Chapter II

Hilbert Functions

Abstract. A well-studied and important numerical invariant of a graded ideal over a graded polynomial ring S is the Hilbert function. It gives the sizes of the graded components of the ideal.

The Hilbert function encodes important information (for example, dimension and multiplicity). Hilbert's insight was that it is determined by finitely many of its values.

In many recent papers and books, Hilbert functions are studied using clever computations with binomials; we mention the binomial-approach briefly and avoid such computations whenever possible. Instead our arguments are founded upon Macaulay's key idea in 1927: *There exist highly structured monomial ideals - lex ideals - which attain all Hilbert functions. Lex ideals play an important role in many results on Hilbert functions.* The pivotal property is that a lex ideal grows as slowly as possible.

Another exciting direction of research is to *parametrize all graded ideals in S with a fixed Hilbert function, and then study their (common) properties and the structure of the parameter space.* Lex ideals play crucial role in Hartshorne's Theorem that Grothendieck's Hilbert scheme is connected.

40 Notation

Let W be a graded finitely generated R-module. It decomposes as a direct sum of its components $W = \oplus_{q\geq 0} W_q$. Its Hilbert function is defined by $q \to \dim_k W_q$. We denote

$$|W_q| \,=\, \dim_k(W_q)\,.$$

Recall that the Hilbert series of W is

$$\mathrm{Hilb}_W(t) = \sum_{q\geq 0} \dim_k(W_q)\, t^q\,.$$

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Throughout this chapter V stands for a graded finitely generated S-module.

41 Lex ideals

Macaulay's Theorem 41.7 characterizes the Hilbert functions of graded ideals in S. The theorem is well-known and has many applications. The key idea is that each Hilbert function is attained by a lex ideal. Lex ideals are highly structured: they are defined combinatorially and it is easy to derive the inequalities characterizing their Hilbert functions. They play other important roles; for example,

- Hartshorne's [Hartshorne 2] proof that the Hilbert scheme is connected uses lex ideals in an essential way.
- The homological properties of lex ideals are combinatorially tractable by Theorem 41.9. This leads to results in Section 47, showing that the lex ideals have greatest Betti numbers.

Notation and Definition 41.1. Recall that S_q is the k-vector space spanned by all monomials in S of degree q. So, S_1 is the k-vector space spanned by the variables. We order the variables lexicographically by $x_1 > \ldots > x_n$. We denote by $\succ_{lex}$ the degree-lex order on the monomials, that is, $m \succ_{lex} m'$ if either $\deg(m) > \deg(m')$ or $\deg(m) = \deg(m')$ and m is lex-greater than m'. Sometimes we say lex-last instead of lex-smallest.

We say that A_q is an S_q***-monomial space*** if it can be spanned by monomials of degree q. We denote by $\{A_q\}$ the set of monomials (non-zero monomials in S_q) contained in A_q. The cardinality of this set is $|A_q| = \dim_k A_q$. By $S_1 A_q$ we mean the k-vector subspace $(A_q)_{q+1}$ of S_{q+1}, (where (A_q) is the ideal generated by the elements in A_q).

The ***lex-segment*** $M_{q,p}$ of length p in degree q is defined as the k-vector space spanned by the lex-greatest p monomials in S_q. An S_q-monomial space M_q is ***lex*** in S_q if there exists a p such that $M_q = M_{q,p}$. The monomial space 0 is lex in S_q by convention. For a monomial space A_q, we say that $M_{q,|A_q|}$ is its S_q***-lexification***.

For an S_q-monomial space A_q sometimes we say for simplicity that A_q is a monomial space in S_q or a monomial space; in the latter case the index q indicates that $A_q \subseteq S_q$.

An S_q-monomial space T_q is ***greater lexicographically*** than an S_q-monomial space A_q if when we order the monomials in $\{T_q\}$ and $\{A_q\}$ lexicographically, and then compare the two ordered sets lexicographically, we get that the first ordered set is greater.

Proposition 41.2. *If a monomial space M_q is lex in S_q, then S_1M_q is lex in S_{q+1}.*

Proof. Let $m \in M_q$ be a monomial and let $u \succ_{lex} x_i m$ be a monomial in S_{q+1}. We have to show that $u \in S_1 M_q$. Write $x_i m = m'z$, where z is the lex-last variable that divides $x_i m$, and $m' = \frac{x_i m}{z}$. It follows that $m' \succ_{lex} m$, so $m' \in M_q$.

Similarly write $u = u'y$, where where y is the lex-last variable that divides u, and $u' = \frac{u}{y}$. Since $u'y = u \succ_{lex} x_i m = m'z$, it follows that $u' \succ_{lex} m'$. As $m' \in M_q$ and M_q is lex, we get that $u' \in M_q$. Therefore, $u = yu' \in S_1 M_q$. $\square$

Proposition 41.3. *Let L be a monomial ideal in S. The following conditions are equivalent.*

1. *For each $q \geq 0$, we have that L_q is lex.*
2. *If m is a monomial, such that $m \succ_{lex} m'$ and* $\deg(m) = \deg(m')$ *for some monomial $m' \in L$, then $m \in L$.*
3. *Let p be a number, such that L has no minimal monomial generators in degrees $> p$. For each $q \leq p$, we have that L_q is lex.*
4. *Let L be minimally generated by the monomials $l_1, \dots, l_r$. If m is a monomial, $m \succ_{lex} l_i$ and* $\deg(m) = \deg(l_i)$ *for some $1 \leq i \leq r$, then $m \in L$.*

Proof. (1) $\iff$ (2) and (3) $\Longrightarrow$ (4) by the definition of lex-segment.

We will show that (4) $\Longrightarrow$ (3) by induction on the degree q. Suppose that L_q is lex; we will prove that L_{q+1} is lex as well.

If L has no minimal monomial generators of degree $q+1$, then by Proposition 41.2 it follows that L_{q+1} is lex.

If u is the lex-last minimal monomial generator of L of degree

$q+1$, then by Proposition 41.2 and (4) it follows that L_{q+1} is the lex monomial space in S_q whose end (that is, whose lex-last monomial) is u.

(1) $\Longrightarrow$ (3). By 41.2 it follows that (3) implies (1). $\square$

Definition 41.4. A monomial ideal L is ***lex*** (or ***lexicographic***) if it satisfies the equivalent conditions in Proposition 41.3.

We usually use (4) in order to show that a given ideal is lex. On the other hand, (1) is the condition usually used in proofs.

Example 41.5. By (4), the ideal $(x_1^2, x_1x_2, x_1x_3, x_2^5, x_2^4x_3, x_2^3x_3^2, x_2^2x_3^3, x_2x_3^6, x_3^9)$ is lex in $k[x_1, x_2, x_3]$.

We are ready to discuss Macaulay's Theorem 41.7, which characterizes the Hilbert functions of graded ideals in S.

Proposition 41.6. *The following properties are equivalent.*

(1) *Let A_q be an S_q-monomial space and L_q be its lexification in S_q. Then $|\,S_1L_q\,| \le |\,S_1A_q\,|$.*
(2) *For every graded ideal J in S there exists a lex ideal L with the same Hilbert function.*

The key property of lex ideals is expressed in (1) above: among all subspaces of the same dimension, the lex monomial space generates as little as possible in the next degree.

Proof. We will prove that (1) and (2) are equivalent. (2) implies (1). Assume that (1) holds. We will prove (2). We can assume that J is a monomial ideal by Gröbner basis theory. For each $q \ge 0$, let L_q be the lexification of J_q. By (1), it follows that $L = \oplus_{q\ge 0} L_q$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as J in all degrees. $\square$

In Section 45, we will prove that (1) holds which will establish Macaulay's Theorem.

Macaulay's Theorem 41.7. *The equivalent properties in Proposition* 41.6 *hold.*

We say that an S_q-monomial space A_q is ***Borel*** if whenever a monomial $x_j m \in A_q$ and $1 \le i \le j$ it follows that $x_i m \in A_q$.

Exercise 41.8. *Every lex ideal is Borel.*

This yields the following result.

Theorem 41.9. *The minimal graded free resolution of a lex ideal is the Eliahou-Kervaire resolution.*

42 Compression

Compression is a technique, introduced by Macaulay in order to study Hilbert functions.

Let $1 \le i \le n$ be an integer. An S_q-monomial space C_q can be written uniquely in the form

$$\{C_q\} = \coprod_{0 \le j \le q} x_i^{q-j}\{L_j\}$$

where L_j is a monomial space in the ring S/x_i.

We say that C_q is ***i-compressed*** if each L_j is lex in S/x_i. Furthermore, we say that C_q is ***S_q-compressed*** (or ***compressed***) if it is i-compressed for all $1 \le i \le n$.

A monomial ideal P is ***i-compressed*** if P_q is i-compressed for all $q \ge 0$. The ideal is ***compressed*** if P_q is compressed for all $q \ge 0$.

Example 42.1. [Mermin-Peeva 2, Example 3.2] We give an example of an ideal P which is compressed but not lex. Consider

$$P = (a^3, a^2b, a^2c, ab^2, abc, b^3, b^2c)$$

in $k[a,b,c]$ with $a > b > c$.

Proposition 42.2. *If a monomial space C_q is i-compressed in S_q, then S_1C_q is i-compressed in S_{q+1}.*

Proof. Consider the disjoint union $\{C_q\} = \coprod_{0 \le j \le q} x_i^{q-j}\{L_j\}$ where each L_j is lex in $\big(S/x_i\big)_j$. In the next degree $q+1$ we get the disjoint union

$$\{S_1 C_q\} = \coprod_{0 \le j \le q+1} x_i^{q-j+1}\{L_j + (S_1/x_i)L_{j-1}\}\,.$$

Since both L_j and $(S_1/x_i)L_{j-1}$ are lex $(S/x_i)_j$-monomial spaces, it follows that $L_j + (S_1/x_i)L_{j-1}$ is the longer of these two lex monomial spaces. □

Exercise 42.3. *Let P be a monomial ideal and p be a number, such that P has no minimal monomial generators in degrees $> p$. If P_q is i-compressed for every $0 \le q \le p$, then P is i-compressed.*

Exercise 42.4. *If an S_q-monomial space L_q is lex, then it is S_q-compressed.*

Structure Lemma 42.5.
(1) *If a monomial space C_q is compressed and $n \ge 3$, then C_q is Borel.*
(2) *If $n \le 2$, then every monomial space is compressed.*

Proof. We will prove (1). Recall that a monomial $m' \in S$ is said to be in the big shadow of a monomial $m \in S$ if $m' = \dfrac{x_i\, m}{x_j}$ for some x_j dividing m and some $i \le j$. Let $m \in \{C_q\}$ and m' be a monomial in its big shadow. Hence $m' = \dfrac{x_i\, m}{x_j}$ for some x_j dividing m and some $i \le j$. As $n \ge 3$, there exists an index $1 \le p \le n$ such that $p \ne i, j$. Note that the monomials m and m' have the same p-exponents. Since C_q is p-compressed and $m' \succ_{lex} m$, it follows that $m' \in \{C_q\}$. Therefore, C_q is Borel. □

Construction 42.6. Fix an $1 \le i \le n$. Let A_q be an S_q-monomial space with disjoint union

$$\{A_q\} = \coprod_{0 \le j \le q} x_i^{q-j} \{U_j\}$$

where each U_j is a monomial space in $(S/x_i)_j$. For each $0 \le j \le q$, let L_j be the lexification of the space U_j in $(S/x_i)_j$. The S_q-monomial space C_q defined by

$$\{C_q\} = \coprod_{0 \le j \le q} x_i^{q-j} \{L_j\}$$

is the ***i*-compression** of A_q. Clearly, $|C_q| = |A_q|$.

Example 42.7. Let A_2 be the S_2-monomial space spanned by $\{x_1^2, x_2x_3, x_2^2, x_3x_4\}$. We have the disjoint union

$$\{A_2\} = x_2^2\{1\} \coprod x_2\{x_3\} \coprod 1\{x_1^2, x_3x_4\}$$

so U_2 is spanned by $\{x_1^2, x_3x_4\}$, U_1 is spanned by $\{x_3\}$, and U_0 is spanned by $\{1\}$. Therefore L_2 is spanned by $\{x_1^2, x_1x_3\}$, L_1 is spanned by $\{x_1\}$, and L_0 is spanned by $\{1\}$. The 2-compression of A_2 is

$$\{C_2\} = x_2^2\{1\} \coprod x_2\{x_1\} \coprod 1\{x_1^2, x_1x_3\}\,.$$

Lemma 42.8. *Let A_q be an S_q-monomial space. Fix an $1 \le i \le n$. Let C_q be the i-compression of A_q. We have that $|C_q| = |A_q|$ and $|S_1C_q| \le |S_1A_q|$.*

Proof. We use induction on the number of variables, and assume that Theorem 41.7(1) holds for $n-1$ variables.

Suppose that A_q is not i-compressed. Set $z = x_i$ and $\mathbf{n} = S_1/z$. Use the notation in Construction 42.6. We have the disjoint unions

$$\{S_1A_q\} = \coprod_{0 \le j \le q+1} z^{q-j+1}\{U_j + \mathbf{n}U_{j-1}\}$$

$$\{S_1C_q\} = \coprod_{0 \le j \le q+1} z^{q-j+1}\{L_j + \mathbf{n}L_{j-1}\}\,.$$

We will show that

$$|L_j + \mathbf{n}L_{j-1}| = \max\Big\{\, |L_j|,\, |\mathbf{n}L_{j-1}| \,\Big\}$$
$$\leq \max\Big\{\, |U_j|,\, |\mathbf{n}U_{j-1}| \,\Big\} \leq |U_j + \mathbf{n}U_{j-1}|\,.$$

The first equality above holds because both L_j and $\mathbf{n}L_{j-1}$ are lex $(S/z)_j$-monomial spaces, so $L_j + \mathbf{n}L_{j-1}$ is the longer of these two lex monomial spaces. The last inequality is obvious. The middle inequality holds since: by construction L_{j-1} is the lexification of U_{j-1}, so $|L_{j-1}| = |U_{j-1}|$ and by induction on the number of variables we can apply Macaulay's Theorem 41.7 to the ring S/z.

Thus, $|L_j + \mathbf{n}L_{j-1}| \leq |U_j + \mathbf{n}U_{j-1}|$ for each j. This implies the desired inequality $|S_1C_q| \leq |S_1A_q|$. □

Compression Lemma 42.9. (Clements-Lindström) *Let A_q be an S_q-monomial space. There exists a compressed monomial space T_q in S_q such that $|T_q| = |A_q|$ and $|S_1T_q| \leq |S_1A_q|$.*

Proof. Suppose that A_q is not i-compressed for some $1 \leq i \leq n$. Let C_q be the i-compression of A_q. By the above lemma, we have that $|C_q| = |A_q|$ and $|S_1C_q| \leq |S_1A_q|$.

Note that $\{C_q\}$ is greater lexicographically than $\{A_q\}$. If C_q is not compressed, we can apply the argument above. After finitely many steps in this way, the process must terminate because at each step we construct a lexicographically greater S_q-monomial space. Thus, after finitely many steps, we reach a compressed monomial space. □

43 Multicompression

In this section we describe a multigraded version of the technique of compression.

Let $\mathcal{A} \subset \{x_1, \ldots, x_n\}$; its ***complement*** is $\mathcal{A}^c = \{x_1, \ldots, x_n\} \setminus \mathcal{A}$. Denote by $\oplus_m$ the direct sum over all monomials m in the variables in $\mathcal{A}^c$. An S_q-monomial space C_q can be written uniquely in the form

$$C_q = \bigoplus_m m\, V_m ,$$

where V_m is a monomial space in the ring $k[\mathcal{A}] = k[x_i \,|\, x_i \in \mathcal{A}]$.

We say that C_q is ***$\mathcal{A}$-multicompressed*** if each V_m is lex in $k[\mathcal{A}]$.

Furthermore, we say that C_q is ***(j)-multicompressed*** if it is $\mathcal{A}$-multicompressed for every set $\mathcal{A}$ of size j. We say that C_q is ***multicompressed*** if it is $\mathcal{A}$-multicompressed for every set $\mathcal{A}$.

A monomial ideal P is ***$\mathcal{A}$-multicompressed*** if P_q is $\mathcal{A}$-multicompressed for all $q \geq 0$. The ideal is ***(j)-multicompressed*** if P_q is (j)-multicompressed for all $q \geq 0$.

Example 43.1. Let $\mathcal{A} = \{x_1, x_3\} \subset \{x_1, x_2, x_3, x_4\}$ and C_2 be spanned by the monomials

$$x_2^2, x_1x_2, x_1^2, x_1x_3, x_4^2, x_1x_4, x_2x_4 \,.$$

We have the decomposition

$$\begin{aligned}\{C_2\} = x_2^2\{1\} \coprod x_2\{x_1\} \coprod 1\{x_1^2, x_1x_3\} \coprod x_4^2\{1\} \\ \coprod x_4\{x_1\} \coprod x_2x_4\{1\} \,.\end{aligned}$$

We see that

$$\begin{aligned}\{V_{x_2^2}\} &= \{1\}, & \{V_{x_2}\} &= \{x_1\}, \\ \{V_{x_4^2}\} &= \{1\}, & \{V_1\} &= \{x_1^2, x_1x_3\}, \\ \{V_{x_2x_4}\} &= \{1\}, & \{V_{x_4}\} &= \{x_1\}\end{aligned}$$

are all lex, so C_2 is $\{x_1, x_3\}$-compressed.

Exercise 43.2. *If C_q is $\mathcal{A}$-multicompressed in S_q, then S_1C_q is $\mathcal{A}$-multicompressed in S_{q+1}.*

Exercise 43.3. *Let P be a monomial ideal and p be a number, such that P has no minimal monomial generators in degrees $> p$. If P_q is $\mathcal{A}$-multicompressed for every $0 \leq q \leq p$, then P is $\mathcal{A}$-multicompressed.*

Exercise 43.4. *If L_q is lex, then it is $\mathcal{A}$-multicompressed for every set $\mathcal{A}$.*

Exercise 43.5. *If $\mathcal{A}'$ is a subset of $\mathcal{A}$ and C_q is $\mathcal{A}$-multicompressed in S_q, then C_q is $\mathcal{A}'$-multicompressed.*

Exercise 43.6. *If C_q is (j)-multicompressed, then it is (i)-multicompressed for every $i \leq j$.*

Structure Theorem 43.7. [Mermin]

(1) *A monomial space C_q is Borel if and only if it is* (2)*-multicompressed.*

(2) *A monomial space C_q is lex if and only if it is* (3)*-multicompressed.*

Proof. First, we prove (1).

Let C_q be (2)-multicompressed. We will prove that it is Borel. Let $x_j m' \in C_q$ be a monomial and fix an $1 \leq i < j$. Set $\mathcal{A} = \{x_i, x_j\}$. Write $x_j m' = x_i^s x_j^t m$ so that m is not divisible by either x_i or x_j. Hence $x_i^s x_j^t \in \{V_m\}$. The monomial $x_i^{s+1} x_j^{t-1}$ is lex-greater than $x_i^s x_j^t$. Since V_m is lex, it follows that $x_i^{s+1} x_j^{t-1} \in \{V_m\}$. Hence $x_i m' \in C_q$.

Let C_q be a Borel monomial space. We will prove that it is (2)-multicompressed. Fix a set $\mathcal{A} = \{x_i, x_j\}$ with $1 \leq i < j$. We will show that each V_m is lex. Let $x_i^s x_j^t \in V_m$. Let $x_i^{s+h} x_j^{t-h}$ be a monomial that is lex-greater than $x_i^s x_j^t$. Since $x_i^s x_j^t m \in C_q$ and C_q is Borel, it follows that $x_i^{s+h} x_j^{t-h} m \in C_q$. Hence $x_i^{s+h} x_j^{t-h} \in V_m$. Therefore, V_m is lex.

Now, we prove (2). If C_q is lex then it is (3)-multicompressed by Exercise 43.4. Suppose that C_q is (3)-multicompressed. We will show that it is lex. By (1) and Exercise 43.6, it follows that C_q is Borel.

Let $u = x_1^{a_1} \ldots x_n^{a_n}$ be a monomial in C_q. Let $v = x_1^{c_1} \ldots x_n^{c_n}$ be a monomial that is lex-greater than u. We will show that $v \in C_q$. Let i be minimal so that $a_i \neq c_i$. Then $a_i < c_i$ since v is lex-greater than u. Set $w = x_1^{a_1} \ldots x_{i-1}^{a_{i-1}}$ and $e = \deg(x_{i+1}^{a_{i+1}} \ldots x_n^{a_n}) = a_{i+1} + \ldots + a_n$.

Since $u \in C_q$, we can use that C_q is Borel in order to conclude that $wx_i^{a_i}x_{i+1}^e \in C_q$. Set $\mathcal{A} = \{x_i, x_{i+1}, x_n\}$. Then $x_i^{a_i}x_{i+1}^e \in V_w$. As C_q is $\{x_i, x_{i+1}, x_n\}$-multicompressed, it follows that V_w is lex. The monomial $x_i^{a_i+1}x_n^{e-1}$ is lex-greater than $x_i^{a_i}x_{i+1}^e$, so $x_i^{a_i+1}x_n^{e-1} \in V_w$. Hence $wx_i^{a_i+1}x_n^{e-1} \in C_q$. As C_q is Borel it follows that $v \in C_q$. □

The following is an immediate corollary.

Structure Theorem 43.8. [Mermin]
(1) *If $n < 3$, then every monomial space is multicompressed.*
(2) *If $n = 3$, then the multicompressed monomial spaces are exactly the Borel spaces.*
(3) *If $n > 3$ then the multicompressed monomial spaces are exactly the lex spaces.*

The following lemma is proved similarly to the Compression Lemma 42.9.

Lemma 43.9. *Let $\mathcal{A} \subset \{x_1, \ldots, x_n\}$. Let A_q be an S_q-monomial space. There exists an $\mathcal{A}$-compressed monomial space T_q in S_q such that $|T_q| = |A_q|$ and $|S_1T_q| \le |S_1A_q|$.*

Lemma 43.10. *Fix a $1 \le j \le n-1$. Let A_q be an S_q-monomial space. There exists a (j)-compressed monomial space C_q in S_q such that $|C_q| = |A_q|$ and $|S_1C_q| \le |S_1A_q|$.*

Proof. Apply Lemma 43.9 repeatedly if necessary. □

44 Green's Theorem

Green's Theorem describes the change in the Hilbert function when we factor out a generic form.

For a monomial m define

$$\max(m) = \max\{i \,|\, x_i \text{ divides } m\}$$
$$\min(m) = \min\{i \,|\, x_i \text{ divides } m\}\,.$$

For an S_q-monomial space A_q set

$$r_{i,j}(A_q) = \Big| \{ m \in \{A_q\} \mid \max(m) \le i \ \text{ and } \ x_i^j \text{ does not divide } m \} \Big|$$

$$t_i(A_q) = \Big| \{ m \in \{A_q\} \mid \max(m) \le i \} \Big| .$$

Lemma 44.1. (Bigatti) *If an S_q-monomial space B_q is Borel, then $\{S_1 B_q\}$ is the set*

$$\mathcal{B} = \coprod_{i=1}^{n} x_i \{ m \in \{B_q\} \mid \max(m) \le i \}$$

and

$$\Big| \{S_1 B_q\} \Big| = \sum_{i=1}^{n} t_i(B_q) .$$

Proof. Let $w \in \{B_q\}$. For $j \ge \max(w)$ we have that $x_j w \in \mathcal{B}$. Let $j < \max(w)$. Then $v = x_j \dfrac{w}{x_{\max(w)}} \in B_q$. So, $x_j w = x_{\max(w)} v \in \mathcal{B}$. $\square$

Lemma 44.2. *Let A_q be a Borel S_q-monomial space. Its n-compression C_q is Borel.*

Proof. We use the notation in Construction 42.6. Consider the disjoint unions

$$\{A_q\} = \coprod_{0 \le j \le q} x_n^{q-j} \{U_j\}$$

$$\{C_q\} = \coprod_{0 \le j \le q} x_n^{q-j} \{L_j\} .$$

Since A_q is Borel, it follows that

$$(S_1/x_n) U_j \subseteq U_{j+1} .$$

We use induction on the number of variables, and assume that Theorem 41.7(1) holds for $n-1$ variables. Since $|L_j| = |U_j|$, by Theorem 41.7(1) it follows that

$$|(S_1/x_n) L_j| \le |(S_1/x_n) U_j| \le |U_{j+1}| = |L_{j+1}| .$$

As both $(S_1/x_n)L_j$ and L_{j+1} are lex monomial spaces, we conclude that $(S_1/x_n)L_j \subseteq L_{j+1}$. Let $x_n^{q-j}m$ be a monomial in C_q and $m \in L_j$. Then for each $1 \le i < n$ we have that $x_i m \in (S_1/x_n)L_j \subseteq L_{j+1}$, so $x_n^{q-j-1}x_i m \in C_q$. If x_p divides m, then for each $1 \le c \le p$ we have that $\frac{x_c m}{x_p} \in L_j$ since L_j is lex. We proved that C_q is Borel. □

The main work for proving the Generalized Green's Theorem 44.5 is in the following lemma.

Lemma 44.3. *Let C_q be an n-compressed Borel S_q-monomial space, and let L_q be a lex monomial space in S_q with $|L_q| \le |C_q|$. For each $1 \le i \le n$ and each $1 \le j$ we have the inequality*

$$r_{i,j}(L_q) \le r_{i,j}(C_q)\,.$$

Proof. Note that both L_q and C_q are Borel and n-compressed.

First, we consider the case $i = n$. Clearly, $r_{n,q+1}(L_q) = |L_q| \le |C_q| = r_{n,q+1}(C_q)$. We induct on j decreasingly. Suppose that the inequality $r_{n,j+1}(L_q) \le r_{n,j+1}(C_q)$ holds by induction.

If $\{C_q\}$ contains no monomial divisible by x_n^j then

$$r_{n,j}(L_q) \le r_{n,j+1}(L_q) \le r_{n,j+1}(C_q) = r_{n,j}(C_q)\,.$$

Suppose that $\{C_q\}$ contains a monomial divisible by x_n^j. Denote by $e = x_1^{e_1}\dots x_n^{e_n}$, with $e_n \ge j$, the lex-last monomial in C_q that is divisible by x_n^j.

Let $0 \le p \le j-1$. Let the monomial $v = x_1^{a_1}\dots x_{n-1}^{a_{n-1}}x_n^p \in S_q$ be lex-greater than e. Since C_q is Borel, it follows that $w = x_{n-1}^{e_n-p}\dfrac{e}{x_n^{e_n-p}} \in C_q$. This is the lex-last monomial that is lex-greater than e and x_n divides it at power p. Since C_q is n-compressed and v is lex-greater (or equal) than w, it follows that $v \in C_q$.

For a monomial u, we denote by $x_n^j \notin u$ the property that x_n^j does not divide u. By what we proved above, it follows that

(∗)

$$\Big|\{u \in \{C_q\} \mid x_n^j \notin u,\ u \succ_{lex} e\}\Big| = \Big|\{u \in \{S_q\} \mid x_n^j \notin u,\ u \succ_{lex} e\}\Big|\,.$$

Therefore,

$$
\begin{aligned}
& r_{n,j}(L_q) \\
& = |\{u \in \{L_q\} \,|\, x_n^j \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_q\} \,|\, x_n^j \notin u,\ u \prec_{lex} e\}| \\
& \le |\{u \in \{S_q\} \,|\, x_n^j \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_q\} \,|\, x_n^j \notin u,\ u \prec_{lex} e\}| \\
& \le |\{u \in \{S_q\} \,|\, x_n^j \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_q\} \,|\, u \prec_{lex} e\}| \\
& \le |\{u \in \{S_q\} \,|\, x_n^j \notin u,\ u \succ_{lex} e\}| + |\{u \in \{C_q\} \,|\, u \prec_{lex} e\}| \\
& = |\{u \in \{S_q\} \,|\, x_n^j \notin u,\ u \succ_{lex} e\}| + |\{u \in \{C_q\} \,|\, x_n^j \notin u,\ u \prec_{lex} e\}| \\
& = |\{u \in \{C_q\} \,|\, x_n^j \notin u,\ u \succ_{lex} e\}| + |\{u \in \{C_q\} \,|\, x_n^j \notin u,\ u \prec_{lex} e\}| \\
& = r_{n,j}(C_q)\,;
\end{aligned}
$$

for the third inequality we used the fact that L_q is a lex monomial space in S_q with $|L_q| \le |C_q|$; for the equality after that we used the definition of e; for the next equality we used $(*)$. Thus, we have the desired inequality in the case $i = n$.

In particular, we proved that

$$(**) \qquad r_{n,1}(L_q) \le r_{n,1}(C_q)\,.$$

Finally, we prove the lemma for all $i < n$. Both $\{C_q/x_n\}$ and $\{L_q/x_n\}$ are lex monomial spaces in S_q/x_n since C_q is n-compressed. By (**) the inequality $r_{n,1}(L_q) \le r_{n,1}(C_q)$ holds, and it implies the inclusion $\{C_q/x_n\} \supseteq \{L_q/x_n\}$. The desired inequalities follow since

$$
\begin{aligned}
r_{i,j}(C_q) &= r_{i,j}\big(C_q/(x_{i+1},\dots,x_n)\big) \\
r_{i,j}(L_q) &= r_{i,j}\big(L_q/(x_{i+1},\dots,x_n)\big)\,. \qquad \square
\end{aligned}
$$

Comparison Theorem 44.4. *Let B_q be a Borel monomial space in S_q. Let L_q be a lex monomial space in S_q with $|L_q| \le |B_q|$. We have the inequalities*

$$t_i(L_q) \le t_i(B_q)$$

$$r_{i,j}(L_q) \le r_{i,j}(B_q)\,.$$

for each $1 \le i \le n$ *and each* $1 \le j$.

Proof. First, note that $t_i(A_q) = r_{i,q+1}(A_q)$ for any monomial space A_q. Thus, it suffices to prove the inequalities $r_{i,j}(L_q) \le r_{i,j}(B_q)$.

We prove the inequalities by decreasing induction on the number of variables n. Let C_q be the n-compression of B_q. Since C_q is Borel and n-compressed by Lemma 44.2, we can apply Lemma 44.3 and we get

$$r_{i,j}(L_q) \le r_{i,j}(C_q)$$

for each $1 \le i \le n$ and each $1 \le j$. It remains to compare $r_{i,j}(C_q)$ and $r_{i,j}(B_q)$. For $i = n$, we have equalities $r_{n,j}(C_q) = r_{n,j}(B_q)$. Let $i < n$. Then $r_{i,j}(C_q) = r_{i,j}(C_q/x_n)$ and $r_{i,j}(B_q) = r_{i,j}(B_q/x_n)$, where $C_q/x_n = L_q$ is lex and $B_q/x_n = U_q$ is Borel in S/x_n. So, by induction the desired inequalities hold. □

Generalized Green's Theorem 44.5. *Let* B_q *be a Borel monomial space in* S_q. *Let* L_q *be a lex monomial space in* S_q *with* $|L_q| \le |B_q|$. *The inequality*

$$\dim_k \Big(S_q/(L_q + x_n^j\, S_{q-j})\Big) \ge \dim_k \Big(S_q/(B_q + x_n^j\, S_{q-j})\Big)$$

holds for each $1 \le j \le q$.

Proof. Note that the desired inequality is equivalent to

$$r_{n,j}(L_q) \le r_{n,j}(B_q)\,.$$

It holds by Theorem 44.4. □

Assume $\mathrm{char}(k) = 0$. Let I be a graded ideal in S and $R = S/I$. Fix an integer j. The affine space R_j is irreducible, so every nonempty Zariski-open subset is dense. We say that a property $\mathcal{P}$ ***holds for*** a ***generic*** j***-form*** if there exists a nonempty Zariski-open subset $\mathcal{U} \subseteq R_j$ such that the property $\mathcal{P}$ holds for every j-form in $\mathcal{U}$.

Lemma 44.6. *Assume* $\mathrm{char}(k) = 0$. *Suppose that* I *is a graded ideal in* S *and* $R = S/I$. *Fix integers* i *and* j. *Let*

$$t = \max\{\dim_k(g\, R_i) \,|\, g \in R_j\}$$

There exists a non-empty Zariski-open set $\mathcal{U} \subseteq R_j$ *such that* $\dim_k(h\, R_i) = t$ *for every generic* j*-form* $h \in \mathcal{U}$.

Proof. Let

$$\mathcal{U} = \{\, v \in R_j \mid \dim_k(vR_i) = t\} \subseteq R_j \,.$$

Choose a basis $f_1, \dots, f_a$ of R_j and a basis $g_1, \dots, g_c$ of R_i. The elements $f_p g_q$ span R_{i+j}, so we can choose a subset that is a basis. Write $v = \sum_{1\le p\le a} \alpha_p f_p$, where the coefficients $\alpha_1, \dots, \alpha_a$ are in k. The multiplication map $v:\ R_i \to R_{i+j}$ has a matrix M whose entries are linear forms in $\alpha_1, \dots, \alpha_a$. A j-form v is in $\mathcal{U}$ if and only if the matrix M has a non-zero $(t\times t)$-minor. When we vary v, we can think of $\alpha_1, \dots, \alpha_a$ as indeterminates which take values in k. Therefore, the complement of $V(I_t(M))$ is a Zariski-open set (here $I_t(M)$ is the ideal generated by all $(t\times t)$-minors of M, and $V(I_t(M))$ is the set on which all elements in $I_t(M)$ vanish). $\square$

Exercise 44.7. *Assume* $\mathrm{char}(k) = 0$. *Let* I *be a graded ideal in* S *and* $R = S/I$. *Fix integers* i *and* j. *Let* $a = \min\{\dim_k((R/g)_i) \mid g \in R_j\}$. *Then* $\dim_k((R/h)_i) = a$ *for a generic* j*-form* h.

In Exercise 44.8 and Green's Theorem 44.9 by a generic j-form, we mean a j-form generic in the sense of Exercise 44.7.

Exercise 44.8. *Assume* $\mathrm{char}(k) = 0$. *Fix an integer* j. *Then* x_n^j *is a generic* j*-form for every Borel ideal in* S.

The following result is a straightforward corollary of Theorem 44.5 and Exercise 44.8.

Green's Theorem 44.9. (Herzog-Popescu), [Gasharov] *Assume that* $\mathrm{char}(k) = 0$. *Let* B_q *be a Borel monomial space in* S_q. *Let* L_q *be a lex monomial space in* S_q *with* $|L_q| \le |B_q|$. *Let* g *be a generic form of degree* $j \ge 1$. *The inequality*

$$\dim_k \Big(S_q/(L_q + g\, S_{q-j}\Big) \ge \dim_k \Big(S_q/(B_q + g\, S_{q-j}\Big)$$

holds.

Green's Hyperplane Restriction Theorem 44.10. [Green]
Assume char$(k) = 0$. *Let J be a graded ideal in S, and L be the lex ideal with the same Hilbert function as J. Let h be a generic linear form. For every $q \geq 0$ we have*

$$\dim_k \Big(S/(L,h)\Big)_q \geq \dim_k \Big(S/(J,h)\Big)_q .$$

Proof. Assume that we work in generic coordinates, so we can take $x_n = h$. Note that when we take the initial ideal with respect to revlex order we get $\mathrm{in}(J, x_n) = (\mathrm{in}(J), x_n)$. Therefore, we can replace J by $B = \mathrm{in}(J)$. By Theorem 28.4, the ideal B is Borel. Hence, Theorem 44.5 yields the desired result. □

Green's Theorem holds without the restriction char$(k) = 0$, see [Gasharov].

45 Proofs of Macaulay's Theorem

We are ready to prove Macaulay's Theorem 41.7; namely, we will prove that (1) in Proposition 41.6 holds. It is straightforward that (1) holds if $n \leq 2$. Consider the case $n \geq 3$. Applying Lemma 43.10, we conclude that there exist a (2)-multicompressed monomial space C_q such that $|C_q| = |A_q|$ and $|S_1 C_q| \leq |S_1 A_q|$. By Theorem 43.7 it follows that C_q is Borel. Let L_q be the lex monomial space for which $|C_q| = |L_q|$. We will prove that $|S_1 L_q| \leq |S_1 C_q|$.

We will present two different proofs. The former uses Green's Theorem. The latter uses the structure theorem for compressed ideals. A third proof by induction is given in [Mermin-Peeva].

Proof.
First Proof. This proof uses Green's Theorem. The monomial space C_q is Borel. For an S_q-monomial space D_q recall that $t_i(D_q) = \Big|\{ m \in \{D_q\} \mid \max(m) \leq i \}\Big|$. We apply Lemma 44.1 to conclude that

$$\Big|\{S_1 C_q\}\Big| = \sum_{i=1}^{n} t_i(C_q) \quad \text{and} \quad \Big|\{S_1 L_q\}\Big| = \sum_{i=1}^{n} t_i(L_q) .$$

Finally, we apply Theorem 44.4 and get $\left|\,\{S_1L_q\}\,\right| \le \left|\,\{S_1C_q\}\,\right|$.

Second Proof. (Mermin) This proof is by compression. Let $n > 3$. Applying Lemma 43.10, we conclude that there exist a (3)-multicompressed monomial space C_q such that $|C_q| = |A_q|$ and $|S_1C_q| \le |S_1A_q|$. By Theorem 43.7 it follows that C_q is lex, and we are done.

Suppose that $n = 3$. Let L_q be the lex S_q-monomial space such that $|L_q| = |C_q|$. As both L_q and C_q are Borel, we have

$$|S_1C_q| = |C_q| + |C_q \cap k[x_1, x_2]| + |C_q \cap k[x_1]|$$
$$|S_1L_q| = |L_q| + |L_q \cap k[x_1, x_2]| + |L_q \cap k[x_1]|$$

by Lemma 44.1. Note that $|L_q| = |C_q|$ by construction, and $|C_q \cap k[x_1]| = |L_q \cap k[x_1]| = 1$ as $\{C_q \cap k[x_1]\} = \{L_q \cap k[x_1]\} = x_1^q$. Therefore, we need to prove that $|L_q \cap k[x_1, x_2]| \le |C_q \cap k[x_1, x_2]|$. We will show that if a monomial $v \in L_q$ is not in C_q, then $v \notin k[x_1, x_2]$. Assume the opposite: let $v = x_1^a x_2^c \in L_q$ and $v \notin C_q$. As $L_q \ne C_q$ we conclude that there exists a monomial $x_1^{a'} x_2^{c'} x_3^{e'} \in C_q$ that is lex-smaller than v. Hence $a' \le a$. Since C_q is Borel, it follows that $v \in C_q$, which is a contradiction. $\square$

46 Compression ideals

Proposition 41.6 makes it possible to work in our arguments by focusing on only two consecutive degrees at a time (instead of dealing with the whole ideal). In this section we show that the compressions can be assembled into an ideal.

Construction 46.1. Fix an $1 \le i \le n$. Let A be a monomial ideal in S. For each $q \ge 0$, let C_q be the i-compression of A_q. We call $C = \oplus_{0 \le q} C_q$ the ***i-compression*** of A.

As a corollary of Macaulay's Theorem, we will prove the following result.

Proposition 46.2. *Let A be a monomial ideal in S and fix an $1 \le i \le n$. Its i-compression C is an ideal.*

Proof. We use the following notation. For each $q \geq 0$ we have a disjoint union

$$\{A_q\} = \coprod_{0 \leq j \leq q} x_i^{q-j} \{U_j^q\}$$

where each U_j^q is a monomial space in $(S/x_i)_j$. Let

$$\{C_q\} = \coprod_{0 \leq j \leq q} x_i^{q-j} \{L_j^q\}$$

be the i-compression of A_q. Thus, L_j^q is the lexification of U_j^q in S/x_i. The i-compression of A is $C = \oplus_{0 \leq q} C_q$.

Fix a $q \geq 0$ and a $0 \leq j \leq q$. Let $m \in x_i^{q-j} L_j$ be a monomial. We will prove that $S_1 m \in C$.

We will show that $(S_1/x_i)L_j^q \subseteq L_{j+1}^{q+1}$. Both $(S_1/x_i)L_j^q$ and L_{j+1}^{q+1} are lex monomial spaces. So, in order to show that $(S_1/x_i)L_j^q \subseteq L_{j+1}^{q+1}$ it suffices to show that $|(S_1/x_i)L_j^q| \leq |L_{j+1}^{q+1}|$. This first inequality below follows from Macaulay's Theorem, and the second inequality holds since A is an ideal:

$$|(S_1/x_i)L_j^q| \leq |(S_1/x_i)U_j^q| \leq |U_{j+1}^{q+1}| = |L_{j+1}^{q+1}|\,.$$

Since $(S_1/x_i)L_j^q \subseteq L_{j+1}^{q+1}$, it follows that $(S_1/x_i)m \in C$.

It remains to prove that $x_i m \in C$. We will show that $L_j^q \subseteq L_j^{q+1}$. Both L_j^q and L_j^{q+1} are lex monomial spaces in $(S/x_i)_j$. So, in order to show that $L_j^q \subseteq L_j^{q+1}$ it suffices to show that $|L_j^q| \leq |L_j^{q+1}|$. Since A is an ideal, we have that $U_j^q \subseteq U_j^{q+1}$. Hence

$$|L_j^q| = |U_j^q| \leq |U_j^{q+1}| = |L_j^{q+1}|.$$

The inclusion $L_j^q \subseteq L_j^{q+1}$ implies that $x_i m \in C$. $\square$

We will see that the situation is similar for multicompression.

Construction 46.3. Fix a set $\mathcal{A} \subset \{x_1, \ldots, x_n\}$. An S_q-monomial space A_q can be written uniquely in the form

$$A_q = \bigoplus_m m\,U_m,$$

where U_m is a monomial space in the ring $k[\mathcal{A}] = k[x_i \,|\, x_i \in \mathcal{A}]$. For each m, let L_m be the lexification of the space U_m in $k[\mathcal{A}]$. The monomial space C_q defined by

$$C_q = \bigoplus_m m\,L_m$$

is the ***$\mathcal{A}$-compression*** of A_q. Clearly, $|C_q| = |A_q|$.

Let A be a monomial ideal in S. For each $q \geq 0$, let C_q be the $\mathcal{A}$-compression of A_q. We call $C = \oplus_{0 \leq q} C_q$ the ***$\mathcal{A}$-compression*** of A.

The following result can be proved similarly to Proposition 46.2.

Proposition 46.4. *Let A be a monomial ideal in S and fix a set $\mathcal{A} \subset \{x_1, \ldots, x_n\}$. The $\mathcal{A}$-compression C of A is an ideal.*

47 Ideals with a fixed Hilbert function

The problem *"What can be said about the properties of ideals with a fixed Hilbert function?"* has received a lot of attention. Evans raised the problem to study the properties of the Betti diagrams of all graded ideals in S with a fixed Hilbert function; since the problem is very complex in general, people focused on maximal and on minimal Betti numbers. We will show that a lex ideal attains the greatest Betti numbers among all ideals with a fixed Hilbert function.

For simplicity, we assume throughout this section that $\mathrm{char}(k) = 0$. If M is a monomial ideal, then $G(M)_j$ stands for the set of monomials of degree j in the minimal system of monomial generators of M, and furthermore we denote by $|G(M)_j|$ the number of monomials in $G(M)_j$.

Let J be a graded ideal in S. By Macaulay's Theorem 41.7, there exists a lex ideal L with the same Hilbert function as J. The next result follows by Proposition 41.6.

Proposition 47.1. *For every $j \geq 0$, the number of elements of degree j in a minimal system of homogeneous generators of J is $\leq |G(L)_j|$.*

This property extends to all graded Betti numbers as follows.

Theorem 47.2. (Bigatti, Hulett) *Assume* $\operatorname{char}(k) = 0$*. Let J be a graded ideal in S. If L is the lex ideal with the same Hilbert function as J, then*

$$b^S_{i,i+j}(J) \leq b^S_{i,i+j}(L) \quad \textit{for all } i, j.$$

Remark 47.3. It is proved in [Pardue] that Theorem 47.2 holds without the assumption $\operatorname{char}(k) = 0$.

Note that the minimal free resolution and the Betti numbers of a lex ideal are given by the Eliahou-Kervaire resolution, see Section 28.

Recall that for an S_q-monomial space A_q we set

$$t_i(A_q) = \Big| \{ m \in \{A_q\} \mid \max(m) \leq i \} \Big| .$$

Set

$$u_i(A_q) = t_i(A_q) - t_{i-1}(A_q) = \Big| \{ m \in \{A_q\} \mid \max(m) = i \} \Big| .$$

Lemma 47.4. *If M is a Borel ideal in S, then $b^S_{i,i+j}(M)$ is equal to*

$$|\, M_j \,| \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(M_j) \binom{p-1}{i-1} - \sum_{p=1}^{n} t_p(M_{j-1}) \binom{p-1}{i} .$$

Proof. By Corollary 28.12, we have that

$$b^S_{i,i+j}(M) = \sum_{m \in G(M)_j} \binom{\max(m) - 1}{i} = \sum_{p=1}^{n} u_p(G(M)_j) \binom{p-1}{i} .$$

Since

$$G(M)_j = \{M_j\} \ \setminus \ \{S_1\, M_{j-1}\}$$

we obtain

$$b^S_{i,i+j}(M) = \sum_{p=1}^{n} u_p(M_j)\binom{p-1}{i} - \sum_{p=1}^{n} u_p(S_1\, M_{j-1})\binom{p-1}{i}.$$

Furthermore, since M is Borel, by Lemma 44.1 we have

$$\{S_1\, M_{j-1}\} = \coprod_{p=1}^{n} x_p\{m \in \{M_{j-1}\} \mid \max(m) \leq p\}\,,$$

and hence $u_p(S_1\, M_{j-1}) = t_p(M_{j-1})$. Therefore,

$$\begin{aligned} &b^S_{i,i+j}(M) \\ &= \sum_{p=1}^{n} \Big(t_p(M_j) - t_{p-1}(M_j)\Big)\binom{p-1}{i} - \sum_{p=1}^{n} t_p(M_{j-1})\binom{p-1}{i} \\ &= |\, M_j\,| \binom{n-1}{i} - \sum_{p=1}^{n-1} t_p(M_j)\binom{p-1}{i-1} - \sum_{p=1}^{n} t_p(M_{j-1})\binom{p-1}{i}. \end{aligned}$$

□

Proof of Theorem 47.2. We will present the proof in [Chardin-Gasharov-Peeva]. Let M be the generic initial ideal of J with respect to a fixed term order (say, revlex). It is Borel, by Theorem 28.4. Thus, there exists a Borel ideal M with the same Hilbert function as J such that

$$b^S_{i,i+j}(J) \leq b^S_{i,i+j}(M) \quad \text{for all } i, j.$$

Both M and L are Borel ideals. Use the formula for the Betti numbers in Lemma 47.4 and apply 44.4 to obtain the inequalities

$$b^S_{i,i+j}(M) \leq b^S_{i,i+j}(L).$$ □

Theorem 47.5. *There is an upper bound on the regularities of all graded ideals with a fixed Hilbert function.*

Proof. By Remark 47.3, it follows that the regularity of the lex ideal with that Hilbert function is the smallest upper bound. □

Problem 47.6. [Geramita-Harima-Shin] *Does there exist an ideal that has greatest graded Betti numbers among all Gorenstein artinian graded ideals with a fixed Hilbert function?*

Next, we will discuss the following question: *Assume* $\mathrm{char}(k) = 0$. *Let J be a graded ideal in S and let L be the lex ideal with the same Hilbert function. How do the graded Betti numbers of J and L differ?* We would like to obtain more precise information than Theorem 47.2.

The Hilbert function can be computed from the graded Betti numbers by Theorem 16.2 and we get

$$\begin{array}{c}\displaystyle\sum_{j=0}^{\infty} \dim_k (S/J)_j\, t^j = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i b_{i,j}^S (S/J)\, t^j}{(1-t)^n} \\ \| \\ \displaystyle\sum_{j=0}^{\infty} \dim_k (S/L)_j\, t^j = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i b_{i,j}^S (S/L)\, t^j}{(1-t)^n}\,.\end{array}$$

These equalities imply that the graded Betti numbers $b_{i,j}^S(S/J)$ and $b_{i,j}^S(S/L)$ are related as described below.

Given a sequence of numbers $\{c_{p,q}\}$, we obtain a new sequence by a ***cancellation*** as follows: fix a q, and choose $p < t$ so that one of the numbers is odd and the other is even; then replace $c_{p,q}$ by $c_{p,q} - 1$, and replace $c_{t,q}$ by $c_{t,q} - 1$. The equalities above imply that the graded Betti numbers $b_{i,j}^S(S/J)$ are related to the graded Betti numbers $b_{i,j}^S(S/L)$ by a sequence of cancellations. This has been observed and applied in order to study the Betti diagrams of ideals with a fixed Hilbert function. Recall the definition of a consecutive cancellation in Section 22.

Theorem 47.7. [Peeva 2] *Let J be a graded ideal and L be the lex ideal in S with the same Hilbert function. The graded Betti numbers*

$b^S_{i,j}(S/J)$ can be obtained from the graded Betti numbers $b^S_{i,j}(S/L)$ by a sequence of consecutive cancellations.

Extending Hartshorne's method [Hartshorne] Pardue proved the next result. There, by a sequence of deformations we mean a composition of deformations (applied one after another).

Theorem 47.8. [Pardue] *Every two graded ideals in a polynomial ring with the same Hilbert function are connected by a sequence of deformations over $\mathbf{A}^1_k$.*

More precisely, in the notation of 47.7 Pardue proved that J and L are connected by a sequence of deformations of the following three types:
(1) generic change of coordinates
(2) deformation between an ideal and an initial ideal; see Theorem 22.8
(3) polarization and then factoring out generic hyperplane sections; more precisely, applying $\sigma'_{\mathbf{L}}$ defined in [Pardue, Section 4].

We are ready to prove Theorem 47.7.

Proof. The graded Betti numbers are preserved under (1). For (2) we apply Theorem 22.9. By Theorem 21.10, we have that (3) preserves the graded Betti numbers as well. □

Theorem 47.7 can be used in order to prove that certain Hilbert functions are not attained within a given class of ideals.

It should be noted that the there are many examples where the existence of possible consecutive cancellations does not imply the existence of an ideal for which those cancellations are realized.

Corollary 47.9. *Let L be a lex ideal. Suppose that L does not have two minimal monomial generators in consecutive degrees. If J is a graded ideal with the same Hilbert function as L, then J has the same graded Betti numbers as L.*

The following can be explored.

Open-Ended Problem 47.10. (folklore) *Let J be a graded ideal in*

S and let L be the lex ideal with the same Hilbert function. Which consecutive cancellations occur as cancellations when we are comparing the graded Betti numbers of L and J, in the case when some additional properties of J (e.g. monomial, artinian, Gorenstein, compressed) are assumed?

Example 47.11. In contrast to Theorem 47.2, there exist examples where no ideal attains smallest Betti numbers among the ideals with a fixed Hilbert function. The following examples are proved in [Dodd-Marks-Meyerson-Richert] and were noted by Gelvin-LaVictore-Reed-Richert. Let

$$J = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_3x_6, x_4x_5)\,.$$

Then:

(1) Among the graded ideals with the same Hilbert function as J, there exists no ideal with smallest Betti numbers.
(2) Among the squarefree monomial ideals with the same Hilbert function as J, there exists no ideal with smallest Betti numbers.

48 Gotzmann's Persistence Theorem

Gotzmann's Persistence Theorem is a major result on Hilbert functions. It shows that once an ideal achieves minimal growth then it grows minimally forever after.

Gotzmann's Persistence Theorem 48.1. (Gotzmann) *Let J be a graded ideal in S, and L be the lex ideal with the same Hilbert function as J. Suppose that q is an integer such that the following two conditions are satisfied:*
(1) *J is generated in degrees $\leq q$.*
(2) $\dim_k(J_{q+1}) = \dim_k(S_1L_q)$.
We have that

$$\dim_k(J_{q+i}) = \dim_k(S_iL_q)$$

for all $i \geq 1$. Equivalently, L is generated in degrees $\leq q$.

Proof. The proof is from [Gahsarov-Murai-Peeva 2]. It uses consecutive cancellations. Assumption (2) means that L has no minimal generator in degree $q+1$. We will show that L has no minimal monomial generator in degree $q+2$. Assume the opposite, then we have that $b^S_{1,q+2}(S/L) \neq 0$. On the other hand, we know that J does not have a minimal monomial generator in degree $q+2$, so $b^S_{1,q+2}(S/J) = 0$. Since $b^S_{1,q+2}(S/J) = 0$ is obtained from $b^S_{1,q+2}(S/L) \neq 0$ by consecutive cancellations by Theorem 47.7, it follows that $b^S_{2,q+2}(S/L) \neq 0$.

The ideal L is Borel, so the minimal free resolution of S/L is the Eliahou-Kervaire resolution 28.6. Since L does not have a minimal monomial generator in degree $q+1$, it follows that $b^S_{2,q+2}(S/L) = 0$. This is a contradiction.

We proved that L does not have a minimal monomial generator in degree $q+2$. Therefore, $\dim_k(J_{q+2}) = \dim_k(S_1 L_{q+1})$. The theorem holds by induction on degree. $\square$

Example 48.2. Consider the ideal $J = (y^2, z^2)$ in $A = k[x, y, z]$. We will compute the lex ideal L with the same Hilbert function as J. The k-vector space J_2 has basis y^2, z^2. Hence the k-vector space L_2 has basis x^2, xy. The k-vector space J_3 has basis y^3, y^2x, y^2z, z^2x, z^2y, z^3, so it is 6-dimensional. Therefore, L_3 has basis x^3, x^2y, x^2z, xy^2, xyz, xz^2. So far we have found that the lex ideal has minimal generators x^2, xy, xz^2.

The k-vector space $(A/J)_4$ has basis x^4, x^3y, x^3z, x^2yz. Hence the k-vector space $(A/L)_4$ has basis y^3z, y^2z^2, yz^3, z^4. Therefore, L_4 is spanned by A_1L_3 and y^4.

In degree 5, the k-vector space $(A/J)_5$ has basis x^5, x^4y, x^4z, x^3yz. Hence, the k-vector space $(A/L)_5$ has basis $y^3z^2, y^2z^3, yz^4, z^5$. Therefore, L_5 is spanned by A_1L_4. Thus, L has no minimal generators in degree 5.

By Gotzmann's Persistence Theorem 48.1 it follows that $L = (x^2, xy, xz^2, y^4)$.

Gotzmann's Regularity Theorem 48.3. *Let J be a graded ideal in S. Let q be an integer such that the following two conditions are satisfied:*

(1) *J is generated in degrees $\leq q$.*

(2) $\dim_k(J_{q+1}) = \dim_k(S_1 L_q)$.
Then $\operatorname{reg}_S(J) \leq q$.

Proof. By Remark 47.3, Corollary 28.13, and Theorem 48.1 we get $\operatorname{reg}_S(J) \leq \operatorname{reg}_S(L) \leq q$. □

Let J be a graded ideal in S, and L be the lex ideal with the same Hilbert function as J. We say that J is a ***Gotzmann ideal*** if the equality

$$\dim_k(S_1 J_q) = \dim_k(S_1 L_q)$$

holds for every $q \geq 0$.

Exercise 48.4. *Let J be a graded ideal in S, and L be the lex ideal with the same Hilbert function as J. The ideal J is Gotzmann if and only if J and L have the same number of minimal generators.*

Theorem 48.5. (Herzog-Hibi) *Let J be a graded Gotzmann ideal in S, and L be the lex ideal with the same Hilbert function as J. We have equalities of graded Betti numbers*

$$b^S_{i,j}(S/J) = b^S_{i,j}(S/L) \quad \textit{for all } i, j \geq 0\,.$$

Proof. Let p be the smallest degree in which L has a minimal monomial generator. For $q \geq p$, denote by $J(q)$ the ideal generated by all monomials in J of degree $\leq q$. Similarly, denote by $L(q)$ the ideal generated by all monomials in L of degree $\leq q$. By Gotzmann's Persistence Theorem 48.1, for each $q \geq p$ the ideals $J(q)$ and $L(q)$ have the same Hilbert function. Furthermore, by Remark 47.3 it follows that the graded Betti numbers of $S/L(q)$ are greater or equal to those of $S/J(q)$.

All Betti numbers in the proof are over S. By Theorem 16.2 the graded Betti numbers $b_{i,j}(S/T)$ for a homogeneous ideal T and its Hilbert function are related by

$$\sum_{j=0}^{\infty} \dim_k(S/T)_j\, t^j = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i b_{i,j}(S/T)\, t^j}{(1-t)^n}\,.$$

Therefore, for each $q \geq p$ we have that

$$(*) \qquad \sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \Big(b_{i,j}(S/J(q)) - b_{i,j}(S/L(q)) \Big) t^j = 0 \, .$$

By induction on q we will show that the graded Betti numbers of $S/L(q)$ are equal to those of $S/J(q)$.

First, consider the case when $q = p$. By the Eliahou-Kervaire resolution, it follows that $L(p)$ has a p-linear minimal free resolution, that is, $b_{i,j}(S/L(p)) = 0$ for $j \neq i + p - 1$. Since the graded Betti numbers of $S/L(p)$ are greater or equal to those of $S/J(p)$, it follows that $b_{i,j}(S/J(p)) = 0$ for $j \neq i+p-1$. By $(*)$ we obtain the equalities of graded Betti numbers

$$b_{i,j}(S/J(p)) = b_{i,j}(S/L(p)) \quad \text{for all } i, j \, .$$

Suppose that the claim is proved for q. Now, we consider the ideals $L(q+1)$ and $J(q+1)$. For $j < i + q$, we have that

$$b_{i,j}(S/L(q+1)) = b_{i,j}(S/L(q)) = b_{i,j}(S/J(q)) \, ,$$

where the first equality follows from the Eliahou-Kervaire resolution and the second equality holds by induction hypothesis. As $J(q+1)_s = J(q)_s$ for $s \leq q$ by construction, and since $b_{i,j}(S/J(q)) = 0$ for $j \geq i + q$, we conclude that $b_{i,j}(S/J(q+1)) = b_{i,j}(S/J(q))$ for $j < i + q$. Hence,

$$\begin{aligned} &b_{i,j}(S/L(q+1)) = b_{i,j}(S/J(q+1)) \text{ for } j < i + q \\ &b_{i,j}(S/L(q+1)) = 0 \text{ for } j > i + q, \text{ by the Eliahou-Kervaire resolution.} \end{aligned}$$

Since the graded Betti numbers of $S/L(q+1)$ are greater or equal to those of $S/J(q+1)$, we conclude that

$$\begin{aligned} &b_{i,j}(S/J(q+1)) = b_{i,j}(S/L(q+1)) \text{ for } j < i + q \\ &b_{i,j}(S/J(q+1)) = b_{i,j}(S/L(q+1)) = 0 \text{ for } j > i + q \, . \end{aligned}$$

By $(*)$ it follows that

$$\sum_{i=0}^{n}(-1)^i\Big(b_{i,i+q}(S/J(q))-b_{i,i+q}(S/L(q))\Big)t^{i+q}=0\,.$$

Hence

$$b_{i,j}(S/J(q+1))=b_{i,j}(S/L(q+1))\quad\text{for all } i,j\,,$$

as desired. □

Let J be a graded ideal in S. We say that J is ***componentwise linear*** if for every $q\geq 0$ the ideal generated by J_q has a q-linear minimal free resolution. By Theorem 48.5, we have that a Gotzmann ideal is componentwise linear.

49 Numerical versions

Since lex ideals are highly structured, it is easy to derive the inequalities characterizing their Hilbert functions. As an application, we discuss numerical versions of some results proved earlier.

Note that by convention $\binom{a}{b}=0$ if $a<b$.

Lemma 49.1. *Let q be a positive integer. For every $p\in\mathbf{N}$ there exist numbers $s_q>\ldots>s_1\geq 0$ such that*

$$p=\binom{s_q}{q}+\binom{s_{q-1}}{q-1}+\ldots+\binom{s_1}{1}.$$

Proof. The proof is by induction. Set

$$s_q=\max\left\{\,j\;\middle|\;\binom{j}{q}\leq p\,\right\}.$$

If $p=\binom{s_q}{q}$, then set $s_i=i-1$ for each $1\leq i<q$. Suppose that $p-\binom{s_q}{q}>0$. By induction, we can find $s_{q-1}>\ldots>s_1\geq 0$ such that

$$p-\binom{s_q}{q}=\binom{s_{q-1}}{q-1}+\ldots+\binom{s_1}{1}.$$

It remains to show that $s_q > s_{q-1}$. Assume the opposite. Then we obtain

$$\begin{aligned}\binom{s_{q-1}}{q-1} \geq \binom{s_q}{q-1} &= \binom{s_q+1}{q} - \binom{s_q}{q} \\ &> p - \binom{s_q}{q} = \binom{s_{q-1}}{q-1} + \ldots + \binom{s_1}{1},\end{aligned}$$

which is a contradiction. $\square$

This is called the q'th ***Macaulay representation*** of p. The numbers $s_1, \ldots, s_q$ are called the q'th ***Macaulay coefficients*** of p.

Example 49.2. The 3'rd Macaulay representation of 14 is

$$14 = \binom{5}{3} + \binom{3}{2} + \binom{1}{1}.$$

Exercise 49.3. *The q'th Macaulay coefficients of p are unique.*

Set $0^{\langle q \rangle} = 0$ and

$$p^{\langle q \rangle} = \binom{s_q+1}{q+1} + \binom{s_{q-1}+1}{q-1+1} + \ldots + \binom{s_1+1}{1+1}.$$

Proposition 49.4. *Let L be an ideal generated by a lex segment in S_q. If $p = \dim_k (S/L)_q$, then $\dim_k (S/L)_{q+1} = p^{\langle q \rangle}$.*

Proof. Set $j = \min\{i | \, x_i^q \notin L\}$. We have that the monomials

$$\{ u \in S_q \text{ is a monomial } | \, u \preceq_{lex} x_j^q \} = k[x_j, \ldots, x_n]_q$$

are non-zero monomials in $(S/L)_q$. The number of such monomials is

$$\dim_k k[x_j, \ldots, x_n]_q = \binom{n-j+q}{q} = \binom{s_q}{q},$$

where $s_q = n - j + q$. Furthermore,

$$(x_j, \ldots, x_n)\{ u \, | \, u \preceq_{lex} x_{j-1}^q \} = k[x_j, \ldots, x_n]_{q+1}$$

are non-zero monomials in $(S/L)_{q+1}$. The number of such monomials is

$$\dim_k k[x_j,\ldots,x_n]_{q+1} = \binom{n-j+q+1}{q+1} = \binom{s_q+1}{q+1}.$$

Let m be the lex-greatest monomial in S_q but not in L. Hence $x_{j-1}^q \in L$ and $x_{j-1}^q \succ_{lex} m \succ_{lex} x_j^q$. Set

$$\mathcal{D} = \{\, u \in \{S_q\} \,|\, m \succeq_{lex} u \succ_{lex} x_j^q \,\}$$
$$\mathcal{F} = \{\, u \in \{S_q\} \,|\, x_j^q \succeq_{lex} u \,\}.$$

All monomials in $\mathcal{D}$ are divisible by x_{j-1}, so we can write $\mathcal{D} = x_{j-1}\mathcal{D}'$. Now,

$$\dim_k (S/L)_q = |\mathcal{D}'| + |\mathcal{F}|$$
$$\dim_k (S/L)_{q+1} = |(x_j,\ldots,x_n)_1\,\mathcal{D}'| + |(x_j,\ldots,x_n)_1\,\mathcal{F}|.$$

We showed that $|\mathcal{F}| = \binom{s_q}{q}$ and $|(x_j,\ldots,x_n)_1\,\mathcal{F}\,| = \binom{s_q+1}{q+1}$. By induction on the degree, we have that

$$|\mathcal{D}'| = \binom{s_{q-1}}{q-1} + \ldots + \binom{s_1}{1}$$

and

$$|(x_j,\ldots,x_n)_1\,\mathcal{D}'| = \binom{s_{q-1}+1}{q} + \ldots + \binom{s_1+1}{1+1}.$$

In order to finish the proof, we need to verify that $s_q,\ldots,s_1$ are the Macaulay coefficients, that is, we have to verify that $s_q > \ldots > s_1$. The inequalities $s_{q-1} > \ldots > s_1$ hold by induction. So we have to check that $s_q > s_{q-1}$.

We have that $s_q = n-j+q$, where $j = \min\{i|\, x_i^q \notin L\,\}$. Similarly, $s_{q-1} = n-c+(q-1)$, where

$$\begin{aligned} c &= \min\{i\,|\, x_i^{q-1} \notin \mathcal{D}'\,\} = \min\{i\,|\, x_{j-1}x_i^{q-1} \notin \mathcal{D}\,\} \\ &= \min\{i\,|\, x_{j-1}x_i^{q-1} \notin L,\, i \geq j\,\}. \end{aligned}$$

Therefore, $c \geq j$. Hence $s_{q-1} = n - c + q - 1 < n - j + q = s_q$. $\square$

The above proposition and Macaulay's Theorem imply the following result.

Numerical Version of Macaulay's Theorem 49.5. *Let J be a graded ideal in S. Then*

$$\dim_k (S/J)_{j+1} \leq (\dim_k (S/J)_j)^{\langle j \rangle} \quad \textit{for} \;\; j \geq 0 .$$

Similarly, the above proposition and Gotzmann's Persistence Theorem 48.1 imply the following result.

Numerical Version of Gotzmann's Theorem 49.6. *Let J be a graded ideal in S. Let q be an integer such that the following two conditions are satisfied:*

(1) *J is generated in degrees $\leq q$.*

(2) $\dim_k (S/J)_{q+1} = \big(\dim_k (S/J)_q\big)^{\langle q \rangle}$.

Then

$$\dim_k (S/J)_{j+1} = \big(\dim_k (S/J)_j\big)^{\langle j \rangle} \quad \textit{for } j \geq q .$$

Numerical Version of Gotzmann's Regularity Theorem 49.7. *Let J be a graded ideal in S. Let q be an integer such that the following two conditions are satisfied:*

(1) *J is generated in degrees $\leq q$.*

(2) $\dim_k (S/J)_{q+1} = \big(\dim_k (S/J)_q\big)^{\langle q \rangle}$.

Then $\mathrm{reg}_S(J) \leq q$.

Let $\alpha = \{ \alpha_0, \alpha_1, \ \ldots \}$ be a sequence of non-negative integer numbers. We say that α is a ***Macaulay sequence*** if $\alpha_0 = 1$ and $\alpha_{q+1} \leq \alpha_q^{\langle q \rangle}$ for each $q \geq 1$.

Corollary 49.8. *Let $\alpha = \{ \alpha_0 = 1, \ \alpha_1 \leq n, \ \alpha_2, \ldots \}$ be a sequence of non-negative integer numbers. There exists a graded ideal J in S with* $\dim_k (S/J)_i = \alpha_i$ *for all $i \geq 0$, if and only if, α is a Macaulay sequence.*

Proof. Note that $\alpha_0 = 1$ as $S_0 = k$. Furthermore, $\alpha_1 \leq n$ since $\dim_k S_1 = n$.

Suppose that there exists a graded ideal J in S with $\dim_k (S/J)_i = \alpha_i$ for all $i \geq 0$. By Macaulay's Theorem, there exists a lex ideal L with $\dim_k (S/L)_i = \alpha_i$ for all $i \geq 0$. Applying Proposition 49.4 we conclude that if L has no minimal monomial generators in degree $q+1$ then we have the equality $\alpha_{q+1} = \alpha_q^{\langle q \rangle}$, and otherwise we have the inequality $\alpha_{q+1} \leq \alpha_q^{\langle q \rangle}$.

Suppose that α is a Macaulay sequence. Let L_q be the lex segment in S_q such that $\dim_k (S/L)_q = \alpha_q$. By Proposition 49.4, it follows that $L_{q+1} \supseteq S_1 L_q$. Hence $L = \oplus_{q \geq 0} L_q$ is an ideal. It has the desired Hilbert function. $\square$

Corollary 49.9. *Let J be a graded ideal in S. The Hilbert polynomial of S/J has the form*

$$h_{S/J}(t) = \binom{t + a_q}{a_q} + \binom{t + a_{q-1}}{a_{q-1}} + \ldots + \binom{t + a_1}{a_1}.$$

for some $a_q \geq \ldots \geq a_1 \geq 0$.

Proof. Let L be the lex ideal with the same Hilbert function as J. Let q be the maximal degree in which L has a minimal monomial generator. Denote by N the ideal generated by L_q. It follows that $\dim_k (J_i) = \dim_k (N_i)$ for $i \geq q$. Hence S/J and S/N have the same Hilbert polynomial.

Let $s_q > \ldots > s_1 \geq 0$ be the Macaulay's coefficients of the q'th Macaulay representation of the number $\dim_k (S/N)_q$.

By Proposition 49.4, it follows that the Hilbert polynomial of S/N is

$$h_{S/N}(q+t) = \binom{s_q + t}{q + t} + \binom{s_{q-1} + t}{q - 1 + t} + \ldots + \binom{s_1 + t}{1 + t}.$$

Set $a_i = s_i - i$ for each i. Hence

$$h_{S/N}(q+t) = \binom{t + q + a_q}{q + t} + \binom{t + q - 1 + a_q}{q - 1 + t} + \ldots + \binom{t + 1 + a_1}{1 + t}.$$

Therefore,

$$\begin{aligned}h_{S/N}(t) =&\binom{t+a_q}{t} + \binom{t+a_{q-1}}{t} + \ldots + \binom{t+a_1}{t}\\ =&\binom{t+a_q}{a_q} + \binom{t+a_{q-1}}{a_{q-1}} + \ldots + \binom{t+a_1}{a_1}.\end{aligned}$$

□

Corollary 49.10. *Suppose that the field k is infinite. Let*

$$g(t) = a_r t^r + \ldots + a_1 t + a_0$$

$g(1) \neq 0$, and $a_i \in \mathbf{Z}$ for all i. There exists a $p \in \mathbf{N}$ such that $\dfrac{g(t)}{(1-t)^p}$ is equal to $\mathrm{Hilb}_{S/J}(t)$ for some graded Cohen-Macaulay J if and only if a_0, $a_1 \leq n$, $\ldots$, a_r is a Macaulay sequence of positive numbers.

Proof. Let $\mathrm{Hilb}_{S/J}(t) = \dfrac{g(t)}{(1-t)^{\dim(S/J)}}$. If S/J is Cohen-Macaulay, then by 20.1 there exists a regular sequence of linear forms of length $\dim(S/J)$. Hence, $g(t)$ is the Hilbert series of an artinian graded quotient of S. Therefore, $a_0, a_1 \leq n, \ldots, a_r$ is a Macaulay sequence of positive numbers.

On the other hand, suppose that $a_0, a_1 \leq n, \ldots, a_r$ is a Macaulay sequence of positive numbers. Therefore, there exists an artinian graded quotient S/J of S with Hilbert series $g(t)$. □

The following problems have been studied, cf. [Valla].

Problems 49.11.

(1) *Characterize the Hilbert functions of graded artinian Gorenstein quotients of S.*

(2) *Characterize the Hilbert functions of graded Cohen-Macaulay domains that are quotients of S.*

(3) *Characterize the Hilbert functions of sets of points in uniform position.*

A problem of this type is also the Eisenbud-Green-Harris Conjecture, discussed in Section 53. Another conjecture of this type is

Fröberg's conjecture.

Fröberg's Conjecture 49.12. (Fröberg) *Let $f_1, \ldots, f_r$ be generic forms in S of degrees $a_1, \ldots, a_r$, and let $T = (f_1, \ldots, f_r)$. The Hilbert series of S/T is*

$$\mathrm{Hilb}_{S/T}(t) = \left| \frac{\prod_{1 \le i \le r} (1 - t^{a_i})}{(1-t)^n} \right| ,$$

where $|\ |$ means that a term $c_i t^i$ in the series is omitted if there exists a term $c_j t^j$ with $j \le i$ and negative coefficient c_j. (Here $r > n$ is the interesting case, since for $r \le n$ we have that $f_1, \ldots, f_r$ is a regular sequence.)

Set $0_{\langle q \rangle} = 0$ and

$$p_{\langle q \rangle} = \binom{s_q - 1}{q} + \binom{s_{q-1} - 1}{q-1} + \ldots + \binom{s_1 - 1}{1} .$$

Exercise 49.13. *Let L be an ideal generated by a lex segment in S_q. If $p = \dim_k (S/L)_q$, then*

$$\dim_k \big(S/(L, x_n)\big)_q = p_{\langle q \rangle} .$$

Green's Hyperplane Restriction Theorem 44.10 and 49.13 imply the next result.

Numerical Version of Green's Hyperplane Restriction Theorem 49.14. *Let J be a graded ideal in S, and h be a generic linear form. If $p = \dim_k (S/J)_q$, then*

$$\dim_k (S/(J,h))_q \le p_{\langle q \rangle} .$$

50 Hilbert functions over quotient rings

The main idea in Macaulay's Theorem is that every Hilbert function is attained by a lex ideal. One can wonder for what quotient rings this idea works out. If I is a monomial or toric ideal, then we can define the notion of a lex ideal in the quotient ring $R = S/I$. There might be other classes of rings for which one can introduce a meaningful notion of lex ideals, that is, *find a class of ideals which attain all Hilbert functions and which are defined in a nice way* (and call such ideals lex ideals).

It is easy to find quotient rings over which Macaulay's Theorem does not hold. For example, there exists no lex ideal with the same Hilbert function as the ideal (ab) in the quotient ring $k[a,b]/(a^2b, ab^2)$ by [Mermin-Peeva 2, Example 2.13]. Since the trouble is sometimes in the degrees of the minimal generators of I, it makes sense to relax the problem to Problem 50.1(1). Furthermore, in view of Hartshorne's Theorem that every graded ideal in S is connected by a sequence of deformations to a lex ideal, it is natural to raise Problem 50.1(2). Problem 50.1(3) is motivated by Theorem 47.2 which shows that a lex ideal attains the greatest graded Betti numbers among all graded ideals in S with the same Hilbert function.

Open-Ended Problems 50.1. [Mermin-Peeva, Mermin-Peeva 2]

(1) *Let p be the maximal degree of an element in a minimal homogeneous system of generators of I. Find classes of graded ideals I so that every Hilbert function over $R = S/I$ of a graded ideal generated in degrees $> p$ is attained by a lex ideal.*

(2) *Let J be a graded ideal in R, and L be a lex ideal with the same Hilbert function. When is J connected to L by a sequence of deformations? What can be said about the structure of the Hilbert scheme that parametrizes all graded ideals in R with the same Hilbert function as L?*

(3) *Let J be a graded ideal in R, and L be a lex ideal with the same Hilbert function. Find conditions on R and/or J so that the Betti numbers of J over R are less than or equal to those of L.*

Furthermore, one can also ask for generalizations or extensions of the Gotzmann's Persistence Theorem and the Lex-Plus-Powers Conjecture.

Open-Ended Problem 50.2. (Peeva) *Find classes of graded ideals*

I so that Gotzmann's Persistence Theorem holds over R.

Open-Ended Problem 50.3. [Mermin-Peeva] *Let J be a graded ideal in R, and L be a lex ideal with the same Hilbert function in R. Denote by $\widetilde{J}$ and $\widetilde{L}$ the preimages of J and L in S. Find conditions on R and/or J so that the Betti numbers of $\widetilde{J}$ over S are less than or equal to those of $\widetilde{L}$.* (We say that $\widetilde{L}$ is a ***lex-plus-I*** ideal.)

As we have seen in Section 51, Hilbert functions over an exterior algebra coincide with f-vectors of simplicial complexes. The situation over an exterior algebra is well-studied and we have the following results.

Theorem 50.4. *Let E be a standard graded exterior algebra on n variables of degree one.*

(1) (Kruskal-Katona) *For every graded ideal in E there exists a lex ideal with the same Hilbert function.*
(2) [Peeva-Stillman 3] *The Hilbert scheme, that parametrizes all graded ideals with a fixed Hilbert function, is connected. Each graded ideal in E is connected by a sequence of deformations to the lex ideal with the same Hilbert function.*
(3) [Aramova-Herzog-Hibi 2] *Each lex ideal in E attains maximal Betti numbers among all graded ideals with the same Hilbert function.*
(4) [Mermin-Peeva-Stillman] *Each lex-plus-$(x_1^2,\dots,x_n^2)$ ideal in S attains maximal Betti numbers among all graded ideals containing $(x_1^2,\dots,x_n^2)$ and with the same Hilbert function.*
(5) [Aramova-Herzog-Hibi 2] *Gotzmann's Persistence Theorem holds over E.*

51 Squarefree ideals plus squares

In this section, we study how the Hilbert function and the minimal free resolution change when we add the squares of the variables to a squarefree monomial ideal. This relates to (4) in Theorem 50.4.

Throughout the section Δ is a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. Set $c = \dim(\Delta) + 1$.

The ***Stanley-Reisner ideal*** (in S) of Δ is

$$I_\Delta \;=\; (x_{i_1} \ldots x_{i_p} \,|\, \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta\,)\,.$$

Each squarefree monomial ideal in S is the Stanley-Reisner ideal of some simplicial complex on vertex set $\{x_1, \ldots, x_n\}$.

The ***Stanley-Reisner ring*** of Δ is $R_\Delta = S/I_\Delta$. The ring

$$Q_\Delta = R_\Delta/\,(x_1^2, \ldots, x_n^2) = S/(I_\Delta + (x_1^2, \ldots, x_n^2))$$

is closely related to R_Δ. We say that $I_\Delta + (x_1^2, \ldots, x_n^2)$ is a ***squarefree-plus-squares*** ideal.

First, we study how the Hilbert function changes when we add the squares of the variables to a squarefree monomial ideal.

Construction 51.1. Consider the correspondence

$$\varphi : x_{i_1} \cdots x_{i_p} \rightarrow \text{ the face with vertices } \{x_{i_1}, \ldots, x_{i_p}\}\,.$$

from the set of squarefree monomials in n variables to the faces of the simplex on n vertices. Clearly, φ is a bijection.

The f***-vector*** of Δ is $(f_{-1}, f_0, \ldots, f_{c-1})$, where f_i is the number of faces of dimension i in Δ. Note that $f_{-1} = 1$, since a simplicial complex has one empty face. The polynomial $f(t) = \sum_{0 \le i \le c} f_{i-1}\, t^i$ is called the f***-polynomial***.

Proposition 51.2.

$$\mathrm{Hilb}_{Q_\Delta}(t) = \sum_{0 \le i \le c} f_{i-1}\, t^i \;= f(t).$$

Proof. The bijection in Construction 51.1 induces the bijection

$$\psi : \text{ the monomials in } Q_\Delta = R_\Delta/\,(x_1^2, \ldots, x_n^2) \quad\longrightarrow\quad \text{the faces of } \Delta\,.$$

$\square$

Proposition 51.3.

$$\mathrm{Hilb}_{R_\Delta}(t) = \mathrm{Hilb}_{Q_\Delta}\left(\frac{t}{1-t}\right).$$

Proof. Let m be a squarefree monomial in Q_Δ of degree i. Denote $\operatorname{supp}(m) = \{x_j \,|\, x_j \text{ divides } m\}$. All monomials in S with the same support as m are monomials in R_Δ and they are exactly the monomials in $m\, k[x_j | x_j \in \operatorname{supp}(m)]$; hence they contribute $\dfrac{t^i}{(1-t)^i}$ to $\operatorname{Hilb}_{R_\Delta}(t)$. Therefore,

$$\operatorname{Hilb}_{R_\Delta}(t) = \sum_{0\le i\le c} f_{i-1}\,\frac{t^i}{(1-t)^i} = \operatorname{Hilb}_{Q_\Delta}\left(\frac{t}{1-t}\right).$$

$\square$

Theorem 51.4. *If Δ and Δ' are simplicial complexes on n vertices, then*

$$\operatorname{Hilb}_{R_\Delta}(t) = \operatorname{Hilb}_{R_{\Delta'}}(t) \qquad \Longleftrightarrow \qquad \operatorname{Hilb}_{Q_\Delta}(t) = \operatorname{Hilb}_{Q_{\Delta'}}(t)\,.$$

Theorem 51.5.

$$\dim(R_\Delta) = \dim(\Delta) + 1\,.$$

Proof. We have that

$$\operatorname{Hilb}_{R_\Delta}(t) = \sum_{0\le i\le c} f_{i-1}\frac{t^i}{(1-t)^i} = \sum_{0\le i\le c} f_{i-1}\frac{t^i(1-t)^{c-i}}{(1-t)^c}\,.$$

Set $h(t) = \sum_{0\le i\le c} f_{i-1}t^i(1-t)^{c-i}$. Then $\operatorname{Hilb}_{R_\Delta}(t) = \dfrac{h(t)}{(1-t)^c}$ and $h(1) = f_{c-1} \neq 0$. Hence $\dim(R_\Delta) = c = \dim(\Delta)+1$. $\square$

Recall that the polynomial $h(t)$ above is called the h-polynomial.

Corollary 51.6. $h(t) = (1-t)^c \cdot f\left(\dfrac{t}{1-t}\right)$.

Next, we study how the Betti numbers change when we add the squares of the variables to a squarefree monomial ideal.

Theorem 51.7. *Let N be a squarefree ideal. Set $P(i) = (x_1^2, \dots, x_i^2)$ and $P(0) = 0$. For each $0 \le i < n$, the mapping cone of the short* *exact sequence*

$$0 \to S/\big((N+P(i)) : x_{i+1}\big) \xrightarrow{x_{i+1}^2} S/\big(N+P(i)\big) \to S/\big(N+P(i+1)\big) \to 0$$

yields a minimal free resolution of $S/\big(N+P(i+1)\big)$.

This theorem shows how to obtain the Betti numbers of $N + (x_1^2, \dots, x_n^2)$ starting from the minimal free resolution of N and adding the squares one after another. At each step, we use a mapping cone.

Proof. First, note that

$$\big((N+P(i)) : x_{i+1}^2\big) = \big((N+P(i)) : x_{i+1}\big)$$

because the ideal $N+P(i)$ is squarefree on the variable x_{i+1}. Thus, the sequence above is exact.

Since the ideal $N+P(i)$ is squarefree on the variable x_{i+1}, by Taylor's resolution, it follows that the Betti numbers of $S/\big(N+P(i)\big)$ are concentrated in multidegrees not divisible by x_{i+1}^2. On the other hand, the first map in the short exact sequence is multiplication by x_{i+1}^2. Therefore, there can be no cancellations in the mapping cone. Hence, the mapping cone yields a minimal free resolution. $\square$

The disadvantage of the above theorem is that we may change the Hilbert function by adding some (but not all) of the squares. That is, if N and N' are two squarefree ideals with the same Hilbert function, then $N+P(i)$ and $N'+P(i)$ may not have the same Hilbert function. There are examples, when $N+(x_1^2)$ and $N'+(x_1^2)$ have different Hilbert functions. For example, consider the polynomial ring $k[a,b,c,e]$ and let T be the ideal generated by the squarefree cubic monomials; the ideal $N=(ab,ac,bc)+T$ is squarefree Borel and the ideal $N'=(ab,ac,ae)+T$ is squarefree lex. The ideals N and N' have the same Hilbert function, but $N+(a^2)$ and $N'+(a^2)$ have different Hilbert functions. The next theorem shows how to use mapping cones while preserving the Hilbert function.

For a $\sigma \subseteq \{x_1, \dots, x_n\}$, we set $\mathbf{x}_\sigma = \prod_{x_i \in \sigma} x_i$.

Theorem 51.8. [Mermin-Peeva-Stillman] *Let* $P = (x_1^2, \dots, x_n^2)$ *and* N *be a squarefree monomial ideal.*

(1) *We have the long exact sequence*

$$(*)\qquad \begin{aligned} 0 &\to \oplus_{|\sigma|=n}\ S/(N:\mathbf{x}_\sigma) \xrightarrow{\varphi_n} \dots \\ &\to \oplus_{|\sigma|=i}\ S/(N:\mathbf{x}_\sigma) \xrightarrow{\varphi_i} \oplus_{|\sigma|=i-1}\ S/(N:\mathbf{x}_\sigma) \to \dots \\ &\to \oplus_{|\sigma|=1}\ S/(N:\mathbf{x}_\sigma) = \oplus_{1\le j\le n} S/(N:x_j) \xrightarrow{\varphi_1} \\ &\to \oplus_{|\sigma|=0}\ S/(N:\mathbf{x}_\sigma) = S/N \to S/(N+P) \to 0 \end{aligned}$$

with maps φ_i the Koszul maps for the sequence $x_1^2,\dots,x_n^2$, and $\sigma \subseteq \{1,\dots,n\}$.

(2) *$S/(N+P)$ is minimally resolved by the iterated mapping cones from $(*)$.*

(3) *Each of the ideals $(N:\mathbf{x}_\sigma)$ in* (1) *is a squarefree monomial ideal.*

(4) *For the graded Betti numbers of $S/(N+P)$ we have*

$$b_{p,s}(S/(N+P)) = \sum_{0\le i\le p}\left(\sum_{|\sigma|=i} b_{p-i,s-2i}(S/(N:\mathbf{x}_\sigma))\right).$$

Proof. First, note that $(N:\mathbf{x}_\sigma^2) = (N:\mathbf{x}_\sigma)$ is squarefree since N is squarefree.

By Construction 14.1 and Theorem 14.7, the exact Koszul complex $\mathbf{K}$ for the sequence $x_1^2,\dots,x_n^2$ has the form

$$\begin{aligned} 0 &\to \oplus_{|\sigma|=n}\ S \xrightarrow{\varphi_n} \dots \\ &\to \oplus_{|\sigma|=i}\ S \xrightarrow{\varphi_i} \oplus_{|\sigma|=i-1}\ S \to \dots \\ &\to \oplus_{|\sigma|=1}\ S = \oplus_{1\le j\le n} S \xrightarrow{\varphi_1} \\ &\to \oplus_{|\sigma|=0}\ S = S \to S/P \to 0\,. \end{aligned}$$

Write $\mathbf{K} = \mathbf{K}' \oplus \mathbf{K}''$, where $\mathbf{K}'$ consists of the components of $\mathbf{K}$ in all multidegrees $m \notin N$, and $\mathbf{K}''$ consists of the components of $\mathbf{K}$ in all multidegrees $m \in N$. Note that both $\mathbf{K}'$ and $\mathbf{K}''$ are exact by 3.7. We will show that $(*)$ coincides with $\mathbf{K}'$.

By 14.1, $\mathbf{K}$ is an exterior algebra on variables $e_1,\dots e_n$. Let $me_{j_1}\wedge\dots\wedge e_{j_i}$ be an element in $\mathbf{K}_i$ and m be a monomial. The multidegree of the variable e_j is x_j^2; hence, the multidegree of $me_{j_1}\wedge$

$\ldots \wedge e_{j_i}$ is $mx_{j_1}^2 \ldots x_{j_i}^2$. Now, $me_{j_1} \wedge \ldots \wedge e_{j_i} \in \mathbf{K}'$ if and only if $mx_{j_1}^2 \ldots x_{j_i}^2 \notin N$, if and only if $mx_{j_1} \ldots x_{j_i} \notin N$, if and only if $m \notin (N : x_{j_1} \ldots x_{j_i})$. Therefore,

$$\mathbf{K}'_i \to \oplus_{|\sigma|=i} S/(N : \mathbf{x}_\sigma)$$
$$me_{j_1} \wedge \ldots \wedge e_{j_i} \mapsto m \in S/(N : x_{j_1} \ldots x_{j_i})$$

is an isomorphism. We proved (1).

We will prove (2). Denote by V_i the kernel of $\varphi_i : \mathbf{K}'_i \to \mathbf{K}'_{i-1}$. We have the short exact sequence

$$0 \to V_i \to \oplus_{|\sigma|=i} S/(N : \mathbf{x}_\sigma) \to V_{i-1} \to 0\,.$$

Each of the ideals $(N : \mathbf{x}_\sigma)$ is squarefree. By Corollary 26.10, the Betti numbers of $\oplus_{|\sigma|=i} S/(N : \mathbf{x}_\sigma)$ are concentrated in squarefree multidegrees. On the other hand, the entries in the matrix of the map φ_i are squares of the variables. Therefore, there can be no cancellations in the mapping cone. Hence, the mapping cone yields a minimal free resolution.

(4) follows from (2). □

Furthermore, the following result is proved in [Mermin-Peeva-Stillman].

Proposition 51.9. *Let N and N' be two squarefree monomial ideals with the same Hilbert function. Fix an integer $1 \leq p \leq n$. The graded modules $\oplus_{|\sigma|=p} (N : \mathbf{x}_\sigma)$ and $\oplus_{|\sigma|=p} (N' : \mathbf{x}_\sigma)$ have the same Hilbert function.*

Proposition 51.10. *Let N be a squarefree monomial ideal, and Δ be its Stanley-Reisner simplicial complex. Let $\sigma \subseteq \{1, \ldots, n\}$. The Stanley-Reisner simplicial complex of $(N : \mathbf{x}_\sigma)$ is*

$$\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta \,|\, \tau \cup \sigma \in \Delta\}\,,$$

(recall 36.17).

Proof.

$$\begin{aligned}
\mathrm{star}_\Delta(\sigma) &= \{\tau \in \Delta \,|\, \tau \cup \sigma \in \Delta\} \\
&= \{\tau \subseteq \{1, \ldots, n\} \,|\, \mathrm{lcm}(\mathbf{x}_\tau, \mathbf{x}_\sigma) \notin N\} \\
&= \{\tau \subseteq \{1, \ldots, n\} \,|\, \mathbf{x}_\tau \mathbf{x}_\sigma \notin N\} \\
&= \{\tau \subseteq \{1, \ldots, n\} \,|\, \mathbf{x}_\tau \notin (N : \mathbf{x}_\sigma)\,\} \\
&= \{\tau \subseteq \{1, \ldots, n\} \,|\, \mathbf{x}_\tau \notin (N : \mathbf{x}_\sigma)\,\}\,.
\end{aligned}$$

□

52 Clements-Lindström rings

Counting faces of simplicial complexes (that is, counting in an exterior algebra) naturally generalizes to counting in multicomplexes. This leads to considering Clements-Lindström rings, which have the form

$$P = S/(x_1^{a_1}, \ldots, x_n^{a_n})\,,$$

where $2 \le a_1 \le \ldots \le a_n$. We will prove the analogue of Theorem 50.4(1), that is, we will prove that Macaulay's Theorem holds over P. The remaining parts (2)-(5) of Theorem 50.4 hold over P as well and are proved in [Gasharov-Murai-Peeva 2] and [Mermin-Murai].

The notions of a P_q-monomial space, compression, Borel, and lex ideals easily extend over P. For example, we say that a P_q-monomial space B_q is ***Borel*** if whenever a non-zero monomial $x_j m \in B_q$ and $1 \le i \le j$ it follows that $x_i m \in B_q$ (note that $x_i m = 0$ is possible since P is a quotient ring).

We will need some lemmas. Minor modifications in the proofs of Structure Lemma 42.5, Lemma 44.1, the Comparison Theorem 44.4, Compression Lemma 42.9, and Proposition 41.6 lead to the following analogs (listed below) over P of these results. See [Mermin-Peeva] for detailed proofs.

Structure Lemma 52.1. *If a P_q-monomial space C_q is compressed and $n \ge 3$, then C_q is Borel.*

Lemma 52.2. *If a P_q-monomial space B_q is Borel, then*

$$\Big|\{P_1 B_q\}\Big| = \sum_{i=1}^{n} r_{i,a_i-1}(B_q)\,.$$

Comparison Theorem 52.3. *Let B_q be a Borel monomial space in P_q. Let L_q be a lex monomial space in P_q with $|L_q| \leq |B_q|$. Then*

$$r_{i,j}(L_q) \leq r_{i,j}(B_q)$$

for each $1 \leq i \leq n$ and each $1 \leq j \leq a_i$.

Compression Lemma 52.4. *Let A_q be a P_q-monomial space. There exists a compressed monomial space T_q in P_q such that $|T_q| = |A_q|$ and $|P_1 T_q| \leq |P_1 A_q|$.*

Proposition 52.5. *The following properties are equivalent.*

(1) *Let A_q be a P_q-monomial space and L_q be its lexification in P_q. Then $|\,P_1 L_q\,| \leq |\,P_1 A_q\,|$.*
(2) *For every graded ideal J in P there exists a lex ideal L with the same Hilbert function.*

Using the above results we will prove Macaulay's Theorem over the Clements-Lindström ring P.

Clements-Lindström's Theorem 52.6. *Let $P = S/(x_1^{a_1}, \ldots, x_n^{a_n})$, where $2 \leq a_1 \leq \ldots \leq a_n$. For every graded ideal in P there exists a lex ideal with the same Hilbert function.*

Proof. We will prove that (1) in Proposition 52.5 holds. We will use the argument in the first proof of Macaulay's Theorem from Section 45.

An easy calculation shows that the theorem holds provided $n = 2$ and we do not have $a_2 \leq q+1 < a_1$. But $a_1 \leq a_2$ by assumption, so the theorem holds for $n = 2$.

Consider the case $n \geq 3$. Applying 52.4, we conclude that there exist a compressed monomial space C_q such that $|C_q| = |A_q|$ and $|P_1 C_q| \leq |P_1 A_q|$. By Lemma 52.1 it follows that C_q is Borel. Let L_q be the lex monomial space for which $|C_q| = |L_q|$. We apply Lemma 52.2

to conclude that

$$\left|\{P_1 C_q\}\right| = \sum_{i=1}^{n} r_{i,a_i-1}(C_q)$$

$$\left|\{P_1 L_q\}\right| = \sum_{i=1}^{n} r_{i,a_i-1}(L_q)\,.$$

Finally, we apply Theorem 52.3 and get the inequality $\left|\{P_1 L_q\}\right| \le \left|\{P_1 C_q\}\right|$. $\square$

53 The Eisenbud-Green-Harris Conjecture

The most exciting currently open conjecture on Hilbert functions is the Eisenbud-Green-Harris Conjecture. It is wide open.

The Eisenbud-Green-Harris Conjecture 53.1. [Eisenbud-Green-Harris 1, Eisenbud-Green-Harris 2] *Let N be a graded ideal in S containing a maximal homogeneous regular sequence in degrees $2 \le e_1 \le \cdots \le e_n$. There exists a monomial ideal T such that N and $T + (x_1^{e_1}, \cdots, x_r^{e_n})$ have the same Hilbert function.*

A monomial ideal $L + (x_1^{e_1}, \ldots, x_r^{e_n})$ is called ***lex-plus-powers*** if it is the preimage of a lex ideal in $S/(x_1^{e_1}, \ldots, x_r^{e_n})$. By Clements-Lindström's Theorem 52.6, it follows that the conjecture can be stated equivalently as follows.

Conjecture 53.2. *Let N be a graded ideal containing a maximal homogeneous regular sequence in degrees $2 \le e_1 \le \cdots \le e_n$. There exists a lex-plus-powers ideal $L + (x_1^{e_1}, \ldots, x_r^{e_n})$ with the same Hilbert function.*

The original conjecture gives a numerical characterization of the Hilbert functions of graded ideals containing a maximal homogeneous regular sequence in degrees $2 \le e_1 \le \cdots \le e_n$. It is well known that the numerical characterization is equivalent to the existence of a lex-plus-powers ideal $L + (x_1^{e_1}, \ldots, x_n^{e_n})$ with the same Hilbert function as the ideal N.

Another equivalent formulation of the conjecture is:

Conjecture 53.3. *Let $f_1, \ldots, f_n$ be a maximal homogeneous regular sequence in S in degrees $2 \le e_1 \le \cdots \le e_n$. Let $\bar{N}$ be a graded ideal in the complete intersection ring $S/(f_1, \ldots, f_n)$. There exists a lex ideal $\bar{L}$ in the Clements-Lindström ring $S/(x_1^{e_1}, \ldots, x_r^{e_n})$ with the same Hilbert function as $\bar{N}$.*

The conjecture is especially interesting in the case $e_1 = \ldots = e_n = 2$ when the regular sequence consists of quadrics.

Next, we focus on problems based on the idea that the lex ideal has the greatest Betti numbers among all ideals with a fixed Hilbert function.

Conjecture 53.4. *Suppose that k is an infinite field (possibly, one should also assume* $\mathrm{char}(k) = 0$*). Let N be a graded ideal containing a homogeneous regular sequence $f_1, \ldots, f_n$ in S in degrees $2 \le e_1 \le \cdots \le e_n$. Suppose that there exists a lex-plus-powers ideal $L + (x_1^{e_1}, \cdots, x_n^{e_n})$ with the same Hilbert function. Then:*

(1) *The Betti numbers of $\bar{N}$ over $S/(f_1, \ldots, f_n)$ are less than or equal to those of $\bar{L}$ over $S/(x_1^{e_1}, \ldots, x_n^{e_n})$, (where $\bar{N}$ and $\bar{L}$ are the images of N and L in the corresponding complete intersection rings).*

(2) **The LPP Conjecture (the lex-plus-powers conjecture).** (Evans) *The Betti numbers of N over S are less than or equal to those of $L + (x_1^{e_1}, \ldots, x_n^{e_n})$.*

The first part of the conjecture is about infinite resolutions, whereas the second part is about finite ones.

The LPP Conjecture was inspired by the Eisenbud-Green-Harris Conjecture. [Francisco-Richert] is an expository paper on the LPP Conjecture.

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