Chapter 2
Local principles

Before we start our walk through the world of local principles, it is useful to give a general idea of what a local principle should be. A local principle will allow us to study invertibility properties of an element of an algebra by studying the invertibility properties of a (possibly large) family of (hopefully) simpler objects. These simpler objects will usually occur as homomorphic images of the given element. To make this more precise, consider a unital algebra $\mathscr{A}$ and a family $\mathscr{W} = (\mathsf{W}_t)_{t \in T}$ of unital homomorphisms $\mathsf{W}_t : \mathscr{A} \to \mathscr{B}_t$ from $\mathscr{A}$ into certain unital algebras $\mathscr{B}_t$. We say that $\mathscr{W}$ forms a *sufficient family of homomorphisms for* $\mathscr{A}$ if the following implication holds for every element $a \in \mathscr{A}$:

$$\mathsf{W}_t(a) \text{ is invertible in } \mathscr{B}_t \text{ for every } t \in T \quad \Longrightarrow \quad a \text{ is invertible in } \mathscr{A}$$

(the reverse implication is satisfied trivially). Equivalently, the family $\mathscr{W}$ is sufficient if and only if

$$\sigma_{\mathscr{A}}(a) \subseteq \cup_{t \in T} \sigma_{\mathscr{B}_t} \qquad \text{for all } a \in \mathscr{A}$$

(again with the reverse inclusion holding trivially). In case the family $\mathscr{W}$ is a singleton, $\{\mathsf{W}\}$ say, then $\mathscr{W}$ is sufficient if and only if W is a symbol mapping in the sense of Section 1.2.1.

Every sufficient family $(\mathsf{W}_t)_{t \in T}$ of homomorphisms for $\mathscr{A}$ gives rise to a symbol mapping W from $\mathscr{A}$ into the direct product $\Pi_{t \in T} \mathscr{B}_t$ of the algebras $\mathscr{B}_t$ via

$$\mathsf{W} : a \mapsto (t \mapsto \mathsf{W}_t(a)).$$

Since symbol mappings preserve spectra, they preserve spectral radii. In the C^*-case (i.e., if all occurring algebras are C^* and the homomorphisms are symmetric), this implies that they also preserve norms of self-adjoint elements (Proposition 1.2.36 (i)), and hence, by the C^*-axiom, the norms of arbitrary elements. Thus, a symmetric symbol mapping $\mathsf{W} : \mathscr{A} \to \mathscr{B}$ between C^*-algebras is nothing but a *-isomorphism between $\mathscr{A}$ and a C^*-subalgebra of $\mathscr{B}$. For general (in particular, Banach) algebras there is a clear distinction between symbol mappings and isomorphisms. Note that $(\mathsf{W}_t)_{t \in T}$ being sufficient only implies that $\cap_{t \in T} \mathrm{Ker}\, \mathsf{W}_t \subseteq \mathscr{R}_{\mathscr{A}}$.

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So, in case of an algebraic environment, “local principle” will mean a process to construct sufficient families of homomorphisms. All local principles encountered in this chapter will use commutativity properties for this construction. It is either the commutativity of the algebra itself (classical Gelfand theory), the presence of a (sufficiently large) center (Allan’s local principle and its relatives due to Douglas, Gohberg-Krupnik, and Simonenko), or the appearance of a completely new identity which replaces the common $ab = ba$ (Krupnik’s principle for *PI*-algebras) which will be employed.

Unless stated otherwise, all algebras and homomorphisms occurring in this chapter are complex.

2.1 Gelfand theory

The Gelfand transform for commutative Banach algebras which we are going to discuss in this section, will not only provide us with the first and simplest local principle; it will also serve as a model for other local principles. Gelfand’s theory associates with every commutative Banach algebra $\mathscr{B}$ with identity a subalgebra of the algebra $C(X)$ of all continuous functions on an appropriately chosen compact Hausdorff space X depending on the internal structure of the algebra. The Gelfand transform is then a homomorphism from $\mathscr{B}$ into $C(X)$, which is a symbol mapping for $\mathscr{B}$.

2.1.1 The maximal ideal space

Let $\mathscr{B}$ be a commutative unital Banach algebra, and let $M_{\mathscr{B}}$ denote the set of all maximal ideals of $\mathscr{B}$. Let x be a maximal ideal[1] of $\mathscr{B}$. Then the quotient algebra $\mathscr{B}/x$ is isomorphic to $\mathbb{C}$ by Theorem 1.3.8. Thus, to each $x \in M_{\mathscr{B}}$ and each $b \in \mathscr{B}$, there is associated a complex number, $\widehat{b}(x)$, which is the image of the coset $b + x$ under the above mentioned isomorphism.

The mapping $b \mapsto \widehat{b}(x)$ is a multiplicative linear functional on $\mathscr{B}$ the kernel of which is x. By Theorem 1.3.19, there are no other multiplicative linear functionals with the same kernel on $\mathscr{B}$, and there is a one-to-one correspondence between the non-trivial multiplicative linear functionals on $\mathscr{B}$ and the maximal ideals of $\mathscr{B}$: the kernel of every non-trivial multiplicative linear functional is a maximal ideal, and every maximal ideal is the kernel of a uniquely determined non-trivial multiplicative linear functional.

[1] The notation “x” for a maximal ideal may seem strange at first. But if one remembers Section 1.4.3 and the relation between the maximal ideals and the multiplicative linear functionals of the algebra $C(X)$, this choice of notation becomes clear.

Definition 2.1.1. Given an element $b \in \mathscr{B}$, the complex-valued function

$$\widehat{b} \, : \, M_{\mathscr{B}} \to \mathbb{C}, \quad x \mapsto \widehat{b}(x)$$

is called the *Gelfand transform of the element* $b \in \mathscr{B}$, and the resulting mapping

$$\widehat{} \, : \, \mathscr{B} \to C(M_{\mathscr{B}}), \quad b \mapsto \widehat{b}$$

is called the *Gelfand transform on* $\mathscr{B}$.

The set $M_{\mathscr{B}}$ can be made into a topological space by the requirement that all Gelfand transforms of elements of $\mathscr{B}$ become continuous functions on it:

Definition 2.1.2. The *Gelfand topology* on $M_{\mathscr{B}}$ is the coarsest topology on $M_{\mathscr{B}}$ that makes all Gelfand transforms $\widehat{b}$ with $b \in \mathscr{B}$ continuous. The set $M_{\mathscr{B}}$ provided with the Gelfand topology is referred to as the *maximal ideal space* of the Banach algebra $\mathscr{B}$.

Equivalently, the sets $\widehat{b}^{-1}(U)$ with b and U running through $\mathscr{B}$ and the open subsets of $\mathbb{C}$, respectively, form a sub-basis of the Gelfand topology.

2.1.2 Classical Gelfand theory

The following two theorems, also known as *Gelfand's representation theorem*, are the central results of classical Gelfand theory. They claim that every complex commutative unital semi-simple Banach algebra is isomorphic to an algebra of complex functions, defined in a certain compact Hausdorff space.

Theorem 2.1.3. *Let $\mathscr{B}$ be a commutative unital Banach algebra. Then:*

(i) *the Gelfand transform is a continuous homomorphism of norm* 1*;*
(ii) *the set $\widehat{\mathscr{B}}$ is a subalgebra of $C(M_{\mathscr{B}})$ which separates the points of $M_{\mathscr{B}}$ and contains the identity of $C(M_{\mathscr{B}})$;*
(iii) *the element $b \in \mathscr{B}$ is invertible if and only if $\widehat{b}(x) \neq 0$ for all $x \in M_{\mathscr{B}}$;*
(iv) *the kernel of the Gelfand transform is the radical of $\mathscr{B}$. Thus, the Gelfand transform is an isomorphism between $\mathscr{B}$ and $\widehat{\mathscr{B}}$ if and only if $\mathscr{B}$ is semi-simple;*
(v) *the spectrum of $b \in \mathscr{B}$ coincides with the image of $\widehat{b}$, and $r(b) = \|\widehat{b}\|_\infty$.*

Proof. The definition of the Gelfand topology in $M_{\mathscr{B}}$ guarantees the continuity of each function $\widehat{b}$. As $\widehat{b_1 + b_2} = \widehat{b_1} + \widehat{b_2}$, $\widehat{\lambda b_1} = \lambda \widehat{b_1}$ and $\widehat{b_1 b_2} = \widehat{b_1}\widehat{b_2}$, for $b_1, b_2 \in \mathscr{B}$ and $\lambda \in \mathbb{C}$, the Gelfand transform is a homomorphism. By the definition of the quotient norm, one also has

$$\left\|\widehat{b}\right\|_\infty = \sup_{x \in M_{\mathscr{B}}} \left|\widehat{b}(x)\right| \leq \|b\|, \tag{2.1}$$

which implies the continuity of the Gelfand transform and that its norm is less than or equal to 1. That $\widehat{\mathscr{B}}$ is a subalgebra of $C(M_{\mathscr{B}})$ is obvious, because the Gelfand transform is a homomorphism. To prove the rest of point (ii) let us suppose that we have two maximal ideals $x_1 \neq x_2$. Choosing an element $b_1 \in x_1$ such that $b_1 \notin x_2$ we obtain $\widehat{b_1}(x_1) = 0$ but $\widehat{b}_1(x_2) \neq 0$. We have also $\widehat{e}(x) = 1$ for all $x \in M_{\mathscr{B}}$, and using point (i) we can now deduce that the norm of $\widehat{\ }$ is 1. The third point is a direct consequence of Theorem 1.3.21. The proof of point (iv) is easy: just remember that $\widehat{b}(x) = 0$ for all $x \in M_{\mathscr{B}}$, if and only if $b \in x$ for all $x \in M_{\mathscr{B}}$.

Finally, regarding (v), we have that $\lambda \in \sigma(b) \Leftrightarrow \lambda e - b$ is not invertible, which is equivalent to $\widehat{\lambda e - b}(x) = 0$ for some $x \in M_{\mathscr{B}}$ by point (iii) above. This is equivalent to $\lambda - \widehat{b}(x) = 0$ for some $x \in M_{\mathscr{B}}$, that is $\lambda \in \widehat{b}(M_{\mathscr{B}})$. By the definition of spectral radius we have $r(b) = \sup_{x \in M_{\mathscr{B}}} |\widehat{b}(x)| = \|\widehat{b}\|_\infty$. ■

In general, neither equality holds in the estimate (2.1), nor is the Gelfand transform $\widehat{\ } : \mathscr{B} \to C(M_{\mathscr{B}})$ injective or surjective. Examples are provided in Exercises 2.1.1 and 2.1.2.

Theorem 2.1.4. *The maximal ideal space $M_{\mathscr{B}}$ of a commutative Banach algebra with identity is a compact Hausdorff space; thus normal.*

Proof. We prepare the proof by recalling some facts from functional analysis. Let X be a Banach space with Banach dual X^*. For each $b \in X$, define a function

$$f_b : X^* \to \mathbb{C}, \quad \phi \mapsto \phi(b).$$

The w^*-topology on X^* is, by definition, the weakest topology on X^* for which all functions f_b with $b \in X$ are continuous. The restriction of the w^*-topology to the closed unit ball

$$E^* := \{\phi : \phi \in X^* \text{ and } \|\phi\| \leq 1\}$$

of X^* makes E^* a compact subset of a Hausdorff space in the w^*-topology (see, for instance, [184, Theorem 49-A]).

Now let $\mathscr{B}$ be a commutative Banach algebra with identity e, $\mathscr{B}^*$ its Banach dual space, and E^* the closed unit ball of $\mathscr{B}^*$, provided with its w^*-topology. Every non-zero multiplicative functional on $\mathscr{B}$ has norm 1 and can thus be considered as an element of E^*, which implies an embedding of the maximal ideal space $M_{\mathscr{B}}$ into E^*. It turns out that the restriction of the w^*-topology on E^* to $M_{\mathscr{B}}$ coincides with the Gelfand topology. Indeed, the restriction of f_b to $M_{\mathscr{B}}$ coincides with the Gelfand transform $\widehat{b}$, because of $f_b(x) = f_b(\phi_x) = \phi_x(b) = \widehat{b}(x)$.

Since E^* is a compact subset of a Hausdorff space with respect to the w^*-topology, it remains only to prove that $M_{\mathscr{B}}$ is a closed subset of E^*. Let $\mathbb{I}$ be a directed set and $(\phi_\alpha)_{\alpha \in \mathbb{I}}$ a net in $M_{\mathscr{B}}$ which converges to $\phi \in E^*$ in the w^*-topology. Then

$$\phi(e) = \lim_\alpha \phi_\alpha(e) = \lim_\alpha 1 = 1$$

and

$$\begin{aligned}\phi(b_1b_2) &= \lim_\alpha \phi_\alpha(b_1b_2)\\ &= \lim_\alpha \phi_\alpha(b_1)\phi_\alpha(b_2) = \lim_\alpha \phi_\alpha(b_1)\lim_\alpha \phi_\alpha(b_2) = \phi(b_1)\phi(b_2)\end{aligned}$$

which shows that $\phi \in M_{\mathscr{B}}$ again. ■

Remark 2.1.5. Using the terminology given at the beginning of the chapter, assertions (i) and (iii) of Theorem 2.1.3 can be rephrased as follows: The Gelfand transform $\widehat{\ } : \mathscr{B} \to C(M_{\mathscr{B}})$ is a symbol mapping for $\mathscr{B}$, and the family of all homomorphisms $\mathsf{W}_x : b \mapsto \widehat{b}(x)$ of $\mathscr{B}$ with $x \in M_{\mathscr{B}}$ is a sufficient family. □

The following result provides a criterion for the injectivity of the Gelfand transform.

Proposition 2.1.6. *Let $\mathscr{B}$ be a commutative unital Banach algebra. The following conditions are equivalent, for any $b \in \mathscr{B}$:*

(i) $\|b^2\| = \|b\|^2$;
(ii) $r(b) = \|b\|$;
(iii) $\|\widehat{b}\| = \|b\|$.

Proof. Condition (i) implies $\|b^{2^k}\| = \|b\|^{2^k}$ for any natural k. Then the formula for the spectral radius (Theorem 1.2.12) gives

$$r(b) = \lim_{n\to\infty} \|b^n\|^{\frac{1}{n}} = \lim_{k\to\infty} \left\|b^{2^k}\right\|^{\frac{1}{2^k}} = \lim_{k\to\infty} \|b\| = \|b\|,$$

which is (ii). Conversely, if $\lambda \in \sigma(b)$ then $\lambda^2 \in \sigma(b^2)$ by Exercise 1.2.5. Hence, $\|b^2\| = r(b^2) = r(b)^2 = \|b\|^2$, showing that (ii) implies (i). The equivalence between (ii) and (iii) comes from (v) in Theorem 2.1.3. ■

For normal elements in a C^*-algebra, condition (i) in the above proposition is always satisfied and we have $\|a^2\| = \|a\|^2$ and $r(a) = \|a\|$, as was seen in Proposition 1.2.36 (i). In a commutative C^*-algebra, all elements are normal. Thus, the Gelfand transform acts as an isometry on commutative C^*-algebras. One can say even more in this case.

Theorem 2.1.7 (Gelfand-Naimark)**.** *Let $\mathscr{B}$ be a commutative unital C^*-algebra. Then the Gelfand transform is an (isometric) *-isomorphism from $\mathscr{B}$ onto the algebra $C(M_{\mathscr{B}})$.*

Proof. Let us first verify that the Gelfand transform is a *-homomorphism. If $h \in \mathscr{B}$ is self-adjoint, then $\sigma(h) \subset \mathbb{R}$ by Proposition 1.2.36, implying that $\operatorname{Im} \widehat{h} \subset \mathbb{R}$ by Theorem 2.1.3 (iii). Consequently, $\overline{\widehat{h}} = \widehat{h} = \widehat{h^*}$. Now let $b \in \mathscr{B}$ be arbitrary. Write b as $b = h + ik$ with h, k self-adjoint. Then

$$\widehat{b^*} = \widehat{(h-ik)} = \widehat{h} - \widehat{ik} = \overline{\widehat{h}} - i\overline{\widehat{k}} = \overline{\widehat{h+ik}} = \overline{\widehat{b}}.$$

Thus, the Gelfand transform is symmetric, and by Propositions 1.2.36 and 2.1.6, it is also an isometry. We are left with proving the surjectivity. The image $\widehat{\mathscr{B}}$ is a closed self-adjoint subalgebra of $C(M_{\mathscr{B}})$ by Theorem 1.1.6, and it separates the points of $M_{\mathscr{B}}$ and contains the constant functions. By the Stone-Weierstrass theorem (see for example [171, Section 5.7]), $\widehat{\mathscr{B}}$ coincides with $C(M_{\mathscr{B}})$. ■

The theorem above justifies thinking of elements of commutative unital C^*-algebras as continuous functions on a compact Hausdorff space, and we shall use this henceforth.

2.1.3 The Shilov boundary

Let X be a compact Hausdorff space. For each function $a \in C(X)$ and each closed non-empty subset E of X we set

$$\|a_{|E}\|_\infty := \max_{x\in E} |a(x)|.$$

Let $\mathscr{C}$ be a subset of $C(X)$. A closed subset F of X is called a *maximizing set for* $\mathscr{C}$ if

$$\|a_{|X}\|_\infty = \|a_{|F}\|_\infty \quad \text{for all } a \in \mathscr{C}.$$

Lemma 2.1.8. *Let $\mathscr{C}$ be a subalgebra of $C(X)$ which contains the constant functions and separates the points of X. Then the intersection of all maximizing sets for $\mathscr{C}$ is a maximizing set for $\mathscr{C}$.*

Proof. Let S denote the intersection of all maximizing sets for $\mathscr{C}$. We claim that every point $x_0 \in X \setminus S$ has an open neighborhood U such that $F \setminus U$ is a maximizing set for $\mathscr{C}$ whenever F is a maximizing set for $\mathscr{C}$.

Indeed, since $x_0 \notin S$, there is a maximizing set F_0 for $\mathscr{C}$ which does not contain x_0. Since $\mathscr{C}$ contains the constant functions and separates the points of X, for each $y \in F_0$, there is a function $a_y \in \mathscr{C}$ with $a_y(x_0) = 0$ and $a_y(y) = 2$. Each set $U_y := \{x \in X : |a_y(x)| > 1\}$ is an open neighborhood of y. Thus, the compact set F_0 can be covered by a finite number of sets of the form U_y. We denote the corresponding functions in $\mathscr{C}$ by $a_1, \ldots, a_r$. Thus, $a_k(x_0) = 0$ for $k = 1, \ldots, r$, and for each $y \in F_0$ there is a $k \in \{1, \ldots, r\}$ such that $a_k(y) > 1$. Let

$$U := \{x : |a_k(x)| < 1 \text{ for every } k = 1, \ldots, r\}.$$

Then U is an open neighborhood of x_0 and $U \cap F_0 = \emptyset$.

Now let F be a maximizing set for $\mathscr{C}$, and suppose that the set $F \setminus U$ is not maximizing for $\mathscr{C}$. Then there is a function $a \in \mathscr{C}$ with

$$\|a_{|X}\|_\infty = 1 > \|a_{|F\setminus U}\|_\infty.$$

Let $M := \max\{\|a_{k|X}\|_\infty : k = 1, \ldots, r\}$ and choose n such that $\|a_{|F\setminus U}\|_\infty^n < 1/M$. Then $\|a^n a_{k|_{F\setminus U}}\|_\infty < 1$ for every $k = 1, \ldots, r$, and one also has $|a^n(x)a_k(x)| < 1$ for all $x \in U$ and $k = 1, \ldots, r$. Since F is maximizing, this implies

$$\|a^n a_{k_{|X}}\|_\infty = \|a^n a_{k|_F}\|_\infty < 1 \quad \text{for every } k = 1, \ldots, r.$$

Since F_0 is a maximizing set, there is a point $y \in F_0$ with $a(y) = 1$. For this point, one gets

$$|a_k(y)| = |a^n(y)a_k(y)| \leq \|a^n a_{k|X}\|_\infty < 1,$$

which implies that $y \in F_0 \cap U$. Hence, $F_0 \cap U$ is not empty; a contradiction. This contradiction proves our claim.

Having the claim at our disposal, the proof of the lemma can be completed as follows. Let $a \in \mathscr{C}$, and let $K := \{x \in X : |a(x)| = \|a_{|_X}\|_\infty\}$. We have to show that $S \cap K$ is not empty. Contrary to what we want to show, assume that $S \cap K = \emptyset$. Then each point $x_0 \in K$ has an open neighborhood U given by the claim. Since K is compact, it is covered by a finite number of these neighborhoods, say $U_1, \ldots, U_n$. The set X is maximizing, and so are the sets

$$X \setminus U_1,\ X \setminus (U_1 \cup U_2),\ \ldots,\ X \setminus (U_1 \cup \ldots \cup U_n) =: E,$$

say. Since $E \cap K = \emptyset$ one obtains $\|a_{|_E}\|_\infty < \|a_{|_X}\|_\infty$ which contradicts the maximality of E. ■

Now let $\mathscr{A}$ be a commutative Banach algebra with identity e. Then the algebra $\mathscr{C}$ of all Gelfand transforms of elements of $\mathscr{A}$ contains the constant functions and separates the points of the maximal ideal space $M_\mathscr{A}$. By the above lemma, the intersection of all maximizing sets for $\mathscr{C}$ is a maximizing set for $\mathscr{C}$. This intersection is called the *Shilov boundary of* $M_\mathscr{A}$. We denote it by $\partial_S M_\mathscr{A}$.

Equivalently, a point $x_0 \in M_\mathscr{A}$ belongs to $\partial_S M_\mathscr{A}$ if and only if, for each open neighborhood $U \subset M_\mathscr{A}$ of x_0, there exists an $a \in \mathscr{A}$ such that

$$\|\widehat{a}_{|M_\mathscr{A}\setminus U}\|_\infty < \|\widehat{a}_{|U}\|_\infty.$$

Theorem 2.1.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be commutative unital Banach algebras.*

(i) *If $\mathsf{W} : \mathscr{A} \to \mathscr{B}$ is a unital homomorphism which preserves spectral radii, i.e., if*

$$r_\mathscr{B}(\mathsf{W}(a)) = r_\mathscr{A}(a) \quad \textit{for all } a \in \mathscr{A},$$

then $\partial_S M_\mathscr{A} \subseteq \mathsf{W}^*(\partial_S M_\mathscr{B})$ (with W^* referring to the dual mapping of W).

(ii) *Now let $\mathscr{A}$ be a unital closed subalgebra of $\mathscr{B}$. Then each maximal ideal in the Shilov boundary of $M_\mathscr{A}$ is contained in some maximal ideal of $\mathscr{B}$.*

Proof. (i) We think of the elements of the maximal ideal space $M_\mathscr{B}$ as non-trivial multiplicative functionals, and thus as elements of the Banach dual $\mathscr{B}^*$. Since W is a homomorphism, its dual W^* sends multiplicative functionals on $\mathscr{B}$ to multiplicative functionals on $\mathscr{A}$. Let $e_{\mathscr{A}}$ and $e_{\mathscr{B}}$ denote the identity elements of $\mathscr{A}$ and $\mathscr{B}$, respectively. From

$$\mathsf{W}^*(\varphi)(e_{\mathscr{A}}) = \varphi(\mathsf{W}(e_{\mathscr{A}})) = \varphi(e_{\mathscr{B}}) = 1$$

one deduces that the image of a non-trivial multiplicative functional φ on $\mathscr{B}$ is non-trivial again. Thus, W^* maps $M_{\mathscr{B}}$ into $M_{\mathscr{A}}$.

The Shilov boundary $\partial_S M_{\mathscr{B}}$ is a compact subset of $M_{\mathscr{B}}$, and $\mathsf{W}^* : M_{\mathscr{B}} \to M_{\mathscr{A}}$ is continuous by the definition of the Gelfand topology. Hence, $\mathsf{W}^*(\partial_S M_{\mathscr{B}})$ is a compact subset of $M_{\mathscr{A}}$. Since $M_{\mathscr{A}}$ is a compact Hausdorff space, the set $\mathsf{W}^*(\partial_S M_{\mathscr{B}})$ is closed in $M_{\mathscr{A}}$. The assertion will follow once we have shown that this set is maximizing for the algebra of all Gelfand transforms of elements of $\mathscr{A}$.

Let $a \in \mathscr{A}$. Then

$$\|\hat{a}_{|\mathsf{W}^*(\partial_S M_{\mathscr{B}})}\|_\infty = \big\|\widehat{\mathsf{W}(a)}_{|\partial_S M_{\mathscr{B}}}\big\|_\infty = r_{\mathscr{B}}(\mathsf{W}(a)) = r_{\mathscr{A}}(a)$$

with r referring to the spectral radius. Thus, $\mathsf{W}^*(\partial_S M_{\mathscr{B}})$ is maximizing and includes the Shilov boundary of the maximal ideal space of $\mathscr{A}$.

(ii) Consider the inclusion map $\mathsf{W} : \mathscr{A} \to \mathscr{B}$. Then W is a homomorphism which preserves spectral radii (use the formula for the spectral radius to check this). Thus, by part (i) of this theorem, every maximal ideal in the Shilov boundary of $\mathscr{A}$ arises as the restriction of some maximal ideal (in the Shilov boundary) of $\mathscr{B}$. ■

2.1.4 Example: SIOs with continuous coefficients

We return now to the SIOs introduced in Section 1.4.4, but will apply Gelfand's representation theorem. Let $\mathscr{B}$ be the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{T}))$ which contains all singular integral operators of the form

$$A = cI + dS + K = fP + gQ + K$$

where $c, d \in C(\mathbb{T})$, K is compact, and $f := c + d$ and $g := c - d$. From Proposition 1.4.12 we infer that the Calkin image $\mathscr{B}^{\mathscr{K}} := \mathscr{B}/\mathscr{K}$ of $\mathscr{B}$ is a commutative and unital Banach algebra which is, consequently, subject to Gelfand's representation theorem. We start with identifying the maximal ideal space of $\mathscr{B}^{\mathscr{K}}$.

Proposition 2.1.10. *All proper ideals of $\mathscr{B}^{\mathscr{K}}$ are contained in ideals of the form*

$$\mathscr{I}_{P,X_0} := \{fP + gQ + \mathscr{K} : f(X_0) = 0\} \quad \text{or} \quad \mathscr{I}_{Q,X_0} := \{fP + gQ + \mathscr{K} : g(X_0) = 0\}$$

with a certain subset X_0 of $\mathbb{T}$.

Proof. Suppose there is an ideal $\mathscr{I}$ of $\mathscr{B}^{\mathscr{K}}$, which does not have the claimed property. Then, for every $x \in \mathbb{T}$, there are functions f_x and $g_x \in C(\mathbb{T})$ such that

$f_x(x) \neq 0$, $g_x(x) \neq 0$, and $f_xP + g_xQ + \mathcal{K} \in \mathcal{I}$. Consequently,

$$A_x := (f_xP + g_xQ + \mathcal{K})(\overline{f}_xP + \overline{g}_xQ + \mathcal{K}) = |f_x|^2P + |g_x|^2Q + \mathcal{K} \in \mathcal{I}.$$

The functions $|f_x|^2$ and $|g_x|^2$ are positive in a certain open neighborhood U_x of x, and the collection of all of these neighborhoods covers $\mathbb{T}$. By compactness, one can extract a finite subcovering $\mathbb{T} = U_{x_1} \cup \ldots \cup U_{x_n}$, say. It is easy to see that then the operator $A = \sum_{k=1}^{n} A_{x_k} \in \mathcal{I}$ is invertible in $\mathcal{B}^{\mathcal{K}}$, which is a contradiction. ∎

Proposition 2.1.10 implies that the maximal ideals of $\mathcal{B}^{\mathcal{K}}$ are necessarily of the form

$$\mathcal{I}_{P,x_0} := \{fP + gQ + \mathcal{K} : f(x_0) = 0\} \text{ and } \mathcal{I}_{Q,x_0} := \{fP + gQ + \mathcal{K} : g(x_0) = 0\}$$

with $x_0 \in \mathbb{T}$. These ideals are closed by Theorem 1.3.5. Thus, there is a bijection between the maximal ideal space of $\mathcal{B}^{\mathcal{K}}$ and two copies of the unit circle $\mathbb{T}$ or, more precisely, between the maximal ideal space of $\mathcal{B}^{\mathcal{K}}$ and the product $\mathbb{T} \times \{0,1\}$. A closer look shows that this bijection is even a homeomorphism, which allows one to identify the maximal ideal space of $\mathcal{B}^{\mathcal{K}}$ with $\mathbb{T} \times \{0,1\}$, provided with the usual product topology. Under this identification, the Gelfand transform of an element $A + \mathcal{K} = fP + gQ + \mathcal{K}$ is given by

$$\widehat{A + \mathcal{K}}(x,n) = \begin{cases} f(x) & \text{if } n = 0, \\ g(x) & \text{if } n = 1. \end{cases}$$

Thus, the coset $A + \mathcal{K}$ is invertible (equivalently, the operator A is Fredholm) if and only if $f(x) \neq 0$ and $g(x) \neq 0$ for all $x \in \mathbb{T}$. It is also easy to see that the radical of $\mathcal{B}^{\mathcal{K}}$ is $\{0\}$.

These results remain valid for curves Γ other than the unit circle, provided that the singular integral operator S_Γ on that curve satisfies $S_\Gamma^2 = I$ and that the commutator of S_Γ with every operator of multiplication by a continuous function on Γ is compact. Examples of such curves will be discussed in Chapter 4.

2.1.5 Exercises

Exercise 2.1.1. Let $\mathcal{B}$ be the algebra of all matrices $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ with complex entries a, b. Determine the maximal ideal space of $\mathcal{B}$ and the Gelfand transform on $\mathcal{B}$. Conclude that the Gelfand transform is not injective. (Evidently, the reason for being not injective is that $\mathcal{B}$ has a non-trivial radical.)

Exercise 2.1.2. Let $\mathbb{A}$ refer to the disk algebra introduced in Example 1.2.23. Characterize $\mathbb{A}$ as the set of all functions $f \in C(\mathbb{T})$ which possess an analytic continuation into the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Show that $\mathbb{A}$ is a commutative Banach algebra with identity and that every character of $\mathbb{A}$ is of the form $f \mapsto f(z)$ with some fixed point $z \in \mathbb{D} \cup \mathbb{T}$. Conclude that the maximal ideal space of $\mathbb{A}$ is homeomorphic to the closed unit disk $\mathbb{D} \cup \mathbb{T}$ with its standard (Euclidean) topology. Show further that the Gelfand transform of $f \in \mathbb{A}$ coincides with the analytic continuation of f into the interior of the unit disk. Thus, the image of the Gelfand transform on $\mathbb{A}$ is a proper subset of $C(M_{\mathbb{A}}) = C(\mathbb{D} \cup \mathbb{T})$.

Exercise 2.1.3. A Banach algebra $\mathscr{B}$ with identity e is *singly generated* if there is an element $b \in \mathscr{B}$ such that the smallest closed subalgebra of $\mathscr{B}$ which contains e and b coincides with $\mathscr{B}$. In this case, b is called a *generator* of $\mathscr{B}$. Prove that the maximal ideal space of a singly generated (by b, say) Banach algebra is homeomorphic to the spectrum $\sigma_{\mathscr{B}}(b)$. Suggestion: under this homeomorphism, the point $\lambda \in \sigma_{\mathscr{B}}(b)$ corresponds to the smallest closed ideal of $\mathscr{B}$ which contains $b - \lambda e$ (see [37, 15.3.6]).

Exercise 2.1.4. We know from Exercise 1.4.8 that the Toeplitz algebra $\mathscr{T}(C)$ contains the ideal $\mathscr{K}(l^2(\mathbb{Z}^+))$ of the compact operators and that the quotient algebra $\mathscr{T}(C)/\mathscr{K}(l^2(\mathbb{Z}^+))$ is commutative. Identify the maximal ideal space of this quotient algebra. Show that the Gelfand transform of $T(a) + \mathscr{K}(l^2(\mathbb{Z}^+))$ can be identified with $a \in C(\mathbb{T})$.

Exercise 2.1.5. Let $\mathscr{B}$ be a commutative Banach algebra with identity, generated by the elements $\{b_1, \ldots, b_n\}$. Show that $M_{\mathscr{B}}$ is homeomorphic to the joint spectrum $\sigma_{\mathscr{B}}(b_1, \ldots, b_n)$ in $\mathscr{B}$.

Exercise 2.1.6. Review Exercise 1.3.9. Let M_{l_∞} be the space of multiplicative linear functionals of l_∞, with the w^* topology. Given $u \in l_\infty$, define $\widehat{u}(\phi)$ as $\phi(u)$ for $\phi \in M_{l_\infty}$. Note that M_{l_∞} is a compact Hausdorff space.

a) Show that M_{l_∞} is extremely disconnected (i.e., the closure of every open set is open).
b) Let $\phi_n(u) := u_n$ for $n \in \mathbb{N}$ and $u \in l_\infty$. Show that the subset $\{\phi_n : n \in \mathbb{N}\}$ of M_{l_∞} is homeomorphic to $\mathbb{N}$ and that, consequently, $\mathbb{N}$ can be identified with a subset of M_{l_∞}. Show that $\mathbb{N}$ is dense in M_{l_∞}.
c) Show that the one point subset $\{\phi_n\}$ of M_{l_∞} is open in M_{l_∞}.
d) Show that $\phi(u) = 0$ when $\phi \in M_{l_\infty} \setminus \mathbb{N}$ and $u \in l_\infty^0$.

Exercise 2.1.7. Determine the Shilov boundaries of the maximal ideal spaces of the disk algebra $\mathbb{A}$ and of the C^*-algebra $C(X)$ where X is a compact Hausdorff space. (Hint: use the Tietze-Uryson extension theorem.)

2.2 Allan's local principle

As we have seen, Gelfand theory associates with each unital commutative Banach algebra $\mathscr{B}$ a compact Hausdorff space $M_{\mathscr{B}}$, called the *maximal ideal space* of $\mathscr{B}$, and with every element $b \in \mathscr{B}$ a continuous function $\widehat{b} : M_{\mathscr{B}} \to \mathbb{C}$, called the *Gelfand transform* of b, such that the mapping

$$\widehat{\ } : \mathscr{B} \to C(M_{\mathscr{B}}), \quad b \mapsto \widehat{b}$$

becomes a contractive algebra homomorphism which preserves spectra. Allan's local principle is a generalization of classical Gelfand theory to unital Banach algebras which are close to commutative algebras in the sense that their centers are non-trivial.

2.2.1 Central subalgebras

Let $\mathscr{A}$ be a unital Banach algebra. Recall that the center of $\mathscr{A}$ is the set Cen $\mathscr{A}$ of all elements $a \in \mathscr{A}$ such that $ab = ba$ for all $b \in \mathscr{A}$. Evidently, Cen $\mathscr{A}$ is a closed commutative subalgebra of $\mathscr{A}$ which contains the identity element. A *central* subalgebra of $\mathscr{A}$ is a closed subalgebra $\mathscr{B}$ of the center of $\mathscr{A}$ which contains the identity element. Thus, $\mathscr{B}$ is a commutative Banach algebra with compact maximal ideal space $M_{\mathscr{B}}$. For each maximal ideal x of $\mathscr{B}$, consider the smallest closed two-sided ideal $\mathscr{I}_x$ of $\mathscr{A}$ which contains x, and let Φ_x refer to the canonical homomorphism from $\mathscr{A}$ onto the quotient algebra $\mathscr{A}/\mathscr{I}_x$.

In contrast to the commutative setting where $\mathscr{B}/x \cong \mathbb{C}$ for all $x \in M_{\mathscr{B}}$, the quotient algebras $\mathscr{A}/\mathscr{I}_x$ will depend on $x \in M_{\mathscr{B}}$ in general. Moreover, it can happen that $\mathscr{I}_x = \mathscr{A}$ for certain maximal ideals x. In this case we *define* that $\Phi_x(a)$ is invertible in $\mathscr{A}/\mathscr{I}_x$ and that $\|\Phi_x(a)\| = 0$ for each $a \in \mathscr{A}$.

2.2.2 Allan's local principle

The proof of Allan's local principle is based on the following observation.

Proposition 2.2.1 (Allan)**.** *Let $\mathscr{B}$ be a central subalgebra of the unital Banach algebra $\mathscr{A}$. If $\mathscr{M}$ is a maximal left, right, or two-sided ideal of $\mathscr{A}$, then $\mathscr{M} \cap \mathscr{B}$ is a (two-sided) maximal ideal of $\mathscr{B}$.*

Proof. For definiteness, let $\mathscr{M}$ be a maximal left ideal of $\mathscr{A}$. Then $\mathscr{M} \cap \mathscr{B}$ is a proper (since $e \in \mathscr{B} \setminus \mathscr{M}$) closed two-sided ideal of $\mathscr{B}$. The maximality of $\mathscr{M} \cap \mathscr{B}$ will follow once we have shown that

$$\text{for all } z \in \mathscr{B} \setminus \mathscr{M}, \text{ there is a } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } z - \lambda e \in \mathscr{M}. \tag{2.2}$$

Indeed, let $\mathscr{I}$ be a two-sided ideal of $\mathscr{B}$ with $\mathscr{M} \cap \mathscr{B} \subset \mathscr{I}$ and $\mathscr{M} \cap \mathscr{B} \neq \mathscr{I}$. Choose $z \in \mathscr{I} \setminus (\mathscr{M} \cap \mathscr{B}) \subset \mathscr{B} \setminus \mathscr{M}$. According to (2.2), there is a $\lambda \in \mathbb{C}$ and an $l \in \mathscr{M} \cap \mathscr{B}$ with $e = \lambda^{-1} z + l$. Hence, $e \in \mathscr{I}$, whence $\mathscr{I} = \mathscr{B}$ and the maximality of $\mathscr{M} \cap \mathscr{B}$.

We are left with verifying (2.2). In a first step we show that every element $z \in \mathscr{B} \setminus \mathscr{M}$ has a unique inverse modulo $\mathscr{M}$. The set $\mathscr{I}_z := \{l + az : l \in \mathscr{M}, a \in \mathscr{A}\}$ is a left ideal of $\mathscr{A}$ which contains $\mathscr{M}$ properly (since $z \notin \mathscr{M}$). Since $\mathscr{M}$ is maximal, we must have $\mathscr{I}_z = \mathscr{A}$. Hence, $e \in \mathscr{I}_z$, and there is an $a \in \mathscr{A}$ with $az - e, za - e \in \mathscr{M}$ (note that $z \in \operatorname{Cen} A$). Thus, a is an inverse of z modulo $\mathscr{M}$.

Further, $\mathscr{K}_z := \{a \in \mathscr{A} : az \in \mathscr{M}\}$ is a proper (since $e \notin \mathscr{K}_z$) left ideal of $\mathscr{A}$ which contains $\mathscr{M}$. Since $\mathscr{M}$ is maximal, we have $\mathscr{K}_z = \mathscr{M}$. In particular, if a_1 and a_2 are inverses modulo $\mathscr{M}$ of z, then $a_1 - a_2 \in \mathscr{M}$. Thus, the inverses modulo $\mathscr{M}$ of z determine a unique element of the quotient space $\mathscr{A}/\mathscr{M}$.

Contrary to (2.2), suppose that $z - \lambda e \notin \mathscr{M}$ for all $\lambda \in \mathbb{C}$. Let $y^\pi(\lambda)$ denote the (uniquely determined) coset of $\mathscr{A}/\mathscr{M}$ containing the inverses modulo $\mathscr{M}$ of $z - \lambda e$. Then $y^\pi : \mathbb{C} \to \mathscr{A}/\mathscr{M}$ is an analytic function. Indeed, let $\lambda_0 \in \mathbb{C}$, and let $y_0 \in y^\pi(\lambda_0)$ be an inverse modulo $\mathscr{M}$ of $z - \lambda_0 e$. Then, for $|\lambda - \lambda_0| < 1/\|y_0\|$, the element $e - (\lambda - \lambda_0)y_0$ is invertible in $\mathscr{A}$, and it is readily verified that $y_0[e - (\lambda - \lambda_0)y_0]^{-1}$ is an inverse modulo $\mathscr{M}$ of $z - \lambda e$. Thus, for $|\lambda - \lambda_0| < 1/\|y_0\|$,

$$y^\pi(\lambda) = y_0[e - (\lambda - \lambda_0)y_0]^{-1} + \mathscr{M},$$

which implies the asserted analyticity. If $|\lambda| > 2\|z\|$, then $z - \lambda e$ is actually invertible in $\mathscr{A}$ and

$$\|y^\pi(\lambda)\| \le \|(z - \lambda e)^{-1}\| = \frac{1}{|\lambda|}\|(e - z/\lambda)^{-1}\| \le \frac{1}{|\lambda|}\frac{1}{1 - \|z\|/|\lambda|} < \frac{1}{\|z\|}.$$

Therefore, y^π is bounded, whence $y^\pi(\lambda) = 0$ for all $\lambda \in \mathbb{C}$ by Liouville's theorem. In particular, $y^\pi(0) = 0$. Thus, there is a $y_0 \in \mathscr{M}$ with $y_0 z - e \in \mathscr{M}$, whence $e \in \mathscr{M}$. This is impossible since $\mathscr{M}$ is a proper ideal of $\mathscr{A}$. This contradiction implies that there is a $\lambda \in \mathbb{C}$ with $z - \lambda e \in \mathscr{M}$. Since $z \notin \mathscr{M}$, one also has $\lambda \neq 0$. ■

Before tackling Allan's principle, let us recall that a function $f : M_{\mathscr{B}} \to \mathbb{R}$ is said to be *upper semi-continuous* at $x_0 \in M_{\mathscr{B}}$ if, for each $\varepsilon > 0$, there exists a neighborhood $U_\varepsilon \subset M_{\mathscr{B}}$ of x_0 such that $f(x) < f(x_0) + \varepsilon$ for any $x \in U_\varepsilon$. The function f is said to be upper semi-continuous on $M_{\mathscr{B}}$ if it is upper semi-continuous at each point of $M_{\mathscr{B}}$. Every upper semi-continuous function defined on a compact set attains its supremum.

Theorem 2.2.2 (Allan's local principle)**.** *Let $\mathscr{B}$ be a central subalgebra of the unital Banach algebra $\mathscr{A}$. Then:*

(i) *an element $a \in \mathscr{A}$ is invertible if and only if the cosets $\Phi_x(a)$ are invertible in $\mathscr{A}/\mathscr{I}_x$ for each $x \in M_{\mathscr{B}}$;*
(ii) *the mapping $M_{\mathscr{B}} \to \mathbb{R}^+$, $x \mapsto \|\Phi_x(a)\|$ is upper semi-continuous for every $a \in \mathscr{A}$;*
(iii) $\|a\| \ge \max_{x \in M_{\mathscr{B}}} \|\Phi_x(a)\|$*;*
(iv) $\cap_{x \in M_{\mathscr{B}}} \mathscr{I}_x$ *lies in the radical of $\mathscr{A}$.*

Proof. To prove (i) we show that $a \in \mathscr{A}$ is left invertible if and only if $\Phi_x(a)$ is left invertible for all $x \in M_{\mathscr{B}}$. The proof for the right invertibility is analogous.

Clearly, $\Phi_x(a)$ is left invertible if a is so. To verify the reverse implication assume the contrary, i.e., suppose $\Phi_x(a)$ to be left invertible in $\mathscr{A}/\mathscr{I}_x$ for all $x \in M_{\mathscr{B}}$

but let a have no left inverse in $\mathscr{A}$. Denote by $\mathscr{M}$ a maximal left ideal of $\mathscr{A}$ which contains the set $\mathscr{I} := \{ba : b \in \mathscr{A}\}$ (note that $e \notin \mathscr{I}$). Put $x = \mathscr{M} \cap \mathscr{B}$. By Proposition 2.2.1, x is a maximal ideal of $\mathscr{B}$. We claim that $\mathscr{I}_x \subseteq \mathscr{M}$. Indeed, if $l = \sum_{k=1}^n a_k x_k b_k$ where $x_k \in x$ and $a_k, b_k \in \mathscr{A}$, then $l = \sum_{k=1}^n a_k b_k x_k$ (because $\mathscr{B}$ is central), and hence $l \in \mathscr{M}$ (because $\mathscr{M}$ is a left ideal). Thus, $\mathscr{I}_x \subseteq \mathscr{M}$. By our assumption, $\Phi_x(a)$ is left invertible in $\mathscr{A}/\mathscr{I}_x$, that is, there exists a $b \in \mathscr{A}$ with $ba - e \in \mathscr{I}_x$, and since $\mathscr{I}_x \subseteq \mathscr{M}$ we have $ba - e \in \mathscr{M}$. On the other hand, $ba \in \mathscr{I} \subseteq \mathscr{M}$. This implies that $e \in \mathscr{M}$ which contradicts the maximality of $\mathscr{M}$.

(ii) Let $x \in M_{\mathscr{B}}$ and $\varepsilon > 0$. We have to show that there is a neighborhood U of x such that

$$\|\Phi_y(a)\| < \|\Phi_x(a)\| + \varepsilon \quad \text{for all } y \in U.$$

Choose elements $a_1, \dots, a_n \in \mathscr{A}$ and $x_1, \dots, x_n \in x$ with

$$\left\| a + \sum_{j=1}^n a_j x_j \right\| < \|\Phi_x(a)\| + \varepsilon/2, \tag{2.3}$$

set $\delta := \sum_{i=1}^n \|a_i\| + 1$, and define an open neighborhood $U \subset M_{\mathscr{B}}$ of x by

$$U := \{y \in M_{\mathscr{B}} : |\widehat{x_j}(y)| < \varepsilon/(2\delta) \text{ for } j = 1, \dots, n\}.$$

Let $y \in U$ and set $y_j := x_j - \widehat{x_j}(y)e$. Then $\widehat{y_j}(y) = \widehat{x_j}(y) - \widehat{x_j}(y)\widehat{e}(y) = 0$, whence $y_j \in y$ and

$$\|\Phi_y(a)\| \leq \left\| a + \sum_{j=1}^n a_j y_j \right\|. \tag{2.4}$$

The estimates (2.3) and (2.4) give

$$\begin{aligned}
\|\Phi_y(a)\| - \|\Phi_x(a)\| &\leq \left\|a + \sum a_j y_j\right\| - \left\|a + \sum a_j x_j\right\| + \varepsilon/2 \\
&\leq \left\|\sum a_j (y_j - x_j)\right\| + \varepsilon/2 \\
&= \left\|\sum \widehat{x_j}(y) a_j\right\| + \varepsilon/2 < \varepsilon,
\end{aligned}$$

whence the upper semi-continuity of $y \mapsto \|\Phi_y(a)\|$ at x.

(iii) By definition, $\|a\| \geq \|\Phi_x(a)\|$ for any $x \in M_{\mathscr{B}}$, which implies that $\|a\| \geq \sup_{x \in M_{\mathscr{B}}} \|\Phi_x(a)\|$. The supremum in this estimate is actually a maximum due to the compactness of $M_{\mathscr{B}}$.

(iv) Let $k \in \bigcap_{x \in M_{\mathscr{B}}} \mathscr{I}_x$. Then, for any $a \in \mathscr{A}$, $\Phi_x(e - ak) = \Phi_x(e)$ and by (i) above $e - ak$ is invertible. Thus, by Proposition 1.3.3, k belongs to the radical of $\mathscr{A}$. ■

2.2.3 Local invertibility and local spectra

As consequences of the upper semi-continuity, we mention the following properties of local invertibility and local spectra.

Let X be a locally compact Hausdorff space with one-point compactification $X \cup \{x_\infty\}$, and let M be a mapping from X into the set of all compact subsets of the complex plane. For each net $y := (y_t)_{t\in T}$ in X with limit x_∞, consider the set $L(y)$ of all limits of convergent nets $(\lambda_t)_{t\in T}$ with $\lambda_t \in M(y_t)$. The *limes superior* (also called the *partial limiting set*) $\limsup_{x\to x_\infty} M(x)$ is defined as the union of all sets $L(y)$, where the union is taken over all nets y in X tending to x_∞. For $X = \mathbb{Z}^+$, this definition coincides with that one given before Proposition 1.2.16. Below we apply this definition when Y is a compact Hausdorff space, $x_\infty \in Y$ and $X := Y \setminus \{x_\infty\}$.

Proposition 2.2.3. *Let the hypothesis be as in Theorem 2.2.2.*

(i) *If $\Phi_x(a)$ is invertible in $\mathscr{A}/\mathscr{I}_x$, then there is a neighborhood U of $x \in M_{\mathscr{B}}$ as well as a neighborhood V of $a \in \mathscr{A}$ such that $\Phi_y(c)$ is invertible in $\mathscr{A}/\mathscr{I}_y$ and*

$$\|\Phi_y(c)^{-1}\| \le 4\,\|\Phi_x(a)^{-1}\| \quad \text{for all } y \in U \text{ and } c \in V.$$

(ii) *For all $a \in \mathscr{A}$ and $x \in M_{\mathscr{B}}$,*

$$\limsup_{y\to x} \sigma(\Phi_y(a)) \subseteq \sigma(\Phi_x(a)).$$

The number 4 in the estimate in assertion (i) can be replaced by any constant greater than 1.

Proof. (i) Let $\Phi_x(a)$ be invertible and choose $b \in \mathscr{A}$ such that

$$\Phi_x(ab-e) = \Phi_x(ba-e) = 0.$$

By Theorem 2.2.2 (ii), the mappings

$$y \mapsto \|\Phi_y(ab-e)\| \quad \text{and} \quad y \mapsto \|\Phi_y(ba-e)\|$$

are upper semi-continuous on the maximal ideal space of $\mathscr{B}$. Hence,

$$\|\Phi_y(ab-e)\| < 1/4 \quad \text{and} \quad \|\Phi_y(ba-e)\| < 1/4$$

for all maximal ideals y in a certain neighborhood U' of x. Further, let V stand for the set of all elements $c \in \mathscr{A}$ with $\|c-a\| < (4\,\|b\|)^{-1}$. Then

$$\Phi_y(c)\Phi_y(b) = \Phi_y(e) + \Phi_y(cb-e) \quad \text{and} \quad \Phi_y(b)\Phi_y(c) = \Phi_y(e) + \Phi_y(bc-e)$$

with

$$\|\Phi_y(cb-e)\| \le \|\Phi_y(ab-e)\| + \|\Phi_y((c-a)b)\| \le 1/4 + \|c-a\|\,\|b\| < 1/2$$

and, analogously, $\|\Phi_y(bc-e)\| \le 1/2$. Since $\Phi(e)$ is the identity element in $\mathscr{A}/\mathscr{I}_y$, a Neumann series argument implies that $\Phi_y(c)$ is invertible in $\mathscr{A}/\mathscr{I}_y$ and that

$$\|\Phi_y(c)^{-1}\| \le 2\,\|\Phi_y(b)\| \quad \text{for all } y \in U' \text{ and } c \in V.$$

Employing the upper semi-continuity once more, one finally gets

$$\|\Phi_y(b)\| \le 2\,\|\Phi_x(b)\| = 2\,\|\Phi_x(a)^{-1}\|$$

for all y in a neighborhood $U \subseteq U'$ of x.

(ii) Let $\lambda \in \limsup_{y\to x} \sigma(\Phi_y(a))$. By definition of the limes superior, there is a net $(y_t)_{t\in T} \in M_{\mathscr{B}}$ with $y_t \to x$ and numbers $\lambda_t \in \sigma(\Phi_{y_t}(a))$ with $\lambda_t \to \lambda$. Consider the elements $a - \lambda_t e$, which converge to $a - \lambda e$ in the norm of $\mathscr{A}$. If the coset $\Phi_x(a-\lambda e)$ was invertible, then the local cosets $\Phi_{y_t}(a-\lambda_t e)$ would be invertible for all sufficiently large t by part (i) of this proposition. This is impossible due to the choice of the points λ_t. Consequently, $\lambda \in \sigma(\Phi_x(a))$. ■

2.2.4 Localization over central C^*-algebras

We will now have a closer look at Allan's local principle in the case that the central subalgebra $\mathscr{B}$ of $\mathscr{A}$ is a C^*-algebra. Thus, we let $\mathscr{A}, \mathscr{B}, M_{\mathscr{B}}, \mathscr{I}_x$ and Φ_x be as in the previous subsection, but in addition we assume that there is an involution on the central subalgebra $\mathscr{B}$ of $\mathscr{A}$ which makes $\mathscr{B}$ to a C^*-algebra with respect to the norm inherited from $\mathscr{A}$. In this context, we will obtain a nice expression for the local norm and a canonical representation for the elements in the local ideals. Further we will briefly discuss some issues related to inverse-closedness.

Proposition 2.2.4. *Let $\mathscr{A}$ be a unital Banach algebra and let $\mathscr{B}$ be a central C^*-subalgebra of $\mathscr{A}$ which contains the identity. Then, for each $a \in \mathscr{A}$ and $x \in M_{\mathscr{B}}$,*

$$\|\Phi_x(a)\| = \inf_b \|ab\|$$

where the infimum is taken over all $b \in \mathscr{B}$ with $0 \le b \le 1$ which are identically 1 in a certain neighborhood of x.

Proof. Denote the infimum on the right-hand side by q. If $b \in \mathscr{B}$ is identically 1 in some neighborhood of x, then $a(b-1)$ belongs to the local ideal $\mathscr{I}_x$, whence

$$\|\Phi_x(a)\| \le \|a + a(b-1)\| = \|ab\|.$$

Taking the infimum over all b with the properties mentioned above gives the estimate $\|\Phi_x(a)\| \le q$.

For the reverse estimate, let $\varepsilon > 0$. Choose functions $b_1, \dots, b_n \in \mathscr{B}$ which vanish at x and non-zero elements $a_1, \dots, a_n \in \mathscr{A}$ such that

$$\|a + a_1b_1 + \ldots + a_nb_n\| \le \|\Phi_x(a)\| + \varepsilon.$$

If b is any function in $\mathscr{B}$ with $0 \le b \le 1$ which is identically 1 in a certain neighborhood of x, then

$$\begin{aligned}\|ab\| &\le \left\|(a + \sum a_ib_i)b\right\| + \left\|\sum a_ib_ib\right\| \\ &\le \left\|a + \sum a_ib_i\right\| + \sum \|a_i\|\,\|b_ib\| \\ &\le \|\Phi_x(a)\| + \varepsilon + \sum \|a_i\|\,\|b_ib\|.\end{aligned}$$

The quantity on the right-hand side becomes smaller than $\|\Phi_x(a)\| + 2\varepsilon$ if b is chosen such that $\|b_ib\| < \varepsilon/(n\|a_i\|)$ for all i. Since $\varepsilon > 0$ is arbitrary, $\|ab\| \le \|\Phi_x(a)\|$, whence the estimate $q \le \|\Phi_x(a)\|$. ■

Here is the aforementioned result on the structure of elements in the local ideals.

Proposition 2.2.5. *Let $\mathscr{A}$ be a Banach algebra with identity e, $\mathscr{B}$ be a central C^*-subalgebra of $\mathscr{A}$ which contains e, and $x \in M_{\mathscr{B}}$. Then*

$$\mathscr{I}_x = \{ca : a \in \mathscr{A} \text{ and } c \in x\}. \tag{2.5}$$

Proof. By definition, the ideal $\mathscr{I}_x$ is the closure in $\mathscr{A}$ of the set of all finite sums

$$\sum_{j=1}^{n} c_ja_j \quad \text{with } c_j \in x \text{ and } a_j \in \mathscr{A}.$$

We claim that each sum of this form can be written as a single product ca with $c_j \in x$ and $a_j \in \mathscr{A}$. Clearly, it is sufficient to prove this fact for $n = 2$. Let $c_1, c_2 \in x$ and $a_1, a_2 \in \mathscr{A}$. We identify the elements of $\mathscr{B}$ with the corresponding Gelfand transforms. Define $c := \sqrt{|c_1| + |c_2|}$. Then $c \in x$, and the point x belongs to the set $N_c := \{y \in M_{\mathscr{B}} : c(y) = 0\}$. For $j = 1, 2$, put

$$g_j(y) := \begin{cases} c_j(y)/c(y) & \text{if } y \notin N_c, \\ 0 & \text{if } y \in N_c. \end{cases}$$

For $y \notin N_c$, one has

$$|g_j(y)| = \frac{|c_j(y)|}{|c(y)|} = \frac{|c_j(y)|}{|c_1(y)| + |c_2(y)|}|c(y)| \le |c(y)|.$$

Since the estimate $|g_j(y)| \le |c(y)|$ holds for $y \in N_c$ as well, the functions g_j are continuous. Thus, they can be identified with elements of $\mathscr{B}$. Since $c_j = cg_j$, it follows that $c_1a_1 + c_2a_2 = c(g_1a_1 + g_2a_2)$ as desired. Hence, the set on the right-hand side of (2.5) forms an ideal of $\mathscr{A}$. We abbreviate this ideal by $\mathscr{I}_x'$ for a moment.

Next we are going to prove that $\mathscr{I}_x'$ is a closed ideal. Let d be in the closure of $\mathscr{I}_x'$. Given any convergent series $\sum_{k=1}^{\infty} \varepsilon_k$ of positive numbers, there are elements

$a_k \in \mathscr{A}$ and $c_k \in x$ such that $\|d - c_k a_k\| < \varepsilon_k/2$ for every k. For each k, there is an open neighborhood $U_k \subset M_{\mathscr{B}}$ of x such that $\|a_k\||c_k(y)| < \varepsilon_k/2$ for all $y \in U_k$. Without loss of generality, one can assume that $\overline{U_{k+1}} \subset U_k$ for every k. Further, let f_k be elements of $\mathscr{B}$ with $0 \le f_k \le 1$ and such that $f_k|_{\overline{U_{k+1}}} = 1$ and $f_k|_{M_{\mathscr{B}} \setminus U_k} = 0$ for all k. Then

$$\|c_k f_k\| = \|c_k f_k\|_\infty \le \sup_{y \in U_k} |c_k(y)|,$$

whence

$$\|f_k d\| \le \|d - c_k a_k\| \|f_k\| + \|a_k\| \|c_k f_k\| < \varepsilon_k.$$

Consequently, the series $d + \sum_{k=1}^\infty f_k d$ is absolutely convergent. Let $d_\infty \in \mathscr{A}$ denote the limit of that series. The estimate

$$\begin{aligned} 0 &\le \left(1 + \sum_{k=1}^{n} f_k(y)\right)^{-1} - \left(1 + \sum_{k=1}^{n+m} f_k(y)\right)^{-1} \\ &\le \begin{cases} (1 + \sum_{k=1}^n f_k(y))^{-1} & \text{for } y \in U_{n+1} \\ 0 & \text{for } y \in M_{\mathscr{B}} \setminus U_{n+1} \end{cases} \le \frac{1}{n+1} \end{aligned}$$

shows that $(1 + \sum_{k=1}^n f_k)^{-1}$ converges in $\mathscr{B}$ as $n \to \infty$ to some element c. Clearly, $c(x) = 0$, whence $c \in x$. Since $d = cd_\infty$ by construction, $\mathscr{I}_x'$ is closed. Thus, $\mathscr{I}_x'$ is a closed ideal of $\mathscr{A}$ which contains the ideal x of $\mathscr{B}$. Since $\mathscr{I}_x$ is the smallest ideal of $\mathscr{A}$ with these properties, the assertion follows. ∎

Finally we will show that every central C^*-subalgebra of a Banach algebra is inverse-closed. Recall that the algebra $\mathscr{B}$ is inverse-closed in $\mathscr{A}$ if every element $b \in \mathscr{B}$ which is invertible in $\mathscr{A}$ possesses an inverse in $\mathscr{B}$. The following definitions make sense in the context of the general local principle (i.e., without assuming the C^*-property of $\mathscr{B}$). Also Lemma 2.2.6 holds in the general context.

Write the maximal ideal space $M_{\mathscr{B}}$ as $M_{\mathscr{B}}^0 \cup M_{\mathscr{B}}^+$ where $M_{\mathscr{B}}^0$ collects those maximal ideals x of $\mathscr{B}$ for which $\mathscr{I}_x \equiv \mathscr{A}$ and where $M_{\mathscr{B}}^+$ is the complement of $M_{\mathscr{B}}^0$ in $M_{\mathscr{B}}$. The set $M_{\mathscr{B}}^0$ is open in $M_{\mathscr{B}}$. Indeed, if $x \in M_{\mathscr{B}}^0$, then $\Phi_x(0)$ is invertible by definition. By Proposition 2.2.3, $\Phi_y(0)$ is invertible for all y in a certain open neighborhood U of x. This is only possible if $y \in M_{\mathscr{B}}^0$.

Lemma 2.2.6. *If $M_{\mathscr{B}}^0 = \emptyset$, then $\mathscr{B}$ is inverse-closed in $\mathscr{A}$.*

Proof. Assume $\mathscr{B}$ is not inverse-closed. Then there is an element $b \in \mathscr{B}$ which is invertible in $\mathscr{A}$ but not in $\mathscr{B}$. Hence, b is contained in some maximal ideal x of $\mathscr{B}$. Then $b \in \mathscr{I}_x$, and $b + \mathscr{I}_x$ is 0 in $\mathscr{A}/\mathscr{I}_x$. But on the other hand, $b + \mathscr{I}_x$ is invertible in $\mathscr{A}/\mathscr{I}_x$. This is possible only if $x \in M_{\mathscr{B}}^0$. Hence, the component $M_{\mathscr{B}}^0$ of $M_{\mathscr{B}}$ is not empty. ∎

Corollary 2.2.7. *Let $\mathscr{A}$ be a Banach algebra with identity e and let $\mathscr{B}$ be a central C^*-subalgebra of $\mathscr{A}$ which contains e. Then:*

(i) *$M^0_{\mathscr{B}}$ is empty;*
(ii) *$\mathscr{B}$ is inverse-closed in $\mathscr{A}$.*

Proof. Let $x \in M^0_{\mathscr{B}}$. Then $\Phi_x(e) = 0$. Proposition 2.2.4 implies that $\inf \|eb\| = \inf \|b\| = 0$, with the infimum taken over all $b \in \mathscr{B}$ with $0 \le b \le 1$ which are identically 1 in a certain neighborhood of x. This is impossible since the Gelfand transform acts as an isometry on $\mathscr{B}$ by the Gelfand-Naimark theorem, whence $\|b\| \ge |\widehat{b}(x)| = 1$. The second assertion follows via Lemma 2.2.6. ■

The most general result regarding inverse-closedness of C^*-algebras in Banach algebras was obtained by Goldstein [78]. By a C^*-subalgebra $\mathscr{B}$ of a Banach algebra $\mathscr{A}$ we mean a (not necessarily closed) subalgebra of $\mathscr{A}$ which carries the structure of a C^*-algebra, i.e., there is an involution and a norm on $\mathscr{B}$ which make $\mathscr{B}$ into a C^*-algebra with respect to the operations inherited from $\mathscr{A}$.

Theorem 2.2.8 (Goldstein)**.** *Let $\mathscr{A}$ be a unital Banach algebra, and let $\mathscr{B}$ be a (not necessarily closed) C^*-subalgebra of $\mathscr{A}$ which contains the identity. Then $\mathscr{B}$ is inverse-closed in $\mathscr{A}$.*

We wish to add a related result. It is well known that the maximal ideal space of a singly generated unital Banach algebra is homeomorphic to the spectrum of its generating element (see Exercise 2.1.3).

Proposition 2.2.9. *Let $\mathscr{A}$ be a unital Banach algebra and let $\mathscr{B}$ be a central closed subalgebra of $\mathscr{A}$ which contains the identity and which is singly generated by an element b and the identity. Identify the maximal ideal space of $\mathscr{B}$ with $\sigma_{\mathscr{B}}(b)$. Then $M^+_{\mathscr{B}} = \sigma_{\mathscr{A}}(b)$.*

Proof. If $x \in M^+_{\mathscr{B}}$, then

$$\Phi_x(b - xe) = \widehat{b}(x)\Phi_x(e) - x\Phi_x(e) = 0\Phi_x(e) = \Phi_x(0)$$

is not invertible in $\mathscr{A}_x$. (Note that if $M_{\mathscr{B}}$ is identified with $\sigma_{\mathscr{B}}(b)$ then the Gelfand transform of b is the identity mapping on $\sigma_{\mathscr{B}}(b)$.) By Allan's local principle, $b - xe$ is not invertible in $\mathscr{A}$, whence $x \in \sigma_{\mathscr{A}}(b)$.

Conversely, let $x \in \sigma_{\mathscr{A}}(b)$. Then $b - xe$ is not invertible in $\mathscr{A}$, and Allan's local principle implies the existence of a point $y \in \sigma_{\mathscr{B}}(b) = M_{\mathscr{B}}$ such that $\Phi_y(b - xe)$ is not invertible. On the other hand, the element

$$\Phi_z(b - xe) = \widehat{b}(z)\Phi_z(e) - x\Phi_z(e) = (z - x)\Phi_z(e)$$

is invertible for every $z \ne x$. Thus, $y = x$, i.e., $\Phi_x(b - xe)$ is not invertible. This implies that $x \in M^+_{\mathscr{B}}$. ■

2.2.5 Douglas' local principle and sufficient families

We are now going to specialize Allan's local principle to the context of C^*-algebras. Allan's local principle provides us with a sufficient family of homomorphisms. We will first discuss sufficient families in the context of C^*-algebras.

Let $\mathscr{A}$ be a unital C^*-algebra, $(\mathscr{B}_t)_{t\in T}$ a family of unital C^*-algebras, and $\mathscr{W} := (\mathsf{W}_t)_{t\in T}$ a family of unital *-homomorphisms $\mathsf{W}_t : \mathscr{A} \to \mathscr{B}_t$. Further, let $\mathscr{F}$ stand for the direct product of the C^*-algebras $\mathscr{B}_t$ with $t \in T$, and let W denote the *-homomorphism

$$\mathsf{W} : \mathscr{A} \to \mathscr{F}, \quad a \mapsto (t \mapsto \mathsf{W}_t(a)). \tag{2.6}$$

Besides sufficient families of homomorphisms it will be convenient to introduce families of homomorphisms which are sufficient in a weaker sense. The family $\mathscr{W}$ is called *weakly sufficient* if the implication

$$\begin{gathered}\mathsf{W}_t(a) \text{ is invertible in } \mathscr{B}_t \text{ for every } t \in T \text{ and } \sup_{t\in T} \|\mathsf{W}_t(a)^{-1}\| < \infty \\ \Rightarrow \quad a \text{ is invertible in } \mathscr{A}\end{gathered}$$

holds for every $a \in \mathscr{A}$.

Theorem 2.2.10. *Let $\mathscr{A}$ be a unital C^*-algebra. The following conditions are equivalent for a family $\mathscr{W} = (\mathsf{W}_t)_{t\in T}$ of unital *-homomorphisms $\mathsf{W}_t : \mathscr{A} \to \mathscr{B}_t$:*

(i) *the family $\mathscr{W}$ is weakly sufficient;*
(ii) *if $\mathsf{W}_t(a) = 0$ for every $t \in T$, then $a = 0$;*
(iii) *for every $a \in \mathscr{A}$, $\|a\| = \sup_{t\in T} \|\mathsf{W}_t(a)\|$;*
(iv) *the homomorphism (2.6) is a symbol mapping for $\mathscr{A}$.*

Proof. For the implication (i) $\Rightarrow$ (ii), let $a \in \mathscr{A}$ be an element such that $\mathsf{W}_t(a) = 0$ for all $t \in T$, and let b be an arbitrary invertible element of $\mathscr{A}$. Then $\mathsf{W}_t(b)$ is invertible for all $t \in T$, and the norms $\|\mathsf{W}_t(b)^{-1}\|$ are uniformly bounded by $\|b^{-1}\|$. Consequently, the elements $\mathsf{W}_t(a+b)$ are invertible for all $t \in T$, and the norms of their inverses are uniformly bounded, too. By (i), the element $a+b$ is invertible. Hence, a belongs to the radical of $\mathscr{A}$ which consists of the zero element only.

If hypothesis (ii) is satisfied, then the gluing mapping (2.6) is a *-homomorphism with kernel $\{0\}$. Hence, W is an isometry, which is equivalent to (iii). The implication (iii) $\Rightarrow$ (iv) follows since every isometry is a symbol mapping.

Finally, for the implication (iv) $\Rightarrow$ (i), let $a \in \mathscr{A}$ be an element for which the operators $\mathsf{W}_t(a)$ are invertible for all $t \in T$ and the norms $\|\mathsf{W}_t(a)^{-1}\|$ are uniformly bounded. Then $\mathsf{W}(a)$ is invertible in the direct product $\mathscr{F}$. Due to the inverse-closedness of C^*-algebras, $\mathsf{W}(a)$ is also invertible in $\mathsf{W}(\mathscr{A})$. Since W is a symbol mapping, a is invertible in $\mathscr{A}$. ■

In the proof of the next result, the concept of the square root of a positive element will be used. The square root of a positive element in a C^*-algebra is defined via the continuous functional calculus, which is an immediate corollary of the Gelfand-Naimark theorem (Theorem 2.1.7).

Theorem 2.2.11. *Let $\mathscr{A}$ be a unital C^*-algebra. The following conditions are equivalent for a family $\mathscr{W} = (\mathsf{W}_t)_{t\in T}$ of unital *-homomorphisms $\mathsf{W}_t : \mathscr{A} \to \mathscr{B}_t$:*

(i) *the family $\mathscr{W}$ is sufficient;*
(ii) *for every $a \in \mathscr{A}$, there is a $t \in T$ such that $\|\mathsf{W}_t(a)\| = \|a\|$.*

Thus, a weakly sufficient family $\mathscr{W}$ is sufficient if and only if the supremum in Theorem 2.2.10 (iii) is attained.

Proof. (i) $\Rightarrow$ (ii): Suppose there is an $a \in \mathscr{A}$ such that

$$\|\mathsf{W}_t(a)\| < \sup_{s\in T} \|\mathsf{W}_s(a)\| \quad \text{for all } t \in T. \tag{2.7}$$

Since

$$\begin{aligned}\|\mathsf{W}_t(a)\|^2 &= \|\mathsf{W}_t(a)^*\mathsf{W}_t(a)\| \\ &= \|(\mathsf{W}_t(a)^*\mathsf{W}_t(a))^{1/2}(\mathsf{W}_t(a)^*\mathsf{W}_t(a))^{1/2}\| \\ &= \|(\mathsf{W}_t(a)^*\mathsf{W}_t(a))^{1/2}\|^2 = \|\mathsf{W}_t((a^*a)^{1/2})\|^2,\end{aligned}$$

one can assume without loss of generality that the element a in (2.7) is self-adjoint and positive. Since the norm of a self-adjoint element b coincides with its spectral radius $r(b)$, (2.7) can be rewritten as

$$r(\mathsf{W}_t(a)) < \sup_{s\in T} r(\mathsf{W}_s(a)) \quad \text{for all } t \in T. \tag{2.8}$$

Denote the supremum on the right-hand side of (2.8) by M and set $c := a - Me$. The elements $\mathsf{W}_t(c) = \mathsf{W}_t(a) - Me_t$ are invertible for all $t \in T$ since $r(\mathsf{W}_t(a)) < M$. Thus, $c = a - Me$ is invertible by hypothesis (i). Since the set of the invertible elements is open, we get the invertibility of $a - me$ for all $m \in \mathbb{R}$ belonging to some neighborhood U of M. On the other hand, by the definition of the supremum, there is an $s_U \in T$ such that $m_U := r(\mathsf{W}_{s_U}(a))$ belongs to U. The element $\mathsf{W}_{s_U}(a) - m_U e_{s_U}$ is not invertible, because the spectral radius of a positive element belongs to the spectrum of that element. Hence, $a - m_U e$ is not invertible. This contradiction proves the assertion.

(ii) $\Rightarrow$ (i): Let $a \in \mathscr{A}$ be not invertible. We claim that there is a $t \in T$ such that $\mathsf{W}_t(a)$ is not invertible.

If a is not invertible, then at least one of the elements aa^* or a^*a is not invertible, say a^*a for definiteness. Since a^*a is non-negative, a clear application of the Gelfand-Naimark theorem yields

$$\big\| \|a^*a\|e - a^*a\big\| = \|a^*a\|. \tag{2.9}$$

Set $b := \|a^*a\|e - a^*a$, and choose $t \in T$ such that $\|\mathsf{W}_t(b)\| = \|b\|$. Then (2.9) implies

$$\big\| \|a^*a\|e_t - \mathsf{W}_t(a^*a)\big\| = \|\mathsf{W}_t(b)\| = \|b\| = \|a^*a\|.$$

Since $\|\mathsf{W}_t(a^*a)\| \le \|a^*a\|$, one can apply the Gelfand-Naimark theorem once more to get the non-invertibility of $\mathsf{W}_t(a^*a)$ and, thus, of $\mathsf{W}_t(a)$. ∎

Theorem 2.2.12 (Douglas' local principle)**.** *Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{B}$ a symmetric closed subalgebra of the center of $\mathscr{A}$ which contains the identity element. Then the assertions of Allan's local principle can be completed as follows:*

(i) $M^0_{\mathscr{B}} = \emptyset$;
(ii) $\|a\| = \max_{x\in M_{\mathscr{B}}} \|\Phi_x(a)\|$ *for each* $a \in \mathscr{A}$;
(iii) $\cap_{x\in M_{\mathscr{B}}} \mathscr{I}_x = \{0\}$.

Proof. Assertion (i) is Corollary 2.2.7 (i). Assertion (ii) follows from Theorem 2.2.11, and (iii) is a consequence of the semi-simplicity of C^*-algebras. ∎

2.2.6 Example: SIOs with piecewise continuous coefficients

Consider the algebra $PC(\mathbb{T})$ of all *piecewise continuous functions* on $\mathbb{T}$, that is, the algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ which possess finite one-sided limits $a(x^\pm)$ at each point of $\mathbb{T}$. For definiteness, let $a(x^+)$ refer to the limit of a at x, taken in the clockwise direction. The algebra $PC(\mathbb{T})$ is naturally embedded in $L^\infty(\mathbb{T})$ and contains $C(\mathbb{T})$. Further, let $\mathscr{A} := \mathrm{alg}(S, PC(\mathbb{T}))$ stand for the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{T}))$ which contains the singular integral operator S, all operators of multiplication by a piecewise continuous function, and the ideal $\mathscr{K}$ of the compact operators.

We are interested in the subalgebra $\mathscr{A}^{\mathscr{K}} := \mathscr{A}/\mathscr{K}$ of the Calkin algebra. Proposition 1.4.12 implies that the set $C(\mathbb{T})+\mathscr{K} = \{fI+\mathscr{K} : f \in C(\mathbb{T})\}$ is a central subalgebra of $\mathscr{A}^{\mathscr{K}}$, and this algebra is isometrically isomorphic to the algebra $C(\mathbb{T})$ by Proposition 1.4.11. The maximal ideal space of $C(\mathbb{T})+\mathscr{K}$ is homeomorphic to $\mathbb{T}$, and the maximal ideal corresponding to $x \in \mathbb{T}$ is $\{(fI)+\mathscr{J} : f \in C(\mathbb{T}), f(x) = 0\}$, as was seen in Section 1.4.3.

Let $\mathscr{I}_x$ denote the smallest closed two-sided ideal in $\mathscr{A}^{\mathscr{K}}$ which contains the maximal ideal x of $C(\mathbb{T})+\mathscr{K}$. Allan's local principle transfers the invertibility problem in $\mathscr{A}^{\mathscr{K}}$ to a family of invertibility problems, one in each local algebra $\mathscr{A}^{\mathscr{K}}_x := \mathscr{A}^{\mathscr{K}}/\mathscr{I}_x$. Let $\Phi^{\mathscr{K}}_x$ stand for the canonical homomorphism from $\mathscr{A}$ to $\mathscr{A}^{\mathscr{K}}_x$. The next results will give some clues about the nature of the local algebras $\mathscr{A}^{\mathscr{K}}_x$.

Lemma 2.2.13. *If $c \in PC(\mathbb{T})$ is continuous at x and $c(x) = 0$, then $\Phi^{\mathscr{K}}_x(cI) = 0$.*

Proof. Given $\varepsilon > 0$, choose $f_\varepsilon \in C(\mathbb{T})$ such that $0 \le f_\varepsilon < 1$ except at x, where $f_\varepsilon(x) = 1$, and that the support of f_ε is contained in the interval $[xe^{-\mathbf{i}\varepsilon}, xe^{+\mathbf{i}\varepsilon}]$ of $\mathbb{T}$. It is easy to see that $\Phi^{\mathscr{K}}_x(f_\varepsilon I)$ is the identity in the local algebra. Consequently,

$$\|\Phi^{\mathscr{K}}_x(cI)\| = \|\Phi^{\mathscr{K}}_x(cI)\Phi^{\mathscr{K}}_x(f_\varepsilon I)\| = \|\Phi^{\mathscr{K}}_x(cf_\varepsilon I)\| \le \|cf_\varepsilon\|_{L^\infty},$$

and the last norm can be as small as desired by choosing ε small enough. ■

For $x \in \mathbb{T}$, define

$$\chi_x(t) := \begin{cases} 0 & \text{if } t \in]xe^{-\mathbf{i}\pi}, x[, \\ 1 & \text{if } t \in [x, xe^{-\mathbf{i}\pi}]. \end{cases}$$

Proposition 2.2.14. *Let $x \in \mathbb{T}$. Every local algebra $\mathscr{A}_x^{\mathscr{K}}$ is unital, and it is generated by the identity element and by two idempotents, namely $\Phi_x^{\mathscr{K}}(\chi_x I)$ and $\Phi_x^{\mathscr{K}}(P)$.*

Proof. It is evident that $\Phi_x^{\mathscr{K}}(I)$ is the identity element of $\mathscr{A}_x^{\mathscr{K}}$ and that $\Phi_x^{\mathscr{K}}(S) = \Phi_x^{\mathscr{K}}(2P - I)$ is a linear combination of the identity element and the idempotent $\Phi_x^{\mathscr{K}}(P)$. Now let $a \in PC(\mathbb{T})$. Then, by Lemma 2.2.13,

$$\begin{aligned} \Phi_x^{\mathscr{K}}(aI) &= \Phi_x^{\mathscr{K}}\left(a(x^-)(1-\chi_x)I + a(x^+)\chi_x I\right) \\ &= a(x^-)\Phi_x^{\mathscr{K}}(I) + \left(a(x^+) - a(x^-)\right)\Phi_x^{\mathscr{K}}(\chi_x I), \end{aligned}$$

representing $\Phi_x^{\mathscr{K}}(aI)$ as a linear combination of the identity element and the idempotent $\Phi_x^{\mathscr{K}}(\chi_x I)$. Since $\Phi_x^{\mathscr{K}}(S)$ together with all cosets $\Phi_x^{\mathscr{K}}(aI)$ generate the algebra $\mathscr{A}_x^{\mathscr{K}}$, the assertion follows. ■

We would like to emphasize once more that the local algebras $\mathscr{A}_x^{\mathscr{K}}$ are generated by two idempotents (and the identity). Algebras generated by (two or more) idempotents appear frequently as local algebras, and Chapter 3 will be devoted to a detailed study of them.

2.2.7 Exercises

Exercise 2.2.1. Show that the family $\{\delta_t\}_{t\in[0,1]}$ of homomorphisms $\delta_t : f \mapsto f(t)$ is sufficient for $C[0, 1]$, whereas $\{\delta_t\}_{t\in[0,1[}$ is weakly sufficient but not sufficient.

Exercise 2.2.2. Describe the center of the algebra $\mathbb{C}^{2\times 2}$. More generally, describe the center of the algebra $\mathscr{L}(E)$ when E is a Banach space.

Exercise 2.2.3. Describe the center of the algebra $M_2([0,1],\mathbb{C})$ of the functions $f : [0,1] \to \mathbb{C}^{2\times 2}$. What is the result of localization of $M_2([0,1],\mathbb{C})$ over its center via Allan's local principle?

Exercise 2.2.4. Localize $C(\mathbb{T})$ with respect to the disk algebra $\mathbb{A}$.

Exercise 2.2.5. Let $\mathscr{T}(PC)$ stand for the smallest closed subalgebra of $\mathscr{L}(l^2(\mathbb{Z}^+))$ which contains all Toeplitz operators with piecewise continuous functions.

(i) Show that $\mathscr{T}(C)/\mathscr{K}(l^2(\mathbb{Z}^+))$ is a central subalgebra of $\mathscr{T}(PC)/\mathscr{K}(l^2(\mathbb{Z}^+))$.
(ii) Using Allan's local principle, localize the algebra $\mathscr{T}(PC)/\mathscr{K}(l^2(\mathbb{Z}^+))$ over the maximal ideal space $\mathbb{T}$ of $\mathscr{T}(C)/\mathscr{K}(l^2(\mathbb{Z}^+))$ (recall Exercise 2.1.4 in

this connection). For $x \in \mathbb{T}$, write Φ_x for the canonical homomorphism from $\mathscr{T}(PC)$ onto the associated local algebra at x. Show that

$$\Phi_x(T(a)) = \Phi_x(a(x^-)T(1-\chi_x) + a(x^+)T(\chi_x))$$

with the notation as in Section 2.2.6. Thus, the local algebra at x is singly generated by $\Phi_x(T(\chi_x))$ (and the identity element).

(iii) Show that the spectrum of $\Phi_x(T(\chi_x))$ is the interval $[0, 1]$.

(iv) Conclude that the Toeplitz operator $T(a)$ with $a \in PC(\mathbb{T})$ is Fredholm on $l^2(\mathbb{Z}^+)$ if and only if the function

$$\hat{a} : \mathbb{T} \times [0, 1] \to \mathbb{C}, \quad (x, t) \mapsto a(x^-)(1-t) + a(x^+)t \tag{2.10}$$

has no zeros.

(v) Conclude from Douglas' local principle that the algebra $\mathscr{T}(PC)/\mathscr{K}(l^2(\mathbb{Z}^+))$ is commutative.

(vi) Show that there is a bijection between the maximal ideal space of the quotient algebra $\mathscr{T}(PC)/\mathscr{K}(l^2(\mathbb{Z}^+))$ and $\mathbb{T} \times [0, 1]$ and that, under the identification of these two sets, the Gelfand transform of the coset $T(a) + \mathscr{K}(l^2(\mathbb{Z}^+))$ is given by (2.10).

Warning: the maximal ideal space of $\mathscr{T}(PC)/\mathscr{K}(l^2(\mathbb{Z}^+))$ and the product $\mathbb{T} \times [0, 1]$ coincide as sets, but the Gelfand topology on $\mathbb{T} \times [0, 1]$ is quite different from the common (Euclidean) product topology. For details see [21, Section 4.88].

Exercise 2.2.6. Let $\mathscr{A}$ be a unital Banach algebra and $p \in \mathscr{A}$ a non-trivial idempotent in the center of $\mathscr{A}$. Then $\operatorname{alg}\{p\}$ and $\operatorname{alg}\{e-p\}$ are maximal ideals of $\operatorname{alg}\{e, p\}$ which we denote by 0 and 1. Show that the local algebras $\mathscr{A}_0$ and $\mathscr{A}_1$ can be identified with $(e-p)\mathscr{A}(e-p)$ and $p\mathscr{A}p$, respectively. (Note that $e-p$ and p are considered as the identity elements of these algebras.)

2.3 Norm-preserving localization

2.3.1 Faithful localizing pairs

Allan's local principle replaces the question of whether an element a in a unital Banach algebra $\mathscr{A}$ is invertible by a whole variety of "simpler" invertibility problems in local algebras $\mathscr{A}/\mathscr{I}_x$. The transition from $\mathscr{A}$ to its local algebras perfectly respects spectral properties: An element in $\mathscr{A}$ is invertible *if and only if* all local representatives of that element are invertible. But if one is interested in the structure of the algebra rather than in the spectra of its elements then this localization can fail. The point is that the intersection of the local ideals $\mathscr{I}_x$ can contain non-zero elements, in which case some structural information gets lost in the process of localization. This cannot happen in the case that $\mathscr{A}$ is a C^*-algebra which is localized over one of its central C^*-subalgebras (compare Douglas' local principle). In this section we are going to establish an effective mixture between Allan and Douglas (likewise, between Banach and C^*-algebras) which combines the advantages of Allan's principle (broad applicability) with those of Douglas' principle (no loss of structural information).

Let $\widehat{b}$ again refer to the Gelfand transforms of the element b of a commutative C^*-algebra.

Definition 2.3.1. Let $\mathscr{A}$ be a unital Banach algebra and $\mathscr{B}$ a subalgebra of $\mathscr{A}$. We say that $(\mathscr{A}, \mathscr{B})$ is a *faithful localizing pair*[2] if:

a) $\mathscr{B}$ is contained in the center of $\mathscr{A}$ and includes the identity element of $\mathscr{A}$;
b) there is an involution $b \mapsto b^*$ in $\mathscr{B}$ that turns $\mathscr{B}$ into a C^*-algebra;
c) $\|a(b_1+b_2)\| \leq \max\{\|ab_1\|, \|ab_2\|\}$ for all elements $a \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$ with $\operatorname{supp}\widehat{b}_1 \cap \operatorname{supp}\widehat{b}_2 = \emptyset$.

Of course, c) is the striking condition in Definition 2.3.1. If, also, the "outer" algebra $\mathscr{A}$ is C^*, then this condition is satisfied automatically.

Proposition 2.3.2. *Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{B}$ be a central and unital C^*-subalgebra of $\mathscr{A}$. Then $(\mathscr{A}, \mathscr{B})$ is a faithful localizing pair.*

Proof. Let $a \in \mathscr{A}$, and let b_1, b_2 be elements of $\mathscr{B}$ such that $\operatorname{supp}\widehat{b}_1 \cap \operatorname{supp}\widehat{b}_2 = \emptyset$. Further, let $r : \mathscr{A} \to \mathbb{R}^+$ denote the spectral radius function. Taking into account that $b_1 b_2 = 0$ one gets

$$\begin{aligned}
\|a(b_1+b_2)\|^2 &= r\left((b_1+b_2)(b_1^*+b_2^*)a^*a\right) \\
&= \lim_{n\to\infty} \|(b_1+b_2)^n(b_1^*+b_2^*)^n(a^*a)^n\|^{1/n} \\
&= \lim_{n\to\infty} \|(b_1^*)^n b_1^n (a^*a)^n + (b_2^*)^n b_2^n (a^*a)^n\|^{1/n} \\
&\leq \lim_{n\to\infty} \left(\|b_1^*a^*ab_1\|^n + \|b_2^*a^*ab_2\|^n\right)^{1/n} \\
&= \max\{\|ab_1\|^2, \|ab_2\|^2\},
\end{aligned}$$

which is the assertion. ■

[2] Formerly we used the notation "$\mathscr{A}$ is KMS with respect to $\mathscr{B}$" instead of "$(\mathscr{A}, \mathscr{B})$ is a faithful localizing pair", and we called c) the "KMS-property" of $\mathscr{A}$. We changed this notation to avoid confusion with a standard abbreviation in C^*-theory where KMS stands for "Kubo/Martin/Schwinger" (and not for political reasons as one might guess: our "KMS" was named after the town "Karl-Marx-Stadt" where the material presented in this section was developed in the eighties; since 1990 this town has again borne its historic name "Chemnitz").

2.3.2 Local norm estimates

Let $(\mathscr{A}, \mathscr{B})$ be a faithful localizing pair. In accordance with Allan's local principle, we localize $\mathscr{A}$ over $\mathscr{B}$ and get local ideals $\mathscr{I}_x$ and local homomorphisms $\Phi_x(a)$ for every x in the maximal ideal space $M_{\mathscr{B}}$ of $\mathscr{B}$. In the present setting it seems to be more convenient to write $\widehat{a}(x)$ instead of $\Phi_x(a)$.

Here is the main result on faithful (norm-preserving) localization.

Theorem 2.3.3. *Let $\mathscr{A}$ be a unital Banach algebra and $\mathscr{B}$ be a central and unital C^*-subalgebra of $\mathscr{A}$. Then $(\mathscr{A}, \mathscr{B})$ is a faithful localizing pair if and only if*

$$\|a\| = \max_{x \in M_{\mathscr{B}}} \|\widehat{a}(x)\|. \tag{2.11}$$

Proof. Let $(\mathscr{A}, \mathscr{B})$ be a faithful localizing pair. By assertion (iii) of Theorem 2.2.2, $\max_{x\in M_{\mathscr{B}}} \|\widehat{a}(x)\| \le \|a\|$. It remains to verify the reverse inequality. Let a be in $\mathscr{A}$. As a consequence of Proposition 2.2.4, given $x \in M_{\mathscr{B}}$ and $\varepsilon > 0$, there is a b in $C(M_{\mathscr{B}})$ such that $0 \le \widehat{b} \le 1$, the support of $\widehat{b}$ is contained in some open neighborhood $U(x)$ of x, and $\|ba\| < \|\widehat{a}(x)\| + \varepsilon$. Consequently, each x in $M_{\mathscr{B}}$ possesses an open neighborhood $U(x)$ such that $\|ba\| < \max_{x\in M_{\mathscr{B}}} \|\widehat{a}(x)\| + \varepsilon$ whenever $b \in C(M_{\mathscr{B}})$, $0 \le \widehat{b} \le 1$, and $\operatorname{supp} \widehat{b} \subseteq U(x)$. Choose a finite number $U_1, \ldots, U_n$ of these neighborhoods which cover $M_{\mathscr{B}}$, fix any (large) positive integer m, and let $k \in \{1, \ldots, m\}$. Further, let $e = f_1 + \cdots + f_n$ be a partition of unity subordinate to the covering $M_{\mathscr{B}} = \cup_{i=1}^n U_i$ (i.e., every $\widehat{f_i}$ is a continuous function with values in $[0, 1]$ and support in U_i), and put

$$V_{ki}^m := \left\{ x \in M_{\mathscr{B}} : \widehat{f}_i(x) \ge \frac{k+1}{n(m+1)},\ \widehat{f}_{i+1}(x) \le \frac{k}{n(m+1)}, \ldots, \widehat{f}_n(x) \le \frac{k}{n(m+1)} \right\}$$

for $i = 1, \ldots, n-1$, and

$$V_{kn}^m := \left\{ x \in M_{\mathscr{B}} : \widehat{f}_n(x) \ge \frac{k+1}{n(m+1)} \right\}.$$

A straightforward check shows that the sets $V_{k1}^m, \ldots, V_{kn}^m$ are closed and pairwise disjoint, that $V_{ki}^m \subset U_i$ for $i = 1, \ldots, n$, and that each x in $M_{\mathscr{B}}$ belongs to at most n of the sets $G_k^m := M_{\mathscr{B}} \setminus \cup_{i=1}^n V_{ki}^m$. Now let $\widehat{g}_{k1}^m, \ldots, \widehat{g}_{kn}^m$ be any functions in $C(M_{\mathscr{B}})$ such that $\widehat{g}_{ki}^m|_{V_{ki}^m} = 1$, $\operatorname{supp} \widehat{g}_{ki}^m \cap \operatorname{supp} \widehat{g}_{kj}^m = \emptyset$ whenever $i \ne j$, $\operatorname{supp} \widehat{g}_{ki}^m \subset U_i$, and $0 \le \widehat{g}_{ki}^m \le 1$. Finally, put $\widehat{g}_k^m = \widehat{g}_{k1}^m + \cdots + \widehat{g}_{kn}^m$. Since $(\mathscr{A}, \mathscr{B})$ is a faithful localizing pair, we have

$$\|g_k^m a\| = \|(g_{k1}^m + \cdots + g_{kn}^m)a\| \le \max_i \|g_{ki}^m a\| < \max_{x\in M_{\mathscr{B}}} \|\widehat{a}(x)\| + \varepsilon$$

(for the last '$<$' recall that $\operatorname{supp} \widehat{g}_{ki}^m \subset U_i$). Hence,

$$\|(g_1^m + \cdots + g_m^m)\, a\| \le m \left(\max_{x\in M_{\mathscr{B}}} \|\widehat{a}(x)\| + \varepsilon \right). \tag{2.12}$$

Put $\widehat{h}_k^m := 1-\widehat{g}_k^m$. Then $0 \le \widehat{h}_k^m \le 1$ and $\operatorname{supp}\widehat{h}_k^m \subseteq G_k^m$, and one has

$$\|(g_1^m+\cdots+g_m^m)a\| = \|ma-(h_1^m+\cdots+h_m^m)a\| \ge m\|a\| - \|h_1^m+\cdots+h_m^m\|\,\|a\|.$$

Because $\operatorname{supp}(\widehat{h}_1^m+\cdots+\widehat{h}_m^m) \subset \cup_{k=1}^m G_k^m$, and since each $x \in M_{\mathscr{B}}$ belongs to at most n of the sets G_k^m, it follows that $\widehat{h}_1^m(x)+\cdots+\widehat{h}_m^m(x) \le n$ for all x in $M_{\mathscr{B}}$. This implies

$$\|(g_1^m+\cdots+g_m^m)\,a\| \ge (m-n)\|a\|. \tag{2.13}$$

Combining (2.12) and (2.13) we arrive at the inequality

$$\|a\| \le \frac{m}{m-n}\left(\max_{x\in M_{\mathscr{B}}}\|\widehat{a}(x)\|+\varepsilon\right).$$

Letting m go to infinity and ε go to zero we obtain the desired inequality.

To prove the reverse implication, let $a \in \mathscr{A}$ and let $b_1, b_2 \in \mathscr{B}$ such that $\operatorname{supp}\widehat{b}_1 \cap \operatorname{supp}\widehat{b}_2 = \emptyset$. Then

$$\|a(b_1+b_2)\|^2 = \max_{x\in M_{\mathscr{B}}}\left\|\widehat{a}(x)\left(\widehat{b}_1(x)+\widehat{b}_2(x)\right)\right\|.$$

For each $x \in M_{\mathscr{B}}$, only one of the values $\widehat{b}_1(x)$ and $\widehat{b}_2(x)$ can be different from 0. Consequently,

$$\left\|\widehat{a}(x)\left(\widehat{b}_1(x)+\widehat{b}_2(x)\right)\right\| = \max\left\{\left\|\widehat{a}(x)\widehat{b}_1(x)\right\|,\left\|\widehat{a}(x)\widehat{b}_2(x)\right\|\right\},$$

and the result follows. ■

Besides the norm computation aspect, faithful localizing pairs are advantageous to investigate local enclosement properties.

Theorem 2.3.4. *Let $\mathscr{A}$ be a unital Banach algebra and $\mathscr{B}$ be a subalgebra of $\mathscr{A}$ such that $(\mathscr{A},\mathscr{B})$ forms a faithful localizing pair. Further, let $\mathscr{C}$ be a closed linear subset of $\mathscr{A}$ such that $bc \in \mathscr{C}$ whenever $b \in \mathscr{B}$ and $c \in \mathscr{C}$ (i.e., $\mathscr{C}$ is a $\mathscr{B}$-module). Let $a \in \mathscr{A}$ and assume that, for each $x \in M_{\mathscr{B}}$, there is an $a_x \in \mathscr{C}$ such that $(\widehat{a-a_x})(x) = (\widehat{a}-\widehat{a}_x)(x) = 0$. Then $a \in \mathscr{C}$.*

Proof. Let $x \in M_{\mathscr{B}}$ and $\varepsilon > 0$. By Theorem 2.2.2 (b), there is a neighborhood $U(x)$ of x such that

$$\left\|(\widehat{a-a_x})(y)\right\| < \varepsilon$$

for all $y \in U(x)$. Choose a finite number $U(x_1),\ldots,U(x_n)$ of these neighborhoods which cover $M_{\mathscr{B}}$, and let $e = f_1+\cdots+f_n$ with $f_k \in \mathscr{B}$ be a partition of unity subordinate to this covering. Put $b_\varepsilon := \sum_{k=1}^n f_k a_{x_k}$. Then

$$
\begin{aligned}
\|a-b_\varepsilon\| &= \left\| \sum_{k=1}^{n} f_k (a - a_{x_k}) \right\| \\
&= \sup_{y \in M_{\mathscr{B}}} \left\| \sum_{k=1}^{n} \widehat{f}_k(y)\,(\widehat{a} - \widehat{a}_{x_k})(y) \right\| \qquad \text{(Theorem 2.3.3)} \\
&= \sup_{y \in M_{\mathscr{B}}} \left\| \sum_{k: y \in U(x_k)} \widehat{f}_k(y)\,(\widehat{a - a_{x_k}})(y) \right\| \\
&\le \sup_{y \in M_{\mathscr{B}}} \sum_{k: y \in U(x_k)} \widehat{f}_k(y)\varepsilon = \varepsilon.
\end{aligned}
$$

The algebraic properties of $\mathscr{C}$ ensure that $b_\varepsilon \in \mathscr{C}$ for every ε. Since $\mathscr{C}$ is closed and $\|a - b_\varepsilon\| < \varepsilon$ for every $\varepsilon > 0$, we have $a \in \mathscr{C}$. ■

As an application of the above result we will get a complete description of the image $\widetilde{\mathscr{A}}$ of $\mathscr{A}$ in the direct product $\mathscr{F}$ of the local algebras $\mathscr{A}/\mathscr{I}_x$ under the mapping

$$\mathscr{A} \to \mathscr{F}, \quad a \mapsto (x \mapsto \Phi_x(a)), \tag{2.14}$$

provided $(\mathscr{A}, \mathscr{B})$ is a faithful localizing pair. To that end, we will have to introduce the concept of *semi-continuity with respect to a given set*. Let X be a compact Hausdorff space. To each point $x \in X$, we associate a Banach algebra $\mathscr{A}_x$ with unit element e_x. Denote the direct product of the algebras $\mathscr{A}_x$ by $\mathscr{F}$. Evidently, this product can be identified with the set of all bounded functions f on X which take at $x \in X$ a value $f(x) \in \mathscr{A}_x$. In particular, if g is a continuous complex-valued function on X, then the function

$$x \mapsto g(x)e_x \tag{2.15}$$

belongs to $\mathscr{F}$. The set $\mathscr{D}$ of all functions of the form (2.15) is a closed subalgebra of the center of $\mathscr{F}$, and it is easy to check that $(\mathscr{F}, \mathscr{D})$ forms a faithful localizing pair. Let $\mathscr{E}$ be a subset of $\mathscr{F}$ which is subject to the following conditions:

(i) $\mathscr{D} \subseteq \mathscr{E}$;
(ii) given $x \in X$ and $a \in \mathscr{A}_x$, there is an $f \in \mathscr{E}$ such that $f(x) = a$;
(iii) the function $x \mapsto \|f(x)\|_{\mathscr{A}_x}$ is upper semi-continuous on X for each $f \in \mathscr{E}$;
(iv) $\mathscr{E}$ is a (not necessarily closed) algebra.

We call a function $g \in \mathscr{F}$ *semi-continuous with respect to* $\mathscr{E}$ if, for each $x_0 \in X$, each $f \in \mathscr{E}$ with $f(x_0) = g(x_0)$, and each $\varepsilon > 0$, there is a neighborhood $U = U(x_0, f, g, \varepsilon)$ of x_0 such that $\|g(x) - f(x)\|_{\mathscr{A}_x} < \varepsilon$ for all $x \in U$. Thus, a function g is semi-continuous with respect to $\mathscr{E}$ if it behaves locally as a function in $\mathscr{E}$.

It is easy to show that the set of all functions which are semi-continuous with respect to $\mathscr{E}$ forms a closed subalgebra of $\mathscr{F}$ which we will denote by $\mathscr{F}(\mathscr{E})$. The following result can be considered as a generalization of the classical Weierstrass theorem (see Exercise 2.3.2).

Theorem 2.3.5. *The smallest closed subalgebra of $\mathscr{F}$ which contains $\mathscr{E}$ coincides with $\mathscr{F}(\mathscr{E})$.*

Proof. The inclusion $\operatorname{clos}\mathscr{E} \subseteq \mathscr{F}(\mathscr{E})$ is an immediate consequence of the fact that, by property (iii) of $\mathscr{E}$, each function in $\mathscr{E}$ is semi-continuous with respect to $\mathscr{E}$. For the reverse inclusion, let $b \in \mathscr{F}(\mathscr{E})$. By property (ii), there is an $f \in \mathscr{E}$ such that $f(x) = b(x)$ or, equivalently, $(\widehat{f-b})(x) = 0$. Thus, the algebras $\mathscr{F}(\mathscr{E})$ and $\mathscr{D}$ and the $\mathscr{D}$-module $\operatorname{clos}\mathscr{E}$ satisfy the assumptions imposed in Theorem 2.3.4 on the algebras $\mathscr{A}$ and $\mathscr{B}$ and the $\mathscr{B}$-module $\mathscr{C}$. Thus, the conclusion follows immediately from that theorem. ■

Let $(\mathscr{A}, \mathscr{B})$ be a faithful localizing pair. As a consequence of the preceding theorem, we get a description of the image $\widetilde{\mathscr{A}}$ of $\mathscr{A}$ in the direct product $\mathscr{F}$ of the local algebras $\mathscr{A}/\mathscr{I}_x$ under the mapping (2.14).

Corollary 2.3.6. *$\widetilde{\mathscr{A}}$ coincides with the closed subalgebra of $\mathscr{F}$ which consists of all functions $f \in \mathscr{F}$ which are semi-continuous with respect to $\widetilde{\mathscr{A}}$.*

Indeed, by Theorem 2.2.2 (b), $\widetilde{\mathscr{A}}$ satisfies the assumptions made for the set $\mathscr{E}$ in Theorem 2.3.5 (with X being the maximal ideal space of $\mathscr{B}$).

Remark 2.3.7. There are more general concepts of faithful localization: in [81], the C^*-compatible norm in Definition 2.3.1 (a) is allowed to be different from the original norm and also (b) is substituted by a more general condition. The price one has to pay is that (2.11) is no longer an equality. Rather, one has

$$C_1\|a\| \leq \max_{x \in M_{\mathscr{B}}} \|\widehat{a}(x)\| \leq C_2\|a\| \quad \text{for all } a \in \mathscr{A}$$

with certain constants C_1, C_2 independent of a. Thus, the exact norm computation aspect of Theorem 2.3.3 gets lost, but the local enclosement Theorem 2.3.4 remains valid without changes. □

2.3.3 Exercises

Exercise 2.3.1. Show that (with the notation of the previous subsection) the set of all functions which are semi-continuous with respect to $\mathscr{E}$ forms a closed subalgebra of $\mathscr{F}$.

Exercise 2.3.2. Let $\mathscr{F} := C[0,1]$ and $\mathscr{E} := \mathbb{C}$ (considered as constant functions). Describe the functions in $\mathscr{F}$ which are semi-continuous with respect to $\mathscr{E}$. Use Theorem 2.3.5 to derive the classical Weierstrass theorem.

2.4 Gohberg-Krupnik's local principle

The local principle by Gohberg and Krupnik has several advantages: its formulation as well as the proofs of its main results are quite elementary, and it works equally well in the case of complex and real algebras.

2.4.1 Localizing classes

Let $\mathscr{A}$ be a (real or complex) Banach algebra with identity e. A subset $M \subset \mathscr{A}$ is called a *localizing class* if it does not contain the element 0 and if for arbitrary elements $f_1, f_2 \in M$, there exists a third element $f \in M$ such that $f_j f = f f_j = f$ for $j = 1, 2$.

Let M be a localizing class. Two elements $a, b \in \mathscr{A}$ are said to be *M-equivalent from the left* (resp. from the right) if

$$\inf_{f \in M} \|(a-b)f\| = 0 \qquad (\text{resp. } \inf_{f \in M} \|f(a-b)\| = 0).$$

Finally, an element $a \in \mathscr{A}$ is called *M-invertible from the left* (resp. from the right) if there exist elements $b \in \mathscr{A}$ and $f \in M$ such that $baf = f$ (resp. $fab = f$).

Proposition 2.4.1. *Let M be a localizing class, and let a_1 and a_2 be elements of $\mathscr{A}$ which are M-equivalent from the left (resp. from the right). Then a_1 is M-invertible from the left (resp. from the right) if and only if a_2 is so.*

Proof. Let a_1 be M-invertible from the left. Choose elements $b_1 \in \mathscr{A}$ and $f \in M$ such that $b_1 a_1 f = f$. Since a_1 and a_2 are M-equivalent from the left, there is a $g \in M$ such that $\|(a_1 - a_2)g\| < \|b_1\|^{-1}$. Let $h \in M$ be such that $fh = gh = h$. Then

$$\begin{aligned} b_1 a_2 h &= b_1 a_1 h - b_1(a_1 - a_2)h \\ &= b_1 a_1 f h - b_1(a_1 - a_2)gh \\ &= fh - b_1(a_1 - a_2)gh \\ &= h - b_1(a_1 - a_2)gh. \end{aligned}$$

Set $u := b_1(a_1 - a_2)g$. Then the above identity can be rewritten as

$$b_1 a_2 h = (e-u)h.$$

Since $\|u\| < 1$, the element $e - u$ is invertible in $\mathscr{A}$. Setting $b_2 := (e-u)^{-1} b_1$, one obtains $b_2 a_2 h = h$. Thus, a_2 is M-invertible from the left. The proof for right M-invertibility is similar. ■

Definition 2.4.2. Let T be a topological space. A system $\{M_\tau\}_{\tau\in T}$ of localizing classes is said to be:

- *covering* if from each choice $\{f_\tau\}_{\tau\in T}$ of elements $f_\tau \in M_\tau$ one can select a finite number of elements $\{f_{\tau_1},\ldots,f_{\tau_m}\}$ the sum of which is invertible in $\mathscr{A}$;
- *overlapping* if each M_τ is a bounded set in $\mathscr{A}$, if $f \in M_{\tau_0}$ for some $\tau_0 \in T$ implies $f \in M_\tau$ for all τ in some open neighborhood of τ_0, and if the elements of $F := \cup_{\tau\in T} M_\tau$ commute pairwise.

Let $\{M_\tau\}_{\tau\in T}$ be an overlapping system of localizing classes. The *commutant* of F is the set $\mathrm{Com}F := \{a \in \mathscr{A} : af = fa\,\forall f \in F\}$. It is easy to verify that $\mathrm{Com}F$ is a closed subalgebra of $\mathscr{A}$. For $\tau \in T$, let Z^τ denote the set of all elements in $\mathrm{Com}F$ which are M_τ-equivalent to zero both from the left and from the right.

Lemma 2.4.3. *The set Z^τ is a proper closed two-sided ideal of* $\mathrm{Com}F$.

Proof. The closedness of Z^τ follows easily from the boundedness of M_τ, and the ideal property of Z^τ can be also straightforwardly checked. The properness of Z^τ can be seen as follows. Suppose the identity element e of $\mathscr{A}$ belongs to Z^τ. Then there exists a sequence of $f_n \in M_\tau$ such that $\|f_n\| \to 0$ as $n \to \infty$. Since there exist non-zero elements $g_n \in M_\tau$ such that $f_n g_n = g_n$, it follows that $\|f_n\| \geq 1$, and we obtained a contradiction. ∎

For $a \in \mathrm{Com}F$, let a^τ denote the coset $a + Z^\tau$ of a in the quotient algebra $\mathrm{Com}F/Z^\tau$.

Proposition 2.4.4. *Let $\{M_\tau\}_{\tau\in T}$ be a system of localizing classes, with each M_τ a bounded set in $\mathscr{A}$. Let $\tau \in T$ and $a \in \mathrm{Com}F$. Then a is M_τ-invertible in $\mathrm{Com}F$ from the left (resp. from the right) if and only if a^τ is left (resp. right) invertible in $\mathrm{Com}F/Z^\tau$.*

Proof. Let a^τ be left invertible in $\mathrm{Com}F/Z^\tau$. Then there is a $b \in \mathrm{Com}F$ such that $ba - e \in Z^\tau$. This implies that ba is M_τ-equivalent from the left to e. Proposition 2.4.1 yields the M_τ-invertibility of ba, and thus of a, from the left. Conversely, if there is $b \in \mathrm{Com}F$ and $f \in M_\tau$ such that $baf = f$, then $(ba-e)f = 0$. Hence $ba - e \in Z^\tau$, and thus $b^\tau a^\tau = e$. The proof for right invertibility is similar. ∎

2.4.2 Gohberg-Krupnik's local principle

The following theorem is a very similar result to Theorem 2.2.2, with the ideals $\mathscr{I}_x$ and the maximal ideal space of the central subalgebra substituted respectively by the ideals Z^τ and the index set T of the system of localizing classes $\{M_\tau\}_{\tau\in T}$. But contrary to Allan's local principle (where complex function theoretic arguments are used in the proof of Proposition 2.2.1), the local principle by Gohberg-Krupnik is valid for real Banach algebras, too.

Theorem 2.4.5 (Gohberg-Krupnik)**.** *Let $\mathscr{A}$ be a Banach algebra with identity and $\{M_\tau\}_{\tau\in T}$ a covering system of localizing classes, the elements of which belong to the center of $\mathscr{A}$. Further, let $a \in \mathscr{A}$ and, for every $\tau \in T$, let a_τ be an element of $\mathscr{A}$ which is M_τ-equivalent from the left to a.*

(i) *The element a is left invertible in $\mathscr{A}$ if and only if a_τ is M_τ-invertible in $\mathscr{A}$ from the left for every $\tau \in T$.*
(ii) *Suppose that each M_τ is a bounded set in $\mathscr{A}$. Then a is left invertible in $\mathscr{A}$ if and only if a^τ is left invertible in $\mathscr{A}/Z^\tau$ for all $\tau \in T$.*
(iii) *If the system $\{M_\tau\}_{\tau\in T}$ is overlapping, then the function $T \to \mathbb{R}^+$, $\tau \mapsto \|a^\tau\|$ is upper semi-continuous.*
(iv) *If $\mathscr{A}$ is a C^*-algebra, then the system $\{M_\tau\}_{\tau\in T}$ is overlapping. If moreover $M_\tau^* = M_\tau$ for all $\tau \in T$, then*

$$\|a\| = \sup_{\tau\in T} \|a^\tau\|.$$

The result remains valid if left *is replaced by* right *everywhere.*

Proof. We will give the proof in case of left equivalence and left invertibility. The proof for right equivalence and right invertibility is, of course, similar.

(i) If a is left invertible, then a is M_τ-invertible in $\mathscr{A}$ ($=$ Com$F$) from the left for all $\tau \in T$. By Proposition 2.4.1, a_τ is M_τ-invertible from the left for all $\tau \in T$. Conversely, suppose a_τ is M_τ-invertible from the left for all $\tau \in T$. Again by Proposition 2.4.1, it follows that a is M_τ-invertible from the left for all $\tau \in T$. Thus there are $b_\tau \in \mathscr{A}$ and $f_\tau \in M_\tau$ such that $b_\tau a f_\tau = f_\tau$. Since $\{M_\tau\}_{\tau\in T}$ is covering, one can choose a finite number of elements $f_{\tau_1}, \dots, f_{\tau_m}$ so that $\sum_{j=1}^m f_{\tau_j}$ is invertible. Put

$$s := \sum_{j=1}^m b_{\tau_j} f_{\tau_j}$$

to obtain

$$sa = \sum_{j=1}^m b_{\tau_j} f_{\tau_j} a = \sum_{j=1}^m b_{\tau_j} a f_{\tau_j} = \sum_{j=1}^m f_{\tau_j}.$$

Thus, $(\sum_{j=1}^m f_{\tau_j})^{-1} s$ is a left inverse of a.

(ii) If a^τ is left invertible in $\mathscr{A}/Z^\tau$ for all $\tau \in T$, then a is left invertible in $\mathscr{A}$ by Proposition 2.4.4. and part (i) above. The converse is trivial.

(iii) Let $\tau_0 \in T$ and $\varepsilon > 0$. Choose $z \in Z^\tau$ so that $\|a+z\| < \|a^{\tau_0}\| + \varepsilon/2$. Since z is M_{τ_0}-equivalent to zero from the left, there is an $f \in M_{\tau_0}$ such that $\|zf\| < \varepsilon/2$. Because $f \in M_{\tau_0}$ implies that $f \in M_\tau$ for all τ in some open neighborhood of τ_0 due to the overlapping property, we deduce that $f \in M_\tau$ for all τ in some open neighborhood U_{τ_0} of τ_0. Put $y := z - zf$. If $\tau \in U_{\tau_0}$, then there exists a $g \in M_\tau$ such that $fg = g$. Consequently, we have that $yg = zg - zfg = zg - zg = 0$. Since $y \in$ Com$F$ (by the definition of Z^τ and due to the overlapping property), it follows that $y \in Z^\tau$ for all $\tau \in U_{\tau_0}$. Hence, $\|a^\tau\| \le \|a+y\|$ for $\tau \in U_{\tau_0}$. Thus, if $\tau \in U_{\tau_0}$, then

$$\|a^\tau\| - \|a^{\tau_0}\| < \|a+y\| - \|a+z\| + \frac{\varepsilon}{2} \le \|y-z\| + \frac{\varepsilon}{2} = \|zf\| + \frac{\varepsilon}{2} < \varepsilon,$$

which proves the upper semi-continuity of the mapping $\tau \mapsto \|a^\tau\|$ at τ_0.
(iv) If Com$F$ and ComF/Z^τ are C^*-algebras then, indeed,

$$\begin{aligned}\|a\|^2 = r(aa^*) &= \sup_{\tau\in T} r\left((aa^*)^\tau\right) \text{ by (ii)} \\ &= \sup_{\tau\in T} r\left(a^\tau (a^\tau)^*\right)) = \sup_{\tau\in T} \|a^\tau\|^2.\end{aligned}$$

■

2.4.3 Exercises

Exercise 2.4.1. Study the algebra $\mathscr{A}^{\mathscr{K}}$ considered in the example of Section 2.2.6, now using Gohberg-Krupnik's local principle, instead of Allan's.

Exercise 2.4.2. Prove Allan's local principle in the special case of localization with respect to a unital central C^*-subalgebra (provided it exists) by means of Gohberg-Krupnik's local principle. Make sure that your proof does not employ properties of *complex* analytic functions. Derive a version of Allan's local principle for *real* Banach algebras. (Hint: see the proof of Theorem 1.21 in [151].)

2.5 Simonenko's local principle

Simonenko's local principle can be viewed as a particular realization of the two previously considered local principles by Allan-Douglas and Gohberg-Krupnik. It is well adapted to the study of Fredholm properties, and its formulation is quite intuitive. In particular, no Banach algebra "infrastructure" is involved.

2.5.1 Spaces and operators of local type

Let X be a locally compact Hausdorff topological space, and let μ be a non-negative (possibly infinite) measure which is defined on a σ-algebra over X which contains all Borel subsets of X. The characteristic function of a measurable subset U of X will be denoted by χ_U in what follows.

Definition 2.5.1. A Banach space E, the elements of which are (equivalence classes of) measurable functions $f : X \to \mathbb{C}$, is called an *ideal Banach space over X* if the characteristic function of every compact subset of X belongs to E and if, for every measurable function $f : X \to \mathbb{C}$ and every function $g \in E$, the inequality

$$|f(x)| \leq |g(x)| \quad \text{a.e. on } X \tag{2.16}$$

implies that $f \in E$ and $\|f\|_E \leq \|g\|_E$.

The archetypal examples of ideal Banach spaces are the Lebesgue spaces $L^p(X)$ with $1 \leq p \leq \infty$. Note also that an ideal Banach space over X contains every bounded measurable function on X with compact support, which follows immediately from the definition.

Lemma 2.5.2. *Let E be an ideal Banach space over X, and let a be a bounded measurable function on X. Then the product af belongs to E for every function $f \in E$. In particular, the operator $aI : E \to E$, $f \mapsto af$ of multiplication by a is well defined. This operator is bounded, and $\|aI\| = \|a\|_\infty$.*

Proof. For every $f \in E$, one has $|a(x)f(x)| \leq |\,\|a\|_\infty f(x)|$ almost everywhere on X, whence

$$\|af\|_E \leq \|\,\|a\|_\infty f\|_E = \|a\|_\infty \|f\|_E. \tag{2.17}$$

The left inequality in (2.17) shows that $af \in E$ (since $\|a\|_\infty f \in E$), and the inequality $\|af\|_E \leq \|a\|_\infty \|f\|_E$ implies the boundedness of aI and the estimate $\|aI\| \leq \|a\|_\infty$. For the reverse estimate, let $\varepsilon > 0$ and choose a compact subset U of X with $\mu(U) > 0$ such that

$$(\|a\|_\infty - \varepsilon)\chi_U(x) \leq |a(x)\chi_U(x)| \quad \text{a.e. on } X.$$

Then the second condition in Definition 2.5.1 implies that

$$(\|a\|_\infty - \varepsilon)\|\chi_U\|_E \leq \|a\chi_U\|_E \leq \|aI\|\,\|\chi_U\|_E$$

which gives the estimate $\|a\|_\infty - \varepsilon \leq \|aI\|$. Letting ε go to zero we obtain the assertion. ∎

Definition 2.5.3. Let E be an ideal Banach space over X. An operator $A \in \mathscr{L}(E)$ is said to be of *local type* if the operator $\chi_{F_1} A \chi_{F_2} I$ is compact for every choice of disjoint closed subsets F_1, F_2 of X. We denote the set of all operators of local type by $\Lambda(E)$.

Let $A \in \mathscr{L}(E)$. The norm of the coset $A + \mathscr{K}(E)$ considered as an element of the Calkin algebra $\mathscr{L}(E)/\mathscr{K}(E)$ is called the *essential norm* of A. We denote it by $|A|$. Thus,

$$|A| := \inf_{K \in \mathscr{K}(E)} \|A + K\|.$$

Two operators $A, B \in \mathscr{L}(E)$ are said to be *essentially equivalent* if $|A - B| = 0$, in which case we write $A \sim B$.

Definition 2.5.4. An ideal Banach space E over X is called a *Banach space of local type over X* if, for each pair $A, B \in \mathscr{L}(E)$ of operators of local type and for each pair F_1, F_2 of disjoint closed subsets of X,

$$|\chi_{F_1} A \chi_{F_1} I + \chi_{F_2} B \chi_{F_2} I| \leq \max\{|A|, |B|\}. \tag{2.18}$$

Example 2.5.5. The Lebesgue spaces $L^p(X) =: E$ are Banach spaces of local type for every $1 \le p \le \infty$. To see this, notice that

$$\|\chi_{F_1} A \chi_{F_1} I + \chi_{F_2} B \chi_{F_2} I\| \le \max\{\|A\|, \|B\|\} \tag{2.19}$$

for each pair of operators $A, B \in \mathscr{L}(E)$ (not necessarily of local type) and for each pair F_1, F_2 of disjoint closed subsets of X. Thus,

$$\begin{aligned}
|\chi_{F_1} A \chi_{F_1} I + \chi_{F_2} B \chi_{F_2} I| &= \inf_{M \in \mathscr{K}(E)} \|\chi_{F_1} A \chi_{F_1} I + \chi_{F_2} B \chi_{F_2} I + M\| \\
&\le \inf_{K, L \in \mathscr{K}(E)} \|\chi_{F_1} A \chi_{F_1} I + \chi_{F_2} B \chi_{F_2} I + \chi_{F_1} K \chi_{F_1} I + \chi_{F_2} L \chi_{F_2} I\| \\
&= \inf_{K, L \in \mathscr{K}(E)} \|\chi_{F_1} (A+K) \chi_{F_1} I + \chi_{F_2} (B+L) \chi_{F_2} I\| \\
&\le \inf_{K, L \in \mathscr{K}(E)} \max\{\|A+K\|, \|B+L\|\} \qquad \text{by (2.19)} \\
&= \max\{|A|, |B|\}.
\end{aligned}$$

□

We proceed with equivalent characterizations of operators of local type which will prove useful in what follows.

Theorem 2.5.6. *Let E be a Banach space of local type over X. The following conditions are equivalent for an operator $A \in \mathscr{L}(E)$:*

(i) *A is of local type;*
(ii) *for each function $f \in C(X)$, the commutator $AfI - fA$ is compact;*
(iii) *for all measurable sets F_1, F_2 of X with $\operatorname{clos} F_1 \subset \operatorname{int} F_2$, one has $\chi_{F_1} A \chi_{F_2} I \sim \chi_{F_1} A$ and $\chi_{F_2} A \chi_{F_1} I \sim A \chi_{F_1} I$.*

Proof. (i) $\Rightarrow$ (ii): Let $f \in C(X)$ be a real-valued function with $0 \le f \le 1$. For $j, n \in \mathbb{N}$ with $1 \le j \le 4n$, define the sets

$$F_j^n := \begin{cases} \left\{x \in X : 0 \le f(x) \le \frac{1}{4n}\right\} & \text{if } j = 1, \\ \left\{x \in X : \frac{j-1}{4n} < f(x) \le \frac{j}{4n}\right\} & \text{if } j \ge 2 \end{cases}$$

and

$$G_j^n := \left\{x \in X : \tfrac{j-2}{4n} \le f(x) \le \tfrac{j+1}{4n}\right\},$$

and set $\chi_j := \chi_{F_j^n}$ as well as $\hat{\chi}_j := \chi_{G_j^n}$. Finally, let $\chi_0 = \chi_{4n+1} := 0$. Being preimages of closed intervals under a continuous function, the sets G_j^n are closed. For each operator $A \in \Lambda(E)$, one has

$$|AfI - fA| = \left| \sum_{j=1}^{4n} \sum_{k=1}^{4n} (\chi_j A \chi_k f I - f \chi_j A \chi_k I) \right| = \left| \sum_{j=1}^{4n} \sum_{k=j-1}^{j+1} (\chi_j A \chi_k f I - f \chi_j A \chi_k I) \right|$$

since $\operatorname{clos} F_j^n \cap \operatorname{clos} F_k^n = \emptyset$ for $|j-k| > 1$ and since A is of local type. A shift of the summation index yields

$$\begin{aligned}
|AfI - fA| &= \left| \sum_{j=1}^{4n} \sum_{k=-1}^{1} (\chi_j A \chi_{j+k} f I - f \chi_j A \chi_{j+k} I) \right| \\
&= \left| \sum_{j=1}^{4n} \sum_{k=-1}^{1} \left(\chi_j A \chi_{j+k} \left(f - \tfrac{j-1}{4n}\right) l - \left(f - \tfrac{j-1}{4n}\right) \chi_j A \chi_{j+k} I \right) \right| \\
&\leq \sum_{k=-1}^{1} \left(\left| \sum_{j=1}^{4n} \chi_j A \chi_{j+k} \left(f - \tfrac{j-1}{4n}\right) I \right| + \left| \sum_{j=1}^{4n} \left(f - \tfrac{j-1}{4n}\right) \chi_j A \chi_{j+k} I \right| \right).
\end{aligned} \tag{2.20}$$

Consider the first of the two inner sums on the right-hand side of (2.20). Taking into account that $G_j^n \cap G_k^n = \emptyset$ for $|j-k| \geq 4$ and $\chi_{j+k} \hat{\chi}_j = \chi_{j+k}$ for $k \in \{-1, 0, 1\}$ and employing the local property of E, we get for $k \in \{-1, 0, 1\}$,

$$\begin{aligned}
\left| \sum_{j=1}^{4n} \chi_j A \chi_{j+k} \left(f - \tfrac{j-1}{4n}\right) I \right| &= \left| \sum_{j=1}^{4n} \hat{\chi}_j \chi_j A \chi_{j+k} \left(f - \tfrac{j-1}{4n}\right) \hat{\chi}_j I \right| \\
&\leq \sum_{r=1}^{4} \left| \sum_{j=0}^{n-1} \hat{\chi}_{4j+r} \chi_{4j+r} A \chi_{4j+r+k} \left(f - \tfrac{4j+r-1}{4n}\right) \hat{\chi}_{4j+r} I \right| \\
&\leq \sum_{r=1}^{4} \max_{0 \leq j \leq n-1} \left| \chi_{4j+r} A \chi_{4j+r+k} \left(f - \tfrac{4j+r-1}{4n}\right) I \right|.
\end{aligned}$$

For $x \in F_{4j+r+k}^n$ one has

$$\frac{4j+r+k-1}{4n} < f(x) \leq \frac{4j+r+k}{4n}$$

which implies

$$\frac{k}{4n} < f(x) - \frac{4j+r-1}{4n} \leq \frac{k+1}{4n}$$

and, consequently,

$$\left| \chi_{4j+r+k} \left(f - \frac{4j+r-1}{4n} \right) \right| \leq \frac{1}{2n}.$$

Thus, the first of the inner sums in (2.20) can be estimated by

$$\frac{1}{2n} \sum_{r=1}^{4} \max_{0 \leq j \leq n-1} |\chi_{4j+r} A| \leq \frac{2}{n} |A|,$$

and a similar estimate for the second sum finally yields

$$|AfI - fA| \leq 3(2/n + 2/n) = 12/n.$$

Letting n go to infinity, we conclude that the operator $AfI - fA$ is compact for every continuous function $f : X \to [0, 1]$. The generalization to arbitrary $f \in C(X)$ is made by scaling and by writing f as the sum of its real, imaginary, positive and negative parts.

(ii) $\Rightarrow$ (iii): Let F_1, F_2 be measurable subsets of X with $\operatorname{clos} F_1 \subset \operatorname{int} F_2$. Then $\operatorname{clos} F_1$ and $\operatorname{clos}(X \setminus F_2)$ are disjoint. Since X is a locally compact Hausdorff space, there is a continuous function f on X such that

$$f(x) = \begin{cases} 0 & \text{if } x \in \operatorname{clos} F_1, \\ 1 & \text{if } x \in \operatorname{clos}(X \setminus F_2). \end{cases}$$

Then

$$\chi_{F_1} A - \chi_{F_1} A \chi_{F_2} I = \chi_{F_1} A \chi_{X \setminus F_2} I = \chi_{F_1} A f \chi_{X \setminus F_2} I \sim \chi_{F_1} f A \chi_{X \setminus F_2} I = 0.$$

The second relation follows in the same way.

(iii) $\Rightarrow$ (i): Let F_1, F_2 be disjoint closed subsets of X. Then $F_1 \subset X \setminus F_2$ which is open. Thus, by (iii),

$$\chi_{F_1} A \chi_{F_2} I = \chi_{F_1} A - \chi_{F_1} A \chi_{X \setminus F_2} I \sim 0$$

which finishes the proof of the theorem. ■

As the first consequence of the above result, we mention the following.

Theorem 2.5.7. *Let E be a Banach space of local type over X. Then $\Lambda(E)$ is a closed and inverse-closed subalgebra of $\mathcal{L}(E)$. The ideal $\mathcal{K}(E)$ of the compact operators is contained in $\Lambda(E)$, and the quotient algebra $\Lambda(E)/\mathcal{K}(E)$ is inverse-closed in the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$. In particular, if A is a Fredholm operator of local type, then each of its regularizers is of local type again.*

Proof. The proof of this theorem becomes straightforward if the equivalence between conditions (i) and (ii) in Theorem 2.5.6 is employed. For example, the identities

$$f(AB) - (AB)fI = (fA - Af)B + A(fB - BfI)$$

and

$$fA^{-1} - A^{-1}fI = A^{-1}(Af - fA)A^{-1}$$

show that $AB \in \Lambda(E)$ whenever A and B are in $\Lambda(E)$ and that $A^{-1} \in \Lambda(E)$ whenever $A \in \Lambda(E)$ is invertible in $\mathcal{L}(E)$. If (A_n) is a sequence of operators of local type which converges to $A \in \mathcal{L}(E)$ in the norm of $\mathcal{L}(E)$, then $fA - AfI$ is the norm limit of the sequence of compact operators $(fA_n - A_n fI)$, and hence compact. It is also evident that $\mathcal{K}(E) \subset \Lambda(E)$. Finally, let $A \in \Lambda(E)$ be a Fredholm operator. Then, there are compact operators K, L, as well as an operator $R \in \mathcal{L}(E)$ such that $AR = I + K$ and $RA = I + L$. For each $f \in C(X)$ one obtains

$$\begin{aligned} fR - RfI &= (RA - L)fR - Rf(AR - K) \\ &= R(Af - fA)R - LfR + RfK \in \mathcal{K}(E) \end{aligned}$$

which shows that $R \in \Lambda(E)$, too. ■

2.5.2 Local equivalence and local norms

The following definitions of local equivalence and of local norms go back to Simonenko.

Definition 2.5.8. Let E be an ideal Banach space over X and let $x \in X$. Then the *local essential norm* of an operator $A \in \Lambda(E)$ at the point x is the quantity

$$|A|_x := \inf_U |\chi_U A|$$

where the infimum is taken over all open neighborhoods U of x. The operators A and B are called *locally equivalent* at x if $|A - B|_x = 0$. Local equivalence at the point x will be denoted by $A \overset{x}{\sim} B$.

Lemma 2.5.9. *Let X, E, A and x be as in the previous definition. Then the local essential norm $|A|_x$ coincides with each of the following quantities:*

(i) $\inf |fA|$, *where the infimum is taken over all continuous functions $f : X \to [0,1]$ which are identically* 1 *in a neighborhood of x;*
(ii) $\inf |fA|$, *where the infimum is taken over all continuous functions f on X with* $f(x) = 1$.

Proof. Let m_1 and m_2 denote the quantities defined in (i) and (ii), respectively. Evidently, $m_2 \le m_1$. Further, given a neighborhood U of x, choose a neighborhood W of x with $\overline{W} \subset U$. Then there is a continuous function $f : X \to [0,1]$ which is identically 1 on W and vanishes outside U. Thus,

$$|fA| = |f\chi_U A| \le |\chi_U A|,$$

whence the estimate $m_1 \le |A|_x$. For the estimate $|A|_x \le m_2$, let $\varepsilon > 0$ and choose a continuous function f with $f(x) = 1$ and $|fA| < m_2 + \varepsilon$. Since f is continuous, there is a neighborhood U of x such that $|f(y) - 1| < \varepsilon$ for all $y \in U$. With this neighborhood, one gets

$$|\chi_U A| \le |\chi_U fA| + |\chi_U(1-f)A| \le |fA| + |\chi_U(1-f)||A| \le m_2 + \varepsilon + \varepsilon|A|.$$

Consequently, $|A|_x \le m_2 + \varepsilon(1 + |A|)$. Letting ε go to zero yields the assertion. ■

Remark 2.5.10. In combination with Theorem 2.5.6, the preceding lemma shows that

$$|A|_x = \inf_U |A\chi_U I|$$

with the infimum taken over all open neighborhoods of x. □

Note that Theorem 2.5.6 offers another way to deal with local properties of operators of local type. The second characterization of $\Lambda(E)$ given in that theorem implies that $\mathscr{B} := C(X) + \mathscr{K}(E)$ is a central subalgebra of $\Lambda(E)/\mathscr{K}(E)$ which contains the identity element. The central subalgebra $\mathscr{B}$ is isometrically isomorphic to the algebra $C(X)$. Indeed, as in Proposition 1.4.11 (where the case $E = L^p(X)$ is considered) one gets that $\|fI + \mathscr{K}(E)\| = \|fI\|$ for each function $f \in C(X)$, and the equality $\|fI\| = \|f\|_\infty$ has been established in Lemma 2.5.2. Thus, the maximal ideal space of the commutative C^*-algebra $\mathscr{B}$ is homeomorphic to X, and one can apply Allan's local principle to localize the algebra $\mathscr{A} := \Lambda(E)/\mathscr{K}(E)$ over X. We let $\mathscr{J}_x$ denote the local ideal of $\mathscr{A}$ which is induced by $x \in X$ and write Φ_x for the canonical homomorphism $\mathscr{A} \to \mathscr{A}/\mathscr{J}_x$. Further we let Ψ_x stand for the homomorphism $\Phi_x \circ \pi$, where π refers to the canonical homomorphism $\Lambda(E) \to \Lambda(E)/\mathscr{K}(E)$. Thus,

$$\Psi_x : \Lambda(E) \to \mathscr{A}/\mathscr{J}_x,\ A \mapsto \Phi_x(A + \mathscr{K}(E)).$$

Proposition 2.5.11. *Let E be a Banach space of local type over X. Then, for each $A \in \Lambda(E)$ and each $x \in X$,*

$$\|\Psi_x(A)\| = |A|_x.$$

Proof. Let $A \in \Lambda(E)$ and $x \in X$. By Lemma 2.5.9, we have to show that

$$\|\Psi_x(A)\| = \inf |fA|, \tag{2.21}$$

where the infimum is taken over all continuous functions $f : X \to [0,1]$ which are identically 1 in a neighborhood of x. We denote this infimum by q. If $f \in C(X)$ is identically 1 in some neighborhood of x, then the coset $(f-1)A + \mathscr{K}(E)$ belongs to $\mathscr{J}_x$. Thus,

$$\|\Psi_x(A)\| \le |A + (f-1)A| = |fA|.$$

Taking the infimum over all f with these properties we get $\|\Psi_x(A)\| \le q$.

For the reverse inequality, given $\varepsilon > 0$, choose functions $f_1, \ldots, f_n \in C(X)$ which vanish at x and operators $B_1, \ldots, B_n \in \Lambda(E)$ such that

$$|A + f_1B_1 + \ldots + f_nB_n| < \|\Psi_x(A)\| + \varepsilon.$$

If f is any function in $C(X)$ with $0 \le f \le 1$ and $f \equiv 1$ in some neighborhood of x, then

$$\begin{aligned}|fA| &\le \left|f\left(A+\sum f_iB_i\right)\right| + \left|\sum ff_iB_i\right| \\ &\le \left|A+\sum f_iB_i\right| + \sum \|ff_i\|_\infty |B_i| \\ &\le \|\Psi_x(A)\| + \varepsilon + \sum \|ff_i\|_\infty |B_i|.\end{aligned}$$

The right-hand side of this estimate becomes smaller than $\|\Psi_x(A)\| + 2\varepsilon$ if f is chosen such that $\|ff_i\|_\infty < \varepsilon/(n|B_i|)$ for every i. Hence, $q \le \|\Psi_x(A)\|$. ∎

Theorem 2.5.12. *Let E be a Banach space of local type over X and let $A \in \mathscr{L}(E)$ be an operator of local type. Then the essential norm of A can be expressed in terms of local norms by*

$$|A| = \max_{x\in X} |A|_x.$$

This theorem will be an immediate consequence of Theorem 2.3.3 and Proposition 2.5.11 once we have shown the following.

Theorem 2.5.13. *$(\Lambda(E)/\mathscr{K}(E), C(X)+\mathscr{K}(E))$ is a faithful localizing pair.*

Proof. We have to show that

$$|(f+g)A| \le \max\{|fA|,\ |gA|\}$$

whenever $A \in \Lambda(E)$ and f and g are functions in $C(X)$ with disjoint supports $M, N \subseteq X$, respectively. Since A is of local type, we conclude from (2.18) that

$$\begin{aligned}\max\{|fA|,\ |gA|\} &\ge |\chi_M fA\chi_M I + \chi_N gA\chi_N I| \\ &= |fA\chi_M I + gA\chi_N I| \\ &= |fA + gA - fA\chi_{X\setminus M} I - gA\chi_{X\setminus N} I| \\ &= |fA + gA|.\end{aligned}$$

Here we used that

$$fA\chi_{X\setminus M} I = (fA - Af)\chi_{X\setminus M} I + Af\chi_{X\setminus M} I = (fA - Af)\chi_{X\setminus M} I$$

and $gA\chi_{X\setminus N} I$ are compact operators. ∎

2.5.3 Local Fredholmness and Fredholmness

Definition 2.5.14. Let E be an ideal Banach space over X and $A \in \mathscr{L}(E)$. An operator $R_l \in \mathscr{L}(E)$ (resp. R_r) is called a *local left (resp. right) regularizer* of the operator A at the point $x \in X$ if there is a neighborhood U of the point x such that $R_l A\chi_U I \sim \chi_U I$ (resp. $\chi_U AR_r \sim \chi_U I$). An operator A is said to be *locally Fredholm at* $x \in X$ if it possesses both a local left and a local right regularizer at that point.

Proposition 2.5.15. *Let E be a Banach space of local type over X and let $A \in \mathscr{L}(E)$ be an operator of local type which is locally Fredholm at $x \in X$. Then A possesses local left and right regularizers at x which are* of local type.

Proof. Let U be an open neighborhood of x and let R_l, $R_r \in \mathscr{L}(E)$ be operators such that

$$R_l A \chi_U I \sim \chi_U I \quad \text{and} \quad \chi_U A R_r \sim \chi_U I.$$

Let g be a continuous function on X which is identically 1 in a neighborhood V of x and which has its support inside U. Since

$$gR_l gA\chi_V I \sim gR_l Ag\chi_V I = gR_l A\chi_U g\chi_V I \sim g\chi_U g\chi_V I = \chi_V I,$$

the operator $gR_l gI$ is a local left regularizer of A at x, too. We claim that $gR_l gI$ is of local type. Let $f \in C(X)$. Then

$$\begin{aligned} fgR_l gI - gR_l gfI &= (fgR_l - gR_l f)g\chi_U I \\ &\sim (fgR_l - gR_l f)g\chi_U AR_r \\ &\sim (fgR_l - gR_l f)AgR_r \\ &\sim (fgR_l A - gR_l Af)gR_r \\ &= (fgR_l A\chi_U - gR_l A\chi_U f)gR_r \\ &\sim (fg - gf)\chi_U gR_r = 0, \end{aligned}$$

which proves the claim. Similarly one gets that $gR_r gI$ is a local right regularizer of A at x which is of local type. ∎

The main result on local Fredholmness reads as follows.

Theorem 2.5.16. *Let E be a Banach space of local type over X. An operator $A \in \mathscr{L}(E)$ of local type is Fredholm if and only if it is locally Fredholm at every point $x \in X$.*

The proof will follow immediately from Allan's local principle (Theorem 2.2.2) and from Theorem 2.5.7 once we have checked the following assertion. The notation is as in Section 2.5.2.

Proposition 2.5.17. *Let E be a Banach space of local type over X. The operator $A \in \Lambda(E)$ is locally Fredholm at $x \in X$ if and only if its local coset $\Psi_x(A)$ is invertible in $\mathscr{A}/\mathscr{J}_x$.*

Proof. If A is locally Fredholm at x, then there are operators R_l and R_r of local type as well as an open neighborhood U of x such that

$$R_l A \chi_U I \sim \chi_U I \quad \text{and} \quad \chi_U A R_r \sim \chi_U I.$$

Let f be a continuous function on X with $f(x) = 1$ and $\operatorname{supp} f \subset U$. Then

$$R_l A f I \sim f I \quad \text{and} \quad f A R_r \sim f I.$$

Applying the local homomorphism Ψ_x to these identities and taking into account that $\Psi_x(f)$ is the identity element of $\mathscr{A}/\mathscr{J}_x$, one gets the invertibility of $\Psi_x(A)$. Conversely, let $\Psi_x(A)$ be invertible in $\mathscr{A}/\mathscr{J}_x$. Let $R \in \Lambda(E)$ be such that $\Psi_x(R)$ is the inverse of $\Psi_x(A)$. Then $\Psi_x(RA-I) = \Psi_x(AR-I) = 0$, whence $|RA-I|_x = 0$ by Proposition 2.5.11. By the definition of the local norm, there is an open neighborhood U of x such that $|RA\chi_U I - \chi_U I| < 1/2$. Equivalently, there is an operator $C \in \mathscr{L}(E)$ with $\|C\| < 1/2$ and a compact operator K such that $RA\chi_U I - \chi_U I = C + K$. Multiplying this equality from both sides by $\chi_U I$ we find

$$\chi_U RA\chi_U I = \chi_U I + \chi_U C\chi_U I + \chi_U K\chi_U I. \tag{2.22}$$

Since $\|\chi_U C\chi_U I\| < 1/2$, the operator $\chi_U I + \chi_U C\chi_U I$ (considered as an operator on the range of the projection $\chi_U I$) is invertible by Neumann series. Let D denote its inverse. Then (2.22) implies

$$D\chi_U RA\chi_U I = \chi_U I + K'$$

with a certain compact operator K'. Consequently, $D\chi_U RA\chi_U I \sim \chi_U I$, i.e., $D\chi_U R$ is a local left regularizer of A at x. The existence of a local right regularizer follows analogously. ■

2.5.4 The envelope of an operator function

Let E be an ideal Banach space over X.

Definition 2.5.18. An operator function $X \to \Lambda(E)$, $x \mapsto A_x$ is said to possess an *envelope* if there is an operator $A \in \Lambda(E)$ such that $A \overset{x}{\sim} A_x$ for all $x \in X$. Each operator A with this property is called an envelope of the function $(A_x)_{x\in X}$.

Definition 2.5.19. The operator function $X \to \Lambda(E)$, $x \mapsto A_x$ is said to be locally semi-continuous if for every point $x_0 \in X$ and every $\varepsilon > 0$ there exists an open neighborhood U of the point x_0 such that every point $x \in U$ has an open neighborhood $V \subset U$ with

$$|(A_{x_0} - A_x)\chi_V I| < \varepsilon \tag{2.23}$$

Theorem 2.5.20. *Let E be a Banach space of local type over X. Then a bounded operator function $X \to \Lambda(E)$, $x \mapsto A_x$ possesses an envelope if and only if it is locally semi-continuous. The envelope is uniquely determined up to a compact operator, and*

$$|A| \le \sup_{x\in X} |A_x|$$

for each envelope A of the given operator function.

Proof. It easy to check that the existence of an envelope implies the local semi-continuity of the operator function. The reverse implication will be shown by having recourse to Theorem 2.3.5 and its application to faithful localizing pairs. Thus, we let $\mathscr{F}$ refer to the set of all bounded functions on X which take, at $x \in X$, a value in $\mathscr{A}/\mathscr{J}_x$ (the notation is as in Subsection 2.5.2). The set $\mathscr{F}$ becomes a Banach algebra by defining elementwise operations and the supremum norm. The Banach algebra $\mathscr{F}$ together with its subalgebra $\mathscr{D}$, consisting of all functions $x \mapsto f(x)\Psi_x(I)$ where $f \in C(X)$, forms a faithful localizing pair. Let $\mathscr{E}$ stand for the set of all functions in $\mathscr{F}$ of the form

$$x \mapsto \Psi_x(A) \quad \text{with} \quad A \in \Lambda(E).$$

Then $\mathscr{E}$ is a subalgebra of $\mathscr{F}$ which satisfies conditions (i) - (iv) on page 89 in Subsection 2.3.2. Moreover, $\mathscr{E}$ is closed. Indeed, let (f_n) with $f_n : x \mapsto \Psi_x(A_n)$ be a Cauchy sequence in $\mathscr{F}$. From

$$\|f_n - f_m\|_{\mathscr{F}} = \sup_{x \in X} \|\Psi_x(A_n - A_m)\| = |A_n - A_m|$$

(by Theorem 2.3.3 and Proposition 2.5.11) we conclude that $(A_n + \mathscr{K}(E))$ is a Cauchy sequence in $\mathscr{A}$. Let $A \in \Lambda(E)$ be such that $A + \mathscr{K}(E)$ is the limit of that sequence. Then $f : x \mapsto \Psi_x(A)$ is a function in $\mathscr{F}$, and $\|f_n - f\|_{\mathscr{F}} \to 0$.

Now let $X \to \Lambda(E)$, $x \mapsto A_x$ be a bounded and locally semi-continuous function on X, and let $x_0 \in X$. Let $A \in \Lambda(E)$ be an operator with $\Psi_{x_0}(A) = \Psi_{x_0}(A_{x_0})$. Due to Allan's local principle (Theorem 2.2.2), the function

$$X \to \mathbb{R}, \quad x \mapsto \|\Psi_x(A - A_{x_0})\|$$

is upper semi-continuous at x_0, i.e., given $\varepsilon > 0$ there is a neighborhood U_1 of x_0 such that $\|\Psi_x(A - A_{x_0})\| < \varepsilon$ for all $x \in U_1$. Further we conclude from (2.23) that there is a neighborhood U_2 of x_0 such that, for $x \in U_2$,

$$\|\Psi_x(A_x) - \Psi_x(A_{x_0})\| = |A_x - A_{x_0}|_x \leq |(A_x - A_{x_0})\chi_V I| < \varepsilon.$$

Hence, for all $x \in U := U_1 \cap U_2$,

$$\|\Psi_x(A_x) - \Psi_x(A)\| \leq \|\Psi_x(A_x) - \Psi_x(A_{x_0})\| + \|\Psi_x(A_{x_0}) - \Psi_x(A)\| \leq 2\varepsilon.$$

Hence, the function $x \mapsto \Psi_x(A_x)$ is semi-continuous with respect to $\mathscr{E}$ in the sense of Section 2.3.2. By Theorem 2.3.5, this function belongs to the closure of $\mathscr{E}$ in $\mathscr{F}$, which actually coincides with $\mathscr{E}$ as we have already seen. Thus, this function is of the form $x \mapsto \Psi_x(B)$ with a certain operator B of local type; in other words, B is an envelope for $(A_x)_{x \in X}$.

For the proof of the norm estimate, we take into account that $|A|_x = |A_x|_x$ by the definition of the local norm $|\cdot|_x$. Then

$$|A| = \max_{x \in X} |A|_x = \max_{x \in X} |A_x|_x \leq \sup_{x \in X} |A_x|$$

where the first equality comes from Theorem 2.5.12. ■

2.5.5 Exercises

Exercise 2.5.1. Prove that local equivalence is a reflexive, symmetric and transitive relation. Moreover,

(i) if A, B, A_x and B_x are local type operators with $A \overset{x}{\sim} A_x$ and $B \overset{x}{\sim} B_x$, then $AB \overset{x}{\sim} A_x B_x$;

(ii) if A and B are local type operators and $x \in X$, then $A \overset{x}{\sim} B$ if and only if for every $\varepsilon > 0$ there exists a neighborhood U of x such that

$$|(A-B)\chi_U I| < \varepsilon \quad \text{and} \quad |\chi_U (A-B)| < \varepsilon;$$

(iii) if A and B are local type operators and U_i, $i = 1, \ldots, 8$ are neighborhoods of $x \in X$ such that $\chi_{U_1} A \chi_{U_2} I \overset{x}{\sim} \chi_{U_3} B \chi_{U_4} I$, then $\chi_{U_5} A \chi_{U_6} I \overset{x}{\sim} \chi_{U_7} B \chi_{U_8} I$;

(iv) if (A_i) and (B_i) are sequences such that $|A_i - A| \to 0$, $|B_i - B| \to 0$, and $A_i \overset{x}{\sim} B_i$ for every i, then $A \overset{x}{\sim} B$.

Exercise 2.5.2. Let E be a Banach space of local type over X and $A \in \mathscr{L}(E)$ a local type operator. Prove that

$$|A| = \sup_{x \in X} |A|_x.$$

Exercise 2.5.3. Show that the equality $|A| = \sup_{x \in X} |A_x|$ does not hold in general.

Exercise 2.5.4. Let A be an operator of local type, $x \in X$, W a neighborhood of x, and R_l (resp. R_r) a local left (resp. right) regularizer of A at x. Prove also that $R_l \chi_W I$ and $\chi_W R_l$ (resp. $R_r \chi_W I$ and $\chi_W R_r$) are local left (resp. right) regularizers of A at x.

Exercise 2.5.5. Let A be an operator of local type, $x \in X$, W a neighborhood of x, and R_l (resp. R_r) a local left (resp. right) regularizer of A at x, and let $f \in C(X)$ be a function which is identically 1 on W. Prove then that $R_l fI$ and fR_l (resp. $R_r fI$ and fR_r) are local left (resp. right) regularizers of A at x.

Exercise 2.5.6. Show that if A is a local type operator which is locally Fredholm at a point $x \in X$, then A possesses a local type regularizer at that point.

Exercise 2.5.7. Let operators A and B be locally equivalent at $x \in X$ and assume that A possesses a left (resp. right) local regularizer at x. Show that then B also possesses a left (resp. right) local regularizer at x. Prove that if A, B and the left (right) local regularizer of A are local type operators, then the left (right) local regularizer of B has the same property.

Exercise 2.5.8. Let operators A and B be locally equivalent at $x \in X$, and let A be locally Fredholm at x. Show that B is also locally Fredholm at x.

Exercise 2.5.9. Let A be a local type operator which possesses a local left (right) local type regularizer for any $x \in X$. Prove that A possesses a global left (right) local type regularizer.

2.6 PI-algebras and QI-algebras

In this section we are going to present another generalization of Gelfand's transform, applicable to special classes of Banach algebras, the so called *PI-* and *QI-algebras*, where *PI* stands for *polynomial identity* and *QI* for *quasi identity*. These algebras are close to commutative algebras in the sense that the defining condition $ab = ba$ of a commutative algebra is replaced by another polynomial identity. For PI- and QI-algebras we will obtain sufficient families of *finite-dimensional* homomorphisms, that is, the algebras $\mathscr{B}_t$ defined in the introduction of the chapter will prove to be matrix algebras, $\mathscr{B}_t = \mathbb{C}^{l(t)\times l(t)}$, with $\sup l(t) < \infty$.

2.6.1 Standard polynomial identities

In this section, $\mathscr{A}$ denotes a unital algebra over an arbitrary field $\mathbb{F}$, and $\mathcal{P}$ is a polynomial of positive degree in n non-commuting variables with coefficients in $\mathbb{F}$. Each time the variables in $\mathcal{P}$ are replaced by elements $a_1, \ldots, a_n$ of the algebra, the result is an element of the algebra which we denote by $\mathcal{P}(a_1, \ldots, a_n)$.

Definition 2.6.1. Let $\mathcal{P}$ be a polynomial of positive degree in n non-commuting variables with coefficients in $\mathbb{F}$. An algebra $\mathscr{A}$ is said to satisfy the *polynomial identity* $\mathcal{P}$ if $\mathcal{P}(a_1, \ldots, a_n) = 0$ for every choice of elements $a_1, \ldots, a_n \in \mathscr{A}$. We then call $\mathscr{A}$ a *$\mathcal{P}$-algebra*. If $\mathscr{A}$ satisfies at least one non-trivial polynomial identity, it is called a *PI-algebra*.

Let Σ_n refer to the permutation group of the set $\{1, \ldots, n\}$. Polynomials of the form

$$\mathcal{P}(a_1, \ldots, a_n) = \sum_{\sigma\in\Sigma_n} \lambda_\sigma a_{\sigma(1)} \ldots a_{\sigma(n)} \tag{2.24}$$

with coefficients $\lambda_\sigma \in \mathbb{F}$ are called *multilinear*. Considered as a mapping from $\mathscr{A}^n$ to $\mathscr{A}$, a multilinear polynomial is indeed linear in each component.

A polynomial $\mathcal{P}$ of positive degree in n non-commuting variables is called *alternating* if any repetition in the choice of elements $a_1, \ldots, a_n$ yields 0, that is, $\mathcal{P}(\ldots, a_j, \ldots, a_j, \ldots) = 0$. Finally, for $1 \le i \ne j \le n$, we let $\mathcal{P}_{ij}$ stand for the polynomial $\mathcal{P}$ with the variables at places i and j interchanged.

Lemma 2.6.2. *Let $\mathcal{P}$ be a multilinear polynomial. Then $\mathcal{P}$ is alternating if and only if $\mathcal{P}_{ij} = -\mathcal{P}$ for each choice of indices $1 \le i \ne j \le n$.*

Proof. Let $\mathcal{P}$ be alternating and $1 \le i \ne j \le n$. Then

$$\begin{aligned} 0 &= \mathcal{P}(\ldots, a_i + a_j, \ldots, a_i + a_j, \ldots) \\ &= \mathcal{P}(\ldots, a_i, \ldots, a_i, \ldots) + \mathcal{P}(\ldots, a_i, \ldots, a_j, \ldots) \\ &\qquad + \mathcal{P}(\ldots, a_j, \ldots, a_i, \ldots) + \mathcal{P}(\ldots, a_j, \ldots, a_j, \ldots) \\ &= \mathcal{P}(\ldots, a_i, \ldots, a_j, \ldots) + \mathcal{P}(\ldots, a_j, \ldots, a_i, \ldots). \end{aligned}$$

Conversely, by interchanging the positions where the repeated element a_j is present, we obtain

$$\mathcal{P}(\dots, a_j, \dots, a_j, \dots) = -\mathcal{P}(\dots, a_j, \dots, a_j, \dots),$$

which implies $\mathcal{P}(\dots, a_j, \dots, a_j, \dots) = 0$. ■

The following definition introduces the process of *multilinearization*. Applied to a (non-multilinear) polynomial, this process results in another polynomial, with one "new" variable and a lesser degree in one of the "old" variables. This process, taken repeatedly, finally allows a multilinear polynomial to be obtained from any polynomial.

Definition 2.6.3. Let $\mathcal{P}\colon \mathscr{A}^n \to \mathscr{A}$ be a function in n variables. For $1 \le i \le n$, we define the function $\Delta_i\mathcal{P}\colon \mathscr{A}^{n+1} \to \mathscr{A}$ by

$$\begin{aligned}\Delta_i\mathcal{P}(a_1, \dots, a_{n+1}) := \mathcal{P}(&a_1, \dots, a_{i-1}, a_i + a_{n+1}, a_{i+1}, \dots, a_n)\\ -\mathcal{P}(&a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)\\ -\mathcal{P}(&a_1, \dots, a_{i-1}, a_{n+1}, a_{i+1}, \dots, a_n).\end{aligned} \tag{2.25}$$

Lemma 2.6.4. *If an algebra $\mathscr{A}$ satisfies a polynomial identity of degree k, then it also satisfies a multilinear identity of degree $\le k$.*

Proof. Let $\mathscr{A}$ satisfy a polynomial identity $\mathcal{P}$ of degree k in n variables. If $\mathcal{P}$ is not linear in the first variable (this happens if the degree of the first variable is greater than 1), then consider

$$\begin{aligned}&\Delta_1\mathcal{P}(a_1, \dots, a_n, a_{n+1})\\ &\quad= \mathcal{P}_n(a_1 + a_{n+1}, a_2, \dots, a_n) - \mathcal{P}(a_1, a_2, \dots, a_n) - \mathcal{P}(a_{n+1}, a_2, \dots, a_n).\end{aligned}$$

Clearly, $\mathscr{A}$ also satisfies the polynomial identity $\Delta_1\mathcal{P}$, and the degree of $\Delta_1\mathcal{P}$ is not greater than that of $\mathcal{P}$. But the degree of the first variable in $\Delta_1\mathcal{P}$ is strictly lower than the degree of the first variable in $\mathcal{P}$. Repeated application of this procedure to every nonlinear variable yields, after a finite number of steps, a multilinear identity of degree not greater than k which is also satisfied by $\mathscr{A}$. ■

Lemma 2.6.5. *The matrix algebra $\mathbb{F}^{n\times n}$ over the field $\mathbb{F}$ does not satisfy a polynomial identity of degree less than $2n$.*

Proof. By the above lemma, we just have to check that $\mathbb{F}^{n\times n}$ does not satisfy any multilinear identity of degree less than $2n$. Suppose that $\mathbb{F}^{n\times n}$ satisfies a multilinear identity $\mathcal{P}_m$ of degree $m < 2n$. Let $E_{pq} \in \mathbb{F}^{n\times n}$ be the matrix with zeros at every entry except at the entry pq, which is 1. Inserting the matrices

$$a_i := \begin{cases} E_{\frac{i+1}{2}\,\frac{i+1}{2}} & \text{if } i \text{ is odd,}\\ E_{\frac{i}{2}\,\frac{i+2}{2}} & \text{if } i \text{ is even}\end{cases}$$

into (2.24), we get immediately that the coefficient associated with the identity permutation is zero. Rearranging the matrices above in $\mathcal{P}_m$, we then conclude that all coefficients must be zero. This contradiction proves the assertion. ■

We are now going to introduce a class of multilinear polynomials which will play a dominant role in what follows.

Definition 2.6.6. Let $\mathscr{A}$ be an algebra and $a_1, \ldots, a_n \in \mathscr{A}$. The *standard polynomial* of degree n is defined by

$$\mathcal{S}_n(a_1, \ldots, a_n) := \sum_{\sigma \in \Sigma_n} \operatorname{sgn}\sigma \, a_{\sigma(1)} \ldots a_{\sigma(n)},$$

where $\operatorname{sgn}\sigma$ takes the value $+1$ if the permutation $\sigma \in \Sigma_n$ is even and -1 if it is odd.

The standard polynomials can also be defined recursively by $\mathcal{S}_1(a_1) := a_1$ and

$$\mathcal{S}_n(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{i-1} a_i \mathcal{S}_{n-1}(a_1, \ldots, \tilde{a}_i, \ldots, a_n)$$

or, equivalently,

$$\mathcal{S}_n(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{n-i} \mathcal{S}_{n-1}(a_1, \ldots, \tilde{a}_i, \ldots, a_n) a_i$$

if $n > 1$ where the tilde indicates that the corresponding element is omitted.

It easy to see that standard polynomials and their scalar multiples are alternating. The next result shows the converse is also true.

Proposition 2.6.7. *Any multilinear alternating polynomial of degree n is a multiple of the standard polynomial* $\mathcal{S}_n$.

Proof. Let $\mathcal{P}$ be a multilinear alternating polynomial of the form (2.24). Since every permutation is a composition of interchanges of variables, one has by Lemma 2.6.2, $\mathcal{P}(a_1, \ldots, a_n) = \operatorname{sgn}\sigma \, \mathcal{P}(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for each permutation $\sigma \in \Sigma_n$. The coefficient of the monomial $a_{\sigma(1)} \ldots a_{\sigma(n)}$ in the polynomial on the left-hand side of this equality is λ_σ; its counterpart on the right-hand side is $\operatorname{sgn}\sigma \lambda_{id}$, with the identity permutation id. Hence, $\lambda_\sigma = \operatorname{sgn}\sigma \lambda_{id}$, which implies

$$\mathcal{P}(a_1, \ldots, a_n) = \lambda_{id} \sum_{\sigma \in \Sigma_n} \operatorname{sgn}\sigma a_{\sigma(1)} \ldots a_{\sigma(n)}.$$

Thus, $\mathcal{P} = \lambda_{id} \mathcal{S}_n$. ■

The following fact will be used in the proof of the Amitsur-Levitzki theorem.

Corollary 2.6.8. *Let* $\mathcal{P}(a_1, \ldots, a_{2n}) = \sum_{\sigma \in \Sigma_{2n}} \operatorname{sgn}\sigma [a_{\sigma_1}, a_{\sigma_2}] \ldots [a_{\sigma_{2n-1}}, a_{\sigma_{2n}}]$, *where* $[a, b]$ *represents the commutator* $ab - ba$. *Then* $\mathcal{P} = 2^n \mathcal{S}_{2n}$.

Proof. It is not difficult to see that $\mathcal{P}$ is a multilinear and alternating polynomial of degree $2n$. By the above proposition, $\mathcal{P}$ is a multiple of the standard polynomial $\mathcal{S}_{2n}$. The polynomial $\mathcal{P}$ is the sum of $2^n(2n)!$ multilinear monomials, none of which cancel. Thus, the constant is 2^n. ∎

Let $\operatorname{tr} a$ denote the trace of the matrix a.

Lemma 2.6.9. *Let $a_1,\ldots,a_{2k} \in \mathbb{F}^{n\times n}$. Then* $\operatorname{tr}\left[\mathcal{S}_{2k}(a_1,\ldots,a_{2k})\right] = 0$.

Proof. For $i = 1, \ldots, 2k$, let $\mathbf{a}^i_{2k}$ denote the $(2k-1)$-tuple $(a_1, \ldots, \tilde{a}_i, \ldots, a_{2k})$ where the tilde indicates that the corresponding element is omitted. Then

$$\begin{aligned}
&2\operatorname{tr}\left[\mathcal{S}_{2k}(a_1, \ldots, a_{2k})\right]\\
&= \operatorname{tr}\left[\sum_{i=1}^{2k}(-1)^{i-1}\, a_i \mathcal{S}_{2k-1}(\mathbf{a}^i_{2k})\right] + \operatorname{tr}\left[\sum_{i=1}^{2k}(-1)^{2k-i}\, \mathcal{S}_{2k-1}(\mathbf{a}^i_{2k})\, a_i\right]\\
&= \sum_{i=1}^{2k}(-1)^{i-1}\operatorname{tr}\left[a_i\mathcal{S}_{2k-1}(\mathbf{a}^i_{2k}) - \mathcal{S}_{2k-1}(\mathbf{a}^i_{2k})a_i\right] = 0,
\end{aligned}$$

since the trace of the commutator of two matrices is zero. ∎

Definition 2.6.10. The algebra $\mathscr{A}$ is said to *satisfy the standard identity of order n* if $\mathcal{S}_n(a_1,\ldots,a_n) = 0$ for any $a_1,\ldots,a_n \in \mathscr{A}$. The family of all algebras with that property will be denoted by SI_n. Further, let SI^m_{2n} stand for the family of all algebras $\mathscr{A}$ with the property that $(\mathcal{S}_{2n}(a_1,\ldots,a_{2n}))^m = 0$ for all choices of elements $a_1,\ldots,a_{2n} \in \mathscr{A}$. Finally, if $\mathscr{A}$ is a (real or complex) Banach algebra, then we call $\mathscr{A}$ a *QI-algebra* and write $\mathscr{A} \in SI^\infty_{2n}$ if there is a number n such that, for any choice of $a_1,\ldots,a_{2n} \in \mathscr{A}$,

$$\lim_{m\to\infty} \|(\mathcal{S}_{2n}(a_1,\ldots,a_{2n}))^m\|^{1/m} = 0.$$

The standard polynomial of order 2 is $\mathcal{S}_2(a_1,a_2) = a_1a_2 - a_2a_1$. Thus an algebra satisfies the standard identity of order 2 if and only if it is commutative.

In Lemma 2.6.5 we have seen that the algebra $\mathbb{F}^{n\times n}$ does not satisfy any polynomial identity of order less than $2n$. The next theorem states that it satisfies the standard identity of degree $2n$.

Theorem 2.6.11 (Amitsur-Levitzki)**.** *The algebra $\mathbb{F}^{n\times n}$ satisfies the standard identity of degree $2n$.*

Proof. First let $\mathbb{F} = \mathbb{Q}$ be the field of the rational numbers and consider a matrix $a \in \mathbb{Q}^{n\times n}$. Newton's formula for the coefficients of the characteristic polynomial $\mathcal{P}_a$ of a,

$$\mathcal{P}_a(\lambda) := \det(\lambda I - a) = \lambda^n + \sum_{k=1}^{n} \alpha_k\lambda^{n-k} \tag{2.26}$$

implies that the coefficients α_k are given as follows. Let Ω_k denote the set of all j-tuples $m = (m_1,\ldots,m_j)$ of integers with $1 \le m_1 \le m_2 \le \cdots \le m_j$ and $m_1 + m_2 +$

$\cdots + m_j = k$. Notice that both the k-tuple $(1, \ldots, 1)$ as well as the 1-tuple (k) belong to Ω_k. Then, for $k = 1, \ldots, n$,

$$\alpha_k = \sum_{m \in \Omega_k} q_m \operatorname{tr}(a^{m_1}) \ldots \operatorname{tr}(a^{m_j})$$

with certain rational numbers q_m (see also [170, Theorem 1.3.19]).

By the Cayley-Hamilton theorem, $\mathcal{P}_a(a) = 0$. We apply the multilinearization process to the mapping $\mathcal{P}: \mathbb{F}^{n \times n} \to \mathbb{F}$, $a \mapsto \mathcal{P}_a(a)$ to get the multivariable function

$$(\Delta\mathcal{P})(a_1, \ldots, a_n) := (\Delta_1)^{n-1}\mathcal{P}(a_1).$$

Clearly, $(\Delta\mathcal{P})(a_1, \ldots, a_n) = 0$. Since the trace is additive, we thereby arrive at the identity

$$\begin{aligned} 0 = (\Delta\mathcal{P})(a_1, \ldots, a_n) &= \sum_{\sigma \in \Sigma_n} a_{\sigma_1} \ldots a_{\sigma_n} + \\ &+ \sum_{k=1}^{n} \sum_{m \in \Omega_k} \sum_{\sigma \in \Sigma_n} q_m \operatorname{tr}(a_{\sigma_1} \ldots a_{\sigma_{m_1}}) \ldots \operatorname{tr}(a_{\sigma_{m_1+\ldots+m_{j-1}+1}} \ldots a_{\sigma_k}) a_{\sigma_{k+1}} \ldots a_{\sigma_n}. \end{aligned}$$

For instance, for $n = 2$ one has

$$0 = \sum_{\sigma \in \Sigma_2} a_{\sigma_1} a_{\sigma_2} - \sum_{\sigma \in \Sigma_2} (\operatorname{tr} a_{\sigma_1}) a_{\sigma_2} + \frac{1}{2} \sum_{\sigma \in \Sigma_2} (\operatorname{tr} a_{\sigma_1})(\operatorname{tr} a_{\sigma_2}) - \frac{1}{2} \sum_{\sigma \in \Sigma_2} \operatorname{tr}(a_{\sigma_1} a_{\sigma_2})$$

due to

$$\mathcal{P}_a(a) = a^2 - (\operatorname{tr} a)a + \frac{1}{2}(\operatorname{tr} a)^2 - \frac{1}{2} \operatorname{tr}(a^2).$$

Back to general n. Given $2n$ matrices $a_1, \ldots, a_{2n} \in \mathbb{F}^{n \times n}$ and a permutation $\sigma' \in \Sigma_{2n}$, we replace each variable a_i in the above identity by $[a_{\sigma'_{2i-1}}, a_{\sigma'_{2i}}]$ and form the sum

$$0 = \sum_{\sigma' \in \Sigma_{2n}} \operatorname{sgn} \sigma' (\Delta\mathcal{P}) \left([a_{\sigma'_1}, a_{\sigma'_2}], \ldots, [a_{\sigma'_{2n-1}}, a_{\sigma'_{2n}}] \right).$$

Using Corollary 2.6.8, we write this identity as

$$0 = 2^n \mathcal{S}_{2n}(a_1, \ldots, a_{2n}) + \mathcal{P}'(a_1, \ldots, a_{2n})$$

where $\mathcal{P}'(a_1, \ldots, a_{2n})$ is a sum of terms of the form

$$\begin{aligned} &q_m \operatorname{tr} \mathcal{S}_{2m_1}(a_{2\sigma_1 - 1}, \ldots, a_{2\sigma_{m_1}}) \times \cdots \times \\ &\qquad \times \operatorname{tr} \mathcal{S}_{2m_j}(a_{2\sigma_{(m_1+\cdots+m_{j-1}+1)}-1}, \ldots, a_{2\sigma_k}) \mathcal{S}_{2(n-k)}(a_{2\sigma_{k+1}-1}, \ldots, a_{2\sigma_n}) \end{aligned}$$

for some $\sigma \in \Sigma_n$. (Again, it is useful to consider the particular case $n = 2$ to understand this construction.) By Lemma 2.6.9, each of these terms is 0. Thus, $\mathcal{P}'(a_1, \ldots a_{2n}) = 0$ and, consequently, $\mathcal{S}_{2n}(a_1, \ldots a_{2n}) = 0$.

Now let $\mathbb{F}$ be an arbitrary field, and $a_i \in \mathbb{F}^{n\times n}$. Each matrix a_i can be written as a linear combination $a_i = \sum_{j,k} a_{jk}^{(i)} E_{jk}$ where the E_{jk} are the unit matrices (i.e., the jk^{th} entry of E_{jk} is equal to 1, and the others are zero). Then, due to the multilinearity of $\mathcal{S}_{2n}$, the matrix $\mathcal{S}_{2n}(a_1,\ldots,a_{2n})$ is a linear combination of the matrices $\mathcal{S}_{2n}(E_{j_1k_1},\ldots,E_{j_{2n}k_{2n}})$ which are all zero by the first part of the proof. ■

It is useful to observe that the argument used in the last part of the above proof also applies to an arbitrary commutative unital algebra $\mathscr{C}$ over $\mathbb{F}$. Thus the Amitsur-Levitzki theorem remains valid if the field $\mathbb{F}$ is replaced by a commutative unital algebra $\mathscr{C}$.

Having this result at our disposal, we can extend the results on matrix algebras from Section 1.1.4 as follows.

Theorem 2.6.12. *Let $\mathscr{A}$ be a unital algebra over a field $\mathbb{F}$, and $\mathscr{C}$ its center. Then $\mathscr{A}$ is isomorphic to $\mathscr{C}^{n\times n}$ if and only if*

(i) *$\mathscr{A} \in SI_{2n}$, and*
(ii) *$\mathscr{A}$ contains a unital subalgebra $\mathscr{A}_0$ which is isomorphic to $\mathbb{F}^{n\times n}$.*

Proof. If $\mathscr{A}$ is isomorphic to $\mathscr{C}^{n\times n}$, then $\mathscr{A} \in SI_{2n}$ by the Amitsur-Levitzki theorem. Hence, (i) is satisfied, and (ii) is obvious. Conversely, let $\mathscr{A}$ satisfy (i) and (ii). Then Theorem 1.1.17 establishes an isomorphism $w = \left[w_{jk}\right]_{j,k=1}^{n}$ between the algebras $\mathscr{A}$ and $\mathscr{D}^{n\times n}$, where $\mathscr{D} = w_{jk}(\mathscr{A})$ is a subalgebra of $\mathscr{A}$. Moreover, the centers of $\mathscr{A}$ and $\mathscr{D}$ coincide. It remains to prove the inclusion $w_{jk}(a) \in \mathscr{C}$ for all $1 \le j, k \le n$, and $a \in \mathscr{A}$. Since $b = \sum_i \sum_m e_{ii} b e_{mm}$ for all $b \in \mathscr{A}$ and $w_{jk}(a) = \sum_{i=1}^n e_{ij} a e_{ki}$ by definition, the commutator $w_{jk}(a)b - bw_{jk}(a)$ is zero only if

$$e_{ij} a e_{ki} b e_{mm} - e_{ii} b e_{mj} a e_{km} = 0$$

for all $a, b \in \mathscr{A}$ and $1 \le i, j, k, m \le n$. But the latter identity is a simple consequence of

$$\begin{aligned} &e_{ij} a e_{ki} b e_{mm} - e_{ii} b e_{mj} a e_{km} \\ &\quad = e_{i1} \mathcal{S}_{2n+1}(e_{1j} a e_{k1}, e_{1i} b e_{m1}, e_{12}, \ldots, e_{n-1,n}, e_{nn}, e_{n,n-1}, \ldots, e_{21}) e_{1m} \end{aligned}$$

and of the fact that each SI_{2n}-algebra also satisfies the standard identity of degree $2n+1$. ■

Corollary 2.6.13. *Let $\mathscr{A} \in SI_{2n}$ be a complex Banach algebra with center $\mathscr{C}$, which contains a unital subalgebra $\mathscr{A}_0$ isomorphic to $\mathscr{C}^{n\times n}$. Then the maximal ideal spaces of $\mathscr{A}$ and $\mathscr{C}$ are homeomorphic.*

2.6.2 Matrix symbols

Let $\mathscr{A}$ be a Banach algebra with identity e over the field $\mathbb{C}$ and $\mathscr{B} \subset \mathscr{A}$ be a subalgebra of $\mathscr{A}$. Let X be an arbitrary set and l a bounded function from X into the set of the positive integers. Set $n := \sup_{x\in X} l(x)$. Assume we are given a family $\{\mu_x\}_{x\in X}$ of representations of $\mathscr{A}$ having the property that $\mu_x(a) \in \mathbb{C}^{l(x)\times l(x)}$ for each $a \in \mathscr{A}$. If it is true for every element $b \in \mathscr{B}$ that b is invertible in $\mathscr{A}$ if and only if $\mu_x(b)$ is invertible for all $x \in X$, then we say that $\{\mu_x\}_{x\in X}$ generates a *matrix symbol of order n for $\mathscr{B}$ in $\mathscr{A}$*. The collection of all subalgebras $\mathscr{B}$ of $\mathscr{A}$ which possess a matrix symbol of order n for $\mathscr{B}$ in $\mathscr{A}$ is denoted by $\mathrm{IS}(n,\mathscr{A})$. In case $\mathscr{A}$ belongs to $\mathrm{IS}(n,\mathscr{A})$ we just say that $\mathscr{A}$ has a matrix symbol of order n.

Let $\mathscr{J}$ be a maximal left ideal of a Banach algebra $\mathscr{A}$, and write E for the linear space $\mathscr{A}/\mathscr{J}$ and $\Phi : \mathscr{A} \to E$ for the canonical linear mapping. Further, let $L^{\mathscr{J}}$ denote the left regular representation of $\mathscr{A}$ induced by $\mathscr{J}$, which was introduced in Example 1.3.12.

Lemma 2.6.14. *Let E_0 be a finite-dimensional linear manifold in E and let $x \in E \setminus E_0$. Then there is an $a \in \mathscr{A}$ such that $L_a^{\mathscr{J}}(E_0) = \{0\}$ and $L_a^{\mathscr{J}}(x) \neq 0$.*

Proof. The proof proceeds by induction with respect to the dimension of E_0. The assertion of the lemma is evidently true if $\dim E_0 = 0$. Suppose that it is true for $\dim E_0 = k$. Choose $y \notin E_0$ and set $E_1 := E_0 + \mathbb{C}y$. Further let $\mathscr{L} := \{L_a^{\mathscr{J}} : L_a^{\mathscr{J}}(E_0) = \{0\}\}$. Consider the set $\{L_a^{\mathscr{J}}(y) : L_a^{\mathscr{J}} \in \mathscr{L}\}$. This set is a non-trivial linear subspace of E which is invariant under all operators $L_b^{\mathscr{J}}$ with $b \in \mathscr{A}$. Since $L^{\mathscr{J}}$ is an algebraically irreducible representation (by Exercise 1.3.7), one has $\{L_a^{\mathscr{J}}(y) : L_a^{\mathscr{J}} \in \mathscr{L}\} = E$.

Now suppose the assertion of the lemma is not valid for E_1. Then there is a $z \in E \setminus E_1$ such that $L_a^{\mathscr{J}}(z) = 0$ for every a satisfying $L_a^{\mathscr{J}}(E_1) = \{0\}$. Consider the operator B that acts on E according to $Bx := L_a^{\mathscr{J}}(z)$, with a chosen such that $L_a^{\mathscr{J}}(y) = x$. It can easily be checked that B is correctly defined and satisfies the conditions of Corollary 1.3.13. Consequently, B is a scalar operator, i.e., $B = \lambda I$ with a complex λ, and for every $L_a^{\mathscr{J}} \in \mathscr{L}$ we have

$$L_a^{\mathscr{J}}(z) = BL_a^{\mathscr{J}}(y) = \lambda L_a^{\mathscr{J}}(y) \Leftrightarrow L_a^{\mathscr{J}}(z - \lambda y) = 0.$$

By hypothesis, if $L_a^{\mathscr{J}}(\xi) = 0$ for all $L_a^{\mathscr{J}} \in \mathscr{L}$, then $\xi \in E_0$. Thus $z - \lambda y \in E_0$, which contradicts the choice of $z \in E \setminus E_1$. ∎

Lemma 2.6.15. *Let $v_1, \ldots, v_n$ and $e_1, \ldots, e_n$ be elements of E, and suppose that the elements e_k are linearly independent. Then there is an element $a \in \mathscr{A}$ with $L_a^{\mathscr{J}} e_k = v_k$ for all $k = 1, \ldots, n$.*

Proof. It follows from Lemma 2.6.14 that, for each $k = 1, \ldots, n$, there is an $a_k \in \mathscr{A}$ such that $L_{a_k}^{\mathscr{J}}(e_k) \neq 0$ and $L_{a_k}^{\mathscr{J}}(e_m) = 0$ for $m \neq k$. Consider the linear manifold $E_k :=$

$\{L_x^{\mathscr{J}} L_{a_k}^{\mathscr{J}}(e_k) : x \in \mathscr{A}\}$. Since $L_a^{\mathscr{J}}(E_k) \subset E_k$ and $L_e^{\mathscr{J}} L_{a_k}^{\mathscr{J}}(e_k) \neq 0$, we have $E_k = E$ for every k. Hence, there are elements $x_k \in \mathscr{A}$ such that $L_{x_k}^{\mathscr{J}} L_{a_k}^{\mathscr{J}}(e_k) = L_{x_k a_k}^{\mathscr{J}}(e_k) = v_k$ and $L_{x_k a_k}^{\mathscr{J}}(e_m) = 0$ for $m \neq k$. The element $a := \sum_{k=1}^n x_k a_k$ has the desired property. ■

Theorem 2.6.16 (Kaplansky). *Every primitive Banach algebra $\mathscr{A} \in SI_{2n}^{\infty}$ is isomorphic to $\mathbb{C}^{l \times l}$ for some $l \leq n$.*

Proof. If the algebra $\mathscr{A}$ is primitive, then it contains a maximal left ideal $\mathscr{J}$ for which the corresponding left regular representation $L^{\mathscr{J}} : \mathscr{A} \to \{L_a^{\mathscr{J}} : a \in \mathscr{A}\}$ is an isomorphism. We show that if $\mathscr{A} \in SI_{2n}^{\infty}$ then $\dim \mathscr{A}/\mathscr{J} \leq n$. Set $E := \mathscr{A}/\mathscr{J}$ and suppose $\dim E > n$. Let $e_1, \ldots, e_{n+1}$ be linearly independent elements in E. For $i, j, k = 1, \ldots, n+1$, set

$$v_k^{(ij)} := \delta_{jk} e_i$$

where δ_{jk} is the Kronecker symbol. By Lemma 2.6.15, there are elements $a_{i,j} \in \mathscr{A}$ such that $L_{a_{i,j}}^{\mathscr{J}} e_k = v_k^{(ij)}$. Then a simple computation yields

$$\mathbb{S}_{2n}\left(L_{a_{n+1,n}}^{\mathscr{J}}, L_{a_{n,n-1}}^{\mathscr{J}}, \ldots, L_{a_{2,1}}^{\mathscr{J}}, L_{a_{1,2}}^{\mathscr{J}}, \ldots, L_{a_{n,n+1}}^{\mathscr{J}}\right) e_{n+1} = e_{n+1},$$

whence

$$\left\| \mathbb{S}_{2n}^m \left(L_{a_{n+1,n}}^{\mathscr{J}}, L_{a_{n,n-1}}^{\mathscr{J}}, \ldots, L_{a_{2,1}}^{\mathscr{J}}, L_{a_{1,2}}^{\mathscr{J}}, \ldots, L_{a_{n,n+1}}^{\mathscr{J}}\right) \right\| \geq 1$$

for all m. Since $L^{\mathscr{J}}$ is continuous, this contradicts our assumption that $\mathscr{A} \in SI_{2n}^{\infty}$. Hence, $\dim E \leq n$. But then, clearly, $\mathscr{L}(E) \equiv \mathbb{C}^{l \times l}$ with some $l \leq n$. ■

The following theorem can be considered as an analog of the Gelfand theory for algebras satisfying a standard polynomial identity.

Theorem 2.6.17. *Let $\mathscr{A} \in SI_{2n}^{\infty}$ be a unital Banach algebra. Then:*

(i) *for each maximal ideal x of $\mathscr{A}$, the quotient algebra $\mathscr{A}_x := \mathscr{A}/x$ is isomorphic to $\mathbb{C}^{l \times l}$ with a certain $l = l(x)$ less than or equal to n;*
(ii) *an element $a \in \mathscr{A}$ is invertible if and only if the matrices $\Phi_x(a) \in \mathbb{C}^{l(x) \times l(x)}$ are invertible for all maximal ideals x of $\mathscr{A}$ (here we set $\Phi_x := \varphi_x \pi_x$ where π_x denotes the canonical homomorphism from $\mathscr{A}$ onto $\mathscr{A}_x$ and φ_x is an arbitrarily chosen isomorphism from $\mathscr{A}_x$ onto $\mathbb{C}^{l(x) \times l(x)}$, which exists by* (i)*);*
(iii) *the radical of $\mathscr{A}$ coincides with the intersection of all maximal ideals of $\mathscr{A}$.*

Proof. (i) If the algebra itself is primitive then $\mathscr{A}$ is isomorphic to $\mathbb{C}^{l \times l}$ with some $l \leq n$ by Theorem 2.6.16. We will exclude this trivial case. For all x in $M_{\mathscr{A}}$, the quotient algebra $\mathscr{A}_x$ is primitive and belongs to SI_{2n}^{∞}. Then (i) follows from Theorem 2.6.16.

(ii) If $a \in \mathscr{A}$ is invertible, then $\Phi_x(a)$ is invertible for every $x \in M_{\mathscr{A}}$. To prove the reverse implication, suppose that the $\Phi_x(a)$ are invertible for every $x \in M_{\mathscr{A}}$. Then

$a+x$ is invertible in $\mathscr{A}_x$. Suppose that a is not left invertible. Then a belongs to a left maximal ideal $\mathscr{J}$ by Proposition 1.3.2. Let $L^{\mathscr{J}}$ be the left regular representation induced by $\mathscr{J}$, and set $\mathscr{I} := \operatorname{Ker} L^{\mathscr{J}}$. It is evident that $\mathscr{I}$ is an ideal which is contained in $\mathscr{J}$, that the quotient algebra $\mathscr{A}^{\mathscr{I}}$ is primitive, and $\mathscr{A}^{\mathscr{I}} \in SI_{2n}^{\infty}$. Theorem 2.6.16 implies that $\mathscr{A}^{\mathscr{I}}$ is isomorphic to $\mathbb{C}^{l\times l}$, with $l \le n$, whence the maximality of $\mathscr{I}$. Since x is a subset of $\mathscr{J}$, the image $\mathscr{J}_x := \pi_x(\mathscr{J})$ is a left ideal again, and $\pi_x(a) \in \mathscr{J}_x$. So $\pi_x(a)$ cannot be invertible in $\mathscr{A}_x$. This contradicts the assumption. Hence, a is left invertible.

Let us prove now that a is also right invertible. Since a is left invertible, there is a $b \in \mathscr{A}$ such that $ba = e$. Thus, $\Phi_x(b)\Phi_x(a) = \Phi_x(e)$. Since $\Phi_x(a)$ is invertible in $\mathscr{A}_x$, it follows that $\Phi_x(a)\Phi_x(b) = \Phi_x(e)$ or, equivalently, $ab - e \in x$ for all $x \in M_{\mathscr{A}}$. Since each maximal left ideal contains a maximal ideal by Lemma 1.3.15, the element $r = ab - e$ belongs to the radical of $\mathscr{A}$. By Proposition 1.3.3, the element $ab = e + r$ is invertible, which implies the right invertibility of a.

(iii) The intersection of all maximal ideals belongs to the radical $\mathscr{R}_{\mathscr{A}}$ of $\mathscr{A}$. Conversely let $r \in \mathscr{R}_{\mathscr{A}}$. Then $r_x := r + x$ belongs to $\mathscr{R}_{\mathscr{A}_x}$. Since $\mathscr{A}_x$ is semi-simple, this implies that $r \in x$ for all maximal ideals x. ■

Remark 2.6.18. The proof of Theorem 2.6.16 made use only of the multilinear property of $\mathbb{S}_{2n}$. Thus the proof holds if instead of $\mathbb{S}_{2n}$ one has any multilinear polynomial. This in particular implies that a version of Theorem 2.6.17 is true if the algebra $\mathscr{A}$ is a PI-algebra, because of Lemma 2.6.4. □

Theorem 2.6.19. *Let $\mathscr{A}$ be a unital Banach algebra. The following assertions are equivalent:*

(i) *$\mathscr{A}/\mathscr{R}_{\mathscr{A}}$ is a PI-algebra;*
(ii) *$\mathscr{A}$ has a matrix symbol of order n;*
(iii) *$\mathscr{A}/\mathscr{R}_{\mathscr{A}} \in SI_{2n}$;*
(iv) *$\mathscr{A} \in SI_{2n}^{\infty}$.*

Proof. Suppose that $\mathscr{A}/\mathscr{R}_{\mathscr{A}}$ is a PI-algebra. Then, $\mathscr{A}/\mathscr{R}_{\mathscr{A}}$ satisfies a multilinear polynomial, and thus has a matrix symbol of some order n (see Remark 2.6.18). Assertion (ii) follows due to the equivalence of the invertibility of an element $a \in cA$, and the invertibility of the coset $a + \mathscr{R}_{\mathscr{A}}$ in $\mathscr{A}/\mathscr{R}_{\mathscr{A}}$. We continue with the implication (ii) $\Rightarrow$ (iii): Assume that there is a family of matrix-valued homomorphisms h_x on $\mathscr{A}$, labeled by the elements of some set X, such that an element $a \in \mathscr{A}$ is invertible in $\mathscr{A}$ if and only if the matrices $h_x(a)$ are invertible for all $x \in X$.

We claim that if $a \in \mathscr{A}$ is invertible, then $a + \mathbb{S}_{2n}(a_1, \ldots, a_{2n})$ is invertible for each choice of elements $a_1, \ldots, a_{2n}$ of $\mathscr{A}$. Since

$$h_x(\mathbb{S}_{2n}(a_1, \ldots, a_{2n})) = \mathbb{S}_{2n}(h_x(a_1), \ldots, h_x(a_{2n}))$$

and all entries $h_x(a_k)$ are $l \times l$ matrices, Theorem 2.6.11 implies that

$$h_x(\mathcal{S}_{2n}(a_1,\ldots,a_{2n})) = 0$$

for all $x \in X$. Hence,

$$h_x(a) = h_x(a + \mathcal{S}_{2n}(a_1,\ldots,a_{2n}))$$

for every $x \in X$. Since the h_x constitute a matrix symbol, the claim follows. Then, by Proposition 1.3.3, $\mathcal{S}_{2n}(a_1,\ldots,a_{2n})$ is in the radical of $\mathscr{A}$, whence the assertion.

The implication (iii) $\Rightarrow$ (iv) is a consequence of the fact that elements of the radical have spectral radius 0 and of Theorem 1.2.12. (iii) also trivially implies (i). The final implication (iv) $\Rightarrow$ (ii) comes directly from Theorem 2.6.17. ■

Example 2.6.20. Let $\mathscr{A}_0$ denote the set of all bounded linear operators A on l^2, such that the coefficients of the matrix representation $(a_{ij})_{i,j=1}^{\infty}$ of A with respect to the standard basis satisfy the following conditions:

- $a_{ij} = 0$ whenever $i > j$;
- $a_{ij} = 0$ whenever $i < j$ with a finite number of exceptions;
- the limit $\lim_{i\to\infty} a_{ii}$ exists and is finite.

The set $\mathscr{A}_0$, provided with the operations inherited from $\mathscr{L}(l^2)$, forms an algebra. Let $\mathscr{A}$ be the closure of $\mathscr{A}_0$ in $\mathscr{L}(l^2)$. Then $\mathscr{A}$ is a QI-algebra in SI_2^{∞}, the mappings

$$\phi_n : \mathscr{A} \to \mathbb{C}, \quad \phi_n(A) := \begin{cases} a_{nn} & \text{if } n \in \mathbb{N}, \\ \lim_{i\to\infty} a_{ii} & \text{if } n = \infty. \end{cases} \tag{2.27}$$

are continuous homomorphisms, and an element $A \in \mathscr{A}$ is invertible in $\mathscr{A}$ if and only if $\phi_n(A) \neq 0$ for all $n \in \mathbb{N} \cup \{\infty\}$. The proof of these facts is left as an exercise. □

We conclude this section with two results on the existence of matrix symbols for algebras with a non-trivial center.

Proposition 2.6.21. *Let $\mathscr{A}$ be a unital Banach algebra, $\mathscr{B}$ be a closed subalgebra and $\mathscr{C}$ a subalgebra of the center of $\mathscr{B}$. If*

$$\sup_{x\in M_{\mathscr{C}}} \dim \mathscr{B}_x =: m < \infty \tag{2.28}$$

then $\mathscr{B} \in \mathrm{IS}(n, \mathscr{A})$ for some $n \leq \sqrt{m}$.

Proof. By hypothesis, $\dim \mathscr{B}_x \leq m$ for any $x \in M_{\mathscr{C}}$. For $b_1,\ldots,b_{m+1} \in \mathscr{B}$, consider $\mathcal{S}_{m+1}(b_1+x,\ldots,b_{m+1}+x)$. Let $e_1,\ldots,e_m$ be a basis of $\mathscr{B}_x$. Since every coset $b_i + x$ can be written as a linear combination of the basis elements $e_1,\ldots,e_m$, the multilinearity of the standard polynomial implies that $\mathcal{S}_{m+1}(b_1+x,\ldots,b_{m+1}+x) = 0$. By Theorem 2.6.17, there exists a positive integer $k \leq \lceil \frac{m+1}{2} \rceil$ such that $\mathscr{B}_x$ has a matrix symbol of order k for all $x \in M_{\mathscr{C}}$. Consider the commutator

$$\mathscr{D} := \{a \in \mathscr{A} : ac = ca \text{ for every } c \in \mathscr{C}\}$$

of the algebra $\mathscr{C}$. It is easy to see that $\mathscr{D}$ is an inverse-closed Banach subalgebra of $\mathscr{A}$ which contains $\mathscr{B}$ and which has $\mathscr{C}$ in its center. Thus, the local algebras $\mathscr{D}_x$ are well defined for $x \in M_{\mathscr{C}}$. Since $\dim \mathscr{B}_x < \infty$, the algebra $\mathscr{B}_x$ is inverse-closed in $\mathscr{D}_x$ (see Exercise 1.2.13). Thus, $\mathscr{B}_x \in \mathrm{IS}(k, \mathscr{D}_x)$ for all $x \in M_{\mathscr{C}}$.

We claim that $\mathscr{B} \in \mathrm{IS}(k, \mathscr{A})$. Indeed, let $\{\Phi_\tau^x\}_{\tau \in \mathscr{T}(x)}$ generate a matrix symbol for $\mathscr{B}_x$ in $\mathscr{D}_x$. For $b \in \mathscr{B}$, set $\Phi_{x,\tau}(b) := \Phi_\tau^x(b_x)$, which defines a homomorphism $\Phi_{x,\tau}$ on $\mathscr{B}$. Since $\mathscr{D}$ is inverse-closed in $\mathscr{A}$, the element $b \in \mathscr{B}$ is invertible in $\mathscr{A}$ if and only if it is invertible in $\mathscr{D}$. By Allan's local principle, b is invertible in $\mathscr{D}$ if and only if b_x is invertible in $\mathscr{D}_x$ for all $x \in M_{\mathscr{C}}$. Thus, b is invertible in $\mathscr{A}$ if and only if $\Phi_{x,\tau}(b)$ is invertible for all $x \in M_{\mathscr{C}}$ and $\tau \in \mathscr{T}(x)$, that is, the family $\{\Phi_{x,\tau}\}$ generates a matrix symbol for $\mathscr{B}$ in $\mathscr{A}$.

The homomorphisms $\Phi_{x,\tau} : \mathscr{A} \mapsto \mathbb{C}^{l(x)\times l(x)}$ obviously satisfy $\dim \mathrm{Im}\, \Phi_{x,\tau} \leq \dim \mathscr{B}_x$. Consequently, $l^2(x) \leq m$, and the result follows. ∎

Corollary 2.6.22. *Let $\mathscr{A}$ be a unital Banach algebra, $\mathscr{B}$ a closed subalgebra of $\mathscr{A}$, and $\mathscr{B}^0$ be a dense subalgebra of $\mathscr{B}$. If $\mathscr{B}^0$ is an m-dimensional module over its center, then $\mathscr{B} \in \mathrm{IS}(n, \mathscr{A})$ with a certain $n \leq \sqrt{m}$.*

2.6.3 Exercises

Exercise 2.6.1. Prove that every finite-dimensional algebra $\mathscr{A}$ with $\dim \mathscr{A} < n$ satisfies the standard identity of order n.

Exercise 2.6.2. Prove that the mappings ϕ_i in Example 2.6.20 are continuous homomorphisms and that an element $A \in \mathscr{A}$ is invertible in $\mathscr{A}$ if and only if $\phi_i(A) \neq 0$ for all $i \in \mathbb{N} \cup \{\infty\}$.

2.7 Notes and comments

The simplest local principle is the classical Gelfand theory for commutative Banach algebras. It was created by Gelfand in 1941. The original paper is [62]. The modern reader can choose between a few textbooks on Banach algebras which present nice introductions to Gelfand theory; see, for example, [162, 171, 202].

Classical Gelfand theory has found a variety of remarkable applications. For instance, it provides an elegant and relatively simple proof of the famous Wiener theorem which states that if f is a non-vanishing function with an absolutely convergent Fourier series expansion, then its inverse f^{-1} has the same property. It is thus nothing but natural that several attempts were made to establish non-commutative versions of Gelfand's theory.

Representation theory for C^*-algebras can be thought of as a non-commutative generalization of the Gelfand theory which carries much of its spirit: the maximal ideal space is replaced by the spectrum of the algebra (which coincides with the space of the primitive ideals in many cases), and the multiplicative functionals correspond to the irreducible representations.

The needs of operator theory, asymptotic spectral theory and numerical analysis show that one has to go beyond C^*-algebras. Already in 1942, Bochner and Phillips [9] generalized Wiener's theorem to functions with values in a Banach algebra. In their paper, they introduced a new tool which can be considered as the first appearance of a (non-commutative) local principle. When speaking about local principles in this text, we mean a generalization of Gelfand's theory to non-commutative Banach algebras which are not too far away from commutative algebras in the sense that they possess a large center, or that a "higher" commutator property is satisfied.

The idea to consider the cosets of an element a modulo the ideals $\mathscr{I}_x$ as a localization of a, as was done in Section 2.2, is quite old. The first reference we know is Glimm's paper [64] from 1960 where he introduced these ideals in the case that $\mathscr{A}$ is a C^*-algebra and $\mathscr{B}$ is the full center of that algebra, and where he already proved the upper semi-continuity of the mapping $x \mapsto \|a + \mathscr{I}_x\|$ (Theorem 2.2.2 (ii) above). In this setting, the ideals $\mathscr{I}_x$ are also known as *Glimm's ideals* in the literature.

The general version of this kind of localization (where $\mathscr{A}$ is a unital Banach algebra and $\mathscr{B}$ a closed central subalgebra) is known as Allan's local principle (Theorem 2.2.2). It appeared in its original form in [2]. Subsections 2.2.3 and 2.2.4 present some additional features related with local invertibility and inverse-closedness. Note, in connection with Subsection 2.2.3, that Allan's local principle can be used to study the continuity of the spectrum of elements in Banach algebras, as proposed in [50]. The combination of Allan's local principle with ideas presented in [9] leads to a piece of non-commutative Gelfand theory which is crucial in the study of one-sided invertibility of Banach algebra-valued holomorphic functions. The interested reader is directed to the monograph [128].

The material presented in Subsection 2.2.4 can be completed by a result which is of interest in connection with Theorem 2.2.8. The result belongs to Hulanicki [89] (see also [52] for a corrected proof) and reads as follows: Assume that $\mathscr{A}$ is an involutive Banach algebra with identity I_H and contained in $\mathscr{L}(H)$ for some Hilbert space H. If

$$r_{\mathscr{A}}(A) = r(A) = \|A\|_{op},$$

for all self-adjoint elements $A \in \mathscr{A}$, then $\mathrm{sp}_{\mathscr{A}} A = \mathrm{sp} A$ for all $A \in \mathscr{A}$. Moreover, the algebra $\mathscr{A}$ is symmetric, that is, $\mathrm{sp}_{\mathscr{A}} A^*A \subset [0, \infty[$ for all $A \in \mathscr{A}$.

Douglas' local principle (Theorem 2.2.12) is the specification of Allan's local principle to C^*-algebras. It appeared independently in [41]. Originally, Allan's local principle was aimed at the study of invertibility properties of holomorphic Banach algebra-valued functions, whereas Douglas' local principle was used in the study of invertibility properties of Toeplitz operators.

Proposition 2.2.5 goes back to Semenyuta and Khevelev ([179]). The proof presented here is an adaptation of the one presented in [14, Proposition 8.6].

The subject of Section 2.3 is perhaps the most important supplement to Allan's local principle: the concept of norm-preserving localization. It appeared for the first time in a paper by Krupnik [107] in the special context of Simonenko's local principle. This local principle is rather specific and works in the algebra of all bounded linear operators of local type acting on spaces $L^p(X,\mu)$, where X is a Hausdorff space of finite dimension. In [107], Krupnik proposed a sharper version which is true without any restriction to the dimension of X and which already offers norm-preserving localization in the Calkin image of the algebra of all operators of local type. What is called a faithful localizing pair in Subsection 2.3.1 first appeared in a joint paper of Böttcher, Krupnik and one of the authors in [18] under the name KMS-algebra. Theorem 2.3.3 is taken from that paper, whereas Theorems 2.3.4 and 2.3.5 are from [169].

Let us mention an independent circle of papers [33, 85, 194, 195] which were published at about the same time as Simonenko's and Allan's local principles appeared. The aim of these papers is to describe C^*-algebras as continuous fields C^*-algebras.

Gohberg-Krupnik's local principle was published in 1973 [70]. It is distinguished by its simplicity and broad applicability. Assertions (i) and (ii) in Theorem 2.4.5 are due to Gohberg and Krupnik, whereas (iii) appeared in [21]. Assertion (iv) is added by the authors.

As already mentioned, Simonenko's local principle, presented in Section 2.5, was originally formulated for operators of local type acting on Lebesgue spaces $L^p(X,\mu)$ with X a Hausdorff space of finite dimension. In the original paper [185, 186], Simonenko had already introduced the notion of an envelope and presented Theorem 2.5.20 under the assumption that the operator function $x \mapsto A_x$ is continuous. The equivalence of (i) and (ii) in Theorem 2.5.6 had already been mentioned by Seeley in his review of Simonenko's paper [185] (see Mathematical Reviews MR0179630). A partial case of Theorem 2.5.20 is contained in [189]. The proofs presented are in the spirit of [18] and [169].

Allan's local principle, completed by the concept of norm-preserving localization, can be regarded as a far-reaching generalization of Simonenko's local principle. For another generalization of Simonenko's principle we refer to Kozak [102].

Operators of local type on spaces $C^n(X)$ and Hölder spaces $H_\alpha(X)$ with $0<\alpha\leq 1$ were studied in the Ph.D. thesis of Pöltz [146] and in the papers [116, 142, 143, 144, 145]. He constructed an analog of Simonenko's theory for the above mentioned spaces, none of which is of "local type" in the sense of Subsection 2.5.1. Unfortunately, the results of Pöltz are almost unknown even among the experts.

In Section 2.6 we mainly follow Krupnik's book [108] together with the paper [54] by Finck and two of the authors. There are several proofs of the Amitsur-Levitzki theorem, but unfortunately they are all rather technical. The proof presented here belongs to Razmyslov [159] (see also [42, 170]). In [108], a different proof can be found. The equivalence between (ii) and (iii) in Theorem 2.6.19 is a natural conclusion of both works [54, 108]. The last two results appeared in [75].

One basic question is left open in our exposition, namely whether it is possible to provide the set of the maximal ideals of a Banach PI-algebra with a topology which has (compactness) properties similar to the Gelfand topology in the commutative case. This questions seems to be delicate. Some partial results can be found in [108]. Regarding functional calculus for Banach PI-algebras, see [115].

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