mathbb{K}$, we have

$$L_{s'}.v_n = \{a(s' + 1) + (u - n)\}v_{n+s'}. \quad (2.3)$$

In particular, if $s' + 1 \equiv u - n \pmod{p}$, then

$$L_{s'}.v_n = (a + 1)(u - n)v_{n+s'}. \quad (2.4)$$

$a \neq -1$ One can find integers $n, s' \in \mathbb{Z}$ such that $n \not\equiv u \pmod{p}$ and $n + s' \equiv u - 1 \pmod{p}$. Hence, $v_n \in M$ for any $n \in \mathbb{Z}$. This is a contradiction, since M is a proper submodule. Hence, $V_{a,b}$ is simple graded.

$a = -1$ By (2.3) and (2.4),

$$V := \bigoplus_{\substack{n \in \mathbb{Z} \\ n \neq b \text{ in } \mathbb{Z}/p\mathbb{Z}}} \mathbb{K}v_n.$$

is a simple graded submodule of $V_{a,b}$. □

For each $a, b \in \mathbb{K}$, we set

$$V'_{a,b} := \begin{cases} V_{a,b} & (a \neq 0, -1 \wedge b \notin \mathbb{Z}/p\mathbb{Z}) \\ V_{a,b}/V & (a = 0 \wedge b \in \mathbb{Z}/p\mathbb{Z}) \\ V & (a = -1 \wedge b \in \mathbb{Z}/p\mathbb{Z}) \end{cases} \quad (2.5)$$

where $V_{a,b}$ and V are as in the above proposition.

Definition 2.3 *The irreducible representations $V'_{a,b}$ ($a, b \in \mathbb{K}$) over $\text{Vir}_{\mathbb{K}}$ are called the **intermediate series**.*

The following is the main result of this chapter.

Theorem 2.1 *Let V be a Harish-Chandra module over $\text{Vir}_{\mathbb{K}}$, where the base field $\mathbb{K}$ is an algebraically closed field of characteristic zero. Then, V is isomorphic to an irreducible highest weight module, an irreducible lowest weight module or one of the intermediate series.*

2.2 Partial Lie Algebras

A partial Lie algebra introduced in [Mat3] plays an essential role in the proof of Theorem 2.1. In this section, we recall its definition and state fundamental properties.

2.2.1 Definition and Main Theorems

First, we introduce the notion of partial Lie algebras and their modules. Let (d, e) be a pair of integers such that $d \leq 0 \leq e$. Let

$$\Gamma := \bigoplus_{d \leq i \leq e} \Gamma_i$$

be a graded $\mathbb{K}$ -vector space. Throughout this section, we always assume that Γ is finite dimensional.

Definition 2.4 *We say that Γ is a **partial Lie algebra** of size (d, e) , if there exists a bilinear map*

$$[\cdot, \cdot] : \Gamma \times \Gamma \longrightarrow \Gamma$$

with the following properties:

1. *for i and j such that $d \leq i, j, i + j \leq e$,*

$$[\Gamma_i, \Gamma_j] \subset \Gamma_{i+j},$$

2. *for i and j such that $d \leq i, j, i + j \leq e$,*

$$[x_i, x_j] + [x_j, x_i] = 0 \quad (x_i \in \Gamma_i, x_j \in \Gamma_j),$$

3. *for i, j and k such that $d \leq i, j, k, i + j, j + k, k + i, i + j + k \leq e$,*

$$\begin{aligned} [x_i, [x_j, x_k]] + [x_k, [x_i, x_j]] + [x_j, [x_k, x_i]] &= 0 \\ (x_i \in \Gamma_i, x_j \in \Gamma_j, x_k \in \Gamma_k). \end{aligned}$$

For a partial Lie algebra $\Gamma = \bigoplus_{d \leq i \leq e} \Gamma_i$, we set

$$\Gamma^- := \bigoplus_{d \leq i < 0} \Gamma_i, \quad \Gamma^+ := \bigoplus_{0 < i \leq e} \Gamma_i,$$

and regard them as partial Lie algebras of size $(d, -1)$ and $(1, e)$ respectively. Then, we have a triangular decomposition $\Gamma = \Gamma^- \oplus \Gamma_0 \oplus \Gamma^+$.

For a $\mathbb{Z}$ -graded Lie algebra $\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i$, we set

$$\text{Par}_d^e \mathcal{L} := \bigoplus_{i=d}^e \mathcal{L}_i.$$

Then, $\text{Par}_d^e \mathcal{L}$ is naturally equipped with a partial Lie algebra structure. We call it the **partial part** of $\mathcal{L}$ of size (d, e) .

Let Γ and Γ' be partial Lie algebras of size (d, e) . A linear map $\phi : \Gamma \rightarrow \Gamma'$ is called a **homomorphism** of partial Lie algebras if the following hold:

1. ϕ is a homomorphism of graded vectors spaces, i.e., for any $d \leq i \leq e$,

$$\phi(\Gamma_i) \subset \Gamma'_i,$$

2. for i and j such that $d \leq i, j, i+j \leq e$,

$$\phi([x_i, x_j]) = [\phi(x_i), \phi(x_j)] \quad (x_i \in \Gamma_i, x_j \in \Gamma_j).$$

A bijective homomorphism of partial Lie algebras is called an **isomorphism**.

Let I be a graded subspace of a partial Lie algebra Γ , i.e., $I = \bigoplus_{d \leq i \leq e} I_i$, where $I_i := I \cap \Gamma_i$. I is called a **subalgebra** (resp. an **ideal**) of Γ , if it satisfies $[I_i, I_j] \subset I_{i+j}$ (resp. $[I_i, \Gamma_j] \subset I_{i+j}$) for any i and j such that $d \leq i, j, i+j \leq e$.

Next, we introduce a partial module over a partial Lie algebra.

Definition 2.5 A pair (V, ρ) of a graded vector space $V = \bigoplus_{d \leq i \leq e} V_i$ and a linear map $\rho : \Gamma \rightarrow \text{End} V$ is called a **partial module** of Γ if it satisfies

1. for i and j such that $d \leq i, j, i+j \leq e$,

$$\rho(x_i)v_j \in V_{i+j} \quad (x_i \in \Gamma_i, v_j \in V_j),$$

2. for i, j and k such that $d \leq i, j, k, i+j, j+k, k+i, i+j+k \leq e$,

$$(\rho(x_i)\rho(x_j) - \rho(x_j)\rho(x_i))v_k = \rho([x_i, x_j])v_k \quad (x_i \in \Gamma_i, x_j \in \Gamma_j, v_k \in V_k).$$

For a $\mathbb{Z}$ -graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ over a $\mathbb{Z}$ -graded Lie algebra $\mathcal{L}$, we define the **partial part** $\text{Par}_d^e M$ of M by

$$\text{Par}_d^e M := \bigoplus_{d \leq i \leq e} M_i,$$

and regard it as a partial module over $\text{Par}_d^e \mathcal{L}$ in a natural way.

Let Γ be a partial Lie algebra of size (d, e) , (V, ρ) and (V', ρ') be partial Γ -modules. A homomorphism $\phi : (V, \rho) \rightarrow (V', \rho')$ of graded vector spaces is a **homomorphism** of partial Γ -modules if

$$\phi(x_i.v_j) = x_i.\phi(v_j) \quad (x_i \in \Gamma_i, v_j \in V_j),$$

for any i and j such that $d \leq i, j, i+j \leq e$. A bijective homomorphism of partial modules is called an **isomorphism**. A partial module is said to be **simple** if there is no non-trivial partial submodule.

The following two theorems are the main results of this section.

Theorem 2.2 Let Γ be a partial Lie algebra of size (d, e) .

1. There exists a unique $\mathbb{Z}$ -graded Lie algebra $\mathcal{L}_{\max}(\Gamma)$ which satisfies the following:

- a. $\text{Par}_d^e \mathcal{L}_{\max}(\Gamma) \simeq \Gamma$ as partial Lie algebra.

- b. For any $\mathbb{Z}$ -graded Lie algebra $\mathfrak{g}$ and a partial Lie algebra homomorphism $\phi : \Gamma \rightarrow \text{Par}_d^e \mathfrak{g}$, there exists a unique Lie algebra homomorphism $\Phi : \mathcal{L}_{\max}(\Gamma) \rightarrow \mathfrak{g}$ whose restriction to Γ is ϕ .
- c. As $\mathbb{Z}$ -graded $\mathbb{K}$ -vector space,

$$\mathcal{L}_{\max}(\Gamma) \simeq \mathcal{L}_{\max}(\Gamma^-) \oplus \Gamma_0 \oplus \mathcal{L}_{\max}(\Gamma^+). \quad (2.6)$$

2. There exists a unique $\mathbb{Z}$ -graded Lie algebra $\mathcal{L}_{\min}(\Gamma)$ which satisfies the following:

- a. $\text{Par}_d^e \mathcal{L}_{\min}(\Gamma) \simeq \Gamma$ as partial Lie algebra.
- b. For any $\mathbb{Z}$ -graded Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}$ is generated by $\text{Par}_d^e \mathfrak{g}$ and a surjective homomorphism $\psi : \text{Par}_d^e \mathfrak{g} \rightarrow \Gamma$, there exists a unique homomorphism $\Psi : \mathfrak{g} \rightarrow \mathcal{L}_{\min}(\Gamma)$ whose restriction to $\text{Par}_d^e \mathfrak{g}$ is ψ .

Theorem 2.3 *Let Γ be a partial Lie algebra of size (d, e) , and let V be a partial module over Γ . Let $\mathcal{L} := \mathcal{L}_{\max}(\Gamma)$ be the $\mathbb{Z}$ -graded Lie algebra given by Theorem 2.2. Then, we have*

1. There exists a unique $\mathbb{Z}$ -graded $\mathcal{L}$ -module $M_{\max}(V)$ such that
 - a. $\text{Par}_d^e M_{\max}(V) \simeq V$ as partial Γ -module.
 - b. For any $\mathbb{Z}$ -graded $\mathcal{L}$ -module M and a homomorphism $\phi : V \rightarrow \text{Par}_d^e M$, there uniquely exists a homomorphism of $\mathcal{L}$ -modules $\Phi : M_{\max}(V) \rightarrow M$ such that its restriction to the partial part coincides with ϕ .
2. There exists a unique $\mathbb{Z}$ -graded $\mathcal{L}$ -module $M_{\min}(V)$ such that
 - a. $\text{Par}_d^e M_{\min}(V) \simeq V$ as partial Γ -module.
 - b. For any $\mathbb{Z}$ -graded $\mathcal{L}$ -module M generated by its partial part as $\mathcal{L}$ -module and a surjective homomorphism $\psi : \text{Par}_d^e M \rightarrow V$, there uniquely exists a homomorphism of $\mathcal{L}$ -modules $\Psi : M \rightarrow M_{\min}(V)$ such that its restriction to the partial part coincides with ψ .

We prove these theorems in the following subsections.

2.2.2 Proof of Theorem 2.2

In this section, we construct $\mathcal{L}_{\max}(\Gamma)$ as a quotient of the free Lie algebra on Γ by the ideal generated by the relations of Γ . Moreover, we show that $\mathcal{L}_{\min}(\Gamma)$ can be realised as a certain quotient of $\mathcal{L}_{\max}(\Gamma)$. Remark that we essentially follow arguments used to construct Kac–Moody algebras in Chapter 1 of [Kac4].

2.2.2.1 Notation

We first introduce some notation for free Lie algebras. Let V be a vector space over $\mathbb{K}$, and let

$$\mathcal{T}(V) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{T}^n(V) \quad \mathcal{T}^n(V) := \begin{cases} V \otimes V \otimes \cdots \otimes V \text{ (} n \text{ copies)} & n \geq 1 \\ \mathbb{K} & n = 0 \end{cases}$$

be the tensor algebra on V . We regard the associative algebra $\mathcal{T}(V)$ as a Lie algebra in the natural way. The free Lie algebra $\mathcal{F}(V)$ on V is the Lie subalgebra of $\mathcal{T}(V)$ generated by V . By definition, we have

Lemma 2.1. *Suppose that V is a module over a Lie algebra $\mathfrak{a}$. Then, $\mathcal{F}(V)$ is an $\mathfrak{a}$ -submodule of $\mathcal{T}(V)$.*

In the case where $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a $\mathbb{Z}$ -graded vector space, $\mathcal{F}(V)$ naturally inherits a $\mathbb{Z}$ -graded Lie algebra structure. We denote this $\mathbb{Z}$ -gradation by

$$\mathcal{F}(V) = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}(V)_i.$$

2.2.2.2 Construction of $\mathcal{G}(\Gamma)$

In the following, let Γ be a partial Lie algebra. Here, we introduce a $\mathbb{Z}$ -graded Lie algebra $\mathcal{G}(\Gamma)$ which plays a role similar to those of $\tilde{\mathfrak{g}}(A)$ in Chapter 1 of [Kac4].

Let $\mathcal{F}(\Gamma)$ be the free Lie algebra on Γ , and let $\mathcal{J}(\Gamma)$ be the ideal of $\mathcal{F}(\Gamma)$ generated by

$$\{u \otimes v - v \otimes u - [u, v]_\Gamma \mid u \in \Gamma_i, v \in \Gamma_j, d \leq i, j, i + j \leq e, ij \leq 0\}, \quad (2.7)$$

where $[\ , \]_\Gamma$ denotes the partial Lie bracket on Γ . We set

$$\mathcal{G}(\Gamma) := \mathcal{F}(\Gamma) / \mathcal{J}(\Gamma). \quad (2.8)$$

By definition, $\mathcal{G}(\Gamma)$ is a $\mathbb{Z}$ -graded Lie algebra. We denote its triangular decomposition by $\mathcal{G}(\Gamma) = \mathcal{G}(\Gamma)^- \oplus \mathcal{G}(\Gamma)_0 \oplus \mathcal{G}(\Gamma)^+$.

We first show the following proposition (cf. Theorem 1.2. in [Kac4]):

Proposition 2.2 *There exists an isomorphism of $\mathbb{Z}$ -graded $\mathbb{K}$ -vector spaces:*

$$\mathcal{G}(\Gamma) \simeq \mathcal{F}(\Gamma^-) \oplus \Gamma_0 \oplus \mathcal{F}(\Gamma^+). \quad (2.9)$$

Moreover, $\mathcal{G}(\Gamma)^\pm \simeq \mathcal{F}(\Gamma^\pm)$ and $\mathcal{G}(\Gamma)_0 \simeq \Gamma_0$ as $\mathbb{Z}$ -graded Lie algebra.

To prove this proposition, we have to introduce an appropriate Lie algebra structure on the direct sum in the right-hand side of (2.9). Here, we show the following general statement.

Proposition 2.3 *Let X_0 be a Lie algebra and let $X^\pm$ be X_0 -modules. We set $X := X^- \oplus X_0 \oplus X^+$ and*

$$\mathcal{X} := \mathcal{F}(X^-) \oplus X_0 \oplus \mathcal{F}(X^+).$$

Let $\phi : X^+ \otimes_{\mathbb{K}} X^- \rightarrow X$ be a homomorphism of X_0 -modules. Then, there exists a Lie algebra structure $[\cdot, \cdot]_{\mathcal{X}}$ on $\mathcal{X}$ which satisfies the following properties:

1. $\mathcal{F}(X^\pm)$ and X_0 are Lie subalgebras of $\mathcal{X}$.
2. For any $x \in X_0$ and $u \in \mathcal{F}(X^\pm)$, $[x, u]_{\mathcal{X}} = x.u$, where the action of X_0 on $\mathcal{F}(X^\pm)$ in the right-hand side is given by Lemma 2.1.
3. For any $u \in X^+$ and $v \in X^-$, $[u, v]_{\mathcal{X}} = \phi(u \otimes v)$ holds.

Proof. We set $X^\geq := X_0 \oplus X^+$ and $X^\leq := X^- \oplus X_0$ for simplicity, and introduce a bilinear map $\phi_X : X^\geq \times X^\leq \rightarrow X$ as follows:

$$\begin{aligned} \phi_X(u, v) &:= \phi(u \otimes v), & \phi_X(w, w') &:= [w, w']_{X_0}, \\ \phi_X(u, w) &:= -w.u, & \phi_X(w, v) &:= w.v, \end{aligned} \tag{2.10}$$

where $u \in X^+$, $v \in X^-$ and $w, w' \in X_0$ and $[\cdot, \cdot]_{X_0}$ denotes the Lie bracket on X_0 . Using this notation, we introduce a $\mathbb{K}$ -associative algebra $\mathcal{U}(X)$ as follows: Let $\mathcal{K}(X)$ be the two-sided ideal of the tensor algebra $\mathcal{T}(X)$ on X generated by

$$\{x \otimes y - y \otimes x - \phi_X(x, y) \mid x \in X^\geq, y \in X^\leq\}.$$

We set

$$\mathcal{U}(X) := \mathcal{T}(X)/\mathcal{K}(X).$$

In the sequel, we naturally regard $\mathcal{U}(X)$ as Lie algebra and realise the Lie algebra $\mathcal{X}$ as a Lie subalgebra of $\mathcal{U}(X)$.

Let $\mathcal{D}(X)$ (resp. $\mathcal{D}(X^\pm)$ and $\mathcal{D}(X_0)$) be the Lie subalgebra of $\mathcal{U}(X)$ generated by X (resp. $X^\pm$ and X_0). We denote the Lie bracket on $\mathcal{D}(X)$ by $[\cdot, \cdot]_{\mathcal{D}}$. By definition, the following commutation relations hold:

$$[x, y]_{\mathcal{D}} = \phi_X(x, y), \quad (\forall x \in X^\geq, \forall y \in X^\leq). \tag{2.11}$$

By using them, we can show that

Lemma 2.2. *There exists an isomorphism of $\mathbb{K}$ -vector spaces:*

$$\mathcal{D}(X) \simeq \mathcal{D}(X^-) \oplus \mathcal{D}(X_0) \oplus \mathcal{D}(X^+).$$

Proof. For simplicity, we set $\tilde{\mathcal{D}}(X) := \mathcal{D}(X^-) \oplus \mathcal{D}(X_0) \oplus \mathcal{D}(X^+)$. Notice that $X \subset \tilde{\mathcal{D}}(X) \subset \mathcal{D}(X)$ and $\mathcal{D}(X)$ is generated by X . Hence, we prove that $\tilde{\mathcal{D}}(X)$ is a Lie subalgebra of $\mathcal{D}(X)$, i.e.,

$$[X, \tilde{\mathcal{D}}(X)]_{\mathcal{D}} \subset \tilde{\mathcal{D}}(X).$$

By the commutation relations (2.11), $[X^\pm, \mathcal{D}(X_0)]_{\mathcal{D}} \subset X^\pm$ and $[X_0, \mathcal{D}(X^\pm)] \subset \mathcal{D}(X^\pm)$ hold. Hence, it suffices to show that

$$[X^+, \mathcal{D}(X^-)] \subset X^\geq \oplus \mathcal{D}(X^-), \quad [X^-, \mathcal{D}(X^+)] \subset X^\leq \oplus \mathcal{D}(X^+).$$

Here, we prove the first inclusion since the second one can be proved similarly.

By definition, $\mathcal{D}(X^-)$ is spanned by elements of the form

$$x = [x_m, [x_{m-1}, [\cdots [x_2, x_1]_{\mathcal{D}}]_{\mathcal{D}} \cdots]_{\mathcal{D}} \quad (x_i \in X^-).$$

By induction on m , one can show that

$$[y, x] \in X^\geq \oplus \mathcal{D}(X^-) \quad (\forall y \in X^+).$$

Hence, we have the first inclusion, and thus, the lemma holds. $\square$

On the other hand, by an argument similar to the proof of the Poincaré–Birkhoff–Witt theorem in [Jac], we can show that

Lemma 2.3. *There exists an isomorphism of $\mathbb{K}$ -vector spaces:*

$$\mathcal{U}(X) \simeq \mathcal{T}(X^-) \otimes_{\mathbb{K}} U(X_0) \otimes_{\mathbb{K}} \mathcal{T}(X^+),$$

where $U(X_0)$ denotes the universal enveloping algebra of X_0 .

Proof. We can prove this lemma in a way similar to the proof of the PBW theorem in Chapter V Section 2 of [Jac]. Hence, we only indicate an outline of the proof.

Let $\{x_i | i \in I^\pm\}$ (resp. $\{x_i | i \in I^0\}$) be $\mathbb{K}$ -bases of $X^\pm$ (resp. X_0), and let $\leq$ be a total order on I^0 . We set $I := I^- \sqcup I^0 \sqcup I^+$. Then, $\{x_i | i \in I\}$ forms a $\mathbb{K}$ -basis of X .

Let us construct a $\mathbb{K}$ -basis of $\mathcal{U}(X)$ by using $\{x_i | i \in I\}$. For each $i, j \in I$, we set

$$\eta(i, j) := \begin{cases} 1 & \text{if } \begin{cases} i \in I^+ \wedge j \in I^0 \\ i \in I^+ \wedge j \in I^- \\ i, j \in I^0 \wedge i > j \\ i \in I^0 \wedge j \in I^- \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

Remark that for a sequence $(i_1, \dots, i_n)$ consisting of elements of I ,

$$\begin{aligned} \eta(i_k, i_{k+1}) &= 0 \quad (\forall k = 1, 2, \dots, n-1) \\ \Leftrightarrow \begin{cases} i_1, \dots, i_{s-1} \in I^- \\ i_s, \dots, i_{t-1} \in I^0 \wedge i_s \leq \dots \leq i_{t-1} \\ i_t, \dots, i_n \in I^+ \end{cases} \quad (\exists s, t, 1 \leq s \leq t \leq n+1). \end{aligned} \tag{2.12}$$

One can easily check that $\mathcal{U}(X)$ is spanned by the vectors of the form

$$x_{i_1} \otimes \cdots \otimes x_{i_n} + \mathcal{K}(X) \quad (n \in \mathbb{Z}_{\geq 0}, \eta(i_k, i_{k+1}) = 0 \ (\forall k)), \quad (2.13)$$

where in the case of $n = 0$, we regard $x_{i_1} \otimes \cdots \otimes x_{i_n} = 1$. Moreover, the following lemma ensures that these vectors are linearly independent. Let $S(X_0)$ be the symmetric algebra on X_0 .

Lemma 2.4. *For each $n \in \mathbb{Z}_{>0}$, there exists a linear map*

$$\sigma_n : \mathcal{T}^{\leq n}(X) \rightarrow \mathcal{T}(X^-) \otimes_{\mathbb{K}} S(X_0) \otimes_{\mathbb{K}} \mathcal{T}(X^+)$$

which satisfies the following conditions:

1. *if $\eta(i_k, i_{k+1}) = 0$ for any k , then*

$$\sigma_n(x_{i_1} \otimes \cdots \otimes x_{i_n}) = (x_{i_1} \otimes \cdots \otimes x_{i_{s-1}}) \otimes (x_{i_s} \cdots x_{i_{t-1}}) \otimes (x_{i_t} \otimes \cdots \otimes x_{i_n}),$$

where s and t are given as in (2.12).

2. *if $\eta(i_k, i_{k+1}) = 1$ for some k , then*

$$\begin{aligned} \sigma_n(x_{i_1} \otimes \cdots \otimes x_{i_n}) - \sigma_n(x_{i_1} \otimes \cdots \otimes x_{i_{k+1}} \otimes x_{i_k} \otimes \cdots \otimes x_{i_n}) \\ = \sigma_n(x_{i_1} \otimes \cdots \otimes \phi_X(x_{i_k}, x_{i_{k+1}}) \otimes \cdots \otimes x_{i_n}), \end{aligned}$$

where $\phi_X : X^{\geq} \times X^{\leq} \rightarrow X$ is the bilinear map defined in (2.10).

Proof. Using the Jacobi identity of X_0 and the facts that $X^{\pm}$ are X_0 -modules and $\phi : X^+ \otimes X^- \rightarrow X$ is an X_0 -module map, one can show this lemma by induction on n . $\square$

The map σ_n induces a linear map

$$\begin{aligned} \bar{\sigma}_n : (\mathcal{T}^{\leq n}(X) + \mathcal{K}(X)) / \mathcal{K}(X) &\longrightarrow \mathcal{T}(X^-) \otimes_{\mathbb{K}} S(X_0) \otimes_{\mathbb{K}} \mathcal{T}(X^+), \\ \bar{\sigma}_n(x_{i_1} \otimes \cdots \otimes x_{i_n} + \mathcal{K}(X)) & \\ = (x_{i_1} \otimes \cdots \otimes x_{i_{s-1}}) \otimes (x_{i_s} \cdots x_{i_{t-1}}) \otimes (x_{i_t} \otimes \cdots \otimes x_{i_n}). & \end{aligned}$$

Hence, the vectors (2.13) are linearly independent, and thus, Lemma 2.3 holds. $\square$

As a corollary of this lemma, we have

$$\mathcal{D}(X^{\pm}) \simeq \mathcal{F}(X^{\pm}), \quad \mathcal{D}(X_0) \simeq X_0,$$

and hence, $\mathcal{D}(X) \simeq \mathcal{X}$ as $\mathbb{K}$ -vector space. Through this isomorphism, we obtain the required Lie algebra structure on $\mathcal{X}$. Thus, we have proved Proposition 2.3. $\square$

Next, we introduce a Lie algebra structure on the direct sum in the right-hand side of (2.9). For simplicity, we denote the direct sum by $\tilde{\mathcal{G}}(\Gamma)$. By Proposition 2.3, there exists a Lie bracket $[\cdot, \cdot]_{\tilde{\mathcal{G}}}$ on $\tilde{\mathcal{G}}(\Gamma)$ which satisfies

$$[x, y]_{\tilde{\mathcal{G}}} = [x, y]_{\Gamma} \quad (x \in \Gamma_i, y \in \Gamma_j) \quad (2.14)$$

for any integers i and j such that $d \leq i, j, i + j \leq e$ and $ij \leq 0$. Hence, there exists a surjective homomorphism of Lie algebras:

$$i_{\mathcal{G}} : \mathcal{G}(\Gamma) \longrightarrow \tilde{\mathcal{G}}(\Gamma). \quad (2.15)$$

Now, Proposition 2.2 follows from the next proposition:

Lemma 2.5. *The homomorphism (2.15) is bijective.*

Proof. By an argument similar to the proof of Lemma 2.2, one can show that $\mathcal{G}(\Gamma)^{\pm}$ and $\mathcal{G}(\Gamma)_0$ are generated by $\Gamma^{\pm}$ and Γ_0 respectively. Hence, the universality of the free Lie algebras $\mathcal{F}(\Gamma^{\pm})$ implies that $i_{\mathcal{G}}$ is injective. $\square$

2.2.2.3 Construction of $\mathcal{L}(\Gamma)$

Here, we introduce a $\mathbb{Z}$ -graded Lie algebra $\mathcal{L}(\Gamma)$ which is the quotient of $\mathcal{F}(\Gamma)$ by the ideal generated by the relations of Γ . We also describe the triangular decomposition of $\mathcal{L}(\Gamma)$ by using that of $\mathcal{G}(\Gamma)$, and in the next subsection, we check that $\mathcal{L}(\Gamma)$ is equipped with the properties required for $\mathcal{L}_{\max}(\Gamma)$.

Let $\mathcal{I}(\Gamma)$ be the ideal of $\mathcal{F}(\Gamma)$ generated by

$$\{u \otimes v - v \otimes u - [u, v]_{\Gamma} | u \in \Gamma_i, v \in \Gamma_j, d \leq i, j, i + j \leq e\}, \quad (2.16)$$

and let $\mathcal{L}(\Gamma)$ be the Lie algebra defined by

$$\mathcal{L}(\Gamma) := \mathcal{F}(\Gamma) / \mathcal{I}(\Gamma). \quad (2.17)$$

By definition, $\mathcal{J}(\Gamma) \subset \mathcal{I}(\Gamma)$, and thus, there exists a canonical projection:

$$\pi_{\mathcal{G}} : \mathcal{G}(\Gamma) \twoheadrightarrow \mathcal{L}(\Gamma).$$

Using this map, we describe the triangular decomposition of $\mathcal{L}(\Gamma)$ explicitly.

To describe $\text{Ker} \pi_{\mathcal{G}}$, we introduce some notation. By replacing Γ with $\Gamma^{\pm}$, we define the ideals $\mathcal{I}(\Gamma^{\pm})$ of $\mathcal{F}(\Gamma^{\pm})$ and set $\mathcal{L}(\Gamma^{\pm}) := \mathcal{F}(\Gamma^{\pm}) / \mathcal{I}(\Gamma^{\pm})$. Moreover, via the isomorphism (2.9), we regard $\mathcal{I}(\Gamma^{\pm}) \subset \mathcal{F}(\Gamma^{\pm}) \subset \mathcal{G}(\Gamma)$. Then, we have

Lemma 2.6. *$\mathcal{I}(\Gamma^{\pm})$ are ideals of the Lie algebra $\mathcal{G}(\Gamma)$.*

Proof. For simplicity, we set $\mathcal{I}^{\pm} := \mathcal{I}(\Gamma^{\pm})$. Since $\mathcal{G}(\Gamma)$ is generated by Γ , it is enough to show that $[\Gamma, \mathcal{I}^{\pm}] \subset \mathcal{I}^{\pm}$. By using commutation relations of $\mathcal{G}(\Gamma)$, $[\Gamma_0, \mathcal{I}^{\pm}] \subset \mathcal{I}^{\pm}$ holds. Hence, we show that $[\Gamma^+, \mathcal{I}^-] \subset \mathcal{I}^-$ and $[\Gamma^-, \mathcal{I}^+] \subset \mathcal{I}^+$. We prove the first inclusion, since the second one can be proved similarly.

We denote the linear span of

$$\{u \otimes v - v \otimes u - [u, v]_{\Gamma} | u \in \Gamma_i, v \in \Gamma_j, d \leq i, j, i + j \leq -1\}$$

by S^- , where $S^- \subset \mathcal{F}(\Gamma^-) \subset \mathcal{G}(\Gamma)$. Then, $\mathcal{I}^-$ is spanned by elements of the form

$$y = [y_m, [y_{m-1}, \dots, [y_1, s] \dots]] \quad (y_i \in \mathcal{F}(\Gamma^-), s \in S^-).$$

Using the Jacobi identity, we have

$$[\Gamma^+, S^-] \subset S^-.$$

Moreover, one can show that

$$[x, y] \in \mathcal{I}^- \quad (\forall x \in \Gamma_k \ (1 \leq k \leq e))$$

by induction on m . Hence, the lemma holds. $\square$

By this lemma, $\mathcal{I}(\Gamma^+) \oplus \mathcal{I}(\Gamma^-)$ is an ideal of $\mathcal{G}(\Gamma)$. Moreover, it coincides with $\text{Ker} \pi_{\mathcal{G}}$, namely, the following holds:

Proposition 2.4 *The following isomorphism of $\mathbb{Z}$ -graded Lie algebras holds:*

$$\mathcal{L}(\Gamma) \simeq \mathcal{G}(\Gamma) / (\mathcal{I}(\Gamma^-) \oplus \mathcal{I}(\Gamma^+)). \quad (2.18)$$

Hence, as $\mathbb{Z}$ -graded vector space,

$$\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Gamma^-) \oplus \Gamma_0 \oplus \mathcal{L}(\Gamma^+). \quad (2.19)$$

Proof. For simplicity, we denote the right-hand side of (2.18) by $\tilde{\mathcal{L}}(\Gamma)$. The inclusion $\mathcal{I}(\Gamma^+) + \mathcal{I}(\Gamma^-) \subset \mathcal{I}(\Gamma) / \mathcal{J}(\Gamma)$ in $\mathcal{G}(\Gamma)$ implies that $\pi_{\mathcal{G}} : \mathcal{G}(\Gamma) \twoheadrightarrow \mathcal{L}(\Gamma)$ factors as follows:

$$\begin{array}{ccc} \mathcal{G}(\Gamma) & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{L}(\Gamma) \\ & \searrow & \nearrow \pi \\ & \tilde{\mathcal{L}}(\Gamma) & \end{array}$$

On the other hand, the kernel of the composition $\mathcal{F}(\Gamma) \rightarrow \mathcal{G}(\Gamma) \twoheadrightarrow \tilde{\mathcal{L}}(\Gamma)$ of canonical projections is an ideal of $\mathcal{F}(\Gamma)$ which contains

$$\{u \otimes v - v \otimes u - [u, v]_T \mid u \in \Gamma_i, v \in \Gamma_j, d \leq i, j, i + j \leq e\}.$$

Hence, this composition factors as follows:

$$\begin{array}{ccccc} \mathcal{F}(\Gamma) & \longrightarrow & \mathcal{G}(\Gamma) & \longrightarrow & \tilde{\mathcal{L}}(\Gamma) \\ & \searrow & & \nearrow \bar{\psi} & \\ & & \mathcal{L}(\Gamma) & & \end{array}$$

By definition, we have

$$\bar{\psi} \circ \pi|_{\text{Par}_d^e \tilde{\mathcal{L}}(\Gamma)} = \text{id}_{\text{Par}_d^e \tilde{\mathcal{L}}(\Gamma)}, \quad \bar{\pi} \circ \bar{\psi}|_{\text{Par}_d^e \mathcal{L}(\Gamma)} = \text{id}_{\text{Par}_d^e \mathcal{L}(\Gamma)}.$$

Hence, $\bar{\pi}$ and $\bar{\psi}$ are isomorphisms, and thus, the proposition follows. $\square$

2.2.2.4 Proof of Theorem 2.2.1

We complete the proof of the first statement of Theorem 2.2. By construction, $\mathcal{L}(\Gamma)$ enjoys the universal property required for $\mathcal{L}_{\max}(\Gamma)$. Moreover, by Proposition 2.4, the triangular decomposition (2.19) holds. Hence, it suffices to show that $\text{Par}_d^e \mathcal{L}(\Gamma) \simeq \Gamma$. The following lemma is a key of the proof.

Lemma 2.7. $\text{Par}_1^e \mathcal{L}(\Gamma^+) \simeq \Gamma^+$ and $\text{Par}_d^{-1} \mathcal{L}(\Gamma^-) \simeq \Gamma^-$ hold.

Proof. Here, we show this lemma for Γ^+ , since the case of Γ^- can be shown similarly. In this case, $\text{Par}_1^e \mathcal{F}(\Gamma^+)$ is spanned by elements of the form

$$x = [x_m, [x_{m-1}, \dots, [x_2, x_1]_{\mathcal{F}} \dots]_{\mathcal{F}}] \quad (x_i \in \Gamma_{n_i}, \quad 0 < n_i \leq e, \quad \sum_{i=1}^m n_i \leq e), \quad (2.20)$$

where $[\cdot, \cdot]_{\mathcal{F}}$ denotes the Lie bracket on $\mathcal{F}(\Gamma^+)$. Hence, there exists a partial Lie algebra homomorphism

$$\psi : \text{Par}_1^e \mathcal{F}(\Gamma^+) \longrightarrow \Gamma^+$$

which sends x of the form (2.20) to

$$[x_m, [x_{m-1}, \dots, [x_2, x_1]_{\Gamma} \dots]_{\Gamma}],$$

where $[\cdot, \cdot]_{\Gamma}$ is the partial Lie bracket on Γ^+ . Moreover, by the definition of $\mathcal{I}(\Gamma^+)$,

$$\text{Par}_1^e \mathcal{I}(\Gamma^+) \subset \text{Ker} \psi.$$

Hence, we have a homomorphism $\bar{\psi} : \text{Par}_1^e \mathcal{L}(\Gamma^+) \rightarrow \Gamma^+$.

On the other hand, there is a homomorphism of partial Lie algebra

$$\phi : \Gamma^+ \longrightarrow \text{Par}_1^e \mathcal{L}(\Gamma^+); \quad x \longmapsto x + \text{Par}_1^e \mathcal{I}(\Gamma^+),$$

which satisfies $\bar{\psi} \circ \phi = \text{id}_{\Gamma^+}$ and $\phi \circ \bar{\psi} = \text{id}_{\text{Par}_1^e \mathcal{L}(\Gamma^+)}$. Thus, we have proved Lemma 2.7. $\square$

Combining this lemma with the triangular decomposition (2.19), we obtain $\text{Par}_d^e \mathcal{L}(\Gamma) \simeq \Gamma$. Now, we have completed the proof of Theorem 2.2.1.

2.2.2.5 Proof of Theorem 2.2.2

Here, we construct $\mathcal{L}_{\min}(\Gamma)$ and prove Theorem 2.2.2. Let $\mathcal{M}(\Gamma)$ be the maximal $\mathbb{Z}$ -graded ideal of $\mathcal{L}(\Gamma)$ such that $\mathcal{M}(\Gamma) \cap \Gamma = \{0\}$.

We show that $\mathcal{L}(\Gamma)/\mathcal{M}(\Gamma)$ gives the Lie algebra $\mathcal{L}_{\min}(\Gamma)$. In fact, the first property

$$\text{Par}_d^e(\mathcal{L}(\Gamma)/\mathcal{M}(\Gamma)) \simeq \Gamma$$

follows by definition. Hence, we show the second property (the universal property).

Let $\mathfrak{g}$ be a $\mathbb{Z}$ -graded Lie algebra and let $\psi : \text{Par}_d^e \mathfrak{g} \rightarrow \Gamma$ be a surjective homomorphism of partial Lie algebras. We show that there exists a homomorphism of $\mathbb{Z}$ -graded Lie algebras $\mathfrak{g} \rightarrow \mathcal{L}(\Gamma)/\mathcal{M}(\Gamma)$ whose restriction to the partial part coincides with ψ .

By Theorem 2.2.1, there exist Lie algebra homomorphisms

$$\Psi_1 : \mathcal{L}(\text{Par}_d^e \mathfrak{g}) \rightarrow \mathcal{L}(\Gamma) \quad \text{and} \quad \Psi_2 : \mathcal{L}(\text{Par}_d^e \mathfrak{g}) \rightarrow \mathfrak{g}.$$

Note that Ψ_1 is surjective, since $\mathcal{L}(\Gamma)$ is generated by Γ . Hence, Ψ_1 maps an ideal of $\mathcal{L}(\text{Par}_d^e \mathfrak{g})$ to that of $\mathcal{L}(\Gamma)$. Since $\Psi_2|_{\text{Par}_d^e \mathfrak{g}} = \text{id}_{\text{Par}_d^e \mathfrak{g}}$, we have $\Psi_1(\text{Ker} \Psi_2) \cap \Gamma = \{0\}$, and thus, $\Psi_1(\text{Ker} \Psi_2) \subset \mathcal{M}(\Gamma)$. Hence, the composition $\mathcal{L}(\text{Par}_d^e \mathfrak{g}) \rightarrow \mathcal{L}(\Gamma) \rightarrow \mathcal{L}(\Gamma)/\mathcal{M}(\Gamma)$ factors as

$$\begin{array}{ccccc} \mathcal{L}(\text{Par}_d^e \mathfrak{g}) & \longrightarrow & \mathcal{L}(\Gamma) & \longrightarrow & \mathcal{L}(\Gamma)/\mathcal{M}(\Gamma) \\ & \searrow & & \nearrow & \\ & & \mathfrak{g} & & \end{array}$$

where the restriction of the homomorphism $\mathfrak{g} \rightarrow \mathcal{L}(\Gamma)/\mathcal{M}(\Gamma)$ to its partial part coincides with ψ .

2.2.3 Proof of Theorem 2.3

To show Theorem 2.3, we introduce an $\mathcal{L}_{\max}(\Gamma)$ -module $M(V)$ associated with a partial Γ -module V and show that it enjoys the properties required for $M_{\max}(V)$.

2.2.3.1 Construction of $M(V)$

To construct $M(V)$, we first introduce the semi-direct product of partial Lie algebra and its partial module. For a partial Lie algebra Γ and its partial module V the semi-direct product $\Gamma \ltimes V$ is defined as follows: We set $\Gamma \ltimes V := \Gamma \oplus V$ (the direct sum as vector space), and define a bilinear operation $[\ , \]$ on $\Gamma \ltimes V$ by

1. $[x_i, x_j] := [x_i, x_j]_\Gamma \ (x_i \in \Gamma_i, x_j \in \Gamma_j)$
2. $[x_i, y_j] := x_i \cdot y_j \ (x_i \in \Gamma_i, y_j \in V_j),$
3. $[y_i, y_j] := 0 \ (y_i \in V_i, y_j \in V_j),$

where $[\cdot, \cdot]_I$ is the partial Lie bracket of I , and $i, j \in \mathbb{Z}$ satisfy $d \leq i, j, i+j \leq e$. By definition, $I \ltimes V$ is a partial Lie algebra of size (d, e) . We should remind the reader that the semi-direct product of a $\mathbb{Z}$ -graded Lie algebra $\mathcal{L}$ and its $\mathbb{Z}$ -graded module M is defined by setting $d := -\infty$ and $e := \infty$ formally.

For simplicity, we set $\mathcal{L} := \mathcal{L}_{\max}(I)$, $\tilde{I} := I \ltimes V$ and $\tilde{\mathcal{L}} := \mathcal{L}_{\max}(\tilde{I})$. Let $\phi_1 : \tilde{I} \rightarrow I$ be the canonical projection. Theorem 2.2 implies that there uniquely exists a homomorphism of $\mathbb{Z}$ -graded Lie algebras

$$\Phi_1 : \tilde{\mathcal{L}} \rightarrow \mathcal{L} \quad (2.21)$$

such that $\Phi_1|_{\text{Par}_d^e \tilde{\mathcal{L}}} = \phi_1$. We set

$$\mathcal{K} := \text{Ker} \Phi_1.$$

Remark that $\mathcal{K}$ is a $\mathbb{Z}$ -graded Lie subalgebra of $\tilde{\mathcal{L}}$.

On the other hand, by Theorem 2.2, the inclusion map $I \hookrightarrow \tilde{I}$ induces a homomorphism of $\mathbb{Z}$ -graded Lie algebras from $\mathcal{L}$ to $\tilde{\mathcal{L}}$. We regard $\mathcal{K}$ as $\mathcal{L}$ -module via the homomorphism $\mathcal{L} \rightarrow \tilde{\mathcal{L}}$. Moreover, $[\mathcal{K}, \mathcal{K}]$ is an $\mathcal{L}$ -submodule of $\mathcal{K}$.

We define the $\mathcal{L}$ -module $M(V)$ by

$$M(V) := \mathcal{K}/[\mathcal{K}, \mathcal{K}]. \quad (2.22)$$

In the following, we check that $M(V)$ satisfies the conditions for $M_{\max}(V)$.

2.2.3.2 Proof of Theorem 2.3.1

We first show that $\text{Par}_d^e M(V) \simeq V$. Since $\text{Par}_d^e \mathcal{K} \simeq V$ by definition, we have to show that $\text{Par}_d^e [\mathcal{K}, \mathcal{K}] \simeq \{0\}$. This fact follows from the following lemma:

Lemma 2.8.

$$[\mathcal{K}, \mathcal{K}] = [\mathcal{K}^+, \mathcal{K}^+] \oplus [\mathcal{K}^-, \mathcal{K}^-].$$

Proof. We first show the following lemma:

Lemma 2.9. *The positive part $\mathcal{K}^+$ (resp. the negative part $\mathcal{K}^-$) of $\mathcal{K}$ coincides with the ideal of $\tilde{\mathcal{L}}^+$ (resp. $\tilde{\mathcal{L}}^-$) generated by V^+ (resp. V^-).*

Proof. Let $\tilde{\mathcal{K}}^+$ (resp. $\tilde{\mathcal{K}}^-$) be the ideal of $\tilde{\mathcal{L}}^+$ (resp. $\tilde{\mathcal{L}}^-$) generated by V^+ (resp. V^-). We set $\tilde{\mathcal{K}} := \tilde{\mathcal{K}}^- \oplus V_0 \oplus \tilde{\mathcal{K}}^+$ and show that $\mathcal{K} = \tilde{\mathcal{K}}$. Since $V \subset \tilde{\mathcal{K}}$, it is enough to show that $\tilde{\mathcal{K}}$ is stable under the adjoint action of $\tilde{I}$.

By an argument similar to the proof of Lemma 2.2, one can prove that

$$[\tilde{I}, \mathcal{K}^+] \subset \mathcal{K}^+ \oplus V^{\leq}, \quad [\tilde{I}, \mathcal{K}^-] \subset \mathcal{K}^- \oplus V^{\geq}. \quad (2.23)$$

Hence, $\mathcal{K} = \tilde{\mathcal{K}}$, and thus, the lemma holds. $\square$

Second, we show the following two facts:

$$[V, \mathcal{K}^\pm] \subset [\mathcal{K}^\pm, \mathcal{K}^\pm], \quad (2.24)$$

$$[\tilde{\Gamma}, [\mathcal{K}^\pm, \mathcal{K}^\pm]] \subset [\mathcal{K}^\pm, \mathcal{K}^\pm]. \quad (2.25)$$

By the Jacobi identity, we have $[\tilde{\Gamma}, [\mathcal{K}^\pm, \mathcal{K}^\pm]] \subset [[\tilde{\Gamma}, \mathcal{K}^\pm], \mathcal{K}^\pm]$. Hence, the second fact follows from the first one and (2.23) and we show the first fact. Here, we prove $[V, \mathcal{K}^-] \subset [\mathcal{K}^-, \mathcal{K}^-]$. It is enough to show that $[V^\geq, \mathcal{K}^-] \subset [\mathcal{K}^-, \mathcal{K}^-]$.

By the above lemma, $\mathcal{K}^-$ is spanned by elements of the form

$$y := [y_m, [y_{m-1}, [\cdots, [y_1, v] \cdots]] \quad (y_i \in \tilde{\Gamma}^-, v \in V^-). \quad (2.26)$$

Noticing this fact, by induction on m , one can show that

$$[u, y] \in [\mathcal{K}^-, \mathcal{K}^-] \quad (\forall u \in V^\geq),$$

and thus, (2.24) holds.

We show the lemma. Since $\mathcal{K} = \mathcal{K}^- \oplus V_0 \oplus \mathcal{K}^+$, it suffices to show

$$[\mathcal{K}^-, \mathcal{K}^+] \subset [\mathcal{K}^-, \mathcal{K}^-] \oplus [\mathcal{K}^+, \mathcal{K}^+]. \quad (2.27)$$

Suppose that $y \in \mathcal{K}^-$ is of the form (2.26) and $x \in \mathcal{K}^+$. Using the Jacobi identity, (2.23), (2.24) and (2.25), one can check

$$[x, y] \in [\mathcal{K}^-, \mathcal{K}^-] \oplus [\mathcal{K}^+, \mathcal{K}^+]$$

by induction on m . Hence, (2.27) holds. We have completed the proof. $\square$

This lemma implies that

$$\text{Par}_1^e[\mathcal{K}^+, \mathcal{K}^+] = \{0\}, \quad \text{Par}_d^{-1}[\mathcal{K}^-, \mathcal{K}^-] = \{0\},$$

since $[V_i, V_j] = \{0\}$ if $d \leq i, j, i + j \leq e$. Hence, we have $\text{Par}_d^e[\mathcal{K}, \mathcal{K}] \simeq \{0\}$.

Next, we state the universal property of $M(V)$. Let M be a $\mathbb{Z}$ -graded $\mathcal{L}$ -module. Suppose that there exists a homomorphism of partial Γ -modules $\phi : V \rightarrow \text{Par}_d^e M$.

Proposition 2.5 *There exists a unique homomorphism of $\mathcal{L}$ -modules $M(V) \rightarrow M$ whose restriction to the partial part V coincides with ϕ .*

Proof. Since $\text{Par}_d^e(\mathcal{L} \ltimes M) = \Gamma \ltimes \text{Par}_d^e M$, the following homomorphism of partial Lie algebras exists:

$$\phi_2 : \tilde{\Gamma} \longrightarrow \text{Par}_d^e(\mathcal{L} \ltimes M); \quad (x, v) \longmapsto (x, \phi(v)) \quad (x \in \Gamma, v \in V).$$

Theorem 2.2 implies that there exists a unique homomorphism of $\mathbb{Z}$ -graded Lie algebras $\Phi_2 : \tilde{\mathcal{L}} \longrightarrow \mathcal{L} \ltimes M$ such that $\Phi_2|_{\tilde{\Gamma}} = \phi_2$.

Let $\Phi_1 : \tilde{\mathcal{L}} \rightarrow \mathcal{L}$ be the homomorphism (2.21), and let $\Phi_3 : \mathcal{L} \ltimes M \rightarrow \mathcal{L}$ be the canonical projection. By definition, we have $\Phi_1|_{\tilde{\Gamma}} = \Phi_3 \circ \Phi_2|_{\tilde{\Gamma}}$, and thus, the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{L}} & \xrightarrow{\Phi_1} & \mathcal{L} \\ & \searrow \Phi_2 \quad \nearrow \Phi_3 & \\ & \mathcal{L} \ltimes M & \end{array}$$

Hence, $\Phi_2(\mathcal{K}) \subset M$ holds, since $\mathcal{K} = \text{Ker}\Phi_1$ and $\text{Ker}\Phi_3 = M$.

Here, we show that $\Phi_2|_{\mathcal{K}} : \mathcal{K} \rightarrow M$ is a homomorphism of $\mathcal{L}$ -modules. Let $\Psi : \mathcal{L} \rightarrow \tilde{\mathcal{L}}$ be the homomorphism induced from the inclusion $\Gamma \hookrightarrow \tilde{\Gamma}$. Since $\Phi_1 \circ \Psi = \text{id}_{\mathcal{L}}$, we see that $\Phi_2 \circ \Psi(x) = (x, 0) \in \mathcal{L} \ltimes M$ for $x \in \mathcal{L}$. Since the $\mathcal{L}$ -module structure of $\mathcal{K}$ is given by Ψ , we have

$$\Phi_2(x.y) = \Phi_2([\Psi(x), y]) = [\Phi_2 \circ \Psi(x), \Phi_2(y)] = x.\Phi_2(y),$$

for $x \in \mathcal{L}$ and $y \in \mathcal{K}$, and thus, $\Phi_2|_{\mathcal{K}}$ is an $\mathcal{L}$ -module homomorphism.

Moreover, we have $\Phi_2([\mathcal{K}, \mathcal{K}]) = \{0\}$, since M is a commutative subalgebra of $\mathcal{L} \ltimes M$. Hence, $\Phi_2|_{\mathcal{K}}$ induces an $\mathcal{L}$ -module homomorphism $\Phi : M(V) \rightarrow M$. By definition, $\Phi|_V = \phi$.

Finally, we show the uniqueness of the homomorphism Φ , namely, if $\phi = 0$, then $\Phi = 0$. In fact, if $\phi = 0$, then $\Phi_2(\tilde{\mathcal{L}}) \subset \mathcal{L}$. Noticing that $\Phi_3|_{\mathcal{L}} = \text{id}_{\mathcal{L}}$, we have $\text{Ker}\Phi_1 = \text{Ker}\Phi_2$. Hence, $\Phi_2|_{\mathcal{K}} = 0$, and thus, $\Phi = 0$. $\square$

2.2.3.3 Proof of Theorem 2.3.2

We construct $M_{\min}(V)$ as a quotient of $M(V)$. Let $J(V)$ be the $\mathbb{Z}$ -graded maximal proper submodule of $M(V)$ such that $J(V) \cap V = \{0\}$. Then, one can show that the quotient module $M(V)/J(V)$ satisfies the conditions for $M_{\min}(V)$ in Theorem 2.3.2 in a way similar to § 2.2.2.5.

2.3 $\mathbb{Z}$ -graded Lie Algebras

In this section, we collect some properties of $\mathbb{Z}$ -graded Lie algebras and $\mathbb{Z}$ -graded modules, which are necessary for the proof of Theorem 2.1. Through this section, let $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ be a $\mathbb{Z}$ -graded Lie algebra over $\mathbb{K}$, and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a $\mathbb{Z}$ -graded $\mathfrak{g}$ -module.

2.3.1 $\mathbb{Z}$ -graded Modules

In this subsection, we give a necessary condition for which there exist highest or lowest degree of a $\mathbb{Z}$ -graded module. As an application, we show that the dimensions of homogeneous component of a simple graded $\text{Vir}_{\mathbb{K}}$ -module without highest or lowest degree are uniformly bounded.

Lemma 2.10. *Suppose that there exist $d, e \in \mathbb{Z}$ ($d \leq 0 \leq e$) such that $\mathfrak{g}$ is generated by its partial part $\Gamma := \text{Par}_d^e \mathfrak{g}$. If there exist $a, b \in \mathbb{Z}$ ($a \leq b$) such that M is generated by $\text{Par}_a^b M$ as $\mathfrak{g}$ -modules, then*

1. *for any $s \geq b$, $\mathfrak{g}^+$ -module $M^{\geq s}$ is generated by $\text{Par}_s^{s+e} M$,*
2. *for any $t \leq a$, $\mathfrak{g}^-$ -module $M^{\leq t}$ is generated by $\text{Par}_{t+d}^t M$.*

Proof. We first notice that, by Theorem 2.2, $\mathfrak{g}^{\pm}$ are generated by $\Gamma^{\pm}$ respectively. Since M is generated by $\text{Par}_a^b M$, M is spanned by the elements of the form

$$x_k x_{k-1} \cdots x_1 y z m$$

where $x_i \in \Gamma^+$, $y \in U(\mathfrak{g}_0)$, $z \in U(\mathfrak{g}^-)$ and $m \in \text{Par}_a^b M$ are homogeneous elements. Suppose that

$$x_k x_{k-1} \cdots x_1 y z m \in M^{\geq s}.$$

In the case $s > b$, we have $k > 0$. Since $1 \leq \text{degree of } x_i \leq e$ is satisfied for each i , there exists k' ($1 \leq k' \leq k$) such that

$$x_{k'} \cdots x_1 y z m \in \text{Par}_s^{s+e} M.$$

Hence, $M^{\geq s}$ is generated by $\text{Par}_s^{s+e} M$. On the other hand, in the case $s = b$, if $k = 0$, then the assertion holds by definition. If $k > 0$, then it follows as above. The other statement for $M^{\leq t}$ can be proved similarly. $\square$

By using Lemma 2.10, we have

Proposition 2.6 *Suppose that a $\mathbb{Z}$ -graded Lie algebra $\mathfrak{g}$ and a $\mathbb{Z}$ -graded $\mathfrak{g}$ -module M satisfy the following conditions:*

- $\mathfrak{g}$ is finite (i.e., $\dim \mathfrak{g}_n < \infty$ for any $n \in \mathbb{Z}$), finitely generated, and $[\mathfrak{g}^-, \mathfrak{g}^{\geq n}] = \mathfrak{g}$ for any $n \in \mathbb{Z}_{>0}$, and
- M is finite (i.e., $\dim M_n < \infty$ for any $n \in \mathbb{Z}$), simple graded, and there exist $s \in \mathbb{Z}$ and $v \in M \setminus \{0\}$ such that $\mathfrak{g}^{\geq s}.v = \{0\}$.

Then, for some $k \in \mathbb{Z}$, $M^{\geq k} = \{0\}$.

Proof. We may assume that v is a homogeneous element without loss of generality.

We define a subspace N of M by

$$N := \{w \in M \mid \mathfrak{g}^{\geq l}.w = \{0\} \text{ for some } l\}.$$

We first show that N is a graded submodule of M . For $w \in N$, let k be a positive integer such that $\mathfrak{g}^{\geq k}.w = \{0\}$. Since for any $g \in \mathfrak{g}_l$, we have $[\mathfrak{g}^{\geq k-l}, g] \subset \mathfrak{g}^{\geq k}$, the following holds:

$$\mathfrak{g}^{\geq k-l} \cdot (g.w) \subset [\mathfrak{g}^{\geq k-l}, g].w + g.\mathfrak{g}^{\geq k-l}.w = \{0\}.$$

Hence, $g.w \in N$ for any $g \in \mathfrak{g}$. Moreover, N is graded by definition. Since $v \in N \neq \{0\}$ and M is simple graded, we have

$$N = M. \quad (2.28)$$

Next, we show that M^+ is a finitely generated $\mathfrak{g}^+$ -module. We may assume that $M_a \neq \{0\}$ for some $a \in \mathbb{Z}_{>0}$. Since M is simple graded, M is generated by M_a . On the other hand, since $\mathfrak{g}$ is finitely generated, there exist integers d, e ($d \leq 0 \leq e$) such that $\mathfrak{g}$ is generated by $T := \text{Par}_d^e \mathfrak{g}$. By Lemma 2.10, $\mathfrak{g}^+$ -module $M^{\geq a}$ is generated by $\text{Par}_a^{a+e} M$. Hence, M^+ is generated by $\text{Par}_1^{a+e} M$. Since M is finite, $\text{Par}_1^{a+e} M$ is finite dimensional. Hence, the $\mathfrak{g}^+$ -module M^+ is finitely generated.

Let X be a set of generators of the $\mathfrak{g}^+$ -module M^+ . We may assume that the cardinality of X is finite. Since $M^+ = N^+$ by (2.28), there exists $s \in \mathbb{Z}_{>0}$ such that

$$\mathfrak{g}^{\geq s}.X = \{0\}.$$

Hence, one can easily show that

$$\mathfrak{g}^{\geq s}.M^+ = \{0\},$$

since $\mathfrak{g}^{\geq s}$ is an ideal of $\mathfrak{g}^+$.

By the assumption that $\mathfrak{g}$ is finite and $[\mathfrak{g}^-, \mathfrak{g}^{\geq n}] = \mathfrak{g}$, we have

$$\mathfrak{g}^+ \subset \mathfrak{g}^{\geq s} + [\text{Par}_{-t}^{-1} \mathfrak{g}, \mathfrak{g}^{\geq s}]$$

for some $t \in \mathbb{Z}_{>0}$. Hence, we obtain

$$\begin{aligned} \mathfrak{g}^+.M^{\geq t} &\subset \{\mathfrak{g}^{\geq s} + [\text{Par}_{-t}^{-1} \mathfrak{g}, \mathfrak{g}^{\geq s}]\}.M^{\geq t} \\ &= \mathfrak{g}^{\geq s}(\text{Par}_{-t}^{-1} \mathfrak{g}).M^{\geq t} \\ &\subset \mathfrak{g}^{\geq s}.M^+ \\ &= \{0\}. \end{aligned}$$

Taking $k \in \mathbb{Z}_{>0}$ such that $k \geq t$ and $M_k \neq \{0\}$, we see that $U(\mathfrak{g}).M_k$ is a non-zero graded submodule of M such that $U(\mathfrak{g}).M_k \subset M^{\leq k}$. Since M is simple graded, we have $M^{>k} = \{0\}$. $\square$

As a corollary of Proposition 2.6, we have the following proposition on simple graded modules over the Virasoro algebra.

Proposition 2.7 *Suppose that the characteristic of $\mathbb{K}$ is zero. Let M be a simple graded $\text{Vir}_{\mathbb{K}}$ -module without highest or lowest degree. Then, the dimensions of the homogeneous components of M are uniformly bounded.*

To show this proposition, a preliminary lemma is necessary.

Lemma 2.11. *For each positive integer $n \in \mathbb{Z}_{>0}$, let $\mathfrak{s}_n$ be a subalgebra of $\text{Vir}_{\mathbb{K}}^+$ generated by $\{L_n, L_{n+1}\}$. Then, the codimension of $\mathfrak{s}_n$ in $\text{Vir}_{\mathbb{K}}^+$ is finite.*

Proof. For each positive integer m , we have

$$L_m \in \mathfrak{s}_n \text{ if } \exists \alpha, \beta \in \mathbb{Z}_{>0} \text{ such that } m = \alpha n + \beta(n+1).$$

Notice that if m satisfies $nk < m < (n+1)k$, then

$$m = \{(n+1)k - m\}n + (m - nk)(n+1).$$

Moreover, we have

- in the case $m = nk$, if $k > n+1$, then $m = (k - n - 1)n + n(n+1)$,
- in the case $m = (n+1)k$, if $k > n$, then $m = (n+1)n + (k - n)(n+1)$.

Hence, we obtain

$$L_m \in \mathfrak{s}_n$$

for any $m \in \mathbb{Z}_{>0}$ such that $m > n(n+1)$, and thus, the codimension of $\mathfrak{s}_n$ in $\text{Vir}_{\mathbb{K}}^+$ is finite. $\square$

PROOF OF PROPOSITION 2.7. We first show that

$$\{\dim M_{-n} | n \in \mathbb{Z}_{>0}\}$$

are uniformly bounded.

By the above lemma, there exists $k \in \mathbb{Z}_{>0}$ such that $\text{Vir}_{\mathbb{K}}^{\geq k} \subset \mathfrak{s}_n$. By Proposition 2.6, if M does not have highest or lowest degree, then $\text{Vir}_{\mathbb{K}}^{\geq k}.v \neq \{0\}$ for any $v \in M$, and thus

$$M^{\mathfrak{s}_n} \subset M^{\text{Vir}_{\mathbb{K}}^{\geq k}} = \{0\}.$$

Hence, we have

$$\text{Ker} \rho(L_n) \cap \text{Ker} \rho(L_{n+1}) = \{0\}, \quad (2.29)$$

where $\rho : \text{Vir}_{\mathbb{K}} \rightarrow \text{End} M$.

On the other hand, we have

$$\begin{aligned} \dim M_0 &\geq \dim \text{Im} \rho(L_n)|_{M_{-n}} = \dim M_{-n} - \dim \text{Ker} \rho(L_n)|_{M_{-n}}, \\ \dim M_1 &\geq \dim \text{Im} \rho(L_{n+1})|_{M_{-n}} = \dim M_{-n} - \dim \text{Ker} \rho(L_{n+1})|_{M_{-n}} \end{aligned}$$

and thus,

$$\begin{aligned} \dim M_0 + \dim M_1 &\geq 2 \dim M_{-n} - \dim \operatorname{Ker} \rho(L_n)|_{M_{-n}} \\ &\quad - \dim \operatorname{Ker} \rho(L_{n+1})|_{M_{-n}}. \end{aligned}$$

Here, (2.29) implies that

$$\begin{aligned} &\dim \operatorname{Ker} \rho(L_n)|_{M_{-n}} + \dim \operatorname{Ker} \rho(L_{n+1})|_{M_{-n}} \\ &= \dim (\operatorname{Ker} \rho(L_n)|_{M_{-n}} \oplus \operatorname{Ker} \rho(L_{n+1})|_{M_{-n}}) \\ &\leq \dim M_{-n}. \end{aligned}$$

Therefore, we see that

$$\dim M_{-n} \leq \dim M_0 + \dim M_1.$$

One can similarly check that

$$\dim M_n \leq \dim M_0 + \dim M_{-1}$$

holds for any $n \in \mathbb{Z}_{>0}$. Now, we have completed the proof. $\square$

2.3.2 Correspondence between Simple $\mathbb{Z}$ -graded Modules and Simple $\mathbb{Z}/N\mathbb{Z}$ -graded Modules

Let $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ be a $\mathbb{Z}$ -graded Lie algebra over the field $\mathbb{K}$. We first introduce some notation. For a $\mathbb{Z}$ -graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $m \in \mathbb{Z}$, we set

$$\operatorname{End}^m V := \{f \in \operatorname{End} V \mid f(V_n) \subset V_{n+m} \ (\forall n \in \mathbb{Z})\}.$$

In the case where V is a $\mathbb{Z}$ -graded $\mathfrak{g}$ -module, we further set

$$\operatorname{End}_{\mathfrak{g}}^m V := \operatorname{End}^m V \cap \operatorname{End}_{\mathfrak{g}} V.$$

For an integer N , one can naturally regard $\mathfrak{g}$ as a $\mathbb{Z}/N\mathbb{Z}$ -graded Lie algebra, i.e.,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{Z}/N\mathbb{Z}} \mathfrak{g}_{\alpha}, \quad \left(\mathfrak{g}_{\alpha} := \bigoplus_{\substack{n \in \mathbb{Z} \\ \alpha = n + N\mathbb{Z}}} \mathfrak{g}_n \right).$$

Let $\mathbf{M} = \bigoplus_{\alpha \in \mathbb{Z}/N\mathbb{Z}} \mathbf{M}_{\alpha}$ be a $\mathbb{Z}/N\mathbb{Z}$ -graded $\mathfrak{g}$ -module. For each $n \in \mathbb{Z}$, we set

$$\tilde{\mathbf{M}} = \bigoplus_{n \in \mathbb{Z}} \tilde{\mathbf{M}}_n, \quad (\tilde{\mathbf{M}}_n := \mathbf{M}_{n+N\mathbb{Z}}),$$

and regard $\tilde{\mathbf{M}}$ as a $\mathbb{Z}$ -graded $\mathfrak{g}$ -module in a natural way.

Definition 2.6 A simple $\mathbb{Z}/N\mathbb{Z}$ -graded $\mathfrak{g}$ -module $\mathbf{M}$ is called **relevant** if $\tilde{\mathbf{M}}$ is a simple $\mathbb{Z}$ -graded $\mathfrak{g}$ -module.

For $\lambda \in \mathbb{K}$, we define $\theta_\lambda \in \text{End}^N \tilde{\mathbf{M}}$ by

$$\theta_\lambda(m) := \lambda m \in \tilde{\mathbf{M}}_{n+N} \quad (\forall m \in \tilde{\mathbf{M}}_n).$$

Remark 2.1 1. $\theta_\lambda \in \text{End}_{\mathfrak{g}} \tilde{\mathbf{M}}$,

2. $\tilde{\mathbf{M}}$ is not $\mathbb{Z}$ -graded simple, since $\text{Im}(\text{id}_{\tilde{\mathbf{M}}} - \theta_\lambda)$ is a non-trivial proper submodule for $\lambda \neq 0$.

3. $\tilde{\mathbf{M}}/\text{Im}(\text{id}_{\tilde{\mathbf{M}}} - \theta_{\lambda_1}) \simeq \tilde{\mathbf{M}}/\text{Im}(\text{id}_{\tilde{\mathbf{M}}} - \theta_{\lambda_2})$ if and only if $\lambda_1 = \lambda_2$.

4. If $\lambda = 1$, then $\mathbf{M} \simeq \tilde{\mathbf{M}}/\text{Im}(\text{id}_{\tilde{\mathbf{M}}} - \theta_\lambda)$.

From now on, we assume that the base field $\mathbb{K}$ is an algebraically closed field.

Proposition 2.8 Suppose that a $\mathfrak{g}$ -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is finite simple $\mathbb{Z}$ -graded and not $\mathbb{Z}$ -graded simple. Then, there exists a positive integer N and an invertible homomorphism $\theta \in \text{End}_{\mathfrak{g}}^N M$ such that

$$\text{End}_{\mathfrak{g}} M = \mathbb{K}[\theta, \theta^{-1}].$$

θ is called a **generating endomorphism** of M .

Proof. We divide the proof into two steps.

Step I: We show that $\text{End}_{\mathfrak{g}}^m M \neq \{0\}$ for some non-zero integer m . For $v \in M$ such that

$$v = v_{k_1} + v_{k_2} + \cdots + v_{k_s} \quad (v_{k_i} \in M_{k_i} \setminus \{0\}),$$

where $k_i \in \mathbb{Z}$ ($1 \leq i \leq s$) and $k_1 < k_2 < \cdots < k_s$, we set

$$\ell(v) := k_s - k_1.$$

Note that if v is a homogeneous vector, then $\ell(v) = 0$.

Since M is not $\mathbb{Z}$ -graded simple, there exists a non-trivial proper submodule M' of M . We set

$$N_0 := \min\{\ell(v) \mid v \in M' \setminus \{0\}\}.$$

Since M is simple $\mathbb{Z}$ -graded, M' does not contain homogeneous vectors. Hence, we see $N_0 > 0$. We fix $w \in M' \setminus \{0\}$, which attains N_0 . Suppose that

$$w = w_{k_1} + w_{k_2} + \cdots + w_{k_s} \quad (w_{k_i} \in M_{k_i} \setminus \{0\}),$$

where $k_i \in \mathbb{Z}$ ($1 \leq i \leq s$) and $k_1 < k_2 < \cdots < k_s$. Notice that $N_0 = k_s - k_1$. Since M is simple $\mathbb{Z}$ -graded, we have $U(\mathfrak{g}).w_{k_1} = M$. Hence, we define $f \in \text{End} M$ by

$$f(x.w_{k_1}) := x.w_{k_2} \quad (x \in U(\mathfrak{g})).$$

Indeed, one can check that f is well-defined as follows: It is enough to see that

$$x.w_{k_1} = y.w_{k_1} \Rightarrow x.w_{k_2} = y.w_{k_2}$$

for any $x, y \in U(\mathfrak{g})$. We may assume that x and y are homogeneous. Since

$$(x - y).w = (x - y).w_{k_2} + \cdots + (x - y).w_{k_s} \in M'$$

and $k_s - k_2 < N_0$, from the assumption on N_0 , we see that $(x - y).w = 0$. Since x and y are homogeneous, we have

$$x.w_{k_2} = y.w_{k_2}.$$

Hence, f is well-defined. By definition, $f \notin \text{Kid}_M$. Thus, we see that $\text{End}_{\mathfrak{g}} M \neq \text{Kid}_M$.

Step II: Remark that, in general,

$$\bigoplus_{m \in \mathbb{Z}} \text{End}^m M \subsetneq \text{End} M.$$

Nevertheless, the following lemma holds:

Lemma 2.12.

$$\text{End}_{\mathfrak{g}} M = \bigoplus_{m \in \mathbb{Z}} \text{End}_{\mathfrak{g}}^m M.$$

Proof. The inclusion $\supset$ is clear. We show $\subset$.

We first show that for $f \in \text{End}_{\mathfrak{g}} M$,

$$f \in \bigoplus_{m \in \mathbb{Z}} \text{End}^m M.$$

Any f can be expressed as

$$f = \sum_{i \in \mathbb{Z}} f_i \quad (f_i \in \text{End}^i M), \quad (2.30)$$

where the sum in the right-hand side is not necessarily finite. Let us take a homogeneous vector $v \in M$ such that $f(v) \neq 0$. It follows from $f(v) \in M = \bigoplus_{i \in \mathbb{Z}} M_i$ that $f_i(v) = 0$ for all but a finite number of $i \in \mathbb{Z}$. Since $f \in \text{End}_{\mathfrak{g}} M$, we have

$$f(x.v) = \sum_{i \in \mathbb{Z}} x.f_i(v). \quad (2.31)$$

Since $U(\mathfrak{g}).v = M$, we conclude that the sum (2.30) is finite.

Moreover, if $x \in U(\mathfrak{g})$ is a homogeneous element, then (2.31) implies that $f_i(x.v) = x.f_i(v)$ for any $i \in \mathbb{Z}$. Hence, $f_i \in \text{End}_{\mathfrak{g}}^i M$ for any i , i.e.,

$$f \in \bigoplus_{m \in \mathbb{Z}} \text{End}_{\mathfrak{g}}^m M.$$

We notice that any $f \in \text{End}_{\mathfrak{g}}^m M \setminus \{0\}$ is invertible. Indeed, since $\text{Ker } f$ and $\text{Im } f$ are $\mathbb{Z}$ -graded submodules of M , $\text{Ker } f = \{0\}$ and $\text{Im } f = M$.

We set

$$N := \min\{m \in \mathbb{Z}_{>0} \mid \text{End}_{\mathfrak{g}}^m M \neq \{0\}\},$$

and fix $\theta \in \text{End}_{\mathfrak{g}}^N M \setminus \{0\}$. Notice that θ is invertible by the above fact. Let us show that

$$\text{End}_{\mathfrak{g}}^m M = \begin{cases} \{0\} & (m \neq kN \text{ for all } k \in \mathbb{Z}) \\ \mathbb{K}\theta^k & (m = kN \text{ for some } k \in \mathbb{Z}) \end{cases}.$$

First, we show the case $m = 0$. Suppose that $f \in \text{End}_{\mathfrak{g}}^0 M$. Since M is finite, we have $\dim M_0 < \infty$. Recall that $\mathbb{K}$ is algebraically closed. Hence, there exists an eigenvalue $\lambda \in \mathbb{K}$ of $f|_{M_0}$. Since $f - \lambda \text{id}_M$ is not invertible, the above fact implies $f = \lambda \text{id}_M$. Thus, $\text{End}_{\mathfrak{g}}^0 = \mathbb{K} \text{id}_M$. The rest of the assertions follows from the minimality of N . $\square$

We complete the proof of Proposition 2.8. $\square$

Remark 2.2 *It follows from the proof of Proposition 2.8 that for a finite simple $\mathbb{Z}$ -graded $\mathfrak{g}$ -module M , if $\text{End}_{\mathfrak{g}} M \neq \mathbb{K} \text{id}_M$, then M is not $\mathbb{Z}$ -graded simple.*

Lemma 2.13. *Let M be a finite simple $\mathbb{Z}$ -graded and not $\mathbb{Z}$ -graded simple $\mathfrak{g}$ -module, and let θ be a generating endomorphism of M . We set*

$$\mathbf{M}_{\theta} := M / \text{Im}(\text{id}_M - \theta).$$

Then, $\mathbf{M}_{\theta}$ is a finite simple $\mathbb{Z}/N\mathbb{Z}$ -graded relevant $\mathfrak{g}$ -module.

Proof. One can check that $\mathbf{M}_{\theta}$ is finite and simple $\mathbb{Z}/N\mathbb{Z}$ -graded. Hence, we show that $\mathbf{M}_{\theta}$ is relevant. Notice that $M \simeq \tilde{\mathbf{M}}_{\theta}$, since

$$M_n \ni x \mapsto x + \text{Im}(\text{id}_M - \theta) \in (\tilde{\mathbf{M}}_{\theta})_n$$

gives an isomorphism of $\mathbb{Z}$ -graded $\mathfrak{g}$ -modules. Hence, $\tilde{\mathbf{M}}_{\theta}$ is simple $\mathbb{Z}$ -graded and thus, $\mathbf{M}_{\theta}$ is relevant. $\square$

Remark 2.3 *Let M be as above. Then, we have*

1. *for any generating endomorphisms θ_i ($i = 1, 2$) of M (by Proposition 2.8, $\theta_1 \propto \theta_2$),*

$$\mathbf{M}_{\theta_1} \simeq \mathbf{M}_{\theta_2} \Leftrightarrow \theta_1 = \theta_2,$$

2. *for any generating endomorphisms θ of M ,*

$$\tilde{\mathbf{M}}_\theta \simeq M$$

as $\mathbb{Z}$ -graded $\mathfrak{g}$ -module.

Let $\mathcal{M}$ be the set of the pairs (M, θ) such that M is a finite simple $\mathbb{Z}$ -graded and not $\mathbb{Z}$ -graded simple $\mathfrak{g}$ -module, and θ is a generating endomorphism of M . We define an equivalence relation $\sim$ on $\mathcal{M}$ by

$$(M, \theta) \sim (M', \theta') \Leftrightarrow \exists f : M \rightarrow M' : \text{an isomorphism of } \mathfrak{g}\text{-modules such that}$$

$$(i) \quad f(M_n) \subset M'_n \quad (\forall n \in \mathbb{Z}),$$

$$(ii) \quad f \circ \theta = \theta' \circ f.$$

Set $\bar{\mathcal{M}} := \mathcal{M} / \sim$. We further denote the set of the isomorphism classes of finite simple $\mathbb{Z}/N\mathbb{Z}$ -graded relevant $\mathfrak{g}$ -modules by $\mathcal{N}$. Then, we have

Proposition 2.9 *There exists a bijective correspondence between the sets $\bar{\mathcal{M}}$ and $\mathcal{N}$ which sends an equivalence class represented by (M, θ) to an isomorphism class represented by $\mathbf{M}_\theta := M / \text{Im}(\text{id}_M - \theta)$.*

Proof. By Lemma 2.13, $\mathbf{M}_\theta$ is a finite simple $\mathbb{Z}/N\mathbb{Z}$ -graded relevant $\mathfrak{g}$ -module. On the other hand, one can show that the correspondence defined from the map $\mathbf{M} \mapsto (\tilde{\mathbf{M}}, \theta)$, where θ is a generating endomorphism of $\tilde{\mathbf{M}}$, gives the inverse of the correspondence by Remark 2.1. $\square$

2.3.3 R -forms

Let R be a subring of the base field $\mathbb{K}$. In this subsection, we state a lemma on R -forms of a $\mathbb{Z}$ -graded Lie algebra, a partial Lie algebra and their modules.

For a finite $\mathbb{Z}$ -graded vector space $M = \bigoplus_{n \in \mathbb{Z}} M_n$, let M_R be a $\mathbb{Z}$ -graded R -submodule of M such that

$$M_R = \bigoplus_{n \in \mathbb{Z}} (M_R)_n, \quad (M_R)_n := M_R \cap M_n.$$

M_R is called an **R -form** of M if

1. $M = \mathbb{K} \otimes_R M_R$,
2. for each $n \in \mathbb{Z}$, $(M_R)_n$ is a finitely generated R -submodule of M_n .

For a finite $\mathbb{Z}$ -graded Lie algebra $\mathfrak{g}$, an R -form $\mathfrak{g}_R$ of $\mathfrak{g}$ as a $\mathbb{Z}$ -graded vector space is called an R -form of $\mathfrak{g}$ if it is an R -Lie subalgebra of $\mathfrak{g}$. Similarly, for a finite $\mathbb{Z}$ -graded $\mathfrak{g}$ -module M , an R -form M_R of M is an R -form of M as a $\mathbb{Z}$ -graded vector space and a $\mathfrak{g}_R$ -submodule of M . For a partial Lie algebra and its partial module, their R -forms are defined similarly.

Lemma 2.14. *Let $\mathfrak{g}$ be a finite $\mathbb{Z}$ -graded Lie algebra generated by its partial part $\Gamma := \text{Par}_d^e \mathfrak{g}$, and let M be a finite $\mathbb{Z}$ -graded $\mathfrak{g}$ -module. Set $V := \text{Par}_d^e M$.*

For an R -form Γ_R of Γ , and an R -form V_R of V , let $\mathfrak{g}_R$ be an R -Lie subalgebra generated by Γ_R , and let M_R be a $\mathfrak{g}_R$ -submodule of M generated by V_R . Then, $\mathfrak{g}_R$ (resp. M_R) is an R -form of $\mathfrak{g}$ (resp. M) with a partial part Γ_R (resp. V_R).

Proof. For simplicity, we set $U^\pm := U(\mathfrak{g}^\pm)$. Let $U_R^\pm$ be R -subalgebras of $U^\pm$ generated by $\Gamma_R^\pm$. Then, one can show that $(U_R^\pm)_n$ is a finitely generated R -module and $\mathbb{K} \otimes_R U_R^\pm = U^\pm$. Hence, $U_R^\pm$ is an R -form of $U^\pm$. Let us introduce filtrations $\{F_n U_R^\pm | n \in \mathbb{Z}_{>0}\}$ as follows:

$$F_1 U_R^\pm := \Gamma_R^\pm \oplus R1, \quad F_n U_R^\pm := \Gamma_R^\pm F_{n-1} U_R^\pm + F_{n-1} U_R^\pm \quad (n \geq 2).$$

We set

$$\mathfrak{g}_R := U_R^- \cdot \Gamma_R^- \oplus (\Gamma_R)_0 \oplus U_R^+ \cdot \Gamma_R^+ \subset \mathfrak{g},$$

where $U_R^\pm$ acts on $\mathfrak{g}$ via the adjoint action. By induction on n in $F_n U_R^\pm$, one can show that $\mathfrak{g}_R$ is an R -Lie subalgebra of $\mathfrak{g}$. By construction, for each $n \in \mathbb{Z}$, $(\mathfrak{g}_R)_n$ is a finitely generated R -module and $\mathbb{K} \otimes_R \mathfrak{g}_R = \mathfrak{g}$. Moreover, by definition, $\text{Par}_d^e \mathfrak{g}_R = \Gamma_R$. Hence, $\mathfrak{g}_R$ is an R -form which satisfies the conditions in this lemma.

Similarly, if we set

$$M_R := U_R^- \cdot V_R^- \oplus (V_R)_0 \oplus U_R^+ \cdot V_R^+ \subset M,$$

then M_R is the desired R -form of M . □

2.4 Lie p -algebra $W(m)$

For the classification of the Harish-Chandra modules over the Virasoro algebra, we use the representations over a Lie p -algebra $W(m)$. In this section, we estimate the dimension of irreducible representations over $W(m)$.

Through this section, let $\mathbb{K}$ be a field whose characteristic is $p > 3$, unless otherwise stated.

2.4.1 Definitions

Let m be a positive integer. We first introduce the Lie p -algebra $W(m)$. Let $\mathbb{K}[t]$ be a polynomial ring in a variable t . We set

$$\begin{aligned} W(m) &:= \text{Der}(\mathbb{K}[t]/(t^{pm})) \\ &= \bigoplus_{i=-1}^{pm-2} \mathbb{K}e_i \quad (e_i := -t^{i+1} \frac{d}{dt}). \end{aligned}$$

For simplicity, we set $e_i := 0$ for $i > pm - 2$. Since

$$(\text{ad} e_i)^p e_j = \prod_{k=-1}^{p-2} (-ik - j) e_{pi+j} = \begin{cases} -j e_{pi+j} & (i \equiv 0 \pmod{p}) \\ 0 & (i \not\equiv 0 \pmod{p}) \end{cases},$$

by Proposition B.2, $W(m)$ is a Lie p -algebra with the p th power operation given by

$$e_i^{[p]} := \begin{cases} e_{pi} & (i \equiv 0 \pmod{p}) \\ 0 & (i \not\equiv 0 \pmod{p}) \end{cases}. \quad (2.32)$$

For study of representations over $W(m)$, we introduce a p -subalgebra and ideals of $W(m)$. We set

$$B(m) := \bigoplus_{i=0}^{pm-2} \mathbb{K} e_i.$$

Then, it is a completely solvable Lie p -algebra (see Definition B.6). Indeed, by setting

$$B(m)_k := \bigoplus_{i=pm-1-k}^{pm-2} \mathbb{K} e_i$$

for $1 \leq k \leq pm - 1$, we have a chain

$$\{0\} =: B(m)_0 \subset B(m)_1 \subset B(m)_2 \subset \cdots \subset B(m)_{pm-1} = B(m) \quad (2.33)$$

of ideals of $B(m)$ such that $\dim B(m)_k = k$. Further, we have

Lemma 2.15. $B(m)_k$ is a p -ideal of $B(m)$.

Proof. It suffices to show that

$$\left(\sum_{j=pm-1-k}^{pm-2} c_j e_j \right)^{[p]} \in B(m)_k$$

for any $c_j \in \mathbb{K}$. By Lemma B.3, for any $x, y \in B(m)_k$,

$$s_i(x, y) \in B(m)_k \quad (i = 1, 2, \dots, p).$$

Hence, combining this fact with (2.32), we obtain the lemma. $\square$

Next, we set $I(m) := B(m)_p \subset W(m)$. By definition, we have

$$[e_{-1}, I(m)] \subset I(m).$$

In addition, Lemma 2.15 implies that $I(m)$ is stable under p th power operation. Hence, $I(m)$ is a p -ideal of $W(m)$. Notice that the p th power of an element of $I(m)$ for $m \geq 2$ is trivial, i.e.,

Lemma 2.16. *Suppose that $m \geq 2$. Then,*

$$x^{[p]} = 0 \quad (\forall x \in I(m)).$$

Proof. By Lemma B.3,

$$s_i(x, y) = 0 \quad (i = 1, 2, \dots, p),$$

since $[x, y] = 0$ for any $x, y \in I(m)$. Hence, from the Lie p -algebra structure (2.32), we obtain the lemma. $\square$

2.4.2 Preliminaries

Until the end of Lemma 2.18, the base field $\mathbb{K}$ is not necessarily of positive characteristic.

Let V be a finite dimensional vector space over $\mathbb{K}$, and let S be a subset of $\text{End}V$ closed under $[\cdot, \cdot]$, i.e., for any $x, y \in S$, $[x, y] := xy - yx \in S$. Here, let us denote by $\langle S \rangle$ the associative subalgebra of $\text{End}V$ generated by S . We first show the following theorem due to N. Jacobson.

Theorem 2.4 ([Jac] Chapter II) *Suppose that any elements of S are nilpotent. Then, $\mathfrak{S} := \langle S \rangle$ is nilpotent, i.e., there exists $k \in \mathbb{Z}_{>0}$ such that $\mathfrak{S}^k = \{0\}$.*

To prove this theorem, we need the following lemma:

Lemma 2.17. *Suppose that a subset T of S is closed under $[\cdot, \cdot]$. Set $\mathfrak{T} := \langle T \rangle$.*

1. *If $x \in S$ satisfies $[T, x] \subset \mathfrak{T}$, then*

$$x\mathfrak{T} \subset \mathfrak{T}x + \mathfrak{T}.$$

2. *Suppose that $\mathfrak{T}$ is nilpotent and $\mathfrak{T} \subsetneq \mathfrak{S}$. Then, there exists $x \in S$ such that $x \notin \mathfrak{T}$ and $[T, x] \subset \mathfrak{T}$.*

Proof. The first assertion follows from $[T, x] \subset \mathfrak{T}$. Hence, we show the second one.

We assume that $[T, x] \not\subset \mathfrak{T}$ for any $x \in S \setminus \mathfrak{T}$, and lead to a contradiction. Let us fix $x \in S \setminus \mathfrak{T}$. By the above assumption, there exists $t_1 \in T$ such that

$$[t_1, x] \notin \mathfrak{T}.$$

Moreover, since $[t_1, x] \in S \setminus \mathfrak{T}$, there exists $t_2 \in T$ such that

$$[t_2, [t_1, x]] \notin \mathfrak{T}.$$

Hence, for any integer i , there exists $t_i \in T$ such that

$$[t_i, [t_{i-1}, \dots, [t_2, [t_1, x]] \dots]] \notin \mathfrak{T}.$$

On the other hand, since $\mathfrak{T}$ is nilpotent, there exists $k \in \mathbb{Z}_{>0}$ such that $\mathfrak{T}^k = \{0\}$. Then, we have

$$(\text{ad}T)^{2k} = \{0\} \quad \text{in } \text{End}V.$$

This is a contradiction. $\square$

PROOF OF THEOREM 2.4. We show this theorem by induction on $\dim V$. In the case where $\dim V = 0$ or $\mathfrak{S} = \{0\}$, the theorem follows by definition. Let us assume that $\dim V > 0$ and $\mathfrak{S} \neq \{0\}$.

We set

$$\Omega := \{S' | S' \subset S, [S', S'] \subset S', \langle S' \rangle : \text{nilpotent}\}.$$

Let T be an element of Ω such that $\dim \langle T \rangle$ is maximal. To show this theorem, it is enough to see that $\mathfrak{S} = \langle T \rangle$. For simplicity, we set $\mathfrak{T} := \langle T \rangle$.

First, we see that $\mathfrak{T} \neq \{0\}$. Indeed, for a non-zero element x of S , if we set

$$X := \langle \{x\} \rangle \cap S,$$

then X is closed under $[\cdot, \cdot]$, and we have

$$\langle X \rangle = \langle \{x\} \rangle = \sum_{i \geq 1} \mathbb{K}x^i.$$

Hence, $X \in \Omega$ and $\dim \langle X \rangle > 0$.

Second, we set $W := \mathfrak{T}.V$. Notice that $W \subsetneq V$, since $\mathfrak{T}$ is nilpotent. Here, we further set

$$S' := \{x \in S | x.W \subset W\}.$$

By definition, $T \subset S'$. We show that $S' \in \Omega$. We may regard $S' \subset \text{End}W$, and consider an associative subalgebra $\mathfrak{S}'_1$ of $\text{End}W$ generated by S' . We also regard $S' \subset \text{End}(V/W)$, and consider an associative subalgebra $\mathfrak{S}'_2$ of $\text{End}(V/W)$ generated by S' . Notice that $\dim W, \dim(V/W) < \dim V$. Hence, by induction hypothesis, both $\mathfrak{S}'_1$ and $\mathfrak{S}'_2$ are nilpotent, i.e., there exist positive integers k_1 and k_2 such that

$$(\mathfrak{S}'_1)^{k_1} = \{0\} \quad \text{and} \quad (\mathfrak{S}'_2)^{k_2} = \{0\}.$$

This means that, if we set $\mathfrak{S}' := \langle S' \rangle \subset \text{End}V$, then

$$(\mathfrak{S}')^{k_2}.V \subset W \quad \text{and} \quad (\mathfrak{S}')^{k_1}.W = \{0\}.$$

Hence, $(\mathfrak{S}')^{k_1+k_2} = \{0\}$, i.e., $\mathfrak{S}'$ is nilpotent. Since S' is closed under $[\cdot, \cdot]$ by definition, we have $S' \in \Omega$.

Third, we assume $\mathfrak{T} \subsetneq \mathfrak{S}$, and lead to a contradiction. By Lemma 2.17.2, there exists $x \in S$ such that $x \notin \mathfrak{T}$ and $[T, x] \subset \mathfrak{T}$. Hence, by Lemma 2.17.1, we get

$$x.W = x\mathfrak{T}.V \subset (\mathfrak{T}x + \mathfrak{T}).V \subset W.$$

Hence, $x \in S'$. Since $x \notin \mathfrak{T}$, we have

$$\dim \mathfrak{S}' \geq \dim \mathfrak{T} + 1.$$

This contradicts the assumption that $\dim \mathfrak{T}$ is maximal. $\square$

Lemma 2.18. *Let $\mathfrak{a}$ be a Lie algebra over an algebraically closed field $\mathbb{K}$, and let V be an irreducible $\mathfrak{a}$ -module. Suppose that for any $x \in [\mathfrak{a}, \mathfrak{a}]$, $\rho(x)$ is nilpotent, where $\rho : \mathfrak{a} \rightarrow \text{End}V$. Then,*

$$\dim V = 1.$$

Proof. If we set

$$S := \{\rho(x) | x \in [\mathfrak{a}, \mathfrak{a}]\},$$

then S satisfies the conditions in Theorem 2.4. Hence, $\mathfrak{S} := \langle S \rangle$ is a nilpotent associative subalgebra of $\text{End}V$. Hence,

$$W := \{v \in V | \rho([\mathfrak{a}, \mathfrak{a}]).v = \{0\}\}$$

satisfies $W \neq \{0\}$. Moreover, W is an $\mathfrak{a}$ -submodule of V . Since V is irreducible, we have $V = W$. This means that V is an irreducible $(\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}])$ -module. Since $(\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}])$ is an abelian Lie algebra and $\mathbb{K}$ is algebraically closed, we conclude that $\dim V = 1$. $\square$

From now on, we assume that $\mathbb{K}$ is a field whose characteristic is $p > 0$ again. Let $\mathfrak{g}$ be a Lie p -algebra over $\mathbb{K}$ with a p th power operation $(\cdot)^{[p]}$. For $x \in \mathfrak{g}$, we say that x is p -nilpotent if there exist $k \in \mathbb{Z}_{>0}$ such that

$$x^{[p^k]} = 0,$$

where for $n \in \mathbb{Z}_{>0}$ we set

$$x^{[p^n]} := ((\dots (x^{[p]})^{[p]})^{[p]} \dots)^{[p]} \quad \underbrace{\hspace{1.5cm}}_{n \text{ times}}.$$

Lemma 2.19. *Let M (and $\rho : \mathfrak{g} \rightarrow \text{End}M$) be a $\mathfrak{g}$ -module with central character $\chi \in \mathfrak{g}^*$. For a p -nilpotent element $x \in \mathfrak{g}$, if*

$$\chi(x^{[p^n]}) = 0$$

for any $n \in \mathbb{Z}_{\geq 0}$, then $\rho(x)$ is nilpotent. In particular, if any $x \in \mathfrak{g}$ are p -nilpotent and M has the trivial central character, then $\rho(x)$ is nilpotent for any $x \in \mathfrak{g}$.

Proof. For $x \in \mathfrak{g}$, let k be an integer such that $x^{[p^k]} = 0$. Note that $x^{[p^{s+1}]}$ commutes with $x^{[p^s]}$, since

$$[x^{[p^{s+1}]}, x^{[p^s]}] = \text{ad}(x^{[p^s]})^p(x^{[p^s]}) = 0.$$

Hence, we have

$$\begin{aligned} & \sum_{s=0}^{k-1} \{(x^{[p^s]})^p - x^{[p^{s+1}]} \} p^{k-1-s} \\ &= \sum_{s=0}^{k-1} \{(x^{[p^s]})^{p^{k-s}} - (x^{[p^{s+1}]})^{p^{k-1-s}} \} \\ &= x^{p^k} - x^{[p^k]} \\ &= x^{p^k}. \end{aligned}$$

On the other hand, since $\chi(x^{[p^s]}) = 0$ for any $s \in \mathbb{Z}_{\geq 0}$,

$$\rho(x^{[p^s]})^p - \rho(x^{[p^{s+1}]}) = \chi(x^{[p^s]})^p = 0.$$

Hence, $\rho(x)^{p^k} = 0$, i.e., $\rho(x)$ is nilpotent. $\square$

Lemma 2.20. *Let $\mathfrak{g}$ be a finite dimensional Lie p -algebra, $\mathfrak{h}$ be a p -subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$ be a p -ideal of $\mathfrak{h}$ such that any $x \in \mathfrak{k}$ are p -nilpotent. Let M be an irreducible $\mathfrak{g}$ -module with the central character $\chi \in \mathfrak{g}^*$ such that*

$$\chi(\mathfrak{k}) = \{0\}.$$

Moreover, assume that there exist $x, y \in \mathfrak{g}$ which satisfy the following conditions:

1. $\mathfrak{g} = \mathbb{K}x \oplus \mathfrak{h}$,
2. $y \in \mathfrak{k}$ and $[x, y], [x, [x, y]] \in \mathfrak{h}$,
3. $\rho([x, y])$ is invertible where $\rho : \mathfrak{g} \rightarrow \text{End} M$.

Then, for any irreducible $\mathfrak{h}$ -submodule M' of M , the following holds:

$$\dim M = p \dim M'.$$

Proof. Since M is an irreducible $\mathfrak{g}$ -module, by Proposition B.4, there exists a surjective map

$$\text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M'; \chi) \twoheadrightarrow M.$$

Further, since $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$, we have $\dim \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M'; \chi) = p \dim M'$. Hence, the inequality

$$\dim M \leq p \dim M' \quad (2.34)$$

holds. Here, we assume that $\dim M < p \dim M'$ and leads to a contradiction.

We first show that

$$\mathfrak{k}.M' = \{0\}. \quad (2.35)$$

By Lemma 2.19, for any $z \in \mathfrak{k}$, $\rho(z)$ is nilpotent. Hence, by Theorem 2.4,

$$\{\rho(z)|_{M'} | z \in \mathfrak{k}\}$$

generates a nilpotent subalgebra of $\text{End} M'$. This means that $\mathfrak{k}.M'$ is a proper $\mathfrak{h}$ -submodule of M' . Since M' is an irreducible $\mathfrak{h}$ -module, we obtain (2.35).

By the assumption $\dim M < p \dim M'$, there exist an integer n ($0 < n < p$) and $u_0, \dots, u_n \in M' \subset M$ such that

$$\sum_{s=0}^n x^s . u_s = 0. \quad (2.36)$$

We fix the minimal integer n such that (2.36) holds. Notice that

$$yx^s = x^s y + \sum_{i=1}^s \binom{s}{i} x^{s-i} [\underbrace{\dots [y, x], x] \dots x}_{i \text{ times}}].$$

Since $[x, [x, y]] \in \mathfrak{h}$ and $\sum_{i=0}^s x^i M'$ is $\mathfrak{h}$ -invariant for $0 \leq s < p$, we have

$$[\dots [y, \underbrace{x], x] \dots x] . M' \subset \{(\text{ad} x)^{s-2} \mathfrak{h}\} . M' \subset \sum_{i=0}^{s-2} x^i M'$$

for $s \geq 2$. Hence, there exist $u'_s \in M'$ ($0 \leq s \leq n$) such that

$$\begin{aligned} 0 &= y . \left(\sum_{s=0}^n x^s . u_s \right) = x^n y u_n + x^{n-1} (y . u_{n-1} + n[y, x] . u_n) + \sum_{s=0}^{n-2} x^s . u'_s \\ &= x^{n-1} n[y, x] . u_n + \sum_{s=0}^{n-2} x^s . u'_s, \end{aligned}$$

since $y . u_n = y . u_{n-1} = 0$ by (2.35). It follows from the choice of n that $n[y, x] . u_n = 0$. Since $[x, y]$ is invertible on M , we obtain $u_n = 0$. This contradicts the choice of n . Now, we have completed the proof. $\square$

2.4.3 Irreducible Representations of $W(m)$ ($m \geq 2$)

In this subsection, we assume that $\mathbb{K}$ is an algebraically closed field. The main result of this subsection is the following theorem:

Theorem 2.5 *Let M be an irreducible faithful representation of $W(m)$. Then,*

$$\dim M \geq p^{\frac{1}{2}(m-1)(p-1)}.$$

For the proof, we need a preliminary lemma. First, we notice that $I(m).M \neq \{0\}$, since M is a faithful representation, and that $I(m).M$ is a submodule of M , since $I(m)$ is an ideal of $W(m)$. Hence, we see that $I(m).M = M$. This implies that there exists an irreducible $B(m)$ -subquotient M' of M such that $I(m).M' \neq \{0\}$. Hence, we have

$$I(m).M' = M'. \quad (2.37)$$

To prove Theorem 2.5, here, we show

$$\dim M' \geq p^{\frac{1}{2}(m-1)(p-1)}.$$

Let $\chi \in B(m)^*$ be the central character of M' . Theorem B.4 ensures that there exist $f \in B(m)^*$ and the Vergne polarisation $\mathfrak{p}$ of $B(m)$ at f constructed from the chain (2.33) such that the induced representation

$$\mathrm{Ind}_{\mathfrak{p}}^{B(m)}(\mathbb{K}_f; \chi)$$

is isomorphic to M' . We have

Lemma 2.21. *1. There exists j ($pm - p - 1 \leq j \leq pm - 2$) such that $\chi(e_j) \neq 0$.
2. $f = \chi$ on $I(m)$.*

Proof. By the definition of central character, for $j \geq pm - p - 1$

$$\chi(e_j) = 0 \Leftrightarrow e_j^p.M' = \{0\},$$

since $e_j^{[p]} = 0$. We assume that $\chi(e_j) = 0$ for $pm - p - 1 \leq j \leq pm - 2$, and lead to a contradiction. We set

$$S := \bigcup_{j=pm-p-1}^{pm-2} \mathbb{K}e_j.$$

Then, S satisfies the conditions in Theorem 2.4. Hence, $\langle S \rangle$ is nilpotent, Since $I(m) \subset \langle S \rangle$ by definition, there exists $k \in \mathbb{Z}_{>0}$ such that

$$I(m)^k.M' = \{0\}.$$

This contradicts (2.37). The first statement follows.

To show the second statement, we notice that $I(m) = B(m)_p$ and

$$[B(m)_p, B(m)_p] = \{0\}.$$

By definition,

$$\mathfrak{p} \supset \mathfrak{c}_{B(m)_p}(f|_{B(m)_p}) = B(m)_p = I(m).$$

On the other hand, as stated in Remark B.2,

$$f(x) - f(x^{[p]})^{\frac{1}{p}} = \chi(x)$$

holds for any $x \in \mathfrak{p}$. Hence, by Lemma 2.16, the second statement follows. $\square$

PROOF OF THEOREM 2.5. We set

$$j := \max\{i | f(e_i) \neq 0\}.$$

By the above lemma, we see that

$$j \geq pm - p - 1.$$

Hence, considering the matrix expression of the form df with respect to the basis $\{e_0, e_1, \dots, e_{pm-2}\} \subset B(m)$, we see that

$$\text{rank} df \geq \text{rank} de_j^*.$$

Further, considering the matrix expression of the form de_j^* , we have

$$\begin{aligned} \text{rank} de_j^* &= \sharp\{(k, l) | de_j^*(e_k, e_l) \neq 0\} \\ &= \sharp\{(k, l) | (e_k \cdot e_j^*)(e_l) \neq 0\} \\ &= \sharp\{k | e_k \cdot e_j^* \neq 0\}, \end{aligned}$$

where $B(m)$ acts on $B(m)^*$ via the coadjoint action, i.e.,

$$e_k \cdot e_j^* = (j - 2k)e_{j-k}^*.$$

Hence, we see that

$$\begin{aligned} \text{rank} de_j^* &= (j + 1) - \sharp\{k | 0 \leq k \leq j \wedge j - 2k \equiv 0 \pmod{p}\} \\ &\geq (pm - p - 1) + 1 - (m - 1) \\ &= (p - 1)(m - 1). \end{aligned}$$

Therefore, by Theorem B.4, we obtain

$$\dim M \geq \dim M' \geq p^{\frac{1}{2}(m-1)(p-1)}.$$

$\square$

2.4.4 Irreducible Representations of $W(1)$

The purpose of this subsection is also to give an effective estimate of dimension for faithful and irreducible $W(1)$ -modules. In this subsection, we assume that $\mathbb{K}$ is algebraically closed.

To study representations over $W(1)$, it is convenient to use the following new basis of $W(1)$: Recall that $W(1) = \text{Der}(\mathbb{K}[t]/(t^p))$. We set

$$\ell_s := -(1+t)^{s+1} \frac{d}{dt} \quad (s = 0, 1, \dots, p-1).$$

Since $(1+t)^p = 1$ in $\mathbb{K}[t]/(t^p)$, we can regard $\{\ell_s\}$ as elements indexed over $\mathbb{Z}/p\mathbb{Z}$. By definition, we have

$$W(1) = \bigoplus_{s \in \mathbb{Z}/p\mathbb{Z}} \mathbb{K}\ell_s.$$

The above basis elements satisfy the following commutation relations:

$$[\ell_r, \ell_s] = (r-s)\ell_{r+s},$$

where $r, s \in \mathbb{Z}/p\mathbb{Z} \subset \mathbb{K}$.

For each $a, b \in \mathbb{K}$, we introduce a $W(1)$ -module $M_{a,b}$. Set

$$M_{a,b} := \bigoplus_{n \in \mathbb{Z}/p\mathbb{Z}} \mathbb{K}v_n, \tag{2.38}$$

and regard $M_{a,b}$ as a $W(1)$ -module via

$$\ell_s.v_n := (as + b - n)v_{n+s} \quad (s, n \in \mathbb{Z}/p\mathbb{Z}).$$

It should be noted that any submodule of $M_{a,b}$ is ℓ_0 -diagonalisable. Hence, by an argument similar to the proof of Proposition 2.1, we obtain

- Lemma 2.22.** 1. If $a \neq 0, -1$ or $b \notin \mathbb{Z}/p\mathbb{Z}$, then $M_{a,b}$ is irreducible.
 2. If $a = 0$ and $b \in \mathbb{Z}/p\mathbb{Z}$, then $M_{0,b}$ contains the trivial representation $\mathbb{K}v_b$ as a submodule, and the quotient module

$$M_{0,b}/\mathbb{K}v_b$$

is irreducible.

3. If $a = -1$ and $b \in \mathbb{Z}/p\mathbb{Z}$, then

$$\bigoplus_{n \neq b} \mathbb{K}v_n$$

is an irreducible submodule of $M_{a,b}$.

In the sequel, for $a, b \in \mathbb{K}$, we set

$$M'_{a,b} := \begin{cases} M_{a,b} & (a \neq 0, -1 \wedge b \notin \mathbb{Z}/p\mathbb{Z}) \\ M_{0,b}/\mathbb{K}v_b & (a = 0 \wedge b \in \mathbb{Z}/p\mathbb{Z}) \\ \bigoplus_{n \neq b} \mathbb{K}v_n & (a = -1 \wedge b \in \mathbb{Z}/p\mathbb{Z}) \end{cases}. \quad (2.39)$$

The main result of this subsection is

Theorem 2.6 ([Ch]) *Let M be an irreducible representation of $W(1)$. Then, one of the following holds:*

1. $\dim M \geq p^2$,
2. $\dim M < p^2$ and

$$M \simeq M'_{a,b} \quad \text{or} \quad M \simeq \mathbb{K}.$$

In particular, if M is faithful, then

$$\dim M \geq p^2 \quad \text{or} \quad \dim M = p, \quad p - 1.$$

Proof (cf. [St]). Let $\chi \in W(1)^*$ be the central character of M . We divide the proof into the following three cases:

Case I: $\chi(e_{p-2}) \neq 0$.

Case II: There exists j ($2 \leq j \leq p-2$) such that $\chi(e_k) = 0$ for all $k \geq j$ and $\chi(e_{j-1}) \neq 0$.

Case III: $\chi(e_k) = 0$ for all $k \geq 1$.

In the following, we show that $\dim M \geq p^2$ in Cases I and II and $\dim M < p^2$ in Case III.

Case I Let M' be an irreducible $B(1)$ -submodule of M . Set $\chi' := \chi|_{B(1)} \in B(1)^*$. Then, χ' is the central character of M' . By Theorem B.4, there exist $f \in B(1)^*$ and the Vergne polarisation $\mathfrak{p}$ of $B(1)$ at f constructed from the chain (2.33) such that

$$M' \simeq \text{Ind}_{\mathfrak{p}}^{B(1)}(\mathbb{K}_f; \chi').$$

We first show that

$$f(e_{p-2}) = \chi(e_{p-2}). \quad (2.40)$$

Notice that $B(1)_1 = \mathbb{K}e_{p-2}$ and $[B(1)_1, B(1)_1] = \{0\}$. By the definition of the Vergne polarisation, we see that

$$\mathfrak{p} \supset \mathfrak{c}_{B(1)_1}(f|_{B(1)_1}) = B(1)_1.$$

Hence, by Remark B.2 and $e_{p-2}^{[p]} = 0$, we obtain (2.40). Hence, by the assumption of Case I, we see that $f(e_{p-2}) \neq 0$. Considering the matrix expression of df with respect to the base $\{e_0, e_1, \dots, e_{p-2}\}$, we see that

$$\text{rank} df \geq \text{rank} de_{p-2}^*.$$

One can directly check that $\text{rank} de_{p-2}^* = p - 1$. By Theorem B.4, we obtain

$$\dim M' \geq p^{\frac{1}{2}(p-1)}.$$

Since $p > 3$, we have the conclusion $\dim M' \geq p^2$.

Case II We use Lemma 2.20. We set

$$\mathfrak{g} := W(1), \quad \mathfrak{h} := B(1), \quad \mathfrak{k} := \bigoplus_{l \geq j} \mathbb{K}e_l,$$

and $x := e_{-1}$, $y := e_j$. Indeed, since $[x, y] = -(j+1)e_{j-1}$ and $e_{j-1}^p = \chi(e_{j-1})^p \text{id}_M$ on M , we see that $[x, y]$ is invertible on M . Applying Lemma 2.20, we see that for any irreducible $B(1)$ -submodule M' of M ,

$$\dim M = p \dim M'. \quad (2.41)$$

We take $f \in B(1)^*$ and the Vergne polarisation $\mathfrak{p}$ of $B(1)$ at f constructed from the chain (2.33) such that

$$M' \simeq \text{Ind}_{\mathfrak{p}}^{B(1)}(\mathbb{K}_f; \chi')$$

(see Theorem B.4). In this case,

$$df \neq 0 \quad (2.42)$$

holds. In fact, if $df = 0$, then

$$\mathfrak{p} \supset \mathfrak{c}_{B(1)}(f) = B(1).$$

Hence, by Remark B.2, $f(e_i) = \chi(e_i)$ for $i \geq 1$. Moreover, $df = 0$ implies $f(e_i) = 0$ for $i \geq 1$. This contradicts the assumption of Case II. Hence, (2.42) holds. Hence, by Theorem B.4, we have

$$\dim M' \geq p.$$

Combining this estimation with (2.41), we obtain the conclusion $\dim M \geq p^2$.

Case III We set

$$B'(1) := [B(1), B(1)] = \bigoplus_{i=1}^{p-2} \mathbb{K}e_i.$$

From the assumption of Case III, we see that χ vanishes on $B'(1)$. By definition, one can show that $x^{[p]} = 0$ for any $x \in B'(1)$. Hence, by Lemma 2.19, any $x \in B'(1)$ are nilpotent on M , and thus, by Lemma 2.18, any irreducible $B(1)$ -submodules are one-dimensional. Hence, there exists a non-zero element $u \in M$ such that $\mathbb{K}u$ is an irreducible $B(1)$ -submodule of M . Since e_i ($i \geq 1$)

are nilpotent on M , there exists $\lambda \in \mathbb{K}$ such that

$$e_i \cdot u = \begin{cases} \lambda u & (i = 0) \\ 0 & (i \geq 1) \end{cases}.$$

If we set

$$N := \text{Ind}_{B(1)}^{W(1)}(\mathbb{K}u; \chi),$$

then there exists a surjective homomorphism of $W(1)$ -modules $N \twoheadrightarrow M$, since M is irreducible.

For this $W(1)$ -module N , we have

Lemma 2.23.

$$N \simeq M_{\lambda-1, \lambda-1}, \quad (2.43)$$

where $M_{a,b}$ is defined in (2.38).

Proof. Note that

$$N = \bigoplus_{i=0}^{p-1} \mathbb{K}(e_{-1})^i \otimes u.$$

By using

$$e_j(e_{-1})^i = (e_{-1})^i e_j + \sum_{k=1}^i \binom{i}{k} (e_{-1})^{i-k} [\cdots [e_j, \underbrace{e_{-1}, e_{-1}, \dots, e_{-1}}_{k \text{ times}}], \cdots],$$

for $j \geq 1$ we have

$$\begin{aligned} e_j \cdot (e_{-1})^i \otimes u &= \binom{i}{j} (e_{-1})^{i-j} (j+1)! e_0 \otimes u \\ &\quad + \binom{i}{j+1} (e_{-1})^{i-j-1} (j+1)! e_{-1} \otimes u \\ &= \frac{i!}{(i-j)!} \{(j+1)\lambda + (i-j)\} (e_{-1})^{i-j} \otimes u. \end{aligned}$$

Notice that this formula still holds for $j = 0, -1$. By setting

$$v_i := \frac{1}{(p-1-i)!} (e_{-1})^{p-1-i} \otimes u \quad (0 \leq i \leq p-1),$$

$\{v_i | 0 \leq i \leq p-1\}$ forms a basis of N . Moreover, by direct computation, one can check that

$$e_j \cdot v_i = \{(\lambda-1)j + (\lambda-1) - i\} v_{i+j}$$

for any $0 \leq i, j \leq p-1$. Moreover, for $0 \leq i \leq p-1$, we set

$$u_i := \sum_{k=0}^i \binom{i}{k} v_k.$$

Identifying the index set of $\{u_i\}$ with $\mathbb{Z}/p\mathbb{Z}$, we have

$$\ell_j.u_i = \{(\lambda - 1)j + (\lambda - 1) - i\}u_{i+j}$$

by direct computation. Hence, the isomorphism (2.43) has been proved. $\square$

Therefore, Case III of Theorem 2.6 follows from Lemma 2.22. Now, we have completed the proof. $\square$

2.4.5 $\mathbb{Z}/N\mathbb{Z}$ -graded Modules over $\text{Vir}_{\mathbb{K}}$

In this subsection, let $\mathbb{K}$ be an algebraically closed field of *positive* characteristic $p \neq 2, 3$.

Let $\mathcal{D}_{\mathbb{K}} := \mathbb{K}[t, t^{-1}] \frac{d}{dt}$ be the Lie algebra with commutation relation

$$[f_1(t) \frac{d}{dt}, f_2(t) \frac{d}{dt}] = (f_1(t)f_2'(t) - f_1'(t)f_2(t)) \frac{d}{dt}.$$

For $g(t) \in \mathbb{K}[t]$ such that $g(0) \neq 0$, we set

$$I(g(t)) := g(t)^p \mathbb{K}[t, t^{-1}] \frac{d}{dt} \subset \mathcal{D}_{\mathbb{K}}. \quad (2.44)$$

Since $\frac{d}{dt}g(t)^p = 0$, $I(g(t))$ is an ideal of $\mathcal{D}_{\mathbb{K}}$. Moreover, since $\mathbb{K}$ is algebraically closed, there exist $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{K} \setminus \{0\}$ and $m_1, m_2, \dots, m_s \in \mathbb{Z}_{>0}$ such that

$$g(t) = (t - \alpha_1)^{m_1} (t - \alpha_2)^{m_2} \cdots (t - \alpha_s)^{m_s}.$$

Now, $(t - \alpha)^{mp} = t^{mp} - \alpha^{mp}$ implies

$$\begin{aligned} \mathbb{K}[t, t^{-1}] \frac{d}{dt} \Big/ (t - \alpha)^{mp} \mathbb{K}[t, t^{-1}] \frac{d}{dt} &\simeq (\mathbb{K}[t]/(t^{mp} - \alpha^{mp})) \frac{d}{dt} \\ &\simeq \text{Der}(\mathbb{K}[t - \alpha]/((t - \alpha)^{mp})) \\ &\simeq W(m). \end{aligned}$$

Hence, we have

$$\mathcal{D}_{\mathbb{K}}/I(g(t)) \simeq W(m_1) \oplus W(m_2) \oplus \cdots \oplus W(m_s).$$

Here, let us denote the canonical projection $\text{Vir}_{\mathbb{K}} \twoheadrightarrow \mathcal{D}_{\mathbb{K}}$ by π . For $g(t) \in \mathbb{K}[t]$ such that $g(0) \neq 0$, we set

$$\mathcal{I}(g(t)) := \pi^{-1}(I(g(t))).$$

Lemma 2.24. 1. For any ideal I of $\mathcal{D}_{\mathbb{K}}$, there exists a polynomial $g(t) \in \mathbb{K}[t]$ such that $I = I(g(t))$.

2. For any ideal $\mathcal{I}$ of $\text{Vir}_{\mathbb{K}}$, $C \in \mathcal{I}$ holds.

Proof. For an ideal I of $\mathcal{D}_{\mathbb{K}}$, we set

$$J_I := \left\{ g(t) \in \mathbb{K}[t, t^{-1}] \left| g(t) \frac{d}{dt} \in I \right. \right\}.$$

We first show that J_I is an ideal of $\mathbb{K}[t, t^{-1}]$. For $g(t) \in \mathbb{K}[t, t^{-1}]$, we express it in the form

$$g(t) = \sum_{i=0}^{p-1} g_i(t^p) t^i \quad (g_i(t) \in \mathbb{K}[t, t^{-1}]).$$

For $g(t) \in J_I$ and $a(t) \in \mathbb{K}[t, t^{-1}]$, $a(t)g'(t) - a'(t)g(t) \in J_I$ holds by definition. By taking $a(t) = t$, we have $tg'(t) \in J_I$. Hence,

$$\sum_{i=0}^{p-1} i^s g_i(t^p) t^i \in J_I \quad (s \geq 0),$$

and thus, $g_i(t^p) t^i \in J_I$ for any i . For $a(t) = t^m$ ($m \in \mathbb{Z}$) and $g(t) = \tilde{g}(t^p) t^n \in \mathbb{K}[t, t^{-1}]$, we have $a(t)g'(t) - a'(t)g(t) = (n - m)\tilde{g}(t^p)t^{m+n-1}$. Specialising it appropriately, at most twice, we see that $g_i(t^p)t^j \in J_I$ for any integer j . Hence, $g(t)t^j \in J_I$, and thus, J_I is an ideal of $\mathbb{K}[t, t^{-1}]$. Moreover, since J_I is generated by elements of the form $\tilde{g}(t^p)$ for $\tilde{g}(t) \in \mathbb{K}[t, t^{-1}]$, we have

$$J_I = (\tilde{g}(t^p))$$

for some $\tilde{g}(t^p) \in \mathbb{K}[t, t^{-1}]$. Since there exists $g(t) \in \mathbb{K}[t, t^{-1}]$ such that $\tilde{g}(t^p) = g(t)^p$, we conclude that $I = I(g(t))$.

Next, we show the second statement. Suppose that there exists a non-trivial ideal $\mathcal{I}$ of $\text{Vir}_{\mathbb{K}}$ such that $C \notin \mathcal{I}$. Let ω be the non-trivial 2-cocycle of $\mathcal{D}_{\mathbb{K}}$ defined by

$$\omega(L_m, L_n) := \delta_{m+n,0} \frac{m^3 - m}{12},$$

where we set $L_m := -t^{m+1} \frac{d}{dt} \in \mathcal{D}_{\mathbb{K}}$ by abuse of notation. Because of the well-definedness of the commutation relations of $\text{Vir}_{\mathbb{K}}/\mathcal{I}$, we see that

$$\omega(I, \mathcal{D}_{\mathbb{K}}) = \{0\},$$

where $I := \pi(\mathcal{I})$. Hence, I is a subset of the radical of ω which is given by $\bigoplus_{n \in p\mathbb{Z}} \{\mathbb{K}L_{n-1} \oplus \mathbb{K}L_n \oplus \mathbb{K}L_{n+1}\}$. By the first statement, we deduce that $I = \{0\}$. This is a contradiction. $\square$

Notice that for any integer N , $\text{Vir}_{\mathbb{K}}$ is naturally $\mathbb{Z}/N\mathbb{Z}$ -graded.

Lemma 2.25. *Let N be a positive integer such that there exist integers u and v satisfying $N = vp^u$, $u \geq 1$ and $(p, v) = 1$. Let $\mathcal{I}$ be a non-trivial and non-central $\mathbb{Z}/N\mathbb{Z}$ -graded ideal of $\text{Vir}_{\mathbb{K}}$. Set $\mathcal{G} := \text{Vir}_{\mathbb{K}}/\mathcal{I}$. Let M be a simple and faithful $\mathcal{G}$ -module. Then, the following three inequalities hold:*

A

$$\dim M \geq p^{\frac{1}{2}v(p-1)(p^{u-1}-1)},$$

B if $u = 1$, then

$$\dim M \geq (p-1)^v,$$

C if $N = p$ and $\dim M \neq p, p-1$, then

$$\dim M \geq p^2.$$

Proof. By Lemma 2.24, there exists a non-constant polynomial $g(t)$ with $g(0) \neq 0$ such that $\mathcal{I} = \mathcal{I}(g(t))$. Since $\mathcal{I}$ is $\mathbb{Z}/N\mathbb{Z}$ -graded, we may assume that $g(t)^p \in \mathbb{K}[t^N]$, and thus, $g(t) \in \mathbb{K}[t^{vp^{u-1}}]$. Hence, there exists $h(t) \in \mathbb{K}[t^v]$ such that $g(t) = h(t)^{p^{u-1}}$. The set of the roots of $h(t) = 0$ is stable under the multiplication of v th roots of unity. Hence, $g(t) = 0$ has at least v roots with multiplicity equal to or greater than p^{u-1} . This implies that

$$\mathcal{G} \simeq \bigoplus_i W(m_i),$$

for $\{m_i\}$ such that $\#\{i | m_i \geq p^{u-1}\} \geq v$. Since M is a faithful $\mathcal{G}$ -module, the inequality in **A** follows from Theorem 2.5, and the inequalities in **B** and **C** follow from Theorem 2.6. $\square$

Proposition 2.10 *Let k be a finite field, and let $\bar{k}$ be its algebraic closure. Let $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ be a finitely generated finite $\mathbb{Z}$ -graded Lie algebra over $\bar{k}$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a simple $\mathbb{Z}$ -graded $\mathfrak{g}$ module which satisfies $\dim M = \infty$ and $\dim M_n$ are uniformly bounded, i.e., there exists $d > 0$ such that $\dim M_n < d$ for any $n \in \mathbb{Z}$. Then, M is not graded simple.*

Proof. Since $\mathfrak{g}$ is finitely generated, there exists $s \in \mathbb{Z}_{>0}$ such that $\mathfrak{g}$ is generated by the partial part $\Gamma := \text{Par}_{-s}^s \mathfrak{g}$. We set $\mathcal{L} := \mathcal{L}_{\max}(\Gamma)$. By Theorem 2.2, there exists a surjective homomorphism

$$\mathcal{L} \twoheadrightarrow \mathfrak{g}.$$

In the sequel, we regard M as $\mathcal{L}$ module via the surjection. Since M is a simple $\mathbb{Z}$ -graded $\mathfrak{g}$ -module, it is a simple $\mathbb{Z}$ -graded $\mathcal{L}$ -module.

We may assume that $M_0 \neq \{0\}$ without loss of generality. We set $\sigma := \text{Par}_{-s}^s M$ and regard it as Γ -module. Then, M is generated by σ as an $\mathcal{L}$ -module, since M is simple graded. By Theorem 2.3, there is a surjection

$$M \rightarrow M_{\min}(\sigma).$$

We fix a basis $\{x_1, \dots, x_a\}$ ($a := \dim \Gamma$) of Γ and $\{v_1, \dots, v_b\}$ ($b := \dim \sigma$) of σ . Let $\{c_{i,j}^l | 1 \leq i, j, l \leq a\}$ and $\{d_{i,j}^l | 1 \leq i \leq a, 1 \leq j, l \leq b\}$ be the structure constants

$$[x_i, x_j] = \sum_{l=1}^a c_{i,j}^l x_l, \quad x_i \cdot v_j = \sum_{l=1}^b d_{i,j}^l v_l.$$

Let K be the extension field $k(c_{i,j}^l, d_{i,j}^l)$. Notice that K is a finite field. We set

$$\Gamma_K := \bigoplus Kx_i, \quad \sigma_K := \bigoplus Kv_i,$$

$\mathcal{L}_K := \mathcal{L}_{\max}(\Gamma_K)$ and $M_K := M_{\min}(\sigma_K)$. Then,

$$\Gamma \simeq \bar{k} \otimes \Gamma_K, \quad \sigma \simeq \bar{k} \otimes \sigma_K, \quad \mathcal{L} \simeq \bar{k} \otimes \mathcal{L}_K, \quad M \simeq \bar{k} \otimes M_K.$$

In particular,

$$\dim(M_K)_n = \dim M_n < d \quad (\forall n \in \mathbb{Z}).$$

Moreover, it is easy to see that M_K is simple $\mathbb{Z}$ -graded.

For each $m \in \mathbb{Z}$, setting

$$\sigma(m)_K := \bigoplus_{|m-n| \leq s} (M_K)_n,$$

we naturally regard it as partial Γ -module. Here, it should be noted that the cardinality of the equivalence classes of the partial Γ -module $\tau = \bigoplus_{|n| \leq s} \tau_n$ such that $\dim \tau_n \leq d$ for any n is finite, since K is a finite field. Moreover, $\sigma(m)_K \neq \{0\}$ for infinitely many integers m . Therefore, there exist $m_1, m_2 \in \mathbb{Z}$ ($m_1 \neq m_2$) such that

$$\sigma(m_1)_K \simeq \sigma(m_2)_K.$$

Here, we allow any degree shift for an isomorphism of partial Lie algebras. This implies that there exists a generating homomorphism

$$\theta \in \text{End}_{\mathcal{L}_K}(M_K)_{m_1-m_2}.$$

By Remark 2.2, M_K is not graded simple, and thus, M is not graded simple. $\square$

2.5 Proof of the Classification of Harish-Chandra Modules

We complete the proof of the classification of Harish-Chandra modules over the Virasoro algebra.

2.5.1 Structure of Simple $\mathbb{Z}$ -graded Modules

In this subsection, we show a proposition on structures of finite simple $\mathbb{Z}$ -graded modules over the Virasoro algebra in positive characteristic.

Proposition 2.11 *Suppose that $\mathbb{K} = \bar{\mathbb{F}}_p$ for $p > 0$ such that $p \neq 2, 3$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finite simple $\mathbb{Z}$ -graded $\text{Vir}_{\mathbb{K}}$ -module such that*

$$\dim M_n < \frac{1}{2p}(p-1)^2 \quad (\forall n \in \mathbb{Z}).$$

Then, there exist $a, b \in \mathbb{K}$ such that the following isomorphism holds:

$$M \simeq V'_{a,b}$$

where $V'_{a,b}$ are defined as in (2.5).

Proof. By Proposition 2.10, M is not graded simple. Hence, by Proposition 2.8, there exists a generating homomorphism $\theta \in \text{End}_{\text{Vir}_{\mathbb{K}}}(M)$ of degree $N > 0$. Here, we use the following notation:

$$\mathbf{M} := M/(\text{id}_M - \theta)M.$$

Let $\mathcal{I}$ be the kernel of $\text{Vir}_{\mathbb{K}} \rightarrow \text{End}_{\mathbb{K}}(\mathbf{M})$. Then, one can show that $\mathcal{I}$ is a non-trivial and non-central $\mathbb{Z}/N\mathbb{Z}$ -graded ideal. Indeed, if $\mathcal{I}$ is central, then $\dim(\text{Vir}_{\mathbb{K}}/\mathcal{I}) = \infty$. On the other hand, since $\mathbf{M}$ is a faithful $(\text{Vir}_{\mathbb{K}}/\mathcal{I})$ -module, $\text{Vir}_{\mathbb{K}}/\mathcal{I} \hookrightarrow \text{End}_{\mathbb{K}}(\mathbf{M})$. But this contradicts $\dim_{\mathbb{K}} \mathbf{M} < \infty$. One can also show that $\mathcal{I}$ is a $\mathbb{Z}/N\mathbb{Z}$ -graded ideal by definition.

Let u and v be integers such that $N = vp^u$ and $(v, p) = 1$. Since θ commutes with L_0 , we see that $p|N$. Hence, $u \geq 1$. By assumption, we have

$$\dim \mathbf{M} < N \frac{1}{2p}(p-1)^2 = \frac{1}{2}vp^{u-1}(p-1)^2.$$

By Lemma 2.25. A, we have

$$p^{\frac{1}{2}v(p-1)(p^{u-1}-1)} < \frac{1}{2}vp^{u-1}(p-1)^2.$$

Hence, $u = 1$. By Lemma 2.25. B, we have

$$(p-1)^v < \frac{1}{2}v(p-1)^2.$$

Hence, $v = 1$ and $N = p$. By Lemma 2.25. C, we have

$$\dim \mathbf{M} = p, \quad p-1.$$

By Lemma 2.24, there exists $m_1, \dots, m_s$ such that

$$\mathrm{Vir}_{\mathbb{K}}/\mathcal{I} \simeq \bigoplus_{i=1}^s W(m_i).$$

Hence, by Theorem 2.5, we obtain $s = 1$ and $m_1 = 1$, and thus

$$\mathrm{Vir}_{\mathbb{K}}/\mathcal{I} \simeq W(1).$$

By Theorem 2.6,

$$\mathbf{M} \simeq M'_{a,b}$$

for some $a, b \in \mathbb{K}$. Hence, by Proposition 2.9, we obtain

$$M \simeq V'_{a,b}. \quad \square$$

2.5.2 Semi-continuity Principle

We recall the definition and some properties of Dedekind domain, which we use in the proof of Theorem 2.1. For definitions and results on commutative algebra, which we omit in this subsection, see [AtM] or [Matsu].

The aim of this section is the following proposition:

Proposition 2.12 *Let K be an algebraic number field, i.e., a finite algebraic extension of $\mathbb{Q}$, and let R' be the ring of integers of K , i.e., the integral closure of $\mathbb{Z}$ in K . Let R be a localisation of R' such that $R \neq K$, and let k be a residue field of R . Let M be a finite dimensional K -vector space, and let M_R be an R -form of M . We set $M_k := k \otimes_R M_R$. Then, we have*

$$\dim M = \dim M_k.$$

This proposition is usually referred to as the **semi-continuity principle**.

Recall the definition of Dedekind domain (see, e.g., [AtM] Chapter 9).

Definition 2.7 *An integral domain A is called a Dedekind domain if*

1. A is Noetherian,
2. every non-zero prime ideal is a maximal ideal of A , and
3. A is integrally closed.

An important example is given by the next proposition:

Proposition 2.13 *The ring of integers in an algebraic number field is a Dedekind domain.*

Proof. Theorem 9.5 in [AtM]. $\square$

To show Proposition 2.12, we recall two facts from commutative algebras:

Lemma 2.26. *Suppose that A is a Dedekind domain. Then, for any non-zero prime ideal $\mathfrak{p}$ of A , $A_{\mathfrak{p}}$ is a principal ideal domain.*

Proof. Proposition 9.2 and Theorem 9.3 in [AtM]. $\square$

Lemma 2.27. *Let A be an integral domain, and let M be an A -module. M is torsion free if and only if $M_{\mathfrak{p}}$ is a torsion free $A_{\mathfrak{p}}$ -module for any prime ideal $\mathfrak{p}$.*

Proof. Exercise 13 in Chapter 3 of [AtM]. $\square$

PROOF OF PROPOSITION 2.12. First, we notice that M_R is torsion free. Hence, by Lemma 2.27, for any non-zero prime ideal $\mathfrak{p}$ of R , $(M_R)_{\mathfrak{p}}$ is a torsion free $R_{\mathfrak{p}}$ -module. On the other hand, it follows from $R \subset R_{\mathfrak{p}} \subset K$ that $(M_R)_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$ -form of M . Since by Lemma 2.26, $R_{\mathfrak{p}}$ is a principal ideal domain, we see that $(M_R)_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank $\dim M$. Hence, we have

$$\dim M_k = \dim M. \quad \square$$

2.5.3 Proof of Theorem 2.1

To complete the proof of the classification theorem, we introduce some notation. We set

$$\Gamma := \bigoplus_{|i| \leq 2} \mathbb{K}L_i \oplus \mathbb{K}C \subset \text{Vir}_{\mathbb{K}},$$

and regard it as a partial Lie algebra.

We denote the kernel of the map $\mathcal{L}_{\max}(\Gamma) \rightarrow \text{Vir}_{\mathbb{K}}$ by $\mathcal{I}$.

Let $V = \bigoplus_{|i| \leq 2} V_i$ be a graded $\mathbb{K}$ -vector space. Let $\mathcal{V}(V, \Gamma)$ be the variety of partial Γ -module structures on V . It is a closed subvariety of $(\text{End} V)^{\dim \Gamma}$. A partial Γ -module σ can be regarded as an element of $\mathcal{V}(V, \Gamma)$, and we will often do so.

For a positive integer f , let $\mathcal{V}(V, \Gamma, \mathcal{I}, f)$ be the subvariety of $\mathcal{V}(V, \Gamma)$ defined by

$$\begin{aligned} \sigma &\in \mathcal{V}(V, \Gamma, \mathcal{I}, f) \\ \Leftrightarrow \sigma &: \text{simple}, \mathcal{I}.M_{\min}(\sigma) = \{0\}, \dim M_{\min}(\sigma)_n \leq f \ (\forall n \in \mathbb{Z}). \end{aligned}$$

By definition, $\mathcal{V}(V, \Gamma, \mathcal{I}, f)$ is a locally closed subvariety.

Let $\mathcal{D} = (D_{-2}, D_{-1}, D_0, D_1, D_2)$ be a quintuple of non-negative integers such that $D_i \leq f$ for $-2 \leq i \leq 2$. Let $V_{\mathcal{D}}$ be a graded $\mathbb{K}$ -vector space $\bigoplus_{|i| \leq 2} V_i$ such that $\dim V_i = D_i$. We set

$$\mathcal{V}(f) := \bigcup_{\mathcal{D}} \mathcal{V}(V_{\mathcal{D}}, \Gamma, \mathcal{I}, f).$$

Moreover, let $\mathcal{X}$ be the subvariety of $\mathcal{V}(f)$ which consists of $\text{Par}_{-2}^2 V'_{a,b}$, where $V'_{a,b}$ is the irreducible $\text{Vir}_{\mathbb{K}}$ -module defined as in (2.5).

By Proposition 2.7, for any Harish-Chandra module

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

without highest or lowest degree, there exists a positive integer f such that $\dim M_n \leq f$ for any $n \in \mathbb{Z}$. Hence, we have

$$\text{Par}_{-2}^2 M \in \mathcal{V}(f).$$

Hence, to show the theorem, it suffices to prove that $\mathcal{X} = \mathcal{V}(f)$.

It should be noted that the varieties $\mathcal{V}(V, \Gamma)$, $\mathcal{V}(V, \Gamma, \mathcal{I}, f)$, $\mathcal{V}(f)$ and $\mathcal{X}$ are defined over $\mathbb{Q}$. Hence, we may assume that the base field $\mathbb{K}$ is $\overline{\mathbb{Q}}$ without loss of generality.

In the sequel, suppose that $\sigma \in \mathcal{V}(f)$ is defined on some localisation R of the ring of integers of a number field.

Remark 2.4 *Let K be the field of the fractions of R . By assumption, there exists a K -basis $\{v_i\}$ of σ such that the structure constants $(c_{i,j}^l)$ defined by*

$$L_i \cdot v_j = \sum_l x_{i,j}^l v_l, \quad C \cdot v_j = \sum_l y_j^l v_l$$

satisfy $(x_{i,j}^l, y_j^l) \subset R$. Hence, $\sigma_K := \bigoplus_j K v_j$ is a partial Γ_K -module, where $\Gamma_K := \bigoplus_{|i| \leq 2} K L_i \oplus K C$. Moreover, $\dim_{\mathbb{K}} \sigma_{\mathbb{K}} = \dim_K \sigma_K$ holds.

By localising again, if necessary, we may assume that all prime numbers p such that

$$\frac{1}{2p}(p-1)^2 \leq f$$

are invertible in R . Moreover,

Lemma 2.28. *By localising R appropriately, if necessary, we may assume that for any residue field k , $\sigma_k := \sigma_R \otimes_R k$ is absolutely simple.*

Proof. For simplicity, we put $r := \dim \sigma$. Here, we denote the set of $r \times r$ matrices whose elements belong to a ring A by $\text{Mat}_r(A)$. To show

σ_k is absolutely simple, it is enough to check that the image of the map $U(\mathcal{L}_{\max}(\Gamma_k)) \rightarrow \text{End}_k(\sigma_k)$ is isomorphic to $\text{Mat}_r(k)$.

Since σ is absolutely simple, the map $\varphi : U(\mathcal{L}_{\max}(\Gamma)) \rightarrow \text{End}_{\mathbb{K}}(\sigma)$ is surjective, i.e., $\text{Im}\varphi \simeq \text{Mat}_r(\mathbb{K})$. Hence, for any $1 \leq i, j \leq r$, there exists an element $X \in U(\mathcal{L}_{\max}(\Gamma_R))$ such that $\varphi(X) \in RE_{i,j}$, where $E_{i,j}$ denotes the matrix unit. Hence, by taking an appropriate localisation R' of R , the image $\varphi(U(\mathcal{L}_{\max}(\Gamma_{R'})))$ is isomorphic to $\text{Mat}_r(R')$ where $\sigma_{R'} := \sigma_R \otimes R'$. Therefore, $\sigma_{k'}$ is absolutely simple, where k' is any residue field of R' . $\square$

From now on, we denote the localisation R' by R for simplicity.

Remark 2.5 *Since $\dim \sigma < \infty$, we may assume that for infinitely many prime integers $q \in \mathbb{Z}$, q is a prime element of R . We use this assumption at the end of this proof.*

Let M_R be the $\mathcal{L}_{\max}(\Gamma_R)$ -submodule of $M := M_{\min}(\sigma)$ generated by σ_R . By Lemma 2.14, M_R is an R -form of M . Set

$$M'_k := M_R \otimes k,$$

where k is a residue field of R . Let N_k be a graded maximal proper submodule of M'_k . We set

$$M_k := M'_k / N_k.$$

Since σ_k is a simple partial module and generates M'_k as $\mathcal{L}_{\max}(\Gamma_k)$ -module, we have $\sigma_k \cap N_k = \{0\}$. Hence,

$$\text{Par}_{-2}^2 M_k \simeq \sigma_k.$$

Moreover, since M_k is generated by σ_k , by Theorem 2.3, there exists a surjection $M_k \twoheadrightarrow M_{\min}(\sigma_k)$. Since M_k is simple $\mathbb{Z}$ -graded, we have

$$M_k \simeq M_{\min}(\sigma_k).$$

By Proposition 2.12 and Remark 2.4, we see that

$$\dim(M_k)_n \leq f.$$

Since the residue field k is finite, and the characteristic p of k satisfies

$$f < \frac{1}{2p}(p-1)^2,$$

it follows from Proposition 2.11 that

$$M_k \otimes \bar{k} \simeq V'_{a,b;\bar{k}}$$

where $V'_{a,b;\bar{k}}$ denotes the irreducible $\text{Vir}_{\bar{k}}$ -module defined as in (2.5), and to avoid confusion, we specify the ground field $\bar{k}$.

In the following, we divide the proof into the following cases:

1. $a \neq 0, -1$ or $b \notin \mathbb{Z}/p\mathbb{Z}$,
2. $a = 0, -1$ and $b \in \mathbb{Z}/p\mathbb{Z}$,

and here, we prove the theorem in the first case only, since the other case can be similarly treated.

Case: $a \neq 0, -1 \vee b \notin \mathbb{Z}/p\mathbb{Z}$ If we set $\sigma_{\bar{k}} := \sigma_k \otimes_k \bar{k}$, then

$$\dim(\sigma_{\bar{k}})_n = 1$$

for any $-2 \leq n \leq 2$. Since, by Proposition 2.12

$$\dim(\sigma)_n = \dim(\sigma_{\bar{k}})_n$$

holds, we have

$$\dim(\sigma)_n = 1 \quad (-2 \leq n \leq 2).$$

Let us fix $v_0 \in (V_{a,b;\mathbb{K}})_0$. Notice that for any $-2 \leq n \leq 2$, there exists $x_n \in U(\text{Vir}_R)$ ($-2 \leq n \leq 2$) such that $x_n.v_0 \in (V_{a,b;\mathbb{K}})_n \setminus \{0\}$. By using the elements $x_{-2}, x_{-1}, \dots, x_2$, we define an R -form of the partial module $\tau := \text{Par}_{-2}^2 V_{a,b}$ by

$$\tau_R := \sum_{n=-2}^2 R x_i v_0,$$

and set $\tau_{\bar{k}} := \tau_R \otimes_R \bar{k}$. Then, by the isomorphism $M_k \otimes_k \bar{k} \simeq V_{a,b;\bar{k}}$, the following isomorphism holds:

$$\tau_{\bar{k}} \simeq \sigma_{\bar{k}}. \quad (2.45)$$

Let w_0 be an element of $(\sigma_R)_0$ such that $(\sigma_R)_0 = R w_0$. (Note that $\text{rank}(\sigma_R)_0 = 1$, since $\dim(\sigma)_0 = 1$.) We define an R -linear map

$$\varphi_R : \tau_R \longrightarrow \sigma_R$$

by $\varphi_R(x_i.v_0) := x_i.w_0$. Let $\varphi_{\bar{k}} : \tau_{\bar{k}} \longrightarrow \sigma_{\bar{k}}$ be the $\bar{k}$ -linear map induced from φ_R . Since both $\tau_{\bar{k}}$ and $\sigma_{\bar{k}}$ are simple and isomorphic to each other, we see that $\varphi_{\bar{k}}$ gives an isomorphism of $\text{Vir}_{\bar{k}}$ -modules. Hence, for any $v \in \tau_R$ and $-2 \leq n \leq 2$,

$$L_n \varphi_R(v) - \varphi_R(L_n.v) \in \left(\bigcap_{\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)} \mathfrak{m} \right) \sigma_R,$$

where we denote the set of the maximal ideals of R by $\mathfrak{m}\text{-Spec}(R)$. Hence, the following lemma implies that φ_R commutes with the action of L_n for $-2 \leq n \leq 2$.

Lemma 2.29.

$$\bigcap_{\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)} \mathfrak{m} = \{0\}.$$

Proof. By Remark 2.5, for infinitely many prime integers $q \in \mathbb{Z}$, $(q) \in \text{Spec} R$. Since R is a Dedekind domain, $(q) \in \mathfrak{m}\text{-Spec}(R)$. This implies the result. $\square$

Thus, we can extend φ_R to the $\mathbb{K}$ -linear map

$$\varphi : \tau \longrightarrow \sigma.$$

Then, since $\dim(\sigma)_n = 1$ for any $-2 \leq n \leq 2$, φ gives an isomorphism of partial module over $\Gamma = \bigoplus_{n=-2}^2 \mathbb{K}L_n \oplus \mathbb{K}C$. Therefore, we have proved that $\sigma \in \mathcal{X}$ for any $\sigma \in \mathcal{V}(f)$. The inverse inclusion $\mathcal{X} \subset \mathcal{V}(f)$ holds by definition, and thus, we have completed the proof of the classification theorem.

2.6 Bibliographical Notes and Comments

In 1981, V. G. Kac [Kac3] proposed some open problems with conjectures. Among them, the problem of classifying Harish-Chandra modules over the Virasoro algebra is the theme of this chapter. Let us briefly recall its history and some relevant topics.

In 1985, I. Kaplansky and L. J. Santharoubane [KaSa] classified the Harish-Chandra modules all of whose weight multiplicities are 1, which is summarised in the appendix of this chapter. Later, in 1988, V. Chari and A. Pressley [CP2] classified the unitarisable Harish-Chandra modules. In 1991, C. Martin and A. Piard [MaP] proved that any indecomposable $\mathbb{Z}$ -graded Vir-module with bounded weight multiplicities contains a $\mathbb{Z}$ -graded submodule whose weight multiplicities are less than or equal to 1. Thus, in particular, their results together with Proposition 2.7 gives another proof of the conjecture mentioned above. All of these results are obtained only by purely characteristic zero arguments but involve complicated computations. The complete proof of the conjecture obtained by O. Mathieu [Mat2], as explained in this book, relies on completely different methods, *partial Lie theory* and representation theory in positive characteristic.

In this direction, a more general result was obtained by J. Germoni [Ger]. In 2001, he showed that every block in the category of $\mathbb{Z}$ -graded Vir-modules whose weight multiplicities are bounded is wild. Moreover, he also proved that the category of finite-length extensions of irreducible highest (resp. lowest) weight Vir-modules is also wild.

Let us explain the original usage of the *partial Lie theory*. In [Kac3], V. G. Kac conjectured that any infinite dimensional simple $\mathbb{Z}$ -graded Lie algebras with finite growth should be isomorphic to either 1) a loop algebra (including a twisted algebra), 2) a Cartan type Lie algebra or 3) the Virasoro algebra. In 1986, O. Mathieu [Mat1] proved this conjecture in the case when the growth

is less than or equal to one, and in 1992, again, O. Mathieu [Mat3] gave a complete affirmative answer to this conjecture. ‘*Partial Lie theory*’ was the key technique in his proof of the classification problem.

Finally, let us make a brief remark on the classification of Harish-Chandra modules over an infinite dimensional simple graded Lie algebra of finite growth. The classification of the Harish-Chandra modules over a (untwisted-)loop algebra was obtained by V. Chari [Char] and that for a twisted loop algebra was obtained by V. Chari and A. Pressley [CP1]. The classification of Harish-Chandra modules over a Cartan type Lie algebra of rank ≥ 2 was obtained by I. A. Kostrikin [Kostr], and the only rank 1 case, i.e., what is called of type W_1 , was treated by O. Mathieu [Mat2]. Therefore, the classification of Harish-Chandra modules over all of the infinite-dimensional simple $\mathbb{Z}$ -graded Lie algebras is complete.

2.A Appendix: Indecomposable $\mathbb{Z}$ -graded Vir-Modules with Weight Multiplicities 1

Here, we state the classification of the indecomposable $\mathbb{Z}$ -graded modules with weight multiplicities 1 following [KaSa] and [Ka].

2.A.1 Definition of $A(\alpha)$ and $B(\beta)$

For $\alpha, \beta \in \mathbb{C}P^1$, the Vir-modules $A(\alpha)$ and $B(\beta)$ are defined as follows:

$$1. A(\alpha) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}w_n; \quad C.w_n = 0,$$

$$i) \alpha \in \mathbb{C},$$

$$L_i.w_j := (-i - j)w_{i+j} \quad (j \neq 0),$$

$$L_i.w_0 := -i(1 + (i + 1)\alpha)w_i.$$

$$ii) \alpha = \infty,$$

$$L_i.w_j := (-i - j)w_{i+j} \quad (j \neq 0),$$

$$L_i.w_0 := -i(i + 1)w_i.$$

$$2. B(\beta) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}w_n; \quad C.w_n = 0,$$

$$i) \beta \in \mathbb{C},$$

$$L_i.w_j := -jw_{j+i} \quad (j \neq -i),$$

$$L_i.w_{-i} := i(1 + (i + 1)\beta)w_0.$$

ii) $\beta = \infty$,

$$\begin{aligned} L_i \cdot w_j &:= -jw_{j+i} \quad (j \neq -i), \\ L_i \cdot w_{-i} &:= i(i+1)w_0. \end{aligned}$$

Notice that in [KaSa], the case $\alpha = \beta = \infty$ was missing owing to an error in the calculation of, what they call, the inverted module of $A(\alpha)$ and $B(\beta)$. These modules are clearly indecomposable. In fact, the next lemma holds:

Lemma 2.30. *For $\alpha, \beta \in \mathbb{CP}^1$,*

1. *$A(\alpha)$ is always reducible and its submodule $\bigoplus_{n \neq 0} \mathbb{C}w_n$ is isomorphic to the non-trivial proper submodule of $V_{-1,0} \cong A(0)$.*
2. *$B(\beta)$ is always reducible and its irreducible quotient $B(\beta)/\mathbb{C}w_0$ is isomorphic to the irreducible quotient of $V_{0,0} \cong B(0)$.*

Remark 1.7 shows the geometric nature of these modules. Indeed, under an isomorphism $V_{a,b} \cong t^{a-b}\mathbb{C}[t, t^{-1}](dt)^{-a}$, we have

$$V_{-1,0}/\bigoplus_{n \neq 0} \mathbb{C}w_n \cong \mathbb{C}\frac{dt}{t}, \quad V_{0,0} \cong \mathbb{C}[t, t^{-1}] \quad (w_0 \longmapsto 1).$$

See also Proposition 1.7.

2.A.2 Classification Theorem

In the previous subsection, we have seen that $\dim \text{Ext}_{\mathcal{C}_{\text{adm}}}^1(\mathbb{C}1, \mathbb{C}[t, t^{-1}]/\mathbb{C}1)$ and $\dim \text{Ext}_{\mathcal{C}_{\text{adm}}}^1(\mathbb{C}[t, t^{-1}]/\mathbb{C}1, \mathbb{C}1)$ are at least 2. In fact, by the theorem due to I. Kaplansky and L. J. Santharoubane [KaSa] which is stated below, it follows that

$$\text{Ext}_{\mathcal{C}_{\text{adm}}}^1(\mathbb{C}1, \mathbb{C}[t, t^{-1}]/\mathbb{C}1) \cong \mathbb{C}^2, \quad \text{Ext}_{\mathcal{C}_{\text{adm}}}^1(\mathbb{C}[t, t^{-1}]/\mathbb{C}1, \mathbb{C}1) \cong \mathbb{C}^2,$$

and each isomorphism class is represented by $A(\alpha)$ (resp. $B(\beta)$). Indeed, they have proved the next stronger statement:

Theorem 2.7 ([KaSa]) *Any $\mathbb{Z}$ -graded indecomposable Vir-module whose weight multiplicities are 1 is one of the following modules:*

$$V_{a,b} \ (a, b \in \mathbb{C}), \quad A(\alpha) \ (\alpha \in \mathbb{CP}^1), \quad B(\beta) \ (\beta \in \mathbb{CP}^1).$$

Their proof is based on straightforward calculations. For its proof see [KaSa] together with [Ka] where the errors in the proof of the first lemma which states that, for a $\mathbb{Z}$ -graded Vir-module $V = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}w_n$,

if $L_{\pm 1} \cdot w_j \neq 0$ ($\forall j \in \mathbb{Z}$), then $\exists a, b \in \mathbb{C}$ such that $V \cong V_{a,b}$, was corrected.

We remark that the modules $A(\alpha)$ and $B(\beta)$ ($\alpha, \beta \in \mathbb{C}P^1$) have the following dualities:

1. The antipode-dual:

$$A(\alpha)^{\sharp a} \cong B(\alpha), \quad B(\beta)^{\sharp a} \cong A(\beta).$$

2. The contragredient dual:

$$A(\alpha)^c \cong B\left(-\frac{\alpha}{2\alpha+1}\right), \quad B(\beta)^c \cong A\left(-\frac{\beta}{2\beta+1}\right).$$

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