

Chapter 2

Minimization Techniques: Compact Problems

Throughout this chapter we show how techniques based on minimization arguments can be used to establish existence results for various types of problems.

Our aim is not to describe the most general results, but to give a series of examples, and to show how simple techniques can be refined to treat more complex cases.

2.1 Coercive Problems

We begin with the following problem, that will provide our guideline through the whole chapter. We want to find a (weak) solution to

$$\begin{cases} -\Delta u + q(x)u = f(u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

In this section, the general framework is specified by the assumptions

(h₁) $\Omega \subset \mathbb{R}^N$ is bounded and open, $q \in L^\infty(\Omega)$ and $q(x) \geq 0$ a.e. in Ω .

(h₂) $h \in L^2(\Omega)$.

We equip $H_0^1(\Omega)$ with the scalar product

$$(u|v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} q(x)uv \, dx, \quad (2.2)$$

and we denote by $\|\cdot\|$ the induced norm, equivalent to the standard one.

Remark 2.1.1 In assumption **(h₁)**, the requirement $q(x) \geq 0$ a.e. is used only to obtain $\lambda_1(-\Delta + q(x)) > 0$, which guarantees that (2.2) is indeed a scalar product and that the induced norm is equivalent to the standard norm of $H_0^1(\Omega)$, see Remark 1.7.5 and Exercise 9 in Chap. 1. Therefore in all the results of this chapter, and similarly in all the subsequent chapters, the assumption $q \geq 0$ could be replaced by the “abstract” condition

$$\lambda_1(-\Delta + q(x)) > 0, \quad (2.3)$$

and everything would work perfectly well with no changes in the proofs. The point is, precisely, that (2.3) is abstract, and nobody knows for which general q 's it is satisfied. We prefer, in this book, to assume an explicit sign condition on q , rather than an indirect one on λ_1 . The reader should however keep in mind this clarification.

We begin by assuming the following hypothesis on the nonlinearity f .

(h₃) $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded.

Setting $F(t) = \int_0^t f(s) ds$, the computations carried out in Example 1.3.20 show that the functional $I : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} q(x) u^2 dx - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx \\ &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx \end{aligned}$$

is differentiable on $H_0^1(\Omega)$. Its critical points are the weak solutions of (2.1).

Note that unless F is concave, which we do not assume, the functional I needs not be convex.

Theorem 2.1.2 *Under the assumptions (h₁)–(h₃), Problem (2.1) admits at least one solution.*

Remark 2.1.3 The leading idea of the proof is that since f is bounded, the term $\int_{\Omega} F(u) dx$ should grow at most linearly with respect to $\|u\|$, as well as the last term. If this is true, the functional I can be seen as an “at most linear” perturbation of the quadratic term $\|u\|^2$. This suggests the existence of a global minimum. Let us see how all this really works.

Proof We break it into two steps. We make repeated use of Hölder and Sobolev inequalities.

Step 1. The functional I is coercive. Note first that since f is bounded, then

$$|F(t)| \leq M|t|$$

for some $M > 0$ and all $t \in \mathbb{R}$. Hence

$$\left| \int_{\Omega} F(u) dx \right| \leq M \int_{\Omega} |u| dx \leq C \|u\|,$$

where the last inequality comes from the continuity of the embedding of $H_0^1(\Omega)$ into $L^1(\Omega)$. This confirms the idea of the linear growth as in the preceding remark. Thus

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx \geq \frac{1}{2} \|u\|^2 - C \|u\| - |h|_2 |u|_2 \\ &\geq \frac{1}{2} \|u\|^2 - C \|u\|, \end{aligned}$$

which shows that I is coercive.

Step 2. The infimum of I is attained. Set

$$m = \inf_{u \in H_0^1(\Omega)} I(u).$$

Step 1 shows that $m > -\infty$, although one does not really need this: it will follow automatically from the fact that it is attained.

Let $\{u_k\}_k \subset H_0^1(\Omega)$ be a minimizing sequence for I ; from Step 1 we immediately see that $\{u_k\}_k$ is bounded in $H_0^1(\Omega)$, and therefore we can assume that there is a subsequence, still denoted u_k , such that

- $u_k \rightharpoonup u$ in $H_0^1(\Omega)$;
- $u_k \rightarrow u$ in $L^2(\Omega)$;
- $u_k(x) \rightarrow u(x)$ a.e. in Ω ;
- there exists $w \in L^2(\Omega)$ such that $|u_k(x)| \leq w(x)$ a.e. in Ω and for all k .

Notice now that since F is continuous we have $F(u_k(x)) \rightarrow F(u(x))$ a.e. in Ω , and due to the growth properties of F , we also have

$$|F(u_k(x))| \leq M |u_k(x)| \leq M w(x)$$

a.e. in Ω and for all k . Since Ω is bounded, $w \in L^1(\Omega)$, and by dominated convergence we obtain $F(u_k) \rightarrow F(u)$ in $L^1(\Omega)$; in particular,

$$\int_{\Omega} F(u_k) dx \rightarrow \int_{\Omega} F(u) dx.$$

We also have, of course,

$$\int_{\Omega} h u_k dx \rightarrow \int_{\Omega} h u dx \quad \text{and} \quad \|u\|^2 \leq \liminf_k \|u_k\|,$$

by weak lower semicontinuity of the norm. Thus

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} h u dx \\ &\leq \liminf_k \frac{1}{2} \|u_k\|^2 - \lim_k \int_{\Omega} F(u_k) dx - \lim_k \int_{\Omega} h u_k dx \\ &= \liminf_k \left(\frac{1}{2} \|u_k\|^2 - \int_{\Omega} F(u_k) dx - \int_{\Omega} h u_k dx \right) = \liminf_k I(u_k) = m. \end{aligned}$$

But $u \in H_0^1(\Omega)$, so that $I(u) \geq m$, which shows that $I(u) = m$. Therefore u is a global minimum for I , and hence it is a critical point, namely a solution to (2.1). $\square$

Remark 2.1.4 Analyzing the preceding proof one sees that what we actually did is to show that I is coercive and weakly lower semicontinuous on $H_0^1(\Omega)$. These are exactly the assumptions that one needs in the (generalized) Weierstrass Theorem to deduce the existence of a global minimum, see Remark 1.5.7.

The boundedness of f in the previous result has been used to show that the nonlinear term $\int_{\Omega} F(u) dx$ does not destroy the growth properties of I inherited by

the term $\|u\|^2$. This occurred because, as we have seen, the nonlinear term grows at most linearly. Now this is not really necessary: it is enough that this term grows *less than quadratically*. Let us see what kind of assumptions we can use in this sense in the next two results.

We begin by replacing the boundedness condition **(h₃)** by the growth assumption

(h₄) $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist $\sigma \in (0, 1)$ and $a, b > 0$ such that

$$|f(t)| \leq a + b|t|^\sigma \quad \forall t \in \mathbb{R}.$$

Thus f is no longer bounded, but is allowed to grow *sublinearly* ($\sigma < 1$). It follows that F grows at most *subquadratically*, in the sense that for some $a_1, b_1 > 0$,

$$|F(t)| \leq a_1 + b_1|t|^{\sigma+1} \quad \forall t \in \mathbb{R}, \quad (2.4)$$

with $\sigma + 1 < 2$.

Theorem 2.1.5 *Under the assumptions **(h₁)**, **(h₂)** and **(h₄)**, Problem (2.1) admits at least one solution.*

Proof Working as in the preceding proof we first show that I is coercive. Using the fact that $\sigma + 1 < 2$ we have

$$\left| \int_{\Omega} F(u) dx \right| \leq a_1 |\Omega| + b_1 \int_{\Omega} |u|^{\sigma+1} dx \leq C_1 + C_2 \|u\|^{1+\sigma},$$

thanks to the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{\sigma+1}(\Omega)$. Then

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} h u dx \geq \frac{1}{2} \|u\|^2 - C_1 - C_2 \|u\|^{\sigma+1} - |h|_2 \|u\|_2 \\ &\geq \frac{1}{2} \|u\|^2 - C_2 \|u\|^{\sigma+1} - C_3 \|u\| - C_1, \end{aligned}$$

and coercivity follows.

Let now $\{u_k\}_k \subset H_0^1(\Omega)$ be a minimizing sequence for I . As in the proof of Theorem 2.1.2 above, $\{u_k\}_k$ is bounded and therefore, up to subsequences, it converges weakly to some $u \in H_0^1(\Omega)$ and satisfies the same properties as in the preceding case. Then, reasoning as we did above, we obtain again

$$\int_{\Omega} F(u_k) dx \rightarrow \int_{\Omega} F(u) dx,$$

so that

$$I(u) \leq \liminf_k \left(\frac{1}{2} \|u_k\|^2 - \int_{\Omega} F(u_k) dx - \int_{\Omega} h u_k dx \right) = \liminf_k I(u_k) = \inf_{H_0^1(\Omega)} I.$$

The function u is a global minimum, hence a critical point of I , and we have found a solution of (2.1). $\square$

In our quest for more general assumptions we now try to go one step further: precisely, can we allow a *linear* growth for f , and then a *quadratic* growth for F ? The

answer is in the affirmative, provided we supply a quantitative control of the linear growth. This control is formulated in terms of the first eigenvalue $\lambda_1 = \lambda_1(-\Delta + q)$ in the following assumption.

(h₅) $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist $a > 0$ and $b \in (0, \lambda_1)$ such that

$$|f(t)| \leq a + b|t| \quad \forall t \in \mathbb{R}.$$

Integrating, it follows immediately that

$$|F(t)| \leq a|t| + \frac{b}{2}|t|^2 \quad \forall t \in \mathbb{R}.$$

Notice the difference with respect to (1.9): this is because we now want to keep the coefficient in front of $|t|^2$ as small as possible.

Theorem 2.1.6 *Under the assumptions (h₁), (h₂) and (h₅), Problem (2.1) admits at least one solution.*

Proof To control the term $\int_{\Omega} F(u) dx$ we use the characterization of the first eigenvalue, Theorem 1.7.6. We have

$$\left| \int_{\Omega} F(u) dx \right| \leq a \int_{\Omega} |u| dx + \frac{b}{2} \int_{\Omega} |u|^2 dx \leq C \|u\| + \frac{b}{2\lambda_1} \|u\|^2,$$

so that

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx \geq \frac{1}{2} \|u\|^2 - C \|u\| - \frac{1}{2} \frac{b}{\lambda_1} \|u\|^2 - |h|_2 |u|_2 \\ &\geq \frac{1}{2} \left(1 - \frac{b}{\lambda_1} \right) \|u\|^2 - C_1 \|u\|. \end{aligned}$$

Since $b < \lambda_1$, the functional is coercive.

The remaining part of the proof works exactly as in the preceding theorems. $\square$

Remark 2.1.7 In the literature, the growth conditions contained in assumptions (h₄) and (h₅) are often written

$$\limsup_{t \rightarrow \pm\infty} \frac{|f(t)|}{|t|^\sigma} < +\infty \quad \text{and} \quad \limsup_{t \rightarrow \pm\infty} \frac{|f(t)|}{|t|} < \lambda_1$$

respectively.

Remark 2.1.8 It is interesting to inspect what happens if we allow $b \geq \lambda_1$ in (h₅). In this case the functional I is *no longer coercive* and may be unbounded from below. In some cases, as for example if we take $f(t) = \lambda_k t$ ($k \geq 1$), Problem (2.1) has *no solution* for some h (see Theorem 1.7.8). Later we will see how to deal with nonlinearities that grow more than quadratically.

We now examine a variant of Problem (2.1), with the aim of showing how the variational information can be of help in establishing existence results. Consider

$$\begin{cases} -\Delta u + q(x)u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.5)$$

If $f(0) = 0$, a frequent case in the applications, then the problem admits $u \equiv 0$ as a solution (called the trivial solution).

Without further assumptions, it may very well be that the trivial solution is the *only* solution. For example, if $f(t)t \leq 0$ for all t , then any weak solution satisfies

$$\|u\|^2 = \int_{\Omega} f(u)u \, dx \leq 0,$$

and hence $u \equiv 0$.

In the next result we show a condition that prevents this fact.

Theorem 2.1.9 *Let (h_1) hold. Assume moreover that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies*

$$f(0) = 0 \quad \text{and} \quad \limsup_{t \rightarrow \pm\infty} \frac{|f(t)|}{|t|} < \lambda_1.$$

Then Problem (2.5) admits at least one solution (which may be trivial).

If in addition f also satisfies

$$\liminf_{t \rightarrow 0^+} \frac{f(t)}{t} > \lambda_1, \tag{2.6}$$

then Problem (2.5) admits at least one nontrivial solution.

Proof The first part is a special case of Theorem 2.1.6. We now show that under condition (2.6) the solution found in the first part is not identically zero. We use a *level argument*, as follows.

First notice that by (2.6), there exists $\beta > \lambda_1$ and $\delta > 0$ such that

$$f(t) \geq \beta t \quad \forall t \in [0, \delta],$$

which implies that

$$F(t) \geq \frac{1}{2}\beta t^2 \quad \forall t \in [0, \delta].$$

Let $\varphi_1 > 0$ be the first eigenfunction of $-\Delta + q(x)$, and take $\varepsilon > 0$ so small that $\varepsilon\varphi_1(x) < \delta$ for almost every x ; this is possible because $\varphi_1 \in L^\infty(\Omega)$, see Theorem 1.7.3.

Then

$$F(\varepsilon\varphi_1(x)) \geq \frac{1}{2}\beta\varepsilon^2\varphi_1^2(x)$$

a.e. in Ω . This implies that

$$\begin{aligned} I(\varepsilon\varphi_1) &= \frac{1}{2}\|\varepsilon\varphi_1\|^2 - \int_{\Omega} F(\varepsilon\varphi_1) \, dx \leq \frac{1}{2}\varepsilon^2\|\varphi_1\|^2 - \frac{1}{2}\beta\varepsilon^2 \int_{\Omega} \varphi_1^2 \, dx \\ &= \frac{1}{2}\varepsilon^2\lambda_1 \int_{\Omega} \varphi_1^2 \, dx - \frac{1}{2}\beta\varepsilon^2 \int_{\Omega} \varphi_1^2 \, dx = \frac{\varepsilon^2}{2}(\lambda_1 - \beta) \int_{\Omega} \varphi_1^2 \, dx < 0, \end{aligned}$$

since $\beta > \lambda_1$. Let u be the solution that minimizes I . Then

$$I(u) = \min_{v \in H_0^1(\Omega)} I(v) \leq I(\varepsilon \varphi_1) < 0.$$

As $I(0) = 0$, u cannot be the trivial solution. $\square$

Remark 2.1.10 It is possible to show that the preceding problem admits a *nonnegative* solution. Indeed it is enough to proceed as in Example 1.7.10.

Since $(\mathbf{h}_3)$ implies $(\mathbf{h}_4)$ that implies $(\mathbf{h}_5)$, it is clear Theorem 2.1.6 implies Theorem 2.1.5 that in turn implies Theorem 2.1.2. As a further example we examine now another case in which we can apply the scheme of the previous results and that leads to a theorem that is independent of the preceding ones. Consider the assumption

(h₆) $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exist $a, b > 0$ such that

$$|f(t)| \leq a + b|t|^{2^*-1} \quad \forall t \in \mathbb{R}.$$

Moreover

$$f(t)t \leq 0 \quad \forall t \in \mathbb{R}.$$

By integration one easily sees that there exist $a_1, b_1 > 0$ such that

$$|F(t)| \leq a_1 + b_1|t|^{2^*} \quad \forall t \in \mathbb{R}$$

and that

$$F(t) \leq 0 \quad \forall t \in \mathbb{R}$$

Notice that $-F$ is allowed to have *critical* growth, but F is not. Moreover the sign condition $f(t)t \leq 0$ prevents, as we have seen, the existence of nontrivial solutions when $h \equiv 0$. In spite of this, Problem (2.1) is solvable.

Theorem 2.1.11 *Under the assumptions $(\mathbf{h}_1)$, $(\mathbf{h}_2)$ and $(\mathbf{h}_6)$, Problem (2.1) admits at least one solution.*

Proof By Example 1.3.20, the usual functional I is differentiable on $H_0^1(\Omega)$. Coercivity is simple consequence of the sign of F :

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx \geq \frac{1}{2}\|u\|^2 - |h|_2|u|_2 \geq \frac{1}{2}\|u\|^2 - C\|u\|.$$

The proof then proceeds exactly as in the previous theorems. $\square$

Remark 2.1.12 This last existence result is similar to the one obtained in Theorem 1.6.6. However here we do not assume the monotonicity of f , so that the functional needs not be convex. This implies that we have to prove the weakly lower semicontinuity, as in the other theorems of this section, and we do not have a uniqueness result.

Remark 2.1.13 If $|F|$ grows more than critically, the functional I is no longer well defined on $H_0^1(\Omega)$, because a function in $H_0^1(\Omega)$ need not be in $L^p(\Omega)$ if $p > 2^*$. This means that the integral of $F(u)$ may be divergent for some u .

2.2 A min–max Theorem

In this section we deal with a much more complex result than those treated so far; the proof of the main theorem is quite long and can be omitted upon a first reading.

First, a heuristic motivation. The results in the previous section show, very roughly speaking, that if the nonlinearity f “does not interact” with the spectrum of the differential operator, then the procedure used for the linear problem (minimization) still works in the nonlinear case: the functional is coercive, bounded below, and has a global minimum.

Let us see what we mean by “does not interact”, at least in a simplified setting. Suppose that f is differentiable on $\mathbb{R}$ and that

$$\sup_{t \in \mathbb{R}} |f'(t)| < \lambda_1.$$

This assumption implies **(h₅)** and in particular implies that the closure of the range of f' is contained in $(-\lambda_1, \lambda_1)$. This is the property that makes the functional coercive (actually, as we have seen, it is enough that the property holds for large t ; one can also check that $\sup_{t \in \mathbb{R}} f'(t) < \lambda_1$ works as well).

The situation changes dramatically, even in the linear case, if we allow that $f'(t)$ lies (for t large) between some eigenvalues of the differential operator. For example, if we take $f(t) = \lambda t$ with $\lambda > \lambda_1$, then the associated functional is no longer coercive, and it is unbounded below. Thus no minimization is possible.

In *some* cases however, certain ideas used in the previous section can be modified to obtain again an existence result. We now describe one of these cases, returning to an assumption that does not require f to be differentiable. We assume that f is defined on $\mathbb{R}$ and

(h₇) There exist an integer $\nu \geq 1$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\lambda_\nu < \alpha \leq \frac{f(s) - f(t)}{s - t} \leq \beta < \lambda_{\nu+1} \quad \forall s, t \in \mathbb{R}.$$

Remark 2.2.1 Clearly **(h₇)** implies that f is globally Lipschitz continuous. Of course if f is differentiable, then **(h₇)** is equivalent to

$$\lambda_\nu < \alpha \leq f'(t) \leq \beta < \lambda_{\nu+1} \quad (2.7)$$

for all $t \in \mathbb{R}$: the closure of the range of f' lies between two eigenvalues.

Remark 2.2.2 We notice for further use that by direct integration we obtain the following growth properties for f and for its primitive F : there exist constants $c \in \mathbb{R}$ such that

- $\alpha t + c \leq f(t) \leq \beta t + c \quad \forall t \geq 0,$
- $\beta t + c \leq f(t) \leq \alpha t + c \quad \forall t \leq 0,$
- $\frac{1}{2}\alpha t^2 + ct \leq F(t) \leq \frac{1}{2}\beta t^2 + ct \quad \forall t \in \mathbb{R}.$

We are going to prove the following result.

Theorem 2.2.3 *Under the assumptions $(\mathbf{h}_1)$, $(\mathbf{h}_2)$ and $(\mathbf{h}_7)$, Problem (2.1), namely*

$$\begin{cases} -\Delta u + q(x)u = f(u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits exactly one solution.

We will prove this result through a series of lemmas, where we always assume $(\mathbf{h}_1)$, $(\mathbf{h}_2)$ and $(\mathbf{h}_7)$. The point is to study the properties of the functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx,$$

which is defined and differentiable on $H_0^1(\Omega)$ in view of the growth conditions on f (Example 1.3.20).

The space $H_0^1(\Omega)$ is endowed with the scalar product (2.2) and the corresponding norm. We denote

$$X_1 = \text{span}\{\varphi_1, \dots, \varphi_\nu\}, \quad \text{and} \quad X_2 = X_1^\perp = \text{cl}\{\text{span}\{\varphi_k \mid k \geq \nu + 1\}\}, \quad (2.8)$$

where φ_k is the eigenfunction associated to λ_k and “cl” denotes closure in $H_0^1(\Omega)$. By definition the subspaces X_1 and X_2 are orthogonal and $H_0^1(\Omega) = X_1 \oplus X_2$.

Remark 2.2.4 In the following, with a slight abuse of notation, we will write any $w \in H_0^1(\Omega)$ indifferently as $w = u + v$ or $w = (u, v)$, with $u \in X_1$ and $v \in X_2$. This simplifies the notation at various stages.

Lemma 2.2.5 *We have*

$$\int_{\Omega} u^2 dx \geq \frac{1}{\lambda_\nu} \|u\|^2 \quad \forall u \in X_1, \quad (2.9)$$

and

$$\int_{\Omega} v^2 dx \leq \frac{1}{\lambda_{\nu+1}} \|v\|^2 \quad \forall v \in X_2. \quad (2.10)$$

Proof With the notation of Remark 1.7.5, if $u \in X_1$ then

$$\int_{\Omega} u^2 dx = \sum_{k=1}^{\nu} \alpha_k^2 = \sum_{k=1}^{\nu} \frac{\beta_k^2}{\lambda_k} \geq \frac{1}{\lambda_\nu} \sum_{k=1}^{\nu} \beta_k^2 = \frac{1}{\lambda_\nu} \|u\|^2,$$

while if $v \in X_2$, then

$$\int_{\Omega} v^2 dx = \sum_{k=\nu+1}^{\infty} \alpha_k^2 = \sum_{k=\nu+1}^{\infty} \frac{\beta_k^2}{\lambda_k} \leq \frac{1}{\lambda_{\nu+1}} \sum_{k=1}^{\nu} \beta_k^2 = \frac{1}{\lambda_{\nu+1}} \|v\|^2. \quad \square$$

This lemma contains the relevant information to deduce the coercivity properties of the functional I . Of course we do not expect coercivity on the subspace X_1 , and indeed on X_1 we have the opposite behavior. The next lemma clarifies this.

We define a functional $J : X_1 \times X_2 \rightarrow \mathbb{R}$ by

$$J(u, v) = I(u + v).$$

Lemma 2.2.6 *For every $v \in X_2$, the functional $J(\cdot, v) : X_1 \rightarrow \mathbb{R}$ is anticoercive (recall that this means that $-J(\cdot, v)$ is coercive). For every $u \in X_1$, the functional $J(u, \cdot) : X_2 \rightarrow \mathbb{R}$ is coercive.*

Proof By the growth properties listed in Remark 2.2.2 we have, for every fixed $v \in X_2$,

$$\begin{aligned} J(u, v) &= I(u + v) = \frac{1}{2} \|u + v\|^2 - \int_{\Omega} F(u + v) dx - \int_{\Omega} h(u + v) dx \\ &\leq \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 - \frac{1}{2} \int_{\Omega} \alpha(u + v)^2 dx \\ &\quad - c \int_{\Omega} (u + v) dx - \int_{\Omega} h(u + v) dx \\ &\leq \frac{1}{2} \|u\|^2 - \frac{\alpha}{2} \int_{\Omega} u^2 dx + c_1 \|u\| + c_2, \end{aligned}$$

where the c_i 's are constants that depend on v, f, h but not on u . Since $u \in X_1$, from Lemma 2.2.5 we obtain

$$J(u, v) \leq \frac{1}{2} \|u\|^2 - \frac{1}{2} \frac{\alpha}{\lambda_v} \|u\|^2 + c_1 \|u\| + c_2 = \frac{1}{2} \left(1 - \frac{\alpha}{\lambda_v} \right) \|u\|^2 + c_1 \|u\| + c_2.$$

As $\alpha > \lambda_v$, this proves the first part.

With the same argument, for every fixed $u \in X_1$ we have

$$\begin{aligned} J(u, v) &\geq \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 - \frac{1}{2} \int_{\Omega} \beta(u + v)^2 dx - c \int_{\Omega} (u + v) dx - \int_{\Omega} h(u + v) dx \\ &\geq \frac{1}{2} \|v\|^2 - \frac{\beta}{2} \int_{\Omega} v^2 dx + c_3 \|v\| + c_4, \end{aligned}$$

and applying Lemma 2.2.5 we conclude that

$$\begin{aligned} J(u, v) &\geq \frac{1}{2} \|v\|^2 - \frac{1}{2} \frac{\beta}{\lambda_{v+1}} \|v\|^2 + c_3 \|v\| + c_4 \\ &\geq \frac{1}{2} \left(1 - \frac{\beta}{\lambda_{v+1}} \right) \|v\|^2 + c_3 \|v\| + c_4, \end{aligned}$$

where the constants do not depend on v . Since $\beta < \lambda_{v+1}$, also the second part is proved. $\square$

The next property is crucial.

Lemma 2.2.7 *For every $u \in X_1$, the functional $J(u, \cdot) : X_2 \rightarrow \mathbb{R}$ is strictly convex and for every $v \in X_2$, the functional $J(\cdot, v) : X_1 \rightarrow \mathbb{R}$ is strictly concave.*

Proof We are going to check the convexity properties through Proposition 1.5.10. Fix $u \in X_1$ and consider the functional $v \mapsto J(u, v) = I(u + v)$. This functional is

differentiable, and, denoting by $\partial_2 J(u, v)$ its differential at $v \in X_2$, we have for all $w \in X_2$,

$$\partial_2 J(u, v)w = I'(u + v)w = (v|w) - \int_{\Omega} f(u + v)w \, dx - \int_{\Omega} h w \, dx.$$

Therefore, for every $v_1, v_2 \in X_2$,

$$\begin{aligned} & [\partial_2 J(u, v_1) - \partial_2 J(u, v_2)](v_1 - v_2) \\ &= (v_1|v_1 - v_2) - \int_{\Omega} f(u + v_1)(v_1 - v_2) \, dx - \int_{\Omega} h(v_1 - v_2) \, dx \\ &\quad - (v_2|v_1 - v_2) + \int_{\Omega} f(u + v_2)(v_1 - v_2) \, dx + \int_{\Omega} h(v_1 - v_2) \, dx \\ &= \|v_1 - v_2\|^2 - \int_{\Omega} (f(u + v_1) - f(u + v_2))(v_1 - v_2) \, dx. \end{aligned}$$

Now the assumption

$$\frac{f(t) - f(s)}{t - s} \leq \beta$$

implies that for all $t, s \in \mathbb{R}$ there results

$$(f(t) - f(s))(t - s) \leq \beta(t - s)^2,$$

so that

$$\begin{aligned} & \int_{\Omega} (f(u + v_1) - f(u + v_2))(v_1 - v_2) \, dx \\ &= \int_{\Omega} (f(u + v_1) - f(u + v_2))((u + v_1) - (u + v_2)) \, dx \\ &\leq \beta \int_{\Omega} ((u + v_1) - (u + v_2))^2 \, dx = \beta \int_{\Omega} (v_1 - v_2)^2 \, dx \leq \frac{\beta}{\lambda_{v+1}} \|v_1 - v_2\|^2, \end{aligned}$$

by Lemma 2.2.5. Thus we obtain

$$[\partial_2 J(u, v_1) - \partial_2 J(u, v_2)](v_1 - v_2) \geq \left(1 - \frac{\beta}{\lambda_{v+1}}\right) \|v_1 - v_2\|^2 > 0 \quad (2.11)$$

for all $v_1 \neq v_2$, because $\beta < \lambda_{v+1}$.

Proposition 1.5.10 applies to show that for every $u \in X_1$ the functional $J(u, \cdot)$ is strictly convex.

The argument to prove the concavity of $J(\cdot, v)$ is essentially the same. Fix $v \in X_2$ and denote by $\partial_1 J(u, v)$ the differential of the functional $u \mapsto J(u, v)$ at u . As above we obtain, for all $z \in X_1$,

$$\partial_1 J(u, v)z = (u|z) - \int_{\Omega} f(u + v)z \, dx - \int_{\Omega} h z \, dx,$$

so that

$$\begin{aligned}
& [\partial_1 J(u_1, v) - \partial_2 J(u_2, v)](u_1 - u_2) \\
&= (u_1 | u_1 - u_2) - \int_{\Omega} f(u_1 + v)(u_1 - u_2) dx - \int_{\Omega} h(u_1 - u_2) dx \\
&\quad - (u_2 | u_1 - u_2) + \int_{\Omega} f(u_2 + v)(u_1 - u_2) dx + \int_{\Omega} h(u_1 - u_2) dx \\
&= \|u_1 - u_2\|^2 - \int_{\Omega} (f(u_1 + v) - f(u_2 + v))(u_1 - u_2) dx.
\end{aligned}$$

Thus, by Lemma 2.2.5,

$$\int_{\Omega} (f(u_1 + v) - f(u_2 + v))(u_1 - u_2) dx \geq \int_{\Omega} \alpha(u_1 - u_2)^2 dx \geq \frac{\alpha}{\lambda_v} \|u_1 - u_2\|^2,$$

and then

$$[\partial_1 J(u_1, v) - \partial_1 J(u_2, v)](u_1 - u_2) \leq \left(1 - \frac{\alpha}{\lambda_v}\right) \|u_1 - u_2\|^2 < 0$$

for all $u_1 \neq u_2$, because $\alpha > \lambda_v$. This shows that the functional $u \mapsto J(u, v)$ is strictly concave. $\square$

The preceding lemmas allow us to minimize J over X_2 .

Lemma 2.2.8 *For every $u \in X_1$ there exists a unique $w = w(u) \in X_2$ such that*

$$J(u, w) = \min_{v \in X_2} J(u, v).$$

Proof The function $v \mapsto J(u, v)$ is continuous, coercive and strictly convex. By Theorems 1.5.6 and 1.5.8, it has a unique global minimum. $\square$

This results makes it possible to define a functional $G : X_1 \rightarrow \mathbb{R}$ by

$$G(u) = \min_{v \in X_2} J(u, v) = J(u, w(u))$$

that we are now going to study.

Lemma 2.2.9 *The functional G is bounded above, namely*

$$\sup_{u \in X_1} G(u) < +\infty.$$

Proof If $\sup_{u \in X_1} G(u) = +\infty$, there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset X_1$ such that $G(u_k) \rightarrow +\infty$ as $k \rightarrow +\infty$. By the growth properties of F , for every $u \in X_1$, we have

$$\begin{aligned}
G(u) &\leq J(u, 0) \leq |J(u, 0)| \leq \frac{1}{2} \|u\|^2 + \int_{\Omega} |F(u)| dx + \int_{\Omega} |h u| dx \\
&\leq c_5 \|u\|^2 + c_6 \|u\|,
\end{aligned}$$

for some positive constants c_5, c_6 . Since $G(u_k) \rightarrow +\infty$, this shows that $\|u_k\| \rightarrow \infty$. But J is antio coercive on X_1 , and then

$$G(u_k) \leq J(u_k, 0) \rightarrow -\infty,$$

contradicting the choice of the sequence $\{u_k\}$. $\square$

Lemma 2.2.10 *The number*

$$s = \sup_{u \in X_1} G(u)$$

is attained in X_1 , namely there exists $u_ \in X_1$ such that $G(u_*) = s$.*

Proof Let $\{u_k\}_{k \in \mathbb{N}} \subset X_1$ be a maximizing sequence for G on X_1 , so that $G(u_k) \rightarrow s$. The argument in the preceding proof shows that $\{u_k\}$ is bounded in X_1 . Since X_1 has finite dimension, there exists $u_* \in X_1$ such that, up to subsequences, $u_k \rightarrow u_*$ in X_1 , and then also in $H_0^1(\Omega)$ (all norms are equivalent on X_1).

Thus, for every fixed $v \in X_2$, we obtain, as $k \rightarrow \infty$,

- $\|u_k + v\|^2 \rightarrow \|u_* + v\|^2$,
- $\int_{\Omega} F(u_k + v) dx \rightarrow \int_{\Omega} F(u_* + v)$,
- $\int_{\Omega} h(u_k + v) dx \rightarrow \int_{\Omega} h(u_* + v) dx$,

and therefore

$$J(u_k, v) \rightarrow J(u_*, v).$$

On the other hand, $J(u_k, v) \geq G(u_k) \rightarrow s$, so that

$$J(u_*, v) \geq s.$$

Since this holds for every $v \in X_2$, we see that

$$G(u_*) = \min_{v \in X_2} J(u_*, v) \geq s = \sup_{u \in X_1} G(u),$$

that is,

$$G(u_*) = s = \max_{u \in X_1} G(u). \quad \square$$

Let $v_* = w(u_*)$ be the unique element in X_2 given by Lemma 2.2.8. Notice that $G(u_*) = J(u_*, v_*)$. The following property is determinant to conclude the proof of Theorem 2.2.3.

Lemma 2.2.11 *The element $(u_*, v_*) \in X_1 \times X_2$ is a “saddle point” for J , in the sense that*

$$J(u, v_*) \leq J(u_*, v_*) \leq J(u_*, v) \quad \forall u \in X_1, \forall v \in X_2.$$

Proof Since $J(u_*, v_*) = G(u_*) = \min_{v \in X_2} J(u_*, v)$, the inequality

$$J(u_*, v_*) \leq J(u_*, v) \quad \forall v \in X_2$$

is trivial, and the proof amounts to establish the other inequality, namely

$$J(u, v_*) \leq J(u_*, v_*) \quad \forall u \in X_1. \quad (2.12)$$

Once more, convexity plays a fundamental role.

For every $t \in (0, 1)$ and for every $u \in X_1$, set

$$w_t = w((1-t)u_* + tu),$$

where as usual $w(\cdot)$ is given by Lemma 2.2.8. Since J is concave in $u \in X_1$, and u_* maximizes G , we have

$$\begin{aligned} G(u_*) &\geq G((1-t)u_* + tu) = J((1-t)u_* + tu, w_t) \\ &\geq (1-t)J(u_*, w_t) + tJ(u, w_t) \\ &\geq (1-t)G(u_*) + tJ(u, w_t), \end{aligned}$$

from which we see that for all $t \in (0, 1)$ and all $u \in X_1$,

$$G(u_*) \geq J(u, w_t). \quad (2.13)$$

For u fixed, the preceding inequality, together with the fact that J is coercive in $v \in X_2$, shows that the set $\{w_t \mid t \in (0, 1)\}$ is bounded in X_2 . Therefore we can take a sequence $\{t_k\}_{k \in \mathbb{N}} \subset (0, 1)$ such that

$$t_k \rightarrow 0 \quad \text{and} \quad w_{t_k} \rightharpoonup \tilde{w} \quad \text{in } X_2.$$

Since the functional $v \mapsto J(u, v)$ is weakly lower semicontinuous (Theorem 1.5.3), we obtain

$$J(u, \tilde{w}) \leq \liminf_k J(u, w_{t_k}) \leq G(u_*) \quad (2.14)$$

by (2.13).

We now show that $\tilde{w} = v_*$; in this case the proof is complete since (2.14) reads

$$J(u, v_*) \leq G(u_*) = J(u_*, v_*), \quad (2.15)$$

and u is arbitrary in X_1 .

By the concavity in $u \in X_1$ and the definition of w , for the same sequence as above we have

$$\begin{aligned} &(1-t_k)J(u_*, w_{t_k}) + t_kJ(u, w_{t_k}) \\ &\leq J((1-t_k)u_* + t_ku, w_{t_k}) \\ &= G((1-t_k)u_* + t_ku) = \inf_{v \in X_2} J((1-t_k)u_* + t_ku, v). \end{aligned}$$

This means that for every $v \in X_2$,

$$(1-t_k)J(u_*, w_{t_k}) + t_kJ(u, w_{t_k}) \leq J((1-t_k)u_* + t_ku, v). \quad (2.16)$$

When $k \rightarrow +\infty$ we have $t_k \rightarrow 0$, so that

$$(1-t_k)u_* + t_ku \rightarrow u_*, \quad w_{t_k} \rightharpoonup \tilde{w},$$

and from (2.16) we obtain, by continuity in u and weak lower semicontinuity in v ,

$$J(u_*, \tilde{w}) \leq J(u_*, v).$$

This holds for every $v \in X_2$, and therefore

$$J(u_*, \tilde{w}) = \inf_{v \in X_2} J(u_*, v) = G(u_*).$$

By Lemma 2.2.8, there exists a unique $w \in X_2$ such that $J(u_*, w) = G(u_*)$, and this element is exactly the one that we called v_* . Summing up, we obtain

$$\tilde{w} = v_*,$$

so that (2.15) is true. $\square$

End of the proof of Theorem 2.2.3 We first show that $u_* + v_*$ is a critical point for I .

For every $u \in X_1$ and every $t \in \mathbb{R}$, let

$$\gamma(t) = J(u_* + tu, v_*) = I(u_* + tu + v_*).$$

The function γ is differentiable and has a maximum point at $t = 0$, by the preceding lemma. Then

$$0 = \gamma'(0) = I'(u_* + v_*)u.$$

In the same way, if for $v \in X_2$ we define

$$\eta(t) = J(u_*, v_* + tv) = I(u_* + v_* + tv),$$

then we see that η is differentiable and has a minimum point at $t = 0$. So,

$$0 = \eta'(0) = I'(u_* + v_*)v.$$

Adding the two equations we obtain

$$I'(u_* + v_*)(u + v) = 0$$

for every $u \in X_1$ and every $v \in X_2$; since $H_0^1(\Omega) = X_1 \oplus X_2$, we conclude that $I'(u_* + v_*) = 0$. The existence of a solution to Problem (2.1) is proved.

We now turn to the uniqueness question.

Assume that u_1, u_2 are two solutions of (2.1), that is,

$$(u_i | \psi) - \int_{\Omega} f(u_i) \psi \, dx - \int_{\Omega} h \psi = 0$$

for $i = 1, 2$ and for all $\psi \in H_0^1(\Omega)$. We define a function $a \in L^\infty(\Omega)$ by

$$a(x) = \begin{cases} \frac{f(u_1(x)) - f(u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x), \\ \alpha & \text{if } u_1(x) = u_2(x). \end{cases}$$

From (h7) we see that $\alpha \leq a(x) \leq \beta$ for a.e. $x \in \Omega$. Subtracting the equation for u_2 from the equation for u_1 and setting $w = u_1 - u_2$, we see that w satisfies

$$(w | \psi) - \int_{\Omega} (f(u_1) - f(u_2)) \psi \, dx = 0$$

for all $\psi \in H_0^1(\Omega)$, that we can also write

$$(w|\psi) - \int_{\Omega} a(x)w\psi \, dx = 0. \quad (2.17)$$

We now write $w = w_1 + w_2$, with $w_i \in X_i$ and we recall from Lemma 2.2.5 that

$$\int_{\Omega} w_1^2 \, dx \geq \frac{1}{\lambda_v} \|w_1\|^2, \quad \int_{\Omega} w_2^2 \, dx \leq \frac{1}{\lambda_{v+1}} \|w_2\|^2.$$

Choosing $\psi = w_1$ and then $\psi = w_2$ in (2.17) we obtain

$$\|w_1\|^2 = \int_{\Omega} a w_1^2 \, dx + \int_{\Omega} a w_1 w_2 \, dx \quad \text{and} \quad \|w_2\|^2 = \int_{\Omega} a w_2^2 \, dx + \int_{\Omega} a w_1 w_2 \, dx,$$

so that

$$\begin{aligned} \|w_1\|^2 &= \int_{\Omega} a w_1^2 \, dx - \int_{\Omega} a w_2^2 \, dx + \|w_2\|^2 \geq \alpha \int_{\Omega} w_1^2 \, dx - \beta \int_{\Omega} w_2^2 \, dx + \|w_2\|^2 \\ &\geq \frac{\alpha}{\lambda_v} \|w_1\|^2 - \frac{\beta}{\lambda_{v+1}} \|w_2\|^2 + \|w_2\|^2, \end{aligned}$$

that can be written

$$\left(1 - \frac{\alpha}{\lambda_v}\right) \|w_1\|^2 \geq \left(1 - \frac{\beta}{\lambda_{v+1}}\right) \|w_2\|^2.$$

Since $\alpha > \lambda_v$ and $\beta < \lambda_{v+1}$, this inequality implies that $\|w_1\| = \|w_2\| = 0$, namely $w = 0$ and thus $u_1 = u_2$. Uniqueness is verified, and the proof of the theorem is complete. $\square$

Remark 2.2.12 The proof of uniqueness uses the equation satisfied by critical points and various inequalities. A more “abstract” proof involves just convexity properties and works as follows. Assume for simplicity that I has a critical point at zero (translating if necessary) and a critical point at $u + v \in X_1 \oplus X_2$. Define $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$\varphi(s, t) = I(su + tv) = J(su, tv).$$

It follows immediately from the properties of J (Lemma 2.2.7) that φ is strictly concave in s and strictly convex in t . Just as easily, φ has a critical point at $(0, 0)$ and another one at $(1, 1)$. This is impossible. Indeed 0 is necessarily a strict global maximum for $s \mapsto \varphi(s, 0)$ and a strict global minimum for $t \mapsto \varphi(0, t)$, while 1 is a strict global maximum for $s \mapsto \varphi(s, 1)$, and a strict global minimum for $t \mapsto \varphi(1, t)$. Then

$$\varphi(0, 0) < \varphi(0, 1) < \varphi(1, 1) < \varphi(1, 0) < \varphi(0, 0),$$

a contradiction. Thus I cannot have more than one critical point.

Remark 2.2.13 One of the interesting aspects of the previous theorem is the *procedure* by which we have found a critical point. Indeed, we have first split the space

orthogonally as $H_0^1(\Omega) = X_1 \oplus X_2$, according to convexity and concavity properties of the functional I ; then, writing the generic element of $H_0^1(\Omega)$ as $u + v$, with $u \in X_1$ and $v \in X_2$, we have found a critical level s for I as

$$s = \max_{u \in X_1} \min_{v \in X_2} I(u + v).$$

This is a first example of a procedure that we will generalize in Chap. 4.

Remark 2.2.14 It is important to notice that many steps of the proof of the theorem work because of convexity (or concavity) properties of the functional I . These properties hold because of the rather strong assumption **(h7)**, that rules the behavior of the nonlinearity f on the entire real line. For example, if f is differentiable and we require that it satisfies (2.7) for $|t|$ large only, then the convexity properties of I fail, and we cannot prove Theorem 2.2.3. This does not mean that we cannot find a solution to the problem, but certainly we cannot repeat the above proof. These cases will be dealt with in later chapters with stronger methods.

Remark 2.2.15 In assumption **(h7)** one cannot allow $\alpha = \lambda_v$ or $\beta = \lambda_{v+1}$. For example, as we have already pointed out, the linear equation $-\Delta u = \lambda_v u + h$ in $H_0^1(\Omega)$ does not admit a solution for every $h \in L^2(\Omega)$.

Remark 2.2.16 As a final remark we point out that if $h = 0$ in Theorem 2.2.3, then the problem admits only the trivial solution when $f(0) = 0$. Indeed in this case $u = 0$ is a critical point of I , and, by uniqueness, it is its *only* critical point.

2.3 Superlinear Problems and Constrained Minimization

Up to now we have only treated problems where the nonlinear term f has an at most linear growth. In case the nonlinearity grows faster than linearly, one speaks of *superlinear* problems. For these types of problems the techniques used so far do not work anymore; for example the functionals associated to these problems are generally unbounded from below and they present a lack of convexity or concavity properties that makes the arguments of the previous sections useless.

Actually for rather general f and h only partial results are known, though a quite rich theory has been developed. We begin to present here some of the simplest cases, in which h is identically zero and f is a power. Further results will be presented in later sections.

We take throughout this section a real number p such that

$$2 < p < 2^* = \frac{2N}{N-2},$$

and we search a function u that satisfies the *superlinear* and *subcritical* problem

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.18)$$

Of course this problem admits the zero solution, which poses an extra difficulty. We will prove the existence of a nontrivial solution by two different methods, which can be extended to cover more general problems. The main result is the following.

Theorem 2.3.1 *Let $p \in (2, 2^*)$ and assume that $(\mathbf{h}_1)$ holds. Then Problem (2.18) admits at least one nonnegative and nontrivial solution.*

From now on we tacitly assume that the hypotheses of Theorem 2.3.1 hold. For the role of $(\mathbf{h}_1)$, see Remark 2.1.1.

The functional whose critical points are the (weak) solutions of (2.18) is $I : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} q(x) u^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx = \frac{1}{2} \|u\|^2 - \frac{1}{p} |u|_p^p,$$

which is differentiable by the results of Example 1.3.20. We notice immediately that I is not bounded below, since for every $u \neq 0$ we have

$$I(tu) = \frac{t^2}{2} \|u\|^2 - \frac{t^p}{p} |u|_p^p \rightarrow -\infty$$

if $t \rightarrow +\infty$, because $p > 2$. Notice also that I is the difference of two strictly convex functionals.

2.3.1 Minimization on Spheres

In the first method that we present the key property is the *homogeneity* of the two terms in I ; indeed the first term is positively homogeneous of degree 2, while the second is positively homogeneous of degree p . This difference of degrees of homogeneity will play a central role at various steps of the proof.

Since the functional I is unbounded below, no minimization is possible on the whole space $H_0^1(\Omega)$. The first step consists in getting rid of this unboundedness by *constraining* the functional on a suitable set where it becomes bounded below. The first choice (a second one will be presented in the next section) is a *sphere* of $L^p(\Omega)$.

If, for every $\beta > 0$, we set

$$\Sigma_{\beta} = \left\{ u \in H_0^1(\Omega) \mid \int_{\Omega} |u|^p dx = \beta \right\},$$

we see that I restricted to Σ_{β} takes the form $I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \beta$, so that it is certainly bounded from below. Now minimizing I on Σ_{β} is equivalent to minimizing just the square of the norm, so we set

$$m_{\beta} = \inf_{u \in \Sigma_{\beta}} \|u\|^2.$$

We are going to show that m_{β} is attained by some function, and that this function gives rise to a solution of Problem (2.18).

Lemma 2.3.2 *For every $\beta > 0$, the level m_β is attained by a nonnegative function, namely there exists $u \in \Sigma_\beta$, $u(x) \geq 0$ a.e. in Ω , such that*

$$\|u\|^2 = m_\beta.$$

Proof Let $\{u_k\}_k \subset \Sigma_\beta$ be a minimizing sequence for $\|u\|^2$. Obviously the sequence $\{|u_k|\}_k$ is still a minimizing sequence in Σ_β (see the properties of H_0^1 in Sect. 1.2.1) and therefore we can assume from the beginning that $u_k(x) \geq 0$ a.e. in Ω and for all k . This minimizing sequence is of course bounded in $H_0^1(\Omega)$, so that, up to subsequences,

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } H_0^1(\Omega), & u_k &\rightarrow u \quad \text{in } L^p(\Omega), \quad \text{and} \\ u_k(x) &\rightarrow u(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

We then obtain immediately, by weak lower semicontinuity of the norm,

$$\|u\|^2 \leq m_\beta,$$

together with

$$\int_{\Omega} |u|^p dx = \beta, \quad \text{and} \quad u(x) \geq 0 \quad \text{a.e. in } \Omega.$$

Thus $u \in \Sigma_\beta$, and this shows that $\|u\|^2 = m_\beta$. Notice also that $u \in \Sigma_\beta$ implies that u does not vanish identically. $\square$

Remark 2.3.3 The preceding proof is very simple. The reader should retain from it that the key assumption is the *subcritical* growth condition $p < 2^*$. For such p 's the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ is *compact*, and this is what allows one to say that $u_k \rightarrow u$ in L^p , and eventually that $u \in \Sigma_\beta$. This is essential to conclude that u is a minimum in Σ_β .

In *critical* problems, namely when $p = 2^*$, the compactness of the embedding fails and the above argument breaks down. Even worse, some problems may have no nontrivial solution. This is the starting point of a line of research, begun with [14], that is still very active today. For some references see [26, 45].

Lemma 2.3.4 *Let u be a minimizer found with the previous lemma. Then u satisfies*

$$\int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} q(x)uv dx = \frac{m_\beta}{\beta} \int_{\Omega} |u|^{p-2}uv dx \quad (2.19)$$

for all $v \in H_0^1(\Omega)$.

Proof Although u minimizes the functional $N(u) = \|u\|^2$ on Σ_β , we cannot conclude that the differential of N vanishes at u , because Σ_β is not a vector space. Concretely, this means that we cannot compare the values of $N(u)$ and of $N(u+v)$, because $u+v$ will not belong to Σ_β , in general. We have to construct small “variations” of u that lie on Σ_β .

To this aim, fix $v \in H_0^1(\Omega)$. For $s \in \mathbb{R}$ small enough, say $s \in (-\varepsilon, \varepsilon)$, the function $u + sv$, is not identically zero. Therefore there exists $t : (-\varepsilon, \varepsilon) \rightarrow (0, +\infty)$ such that

$$\int_{\Omega} |t(s)(u + sv)|^p dx = \beta;$$

precisely,

$$t(s) = \left(\frac{\beta}{\int_{\Omega} |u + sv|^p dx} \right)^{1/p}.$$

Notice that the application $s \mapsto t(s)(u + sv)$ defines a *curve* on Σ_{β} that passes through u when $s = 0$. The function t is differentiable on $(-\varepsilon, \varepsilon)$, and

$$t'(s) = -\beta^{1/p} \left(\int_{\Omega} |u + sv|^p dx \right)^{-\frac{1}{p}-1} \int_{\Omega} |u + sv|^{p-2} (u + sv) v dx.$$

Then we have

$$t(0) = 1 \quad \text{and} \quad t'(0) = -\beta^{-1} \int_{\Omega} |u|^{p-2} uv dx.$$

We define $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ as

$$\gamma(s) = N(t(s)(u + sv)) = \|t(s)(u + sv)\|^2.$$

Since $t(s)(u + sv) \in \Sigma_{\beta}$ for every $s \in (-\varepsilon, \varepsilon)$, the point $s = 0$ is a local minimum for γ . The function γ is differentiable and

$$\gamma'(s) = 2(t(s)(u + sv) | t'(s)(u + sv) + t(s)v),$$

so that

$$0 = \gamma'(0) = 2t(0)t'(0)\|u\|^2 + 2t^2(0)(u|v) = -2\frac{m\beta}{\beta} \int_{\Omega} |u|^{p-2} uv dx + 2(u|v).$$

We have thus shown that

$$(u|v) = \frac{m\beta}{\beta} \int_{\Omega} |u|^{p-2} uv dx,$$

for every $v \in H_0^1(\Omega)$, namely (2.19). □

Remark 2.3.5 The reader with some knowledge of differential geometry will have noticed that the preceding proof amounts to a direct check of the fact that the differential of N at u vanishes on the tangent space to Σ_{β} at u , as is always the case for minimizers of differentiable functionals defined on differentiable manifolds. Indeed the previous result can be obtained “abstractly” through the Lagrange Theorem on constrained extrema: setting

$$G(u) = \int_{\Omega} |u|^p dx,$$

we see that our problem consists in minimizing N constrained on $G^{-1}(\beta) = \Sigma_\beta$. Now G is of class C^1 and for $u \in G^{-1}(\beta)$, we have that

$$G'(u)u = pG(u) = p\beta \neq 0.$$

Therefore, by the Implicit Function Theorem, $G^{-1}(\beta)$ is a C^1 manifold. The Lagrange Theorem then says that if u minimizes N on $G^{-1}(\beta)$, there exists $\lambda \in \mathbb{R}$ (the *Lagrange multiplier*) such that

$$N'(u) = \lambda G'(u).$$

This equation is precisely (2.19), with $\lambda = m_\beta/\beta$. We do not carry out a more general theory of constrained critical points; the interested reader can consult [2, 17].

End of the proof of Theorem 2.3.1 The last step consists in getting rid of the Lagrange multiplier m_β/β , and this is where homogeneity plays again a fundamental role.

Let u be the minimum of N over Σ_β . Set $u = cw$, with $c \in \mathbb{R}$ to be determined. By the previous lemma, w satisfies

$$c(w|v) = \frac{m_\beta}{\beta} c^{p-1} \int_{\Omega} |w|^{p-2} wv \, dx$$

for all $v \in H_0^1(\Omega)$. Choosing $c = (\beta/m_\beta)^{\frac{1}{p-2}}$, we see that w (is nonnegative and) satisfies

$$(w|v) = \int_{\Omega} |w|^{p-2} wv \, dx$$

for all $v \in H_0^1(\Omega)$, namely is a weak (nontrivial) solution of (2.18). $\square$

Remark 2.3.6 We have obtained a solution by minimizing N on Σ_β . It is tempting to constrain I or N on Σ_γ with $\gamma \neq \beta$ and repeat the argument. However this does not produce a new solution. Indeed it is very easy to see that

$$\frac{m_\beta}{\beta^{2/p}} = \frac{m_\gamma}{\gamma^{2/p}}$$

and that u is a minimum of I on Σ_β if and only if $v = (\gamma/\beta)^{1/p}u$ is a minimum of I on Σ_γ . These two functions give rise to the *same* solution of (2.18). For this reason one normally chooses $\beta = 1$.

2.3.2 Minimization on the Nehari Manifold

We present a second approach to the search of solutions to (2.18), still based on constrained minimization. This approach is slightly more complicated than minimization on spheres, but has the advantage that it does not require the nonlinearity to be homogeneous, and can thus be applied to a wider class of problems (under

convenient assumptions). To illustrate it we concentrate again on Problem (2.18), and we recall that the functional associated to it is

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} q(x) u^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx = \frac{1}{2} \|u\|^2 - \frac{1}{p} |u|_p^p,$$

defined and differentiable on $H_0^1(\Omega)$. Recall that I is unbounded below; once again we restrict I to a suitable set in order to get rid of this problem. In the previous section we have constrained I on a sphere, now we use the set

$$\begin{aligned} \mathcal{N} &= \{u \in H_0^1(\Omega) \mid u \neq 0, I'(u)u = 0\} \\ &= \left\{ u \in H_0^1(\Omega) \mid u \neq 0, \|u\|^2 = \int_{\Omega} |u|^p dx \right\}. \end{aligned} \quad (2.20)$$

This set is called the *Nehari manifold*, and indeed it can be proved, under certain assumptions, that it is a differential manifold diffeomorphic to the unit sphere of $H_0^1(\Omega)$, see [2] or [48]. We do not prove nor use these properties, but we confine ourselves to an “elementary” approach.

Remark 2.3.7 By definition, the Nehari manifold contains all the nontrivial critical points of I .

Notice that on $\mathcal{N}$ the functional I reads

$$I(u) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 = \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} |u|^p dx.$$

This shows at once that I is coercive on $\mathcal{N}$, in the sense that if $\{u_k\}_k \subset \mathcal{N}$ satisfies $\|u_k\| \rightarrow \infty$, then $I(u_k) \rightarrow \infty$.

We define

$$m = \inf_{u \in \mathcal{N}} I(u),$$

and we show, through a series of lemmas, that m is attained by some $u \in \mathcal{N}$ which is a critical point of I considered on the whole space $H_0^1(\Omega)$, and therefore a solution to (2.18).

We begin with some basic properties of $\mathcal{N}$ and I .

Lemma 2.3.8 *The Nehari manifold is not empty.*

Proof For every not identically zero $u \in H_0^1$, one sees immediately that $tu \in \mathcal{N}$ for some $t > 0$. Indeed, $tu \in \mathcal{N}$ is equivalent to

$$\|tu\|^2 = \int_{\Omega} |tu|^p dx,$$

which is solved by

$$t = \left(\frac{\|u\|^2}{\int_{\Omega} |u|^p dx} \right)^{\frac{1}{p-2}} > 0. \quad \square$$

Remark 2.3.9 Even under more general assumptions on the nonlinearity (now we are considering only the model case $s \rightarrow |s|^{p-2}s$) one can prove that for every $u \neq 0$, there exists a *unique* $t = t(u) > 0$ such that $t(u)u \in \mathcal{N}$. Thus one can define a map ψ from the unit sphere of $H_0^1(\Omega)$ to $\mathcal{N}$ as $\psi(u) = t(u)u$. In many concrete cases this map is the diffeomorphism between the unit sphere and the Nehari manifold that we mentioned above, see again [2] or [48].

Lemma 2.3.10 *We have*

$$m = \inf_{u \in \mathcal{N}} I(u) > 0.$$

Proof If $u \in \mathcal{N}$, by the Sobolev inequalities we have

$$\|u\|^2 = \int_{\Omega} |u|^p dx \leq C \|u\|^p,$$

for some $C > 0$. Since $\|u\| \neq 0$ and $p > 2$ we obtain

$$\|u\| \geq \left(\frac{1}{C}\right)^{\frac{1}{p-2}}, \quad (2.21)$$

for every $u \in \mathcal{N}$, so that

$$m = \inf_{u \in \mathcal{N}} I(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \inf_{u \in \mathcal{N}} \|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{1}{C}\right)^{\frac{1}{p-2}} > 0. \quad \square$$

Lemma 2.3.11 *The level m is attained by a nonnegative function, namely there exists $u \in \mathcal{N}$, $u(x) \geq 0$ a.e. in Ω , such that*

$$I(u) = m.$$

Proof Let $\{u_k\}_k \subset \mathcal{N}$ be a minimizing sequence for I , namely such that $I(u_k) \rightarrow m$. Clearly $|u_k| \in \mathcal{N}$ and $I(|u_k|) = I(u_k)$, so that $\{|u_k|\}_k$ is another minimizing sequence; for this reason we assume straight away that $u_k(x) \geq 0$ a.e. in Ω for all k . We have already observed that I is coercive on $\mathcal{N}$; this implies that the sequence $\{u_k\}_k$ is bounded in $H_0^1(\Omega)$, and as usual this means that, up to subsequences,

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } H_0^1(\Omega), & u_k &\rightarrow u \quad \text{in } L^p(\Omega), \quad \text{and} \\ u_k(x) &\rightarrow u(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Then we have $u \geq 0$ a.e. and, by weak lower semicontinuity,

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{p} |u|_p^p \leq \liminf_k \left(\frac{1}{2} \|u_k\|^2 - \frac{1}{p} |u_k|_p^p \right) \\ &= \liminf_k I(u_k) = m. \end{aligned} \quad (2.22)$$

Since $u_k \in \mathcal{N}$, we have $\|u_k\|^2 = \int_{\Omega} |u_k|^p dx$. By (2.21) it cannot be $\|u_k\| \rightarrow 0$, and therefore $\int_{\Omega} |u_k|^p dx$ cannot tend to zero; thus, by strong convergence, $\int_{\Omega} |u|^p dx \neq 0$, which shows that $u \neq 0$. Passing to the limit we obtain

$$\|u\|^2 \leq \int_{\Omega} |u|^p dx. \quad (2.23)$$

If $\|u\|^2 = \int_{\Omega} |u|^p dx$, then $u \in \mathcal{N}$ and (2.22) shows that u is the required minimizer. Since (2.23) holds, we only have to treat the case where

$$\|u\|^2 < \int_{\Omega} |u|^p dx. \quad (2.24)$$

We now show that if this happens, we reach a contradiction. Indeed, take $t > 0$ such that $tu \in \mathcal{N}$, namely

$$t = \left(\frac{\|u\|^2}{\int_{\Omega} |u|^p dx} \right)^{\frac{1}{p-2}}.$$

Since we are assuming (2.24), we deduce that $0 < t < 1$. But $tu \in \mathcal{N}$, so that

$$\begin{aligned} 0 < m &\leq I(tu) = \left(\frac{1}{2} - \frac{1}{p} \right) \|tu\|^2 = t^2 \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 \\ &\leq t^2 \liminf_k \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|^2 = t^2 \liminf_k I(u_k) = t^2 m < m. \end{aligned}$$

This is impossible, and the proof is complete. $\square$

Remark 2.3.12 Once again, the key property is the compactness of the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, that has been used several times in the preceding proofs.

End of the proof of Theorem 2.3.1 Let $u \in \mathcal{N}$ be a minimizer for I found in the previous lemma. We show that $I'(u)v = 0$ for all $v \in H_0^1(\Omega)$, so that u is the required solution.

Notice that as in Lemma 2.3.4, we cannot conclude this directly because we have minimized I with the constraint $u \in \mathcal{N}$.

Take any $v \in H_0^1(\Omega)$. For every s in some small interval $(-\varepsilon, \varepsilon)$ certainly the function $u + sv$ does not vanish identically. Let $t(s) > 0$ be a function such that $t(s)(u + sv) \in \mathcal{N}$, namely

$$t(s) = \left(\frac{\|u + sv\|^2}{\int_{\Omega} |u + sv|^p dx} \right)^{\frac{1}{p-2}}.$$

The function $t(s)$ is a composition of differentiable functions, so it is differentiable; the precise expression of t' does not matter here. Notice also that $t(0) = 1$.

The map $s \mapsto t(s)(u + sv)$ defines a curve on $\mathcal{N}$ along which we evaluate I . Thus we define $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ as

$$\gamma(s) = I(t(s)(u + sv)).$$

By construction, $s = 0$ is a minimum point for γ . Therefore

$$0 = \gamma'(0) = I'(t(0)u)[t'(0)u + t(0)v] = t'(0)I'(u)u + I'(u)v = I'(u)v.$$

For the last equality we have used the fact that $I'(u)u = 0$ because $u \in \mathcal{N}$. We have obtained $I'(u)v = 0$ for all $v \in H_0^1(\Omega)$, proving that u is a solution of (2.18). $\square$

Remark 2.3.13 The last part of the proof shows that a minimum of I constrained on the Nehari manifold $\mathcal{N}$ is actually a *free* critical point of I , on the whole space $H_0^1(\Omega)$. This remarkable fact is expressed by saying that the Nehari manifold is a *natural constraint* for I .

2.4 A Perturbed Problem

As we have anticipated the method of minimization on the Nehari manifold can be extended to cover more general problems than that of the preceding section.

We begin with an existence result for a *perturbed* problem, in the sense that we consider a power nonlinearity *plus* a fixed function $h \in L^2(\Omega)$. The result contained in this section is quite delicate, and can be omitted upon first reading.

We are going to prove that under suitable assumptions, and particularly if h is small, the problem admits *two* solutions. Let us make this more precise.

We consider, for $h \in L^2(\Omega)$ and $p \in (2, 2^*)$, the Dirichlet problem

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.25)$$

Our aim is to show that if $|h|_2$ is sufficiently small, then (2.25) admits at least two non-trivial solutions. This is not really surprising since the unperturbed problem ($h \equiv 0$) also admits two solutions: the one found with Theorem 2.3.1 and the trivial solution $u \equiv 0$. If h does not vanish identically, the trivial solution is replaced by a “true” nonzero solution. Thus one can think that for h small the two solutions are “perturbations” of the two solutions already present in the autonomous case.

We add that the solution corresponding in this scheme to the trivial solution of the unperturbed case can be found very easily, while most of the work must be devoted to the search of the analogue of the solution found in Theorem 2.3.1 by minimization on the Nehari manifold.

The functional associated to (2.25) is

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p} \int_{\Omega} |u|^p dx - \int_{\Omega} hu dx = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p - \int_{\Omega} hu dx,$$

which is of course differentiable on $H_0^1(\Omega)$. The main result is the following.

Theorem 2.4.1 *Let $p \in (2, 2^*)$ and assume that $(\mathbf{h}_1)$ holds. Then there exists $\varepsilon^* > 0$ such that for every $h \in L^2(\Omega)$, with $|h|_2 \leq \varepsilon^*$, Problem (2.25) admits at least two solutions.*

We will prove this result going through a series of lemmas, as usual, and we start with the argument leading to the analogue of the solution found in Theorem 2.3.1, which is the hard part.

We introduce some notation. We denote by m the same constant as in the previous section, namely

$$m = \left(\frac{1}{2} - \frac{1}{p} \right) \inf_{u \in \mathcal{N}} \|u\|^2,$$

where $\mathcal{N}$ is the Nehari manifold defined in (2.20). Notice that $\mathcal{N}$ is *not* the Nehari manifold associated to J , but the “unperturbed” one, relative to the functional I of the previous section.

Next, we denote by S_p the best constant for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, that is,

$$S_p = \inf\{C > 0 \mid |u|_p \leq C\|u\| \ \forall u \in H_0^1(\Omega)\}.$$

Lastly, we define the positive numbers

$$d_1 = \left(\frac{1}{(p-1)S_p^p} \right)^{\frac{1}{p-2}}, \quad d_2 = \left(\frac{1}{2} d_1^2 \right)^{1/p}, \quad d_3 = \frac{1}{2} \min \left\{ d_1, \frac{d_2}{S_p} \right\}.$$

Notice, for further reference, that $d_3 < d_1$.

Lemma 2.4.2 *There exists $\varepsilon_1 > 0$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leq \varepsilon_1$ and for every $u \in H_0^1(\Omega)$, if*

$$\|u\|^2 = \int_{\Omega} |u|^p dx + \int_{\Omega} hu dx = |u|_p^p + \int_{\Omega} hu dx, \quad (2.26)$$

then either

$$\|u\| < d_3 \quad \text{or} \quad \|u\| > d_1. \quad (2.27)$$

If the second case occurs, we have also

$$|u|_p \geq d_2. \quad (2.28)$$

Remark 2.4.3 Interpretation. Condition (2.26) expresses the fact that u belongs to the Nehari manifold relative to J (we will introduce it later). Then the meaning of the lemma is that, for $|h|_2$ small, if u is in the Nehari manifold, then either u is small ($\|u\| < d_3$) or u is large ($\|u\| > d_1$). The region $\|u\| \in [d_3, d_1]$ is forbidden.

Proof We begin by showing that if $|h|_2$ is small, one of the two inequalities in (2.27) must hold. Assuming (2.26), we obtain

$$\|u\|^2 \leq S_p^p \|u\|^p + |h|_2 |u|_2 \leq S_p^p \|u\|^p + c |h|_2 \|u\|,$$

where $c = (\frac{1}{\lambda_1})^{1/2}$. Then, if $u \neq 0$,

$$\|u\| - S_p^p \|u\|^{p-1} - c |h|_2 \leq 0. \quad (2.29)$$

Consider now the function $\gamma : [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\gamma(t) = t - S_p^p t^{p-1} - c|h|_2.$$

Since

$$\gamma'(t) = 1 - (p-1)S_p^p t^{p-2},$$

the function γ has a unique global maximum point located at $t = d_1$. Moreover γ is strictly increasing on $(0, d_1)$, strictly decreasing on $(d_1, +\infty)$ and it satisfies $\gamma(0) < 0$ and $\lim_{t \rightarrow +\infty} \gamma(t) = -\infty$. With a simple computation one finds that

$$\gamma(d_1) = \frac{1}{S_p^{\frac{p}{p-2}}} \left(\frac{1}{p-1} \right)^{\frac{1}{p-2}} \frac{p-2}{p-1} - c|h|_2 =: \alpha_1 - c|h|_2.$$

Since $p > 2$, there results $\alpha_1 > 0$, and if we take $|h|_2 \leq \frac{\alpha_1}{2c}$, then

$$\gamma(d_1) \geq \frac{\alpha_1}{2} > 0.$$

This argument shows that if we assume

$$|h|_2 \leq \frac{\alpha_1}{2c},$$

then the function γ has exactly two zeros t_1, t_2 such that $t_1 < d_1 < t_2$, and $\gamma(t) > 0$ in (t_1, t_2) , while $\gamma(t) < 0$ in $[0, t_1) \cup (t_2, +\infty)$.

We notice that t_1 satisfies

$$c|h|_2 = t_1 - S_p^p t_1^{p-1} = t_1(1 - S_p^p t_1^{p-2}).$$

Since $t_1 < d_1$, we deduce from this and the definition of d_1 that

$$c|h|_2 \geq t_1(1 - S_p^p d_1^{p-2}) = t_1 \left(1 - \frac{1}{p-1} \right) = t_1 \frac{p-2}{p-1},$$

so that

$$t_1 \leq \frac{p-1}{p-2} c|h|_2.$$

But then, if we take

$$|h|_2 < \frac{p-2}{p-1} \frac{d_3}{c},$$

we get $t_1 < d_3$.

Summing up, if we choose

$$|h|_2 < \min \left\{ \frac{p-2}{p-1} \frac{d_3}{c}, \frac{\alpha_1}{2c} \right\},$$

we obtain that $\gamma(t) \leq 0$ implies $t < d_3$ or $t > d_1$.

Therefore, with this choice of $|h|_2$, the inequality (2.29) implies (2.27) and thus also (2.26) implies (2.27). The first part is proved.

We still have to check (2.28). If (2.26) holds and $\|u\| > d_1$, we can write

$$|u|_p^p = \|u\|^2 - \int_{\Omega} hu \, dx \geq d_1^2 - |h|_2 |u|_2 \geq d_1^2 - d|h|_2 |u|_p,$$

where $d = |\Omega|^{\frac{p-2}{2p}}$. We obtain

$$|u|_p^p + d|h|_2 |u|_p - d_1^2 \geq 0. \quad (2.30)$$

Consider the function $\eta : [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\eta(t) = t^p + d|h|_2 t - d_1^2.$$

We have

$$\eta(d_2) = \frac{1}{2} d_1^2 + d|h|_2 d_2 - d_1^2 = d|h|_2 d_2 - \frac{1}{2} d_1^2,$$

so that if

$$|h|_2 < \frac{d_1^2}{2dd_2},$$

there results $\eta(d_2) < 0$. Since $\eta'(t) > 0$ for all $t > 0$, we deduce that $\eta(t) < 0$ for $t \in [0, d_2]$, so that the inequality (2.30) implies $|u|_p \geq d_2$.

The proof is then complete, with the choice

$$\varepsilon_1 = \min \left\{ \frac{p-2}{p-1} \frac{d_3}{c}, \frac{\alpha_1}{2c}, \frac{d_1^2}{2dd_2} \right\}. \quad \square$$

From now on we always assume $|h|_2 < \varepsilon_1$; further restrictions will be imposed later.

We define the set

$$\begin{aligned} \mathcal{N}_h &= \{u \in H_0^1(\Omega) \mid J'(u)u = 0, \|u\| > d_1\} \\ &= \left\{ u \in H_0^1(\Omega) \mid \|u\|^2 = \int_{\Omega} |u|^p \, dx + \int_{\Omega} hu \, dx, \|u\| > d_1 \right\} \end{aligned}$$

and the value

$$m_h = \inf_{u \in \mathcal{N}_h} J(u).$$

Notice that $\mathcal{N}_h$ is not the complete Nehari manifold associated to J , but only a subset of it, containing “large” functions ($\|u\| > d_1$). On $\mathcal{N}_h$, the functional J takes the form

$$J(u) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 - \left(1 - \frac{1}{p} \right) \int_{\Omega} hu \, dx.$$

First we investigate under which conditions $\mathcal{N}_h$ is not empty.

Lemma 2.4.4 *There exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that for every $h \in L^2(\Omega)$ with $|h|_2 \leq \varepsilon_2$, there results $\mathcal{N}_h \neq \emptyset$.*

Proof Let $u \in H_0^1(\Omega) \setminus \{0\}$. We study the behavior of the function

$$\begin{aligned} t \mapsto J'(tu)tu &= t^2 \|u\|^2 - t^p \int_{\Omega} |u|^p dx - t \int_{\Omega} hu dx \\ &= t \left[t \|u\|^2 - t^{p-1} |u|_p^p - \int_{\Omega} hu dx \right] \end{aligned}$$

for $t > 0$ by analyzing the function

$$\gamma(t) = t \|u\|^2 - t^{p-1} |u|_p^p - \int_{\Omega} hu dx.$$

Since

$$\gamma'(t) = \|u\|^2 - (p-1)t^{p-2} |u|_p^p,$$

the function γ has a global maximum at

$$\tilde{t} = \left(\frac{\|u\|^2}{(p-1)|u|_p^p} \right)^{\frac{1}{p-2}}.$$

With an easy computation we see that

$$\gamma(\tilde{t}) = \frac{\|u\|^{2\frac{p-1}{p-2}}}{|u|_p^{\frac{p}{p-2}}} \alpha - \int_{\Omega} hu dx,$$

where

$$\alpha = \frac{1}{(p-1)^{\frac{1}{p-2}}} \frac{p-2}{p-1} > 0.$$

Then we obtain

$$\begin{aligned} \gamma(\tilde{t}) &\geq \frac{\|u\|^{2\frac{p-1}{p-2}}}{\|u\|^{\frac{p}{p-2}} S_p^{\frac{p}{p-2}}} \alpha - \int_{\Omega} hu dx = \|u\| \frac{1}{S_p^{\frac{p}{p-2}}} \alpha - |h|_2 |u|_2 \\ &\geq \|u\| \frac{1}{S_p^{\frac{p}{p-2}}} \alpha - c \|u\| |h|_2 = \|u\| \left(\frac{\alpha}{S_p^{\frac{p}{p-2}}} - c |h|_2 \right). \end{aligned}$$

If

$$|h|_2 \leq \frac{\alpha}{2c S_p^{\frac{p}{p-2}}},$$

we have $\gamma(\tilde{t}) > 0$. For such values of $|h|_2$, the function γ has then the following properties: $\gamma(t)$ is strictly increasing in $(0, \tilde{t})$, strictly decreasing in $(\tilde{t}, +\infty)$ and it satisfies $\gamma(\tilde{t}) > 0$ and $\lim_{t \rightarrow +\infty} \gamma(t) = -\infty$.

This shows that γ has at least one zero $t_1 \in (\tilde{t}, +\infty)$, which implies that the function $v = t_1 u$ satisfies (2.26). Moreover,

$$\|v\| = t_1 \|u\| > \tilde{t} \|u\| = \left(\frac{\|u\|^2}{(p-1)|u|_p^p} \right)^{\frac{1}{p-2}} \|u\| \geq \frac{1}{(p-1)^{\frac{1}{p-2}} S_p^{\frac{p}{p-2}}} = d_1,$$

and this shows that $v \in \mathcal{N}_h$. The proof is then complete provided we choose

$$\varepsilon_2 = \min \left\{ \varepsilon_1, \frac{\alpha}{2cS_p^{\frac{p}{p-2}}} \right\}. \quad \square$$

We now prove that for $|h|_2$ small, the levels m_h are uniformly bounded from above. A uniform bound from below will be obtained in Lemma 2.4.7.

Lemma 2.4.5 *Let $\varepsilon_3 = \min\{1, \varepsilon_2\}$. There exists $c_2 > 0$ such that for every $|h|_2 < \varepsilon_3$ there results $m_h \leq c_2$.*

Proof We denote by u_0 and m_0 , respectively, the solution and the level of the solution of the unperturbed problem ($h \equiv 0$) solved in the preceding section, that is,

$$u_0 \in \mathcal{N}, \quad I(u_0) = \min_{u \in \mathcal{N}} I(u) = m_0.$$

From Lemma 2.4.4 we know that if $|h|_2 < \varepsilon_3$, there exists $t_h > 0$ such that $t_h u_0 \in \mathcal{N}_h$. Since $u_0 \in \mathcal{N}$, it satisfies $\|u_0\|^2 = |u_0|_p^p$, so that the condition $t_h u_0 \in \mathcal{N}_h$ is equivalent to

$$(t_h^2 - t_h^p) \|u_0\|^2 = t_h \int_{\Omega} h u_0 dx,$$

namely to

$$(t_h - t_h^{p-1}) \|u_0\|^2 = \int_{\Omega} h u_0 dx.$$

This last condition implies

$$(t_h - t_h^{p-1}) \|u_0\|^2 \geq -c|h|_2 \|u_0\|,$$

i.e.

$$t_h - t_h^{p-1} \geq -\frac{c|h|_2}{\|u_0\|}.$$

If $|h|_2 < \varepsilon_3 \leq 1$ we obtain

$$t_h - t_h^{p-1} \geq -\frac{c}{\|u_0\|}. \quad (2.31)$$

Since the function $t \mapsto t - t^{p-1}$ tends to $-\infty$ as $t \rightarrow +\infty$, the inequality (2.31) implies the existence of $c_3 > 0$, independent of h , such that $t_h \leq c_3$. It is now simple to conclude. Since $t_h u_0 \in \mathcal{N}_h$ we have

$$\begin{aligned} m_h &\leq J(t_h u_0) = \left(\frac{1}{2} - \frac{1}{p} \right) \|t_h u_0\|^2 - \left(1 - \frac{1}{p} \right) \int_{\Omega} h t_h u_0 dx \\ &\leq \left(\frac{1}{2} - \frac{1}{p} \right) c_3^2 \|u_0\|^2 + \left(1 - \frac{1}{p} \right) c_3 c |h|_2 \|u_0\| \\ &\leq \left(\frac{1}{2} - \frac{1}{p} \right) c_3^2 \|u_0\|^2 + \left(1 - \frac{1}{p} \right) c_3 c \|u_0\|. \end{aligned}$$

The last term of this inequality is a positive number that does not depend on h . We call it c_2 and the proof is complete. $\square$

We go on with a uniform bound on minimizing sequences.

Lemma 2.4.6 *There exists $c_4 > 0$ independent of h , such that if $|h|_2 < \varepsilon_3$, then there exists a minimizing sequence $\{u_k\}_k$ for m_h such that $\|u_k\| \leq c_4$ for all k , and hence also $|u_k|_p \leq S_p c_4$ for all k .*

Proof Let $|h|_2 < \varepsilon_3 \leq 1$ and let $\{v_k\}_k$ be a minimizing sequence for m_h , namely

$$v_k \in \mathcal{N}_h \quad \text{and} \quad J(v_k) \rightarrow m_h.$$

Since $m_h \leq c_2$, there exists $\tilde{k}$ such that for every $k \geq \tilde{k}$ there results $J(v_k) \leq 2c_2$, and this implies

$$\begin{aligned} 2c_2 &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|v_k\|^2 - \left(1 - \frac{1}{p}\right) \int_{\Omega} h v_k dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|v_k\|^2 - \left(1 - \frac{1}{p}\right) c |h|_2 \|v_k\| \\ &\geq a_1 \|v_k\|^2 - a_2 \|v_k\|, \end{aligned}$$

where $a_1 = (\frac{1}{2} - \frac{1}{p})$ and $a_2 = (1 - \frac{1}{p})c$. From this inequality one easily sees that

$$\|v_k\| \leq \frac{a_2 + \sqrt{a_2^2 + 8a_1 c_2}}{2a_1}.$$

Then it is enough to set $c_4 = \frac{a_2 + \sqrt{a_2^2 + 8a_1 c_2}}{2a_1}$ and $u_k = v_{\tilde{k}+k}$ to complete the proof. $\square$

The next result completes the estimate of Lemma 2.4.5.

Lemma 2.4.7 *There exists $\varepsilon_4 \leq \varepsilon_3$ such that if $|h|_2 < \varepsilon_4$, then $m_h \geq \frac{1}{2}m_0 > 0$.*

Proof In this proof we denote by J_h the functional J , to stress its dependence on h . We also recall that I is the functional associated to the “unperturbed” problem dealt with in the previous section and that $\mathcal{N}$ is the corresponding Nehari manifold.

For every h with $|h|_2 < \varepsilon_3$, we consider the minimizing sequence $\{u_k\}_k$ obtained in Lemma 2.4.6. Let t_k be such that $t_k u_k \in \mathcal{N}$ (notice that both u_k and t_k depend on h , but we do not denote it explicitly to have lighter notation).

As we know from the preceding section,

$$t_k = \left(\frac{\|u_k\|^2}{|u_k|_p^p} \right)^{\frac{1}{p-2}}.$$

On the other hand $u_k \in \mathcal{N}_h$, so that $\|u_k\|^2 = |u_k|_p^p + \int_{\Omega} h u_k dx$, and hence

$$t_k = \left(1 + \frac{\int_{\Omega} h u_k dx}{|u_k|_p^p} \right)^{\frac{1}{p-2}}. \quad (2.32)$$

Then we can write

$$\begin{aligned}
 m_0 &\leq I(t_k u_k) = \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2 \|u_k\|^2 \\
 &= \left(\frac{1}{2} - \frac{1}{p}\right) t_k^2 \|u_k\|^2 - \left(1 - \frac{1}{p}\right) t_k^2 \int_{\Omega} h u_k \, dx + \left(1 - \frac{1}{p}\right) t_k^2 \int_{\Omega} h u_k \, dx \\
 &= t_k^2 J_h(u_k) + \left(1 - \frac{1}{p}\right) t_k^2 \int_{\Omega} h u_k \, dx.
 \end{aligned} \tag{2.33}$$

We want to obtain a uniform bound from below on $J_h(u_k)$. First of all we estimate t_k from above. To this aim, by the estimates obtained in Lemmas 2.4.2 and 2.4.6 we have

$$\frac{|\int_{\Omega} h u_k \, dx|}{|u_k|_p^p} \leq |h|_2 \frac{c \|u_k\|}{|u_k|_p^p} \leq |h|_2 \frac{c c_4}{d_2^p},$$

so that if

$$|h|_2 \leq \frac{d_2^p}{c c_4} \left[\left(\frac{4}{3} \right)^{\frac{p-2}{2}} - 1 \right],$$

we obtain

$$\frac{|\int_{\Omega} h u_k \, dx|}{|u_k|_p^p} \leq \left(\frac{4}{3} \right)^{\frac{p-2}{2}} - 1$$

and hence, from (2.32),

$$t_k \leq \left(\frac{4}{3} \right)^{1/2},$$

which is the desired estimate on t_k . Next we analyze the last term in (2.33). We see immediately that

$$\left| \left(1 - \frac{1}{p} \right) t_k^2 \int_{\Omega} h u_k \, dx \right| \leq \left(1 - \frac{1}{p} \right) t_k^2 |h|_2 c \|u_k\| \leq \frac{4}{3} |h|_2 \left(1 - \frac{1}{p} \right) c c_4.$$

We now choose

$$|h|_2 < \frac{m_0}{4(1 - \frac{1}{p}) c c_4},$$

so that we obtain

$$\left| \left(1 - \frac{1}{p} \right) t_k^2 \int_{\Omega} h u_k \, dx \right| \leq \frac{m_0}{3}.$$

From this and from (2.33), we deduce that

$$t_k^2 J_h(u_k) \geq \frac{2}{3} m_0,$$

which implies first of all that $J_h(u_k) > 0$. Since we also have $t_k^2 \leq \frac{4}{3}$, we finally get to

$$\frac{2}{3} m_0 \leq t_k^2 J_h(u_k) \leq \frac{4}{3} J_h(u_k),$$

namely $\frac{1}{2}m_0 \leq J_h(u_k)$. Passing to the limit as $k \rightarrow \infty$, we see that

$$\frac{1}{2}m_0 \leq m_h.$$

All this holds with the bounds we described on $|h|_2$. The lemma is thus proved with

$$\varepsilon_4 = \min \left\{ \varepsilon_3, \frac{d_2^p}{cc_4} \left[\left(\frac{4}{3} \right)^{\frac{p-2}{2}} - 1 \right], \frac{m_0}{4(1 - \frac{1}{p})cc_4} \right\}. \quad \square$$

We have now everything we need to conclude the argument. First of all we show that the minimum on $\mathcal{N}_h$ is attained.

Lemma 2.4.8 *There exists $\varepsilon_5 \leq \varepsilon_4$ such that for every $|h|_2 < \varepsilon_5$ the infimum m_h is attained by some $u \in \mathcal{N}_h$.*

Proof To begin with, let $|h|_2 < \varepsilon_4$ and let $\{u_k\}$ be a minimizing sequence for m_h constructed as in Lemma 2.4.6. Being bounded, we can assume that, up to subsequences, there exists $u \in H_0^1$ such that

$$u_k \rightharpoonup u \quad \text{in } H_0^1(\Omega), \quad u_k \rightarrow u \quad \text{in } L^p(\Omega) \text{ and in } L^2(\Omega).$$

It is then immediate to deduce that

$$J(u) \leq \liminf_k J(u_k) = m_h, \quad (2.34)$$

and that

$$\|u\|^2 \leq |u|_p^p + \int_{\Omega} hu \, dx. \quad (2.35)$$

Let us suppose first of all that in (2.35) equality holds, that is,

$$\|u\|^2 = |u|_p^p + \int_{\Omega} hu \, dx. \quad (2.36)$$

In Lemma 2.4.2 we have seen that if (2.36) holds, then either $\|u\| > d_1$ or $\|u\| < d_3$. In the first case, $u \in \mathcal{N}_h$, and then (2.34) implies that u is the minimum we are looking for. If, on the contrary, $\|u\| < d_3$, then recalling the definition of d_3 we obtain

$$|u|_p \leq S_p \|u\| \leq S_p d_3 < S_p \frac{d_2}{S_p} = d_2.$$

But Lemma 2.4.2 says that $|u_k|_p \geq d_2$, and, by the strong convergence in $L^p(\Omega)$ we obtain $|u|_p \geq d_2$. This contradiction shows that if (2.36) holds, it cannot be $\|u\| < d_3$ and hence, as we have seen, u is a minimum.

Let us suppose now that in (2.35) the strict inequality holds, namely

$$\|u\|^2 < |u|_p^p + \int_{\Omega} hu \, dx. \quad (2.37)$$

We know from Lemma 2.4.4 that there exists $t^* > 0$ such that $t^*u \in \mathcal{N}_h$ and

$$t^* > \tilde{t} = \left(\frac{\|u\|^2}{(p-1)|u|_p^p} \right)^{\frac{1}{p-2}}.$$

Since (2.35) holds, we have

$$\tilde{t} \leq \left(\frac{|u|_p^p + \int_{\Omega} hu \, dx}{(p-1)|u|_p^p} \right)^{\frac{1}{p-2}} = \left(\frac{1}{p-1} + \frac{\int_{\Omega} hu \, dx}{(p-1)|u|_p^p} \right)^{\frac{1}{p-2}}.$$

But on the other hand,

$$\frac{|\int_{\Omega} hu \, dx|}{(p-1)|u|_p^p} \leq |h|_2 c \frac{\|u\|}{(p-1)|u|_p^p} \leq |h|_2 c c_4 \frac{1}{(p-1)d_2^p}.$$

Since $\frac{1}{p-1} < 1$, it is clear that we can choose $\varepsilon_5 = \min\{\frac{(p-2)(p-1)d_2^p}{2cc_4}, \varepsilon_4\}$, to obtain that if $|h|_2 < \varepsilon_5$, then $\tilde{t} < 1$.

Let us consider now the function

$$\gamma(t) = t\|u\|^2 - t^{p-1}|u|_p^p - \int_{\Omega} hu \, dx$$

that we have already studied. We know that γ is decreasing in $(\tilde{t}, +\infty)$. Since $t^*u \in \mathcal{N}_h$, we have $\gamma(t^*) = 0$. The inequality (2.37) is equivalent to $\gamma(1) < 0$. Since $\tilde{t} < 1$ and $\tilde{t} < t^*$, we must necessarily have $t^* < 1$. This last estimate allows us to conclude. Indeed, since $t^*u \in \mathcal{N}_h$ we obtain

$$\begin{aligned} m_h &\leq J(t^*u) = (t^*)^2 \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 - t^* \left(1 - \frac{1}{p} \right) \int_{\Omega} hu \, dx \\ &\leq (t^*)^2 \liminf_k \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|^2 - t^* \lim_k \left(1 - \frac{1}{p} \right) \int_{\Omega} hu_k \, dx \\ &\leq t^* \liminf_k \left[\left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|^2 - \left(1 - \frac{1}{p} \right) \int_{\Omega} hu_k \, dx \right] \\ &= t^* \liminf_k J(u_k) = t^* m_h < m_h. \end{aligned}$$

The last strict inequality holds because $t^* \in (0, 1)$ and $m_h > 0$. The contradiction shows that (2.37) cannot hold, so that (2.36) is true. In this case, as we have seen, u is the required minimum. $\square$

The last step consists in proving that the minimum is a critical point of J on the whole space $H_0^1(\Omega)$.

Lemma 2.4.9 *There exists $\varepsilon_6 \in (0, \varepsilon_5)$ such that if $|h|_2 < \varepsilon_6$, then the minimum u of J on $\mathcal{N}_h$ satisfies $J'(u)v = 0$ for all $v \in H_0^1(\Omega)$.*

Proof Fix $v \in H_0^1(\Omega)$ and consider the function $\varphi : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(s, t) = t^2\|u + sv\|^2 - t^p|u + sv|^p - t \int_{\Omega} h(u + sv) \, dx.$$

Since $u \in \mathcal{N}_h$ we have $\varphi(0, 1) = 0$. Clearly φ is of class C^1 and

$$\frac{\partial \varphi}{\partial t}(0, 1) = 2\|u\|^2 - p|u|_p^p - \int_{\Omega} hu \, dx = (2 - p)\|u\|^2 + (p - 1) \int_{\Omega} hu \, dx.$$

If it were $\frac{\partial \varphi}{\partial t}(0, 1) = 0$, then we would have

$$\|u\|^2 = \frac{p-1}{p-2} \int_{\Omega} hu \, dx \leq \frac{p-1}{p-2} |h|_{2c} \|u\|,$$

namely,

$$\|u\| \leq \frac{p-1}{p-2} |h|_{2c}. \quad (2.38)$$

However, we know that $\|u\| > d_1$. If we take

$$|h|_2 < \frac{p-2}{c(p-1)} d_1,$$

then (2.38) yields $\|u\| < d_1$, a contradiction. Therefore, for such choices of h it must be $\frac{\partial \varphi}{\partial t}(0, 1) \neq 0$. We can then apply the Implicit Function Theorem, obtaining that there exist a number $\delta > 0$ and a C^1 function $t(s) : (-\delta, \delta) \rightarrow \mathbb{R}$ such that $\varphi(s, t(s)) = 0$ for every $s \in (-\delta, \delta)$, and $t(0) = 1$. Since $\|u\| > d_1$, we can also take δ so small that $t(s)(u + sv) > d_1$. We have thus constructed a differentiable curve $s \mapsto \mathcal{N}_h$ passing through u when $s = 0$. We now evaluate J along this curve, by considering the function

$$\gamma(s) = J(t(s)(u + sv)).$$

The function γ is differentiable and has a local minimum at $s = 0$. Then

$$0 = \gamma'(0) = J'(u)[t'(0)u + t(0)v] = t'(0)J'(u)u + J'(u)v = J'(u)v,$$

because $u \in \mathcal{N}_h$. Since this holds for every $v \in H_0^1(\Omega)$, it is enough to take a positive ε_6 such that

$$\varepsilon_6 < \min \left\{ \varepsilon_5, \frac{(p-2)d_1}{c(p-1)} \right\}$$

to conclude. □

We have found a solution to Problem (2.25). We now complete the proof of the theorem by constructing a second solution.

Lemma 2.4.10 *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|h|_2 < \delta$, then Problem (2.25) admits a solution u_h satisfying $\|u_h\| < \varepsilon$.*

Proof Let as usual $I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p$; we have

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{S_p^p}{p}\|u\|^p.$$

The function

$$g(t) = \frac{1}{2}t^2 - \frac{S_p^p}{p}t^p$$

is strictly increasing in a right neighborhood of 0, is continuous and satisfies $g(0) = 0$; therefore there certainly exists $\varepsilon' \leq \varepsilon$ such that for all $t \in (0, \varepsilon')$ there results $g(t) > 0$.

Then fixing any $\eta \in (0, \varepsilon')$, we obtain $I(u) \geq g(\eta) > 0$ if $\|u\| = \eta$. We deduce that

$$J(u) = I(u) - \int_{\Omega} hu \, dx \geq g(\eta) - |h|_2 c \eta.$$

Choosing

$$\delta = \frac{g(\eta)}{2c\eta} \quad \text{and} \quad |h|_2 < \delta,$$

we conclude that, for $\|u\| = \eta$,

$$J(u) \geq \frac{g(\eta)}{2} > 0.$$

Let now

$$B_{\eta} = \{u \in H_0^1(\Omega) \mid \|u\| \leq \eta\} \quad \text{and} \quad n_{\eta} = \inf_{u \in B_{\eta}} J(u).$$

Clearly $n_{\eta} \neq \pm\infty$. Moreover $n_{\eta} \leq J(0) = 0$. It is then straightforward to check, with arguments already used many times in the first part of this chapter, that the value n_{η} is attained by some $u_h \in B_{\eta}$. Since $J(u_h) = n_{\eta} \leq 0$, it cannot be $\|u_h\| = \eta$, and therefore u_h lies in the interior of the ball B_{η} . The function u_h is thus a local minimum for J and solves (2.25). Moreover, $\|u_h\| < \varepsilon$. $\square$

End of the proof of Theorem 2.4.1 Thanks to Lemma 2.4.9 we know that for every $|h|_2 < \varepsilon_6$, there exists a solution u_1 of Problem (2.25) with $\|u_1\| > d_1$. By Lemma 2.4.10, choosing $\varepsilon = d_1$, we can fix $\delta > 0$ such that for every $|h|_2 < \delta$ there exists a solution u_h of Problem (2.25) with $\|u_h\| < d_1$. If

$$|h|_2 < \varepsilon^* := \min\{\varepsilon_6, \delta\},$$

we obtain two *distinct* solutions of Problem (2.25). The proof is complete. $\square$

2.5 Nonhomogeneous Nonlinearities

The method of minimization on the Nehari Manifold can also be extended to cases where the nonlinearity is not homogeneous, under suitable assumptions. In this section we provide some examples in this sense.

We begin with the following problem: find u such that

$$\begin{cases} -\Delta u + q(x)u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.39)$$

where the nonlinearity f satisfies the following assumptions. We recall that we denote $F(t) = \int_0^t f(s)ds$.

- (f₁) $f : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^1 and is odd.
- (f₂) $f'(0) = 0$ and there exists $p \in (2, 2^*)$ such that $\limsup_{t \rightarrow +\infty} \frac{|f'(t)|}{t^{p-2}} < +\infty$.
- (f₃) There exists $\mu > 2$ such that $f(t)t \geq \mu F(t)$ for all $t \in \mathbb{R}$.
- (f₄) For every $t \in (0, +\infty)$ there results $f'(t)t > f(t)$.

Remark 2.5.1 Notice that the power nonlinearity $f(u) = |u|^{p-2}u$, with $p \in (2, 2^*)$, treated in Sect. 2.3 satisfies all the above assumptions.

Actually, assumptions (f₃) and (f₄) imply that the functions

$$t \mapsto \frac{F(t)}{t^\mu} \quad \text{and} \quad t \mapsto \frac{f(t)}{t}$$

are increasing, as one checks immediately by differentiation. This provides some bounds at zero and at infinity. For example, comparing with $t = 1$ one finds that $F(t) \leq F(1)t^\mu$ for $t \in [0, 1]$ and $F(t) \geq F(1)t^\mu$ for $t \geq 1$, and similarly for f .

We will prove the following result.

Theorem 2.5.2 *Assume that (h₁) and (f₁)–(f₄) hold. Then Problem (2.39) admits at least one nontrivial and nonnegative solution.*

As usual, the proof will be split in a series of lemmas, in each of which the assumptions of Theorem 2.5.2 will be taken for granted.

We begin with the description of some properties of $f(t)$ and $F(t)$ that can be deduced from the assumptions (f₁)–(f₄) and that will be used during the proof.

Lemma 2.5.3 *We have*

- (1) $\lim_{t \rightarrow 0} \frac{F(t)}{t^2} = 0$.
- (2) *There exist positive constants M_1, M_2 such that*

$$|f'(t)| \leq M_1 |t|^{p-2} \quad \text{for } |t| > M_2.$$

- (3) *There exist positive constants M_3, M_4 such that*

$$|F(t)| + |f(t)t| \leq M_3 |t|^p \quad \text{for } |t| > M_4.$$

- (4) *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|F(t)| + |f(t)t| \leq \varepsilon t^2 + C_\varepsilon |t|^p \quad \forall t \in \mathbb{R}. \quad (2.40)$$

- (5) *There exists a positive constant D such that for every $t \geq 1$,*

$$f(t)t \geq Dt^\mu \quad \text{and} \quad F(t) \geq Dt^\mu. \quad (2.41)$$

Proof The first statement follows by applying twice the de l'Hôpital rule. Point (2) is a direct consequence of (f₂) and the oddness of f . Integrating, one obtains (3).

Likewise, the fact that $f'(0) = 0$ and $(\mathbf{f}_2)$ imply (4), after integration. Lastly, (5) follows from Remark 2.5.1 and $(\mathbf{f}_3)$. $\square$

Weak solutions to Problem (2.39) are critical points of the functional

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx$$

where the norm is the same as in Sect. 2.3. The Nehari manifold in the present case is

$$\begin{aligned} \mathcal{N} &= \{u \in H_0^1(\Omega) \mid I'(u)u = 0, u \neq 0\} \\ &= \left\{ u \in H_0^1(\Omega) \mid \|u\|^2 = \int_{\Omega} f(u)u dx, u \neq 0 \right\}. \end{aligned}$$

As usual, we begin by checking that working on the Nehari manifold makes sense.

Lemma 2.5.4 *The Nehari manifold $\mathcal{N}$ is not empty.*

Proof We prove that for every $u \in H_0^1(\Omega)$, $u \not\equiv 0$, there exists $t > 0$ such that $tu \in \mathcal{N}$. We begin by assuming that $u(x) \geq 0$ a.e., and we consider the function

$$\gamma(t) = I'(tu)tu = t^2 \|u\|^2 - \int_{\Omega} f(tu)tu dx.$$

We want to study the behavior of γ for $t \rightarrow 0^+$ and for $t \rightarrow +\infty$. The aim is to find a zero of γ .

Let

$$\varphi(t) = \int_{\Omega} f(tu(x))tu(x) dx.$$

By (2.40), for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(tu(x))tu(x)| \leq \varepsilon t^2 u(x)^2 + C_\varepsilon t^p u(x)^p \quad \text{for all } t > 0 \text{ and for a.e. } x \in \Omega.$$

Therefore

$$\varphi(t) = \left| \int_{\Omega} f(tu(x))tu(x) dx \right| \leq \varepsilon t^2 \int_{\Omega} u(x)^2 dx + C_\varepsilon t^p \int_{\Omega} u(x)^p dx.$$

Since u is fixed, ε is arbitrary and $p > 2$, this shows that $\varphi(t) = o(t^2)$ as $t \rightarrow 0^+$. We have thus proved that

$$\gamma(t) = t^2 \|u\|^2 + o(t^2) \quad \text{as } t \rightarrow 0^+.$$

Next we study γ for t large. We begin by choosing t_0 so large that the set

$$A = \{x \in \Omega \mid t_0 u(x) \geq 1\}$$

has positive measure; this is possible since $u(x) \geq 0$ a.e. and $u \not\equiv 0$. Notice that if $t \geq t_0$, then $tu(x) \geq 1$ a.e. on A . Now we apply (2.41) with $t \geq t_0$, also recalling that f is positive on positive arguments:

$$\begin{aligned} \int_{\Omega} f(tu(x))tu(x) dx &= \int_A f(tu(x))tu(x) dx + \int_{\Omega \setminus A} f(tu(x))tu(x) dx \\ &\geq Dt^{\mu} \int_A u(x)^{\mu} dx. \end{aligned}$$

Notice that (3) and (5) of Lemma 2.5.3 imply that $\mu \leq p$, so that the last integral is certainly finite.

The preceding inequality shows that

$$\gamma(t) \leq t^2 \|u\|^2 - Dt^{\mu} \int_A u(x)^{\mu} dx \quad \text{for every } t \geq t_0.$$

The two properties of γ that we have established tell us that $\gamma(t)$ is strictly positive for positive and small values of t , while $\gamma(t) \rightarrow -\infty$ if $t \rightarrow +\infty$.

Then there exists $\tilde{t} > 0$ such that $\gamma(\tilde{t}) = 0$. Since $\tilde{t}u \neq 0$, we conclude that $\tilde{t}u \in \mathcal{N}$.

Finally, if u is not almost everywhere positive, we argue on the function $v = |u|$: we find again $\tilde{t} > 0$ such that

$$\|\tilde{t}v\|^2 - \int_{\Omega} f(\tilde{t}v)\tilde{t}v dx = 0.$$

Since $f(t)t$ is even, of course

$$\int_{\Omega} f(\tilde{t}v)\tilde{t}v dx = \int_{\Omega} f(\tilde{t}u)\tilde{t}u dx \quad \text{and} \quad \|\tilde{t}v\|^2 = \|\tilde{t}u\|^2,$$

so that also $\|\tilde{t}u\|^2 - \int_{\Omega} f(\tilde{t}u)\tilde{t}u dx = 0$, and $\tilde{t}u \in \mathcal{N}$. □

We can now define a candidate critical level as

$$m = \inf_{u \in \mathcal{N}} I(u)$$

and we try to see if m is attained on $\mathcal{N}$.

Lemma 2.5.5 *There results*

$$\inf_{u \in \mathcal{N}} \|u\|^2 > 0.$$

Proof Let λ_1 be the first eigenvalue of the operator $-\Delta + q(x)$, and choose a number $C_1 > 0$ such that

$$|f(t)t| \leq \frac{1}{2}\lambda_1 t^2 + C_1 |t|^p \quad \forall t;$$

this is possible by (2.40).

For every $u \in \mathcal{N}$ we have

$$\|u\|^2 = \int_{\Omega} f(u)u dx \leq \frac{\lambda_1}{2} \int_{\Omega} u^2 dx + C_1 \int_{\Omega} |u|^p dx \leq \frac{1}{2}\|u\|^2 + C_2 \|u\|^p,$$

so that

$$\|u\| \geq \left(\frac{1}{2C_2} \right)^{\frac{1}{p-2}},$$

which proves the claim. $\square$

Lemma 2.5.6 *The functional I is coercive on $\mathcal{N}$ and $m > 0$.*

Proof For every $u \in \mathcal{N}$, by **(f₃)**, we have

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx = \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 + \frac{1}{\mu} \|u\|^2 - \int_{\Omega} F(u) dx \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 + \int_{\Omega} \left(\frac{1}{\mu} f(u)u - F(u) \right) dx \geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2, \end{aligned}$$

which shows that I is coercive on $\mathcal{N}$ and that

$$m = \inf_{u \in \mathcal{N}} I(u) \geq \inf_{u \in \mathcal{N}} \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(\frac{1}{2C_2} \right)^{\frac{2}{p-2}} > 0,$$

by Lemma 2.5.5. $\square$

Lemma 2.5.7 *There exists $u \in \mathcal{N}$ such that $I(u) = m$ and $u \geq 0$ a.e. in Ω .*

Proof Let $\{u_k\}_k \subset \mathcal{N}$ be a minimizing sequence for I . The fact that $F(t)$ and $f(t)t$ are even implies that also $\{|u_k|\}_k$ is a minimizing sequence, so that we can assume from the beginning that $u_k(x) \geq 0$ a.e. in Ω . Since I is coercive on $\mathcal{N}$, the sequence $\{u_k\}_k$ is bounded and then, up to subsequences,

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \quad u_k \rightarrow u \quad \text{in } L^q(\Omega) \quad \forall q \in [2, 2^*), \\ u_k(x) &\rightarrow u(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

This shows that $u(x) \geq 0$ a.e. and, with arguments already used many times,

$$\begin{aligned} \int_{\Omega} F(u_k) dx &\rightarrow \int_{\Omega} F(u) dx, \\ \int_{\Omega} f(u_k)u_k dx &\rightarrow \int_{\Omega} f(u)u dx, \quad \|u\|^2 \leq \liminf_{k \rightarrow +\infty} \|u_k\|^2. \end{aligned}$$

Clearly the assumption $p < 2^*$ is essential in the first two limits.

Thus we deduce that

$$I(u) \leq \liminf_k I(u_k) = m \quad \text{and} \quad \|u\|^2 \leq \int_{\Omega} f(u)u dx.$$

Now we notice that from $\|u_k\|^2 = \int_{\Omega} f(u_k)u_k dx$ we have

$$\int_{\Omega} f(u)u dx = \lim_k \|u_k\|^2.$$

Since

$$\|u_k\|^2 \geq \left(\frac{1}{2C_2}\right)^{\frac{2}{p-2}} > 0$$

for all k we obtain

$$\int_{\Omega} f(u)u \, dx > 0,$$

which shows that it cannot be $u \equiv 0$.

If $\|u\|^2 = \int_{\Omega} f(u)u \, dx$, then $u \in \mathcal{N}$, u is the required minimum and the proof is complete. It remains to show that

$$\|u\|^2 < \int_{\Omega} f(u)u \, dx \quad (2.42)$$

leads to a contradiction.

To this aim, assume that (2.42) holds and consider the function

$$\gamma(t) = t^2 \|u\|^2 - \int_{\Omega} f(tu)tu \, dx.$$

We already know that $\gamma(t) > 0$ in a right neighborhood of 0. The inequality (2.42) tells us that $\gamma(1) < 0$. Then there exists $t^* \in (0, 1)$ such that $\gamma(t^*) = 0$, which means that $t^*u \in \mathcal{N}$. We then obtain

$$m \leq I(t^*u) = \int_{\Omega} \left[\frac{1}{2} f(t^*u)t^*u - F(t^*u) \right] dx.$$

From (f₄) it is easy to see that the function

$$s \rightarrow \frac{1}{2} f(s)s - F(s)$$

is strictly increasing in $(0, +\infty)$, and then, since $u \geq 0$ and $t^* \in (0, 1)$, we see that

$$\begin{aligned} m &\leq \int_{\Omega} \left[\frac{1}{2} f(t^*u)t^*u - F(t^*u) \right] dx < \int_{\Omega} \left[\frac{1}{2} f(u)u - F(u) \right] dx \\ &= \lim_k \int_{\Omega} \left[\frac{1}{2} f(u_k)u_k - F(u_k) \right] dx = \lim_k I(u_k) = m. \end{aligned}$$

This contradiction shows that (2.42) cannot hold and concludes the proof. □

The last step consists in showing that u is a critical point for I .

Lemma 2.5.8 *Let $u \in \mathcal{N}$ be such that $I(u) = m$. Then $I'(u) = 0$.*

Proof For every $v \in H_0^1(\Omega)$ there exists $\varepsilon > 0$ such that $u + sv \not\equiv 0$ for all s in $(-\varepsilon, \varepsilon)$. We know that there exists $t(s) \in \mathbb{R}$ such that $t(s)(u + sv) \in \mathcal{N}$. We now deduce some properties of the function $t(s)$. To this aim, consider the function

$$\varphi(s, t) = t^2 \|u + sv\|^2 - \int_{\Omega} f(t(u + sv))t(u + sv) \, dx,$$

defined for $(s, t) \in (-\varepsilon, \varepsilon) \times \mathbb{R}$. Since $u \in \mathcal{N}$, we have

$$\varphi(0, 1) = \|u\|^2 - \int_{\Omega} f(u)u \, dx = 0.$$

Moreover, by **(f₂)** and **(f₄)**, φ is a C^1 function and

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(0, 1) &= 2\|u\|^2 - \int_{\Omega} f(u)u \, dx - \int_{\Omega} f'(u)u^2 \, dx \\ &= \int_{\Omega} [f(u)u - f'(u)u^2] \, dx < 0. \end{aligned}$$

Then, by the Implicit Function Theorem, for ε small there exists a C^1 function $t(s)$ such that $\varphi(s, t(s)) = 0$ and $t(0) = 1$. This also shows that $t(s) \neq 0$, at least for ε very small. Therefore $t(s)(u + sv) \in \mathcal{N}$.

Then setting

$$\gamma(s) = I(t(s)(u + sv)),$$

we have that γ is differentiable and has a minimum at $s = 0$; therefore

$$0 = \gamma'(0) = I'(t(0)u)(t'(0)u + t(0)v) = t'(0)I'(u)u + I'(u)v = I'(u)v.$$

Since v is arbitrary in $H_0^1(\Omega)$, we deduce that $I'(u) = 0$. $\square$

Example 2.5.9 We list a couple of examples of nonlinearities that satisfy assumptions **(f₁)**–**(f₄)**. These are functions that behave “like powers” without being exact powers. Since we want odd functions, we define them only for $t > 0$, and we extend them by oddness.

- $f(t) = t^{p-1} + t^{q-1}$, with $p, q \in (2, 2^*)$ and $p \neq q$.
- Slightly more generally, $f(t) = \sum_{i=1}^M a_i t^{p_i-1}$ with $a_i > 0$ and $p_i \in (2, 2^*)$.
- $f(t) = \frac{t^{q-1}}{1+t^{q-p}}$, with $p \in (2, 2^*)$ and $q > 2$.

Remark 2.5.10 Let $2 < q < p < 2^*$; the function

$$f(t) = t^{p-1} - t^{q-1} \tag{2.43}$$

for $t > 0$ and extended by oddness to $\mathbb{R}$, does not satisfy the assumptions **(f₃)** and **(f₄)**. However if one repeats the arguments carried out in the previous lemmas, one sees that they still work, with minor changes. Therefore Problem (2.39) admits a solution also for this particular f . We now try to generalize this remark to a wider class of nonlinearities that contains (2.43).

We consider the problem

$$\begin{cases} -\Delta u + q(x)u = f(u) - g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.44}$$

We set $G(t) = \int_0^t g(s)ds$ and we introduce the following assumptions. Here $\mu > 2$, as in **(f₃)**.

- (g₁) $g : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^1 and is odd;
 (g₂) $g'(0) = 0$ and there exists $\theta \in (2, \mu)$ such that $\limsup_{t \rightarrow +\infty} \frac{|g'(t)|}{t^{\theta-2}} < +\infty$;
 (g₃) the inequalities $g(t) \geq 0$ and $g'(t)t \leq (\mu - 1)g(t)$ hold for every $t \in (0, +\infty)$.

As far as f is concerned, we have to replace (f₃) and (f₄) with the stronger assumption

- (f₅) the inequality $f'(t)t \geq (\mu - 1)f(t) > 0$ holds for every $t \in (0, +\infty)$.

We can now prove the following result.

Theorem 2.5.11 *Assume that (h₁), (f₁), (f₂), (f₅) and (g₁)–(g₃) hold. Then Problem (2.44) admits at least one nontrivial and nonnegative solution.*

The proof follows exactly the same lines as the previous one. We start by noticing that since (f₅) implies (f₃) and (f₄), all the properties of the functions $F(t)$ and $f(t)t$ obtained earlier are still valid. Concerning g and G , we list the properties that can be obtained working as in Lemma 2.5.3.

Lemma 2.5.12 *We have*

- (1) $\lim_{t \rightarrow 0} \frac{G(t)}{t^2} = 0$.
 (2) *There exist positive constants M_5, M_6 such that*

$$|g'(t)| \leq M_5 |t|^{\theta-2} \quad \text{for } |t| > M_6.$$

- (3) *There exist positive constants D_1, D_2 such that*

$$|G(t)| + |g(t)t| \leq D_1 t^2 + D_2 |t|^\theta \quad \forall t \in \mathbb{R}. \quad (2.45)$$

Solutions of Problem (2.44) are critical points of the functional

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx + \int_{\Omega} G(u) dx$$

and the corresponding Nehari manifold is

$$\begin{aligned} \mathcal{N} &= \{u \in H_0^1(\Omega) \mid I'(u)u = 0, u \neq 0\} \\ &= \left\{ u \in H_0^1(\Omega) \mid \|u\|^2 = \int_{\Omega} f(u)u dx - \int_{\Omega} g(u)u dx, u \neq 0 \right\}. \end{aligned}$$

The first step consist of course in checking that we can work in $\mathcal{N}$.

Lemma 2.5.13 *The Nehari manifold $\mathcal{N}$ is not empty.*

Proof Repeating the argument of Lemma 2.5.4, we fix $u \in H_0^1(\Omega)$ with $u \not\equiv 0$ and $u \geq 0$, and we consider the function

$$\gamma(t) = I'(tu)tu = t^2 \|u\|^2 - \int_{\Omega} f(tu)tu dx + \int_{\Omega} g(tu)tu dx.$$

We already know that

$$\int_{\Omega} f(tu)tu \, dx = o(t^2)$$

as $t \rightarrow 0$, and the same ideas show that also

$$\int_{\Omega} g(tu)tu \, dx = o(t^2),$$

so that

$$\gamma(t) = t^2\|u\|^2 + o(t^2)$$

as $t \rightarrow 0$.

On the other hand, from the properties of g we deduce that for a suitable $C > 0$,

$$\left| \int_{\Omega} g(tu)tu \, dx \right| \leq Ct^2 + Ct^{\theta} = o(t^{\mu}),$$

when $t \rightarrow +\infty$. From this and the proof of Lemma 2.5.4 we immediately obtain

$$\gamma(t) \leq t^2\|u\|^2 - Dt^{\mu} \int_{\Omega} u^{\mu} \, dx + o(t^{\mu})$$

as $t \rightarrow +\infty$. This proves the existence of $t > 0$ such that $tu \in \mathcal{N}$. We conclude the proof as in Lemma 2.5.4 when u has arbitrary sign. $\square$

Defining as usual

$$m = \inf_{u \in \mathcal{N}} I(u),$$

we proceed to prove that m is attained by a function that solves Problem (2.44).

Lemma 2.5.14 *There results*

$$\inf_{u \in \mathcal{N}} \|u\|^2 > 0.$$

Proof Working as in Lemma 2.5.5 and using the fact that $g(u)u \geq 0$, we obtain for every $u \in \mathcal{N}$,

$$\begin{aligned} \|u\|^2 &= \int_{\Omega} f(u)u \, dx - \int_{\Omega} g(u)u \, dx \leq \frac{1}{2}\lambda_1 \int_{\Omega} u^2 \, dx + C_1 \int_{\Omega} |u|^p \, dx \\ &\leq \frac{1}{2}\|u\|^2 + C_2\|u\|^p, \end{aligned}$$

which allows us to conclude exactly as above. $\square$

Lemma 2.5.15 *The functional I is coercive on $\mathcal{N}$ and $m > 0$.*

Proof From (g₃) we deduce that $\mu G(u) \geq g(u)u$. Then, if $u \in \mathcal{N}$, we have

$$\begin{aligned}
I(u) &= \frac{1}{2} \|u\|^2 - \int_{\Omega} F(u) dx + \int_{\Omega} G(u) dx \\
&= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 + \int_{\Omega} \left(\frac{1}{\mu} f(u)u - F(u) \right) dx \\
&\quad + \int_{\Omega} \left(G(u) - \frac{1}{\mu} g(u)u \right) dx \geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2,
\end{aligned}$$

and we conclude as in Lemma 2.5.6. $\square$

We can now prove that the infimum of I on $\mathcal{N}$ is attained.

Lemma 2.5.16 *There exists $u \in \mathcal{N}$ such that $I(u) = m$ and $u \geq 0$.*

Proof Let $\{u_k\}_k \subset \mathcal{N}$ be a minimizing sequence for I . As above, we can assume that $u_k \geq 0$. By Lemma 2.5.15, I is coercive on $\mathcal{N}$, and hence $\{u_k\}_k$ is bounded. Up to subsequences,

$$\begin{aligned}
u_k &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \quad u_k \rightarrow u \quad \text{in } L^q(\Omega) \quad \forall q \in [2, 2^*), \\
u_k(x) &\rightarrow u(x) \quad \text{a.e. in } \Omega.
\end{aligned}$$

Then $u(x) \geq 0$, and with the usual arguments, as $k \rightarrow \infty$,

$$\begin{aligned}
\int_{\Omega} F(u_k) dx &\rightarrow \int_{\Omega} F(u) dx, \quad \int_{\Omega} f(u_k)u_k dx \rightarrow \int_{\Omega} f(u)u dx, \\
\int_{\Omega} G(u_k) dx &\rightarrow \int_{\Omega} G(u) dx, \\
\int_{\Omega} g(u_k)u_k dx &\rightarrow \int_{\Omega} g(u)u dx, \quad \|u\|^2 \leq \liminf_{k \rightarrow +\infty} \|u_k\|^2.
\end{aligned}$$

These yield immediately

$$I(u) \leq \liminf_k I(u_k) = m \quad \text{and} \quad \|u\|^2 \leq \int_{\Omega} f(u)u dx - \int_{\Omega} g(u)u dx.$$

Since

$$\int_{\Omega} f(u)u dx - \int_{\Omega} g(u)u dx = \lim_k \|u_k\|^2 \geq \left(\frac{1}{2C_2} \right)^{\frac{2}{p-2}} > 0,$$

we obtain

$$\int_{\Omega} f(u)u dx - \int_{\Omega} g(u)u dx > 0,$$

which shows that it cannot be $u \equiv 0$.

Now, if

$$\|u\|^2 = \int_{\Omega} f(u)u dx - \int_{\Omega} g(u)u dx,$$

then $u \in \mathcal{N}$, and u is the required minimum. We must therefore show that it cannot be

$$\|u\|^2 < \int_{\Omega} f(u)u \, dx - \int_{\Omega} g(u)u \, dx. \quad (2.46)$$

To this aim, we consider the function

$$\gamma(t) = t^2 \|u\|^2 - \int_{\Omega} f(tu)tu \, dx + \int_{\Omega} g(tu)tu \, dx.$$

We have already noticed that $\gamma(t) > 0$ in a right neighborhood of zero. Assuming (2.46) means that $\gamma(1) < 0$, hence there exists $t^* \in (0, 1)$ such that $\gamma(t^*) = 0$, namely $t^*u \in \mathcal{N}$. Then we see that

$$\begin{aligned} m &\leq I(t^*u) = \frac{1}{2} \|t^*u\|^2 - \int_{\Omega} F(t^*u) \, dx + \int_{\Omega} G(t^*u) \, dx \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|t^*u\|^2 + \int_{\Omega} \left(\frac{1}{\mu} f(t^*u)t^*u - F(t^*u) \right) \, dx \\ &\quad + \int_{\Omega} \left(G(t^*u) - \frac{1}{\mu} g(t^*u)t^*u \right) \, dx. \end{aligned}$$

From assumptions **(f₅)** and **(g₃)** we deduce immediately that the functions

$$s \mapsto \frac{1}{\mu} f(s)s - F(s) \quad \text{and} \quad s \mapsto G(s) - \frac{1}{\mu} g(s)s$$

are nondecreasing in $(0, +\infty)$, so that being $u \geq 0$ and $t^* \in (0, 1)$, we have

$$\begin{aligned} m &\leq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|t^*u\|^2 + \int_{\Omega} \left(\frac{1}{\mu} f(t^*u)t^*u - F(t^*u) \right) \, dx \\ &\quad + \int_{\Omega} \left(\frac{1}{\mu} G(t^*u) - g(t^*u)t^*u \right) \, dx \\ &< \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 + \int_{\Omega} \left(\frac{1}{\mu} f(u)u - F(u) \right) \, dx + \int_{\Omega} \left(\frac{1}{\mu} G(u) - g(u)u \right) \, dx \\ &\leq \liminf_k \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|^2 + \lim_k \int_{\Omega} \left[\frac{1}{\mu} f(u_k)u_k - F(u_k) \right] \, dx \\ &\quad + \lim_k \int_{\Omega} \left[G(u_k) - \frac{1}{\mu} g(u_k)u_k \right] \, dx \\ &= \liminf_k I(u_k) = m. \end{aligned}$$

This contradiction shows that (2.46) cannot hold, and the proof is complete. $\square$

Lastly we prove that the minimizer u is a critical point of I on $H_0^1(\Omega)$.

Lemma 2.5.17 *Let $u \in \mathcal{N}$ be such that $I(u) = m$. Then $I'(u) = 0$.*

Proof The technique is the same as in Lemma 2.5.13. Take $v \in H_0^1(\Omega)$ and let $\varepsilon > 0$ be so small that $u + sv \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. We know that there exists $t(s) \in \mathbb{R}$ such that $t(s)(u + sv) \in \mathcal{N}$. Consider the function

$$\begin{aligned} \varphi(s, t) &= t^2 \|u + sv\|^2 - \int_{\Omega} f(t(u + sv))t(u + sv) dx \\ &\quad + \int_{\Omega} g(t(u + sv))t(u + sv) dx, \end{aligned}$$

defined for $(s, t) \in (-\varepsilon, \varepsilon) \times \mathbb{R}$. Since $u \in \mathcal{N}$, we have

$$\varphi(0, 1) = \|u\|^2 - \int_{\Omega} f(u)u dx + \int_{\Omega} g(u)u dx = 0.$$

On the other hand, by (f₅) and (g₃),

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(0, 1) &= 2\|u\|^2 - \int_{\Omega} f(u)u dx - \int_{\Omega} f'(u)u^2 dx + \int_{\Omega} g(u)u dx \\ &\quad + \int_{\Omega} g'(u)u^2 dx \\ &\leq 2\|u\|^2 - \int_{\Omega} f(u)u dx - (\mu - 1) \int_{\Omega} f(u)u dx + \int_{\Omega} g(u)u dx \\ &\quad + (\mu - 1) \int_{\Omega} g(u)u dx \\ &= 2\|u\|^2 - \mu \int_{\Omega} f(u)u dx + \mu \int_{\Omega} g(u)u dx = (2 - \mu)\|u\|^2 < 0. \end{aligned}$$

So, by the Implicit Function Theorem, for ε small enough we can determine a function $t \in C^1(-\varepsilon, \varepsilon)$ such that $f(s, t(s)) = 0$ and $t(0) = 1$. This says that $t(s) \neq 0$, at least for ε very small, and then $t(s)(u + sv) \in \mathcal{N}$.

Setting

$$\gamma(s) = I(t(s)(u + sv)),$$

we obtain that γ is differentiable and has a minimum point at $s = 0$; thus

$$0 = \gamma'(0) = I'(t(0)u)(t'(0)u + t(0)v) = t'(0)I'(u)u + I'(u)v = I'(u)v.$$

Since $v \in H_0^1(\Omega)$, is arbitrary, we conclude that $I'(u) = 0$. □

Example 2.5.18 Also in this case it is easy to produce examples of nonlinearities that satisfy the assumption of Theorem 2.5.11. As usual we define the functions for $t > 0$ and we extend them by oddness.

- $f(t) = t^{p-1}$, $g(t) = t^{q-1}$, where $2 < q < p < 2^*$.

Slightly more generally, $f(t) = \sum_{i=1}^M a_i t^{p_i-1}$ and $g(t) = \sum_{i=1}^N b_i t^{q_i-1}$ with $a_i, b_i > 0$, $p_i, q_i \in (2, 2^*)$ and $\min\{p_i\} > \max\{q_i\}$.

- $f(t) = \frac{t^{q-1}}{1+s^{q-p}}$, where $p \in (2, 2^*)$, $q > 2$ and g as in the preceding example, with $\max\{q_i\} < \min\{p, q\}$.

2.6 The p -Laplacian

In this section we leave temporarily the semilinear equations studied up to now to give an example of how the techniques introduced so far can be successfully applied, and with almost no changes, to more general *quasilinear* equations. Although quasilinear equations are in general much more difficult than semilinear ones, a certain number of results can be proved by the application of variational methods with surprisingly little effort.

This section should be considered as an *example* and can be skipped on first reading.

The particular type of equations we consider are the so-called p -Laplace equations, that are commonly seen as the simplest generalizations of the Laplacian to the quasilinear context.

The p -Laplacian is the second order non linear differential operator defined as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

Here $p > 1$, and of course when $p = 2$ the p -Laplacian is the usual Laplacian.

Notice that this operator is *linear* in the second derivatives, but contains terms which are *nonlinear* functions of the *gradient* of the unknown.

When u is regular enough an easy computation shows that

$$\Delta_p u = |\nabla u|^{p-2} \Delta u + (p-2) |\nabla u|^{p-4} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j},$$

a rather complicated expression. For this reason, it is much more convenient to use some weak formulation for this operator. To write down such a weak form we need, as in the previous chapters, to introduce suitable function spaces. We will list a number of definitions and results that generalize those we have seen in Chap. 1 for the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$.

- $W^{1,p}(\Omega)$, for $p \in [1, +\infty)$, is the Sobolev space defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, \dots, N \right\},$$

where the derivative $\frac{\partial u}{\partial x_i}$ is in the sense of distributions. The spaces $W^{1,p}(\Omega)$ are reflexive Banach spaces when endowed with the norm

$$\|u\|_p = |u|_p + \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|_p.$$

Recall that $|\cdot|_p$ is the L^p norm.

- $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.
- If $u \in W_0^{1,p}(\Omega)$, then $|u|, u^+, u^- \in W_0^{1,p}(\Omega)$ and

$$\int_{\Omega} |\nabla |u||^p dx = \int_{\Omega} |\nabla u|^p dx.$$

We also recall, as in the previous chapters, some embedding theorems.

Theorem 2.6.1 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and have smooth boundary. Let $p \geq 1$.*

- *If $p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, \frac{Np}{N-p}]$; the embedding is compact for every $q \in [1, \frac{Np}{N-p})$.*
- *If $p = N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, +\infty)$; the embedding is compact.*
- *If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$; the embedding is compact.*

The following particular case is the most used in our context, and we state it separately.

Theorem 2.6.2 *Let Ω be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 3$. Let $1 < p < N$. Then*

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for every } q \in \left[1, \frac{Np}{N-p}\right].$$

The embedding is compact for every $q \in [1, \frac{Np}{N-p})$.

The number $\frac{Np}{N-p}$ is denoted by p^* and is called the *critical Sobolev exponent* for the embedding of $W_0^{1,p}(\Omega)$ into L^q .

Theorem 2.6.3 (Poincaré inequality for $W_0^{1,p}$) *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Then there exists a constant $C > 0$, depending only on Ω , such that*

$$\int_{\Omega} |u|^p dx \leq C \int_{\Omega} |\nabla u|^p dx \quad \forall u \in W_0^{1,p}(\Omega).$$

Therefore, if Ω is a bounded open set, $(\int_{\Omega} |\nabla u|^p dx)^{\frac{1}{p}}$ is a norm on $W_0^{1,p}(\Omega)$, equivalent to the standard one.

2.6.1 Basic Theory

We begin by studying the differentiability of some relevant functionals on $W_0^{1,p}(\Omega)$. We start with the norm.

Theorem 2.6.4 *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, be an open set. For $p \in (1, +\infty)$, define a functional $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by*

$$J(u) = \int_{\Omega} |\nabla u|^p dx.$$

Then J is differentiable in $W_0^{1,p}(\Omega)$ and

$$J'(u)v = p \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

Proof Consider the function $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$, defined by $\varphi(x) = |x|^p$. It is a C^1 function and $\nabla \varphi(x) = p|x|^{p-2}x$, so that, for all $x, y \in \mathbb{R}^N$,

$$\lim_{t \rightarrow 0} \frac{\varphi(x + ty) - \varphi(x)}{t} = p|x|^{p-2}x \cdot y.$$

As a consequence,

$$\lim_{t \rightarrow 0} \frac{|\nabla u(x) + t \nabla v(x)|^p - |\nabla u(x)|^p}{t} = p|\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \quad \text{a.e. in } \Omega.$$

By the Lagrange Theorem there exists $\theta \in \mathbb{R}$ such that $|\theta| \leq |t|$ and

$$\begin{aligned} \left| \frac{|\nabla u + t \nabla v|^p - |\nabla u|^p}{t} \right| &\leq p |\nabla u + \theta \nabla v|^{p-2} (\nabla u + \theta \nabla v) \cdot \nabla v \\ &\leq C (|\nabla u|^{p-1} |\nabla v| + |\nabla v|^p) \in L^1(\Omega). \end{aligned}$$

By dominated convergence we obtain

$$\lim_{t \rightarrow 0} \int_{\Omega} \frac{|\nabla u + t \nabla v|^p - |\nabla u|^p}{t} \, dx = p \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx,$$

so that J is Gâteaux differentiable and

$$J'_G(u)v = p \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

We have now to prove that $J'_G : W_0^{1,p}(\Omega) \rightarrow [W^{1,p}(\Omega)]'$ is continuous. To this aim we take a sequence $\{u_k\}_k$ in $W_0^{1,p}(\Omega)$ such that $u_k \rightarrow u$ in $W_0^{1,p}(\Omega)$. In particular we can assume, as usual, that up to subsequences,

- $\nabla u_k \rightarrow \nabla u$ in $(L^p(\Omega))^N$ as $k \rightarrow \infty$;
- $\nabla u_k(x) \rightarrow \nabla u(x)$ a.e. as $k \rightarrow \infty$;
- there exists $w \in L^1(\Omega)$ such that $|\nabla u_k(x)|^p \leq w(x)$ a.e. in Ω , and for all $k \in \mathbb{N}$.

We have

$$(J'_G(u) - J'_G(u_k))v = p \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_k|^{p-2} \nabla u_k) \cdot \nabla v \, dx$$

and

$$\begin{aligned} &\left| \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_k|^{p-2} \nabla u_k) \cdot \nabla v \, dx \right| \\ &\leq \left(\int_{\Omega} \left| |\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u \right|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |\nabla v|^p \, dx \right)^{1/p} \\ &\leq \left(\int_{\Omega} \left| |\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u \right|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \|v\|, \end{aligned}$$

so that

$$\begin{aligned} \|J'_G(u) - J'_G(u_k)\| &= \sup \{ |(J'_G(u) - J'_G(u_k))v| \mid v \in W^{1,p}(\Omega), \|v\| = 1 \} \\ &\leq \left(\int_{\Omega} ||\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u|^{p-1} dx \right)^{\frac{p-1}{p}}. \end{aligned}$$

Now we know that

$$|\nabla u_k(x)|^{p-2} \nabla u_k(x) \rightarrow |\nabla u(x)|^{p-2} \nabla u(x)$$

almost everywhere in Ω , and that

$$||\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u|^{\frac{p}{p-1}} \leq C(|\nabla u_k|^p + |\nabla u|^p) \leq w + |\nabla u|^p \in L^1(\Omega).$$

By dominated convergence we then obtain

$$\int_{\Omega} ||\nabla u_k|^{p-2} \nabla u_k - |\nabla u|^{p-2} \nabla u|^{\frac{p}{p-1}} dx \rightarrow 0,$$

and hence $\|J'_G(u_k) - J'_G(u)\| \rightarrow 0$. This holds for a subsequence of the original sequence $\{u_k\}$, but we can argue as before to obtain that J'_G is a continuous function, so that J is differentiable with differential given by

$$J'(u)v = p \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx. \quad \square$$

Remark 2.6.5 When $p \neq 2$, $W_0^{1,p}(\Omega)$ is *not* a Hilbert space. So here is an example of a Banach space where the norm is differentiable, see Remark 1.3.15.

Theorem 2.6.6 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set. Take a number $1 < p < N$ and let $p^* = \frac{Np}{N-p}$ be the critical exponent for the embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and assume that there exist $a, b > 0$ such that*

$$|f(t)| \leq a + b|t|^{p^*-1} \quad (2.47)$$

for all $t \in \mathbb{R}$. Define

$$F(t) = \int_0^t f(s) ds$$

and consider the functional $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$J(u) = \int_{\Omega} F(u(x)) dx.$$

Then J is differentiable on $W_0^{1,p}(\Omega)$ and

$$J'(u)v = \int_{\Omega} f(u(x))v(x) dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

We do not write down the proof of this result, because the argument is very similar to the one of Example 1.3.20.

We can now state a definition of weak solution for equations involving the p -Laplacian. The idea for a weak formulation can be obtained, as in the usual case $p = 2$, using integration by parts. Let us assume that u and v are functions defined on a bounded open set Ω , that they vanish on $\partial\Omega$, and are smooth enough for the following computations to make sense. Integrating by parts we obtain

$$\int_{\Omega} v \Delta_p u \, dx = \int_{\Omega} v \operatorname{div}(|\nabla u|^{p-2} \nabla u) \, dx = - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

This suggests that a good candidate as a weak form for the p -Laplacian is the operator

$$v \mapsto - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

We now give a precise formulation to the preceding heuristic discussion, at least for the cases that we want to treat. Let us consider $1 < p < N$ and take $h \in L^{p'}$, where $p' = \frac{p}{p-1}$ is the conjugate exponent of p . Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the growth assumption (2.47). We will say that u is a weak solution of the boundary value problem

$$\begin{cases} -\Delta_p u = f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.48)$$

if $u \in W_0^{1,p}(\Omega)$ and

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f(u) v \, dx + \int_{\Omega} h v \, dx \quad \forall v \in W_0^{1,p}(\Omega).$$

Notice that, as in the case of the Laplacian, the boundary conditions are englobed in the function space $W_0^{1,p}(\Omega)$. Thanks to the previous results about differentiation of functionals, it is easy to spot the link between weak solutions and critical points. Indeed, denoting $F(t) = \int_0^t f(s) \, ds$, we can define a functional $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by

$$J(u) = \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} F(u) \, dx - \int_{\Omega} h u \, dx.$$

We know that J is differentiable on $W_0^{1,p}(\Omega)$, with

$$J'(u)v = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \int_{\Omega} f(u) v \, dx - \int_{\Omega} h v \, dx.$$

Hence, a critical point of J is exactly a weak solution of (2.48) in the sense previously stated.

Remark 2.6.7 The procedure to obtain a classical solution from a weak one is more problematic in the case of quasilinear equations. Indeed, a weak solution which is regular enough is also a classical solution, but the regularity results for weak solutions of p -Laplacian equations that have been obtained so far are not completely satisfactory: weak solution may be not regular enough to be also classical solutions.

We conclude with a simple existence result for equations with right-hand-side independent of u .

Theorem 2.6.8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $p > 1$. Let $p' = \frac{p}{p-1}$ be the conjugate exponent of p . Then, for every $h \in L^{p'}(\Omega)$, the problem*

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.49)$$

has a unique (weak) solution.

Proof Define a functional $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} hu dx.$$

We know that J is differentiable on $W_0^{1,p}(\Omega)$ with

$$J'(u)v = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \int_{\Omega} hv dx.$$

It is well known that the function $x \rightarrow |x|^p$, $x \in \mathbb{R}^N$, is strictly convex. From this, it is easy to deduce that J is strictly convex. Moreover,

$$J(u) \geq \frac{1}{p} \|u\|^p - |h|_{p'} \|u\|_p \geq \frac{1}{p} \|u\|^p - C \|u\|,$$

so that J is coercive because $p > 1$. Applying Theorems 1.5.6 and 1.5.8, we find that J has a unique critical point which is the unique (weak) solution of Problem (2.49). $\square$

Remark 2.6.9 Notice that the previous theorem holds for any $p > 1$.

Remark 2.6.10 In Sect. 1.6 we have seen similar results. In that case we dealt with linear equations, and it is well known that they can be solved by different methods, for example by a straightforward application of the Riesz Theorem 1.2.9. On the contrary, (2.49), being quasilinear, cannot be treated by the standard methods of the linear theory, available for example for (1.22) and (1.23). Variational methods give an unified treatment for linear and nonlinear problems.

2.6.2 Two Applications

We start with a nonlinear eigenvalue problem. Indeed, it is possible to generalize part of the theory and of the results concerning linear eigenvalue problems to nonlinear problems. Nonlinear eigenvalue problems are a largely studied subject in recent research. For some references on these topics, start for example from [20].

Recall that in the Banach space $W_0^{1,p}(\Omega)$ we use the norm $\|u\|^p = \int_{\Omega} |\nabla u|^p dx$.

Theorem 2.6.11 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set and let $1 < p < N$. Define*

$$\mu = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p}{|u|_p^p}.$$

Then μ is positive and is achieved by some $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. The function u can be chosen nonnegative and satisfies (weakly)

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \mu u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.50)$$

Proof For $u \in W_0^{1,p}(\Omega)$, set $I(u) = \|u\|^p$, $J(u) = |u|_p^p$ and define the quotient functional $Q : W_0^{1,p}(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ as

$$Q(u) = \frac{I(u)}{J(u)}.$$

Then

$$\mu = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} Q(u).$$

The Poincaré inequality immediately gives $\mu > 0$. Let $\{u_k\}_k$ be a minimizing sequence. Clearly $|u_k|$ is also a minimizing sequence for Q , so we can assume that $u_k(x) \geq 0$ a.e. in Ω . As the functional Q is homogeneous of degree zero, i.e. $Q(\lambda u) = Q(u)$ for every $\lambda \in \mathbb{R}$, we can normalize u_k by setting $|u_k|_p = 1$ for every k . Then we see that $J(u_k)$, namely the $W^{1,p}$ norm of u_k , must be bounded independently of k .

By the Sobolev embeddings (Theorem 2.6.2), we deduce that, up to subsequences,

- $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$;
- $u_k \rightarrow u$ in $L^p(\Omega)$;
- $u_k(x) \rightarrow u(x)$ a.e. in Ω ;

In particular, $J(u) = 1$ and $u(x) \geq 0$ a.e. Then, by weak lower semicontinuity of the norm,

$$Q(u) = I(u) \leq \liminf_k I(u_k) = \liminf_k Q(u_k) = \mu,$$

so $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $Q(u) = \mu$.

We have seen in the preceding subsection that I and J are differentiable. Then so is Q , and

$$Q'(u)v = \frac{1}{J(u)} (I'(u)v - Q(u)J'(u)v).$$

Since u is a minimum point for Q , we obtain $Q'(u) = 0$ and therefore

$$I'(u)v = Q(u)J'(u)v = \mu J'(u)v$$

for every $v \in W_0^{1,p}(\Omega)$, which is nothing but the weak form of (2.50). □

We give a last result, concerning existence of a minimum for a coercive functional involving the p -Laplacian. This result generalizes Theorem 2.1.5.

Theorem 2.6.12 *Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 3$, and let $1 < p < N$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that*

$$|f(t)| \leq a + b|t|^{q-1} \quad \forall t \in \mathbb{R},$$

where $a, b > 0$ and $1 \leq q < p$. Then, for all $h \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$, there exists a weak solution of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.51)$$

Proof We consider the functional $I : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx \\ &= \frac{1}{p} \|u\|^p - \int_{\Omega} F(u) dx - \int_{\Omega} hu dx, \end{aligned}$$

where as usual $F(t) = \int_0^t f(s) ds$. The hypotheses on f immediately imply that $F(t) \leq a_1 + b_1|t|^q$, and therefore, by the Hölder and Sobolev inequalities,

$$I(u) \geq \frac{1}{p} \|u\|^p - C_1 - C_2 |u|_q^q - C_3 |h|_{p'} |u|_p \geq C(\|u\|^p - \|u\|^q - \|u\| - 1),$$

which shows that I is coercive because $q < p$. Thus, any minimizing sequence u_k for I is bounded in $W_0^{1,p}(\Omega)$. Then we can extract a subsequence, still denoted u_k , that converges weakly to some $u \in W_0^{1,p}(\Omega)$; by compactness of the Sobolev embedding, we can make sure that it also converges strongly in $L^q(\Omega)$. Weak lower semicontinuity of the norm and the usual arguments show that the weak limit u is a minimum point for I . This gives $I'(u) = 0$, which is the weak form of (2.51). $\square$

2.7 Exercises

1. Let X be a complete normed space, and let $a : X \times X \rightarrow \mathbb{R}$ be a continuous bilinear form. Let $\phi : X \rightarrow \mathbb{R}$ be the associated quadratic form, $\phi(u) = a(u, u)$. Consider the following statements.

- (a) $\forall u \neq 0, \phi(u) > 0$.
- (b) $\lim_{\|u\| \rightarrow \infty} \phi(u) = +\infty$.
- (c) There exists $\alpha > 0$ such that for every $u \in X$, $\phi(u) \geq \alpha \|u\|^2$.

Prove that if X has finite dimension, then (a), (b) and (c) are equivalent. Prove that if X is infinite dimensional, then (c) $\iff$ (b) $\implies$ (a), but (a) needs not imply (b) or (c).

2. Let A be an $N \times N$ symmetric matrix. Prove that A has at least one eigenvalue by maximizing the function $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by $\phi(x) = Ax \cdot x$ constrained on the unit sphere of $\mathbb{R}^N$.
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy $f(t) = 0$ if $t \leq 0$. Assume that

$$t \mapsto \frac{f(t)}{t} \text{ is strictly decreasing on } (0, +\infty)$$

and denote $\alpha = \lim_{t \rightarrow 0^+} \frac{f(t)}{t}$ and $\beta = \lim_{t \rightarrow +\infty} \frac{f(t)}{t}$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Prove that if the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, then $\alpha > \lambda_1$ and $\beta < \lambda_1$. Hint: multiply the equation by φ_1 .

4. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be open and bounded, $h \in L^\infty(\Omega) \setminus \{0\}$ and $2 < p < 2^*$, and consider the problem

$$\begin{cases} -\Delta u = h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- (a) Prove that if $h(x) \leq 0$ a.e. in Ω , the problem has no nontrivial solutions.
- (b) If $h(x) > 0$ on a set of positive measure, prove that the problem has at least one nontrivial solution. Hint: consider the set

$$M = \left\{ u \in H_0^1(\Omega) \mid \int_{\Omega} h(x)|u|^p dx = 1 \right\},$$

prove that $M \neq \emptyset$ and argue as in Sect. 2.3.1.

5. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be open and bounded, and let $a \in L^\infty(\Omega)$ satisfy $a(x) > 0$ a.e. in Ω . Take $p \in (2, 2^*)$. Assume that the differentiable functional $Q : H_0^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ defined by

$$Q(u) = \frac{\int_{\Omega} |\nabla u|^2 dx}{(\int_{\Omega} a(x)|u|^p dx)^{2/p}}$$

has a critical point v . Show that a multiple of v is a weak solution of the problem

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hint: recall Example 1.3.16.

6. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be open and bounded, let $a, b \in C(\overline{\Omega}) \setminus \{0\}$ and let $2 < p < q < 2^*$. Assume that $b \geq 0$ in Ω and that for a.e. x , $a(x) \neq 0$ implies $b(x) \neq 0$. Consider the problem

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.52)$$

- (a) Write down an energy functional I such that $I'(u) = 0$ if and only if u is a weak solution for this problem.

(b) Prove that the set

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) \mid I'(u)u = 0, \int_{\Omega} b(x)|u|^q dx > 0 \right\}$$

is not empty.

(c) Prove that there exists a minimum point u for I on $\mathcal{N}$ such that $u \geq 0$.

(d) Prove that the minimum is a weak solution of (2.52).

7. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be open and bounded, and take numbers r, p such that $1 < r < 2 < p < 2^*$. Consider the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u - |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Define the energy functional and the Nehari manifold for this problem, and prove the existence of a non trivial solution via minimization on the Nehari manifold.

8. Let Ω be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 3$. Take $p \in (2, 2^*]$ and define

$$S_p = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2 dx}{(\int_{\Omega} |u|^p dx)^{2/p}}$$

This number is positive by the Sobolev embedding Theorem 1.2.1. Consider the functional $I : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx$$

and let $\mathcal{N}$ be the Nehari manifold associated to I . Prove that

$$\inf_{u \in \mathcal{N}} I(u) = \left(\frac{1}{2} - \frac{1}{p} \right) S_p^{\frac{p}{p-2}}.$$

9. Prove Theorem 2.6.6 by modifying the argument of Example 1.3.20.
10. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be open and bounded, and let $1 < p < N$ and $p < r < q < p^*$. Consider the p -Laplacian problem

$$\begin{cases} -\Delta_p u = |u|^{r-2}u + |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Define the energy functional and the Nehari manifold for this problem, and prove the existence of a non trivial solution via minimization on the Nehari manifold.

2.8 Bibliographical Notes

- Section 2.1: The techniques presented in this section apply to much more general cases, for instance to quasilinear problems. The direct methods of the Calculus of

Variations, built upon the two notions of coercivity and weak lower semicontinuity have been developed enormously and allow one to deal with general functionals depending on x , u and ∇u . See Chap. I.1 in Struwe [45], or Giusti [21] for a vast description of these problems.

- Section 2.2: The abstract structure behind it is a particular case of the Ky Fan–Von Neumann Theorem. The content of the section is a simplified version of a result from Kavian [26]. Min–maximization under convexity and concavity assumptions is one of central themes of *convex analysis*. A comprehensive treatment of it, with applications to differential problems, can be found in the book [19] by Ekeland and Temam.
- Section 2.3: The failure of the direct method (minimization) for problems with superlinear growth due to the fact that the functionals are not bounded from below led to the idea of constraints. Some problems in mechanics and geometry were the main motivations for the introduction of artificial constraints. The Nehari manifold was introduced in Nehari [36] in the context of ordinary differential equations. A modern review on superlinear Dirichlet problems is Bartsch, Wang, and Willem [8].
- Section 2.4: The question whether “perturbations” can destroy existence results has been widely analyzed in the last few years, especially when the unperturbed problem possesses infinitely many solutions (as is the case for odd nonlinearities). The interested reader can read Rabinowitz [43], or Chap. II.7 of Struwe [45] and the references therein. The reading however requires more sophisticated techniques than those described in these notes.
- Section 2.5: The method of minimization on the Nehari manifold is particularly useful for nonhomogeneous nonlinearities, for which minimization on spheres does not apply. A rather detailed exposition can be found in the books by Willem [48], and Kuzin–Pohožaev [27]; the applications given in [27] however concern problems on unbounded domains, which we treat in Chap. 3.
- Section 2.6: There is by now a vast literature on problems involving the p -Laplacian. A good starting point is Garcia Azorero and Peral [20]. A systematic treatment of many questions concerning the p -Laplacian can be found in the book by Lindqvist [30].

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