Chapter 2
Arbitrary Switching

2.1 Preliminaries

For switched dynamical systems, the switching may be induced by unpredictable change of system dynamics, such as a sudden change of system structure due to the failure of a component. In these cases, in order to keep the system working, it is necessary for the system to be stable under arbitrary switching. That is to say, the system should be stable under any possible switching law. Therefore, the stability is absolute or guaranteed regardless of the switching law. As the term "absolute stability" has been used to describe the global asymptotic stability for Lur'e systems with sector-bounded nonlinearities, here we use the term "guaranteed stability" to describe the stability of switched systems under arbitrary switching.

In this chapter, we consider the switched dynamical system given by

$$x^+(t) = f\big(x(t), \sigma(t)\big), \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the continuous state, $\sigma(t) \in M \stackrel{\text{def}}{=} \{1, \ldots, m\}$ is the discrete state, and $f : \mathbf{R}^n \times M \mapsto \mathbf{R}^n$ is a vector field with $f(\cdot, i)$ Lipschitz continuous for any $i \in M$.

It is clear that system (2.1) includes both continuous evolution and discrete elements. To emphasize the hybrid nature of the system, let us define new vector fields $f_i : \mathbf{R}^n \mapsto \mathbf{R}^n$ by

$$f_i(x) = f(x, i), \quad i \in M.$$

Then, the system can be rewritten as

$$x^+(t) = f_{\sigma(t)}\big(x(t)\big). \tag{2.2}$$

The issue of this chapter is the guaranteed stability analysis of switched dynamical system (2.2). For this, we assume that

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(1) $f_i(0) = 0$ for all $i \in M$, which implies that the origin is an equilibrium.
(2) The system is globally Lipschitz continuous, that is, there exists a positive constant L such that

$$\left|f_i(x) - f_i(y)\right| \le L|x - y| \quad \forall x, y \in \mathbf{R}^n,\ i \in M, \tag{2.3}$$

which guarantees the well-definedness of the switched system.

For clarity, we denote by $\phi(t; t_0, x_0, \sigma)$ the continuous state motion of system (2.2) at time t with initial condition $x(t_0) = x_0$ and switching path σ. By abuse of notation, we also use $\phi(t; x_0, \sigma)$ to denote the solution when $t_0 = 0$. The state evolution can be explicitly expressed in terms of the vector fields f_i, $i \in M$. Indeed, for any initial condition $x(t_0) = x_0$ and time $t > t_0$, in discrete time we have

$$\phi(t; t_0, x_0, \sigma) = f_{\sigma(t-1)} \circ \cdots \circ f_{\sigma(t_0+1)} \circ f_{\sigma(t_0)}(x_0), \tag{2.4}$$

where $\circ$ denotes the composition of functions, that is, $f_1 \circ f_2(x) \stackrel{\text{def}}{=} f_1(f_2(x))$. For continuous-time switched systems, the state evolution is

$$\phi(t; t_0, x_0, \sigma) = \Phi_{t-t_s}^{f_{i_s}} \circ \Phi_{t_s-t_{s-1}}^{f_{i_{s-1}}} \circ \cdots \circ \Phi_{t_2-t_1}^{f_{i_1}} \circ \Phi_{t_1-t_0}^{f_{i_0}}(x_0), \tag{2.5}$$

where $\Phi_t^f(x_0)$ denotes the value at t of the integral curve of f passing through $x(0) = x_0$, and $(t_0, i_0), \ldots, (t_s, i_s)$ is the switching sequence of σ in $[t_0, t)$. However, as the analytic expression of the curve $\Phi_t^f(x_0)$ is not available in general, the expression in (2.5) does not provide a sound basis for further analysis.

To present stability definitions for switched systems, we need more notation. Let $d(x, y)$ denote the Euclidean distance between vectors x and y. For a set $\Omega \subset \mathbf{R}^n$ and a vector $x \in \mathbf{R}^n$, let $|x|_\Omega = \inf_{y \in \Omega} d(x, y)$, and the normal norm $|x|_{\{0\}}$ is denoted by $|x|$ in short. For a set $\Omega \subset \mathbf{R}^n$ and a positive real number τ, let $\mathbf{B}(\Omega, \tau)$ be the τ-neighborhood of Ω, that is,

$$\mathbf{B}(\Omega, \tau) = \left\{x \in \mathbf{R}^n : |x|_\Omega \le \tau\right\}.$$

Similarly, let $\mathbf{H}(\Omega, \tau)$ be the τ-sphere of Ω, that is,

$$\mathbf{H}(\Omega, \tau) = \left\{x \in \mathbf{R}^n : |x|_\Omega = \tau\right\}.$$

In particular, the closed ball $\mathbf{B}(\{0\}, \tau)$ will be denoted by $\mathbf{B}_\tau$ in short, and the sphere $\mathbf{H}(\{0\}, \tau)$ by $\mathbf{H}_\tau$ in short.

Definition 2.1 The origin equilibrium for system (2.2) is said to be

(1) *guaranteed globally attractive* if

$$\lim_{t \to +\infty} \left|\phi(t; x, \sigma)\right| = 0 \quad \forall x \in \mathbf{R}^n,\ \sigma \in \mathcal{S}$$

(2) *guaranteed globally uniformly attractive* if for any $\delta > 0$ and $\epsilon > 0$, there exists $T > 0$ such that

$$\left|\phi(t; x, \sigma)\right| < \epsilon \quad \forall t \in \mathcal{T}_T,\ |x| \leq \delta,\ \sigma \in \mathcal{S}$$

(3) *guaranteed stable* if for any $\epsilon > 0$ and $\sigma \in \mathcal{S}$, there exists $\delta > 0$ such that

$$\left|\phi(t; x, \sigma)\right| \leq \epsilon \quad \forall t \in \mathcal{T}_0,\ |x| \leq \delta$$

(4) *guaranteed uniformly stable* if there exist $\delta > 0$ and a class $\mathcal{K}$ function γ such that

$$\left|\phi(t; x, \sigma)\right| \leq \gamma\big(|x|\big) \quad \forall t \in \mathcal{T}_0,\ |x| \leq \delta,\ \sigma \in \mathcal{S}$$

(5) *guaranteed globally asymptotically stable* if it is both guaranteed stable and guaranteed globally attractive
(6) *guaranteed globally uniformly asymptotically stable* if it is both guaranteed uniformly stable and guaranteed globally uniformly attractive
(7) *guaranteed globally exponentially stable* if for any $\sigma \in \mathcal{S}$, there exist $\alpha > 0$ and $\beta > 0$ such that

$$\left|\phi(t; x, \sigma)\right| \leq \beta e^{-\alpha t}|x| \quad \forall t \in \mathcal{T}_0,\ x \in \mathbf{R}^n$$

and
(8) *guaranteed globally uniformly exponentially stable* if there exist $\alpha > 0$ and $\beta > 0$ such that

$$\left|\phi(t; x, \sigma)\right| \leq \beta e^{-\alpha t}|x| \quad \forall t \in \mathcal{T}_0,\ x \in \mathbf{R}^n,\ \sigma \in \mathcal{S}$$

Note that the uniformity is referred to the switching signals rather than the initial time. By abuse of notation, we say that the system is guaranteed stable/attractive if the origin equilibrium is guaranteed stable/attractive. As we focus on the guaranteed stabilities with possible global attractivity in this chapter, the restrictive words "guaranteed" and "global" will be dropped for short in the sequel. Note also that the uniform asymptotic/exponential stability is consistent with that defined in Definition 1.1.

2.2 Switched Nonlinear Systems

In this section, we investigate the stability issue for switched nonlinear system

$$x^+(t) = f_{\sigma(t)}\big(x(t)\big) \tag{2.6}$$

under arbitrary switching. We assume that each vector field f_i is continuously differentiable.

2.2.1 Common Lyapunov Functions

The direct Lyapunov method provides a rigorous approach for studying stability of dynamical systems. Here, we review the preliminaries of Lyapunov stability theory.

A continuous function $V(x): \mathbf{R}^n \mapsto \mathbf{R}$ with $V(0) = 0$ is:

- *positive definite* ($V(x) \succ 0$) if $V(x) > 0\ \forall x \in \mathbf{R}^n - \{0\}$
- *positive semi-definite* ($V(x) \succeq 0$) if $V(x) \geq 0\ \forall x \in \mathbf{R}^n$
- *radially unbounded* if there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that $V(x) \geq \alpha(|x|)\ \forall x \in \mathbf{R}^n$

Definition 2.2 Let Ω be a neighborhood of the origin. A function $V: \Omega \mapsto \mathbf{R}$ is said to be a *common weak Lyapunov function* (CWLF) for switched system (2.6) if

(1) it is lower semi-continuous in Ω
(2) it admits class $\mathcal{K}$ bounds, that is, there are class $\mathcal{K}$ functions α_1 and α_2 such that

$$\alpha_1\big(|x|\big) \leq V(x) \leq \alpha_2\big(|x|\big) \quad \forall x \in \Omega$$

and
(3) the upper Dini derivative of V along each vector f_i is nonpositive; that is, for all $x \in \Omega$ and $i \in M$, we have

$$\mathcal{D}^+ V(x)|_{f_i} \overset{\text{def}}{=} \limsup_{\tau \to 0^+} \frac{V(\phi(\tau; 0, x, \hat{i})) - V(x)}{\tau} \leq 0$$

in continuous time, where $\hat{i}$ stands for the constant switching signal $\sigma(t) = i\ \ \forall t$, and

$$\mathcal{D}^+ V(x)|_{f_i} \overset{\text{def}}{=} V\big(f_i(x)\big) - V(x) \leq 0$$

in discrete time

Remark 2.3 Note that we do not require that the common weak Lyapunov function is continuous. As a matter of fact, even a uniformly stable nonlinear system $\dot{x} = f(x)$ with f being sufficiently smooth may not admit any continuous weak Lyapunov function [14].

Definition 2.4 A function $V: \mathbf{R}^n \mapsto \mathbf{R}$ is said to be a *common* (*strong*) *Lyapunov function* (CLF) for switched system (2.6) if

(1) it is continuous everywhere and continuously differentiable except possibly at the origin
(2) it admits class $\mathcal{K}_\infty$ bounds, that is, there are class $\mathcal{K}_\infty$ functions α_1 and α_2 such that

$$\alpha_1\big(|x|\big) \leq V(x) \leq \alpha_2\big(|x|\big) \quad \forall x \in \mathbf{R}^n,$$

and

(3) there is a class $\mathcal{K}$ function $\alpha_3 \colon \mathbf{R}^n \mapsto \mathbf{R}_+$ such that

$$\mathcal{D}^+ V(x)|_{f_i} \le -\alpha_3\bigl(|x|\bigr) \quad \forall x \in \mathbf{R}^n,\ i \in M \tag{2.7}$$

Remark 2.5 In continuous time, it can be seen that

$$\mathcal{D}^+ V(x)|_{f_i} = \limsup_{\tau \to 0^+} \frac{V(x + f_i(x)\tau) - V(x)}{\tau} \quad \forall x \in \mathbf{R}^n,\ i \in M, \tag{2.8}$$

due to the local Lipschitz continuity of V. For a continuously differentiable function V, its Dini derivative coincides with the (Lie) derivative of V along the vector field

$$\mathcal{D}^+ V(x)|_f = \frac{d}{dt} V(x) \stackrel{\text{def}}{=} L_f V(x) = \frac{\partial}{\partial x} V(x) f(x).$$

Suppose that system (2.6) admits a common weak Lyapunov function V. Then for any state trajectory $x(t) = \phi(t; x_0, \sigma)$ in Ω, we have $V(x(t)) \le V(x_0)$ for all $t \ge 0$. For any $\epsilon > 0$, choose δ such that

$$\mathbf{B}_\delta \subset \Omega, \quad \bigl\{x \colon V(x) \le \delta\bigr\} \subset \mathbf{B}_\epsilon.$$

Then, we have $|x(t)| \le \epsilon$ for all $t \ge 0$ if $x_0 \in \mathbf{B}_\delta$. This shows that the system is uniformly stable.

Next, suppose that system (2.6) admits a common Lyapunov function V. We are to show that the system is uniformly asymptotically stable. For this, we assume that the system is in continuous time, and the discrete-time case can be treated in a similar way. Fix an initial state $x_0 \ne 0$ and a switching signal σ, and denote $x(t) = x(t; x_0, \sigma)$. It follows from Definition 2.4 that

$$\limsup_{\tau \to 0^+} \frac{V(x(t+\tau)) - V(x)}{\tau} \le -\alpha_4\Bigl(V\bigl(x(t)\bigr)\Bigr) \quad \forall t \in \mathcal{T}_0, \tag{2.9}$$

where $\alpha_4 \stackrel{\text{def}}{=} \alpha_3 \circ \alpha_2^{-1}$. Define the function $\eta \colon \mathbf{R}^+ \mapsto \mathbf{R}$ by

$$\eta(t) = \begin{cases} -\int_1^t \frac{1}{\min(\tau, \alpha_4(\tau))}\, d\tau, & t \in (0,1), \\ -\int_1^t \frac{1}{\alpha_4(\tau)}\, d\tau, & t \ge 1. \end{cases}$$

It is clear that η is strictly decreasing, differentiable, and $\lim_{t \downarrow 0} \eta(t) = +\infty$. From (2.9) it can be seen that

$$\begin{aligned} \eta\Bigl(V\bigl(x(t)\bigr)\Bigr) - \eta\bigl(V(x_0)\bigr) &= \int_0^t \dot{\eta}\Bigl(V\bigl(x(\tau)\bigr)\Bigr)\, dV\bigl(x(s)\bigr) \\ &\ge \int_0^t 1\, ds = t \quad \forall t \ge 0. \end{aligned} \tag{2.10}$$

Define the function $\pi : \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ by

$$\pi(s,t) = \begin{cases} 0, & s = 0, \\ \eta^{-1}(\eta(s)+t), & s > 0, \end{cases}$$

which can be verified to be a $\mathcal{KL}$-function. As η is strictly decreasing, it follows from (2.10) that

$$V\big(x(t)\big) \le \pi\big(V(x_0),t\big) \quad \forall t \ge 0.$$

Define the other $\mathcal{KL}$-function β by

$$\beta(s,t) = \alpha_1^{-1}\big(\pi\big(\alpha_2(s),t\big)\big), \quad s,t \in \mathbf{R}_+.$$

It can be seen that

$$\big|x(t)\big| \le \beta\big(|x_0|,t\big) \quad \forall t \ge 0. \tag{2.11}$$

This, together with the fact that the origin is an equilibrium of the switched system, implies that the system is uniformly asymptotically stable. Indeed, to achieve uniform stability, for any $\epsilon > 0$, let $\delta = \bar{\beta}^{-1}(\epsilon)$, where $\bar{\beta}(\cdot) \overset{\text{def}}{=} \beta(\cdot,0)$. Then, $|\phi(t;0,x,\sigma)| \le \epsilon$ for any $x_0 \in \mathbf{B}_\delta$, $t \in \mathcal{T}_0$, and $\sigma \in \mathcal{S}$. Similarly, to achieve uniform attractivity, for any $\epsilon > 0$ and $\delta > 0$, let $T = \hat{\beta}^{-1}(\epsilon)$ where $\hat{\beta}(\cdot) \overset{\text{def}}{=} \beta(\delta,\cdot)$; then $|\phi(t;0,x,\sigma)| \le \epsilon$ for any $x_0 \in \mathbf{B}_\delta$, $t \in \mathcal{T}_T$, and $\sigma \in \mathcal{S}$.

To summarize, we have the following proposition.

Proposition 2.6 *Switched system* (2.6) *is uniformly stable if it admits a common weak Lyapunov function, and it is uniformly asymptotically stable if it admits a common Lyapunov function.*

Example 2.7 For the planar continuous-time two-form switched system with

$$\frac{d}{dt}x(t) = f_\sigma\big(x(t)\big),$$
$$f_1(x) = \begin{pmatrix} -x_2 \\ x_1 - x_2^k \end{pmatrix}, \qquad f_2(x) = \begin{pmatrix} x_2 \\ -x_1 - x_2^k \end{pmatrix}, \tag{2.12}$$

where k is any odd natural number, it can be verified that $V(x_1,x_2) = x_1^2 + x_2^2$ is a common weak Lyapunov function. By Proposition 2.6, the system is uniformly stable. However, the system does not admit any common Lyapunov function. Indeed, if the system admits a common Lyapunov function, then it follows that any convex combination system of the subsystems is asymptotically stable, that is, the dynamical system

$$\dot{x}(t) = \omega f_1\big(x(t)\big) + (1-\omega) f_2\big(x(t)\big)$$

is asymptotically stable for any $\omega \in [0,1]$. Let $\omega = \frac{1}{2}$. It can be verified that any state on the x_1-axis is an equilibrium. As a result, the convex combination system is not asymptotically stable. The contradiction exhibits that the switched system does not admit any common Lyapunov function.

2.2.2 Converse Lyapunov Theorem

While the existence of a common weak/strong Lyapunov function guarantees the uniform/asymptotic stability of the switched system, one may naturally ask whether the converse is also true. The answer is confirmative, as shown in the following proposition.

Proposition 2.8 *Any uniformly stable switched system admits a common weak Lyapunov function, and any uniformly asymptotically stable switched system admits a common Lyapunov function.*

Proof To prove the first half of the statement, suppose that the system is uniformly stable. Fix an $\epsilon > 0$, and let $\delta > 0$ be the number as in Definition 2.1. As in the standard Lyapunov method, define the function $V : \mathbf{B}_\delta \mapsto \mathbf{R}_+$ by

$$V(x) = \sup_{t \in \mathcal{T}_0, \sigma \in \mathcal{S}} \left|\phi(t; 0, x, \sigma)\right|. \tag{2.13}$$

It is clear that the function is well defined and positive definite in $\mathbf{B}_\delta$. Let $\Omega = \mathbf{B}_\delta^o$, where D^o denotes the interior of set D. We are to prove that V is a common weak Lyapunov function for the system.

To prove the lower semi-continuity of function V, fix an $x \in \mathbf{B}_\delta$. For any $\varepsilon > 0$, there exist a time $t_x \geq 0$ and $\sigma_x \in \mathcal{S}$ such that

$$\left|\phi(t_x; 0, x, \sigma_x)\right| \geq V(x) - \varepsilon.$$

Let $\eta = e^L$ in continuous time and $\eta = L$ in discrete time. It follows from the Lipschitz assumption (2.3) that

$$\begin{aligned} V(y) &\geq \left|\phi(t_x; 0, y, \sigma_x)\right| \geq \left|\phi(t_x; 0, x, \sigma_x)\right| - \left|\phi(t_x; 0, y, \sigma_x) - \phi(t_x; 0, x, \sigma_x)\right| \\ &\geq \left|\phi(t_x; 0, x, \sigma_x)\right| - \eta^{t_x}|x - y| \geq V(x) - 2\varepsilon \quad \forall y \in \mathbf{B}\left(x, \frac{\varepsilon}{\eta^{t_x}}\right) \cap \mathbf{B}_\delta. \end{aligned}$$

As a result, V is lower semi-continuous.

Next, for any $x \in \mathbf{B}_\delta$, $s \in \mathcal{T}_0$, and $i \in M$, we have

$$\begin{aligned} V\big(\phi(s; 0, x, \hat{i})\big) &= \sup_{t \in \mathcal{T}_0, \sigma \in \mathcal{S}} \left|\phi\big(t; 0, \phi(s; 0, x, \hat{i}), \sigma\big)\right| \leq \sup_{t \in \mathcal{T}_0, \sigma \in \mathcal{S}} \left|\phi(t + s; 0, x, \sigma)\right| \\ &\leq \sup_{(t+s) \in \mathcal{T}_0, \sigma \in \mathcal{S}} \left|\phi(t + s; 0, x, \sigma)\right| = V(x). \end{aligned}$$

It follows that $\mathcal{D}^+ V(x)|_{f_i} \leq 0$ for any $i \in M$.

Finally, it is clear that $V(x) \geq |x|$ for any $x \in \mathbf{B}_\delta$, and hence it admits a class $\mathcal{K}$ lower bound. On the other hand, the uniform stability means the existence of a class $\mathcal{K}$ function α_2 such that $|\phi(t; 0, x, \sigma)| \leq \alpha_2(|x|)$ for any $t \in \mathcal{T}_0$, $x \in \mathbf{B}_\delta$, and $\sigma \in \mathcal{S}$. As a result, $V(x) \leq \alpha_2(|x|)$ for all $x \in \mathbf{B}_\delta$.

The above reasonale shows that the function V defined in (2.13) is indeed a common weak Lyapunov function for the switched system.

The proof of the second half of Proposition 2.8 is involved as we have to smooth the above function while preserving its properties such as decreasing along the state trajectories. We will not go into the details; instead we cite the following support lemma, which is a simper version of the well-known Lin–Sontag–Wang converse Lyapunov theorem [152].

Lemma 2.9 *For the uncertain system*

$$x^{+}(t) = f\big(x(t), d(t)\big), \quad x(t) \in \mathbf{R}^n,\ d(t) \in \mathcal{D} \subset \mathbf{R}^p, \tag{2.14}$$

suppose that f is locally Lipschitz continuous in x uniformly in d, $\mathcal{D}$ is a compact set, and $d \in PC(\mathcal{T}_0, \mathcal{D})$, the set of piecewise continuous functions mapping from $\mathcal{T}_0$ to $\mathcal{D}$. Assume that the system is forward complete and the compact subset $\mathcal{X}$ of $\mathbf{R}^n$ is a forward invariant set in the sense that $x(t) \in \mathcal{X}$ for all $x_0 \in \mathcal{X}$ and $d \in PC(\mathcal{T}_0, \mathcal{D})$. If there is a $\mathcal{KL}$ function β such that

$$\big|x(t)\big|_{\mathcal{X}} \le \beta\big(|x_0|_{\mathcal{X}}, t\big) \quad \forall x_0 \in \mathbf{R}^n - \mathcal{X},\ t \ge 0,\ d \in PC(\mathcal{T}_0, \mathcal{D}), \tag{2.15}$$

then, there is a Lyapunov function V for system (2.14), i.e., V is continuous everywhere and infinitely differentiable on $\mathbf{R}^n - \mathcal{X}$, and there are $\mathcal{K}_\infty$ functions α_1, α_2 and a class $\mathcal{K}$ function α_3 such that

$$\alpha_1\big(|x|_{\mathcal{X}}\big) \le V(x) \le \alpha_2\big(|x|_{\mathcal{X}}\big)$$

and

$$L_f V(x) \le -\alpha_3\big(|x|_{\mathcal{X}}\big) \quad \forall x \in \mathbf{R}^n,\ d \in \mathcal{D}.$$

The lemma provides a general converse Lyapunov theorem, which is very important in the development of the stability and robust analysis and design. The proof of the lemma is quite technically involved, and the reader is referred to [124, 152] for details.

With the help of Lemma 2.9, the proof for the necessity of Proposition 2.8 is quite straightforward. Indeed, in Lemma 2.9, let $\mathcal{D} = M$ and $\mathcal{X} = \{0\}$. Then, Proposition 2.8 follows immediately from the lemma due to the fact that asymptotic stability implies (2.15). □

To summarize, in terms of the relationship between the uniform stability and the existence of a common Lyapunov function, the following conclusion can be drawn.

Theorem 2.10 *Switched system (2.6) is uniformly stable iff it admits a common weak Lyapunov function, and it is uniformly asymptotically stable iff it admits a common Lyapunov function.*

The theorem clearly brings the stability verification to the search of an appropriate common (weak) Lyapunov function and extends the conventional Lyapunov theory to the more general setting of switched systems. As in the conventional Lyapunov theory, there is generally no systematic way of finding the Lyapunov functions. Nevertheless, for some classes of systems with special structures or properties, the search of a Lyapunov function is tractable. We will come back to this topic in Sect. 2.3.5.

2.3 Switched Linear Systems

In this section, we focus on a special but very important class of switched systems where all the subsystems are linear time-invariant. These systems are termed as *switched linear systems* and are mathematically represented by

$$x^+(t) = A_{\sigma(t)}x(t), \qquad x(0) = x_0, \tag{2.16}$$

where $A_k \in \mathbf{R}^{n\times n}$, $k \in M$, are constant matrices.

Let $\mathbf{A} = \{A_1, \ldots, A_m\}$. $\mathbf{A}$ can be seen as the system matrix set for the switched linear system. For briefness, we term the switched linear system as system $\mathbf{A}$.

Due to the linear nature of the subsystems, the state solution can be given in an analytic way. In fact, the solution is given by

$$\phi(t; t_0, x_0, \sigma) = \Phi(t; t_0, \sigma)x_0, \tag{2.17}$$

where $\Phi(t; t_0, \sigma)$ is known to be the *state transition matrix*. In discrete time, the state transition matrix is

$$\Phi(t; t_0, \sigma) = A_{\sigma(t-1)} \cdots A_{\sigma(t_0)},$$

while in continuous time, it is

$$\Phi(t; t_0, \sigma) = e^{A_{i_s}(t-t_s)} e^{A_{i_{s-1}}(t_s-t_{s-1})} \cdots e^{A_{i_1}(t_2-t_1)} e^{A_{i_0}(t_1-t_0)},$$

where $t_0, t_1, \ldots, t_s$ and $i_0, i_1, \ldots, i_s$ are the switching time/index sequences in $[t_0, t)$, respectively.

It follows from the above expressions that

$$\phi(t; t_0, \lambda x_0, \sigma) = \lambda\phi(t; t_0, x_0, \sigma) \quad \forall t, t_0, x_0, \sigma,\ \forall \lambda \in \mathbf{R}, \tag{2.18}$$

which we term as the *radial linearity property*, and

$$\phi(t; t_0, x_0, \sigma) = \phi(t - t_0; 0, x_0, \sigma') \quad \forall t, t_0, x_0, \sigma, \tag{2.19}$$

where the switching path σ' is the time transition of σ, that is, $\sigma'(t) = \sigma(t + t_0)$ for all t. The latter property is known as the (*time*) *transition invariance property*.

Due to the above two invariance properties, for switched linear systems, it is clear that local attractivity implies (and is equivalent to) global attractivity, and the initial time can always be taken as $t_0 = 0$ without loss of generality.

2.3.1 Relaxed System Frameworks

In the stability analysis of switched linear systems, the switching signals can be arbitrary. Taking the switching mechanism as the uncertainty, the guaranteed stability requires that the system is robust w.r.t. the uncertainty. As the switching signals are piecewise constant taking values from a finite discrete set, it is natural to "smooth" them in some sense so that the conventional perturbation analysis approaches apply. This leads to the relaxed or extended system frameworks as follows.

Let

$$\mathcal{W} = \left\{ w \in \mathbf{R}^m : w_i \geq 0,\ i = 1, \ldots, m,\ \sum_{i=1}^{m} w_i \leq 1 \right\}$$

$$A(w) = \sum_{i=1}^{m} w_i A_i, \quad w \in \mathcal{W}, \tag{2.20}$$

and

$$\mathcal{A}(x) = \big\{ A(w)x : w \in \mathcal{W} \big\}, \quad x \in \mathbf{R}^n. \tag{2.21}$$

Let us consider the (*convex*) *differential/difference inclusion system*

$$x^+(t) \in \mathcal{A}\big(x(t)\big), \tag{2.22}$$

and the *polytopic linear uncertain system*

$$x^+(t) = A\big(w(t)\big)x(t), \tag{2.23}$$

where $w(\cdot) \in \mathcal{W}$ is a piecewise continuous function. For convenience, system (2.22) is called (*relaxed*) *differential/difference inclusion*. Note that both the differential/difference inclusion and the polytopic linear uncertain system can be connected to the switched system in a one-to-one manner, and the switched system can be seen as the extreme system of the others.

A solution of (2.22) is a vector flow $x : [0, +\infty) \mapsto \mathbf{R}^n$ with absolutely continuous entries that satisfies (2.22) almost everywhere. Solutions of the polytopic system can be understood in the same way.

For comparison, let Γ_s denote the set of solutions of the switched linear system, Γ_p the set of solutions of the polytopic system, and Γ_d the set of solutions of the differential inclusion system. It is readily seen that

$$\Gamma_s \subset \Gamma_p \subset \Gamma_d$$

and the first subset relationship is strict. However, under mild assumptions, each solution of the relaxed differential inclusion system can be approximated by a trajectory of the switched linear system in the sense specified below.

Lemma 2.11 (See [117]) *Fix $\xi \in \mathbf{R}^n$ and let $z\colon [0,+\infty) \mapsto \mathbf{R}^n$ be a solution of*

$$\dot{z}(t) \in \mathcal{A}\bigl(z(t)\bigr), \qquad z(0) = \xi.$$

Let $r\colon [0,+\infty) \mapsto \mathbf{R}$ be a continuous function satisfying $r(t) > 0$ for all $t \geq 0$. Then, there exist η with $|\eta - \xi| \leq r(0)$ and a solution $x\colon [0,+\infty) \mapsto \mathbf{R}^n$ of

$$\dot{x}(t) \in \bigl\{A_1 x(t), \dots, A_m x(t)\bigr\}, \qquad x(0) = \eta,$$

such that

$$\bigl|z(t) - x(t)\bigr| \leq r(t) \quad \forall t \in [0,+\infty).$$

This lemma sets up a connection between stability of a switched linear system and stability of its relaxed system. Indeed, suppose that each solution of the switched linear system is convergent; then, each solution of the differential inclusion (2.22) is also convergent. For discrete-time systems, the relationship also holds.

Corollary 2.12 *The following statements are equivalent*:

(1) *The switched linear system is attractive.*
(2) *The polytopic linear uncertain system is attractive.*
(3) *The differential inclusion system is attractive.*

For linear systems, it is well known that attractivity implies (and is equivalent to) exponential stability. For switched linear systems, it can be proven that the same property also holds as follows.

Proposition 2.13 *The following statements are equivalent*:

(1) *The switched linear system is attractive.*
(2) *The switched linear system is uniformly attractive.*
(3) *The switched linear system is asymptotically stable.*
(4) *The switched linear system is uniformly asymptotically stable.*
(5) *The switched linear system is exponentially stable.*
(6) *The switched linear system is uniformly exponentially stable.*

Proof First, it is clear that the uniform exponential stability implies any other stability in the proposition, and attractivity is implied by any other stability. As a result, we only need to prove that the attractivity implies the uniform exponential stability.

Second, for any state x on the unit sphere, it follows from the attractivity that there is a time t^x such that

$$\sup_{\sigma \in \mathcal{S}} \bigl|\phi\bigl(t^x; 0, x, \sigma\bigr)\bigr| < \frac{1}{2}. \tag{2.24}$$

For any fixed $x \in \mathbf{H}_1$, we are to prove that $\sup_{\sigma \in \mathcal{S}} |\phi(t^x; 0, y, \sigma)| \leq \frac{1}{2}$ if y is sufficiently close to x. Indeed, let $\eta = \max_{i \in M} |A_i|$. It follows from (2.24) and (2.17) that

$$\begin{aligned}\left|\phi\left(t^x;0,y,\sigma\right)\right| &= \left|\phi\left(t^x;0,x,\sigma\right)+\phi\left(t^x;0,y-x,\sigma\right)\right| \\ &\le \left|\phi\left(t^x;0,x,\sigma\right)\right|+\left|\phi\left(t^x;0,x-y,\sigma\right)\right| \\ &\le \left|\phi\left(t^x;0,x,\sigma\right)\right|+e^{\eta t^x}|y-x| \\ &\le \frac{1}{2} \quad \forall\sigma\in\mathcal{S},\ |y-x|\le e^{-\eta t^x}\left(\frac{1}{2}-\sup_{\sigma\in\mathcal{S}}\left|\phi\left(t^x;0,x,\sigma\right)\right|\right).\end{aligned}$$

This implies that, for any $x\in\mathbf{H}_1$, there is a neighborhood N_x of x such that

$$\sup_{\sigma\in\mathcal{S}}\left|\phi\left(t^x;0,y,\sigma\right)\right|\le\frac{1}{2} \quad \forall y\in N_x.$$

Third, letting x vary along the unit sphere, it is obvious that

$$\bigcup_{x\in\mathbf{H}_1} N_x\supseteq\mathbf{H}_1.$$

As the unit sphere is a compact set, by the Finite Covering Theorem, there exist a finite number l and a set of states $x_1,\dots,x_l$ on the unit sphere such that

$$\bigcup_{i=1}^{l} N_{x_i}\supseteq\mathbf{H}_1.$$

Accordingly, we can partition the unit sphere into l regions $R_1,\dots,R_l$ such that

(a) $\bigcup_{i=1}^{l} R_i=\mathbf{H}_1$, and $R_i\cap R_j=\emptyset$ for $i\ne j$; and
(b) for each i, $1\le i\le l$, $x_i\in R_i$, and

$$\sup_{\sigma\in\mathcal{S}}\left|\phi\left(t^{x_i};0,y,\sigma\right)\right|\le\frac{1}{2}, \quad \forall y\in R_i.$$

Define the cones

$$\Omega_i=\{x\in\mathbf{R}^n:\exists\lambda\ne 0 \text{ and } y\in R_i \text{ such that } x=\lambda y\}, \quad i=1,\dots,l.$$

Let $\Omega_0=\{0\}$. It can be seen that $\bigcup_{i=0}^{l}\Omega_i=\mathbf{R}^n$ and $\Omega_i\cap\Omega_j=\emptyset$ for $i\ne j$. In particular, Ω_0 is invariant under arbitrary switching and forms an invariant equilibrium of the system.

Fourth, for any $i=1,\dots,l$ and $x\in\Omega_i$, let $t_x=t^{x_i}$. It is clear that

$$\max_{x\ne 0} t_x=\max_{i=1}^{l} t^{x_i}\stackrel{\text{def}}{=}T_1<+\infty. \tag{2.25}$$

According to properties (a) and (b), for any $x\in\Omega_i$, $i=1,\dots,l$, any switching signal σ will bring x into the ball $\mathbf{B}_{\frac{|x|}{2}}$ at time t_x.

Finally, for any initial state x_0 and switching path σ, define recursively a sequence of times and states

$$
\begin{aligned}
s_0 &= 0, \\
z_0 &= x_0, \\
s_k &= s_{k-1} + t_{z_{k-1}}, \\
z_k &= \phi(s_k; 0, x_0, \sigma), \quad k = 1, 2, \ldots.
\end{aligned}
$$

It is readily seen that

$$
s_k \le kT_1, \qquad |z_{k+1}| \le \frac{|z_k|}{2}, \quad k = 0, 1, \ldots, \tag{2.26}
$$

which implies that

$$
\left|\phi(s_k; 0, x_0, \sigma)\right| \le \frac{|x_0|}{2^k} \le e^{-\gamma s_k}|x_0|, \quad k = 0, 1, 2, \ldots,
$$

where $\gamma \overset{\text{def}}{=} \frac{\ln 2}{T_1}$. On the other hand, let $\eta = 2\exp(T_1 \max\{\|A_1\|, \ldots, \|A_m\|\})$. Then, we have

$$
\left|\phi(t; 0, x_0, \sigma)\right| \le \eta e^{-\gamma t}|x_0| \quad \forall t \ge 0. \tag{2.27}
$$

Note that the inequality holds for any x_0 and σ, and the parameters γ and η are independent of x_0 and σ. This clearly shows that the system is uniformly exponentially stable. The proof is completed. □

Remark 2.14 While the theorem establishes a nice property for switched linear systems, the proof itself does not provide a constructive approach for calculating or estimating the convergence rate. For special classes of switched linear systems with additional structure information, however, it is possible to compute the convergence rates explicitly. This issue will be addressed in Sect. 2.3.5.

Simple observation exhibits that the above analysis can be slightly adopted to prove the equivalence (between attractivity and exponential stability) for the polytopic system and the differential inclusion system, respectively. Together with Corollary 2.12 and Theorem 2.10, we reach the following conclusion.

Theorem 2.15 *The following statements are equivalent for the switched linear system, the polytopic system, and the differential/difference inclusion system, respectively*:

(1) *The system is attractive.*
(2) *The system is asymptotically stable.*
(3) *The system is exponentially stable.*
(4) *The switched system admits a common Lyapunov function.*

Remark 2.16 The theorem bridges the stability analysis for various classes of systems with different backgrounds. Indeed, the stability analysis for polytopic systems, for linear differential inclusions, and for switched linear systems is mostly independent of each other until quite recently. The theorem assures that the stability criteria developed for one class of systems are also applied to the others. This greatly enriches the stability theory for the systems.

Theorem 2.15 allows us to define the stabilities in a more refined manner, as in the linear time-invariant case. For this, define

$$\varrho(\mathbf{A}) = \limsup_{t\to+\infty,\sigma\in\mathcal{S},|x|=1} \frac{\ln|\phi(t;0,x,\sigma)|}{t}, \tag{2.28}$$

which is the largest Lyapunov exponent that specifies the highest possible rate of state divergence, and

$$R(\mathbf{A}) = \left\{\phi(t;0,x,\sigma) : t \in \mathcal{T}_0,\ x \in \mathbf{H}_1,\ \sigma \in \mathcal{S}\right\}, \tag{2.29}$$

which is the attainability set of the system from the unit sphere.

Definition 2.17 A switched linear system $\mathbf{A}$ is said to be

(1) (*exponential*) *stable* if $\varrho(\mathbf{A}) < 0$
(2) *marginally stable* if $\varrho(\mathbf{A}) = 0$ and the set $R(\mathbf{A})$ is bounded
(3) *marginally unstable* if $\varrho(\mathbf{A}) = 0$ and the set $R(\mathbf{A})$ is unbounded
(4) (*exponentially*) *unstable* if $\varrho(\mathbf{A}) > 0$

It is clear that the notion of stability here is abused with the one defined for the nonlinear setting as it is referred to the situation of exponential convergence only. Similarly, the notion of instability is referred to exponential divergence only. In the sequel, unless otherwise stated, all the stability notions are referred in accordance with Definition 2.17.

2.3.2 Universal Lyapunov Functions

Through this subsection we assume, unless otherwise stated, that the switched linear system is stable. According to Theorem 2.10, the system admits a common Lyapunov function that is smooth ($\mathcal{C}^1$). As a result, the set of smooth positive definite functions is universal for switched linear systems. Here by universal Lyapunov functions we mean a set of functions such that each stable switched linear system admits a common Lyapunov function which belongs to the set. Due to the linear structure of the subsystems, it seems reasonable to expect a more restricted set of universal Lyapunov functions, e.g., the set of polynomials. In particular, as a stable linear time-invariant system always admits a quadratic Lyapunov function, it is natural to conjecture that the set of quadratic functions is also universal for switched linear systems. If so, then it is possible to develop a constructive approach for calculating a common quadratic Lyapunov function. Unfortunately, this conjecture finally was disproved through a counterexample, which exhibits that the stability analysis of switched linear systems is much more difficult than that of linear systems.

There are quite a few efforts in the literature to reveal the universal sets of common Lyapunov functions for switched linear systems or, equivalently, for polytopic systems or linear convex differential inclusions. Due to the nonsmooth nature of switched system, nonsmooth functions are also considered as Lyapunov function candidates.

The following theorem provides several sets of universal common Lyapunov functions for switched linear systems.

Theorem 2.18 *Each of the following function sets provides universal Lyapunov functions for stable switched linear systems.*

(1) *Convex and homogeneous functions of degree* 2.
(2) *Polynomials.*
(3) *Piecewise linear functions.*
(4) *Piecewise quadratic functions.*
(5) *Norms, that is, positive definite functions* $N: \mathbf{R}^n \mapsto \mathbf{R}_+$ *such that* $N(\lambda x) = |\lambda| N(x)$ *for any* $\lambda \in \mathbf{R}$ *and* $N(x+y) \le N(x) + N(y)$ *for any* $x, y \in \mathbf{R}^n$.

The key ideas of proving the existence of universal Lyapunov functions are outlined as follows. First, as in the conventional Lyapunov approach, for a stable switched linear system, we define the function $V: \mathbf{R}^n \mapsto \mathbf{R}_+$ by

$$V(x) = \sup_{\sigma \in \mathcal{S}} \int_0^{+\infty} \big|\phi(t; 0, x, \sigma)\big|^2 \, dt \tag{2.30}$$

in continuous time and

$$V(x) = \sup_{\sigma \in \mathcal{S}} \sum_{t=0}^{+\infty} \big|\phi(t; 0, x, \sigma)\big|^2 \tag{2.31}$$

in discrete time. It can be verified that the function is continuous, positive definite, strictly convex, and homogeneous of degree 2. In addition, for any nontrivial state trajectory $y(\cdot)$ of the switched system, we have

$$\begin{aligned} V\big(y(t_2)\big) &= \sup_{\sigma \in \mathcal{S}} \int_0^{+\infty} \big|\phi\big(t; 0, y(t_2), \sigma\big)\big|^2 \, dt \\ &= \sup_{\sigma \in \mathcal{S}} \int_0^{+\infty} \big|\phi\big(t + t_2 - t_1; 0, y(t_1), \sigma\big)\big|^2 \, dt \\ &= \sup_{\sigma \in \mathcal{S}} \int_{t_2 - t_1}^{+\infty} \big|\phi\big(t; 0, y(t_1), \sigma\big)\big|^2 \, dt \end{aligned}$$

$$
\begin{aligned}
&< \sup_{\sigma \in \mathcal{S}} \int_0^{+\infty} \left|\phi\big(t; 0, y(t_1), \sigma\big)\right|^2 dt \\
&= V\big(y(t_1)\big) \quad \forall t_2 > t_1,
\end{aligned}
$$

which clearly exhibits that the function V is strictly decreasing along the trajectory. Note that the function is continuous but may be nondifferentiable. The next step is to smooth the function by introducing the integral

$$\tilde{V}(x) = \int_{SO(n)} f(R)V(Rx)\,dR, \quad x \in \mathbf{R}^n,$$

where $SO(n)$ is the set of $n \times n$ orthogonal matrices with positive determinants, $f: SO(n) \mapsto \mathbf{R}_+$ is a smooth function with support in a small neighborhood of the identity matrix, and $\int_{SO(n)} f(R)\,dR = 1$. It can be shown that the function $\tilde{V}$ is continuously differentiable ($\mathcal{C}^1$) except possibly at the origin. A smooth function of $\mathcal{C}^k$ with any k can be obtained iteratively in the same manner. The newly defined function preserves the properties of convexity and homogeneity of degree 2. Moreover, the function strictly decreases along any nontrivial state trajectory of the switched system. By definition, the function is a common Lyapunov function of the system, and Item (1) of the theorem is established. The existence of other classes of universal Lyapunov functions can be guaranteed by the fact that a convex level set can be approximated to any degree by the level sets of polynomials, piecewise linear functions, and piecewise quadratic functions, respectively. We omit the technical details for briefness.

By the first statement of the theorem, there exists a common Lyapunov function of the form

$$V(x) = x^T P(x)x \quad \text{with } P(\lambda x) = P(x) > 0 \quad \forall \lambda \neq 0,\ x \neq 0.$$

Note that this function is in the quadratic form, and positive definite matrix $P(x)$ is homogeneous of degree zero; hence it is uniquely characterized by its image on the unit sphere. A special case is that $P(x)$ is independent of the state, which corresponds to a quadratic Lyapunov function. When this is the case, the search of an appropriate Lyapunov function can be reduced to solving the linear matrix inequalities (LMIs)

$$A_i^T P + P A_i < 0, \quad i = 1, \ldots, m, \tag{2.32}$$

which is computationally tractable.

A piecewise linear function is of the form

$$V(x) = \max\{l_i^T x : i = 1, \ldots, s\},$$

where l_i's are column vectors in $\mathbf{R}^n$. It is convex and positively homogeneous of degree 1. Its level set

$$\Gamma = \{x : l_i^T x \leq 1,\ i = 1, \ldots, s\}$$

is a convex and compact set containing the origin as an interior point. We call such a set *C-set*. The level set Γ is also a polyhedron, and it induces the gauge function, known as the *Minkowski function of* Γ, which is defined as

$$\mathrm{MF}_\Gamma(x) = \inf\{\mu \in \mathbf{R}_+ : x \in \mu\Gamma\},$$

where $\mu\Gamma = \{\mu y : y \in \Gamma\}$. By definition, function MF_Γ is exactly the function V, that is, $\mathrm{MF}_\Gamma(x) = V(x)$ for all $x \in \mathbf{R}^n$. The Minkowski function is a norm iff the level set is 0-symmetric, i.e., $x \in \Gamma$ implies $-x \in \Gamma$. The universal of piecewise linear functions as Lyapunov candidates implies that any stable switched linear system admits a polyhedral C-set as its attractive level set, that is, $D^+V(x) \le -\beta$ for any x on the boundary of the level set, where β is some positive real number. As a result, the stability verification reduces to the search of a proper attractive polyhedral C-set.

A particular and interesting situation is that the common Lyapunov function is a norm. In this case, it can be seen that the norm is exponentially contractive along any nontrivial state trajectory of the switched system.

A piecewise quadratic function is in the form

$$V_{\max}(x) = \max\left\{x^T P_i x : i = 1, \dots, s\right\},$$

where P_i's are symmetric and positive definite matrices. It is clear that the functions are positive definite and homogeneous of degree 2. A level set of $V_{\max}$ is an intersection of a number of ellipsoids and is strict convex. As a result, the function is also strictly convex and thus can be classified into the first class of the theorem.

As a corollary of Theorem 2.18, we have the following useful lemma that establishes the connection between a stable continuous-time system and its discrete-time Euler approximating system.

Lemma 2.19 *Suppose that the continuous-time switched linear system is stable. Then, the Euler approximating system defined as*

$$x(t+1) = (I_n + \tau A_\sigma)x(t) \tag{2.33}$$

is also stable for sufficiently small τ.

Proof The lemma can be proved based on the fact that each stable switched linear system admits a polyhedral common Lyapunov function, V. For the level set $\Gamma = \{x : V(x) \le 1\}$, define $\beta = \min_{x \in \partial\Gamma} \alpha_3(|x|)$, where α_3 is the class $\mathcal{K}$ function as in (2.7), and $\partial\Gamma$ stands for the boundary of set Γ. It follows from (2.7) that $\mathcal{D}^+V(x) \le -\beta$ for any $x \in \partial\Gamma$. This clearly indicates that the set Γ is also attractive for discrete-time system (2.33) when τ is sufficiently small, which in turn implies stability of the discrete-time system for a sufficiently small τ. □

Remark 2.20 An interesting question is whether or not the degrees of the common polynomial Lyapunov functions are uniformly bounded? In other word, does it exist a map $\chi : \mathbf{N}^+ \times \mathbf{N}^+ \mapsto \mathbf{N}^+$ such that each m-form attractive switched linear system of order n admits a common polynomial Lyapunov function of degree $\chi(n, m)$ or less? If the answer is confirmative, then, it is possible to numerically verify the stability by checking that, among all polynomials of degree $\chi(n, m)$ or less, whether there is a common Lyapunov function or not. Unfortunately, such a bound does not generally exist, even for planar switched linear systems. The reader is referred to [168] for detailed analysis.

Finally, we turn to marginal stability and propose a universal set of common weak Lyapunov functions.

Proposition 2.21 *A marginally stable switched linear system admits a norm as its common weak Lyapunov function.*

Proof We are to prove that the common weak Lyapunov function V defined in (2.13) is a norm. In fact, as

$$\begin{aligned} V(x) + V(y) &\geq \sup_{t \in \mathcal{T}_0, \sigma \in \mathcal{S}} \left\{ \left| \phi(t; x, \sigma) \right| + \left| \phi(t; y, \sigma) \right| \right\} \\ &\geq \sup_{t \in \mathcal{T}_0, \sigma \in \mathcal{S}} \left\{ \left| \phi(t; x, \sigma) + \phi(t; y, \sigma) \right| \right\} \geq V(x + y) \quad \forall x, y \in \mathbf{R}^n, \end{aligned}$$

V is convex. On the other hand, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$\left| \phi(t; 0, x, \sigma) \right| \leq \epsilon \quad \forall t \in \mathcal{T}_0, \ x \in \mathbf{B}_\delta, \ \sigma \in \mathcal{S}.$$

It follows from the radial linearity property (2.18) that $V(x) \leq \frac{\epsilon}{\delta}|x|$ for all $x \in \mathbf{R}^n$. This, together with the convexity, implies that

$$\left| V(x) - V(y) \right| \leq V(x - y) \leq \frac{\epsilon}{\delta}|x - y| \quad \forall x, y \in \mathbf{R}^n.$$

As a result, V is globally Lipschitz continuous. It is obvious that V is 0-symmetric, and positively homogeneous of degree one. Thus, it is indeed a norm. □

By Theorem 2.18 and Proposition 2.21, a stable (marginally stable) switched system always admits a norm as its common (weak) Lyapunov function.

A switched linear system $\mathbf{A}$ is said to be *regular* if $\varrho(\mathbf{A})$ is finite, where $\varrho(\mathbf{A})$ is the largest divergence rate defined in (2.28). It is clear that any continuous-time switched linear system is regular and that a discrete-time system is regular if $\varrho(\mathbf{A}) \neq -\infty$. For a continuous-time switched linear system $\mathbf{A} = \{A_1, \ldots, A_m\}$, the *normalized switched system* is the switched system $\underline{\mathbf{A}} = \{A_1 - \varrho(\mathbf{A})I_n, \ldots, A_m - \varrho(\mathbf{A})I_n\}$. Similarly, for a regular discrete-time system $\mathbf{A} = \{A_1, \ldots, A_m\}$, its normalized system is defined to be the switched system $\underline{\mathbf{A}} = \{A_1/e^{\varrho(\mathbf{A})}, \ldots, A_m/e^{\varrho(\mathbf{A})}\}$. It is clear that any normalized system is either marginally stable or marginally unstable.

2.3.3 Algebraic Criteria

For a linear time-invariant system, it is well known that the stability is characterized by the location of the poles. However, such a concise characteristic is still missing for switched linear systems. Nevertheless, much effort has been paid to find algebraic criteria for stability of the switched systems. In this subsection, we present some of the criteria which provide necessary and sufficient algebraic conditions.

For discrete-time systems, the stability is closely related to the convergence of the transition matrices, for which numerous algebraic criteria were developed mainly by researchers in the mathematics community. In the following, we first address the stability and then move to other related properties.

Suppose that $\mathbf{A} = \{A_1, \dots, A_m\}$ is a finite set of matrices in $\mathbf{R}^{n\times n}$. For $k \in \mathbf{N}^+$, denote by $\Pi_k(\mathbf{A})$ the set of length-k products of $\mathbf{A}$, that is,

$$\Pi_k(\mathbf{A}) = \{A_{i_1} \dots A_{i_k} : i_1, \dots, i_k \in M\},$$

and further

$$\Pi(\mathbf{A}) = \bigcup_{k\in\mathbf{N}^+} \Pi_k(\mathbf{A}),$$

which is the set of all products whose factors are elements of $\mathbf{A}$.

Let $|\cdot|$ be any induced norm, and $\rho(\cdot)$ be the spectral radius of a matrix. Define

$$\hat{\rho}_k(\mathbf{A}) = \max\bigl\{\|P\| : P \in \Pi_k(\mathbf{A})\bigr\}, \tag{2.34}$$

which represents the largest possible norm of all products of k matrices chosen in set $\mathbf{A}$. In particular, denote $\hat{\rho}_1(\mathbf{A})$ by $\|\mathbf{A}\|$. The *joint spectral radius* of $\mathbf{A}$ is then defined as

$$\hat{\rho}(\mathbf{A}) = \limsup_{k\to+\infty} \hat{\rho}_k(\mathbf{A})^{1/k}, \tag{2.35}$$

which is the maximal asymptotic norm of the products of matrices. It is clear that $\hat{\rho}(\mathbf{A})$ is norm-independent due to the equivalence of the norms. Analogously, define the number

$$\bar{\rho}_k(\mathbf{A}) = \max\bigl\{\rho(P) : P \in \Pi_k(\mathbf{A})\bigr\},$$

which represents the largest possible spectral radius of all products of k matrices in a set $\mathbf{A}$. Furthermore, define the *generalized spectral radius* of $\mathbf{A}$ as

$$\bar{\rho}(\mathbf{A}) = \limsup_{k\to+\infty} \bar{\rho}_k(\mathbf{A})^{1/k}, \tag{2.36}$$

which is the maximal asymptotic spectral radius of the products of matrices.

Recall that, for a matrix A, we have

$$\lim_{k\to+\infty} \|A^k\|^{1/k} = \lim_{k\to+\infty} \rho(A^k)^{1/k} = \rho(A).$$

Therefore, both the joint spectral radius and generalized spectral radius degenerate into the standard spectral radius when $\mathbf{A}$ is a singleton.

We are to present some properties of the joint spectral radius and generalized spectral radius. To this end, let $\varUpsilon$ be the set of vector norms in $\mathbf{R}^n$. For a set of matrices $\mathbf{A} = \{A_1, \dots, A_m\}$ and a norm $|\cdot| \in \varUpsilon$, define the (*induced*) *norm* of $\mathbf{A}$ to be

$$\|\mathbf{A}\| = \max_{x \neq 0, i \in M} |A_i x|/|x|.$$

The *least norm* of $\mathbf{A}$ is defined to be

$$\mathrm{LN}_{\mathbf{A}} = \inf_{|\cdot| \in \varUpsilon} \|\mathbf{A}\|.$$

A norm $\|\cdot\|^*$ is an *extreme norm* of $\mathbf{A}$ if

$$\|\mathbf{A}\|^* = \mathrm{LN}_{\mathbf{A}}\,.$$

For a real number μ, $\mu\mathbf{A}$ denotes the matrix set $\{\mu A_1, \dots, \mu A_m\}$.

Lemma 2.22 *Suppose that* $\mathbf{A}$ *is a set of real matrices. Then, the following statements hold.*

(1) $\bar{\rho}(\mathbf{A}) \le \hat{\rho}(\mathbf{A}) \le \mathrm{LN}_{\mathbf{A}}$.
(2) $\bar{\rho}(\mu\mathbf{A}) = |\mu|\bar{\rho}(\mathbf{A})$ *and* $\hat{\rho}(\mu\mathbf{A}) = |\mu|\hat{\rho}(\mathbf{A})$ $\forall \mu \in \mathbf{R}$.
(3) $\bar{\rho}(\mathbf{A}) < 1$ *implies the stability of* $\mathbf{A}$.

Proof First, note that $\rho(A) \le \|A\|$ for any norm $\|\cdot\|$. As a result, $\bar{\rho}_k(\mathbf{A}) \le \hat{\rho}_k(\mathbf{A})$, which implies that $\bar{\rho}(\mathbf{A}) \le \hat{\rho}(\mathbf{A})$. On the other hand, it follows from the norm submultiplicativity property that

$$\hat{\rho}(\mathbf{A}) \le \|\mathbf{A}\|,$$

which leads to the inequality $\hat{\rho}(\mathbf{A}) \le \mathrm{LN}_{\mathbf{A}}$. This proves the first statement.

The second statement trivially follows from the linearity of the spectral radius. To establish the third one, observe that $\bar{\rho}(\mathbf{A}) < 1$ implies that the state transition matrix $\Phi(t, 0, \sigma)$ approaches zero as t approaches infinity for any switching signal σ, which further implies that the switched linear system is attractive and hence stable. □

Based on the lemma and the converse Lyapunov theorem presented in Sect. 2.2.2, we are able to establish the equivalence among several fundamental indices.

Theorem 2.23 *For any switched linear system* $\mathbf{A}$, *we have*

$$\bar{\rho}(\mathbf{A}) = \hat{\rho}(\mathbf{A}) = \mathrm{LN}_{\mathbf{A}} = \exp\big(\varrho(\mathbf{A})\big). \tag{2.37}$$

Proof By Lemma 2.22, to establish $\bar{\rho}(\mathbf{A}) = \hat{\rho}(\mathbf{A}) = \mathrm{LN}_{\mathbf{A}}$, we only need to prove $\bar{\rho}(\mathbf{A}) \geq \mathrm{LN}_{\mathbf{A}}$. For this, suppose by contradiction that $\bar{\rho}(\mathbf{A}) < \mathrm{LN}_{\mathbf{A}}$. Fix a real number μ with $\bar{\rho}(\mathbf{A}) < \mu < \mathrm{LN}_{\mathbf{A}}$. Denote $B_i = A_i/\mu$ for $i = 1, \dots, m$, and further $\mathbf{B} = \{B_1, \dots, B_m\}$. It follows from Lemma 2.22 that

$$\bar{\rho}(\mathbf{B}) < 1 < \mathrm{LN}_{\mathbf{B}}. \tag{2.38}$$

Applying Lemma 2.22 once again, the switched linear system $\mathbf{B}$ is stable. By Theorem 2.18, there is a norm V that serves as a common Lyapunov function of the switched system, which is contractive. It is clear that

$$\mathrm{LN}_{\mathbf{B}} \leq \|\mathbf{B}\|_V \leq 1,$$

which contradicts inequality (2.38). This establishes

$$\bar{\rho}(\mathbf{A}) = \hat{\rho}(\mathbf{A}) = \mathrm{LN}_{\mathbf{A}}.$$

By the definition of the largest Lyapunov exponent as in (2.28), it is clear that

$$\exp\big(\varrho(\mathbf{A})\big) \leq \mathrm{LN}_{\mathbf{A}}.$$

Using a similar idea as in the former part of the proof, we arrive at the conclusion that the equality relation must hold. This completes the proof. □

The equality between the generalized spectral radius and the joint spectral radius allows us to term the quantity as the *spectral radius* of matrix set $\mathbf{A}$, denoted $\rho(\mathbf{A})$.

Corollary 2.24 *The discrete-time switched linear system is stable iff its spectral radius is less than one. It is unstable iff its spectral radius is greater than one.*

The corollary provides a new criterion for stability of a switched linear system in terms of the spectral radius, which extends the well-known spectral radius criterion for linear time-invariant systems. A semi-decidable verification procedure can be developed based on the criterion, which will be presented in Sect. 2.4.

Proposition 2.25 *A discrete-time switched linear system* $\mathbf{A}$ *is marginally stable iff it admits an extreme norm with* $\|\mathbf{A}\|^* = 1$.

Proof By Proposition 2.21, marginal stability implies the existence of a common weak Lyapunov norm V. From the definitions for the common (weak) Lyapunov function, we have $V(A_i x) \leq V(x)$ for all $i \in M$ and $x \in \mathbf{R}^n$, which implies that $\|\mathbf{A}\|_V \leq 1$. If $\mathrm{LN}_{\mathbf{A}} < 1$, then the switched system is asymptotically stable, which contradicts the assumption of marginal stability. As a result, we have $\mathrm{LN}_{\mathbf{A}} \geq 1$. As $\mathrm{LN}_{\mathbf{A}} \leq \|\mathbf{A}\|_V$, we have $\mathrm{LN}_{\mathbf{A}} = \|\mathbf{A}\|_V = 1$, which clearly shows that $\|\cdot\|_V$ is an extreme norm for the switched system.

Conversely, suppose that the switched system admits an extreme norm $\|\cdot\|$ with $\|\mathbf{A}\| = \mathrm{LN}_{\mathbf{A}} = 1$. It is clear that the system is either stable or marginally stable. If the system is stable, then, there is a common Lyapunov norm V_0 such that

$$V_0(A_i x) - V_0(x) \leq -\omega(x) \quad \forall x \in \mathbf{R}^n, \ i \in M,$$

where ω is a continuous positive definite function. Let $\beta = \min_{V_0(x)=1} \omega(x)$. It follows that

$$V_0(A_i x) - V_0(x) \leq -\beta V_0(x) \quad \forall x \in \mathbf{R}^n, \ i \in M,$$

which further implies that $\|\mathbf{A}\|_{V_0} \leq 1 - \beta$, a contradiction. Therefore, the switched system must be marginally stable. □

Next, we move to the case of continuous time. For any norm $|\cdot|$ in $\mathbf{R}^n$, the *induced matrix measure* on $\mathbf{R}^{n \times n}$ is defined as

$$\mu_{|\cdot|}(A) = \limsup_{\tau \to 0^+, |x|=1} \frac{|x + \tau A x| - |x|}{\tau}. \tag{2.39}$$

It is clear that the matrix measure possesses the following properties (see, e.g., [253]. The subscript $|\cdot|$ is dropped for briefness):

(1) *Well-definedness.* The matrix measure is well defined for any vector norm.
(2) *Positive homogeneousness.* $\mu(\alpha A) = \alpha \mu(A)$ for all $\alpha \geq 0$ and $A \in \mathbf{R}^{n \times n}$.
(3) *Convexity.* $\mu(\alpha A + (1-\alpha)B) \leq \alpha \mu(A) + (1-\alpha)\mu(B)$ for all $\alpha \in [0, 1]$ and $A, B \in \mathbf{R}^{n \times n}$.
(4) *Exponential estimation.* $|e^{At} x| \leq e^{\mu(A)t} |x|$ for all $t \geq 0, \ x \in \mathbf{R}^n$, and $A \in \mathbf{R}^{n \times n}$.

The definition of the matrix measure is extendable to a set of matrices $\mathbf{A} = \{A_1, \ldots, A_m\}$ as follows. For a set of matrices $\mathbf{A} = \{A_1, \ldots, A_m\}$ and a norm $|\cdot|$ in $\mathbf{R}^n$, the (*induced*) *measure* of $\mathbf{A}$ w.r.t. $|\cdot|$ is defined as

$$\mu_{|\cdot|}(\mathbf{A}) = \max\bigl\{\mu_{|\cdot|}(A_1), \ldots, \mu_{|\cdot|}(A_m)\bigr\}. \tag{2.40}$$

It can be verified that the measure possesses the positive homogeneity and convexity properties as the matrix measure. As for the exponential estimation property, it can be seen that

$$\bigl|\phi(t; 0, x, \sigma)\bigr| \leq e^{\mu_{|\cdot|}(\mathbf{A})t} |x| \quad \forall t \geq 0, \ x \in \mathbf{R}^n, \ \sigma \in \mathcal{S}. \tag{2.41}$$

Furthermore, for a set of matrices $\mathbf{A} = \{A_1, \ldots, A_m\}$, define the *least measure value* as

$$\nu(\mathbf{A}) = \inf_{|\cdot| \in \Upsilon} \mu_{|\cdot|}(\mathbf{A}). \tag{2.42}$$

Any matrix set measure μ with $\mu(\mathbf{A}) = \nu(\mathbf{A})$ is said to be an *extreme measure* for $\mathbf{A}$. It is well known that the switched linear system $\mathbf{A}$ is (exponentially) stable if there exists a norm $|\cdot|$ such that its matrix measure is negative. As an implication, when the least measure is negative, then the switched system is exponentially stable. In the following, we are to establish that the converse is also true.

Theorem 2.26 *For any continuous-time switched linear system* $\mathbf{A}$, *we have*

$$\nu(\mathbf{A}) = \varrho(\mathbf{A}). \tag{2.43}$$

In addition, the switched system admits (at least) one extreme measure iff its normalized system is marginally stable.

Proof Firstly, it can be verified that

$$\mu_{|\cdot|}(\mathbf{A} + \lambda I_n) = \lambda + \mu_{|\cdot|}(\mathbf{A}) \quad \forall |\cdot| \in \Upsilon, \ \lambda \in \mathbf{R},$$

where $\mathbf{A} + \lambda I_n$ denotes the switched linear system $(A_1 + \lambda I_n, \ldots, A_m + \lambda I_n)$. As a result, we have

$$\nu(\mathbf{A} + \lambda I_n) = \lambda + \nu(\mathbf{A}) \quad \forall \lambda \in \mathbf{R}.$$

This, together with the fact that

$$\varrho(\mathbf{A} + \lambda I_n) = \lambda + \varrho(\mathbf{A}) \quad \forall \lambda \in \mathbf{R},$$

indicates that (2.43) holds for general case if it holds when $\varrho(\mathbf{A}) = 0$, that is, the switched system is either marginally stable or marginally unstable.

Secondly, suppose that the system is either stable or marginally stable. Fix a vector norm $|\cdot|$ in $\mathbf{R}^n$ and define the function $V : \mathbf{R}^n \mapsto \mathbf{R}_+$ by

$$V(x) = \sup_{t \in \mathbf{R}_+, \sigma \in \mathcal{S}} \left|\phi(t; 0, x, \sigma)\right|. \tag{2.44}$$

It can be seen that the function is well defined, positive definite, convex, 0-symmetric, and positively homogeneous of degree one. In addition, there is a positive real number L such that

$$|x| \le V(x) \le L|x| \quad \forall x \in \mathbf{R}^n.$$

This, together with the radial linearity property (2.18), implies that V is globally Lipschitz continuous. As a result, the function V in fact forms a vector norm of $\mathbf{R}^n$.

Thirdly, for any $x \in \mathbf{R}^n$, $s \in \mathbf{R}_+$, and $i \in M$, we have

$$\begin{aligned} V\big(\phi(s; 0, x, \hat{i})\big) &= \sup_{t \in \mathbf{R}_+, \sigma \in \mathcal{S}} \left|\phi\big(t; 0, \phi(s; 0, x, \hat{i}), \sigma\big)\right| \\ &\le \sup_{t \in \mathbf{R}_+, \sigma \in \mathcal{S}} \left|\phi(t + s; 0, x, \sigma)\right| \\ &\le \sup_{(t+s) \in \mathbf{R}_+, \sigma \in \mathcal{S}} \left|\phi(t + s; 0, x, \sigma)\right| \\ &= V(x), \end{aligned}$$

where $\hat{i}$ stands for the switching signal $\hat{i}(t) \equiv i$. As V is Lipschitz continuous, we further have

$$\begin{aligned}\mu_V(A_i) &= \limsup_{\tau \to 0^+, x \neq 0} \frac{V(x + \tau A_i x) - V(x)}{\tau V(x)} \\ &= \limsup_{\tau \to 0^+, x \neq 0} \frac{V(\phi(\tau; 0, x, \hat{i})) - V(x)}{\tau V(x)} \\ &\leq 0 \quad \forall i \in M.\end{aligned}$$

That is, the norm V induces a matrix set measure that satisfies

$$\mu_V(\mathbf{A}) \leq 0, \tag{2.45}$$

which further implies that $\nu(\mathbf{A}) \leq 0$.

Fourthly, we focus on the situation that the switched system is marginally stable. We claim that the least measure value is exactly zero. Indeed, if it is negative, then, it follows from relationship (2.41) that the system is (exponentially) stable, which yields a contradiction. This means that the matrix set measure in (2.45) is an extreme measure.

Finally, consider the case that the switched system is marginally unstable. It is clear that the least measure value is nonnegative. Assume that it is positive. Then, there is a positive real number ϵ with $\epsilon < \nu(\mathbf{A})$ such that

$$\nu_{|\cdot|}(\mathbf{A} - \epsilon I_n) > 0,$$

which further means that the switched system $\mathbf{A} - \epsilon I_n$ is either unstable or marginally unstable. This is a contradiction since $\varrho(\mathbf{A} - \epsilon I_n) = -\epsilon < 0$. The contradiction means that the least measure value is zero. On the other hand, when the least measure value is zero, the existence of an extreme measure implies the boundedness of the attainability set $R(\mathbf{A})$, and hence the switched system is either stable or marginally stable. As a result, a marginally unstable switched system does not admit any extreme measure.

To summarize, marginal stability implies the existence of an extreme measure of zero value, and marginal instability implies zero least measure value but does not admit any extreme measure. As the normalization does not alter the existence of an extreme measure, the second statement of the theorem follows. □

Remark 2.27 For a real matrix, its largest divergence rate is the largest real part of its eigenvalues, which is exactly the least measure value. This fact was pointed out in [274]. Theorem 2.26 extends the fact to the case of switched linear systems, though the concept of system spectrum is missing for switched linear systems.

Remark 2.28 The function V in the proof is in fact a (weak) common Lyapunov function for the switched linear system (cf. Proposition 2.21). The observation that it serves as a vector norm is crucial in the development.

With the help of Theorem 2.26, we can fully characterize the stabilities in terms of matrix set measure.

Corollary 2.29 *For a continuous-time switched linear system, we have the following statements*:

(1) *The system is stable iff its least measure value is negative.*
(2) *The system is marginally stable iff its least measure value is zero and it admits an extreme measure.*
(3) *The system is marginally unstable if and only if its least measure value is zero and it does not admit any extreme measure.*
(4) *The systems is unstable iff its least measure value is positive.*

2.3.4 Extended Coordinate Transformation and Set Invariance

In linear systems theory, coordinate transformation and system equivalence are powerful tools in stability analysis. For switched linear systems, coordinate transformation also plays an important role in converting a switched system into a new one with clearer and/or simpler structural information, which enables us to analyze the stability properties in a more convenient manner. However, the standard notion of equivalence coordinate change does not directly work, and we need to extend the notion in a way that it is capable of rigorous stability analysis.

Definition 2.30 Suppose that $T \in R^{n\times r}$ is a constant matrix. The linear coordinate change $x = Ty$ is said to be an *extended coordinate transformation* for the switched linear system if the matrix T is of full row rank, that is, rank $T = n$.

It is clear that the transforming matrix could be nonsquare in that the number of columns might be larger than that of rows. Under the extended coordinate change, the switched system is converted into

$$y^+ = T^+ A_\sigma T y, \tag{2.46}$$

where T^+ denotes the Moore–Penrose pseudo-inverse of a matrix T. Note that the transformed system is r-dimensional. The system is said to be the *extended transformed system* of the original switched linear system. Note that the process of extended coordinate transformation is nonreversible, that is, the original system is not necessarily an extended transformed system of (2.46).

Recall that a matrix $A = (a_{i,j})$ in $\mathbf{R}^{n\times n}$ is said to be *strictly* (*column*) *negatively diagonal dominant* if

$$a_{ii} + \sum_{j\neq i} |a_{ji}| < 0, \quad i = 1, \ldots, n.$$

Note that this can be equivalently characterized by $\mu_1(A) < 0$, where μ_1 is the matrix measure corresponding to the ℓ_1-norm.

Theorem 2.31 *The switched linear system is stable iff there exists an extended coordinate transformation such that the extended transformed system admits a contractive* 1*-norm in discrete time or a negative* 1*-measure in discrete time.*

Proof Note that, if the extended transformed system admits a contractive 1-norm in discrete time or a negative 1-measure in discrete time, then the extended transformed system is stable. It follows from $x = Ty$ that the original switched system is stable. To establish the converse relationship, we first consider the discrete-time case. As the system is stable, it follows that it is also exponentially convergent. Therefore, there is a piecewise linear function that serves as a common Lyapunov function, $V(x) = |Fx|_\infty$. The Lyapunov function can be rewritten in the dual representation

$$V(x) = \min\{|h|_1 : x = Xh,\ h \in \mathbf{R}^r\},$$

where X is a matrix of full row rank. The duality relationship comes from the fact that the set of column vectors of $[X, -X]$ is the set of vertices of the level set $\Gamma = \{x \in \mathbf{R}^n : V(x) \le 1\}$. Let $\{x_j : j = 1, \ldots, s\}$ be the set of vertices of Γ. By the symmetry of Γ, we have that $s = 2r$, and we can reindex the vertices such that $x_{k+r} = -x_k$ for $k = 1, \ldots, r$. As the level set is attractive w.r.t. the switched system, there is $\lambda \in [0, 1)$ such that

$$A_i x_j \in \lambda\Gamma \quad \forall i \in M,\ j = 1, \ldots, s. \tag{2.47}$$

This is equivalent to the existence of vectors p_j^i with $|p_j^i|_1 \le \lambda$ such that

$$A_i x_j = X p_j^i \quad \forall i \in M,\ j = 1, \ldots, r. \tag{2.48}$$

Define

$$P_i = [p_1^i, \ldots, p_r^i], \quad i = 1, \ldots, m.$$

It is clear that $\|P_i\|_1 \le \lambda$. Equations (2.48) can be rewritten as

$$A_i X = X P_i, \quad i \in M,$$

which leads to the first statement of the theorem.

The case of continuous time can be treated based on the above reasonale for discrete time. In fact, by Lemma 2.19, there always exists a sufficiently small positive real number τ such that the discrete-time Euler approximating system

$$x(t+1) = (I_n + \tau A_\sigma) x(t)$$

is also stable. This, together with the fact that the extended transformed system admits a contractive 1-norm, yields

$$(I_n + \tau A_i) X = X P_i, \quad i = 1, \ldots, m \tag{2.49}$$

for some full row rank matrix $X \in \mathbf{R}^{n\times r}$ and matrices P_i with $\|P_i\|_1 < 1$. It is clear that (2.49) can be equivalently expressed by

$$A_i X = X(P_i - I_n)/\tau, \quad i = 1, \dots, m,$$

which directly leads to the conclusion with $H_i = (P_i - I_n)/\tau$. □

Remark 2.32 It is interesting to note that, for a linear time-invariant system $x^+ = Ax$, the criterion degenerates into the matrix relation $AX = XP$ in discrete time and $AX = XH$ in continuous time, where X is a matrix of full row rank, P is a square matrix with $\|P\|_1 < 1$, and H is a strictly negatively diagonal dominant matrix. This criterion for stability contains interesting information as discussed below. The equality $AX = XP$ can be reasonably seen as the generalized similarity between A and P. In this sense, a matrix is Hurwitz iff it is generalized similar to a strictly negatively diagonal dominant matrix. For the switched linear system, it is stable iff the subsystem matrices are simultaneously generalized similar to strictly negatively diagonal dominant matrices. In discrete time, it is interesting to note that any stable switched system is generalized similar to a system which admits the norm $|\cdot|_1$ as its common Lyapunov function.

To further identify the subtle properties of marginal stability and marginal instability, we take a view of invariant sets that start from an origin-symmetric polyhedron Λ_0 which contains the origin as an interior point. An example of Λ_0 is the polyhedron with extreme points whose entries are either 1 or -1. Define the set

$$\Lambda_\infty = \{x \in \mathbf{R}^n : \phi(t; 0, x, \sigma) \in \Lambda_0 \ \forall t \in \mathcal{T}_0, \ \sigma \in \mathcal{S}\}. \tag{2.50}$$

It can be seen that Λ_∞ is the largest (positively) invariant set contained in Λ_0 for the switched system.

Proposition 2.33 *The following statements hold*:

(1) *If switched linear system* $\mathbf{A}$ *is stable or marginally stable, then,* Λ_∞ *contains the origin as an interior point, and* $\Lambda_\infty \cap \partial\Lambda_0 \neq \emptyset$, *where* $\partial\Lambda_0$ *is the boundary of* Λ_0. *Conversely, if* Λ_∞ *contains the origin as an interior point, then, the system is stable or marginally stable, and* $\Lambda_\infty \cap \partial\Lambda_0 \neq \emptyset$.
(2) *If the system is marginally unstable, then,* $\Lambda_\infty \cap \partial\Lambda_0 \neq \emptyset$, *and* $\Lambda_\infty \subset \Lambda_0 \cap H$, *where* H *is a nontrivial subspace of* $\mathbf{R}^n$.
(3) *The switched system is unstable if* $\Lambda_\infty = \{0\}$.

Proof It is clear that Λ_∞ is origin-symmetric. Furthermore, it can be seen that it is convex and closed. Another observation is that, as Λ_∞ is an invariant set for the system, $\lambda\Lambda_\infty$ is also an invariant set for any $\lambda \geq 0$. Therefore, $\Lambda_\infty \cap \partial\Lambda_0 \neq \emptyset$ iff $\Lambda_\infty \neq \{0\}$.

For the first statement, note that the (marginally) stable system admits a common weak Lyapunov function, V. Define

$$v = \max\{r \in \mathbf{R} : V(x) \leq r \Rightarrow x \in \Lambda_0\},$$

$$\Lambda_V = \left\{x \in \mathbf{R}^n : V(x) \leq v\right\}.$$

As Λ_V is an invariant set contained in Λ_0, we have that $\Lambda_\infty \supset \Lambda_V$, and thus it contains the origin as an interior point. Conversely, suppose that Λ_∞ contains the origin as an interior point. As Λ_∞ is invariant w.r.t. the system, the system is stable or marginally stable.

For the third statement, suppose that $\Lambda_\infty = \{0\}$. Define

$$\Lambda_t = \left\{x \in \mathbf{R}^n : \phi(s; 0, x, \sigma) \in \Lambda_0 \ \forall s \in [0, t],\ \sigma \in \mathcal{S}\right\}, \quad t \in \mathcal{T}_0,$$

and further

$$\eta(t) = \sup\left\{|x| : x \in \Lambda_t\right\}, \quad t \in \mathcal{T}_0.$$

It can be seen that the function η is decreasing and approaching to zero. As the function is continuous, there is a time $\tau \in \mathcal{T}_0$ such that

$$\eta(\tau) \leq \frac{1}{2} \inf\{x : x \in \partial \Lambda_0\}.$$

That is, for any $\lambda \leq 2$, we have $\lambda \Lambda_\tau \subset \Lambda_0$. Define

$$\Lambda_\tau^\lambda = \left\{x \in \mathbf{R}^n : \phi^\lambda(s; 0, x, \sigma) \in \Lambda_0 \ \forall s \in [0, \tau],\ \sigma \in \mathcal{S}\right\},$$

where ϕ^λ denotes the state for switched system $\{(1-\lambda)A_1, \ldots, (1-\lambda)A_m\}$ in discrete time and for $\{A_1 - \lambda I_n, \ldots, A_m - \lambda I_n\}$ in continuous time, where I_n is the nth-order identity matrix. It can be seen that $\Lambda_\tau^\lambda \subset (\Lambda_0)^o$ when λ is a sufficiently small positive real number. This means that the above matrix set is neither stable nor marginal stable, which further implies the instability of the original system.

Finally, we prove the second statement. As $\Lambda_\infty = \{0\}$ implies instability as proved previously, $\Lambda_\infty \neq \{0\}$ for marginal instability. As Λ_∞ is convex, 0-symmetric, and containing the origin as a boundary point (w.r.t. $\mathbf{R}^n$), it induces the nontrivial subspace H of $\mathbf{R}^n$ given by

$$H \stackrel{\text{def}}{=} \bigcup_{\lambda \geq 0} \lambda \Lambda_\infty. \tag{2.51}$$

It is clear that $\Lambda_\infty \subset H$, and this completes the proof. □

A byproduct of Proposition 2.33 is the following system decomposition lemma.

Lemma 2.34 *A marginally unstable switched linear system is simultaneously block-triangularizable. That is, there exist a nonsingular matrix T and two positive integers n_1, n_2 with $n_1 + n_2 = n$ such that*

$$\bar{A}_i \stackrel{\text{def}}{=} T^{-1} A_i T = \begin{bmatrix} \bar{A}_i^1 & \bar{A}_i^3 \\ 0 & \bar{A}_i^2 \end{bmatrix}, \ i \in M, \tag{2.52}$$

where $\bar{A}_i^1$ and $\bar{A}_i^2$ are $n_1 \times n_1$ and $n_2 \times n_2$, respectively. In addition, both $\bar{\mathbf{A}}^1 = \{\bar{A}_1^1, \ldots, \bar{A}_m^1\}$ and $\bar{\mathbf{A}}^2 = \{\bar{A}_1^2, \ldots, \bar{A}_m^2\}$ are marginally stable as switched linear systems of dimensions n_1 and n_2, respectively.

Proof Let H be the subspace defined in (2.51), and let further $n_1 = \dim H$ and $n_2 = n - n_1$. The nontriviality of H implies that $1 \le n_1 < n$. Let $H^\perp$ be the subspace orthogonal to subspace H with $H \oplus H^\perp = \mathbf{R}^n$. Let T be a nonsingular matrix whose first n_1 columns belong to H and the others belong to $H^\perp$. The block-triangular structure of (2.52) comes from the fact that H is A_i-invariant for all $i \in M$. Indeed, for any $x \in H$, there exist $y \in \varLambda_\infty$ and $\lambda \in \mathbf{R}_+$ such that $x = \lambda y$. As $\varLambda_\infty$ is A_i-invariant, we have

$$A_i x = \lambda A_i y \in \lambda \varLambda_\infty \subset H.$$

It is clear that the switched system $\bar{\mathbf{A}}^1 = \{\bar{A}_1^1, \ldots, \bar{A}_m^1\}$ is either stable or marginally stable, and the marginal stability of the switched system $\bar{\mathbf{A}}^2 = \{\bar{A}_1^2, \ldots, \bar{A}_m^2\}$ can be derived as follows. If the switched system $\bar{\mathbf{A}}^2$ is marginally unstable, then it also admits a nontrivial invariant subspace. In this case, it can be seen that, besides the subspace $T^{-1}H$, the switched block-triangular system $\bar{\mathbf{A}} = \{\bar{A}_1, \ldots, \bar{A}_m\}$ admits another nontrivial invariant subspace, denoted H'. We can prove that $T^{-1}H + H'$ is $\bar{A}_i$-invariant for all $i \in M$, which further implies that $H + TH'$ is A_i-invariant for all $i \in M$. This contradicts the definition of H as it is the largest invariant subspace under A_i for $i \in M$. The contradiction exhibits that the switched system $\bar{\mathbf{A}}^2$ is either stable or marginally stable. However, the (exponential) stability of the switched system $\bar{\mathbf{A}}^2$ would imply marginal stability of the switched system $\bar{\mathbf{A}}$. Thus the switched system $\bar{\mathbf{A}}^2$ must be marginally stable. Finally, note that the original system is marginally stable if the switched system $\bar{\mathbf{A}}^1$ is stable, and thus the switched system $\bar{\mathbf{A}}^1$ must be marginally stable. □

Finally, by means of the triangular structure in (2.52), it is possible to estimate the rate of growth for state norm.

Proposition 2.35 *Suppose that the switched linear system* $\mathbf{A}$ *is marginally unstable. Then, there is a polynomial P with degree less than n such that*

$$\left|\phi(t; 0, x, \sigma)\right| \le P(t)|x| \quad \forall x \in \mathbf{R}^n,\ t \in \mathcal{T}_0,\ \sigma \in \mathcal{S}. \tag{2.53}$$

Proof For continuous time, write $e^{\bar{A}_i t} = \begin{bmatrix} e^{\bar{A}_i^1 t} & G_i(t) \\ 0 & e^{\bar{A}_i^2 t} \end{bmatrix}$. As both $\bar{\mathbf{A}}^1 = \{\bar{A}_1^1, \ldots, \bar{A}_m^1\}$ and $\bar{\mathbf{A}}^2 = \{\bar{A}_1^2, \ldots, \bar{A}_m^2\}$ are marginally stable as switched linear systems, each $\bar{A}_i^j$ is either stable or marginally stable for $i \in M$ and $j = 1, 2$. As a result, there exists a polynomial vanishing at zero and with degree less than n, denoted $P_1(t) = p_1 t + \cdots + p_{n-1} t^{n-1}$, such that the absolute value of each entry of $G_i(t)$ is upper bounded by $P_1(t)$ for all $i \in M$ and $t \in \mathcal{T}_0$. Without loss of generality, we assume that all the coefficients of P_1 are nonnegative. Fix an arbitrarily given switching signal $\sigma \in \mathcal{S}$ and a time $t \in \mathcal{T}_0$, and let $t_0 = 0, \ldots, t_s$ be the switching time sequence in $[t_0, t)$ and

$i_0, \dots, i_s$ be the corresponding switching index sequence. It is clear that the state transition matrix $\bar{\Phi}(t; t_0, \sigma) = e^{\bar{A}_{i_s}(t-t_s)} \dots e^{\bar{A}_{i_0}(t_1-t_0)}$ is of the form

$$\bar{\Phi}(t; t_0, \sigma) = \begin{bmatrix} e^{\bar{A}^1_{i_s}(t-t_s)} \dots e^{\bar{A}^1_{i_0}(t_1-t_0)} & Q \\ 0 & e^{\bar{A}^2_{i_s}(t-t_s)} \dots e^{\bar{A}^2_{i_0}(t_1-t_0)} \end{bmatrix},$$

where

$$\begin{aligned} Q &= G_{i_s}(t-t_s)e^{\bar{A}^2_{i_{s-1}}(t_s-t_{s-1})} \dots e^{\bar{A}^2_{i_0}(t_1-t_0)} \\ &\quad + e^{\bar{A}^1_{i_s}(t-t_s)} G_{i_{s-1}}(t_s-t_{s-1}) e^{\bar{A}^2_{i_{s-2}}(t_{s-1}-t_{s-2})} \dots e^{\bar{A}^2_{i_0}(t_1-t_0)} \\ &\quad + \dots + e^{\bar{A}^1_{i_s}(t-t_s)} \dots e^{\bar{A}^1_{i_2}(t_3-t_2)} G_{i_1}(t_2-t_1) e^{\bar{A}^2_{i_0}(t_1-t_0)} \\ &\quad + e^{\bar{A}^1_{i_s}(t-t_s)} \dots e^{\bar{A}^1_{i_1}(t_2-t_1)} G_{i_0}(t_1-t_0). \end{aligned}$$

It can be seen that the marginal stability of $\bar{\mathbf{A}}^1$ and $\bar{\mathbf{A}}^2$ implies the existence of an upper bound, denoted κ, for the entries of the corresponding state transition matrices. It follows that each entry of Q satisfies

$$\begin{aligned} \left|Q(j,l)\right| &\le n\kappa^2\big(P_1(t-t_s) + P_1(t_s-t_{s-1}) + \dots + P_1(t_1-t_0)\big) \\ &\le n\kappa^2 P_1(t-t_0), \quad j = 1, \dots, n_1,\ l = 1, \dots, n_2, \end{aligned}$$

where the latter inequality comes from the fact that P_1 vanishes at the origin and its coefficients are nonnegative. It follows that the norm of $\bar{\Phi}(t; t_0, \sigma)$ is upper bounded by $n^2\kappa^2(1 + P_1(t-t_0))$. As a result, the norm of the state transition matrix $\Phi(t; t_0, \sigma)$ is upper bounded by $P(t-t_0) \overset{\text{def}}{=} \|T\| \|T^{-1}\| n^2\kappa^2(1 + P_1(t-t_0))$, which is a polynomial with degree less than n. This clearly leads to the conclusion.

The discrete-time case can be proceeded in a similar manner, and the details are left to the reader. □

The proposition reveals the fact that there is a gap between exponential divergence of instability and divergence rate of marginal instability which is bounded by a polynomial with degree less than the system dimension.

Corollary 2.36 *For any n-dimensional regular switched linear system* $\mathbf{A}$, *there is a polynomial P with degree less than n such that*

$$\left|\phi(t; 0, x, \sigma)\right| \le P(t)e^{\varrho(\mathbf{A})t}|x| \quad \forall x \in \mathbf{R}^n,\ t \in \mathcal{T}_0,\ \sigma \in \mathcal{S}. \tag{2.54}$$

Moreover, P can be chosen to be of degree zero iff switched system $\underline{\mathbf{A}}$ is marginally stable.

Example 2.37 Let us examine the continuous-time two-form switched linear system with subsystem matrices

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & \alpha \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix},$$

where α is a real number. It is clear that both subsystems are stable, and the switched linear system admits a block triangular structure,

$$A_i = \begin{bmatrix} A_{i,1} & A_{i,3} \\ 0 & A_{i,2} \end{bmatrix}, \qquad i = 1, 2.$$

When $\alpha = 0$, it can be verified that the quadratic function

$$V(x) = x^T P x, \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

is a common weak Lyapunov function. By Proposition 2.6, the switched system is stable or marginally stable. On the other hand, it can be seen that the convex combination $\frac{1}{2}(A_1 + A_2)$ is marginally stable, and hence the switched system is not (asymptotically) stable [80]. As a result, the system is marginally stable.

When $\alpha \neq 0$, the switched system is marginally unstable due to the coupling between the two marginally stable modes. It is readily seen that the system is already in the form specified by Lemma 2.34. Let $\Lambda_0 = \{x \in \mathbf{R}^4 \colon \sum_{k=1}^4 |x_i| \le 1\}$. The largest invariant set Λ_∞ contained in Λ_0 satisfies

$$\Lambda_{V_2} \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} : V_2(x) \le 1 \right\} \subset \Lambda_\infty \subset \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} : |x_3| + |x_4| \le 1 \right\},$$

where $V_2(x) = 3x_3^2 + 3x_4^2 - 2x_3x_4$. Figure 2.1 depicts the sets in the $x_3 - x_4$ plane.

Finally, due to the block-triangular structure of the switched system, the system solution satisfies

$$\begin{aligned} &\left|x^1(t)\right| \le \kappa_1\left|x^1(0)\right| + \int_0^t \kappa_1\left|\alpha x_4(\tau)\right| d\tau \le \kappa_1\left|x^1(0)\right| + \alpha\kappa_1\kappa_2 t\left|x^2(0)\right|, \\ &\left|x^2(t)\right| \le \kappa_2\left|x^2(0)\right|, \end{aligned}$$

where $x^1 = [x_1, x_2]^T$, $x^2 = [x_3, x_4]^T$, and κ_1 and κ_2 are the largest possible norms of the state transition matrices w.r.t. subsystems $\{A_{i,1}\}$ and $\{A_{i,2}\}$, respectively. It is clear that the state norm is bounded by a polynomial of degree one, which is consistent with Proposition 2.35. Figure 2.2 presents a sample state trajectory with $\alpha = 1$. It is clear that the state norm grows linearly.

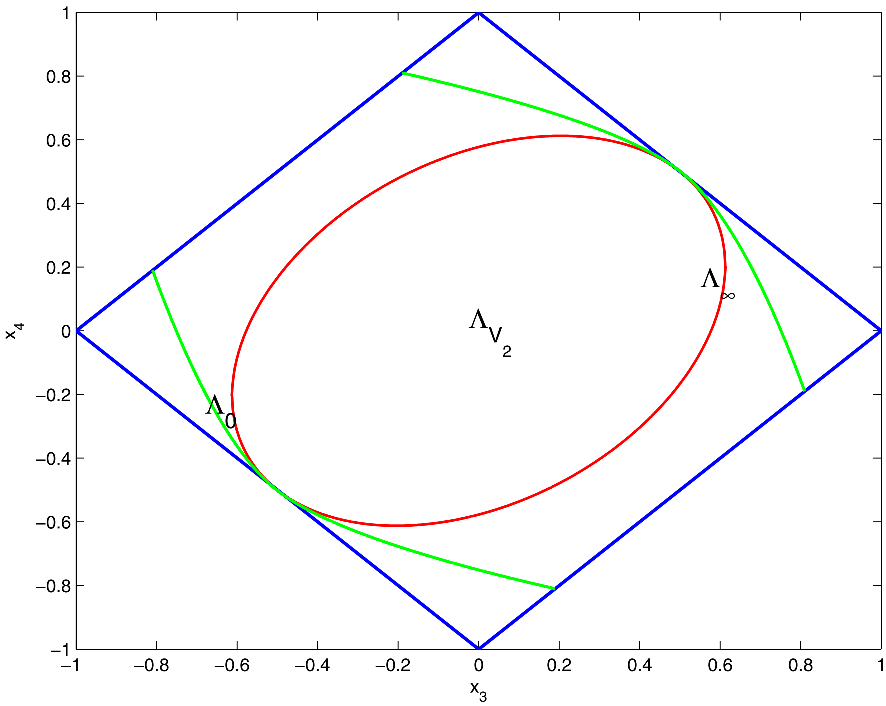

Fig. 2.1 Sets Λ_0, Λ_∞, and Λ_{V_2}

2.3.5 Triangularizable Systems

In this subsection, we focus on a special class of switched linear systems which either possess an upper (lower) triangular structure or are simultaneously equivalent to triangular systems. Triangular systems are interesting because they have simple structures, and many nontriangular systems can be made to be triangular by means of equivalence transformations (simultaneous triangularization).

Definition 2.38 The switched system is said to be *simultaneously* (*upper*) *triangularizable* if the matrix set $\mathbf{A} = \{A_1, \ldots, A_m\}$ is simultaneously triangularizable, that is, there exists a complex nonsingular matrix $T \in \mathbf{C}^{n\times n}$ such that $B_k \stackrel{\text{def}}{=} T^{-1}A_kT$, $k \in M$, are of the upper triangular form,

$$B_k = \begin{bmatrix} b_k(1,1) & \ldots & b_k(1,n) \\ & \ddots & \\ 0 & \ldots & b_k(n,n) \end{bmatrix} \in \mathbf{C}^{n\times n}, \quad k \in M. \tag{2.55}$$

Note that we allow complex matrices as the equivalence transformation. For a simultaneously triangularizable matrix set, we can transform it into the following real normal form.

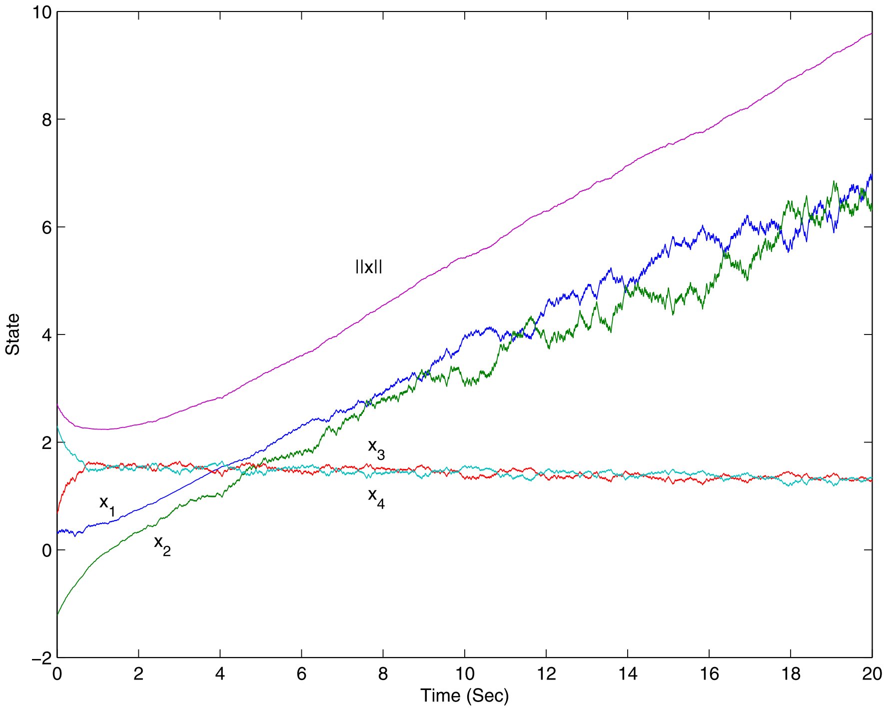

Fig. 2.2 Sample state trajectory

Lemma 2.39 *Suppose that a system* $\mathbf{A} = \{A_1, \ldots, A_m\}$ *is simultaneously triangularizable. Then, there exists a real nonsingular matrix* G, *such that* $G^{-1}A_kG$ *is of the normal form*

$$\bar{A}_k \stackrel{\text{def}}{=} G^{-1}A_kG = \begin{bmatrix} A_{1k} & * & \ldots & * \\ 0 & A_{2k} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{lk} \end{bmatrix}, \tag{2.56}$$

where $l \le n$, A_{jk} *is either a* 1×1 *or* 2×2 *block, and the size of the* j*th block* A_{jk} *is the same for all* $k \in M$. *In addition, if* A_{jk} *is of* 2×2, *then it is of the form*

$$A_{jk} = \begin{bmatrix} \mu_{jk} & \omega_{jk} \\ -\omega_{jk} & \mu_{jk} \end{bmatrix}. \tag{2.57}$$

Proof As the matrix set $\{A_1, \ldots, A_m\}$ is simultaneously triangularizable, for any polynomial $p(y_1, \ldots, y_m)$ over $\mathbf{R}$, the eigenvalues of $p(A_1, \ldots, A_m)$ are $p(b_1(i,i), \ldots, b_m(i,i))$, $i = 1, \ldots, n$. This exhibits that the matrix set possesses Property III in [85, p. 442]. By Theorems 1 and 9 in [85], there is an orthogonal matrix $H \in \mathbf{R}^{n\times n}$ such that

$$\bar{B}_k \stackrel{\text{def}}{=} H^{-1}A_kH = \begin{bmatrix} B_{1k} & * & \dots & * \\ 0 & B_{2k} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{lk} \end{bmatrix}, \tag{2.58}$$

where $l \le n$, and for any fixed $j \le l$, we have

(i) $B_{j1}, \dots, B_{jm}$ are 1×1; or
(ii) $B_{j1}, \dots, B_{jm}$ are 2×2, with one of these matrices, say, B_{jq}, of the form

$$B_{jq} = \begin{bmatrix} r_{jq} & u_{jq} \\ -v_{jq} & r_{jq} \end{bmatrix}, \quad u_{jq} > 0,\ v_{jq} > 0,$$

and each of $B_{jk},\ k \in M$, is a real linear polynomial g_{jk} in B_{jq}, that is, $B_{jk} = g_{jk}(B_{jq})$.

As B_{jq} in (ii) possesses a pair of (conjugated) complex eigenvalues, it follows from the standard matrix theory that there exists a real nonsingular matrix $T_j \in \mathbf{R}^{2\times 2}$ such that

$$T_j^{-1}B_{jq}T_j = \begin{bmatrix} \mu_{jq} & \omega_{jq} \\ -\omega_{jq} & \mu_{jq} \end{bmatrix},$$

where $\mu_{jq}, \omega_{jq} \in \mathbf{R}$. Furthermore, as any polynomial of the matrix $\left[\begin{smallmatrix} \mu_{jq} & \omega_{jq} \\ -\omega_{jq} & \mu_{jq} \end{smallmatrix}\right]$ is still of the same form, we have

$$T_j^{-1}B_{jk}T_j = g_{jk}\big(T_j^{-1}B_{jq}T_j\big) = \begin{bmatrix} \mu_{jk} & \omega_{jk} \\ -\omega_{jk} & \mu_{jk} \end{bmatrix}, \quad \mu_{jk}, \omega_{jk} \in \mathbf{R},\ k \in M.$$

Define $K = \operatorname{diag}[K_1, \dots, K_l]$, where $K_j = 1$ if the corresponding block in (2.58) is 1×1, and $K_j = T_j$ if the block is 2×2. Letting $G = HK$, the theorem follows. □

Remark 2.40 Simultaneous triangularization of matrix sets has been investigated extensively; see, for example, [5, 176, 195, 204] and the references therein. In particular, the following classes of matrix sets (and the corresponding switched systems) have been proven to be simultaneously triangularizable:

(a) The system matrices are commutative pairwise, that is, $A_iA_j = A_jA_i$, $i, j \in M$ [182].
(b) The Lie algebra generated by the system matrices is solvable [1, 147].
(c) $\mathcal{A} = \{A_1, A_2\}$ and $\operatorname{rank}(A_1A_2 - A_2A_1) = 1$ [137].

For switched linear systems that are simultaneously triangularizable, it is possible to judge the stability directly upon the eigenvalues of the subsystems. In fact, we can go further to characterize the largest rate of convergence (cf. (2.28)) explicitly.

Theorem 2.41 *For any simultaneously triangularizable switched linear system* $\mathbf{A}$, *the largest divergence rate is*

$$\varrho(\mathbf{A}) = \max_{k\in M} \max_{i=1}^{n} \Re\lambda_i(A_k) \tag{2.59}$$

in continuous time and

$$\varrho(\mathbf{A}) = \max_{k\in M} \max_{i=1}^{n} \big|\lambda_i(A_k)\big| \tag{2.60}$$

in discrete time.

Remark 2.42 Note that (2.59) and (2.60) only involve eigenvalues of the system which are easily calculated. To apply the theorem, we do not necessarily need to find a linear transformation that converts the system into the triangular form. Instead, we only need to confirm that the system is simultaneously triangularizable.

To prove this theorem, we need the following technical lemma.

Lemma 2.43 *A switched system* $\bar{\mathbf{A}} = \{\bar{A}_1, \dots, \bar{A}_m\}$, *where* $\bar{A}_k$ *is in normal form* (2.56), *is stable iff each* $\bar{A}_k$ *is stable.*

Proof Suppose that the switched system is stable. Then it is clear that each subsystem is also stable.

On the other hand, if each $\bar{A}_k$ is Hurwitz in continuous time, then we can construct a common Lyapunov function of the form $v(x) = x^T P x$ for the switched system. Indeed, for the $\bar{B}_k$ defined in (2.58), let $j_1, \dots, j_l$ denote the size of $\bar{B}_k(1,1), \dots, \bar{B}_k(l,l)$, respectively. Define

$$P = \mathrm{diag}[I_{j_1}, q_2 I_{j_2}, \dots, q_l I_{j_l}],$$

where positive numbers $q_2, \dots, q_l$ are chosen so that the minors of the matrix $-(\bar{A}_k^T P + P\bar{A}_k)$ of order $1, \dots, n$ are positive for all $k \in M$. In this way, the switched system possesses a common Lyapunov function and thus is stable.

The discrete-time counterpart can be established in a similar manner, and the details are omitted. □

Proof of Theorem 2.41 We proceed with the continuous-time case, and the discrete-time case could be proven in a similar manner. Suppose that there exists a real nonsingular matrix G such that for each $k \in M$, the matrix $\bar{A}_k = G^{-1} A_k G$ is of the triangular form (2.56).

Consider the switched system $\mathbf{A} - \gamma\mathbf{I} = \{A_1 - \gamma I_n, \dots, A_m - \gamma I_n\}$, where γ is any given real number. Let $\lambda_k^{\max}$ denote the largest real part of the matrix A_k. Suppose that $\gamma > \max_{k\in M} \lambda_k^{\max}$, then it can be seen that each $\bar{A}_k - \gamma I_n$ is Hurwitz. It follows from Lemma 2.43 that the switched system $\bar{\mathbf{A}} - \gamma\mathbf{I_n} = \{\bar{A}_1 - \gamma I_n, \dots, \bar{A}_m - \gamma I_n\}$ is stable. This in turn implies that

$$\varrho(\bar{\mathbf{A}} - \gamma\mathbf{I_n}) < 0$$

and

$$\varrho(\mathbf{A}) = \varrho(\bar{\mathbf{A}}) < \gamma.$$

Accordingly, we have

$$\varrho(\mathbf{A}) \leq \max_{k \in M} \lambda_k^{\max}.$$

On the other hand, it can be seen that

$$\varrho(\mathbf{A}) \geq \max_{k \in M} \varrho(A_k) = \max_{k \in M} \lambda_k^{\max}.$$

As a result, we have

$$\varrho(\mathbf{A}) = \max_{k \in M} \lambda_k^{\max}. \qquad \square$$

Corollary 2.44 *A simultaneously triangularizable switched linear system is asymptotically stable iff each subsystem is stable.*

Example 2.45 For the continuous-time switched system $\mathbf{A} = \{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} -2 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 2 & 0 \\ -2 & -1 & 0 \\ 1 & -2 & -1 \end{bmatrix},$$

it can be verified that the eigenvalues are $\{-1, -1, -3\}$ for A_1 and $\{-1, -0.5 + 1.9365\sqrt{-1}, -0.5 - 1.9365\sqrt{-1}\}$ for A_2, and $A_1A_2 - A_2A_1$ is of rank one. It follows that the switched linear system is simultaneously triangularizable (cf. Remark 2.40), which further implies that the switched system is exponentially convergent with the rate of -0.5.

2.4 Computational Issues

In the previous section, we presented quite a few stability criteria, mostly for stability and some for marginal stability. In this section, we briefly discuss the possibility of verifying the conditions of the criteria in terms of appropriate computational procedures.

2.4.1 Approximating the Spectral Radius

For discrete-time switched linear systems, it follows from Corollary 2.24 that the verification of asymptotic stability can be reduced to the calculation of the spectral radius. Indeed, if the spectral radius can be exactly calculated in a finite time, then the verification problem is decidable, that is, there exists a computational procedure that produces either "yes" or "no" answer in a finite time. In the literature, it was conjectured that, for any finite matrix set $\mathbf{A}$, there exists a finite k such that $\bar{\rho}(\mathbf{A}) = \bar{\rho}_k(\mathbf{A})^{1/k}$. This conjecture is well known as the *Finiteness Conjecture*, which was finally disproved by counterexamples. As a result, it remains an open problem for the verification of asymptotic stability. Nevertheless, by Corollary 2.24 and the fact that $\rho(\mathbf{A}) = \inf_{k\in\mathbf{N}^+} \hat{\rho}_k(\mathbf{A})^{1/k}$, the verification problem is semi-decidable by verifying the relationship $\hat{\rho}_k(\mathbf{A}) < 1$ for $k = 1, 2, \ldots$ and terminating at the first confirmative instant. On the other hand, the problem of instability verification is also semi-decidable by verifying the relationship $\bar{\rho}_k(\mathbf{A}) > 1$ for $k = 1, 2, \ldots$ and terminating at the first confirmative instant. The problem of verifying the marginal stability, however, was shown to be undecidable.

While it is hard to exactly calculate the spectral radius of a matrix set, it is possible to compute an approximation as follows.

Proposition 2.46 *For any norm and natural number k, we have*

$$\max_{B\in\Pi_k(\mathbf{A})} \rho(B)^{1/k} \le \rho(\mathbf{A}) \le \max_{B\in\Pi_k(\mathbf{A})} \|B\|^{1/k}, \quad k \in \mathbf{N}_+. \tag{2.61}$$

Proof First, observe that, for any natural number j, we have

$$\max_{B\in\Pi_k(\mathbf{A})} \rho(B)^j \le \max_{B\in\Pi_{kj}(\mathbf{A})} \rho(B). \tag{2.62}$$

Indeed, the set of matrix products that can be expressed as the jth power of an element in $\Pi_k(\mathbf{A})$ is a subset of $\Pi_{kj}(\mathbf{A})$. Inequality (2.62) follows from the equality $\rho(A^j) = \rho(A)^j$. Rewrite (2.62) to be

$$\max_{B\in\Pi_k(\mathbf{A})} \rho(B)^{1/k} \le \max_{B\in\Pi_{kj}(\mathbf{A})} \rho(B)^{1/kj}, \tag{2.63}$$

which implies that

$$\max_{B\in\Pi_k(\mathbf{A})} \rho(B)^{1/k} \le \limsup_{j\to+\infty} \max_{B\in\Pi_{kj}(\mathbf{A})} \rho(B)^{1/kj}.$$

On the other hand, it follows from $\Pi_{kj}(\mathbf{A}) \subset \bigcup_{i=1}^{+\infty} \Pi_i(\mathbf{A})$ that

$$\limsup_{j\to+\infty} \max_{B\in\Pi_{kj}(\mathbf{A})} \rho(B)^{1/kj} \le \limsup_{i\to+\infty} \max_{B\in\Pi_i(\mathbf{A})} \rho(B)^{1/i},$$

which, together with inequality (2.63), leads to the first inequality of (2.61).

To establish the second inequality of (2.61), define

$$\zeta = \max\big\{\|A_1\|, \ldots, \|A_m\|\big\}.$$

For any nature number l, there are nonnegative integers ν and j with $j < k$ such that $l = k\nu + j$. For any index sequence $i_1, \ldots, i_l$ in M, we have

$$\|A_{i_1}\cdots A_{i_l}\| \le \prod_{\iota=0}^{\nu-1}\big(\|A_{k\iota+1}A_{k\iota+2}\cdots A_{k\iota+k}\|\big)\prod_{\iota=1}^{j}\|A_{k\nu+\iota}\|$$
$$\le \Big(\max_{B\in\Pi_k(\mathbf{A})}\|B\|\Big)^{\nu}\zeta^{j}.$$

It follows that

$$\Big(\max_{B\in\Pi_l(\mathbf{A})}\|B\|\Big)^{1/l} \le \Big(\max_{B\in\Pi_k(\mathbf{A})}\|B\|\Big)^{1/k-j/kl}\zeta^{j/l},$$

which, by taking $l\to+\infty$, yields

$$\rho(\mathbf{A}) \le \max_{B\in\Pi_k(\mathbf{A})}\|B\|^{1/k}.$$

□

Remark 2.47 The proposition provides two-side bounds for the spectral radius. Based on this relationship, a computational procedure is readily developed to approximate the spectral radius. With a sufficiently large k, the approximation can be made arbitrarily accurate. For stability verification, it suffices to terminate the procedure when either $\max_{B\in\Pi_k(\mathbf{A})}\rho(B)>1$, which implies instability, or $\max_{B\in\Pi_k(\mathbf{A})}\|B\|<1$ that implies stability. While a merit of this approximation is that at each step the approximating accuracy can be estimated, the procedure is not computationally efficient as the cardinality of $\Pi_k(\mathbf{A})$ grows exponentially with k. Another drawback of the estimate is that both sequences $\max_{B\in\Pi_k(\mathbf{A})}\rho(B)^{1/k}$ and $\max_{B\in\Pi_k(\mathbf{A})}\|B\|^{1/k}$ are not necessarily monotone w.r.t. k, which means that the approximate accuracy of estimate (2.61) is not necessarily increasing as k increases. The following example clearly illustrates this.

Example 2.48 Let $M=\{1,2\}$, and

$$A_1=\begin{bmatrix}-1 & -\sqrt{3}\\ -0.9\sqrt{3} & -0.9\end{bmatrix},\qquad A_2=\begin{bmatrix}-0.9 & -0.9\sqrt{3}\\ \sqrt{3} & -1\end{bmatrix}.$$

Table 2.1 shows the calculated joint/generalized spectral radii and the differences that represent the accuracy errors, where $\bar{\rho}_k=\max_{B\in\Pi_k(\mathbf{A})}\rho(B)^{1/k}$, $\hat{\rho}_k=\max_{B\in\Pi_k(\mathbf{A})}\|B\|^{1/k}$, and $e_k=\hat{\rho}_k-\bar{\rho}_k$. It is clear that all the sequences are oscillating.

To obtain a guaranteed precision of a spectral radius estimate, we take a polynomial common Lyapunov approach, which could provide guaranteed accuracy by

Table 2.1 Estimated spectral radii and error bounds

k	1	2	3	4	5	6	7
$\bar{\rho}_k$	1.8974	1.8974	1.9652	1.8974	1.8974	1.9652	1.8974
$\hat{\rho}_k$	2.0000	1.9860	1.9654	1.9739	1.9709	1.9654	1.9702
e_k	0.1026	0.0886	0.0002	0.0765	0.0735	0.0002	0.0728

searching a proper common Lyapunov function in a preassigned set of polynomials. For this, we first restrict ourselves to the case of quadratic common Lyapunov functions and then extend the searching to sum-of-squares polynomials.

If the switched linear system admits a quadratic Lyapunov function, then, by solving the linear matrix inequalities

$$A_i^T P + P A_i < 0, \quad i = 1, \dots, m$$

for symmetric matrix P, the asymptotic stability is decidable via effective algorithms. Based on this idea, we define the *ellipsoid norm* for matrix set $\mathbf{A}$ by

$$\rho_L(\mathbf{A}) = \inf\bigl\{\mu \in \mathbf{R}^+ : \exists\, P > 0 \text{ s.t. } A_i^T P A_i \le \mu^2 P,\ i = 1, \dots, m\bigr\}.$$

It can be seen that the ellipsoid norm could be equivalently defined to be

$$\rho_L(\mathbf{A}) = \inf_{P>0} \max_{i=1}^{m} \|A_i\|_P, \tag{2.64}$$

where $\|A_i\|_P = \max_{x \ne 0} \sqrt{x^T A_i^T P A_i x} / \sqrt{x^T P x}$ is the P-norm.

Proposition 2.49 $\frac{\rho_L(\mathbf{A})}{\sqrt{n}} \le \rho(\mathbf{A}) \le \rho_L(\mathbf{A})$.

To proceed with the proof, we need the following supporting lemma that is part of the well-known *John's theorem* [128].

Lemma 2.50 *Suppose that $\Omega \subset \mathbf{R}^n$ is an origin-symmetric compact convex set with nonempty interior. Then, there is an origin-centered ellipsoid E such that*

$$E \subset \Omega \subset \sqrt{n} E. \tag{2.65}$$

Proof of Proposition 2.49 It follows from the definition of $\rho_L(\mathbf{A})$ that $\rho(\mathbf{A}) \le \rho_L(\mathbf{A})$. Therefore, we need only to establish that $\rho(\mathbf{A}) \ge \frac{\rho_L(\mathbf{A})}{\sqrt{n}}$.

By Theorem 2.23, for any positive real number ϵ, there is a norm $|\cdot|$ such that

$$\|\mathbf{A}\| \le \rho(\mathbf{A}) + \epsilon. \tag{2.66}$$

It is clear that the unit ball $\Omega = \{x \in \mathbf{R}^n : |x| \le 1\}$ is origin-symmetric, compact, and convex. It follows from Lemma 2.50 that there exists an ellipsoid $E = \{x \in \mathbf{R}^n : x^T P x \le \frac{1}{n}\}$ such that relation (2.65) holds. This implies that

$$\sqrt{x^T P x} \le |x| \le \sqrt{n}\sqrt{x^T P x},$$

which, together with inequality (2.66), further implies that

$$A_i^T P A_i \le \bigl(\rho(\mathbf{A}) + \epsilon\bigr) n P.$$

This leads to

$$\max_{i \in M} \|A_i\|_P \le \sqrt{n}\big(\rho(\mathbf{A}) + \epsilon\big).$$

By the arbitrariness of ϵ, we have

$$\rho_L(\mathbf{A}) = \inf_{P>0} \max_{i \in M} \|A_i\|_P \le \sqrt{n}\rho(\mathbf{A}).$$

This completes the proof. □

Remark 2.51 Proposition 2.49 establishes that the ellipsoid norm approximation admits a guaranteed precision that relies on the system dimension. In particular, the approximation is exact for scalar systems. For higher-order systems, the approximation is tight in that Proposition 2.49 does not generally hold when $\sqrt{n}$ is substituted by a smaller number. Therefore, finding a common Lyapunov function becomes more and more difficult as the system dimension increases.

To improve the approximation precision, a natural idea is to use higher-order polynomials as common Lyapunov functions. A suitable class of polynomial Lyapunov functions is the set of homogeneous polynomials that can be expressed as sums of squares.

A polynomial p is said to admit a *sum-of-squares representation* if there are polynomials $p_1, \dots, p_k$ such that

$$p(x) = \sum_{i=1}^{k} \big(p_i(x)\big)^2 \quad \forall x \in \mathbf{R}^n.$$

A polynomial is said to be a sum-of-squares if it admits a sum-of-squares representation.

It is clear that a sum-of-squares is always positive semi-definite, and it is positive definite if polynomials p_i, $i = 1, \dots, k$, do not admit a common root. Moreover, a homogeneous sum-of-squares admits a quadratic representation as stated below.

Lemma 2.52 *A homogeneous polynomial $p(x)$ of degree $2d$ is a sum-of-squares if and only if*

$$p(x) = \big(x^{[d]}\big)^T P x^{[d]}, \tag{2.67}$$

where $x^{[d]}$ is a vector whose entries are monomials of degree d in x, and $P \ge 0$.

For a proof of the lemma, the reader is referred to [184, 203].

Example 2.53 Suppose that we try to find a sum-of-square representation for the polynomial

$$p(x_1, x_2) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4.$$

For this, let $x^{[d]} = [x_1^2, x_1x_2, x_2^2]^T$. Then, try the representation as in (2.67) that reads

$$\begin{bmatrix} x_1^2 & x_1x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$
$$= p_{11}x_1^4 + 2p_{12}x_1^3x_2 + 2p_{23}x_1x_2^3 + (p_{22} + 2p_{13})x_1^2x_2^2 + p_{33}x_2^4.$$

Solving $p(x) = (x^{[d]})^T P x^{[d]}$ gives

$$p_{11} = 2, \qquad p_{12} = 1, \qquad 2p_{13} + p_{22} = -1, \qquad p_{23} = 0, \qquad p_{33} = 5. \tag{2.68}$$

To obtain a positive semi-definite P satisfying the equalities, we can use the semi-definite programming technique, which corresponds to the optimization of a linear function over the intersection of an affine subspace and the cone of positive semi-definite matrices. Specifically, we take the following semi-definite programming:

$$\begin{aligned} &\text{minimize} && 0 \\ &\text{subject to} && \operatorname{tr}(B_i P) = b_i, \quad i = 1, \ldots, 5, \\ & && P \geq 0, \end{aligned}$$

where B_i and b_i are chosen to represent the ith equation, $i = 1, \ldots, 5$. For instance, to represent the third equation, $2p_{13} + p_{22} = -1$, we should choose

$$B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad b_3 = -1.$$

It turns out that a feasible solution of the semi-definite programming is

$$P = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \\ -3 & 0 & 5 \end{bmatrix}.$$

In this way, polynomial p is represented by a sum-of-squares as

$$p(x) = \frac{1}{2}\left[\left(2x_1^2 - 3x_2^2 + x_1x_2\right)^2 + \left(x_2^2 + 3x_1x_2\right)^2\right].$$

To utilize the sums of squares to approximate the spectral radius, we need the following lemma.

Lemma 2.54 *Suppose that p is a positive definite homogeneous polynomial of degree $2d$ that satisfies*

$$p(A_i x) \leq \gamma p(x) \quad \forall x \in \mathbf{R}^n, \ i \in M, \tag{2.69}$$

for some $\gamma > 0$. Then, we have $\rho(\mathbf{A}) \leq \gamma^{\frac{1}{2d}}$.

Proof By the positive definiteness and continuity of p, we have

$$0 < \min_{|x|=1} p(x) \stackrel{\text{def}}{=} \alpha_1 \le \max_{|x|=1} p(x) \stackrel{\text{def}}{=} \alpha_2 < +\infty,$$

which, together with the homogeneity, implies that

$$\alpha_1 |x|^{2d} \le p(x) \le \alpha_2 |x|^{2d}.$$

For any natural number k and $B \in \Pi_k(\mathbf{A})$, there exist an index sequence $i_1, \ldots, i_k$ with elements in M such that $B = A_{i_k} \cdots A_{i_1}$. It is clear that

$$\begin{aligned}
\|A_{i_k} \cdots A_{i_1}\| &= \max_{x \neq 0} \frac{|A_{i_k} \cdots A_{i_1} x|}{|x|} \\
&\le \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2d}} \max_{x \neq 0} \frac{p(A_{i_k} \cdots A_{i_1} x)}{p(x)} \\
&\le \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2d}} \gamma^{k/2d}.
\end{aligned}$$

As a result, we have

$$\rho(\mathbf{A}) = \lim_{k \to +\infty} \sup \max_{B \in \Pi_k(\mathbf{A})} \|B\|^{1/k} \le \gamma^{\frac{1}{2d}}.$$

□

It is clear that, for any homogeneous polynomial p, there always exists a positive real number γ that satisfies inequality (2.69).

Example 2.55 For the planar discrete-time two-form switched linear system $\mathbf{A}$ with

$$A_1 = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & a \\ 0 & -a \end{bmatrix},$$

where a is a positive real number, simple calculation gives

$$\max_{B \in \Pi_k(\mathbf{A})} \rho(B)^{1/k} = a, \quad k = 1, 2, \ldots.$$

As a result, the spectral radius of the switched system is a. Using a common quadratic Lyapunov function, the upper bound on the spectral radius is equal to $\sqrt{2}a$. As a result, only when $a \le \frac{\sqrt{2}}{2}$, we can find a common quadratic Lyapunov function for the switched system, which implies the (marginal) stability of system. On the other hand, take the sum-of-squares homogeneous polynomial of degree four

$$V(x) = \left(x_1^2 - x_2^2\right)^2 + \epsilon\left(x_1^2 + x_2^2\right)^2,$$

where ϵ is a positive real number. We could verify that, for any $b \le \frac{4}{3}$, there is a sufficiently small ϵ such that $ba^4 V(x) - V(A_i x)$ is a sum-of-squares. It follows

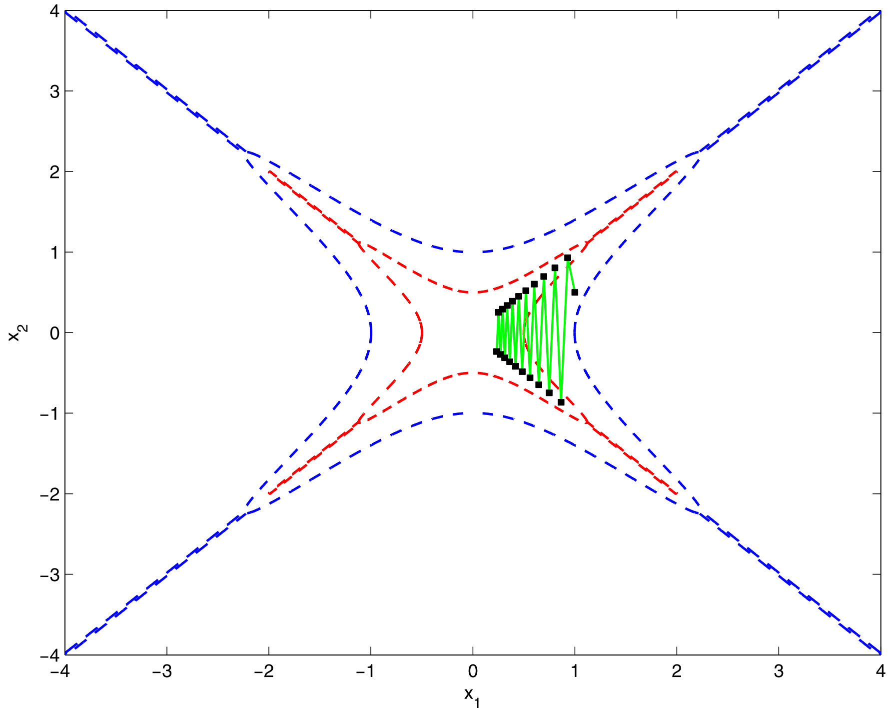

Fig. 2.3 Sample phase portrait and level sets

from Lemma 2.54 that

$$\rho(\mathbf{A}) \le \left(\frac{4}{3}\right)^{1/4} a.$$

In particular, when $a < (\frac{3}{4})^{1/4} \approx 0.9306$, we conclude that the switched system is stable. It is interesting to note that the level sets of $V(x)$ are nonconvex, as shown in Fig. 2.3, where $a = 0.93$ and $\epsilon = 0.01$.

The following lemma, presented in [21], characterizes the ability of sums of squares to approximate a norm.

Lemma 2.56 *For any norm $|\cdot|$ and natural number d, there exists a homogeneous polynomial $p(x)$ of degree $2d$ such that*

(1) *p is a sum-of-squares and*
(2) *for all $x \in \mathbf{R}^n$, we have*

$$p(x)^{\frac{1}{2d}} \le |x| \le \kappa_n^d \, p(x)^{\frac{1}{2d}}, \tag{2.70}$$

where $\kappa_n^d = \binom{n+d-1}{d}^{\frac{1}{2d}}$.

Recall that $\binom{k}{j}$ is the number of combinations, $\binom{k}{j} = \frac{k!}{j!(k-j)!}$, where $k!$ is a factorial. When n is fixed and d is sufficiently large, we have

$$\kappa_n^d \approx 1 + \frac{2d}{(n-1)\ln d}.$$

This means that, for any positive real number ϵ, there is a sufficient large integer d such that $\kappa_n^d \leq 1 + \epsilon$. As an implication, by choosing sufficiently large d, the estimate in (2.70) can achieve any preassigned accuracy.

Let HP_k^n be the set of homogeneous polynomials of degree k defined on $\mathbf{R}^n$, and SOS_k^n be the subset of HP_k^n that are sums-of-squares. Define the quantity

$$\begin{aligned}
&\rho_S^{2d}(\mathbf{A}) = \inf\nolimits_{p\in \mathrm{HP}_k^n} \gamma^{\frac{1}{2d}} \\
&\text{subject to} \quad p \in \mathrm{SOS}_k^n, \\
&\qquad\qquad\qquad \gamma p(x) - p(A_i x) \in \mathrm{SOS}_k^n .
\end{aligned}$$

With the help of the above notation, we are ready to state the main result of the subsection.

Theorem 2.57 *Suppose that d is a natural number. Then, we have*

$$\frac{\rho_S^{2d}(\mathbf{A})}{\kappa_n^d} \leq \rho(\mathbf{A}) \leq \rho_S^{2d}(\mathbf{A}). \tag{2.71}$$

To proceed with a proof of the theorem, we need some further auxiliary material. First, it follows from Lemma 2.54 that

$$\rho(\mathbf{A}) \leq \rho_S^{2d}(\mathbf{A}). \tag{2.72}$$

Another observation is that, when $d = 1$, HP_2^n is exactly the set of quadratic polynomials, and SOS_2^n is exactly the set of positive (semi-)definite quadratic polynomials. For any positive real number ϵ, there is a norm $|\cdot|$ such that

$$\rho(\mathbf{A}) \geq \|\mathbf{A}\| - \epsilon.$$

By Lemma 2.56, we have

$$\rho(\mathbf{A}) \geq \frac{1}{\sqrt{n}} \rho_S^2(\mathbf{A}) - \epsilon,$$

where the equality $\binom{n}{1}^{1/2} = \sqrt{n}$ is used. By the arbitrariness of ϵ, we obtain

$$\rho(\mathbf{A}) \geq \frac{1}{\sqrt{n}} \rho_S^2(\mathbf{A}). \tag{2.73}$$

Note that the approximation is the same as in Proposition 2.49.

Next, for a vector $x \in \mathbf{R}^n$ and a natural number k, define the k-lift of x, denoted $x^{[k]}$, to be the vector with components $\{\sqrt{\alpha!}x^\alpha\}_\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_+^n$, $\sum_{i=1}^n \alpha_i = k$, and $\alpha!$ denotes the multinomial coefficient $\alpha! = \frac{k!}{\alpha_1! \cdots \alpha_n!}$. For example, when $n = 2$, we have

$$x^{[1]} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x^{[2]} = \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{bmatrix}, \qquad x^{[3]} = \begin{bmatrix} x_1^3 \\ \sqrt{3}x_1^2x_2 \\ \sqrt{3}x_1x_2^2 \\ x_2^3 \end{bmatrix}.$$

It is clear that $x^{[k]}$ is of dimension $N_n^k = \binom{n+k-1}{k}$. For standard Euclidean norm, it can be verified that

$$\left|x^{[k]}\right| = |x|^k, \quad k = 1, 2, \ldots. \tag{2.74}$$

On the other hand, for any matrix $A \in \mathbf{R}^{n \times n}$, there is an induced matrix $A^{[k]} \in \mathbf{R}^{N_n^k \times N_n^k}$ satisfying

$$A^{[k]}x^{[k]} = (Ax)^{[k]} \quad \forall x \in \mathbf{R}^n. \tag{2.75}$$

It can be shown that the operation defines an algebra homomorphism that preserves the structure of matrix multiplication. In particular, for any $n \times n$ matrices A and B, we have

$$(AB)^{[k]} = A^{[k]}B^{[k]}, \quad k = 1, 2, \ldots. \tag{2.76}$$

Integrating properties (2.74), (2.75) and the definition of spectral radius, we can obtain the following lemma.

Lemma 2.58 *For a matrix set $\{A_1, \ldots, A_m\}$ and a natural number k, we have*

$$\rho\big(A_1^{[k]}, \ldots, A_m^{[k]}\big) = \rho(A_1, \ldots, A_m)^k. \tag{2.77}$$

Finally, with the help of the above preparations, we are ready to prove Theorem 2.57.

Proof of Theorem 2.57 Note that

$$\begin{aligned} \rho_S^2\big(A_1^{[d]}, \ldots, A_1^{[d]}\big) &= \inf\big\{\gamma : P > 0, \gamma^{2d}P - \big(A_i^{[d]}\big)^T PA_i^{[d]} \geq 0,\ i = 1, \ldots, m\big\} \\ &= \inf\big\{\gamma : p(x) > 0,\ \gamma^{2d}p(x) - p\big(A_i^{[d]}\big) \geq 0,\ i = 1, \ldots, m\big\} \\ &\geq \big(\rho_S^{2d}(\mathbf{A})\big)^d, \end{aligned} \tag{2.78}$$

where the notation $p(x) = (A_i^{[d]})^T PA_i^{[d]}$ and the fact that $\gamma^{2d}p(x) - p(A_i^{[d]}) \in \mathrm{SOS}_2^{N_n^d}$ have been used. Combining Lemma 2.58 with inequalities (2.73) and (2.78) yields

$$\begin{aligned}\rho(\mathbf{A})^d &= \rho\big(A_1^{[d]}, \dots, A_m^{[d]}\big)\\ &\geq \frac{1}{\sqrt{N_n^d}}\rho_S^2\big(A_1^{[d]}, \dots, A_m^{[d]}\big)\\ &\geq \frac{1}{\sqrt{N_n^d}}\big(\rho_S^{2d}(\mathbf{A})\big)^d.\end{aligned}$$

As a result, we have

$$\rho(\mathbf{A}) \geq \frac{\rho_S^{2d}(\mathbf{A})}{\kappa_n^d},$$

which, together with inequality (2.72), leads to inequalities (2.71). The proof of Theorem 2.57 is completed. □

Example 2.59 For the four-dimensional three-form switched system with

$$A_1 = \begin{bmatrix} 0 & 1 & 7 & 4\\ 1 & 6 & -2 & -3\\ -1 & -1 & -2 & -6\\ 3 & 0 & 9 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3 & 3 & 0 & -2\\ -2 & 1 & 4 & 9\\ 4 & -3 & 1 & 1\\ 1 & -5 & -1 & -2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 4 & 5 & 10\\ 0 & 5 & 1 & -4\\ 0 & -1 & 4 & 6\\ -1 & 5 & 0 & 1 \end{bmatrix},$$

it can be calculated that $\rho_S^2 = 9.761$ and $\rho_S^4 = \rho_S^6 = 8.92$. As $\rho(A_1A_3)^{\frac{1}{2}} = 8.915$, we conclude that the spectral radius is between 8.915 and 8.92. It is clear that ρ_S^4 provides a much closer upper bound for the spectral radius than ρ_S^2.

2.4.2 An Invariant Set Approach

An approach for verifying the stability of the switched system is to find an appropriate piecewise linear common Lyapunov function.

Let us start from an initial polyhedron Λ_0 which is origin-symmetric. An example is the polyhedron with extreme points whose entries are either 1 or -1. Recall that for any 0-symmetric polyhedral set Λ, there is a full column rank matrix F_Λ such that $\Lambda = \{x\colon |F_\Lambda x|_\infty \leq 1\}$. Another representation is $\Lambda = \{X_\Lambda \alpha\colon |\alpha|_1 \leq 1\}$, where $X_\Lambda = [x_1, \dots, x_r]$ is full row rank and $\{\pm x_1, \dots, \pm x_r\}$ are vertices of Λ. For a 0-symmetric polyhedral set Λ, define the set

$$C(\Lambda) = \{x\colon A_i x \in \Lambda,\ i = 1, \dots, m\}.$$

It is clear that

$$C(\Lambda) = \{x : |F_\Lambda A_i x|_\infty \le 1, \ i = 1, \dots, m\},$$

which is also a 0-symmetric polyhedron. Define recursively a set of polyhedra by

$$\Lambda_k = C(\Lambda_{k-1}) \cap \Lambda_0, \quad k = 1, 2, \dots. \tag{2.79}$$

The largest invariant set contained in Λ_0, as defined in (2.50), is the set

$$\Lambda_\infty = \bigcap_{k \in \mathbf{N}_+} \Lambda_k.$$

Note that, if $\Lambda_k = \Lambda_{k+1}$ for some $k \in \mathbf{N}_+$, then $\Lambda_\infty = \Lambda_k$, and the set is tractably computed.

Proposition 2.60 *For any initial polyhedron Λ_0, we have the following statements*:

(1) *If the discrete-time switched linear system $\mathbf{A}$ is stable, then, there is a finite number k such that $\Lambda_k = \Lambda_{k-1}$, which means that $\Lambda_k = \Lambda_\infty$. Conversely, if $\Lambda_k = \Lambda_{k-1}$ for some $k < +\infty$, then, $\Lambda_k = \Lambda_\infty$, and the system is stable or marginally stable.*
(2) *If there is a finite number k such that Λ_k is interior to set Λ_0, then $\Lambda_\infty = \{0\}$, and the switched system is unstable.*

Proof For the former statement, let $x(\cdot)$ be a state trajectory of the switched system. The exponential stability implies the existence of a time T with $x(t) \in \Lambda_0$ for $t \ge T$ whenever $x(0) \in \Lambda_0$. For any $k = 0, 1, \dots, T$, Λ_k has the property that $x(0) \in \Lambda_k$ implies that $x(k) \in \Lambda_0$, and visa versa. This means that $\Lambda_{T+1} = \Lambda_T$. Indeed, if this is not true, then, there is a state trajectory $x(\cdot)$ with $x(0) \in \Lambda_T / \Lambda_{T+1}$, which implies that $x(T+1) \notin \Lambda_0$, which is a contradiction. Conversely, if $\Lambda_k = \Lambda_{k-1}$ for some $k < +\infty$, then, it is clear that $\Lambda_k = \Lambda_\infty$ is a polyhedral C-set that is an invariant set for the system. Thus the system is stable or marginally stable.

For the latter statement, we only need to show that $\Lambda_\infty = \{0\}$. As Λ_∞ is an invariant set for the system, it can be seen that $\lambda \Lambda_\infty$ is also an invariant set for any $\lambda \ge 0$. As the set Λ_∞ is interior to the set Λ_0, there is $\lambda > 1$ such that $\lambda \Lambda_\infty \subset \Lambda_0$. This in turn implies that $\lambda \Lambda_\infty \subset \Lambda_k$ for all $k \in \mathbf{N}_+$, which yields $\lambda \Lambda_\infty \subset \Lambda_\infty$. As a result, we have $\Lambda_\infty = \{0\}$. □

It is interesting to notice that $\Lambda_k = \Lambda_\infty$ does not necessarily implies the (exponential) stability. A particular example is the case that $\mathbf{A} = \{I_n\}$, which produces $\Lambda_k = \Lambda_\infty = \Lambda_0$ for any $k \in \mathbf{N}^+$. In fact, if we add the identity matrix to any stable system, we have a marginally stable system which produces the same polyhedral set sequence $\Lambda_0, \Lambda_1, \dots$. To further distinguish between stability and marginal stability, one possible way is to test through the recursive procedure by setting $\mathbf{A} = \{\lambda A_1, \dots, \lambda A_m\}$, where $\lambda > 1$, but $\lambda - 1$ is sufficiently small. If $\Lambda_k = \Lambda_{k-1}$ for a finite k, then, the original system must be stable. Otherwise, the original system is either marginally stable or stable but "nearly marginally stable".

Based on the above discussion, we can develop a computational algorithm that verifies the stability of the discrete-time switched linear system.

Algorithm for calculating the largest invariant set (ACLIS)

Initiation. Set $F^0 := F_{\Lambda_0}$, $X^0 := X_{\Lambda_0}$, $flag := 0$, and $k := 0$. Prespecify a natural number $kmax$.

Step 1. Set $F := F^k$, $X := X^k$, and compute the matrix

$$G = \left[(F_k A_1)^T, \ldots, (F_k A_m)^T\right]^T.$$

Step 2. For each row G^i of G, check if $\|G^i X\|_\infty \le 1$. If yes, remove the row from matrix G.

Step 3. If $flag = 0$ and G is the vacant matrix, that is, it is 0×0, then output the matrix X and the "stability or marginal stability" message, set $flag := 1$ and go to Step 7. If $flag = 1$ and G is the vacant matrix, then terminate with the message "stability".

Step 4. Set $H := [F^T, G^T]^T$. Compute $\Lambda^{k+1} = \{x \in \mathbf{R}^n : |Hx|_\infty \le 1\}$ and set $F^{k+1} := F_\Lambda^{k+1}$ and $X^{k+1} := X_{\Lambda^{k+1}}$.

Step 5. If $flag = 0$ and $\|F^0 X^{k+1}\|_\infty < 1$, then, terminate with the message "instability." If $flag = 1$ and $\|F^0 X^{k+1}\|_\infty < 1$, then, terminate with the message "marginal stability or nearly marginal stability."

Step 6. Set $k := k + 1$. If $k \ge kmax$, terminate with message "time is out." Otherwise, go to Step 1.

Step 7. Set a sufficiently small positive number λ, and $A_i = (1 + \lambda)A_i$ for $i = 1, \ldots, m$. Set $F^0 := F$, and go to Step 1.

It can be seen that the algorithm is not efficient as the number of extreme points of polyhedra Λ_k may grow exponentially. Within a recursive loop, the main computation load is in Step 4, which computes the various representations of the new polyhedron. Fortunately, this can be implemented by commercial softwares (e.g. Matlab Geometric Bounding Toolbox [252]).

For continuous-time switched linear systems, it was established in Lemma 2.19 that, if the system is asymptotically stable, then, its *Euler approximating system*

$$x(t + 1) = (I_n + \tau A_{\sigma(t)})x(t) \tag{2.80}$$

is also asymptotically stable for sufficiently small τ. A verification procedure can thus been outlined as follows.

Step 1. Choose a sufficiently small positive real number τ.

Step 2. Run the Algorithm ACLIS for system (2.80). If the algorithm terminate with stability, then, set $\Lambda_\infty = \Lambda_k$ and go to Step 4. Otherwise, go to Step 3.

Step 3. Set $\tau = \tau/2$ and go to Step 2.

Step 4. Check if Λ_∞ is positively invariant for the original system. If yes, then terminate with the message "the continuous-time system is stable". Otherwise, go to Step 3.

Note that the fourth step can be conducted with the help of Theorem 2.31. Generally, there is no guarantee that the procedure terminates in a finite time even for stable switched systems. As a matter of fact, the stability verification for continuous-time switched systems is still an open problem for further investigation.

2.5 Notes and References

This chapter introduced the fundamental issues which are relevant to the development of guaranteed stability theory for switched dynamical systems. As a matter of fact, the development of stability theory for switched systems is not isolated. On the contrary, the progress actively interacted with stability and robustness issues for several different system frameworks of various backgrounds, as briefly discussed in Sect. 2.3.1. Indeed, when the switching path is taken as a perturbation variable, the stability of a switched system is in fact robustness against a class of time-varying uncertainties. This can be seen from expression (2.1) where the switching signal $\sigma(t)$ is an unknown time-varying perturbation. A unique feature of the perturbation is that its image set is finite and thus isolated. By taking convex linear combinations as in (2.20) and (2.21), the switched system is naturally connected to the polytopic uncertain system and the relaxed differential inclusion. As these classes of dynamical systems share the same stability properties, it is natural that the stability theory for switched systems has been deeply interacted with that of other system frameworks. To fully understand the major progress in the stability analysis of switched systems, it is important to highlight the various sources of literature from the related disciplines.

The first source of literature is the absolute stability analysis for Lur'e systems. A Lur'e system is a linear plant with a sector-bounded nonlinear output feedback. Specifically, a SISO Lur'e system is mathematically described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\varphi(y), \quad && x \in \mathbf{R}^n, \ y \in \mathbf{R}, \\ y(t) &= cx(t), && k_1 y^2 \le y\varphi(y) \le k_2 y^2. \end{aligned} \tag{2.81}$$

The problem of absolute stability is to determine the exact bound of k_1 and k_2 that guarantee global asymptotic stability of the system. The research on this problem could be traced back to the 1940s, and the early pioneers were mainly from the Russian applied mathematics community (Lur'e [157], Aizerman [4], Yakubovich [271], etc.). Aizerman [3] conjectured that, if for each $k \in [k_1, k_2]$, the matrix $A + kbc$ is Hurwitz, then the Lur'e system is $[k_1, k_2]$-absolutely stable. This is equivalent to the statement that the stability of each convex combination of $A + k_1bc$ and $A + k_2bc$ implies the stability of the switched linear system $\mathbf{A} = \{A + k_1bc, A + k_2bc\}$. While this conjecture was disproved by counterexamples, it did greatly stimulate the study on the problem of absolute stability. Quadratic Lyapunov functions (sometimes plus an integral of nonlinearity) were sought by Lur'e himself and many other researchers. A quadratic Lyapunov function for the

Lur'e system is in fact a common Lyapunov function for the extreme systems. That is, the existence of a common Lyapunov function for $A + k_1bc$ and $A + k_2bc$ implies $[k_1, k_2]$-absolute stability of the Lur'e system. In the 1970–1980s, many researchers realized that the quadratic Lyapunov functions are not universal for absolute stability, and they turned to more general nonquadratic Lyapunov functions. Piecewise quadratic Lyapunov functions were proved to be universal for absolute stability [170–172].

The second source of literature is the boundedness analysis for the infinite products of a set of (complex or real) matrices $\mathbf{A} = \{A_1, \ldots, A_m\}$. This topic has been quite active since the 1990s, and the literature can be traced back to the 1950s [260]. It was widely recognized that the joint spectral radius is an index that is closely related to the boundedness of the matrix semigroup [199]. The joint spectral radius and the generalized spectral radius were proved to be equal for any finite set of matrices [25, 69]. Berger and Wang [25] established that a discrete-time switched system is asymptotically stable iff the spectral radius is less than one. As the spectral radius is equal to the least possible induced norms [69], the asymptotic stability is equivalent to the existence of a contractive norm. This norm is in fact a common Lyapunov function for the matrix set. To verify the boundedness, Lagarias and Wang [138] conjectured the existence of a finite k such that the generalized spectral radius $\bar{\rho}(\mathbf{A}) = \bar{\rho}_k(\mathbf{A})^{1/k}$ for any given finite matrix set $\mathbf{A}$. This well-known Finiteness Conjecture was finally disproved [39, 45], which indicates that the exact computation of the spectral radius might be very involved. On the other hand, Brayton and Tong [47, 48] developed constructive procedures for stability verification. The procedures search a piecewise linear common Lyapunov function by recursively approximating its level set. While the computational algorithms are not efficient, the method itself is valuable as it clarifies some useful properties that benefit the forthcoming investigations. Besides the mathematical characteristics, there has been much effort to establish the connections among various notions from the dynamical system's viewpoints. In particular, the notions of vanishing-step(VS)-stability, bounded-variation(BV)-stability, and para-contractility were introduced, and their relationships with the left-convergent-products (LCP) property (all left-infinite products converge) were established [255]. There were also a few works focusing on the more subtle situation that the spectral radius is one, which corresponds to either marginal stability or marginal instability. Among these, the notion of defectivity and its properties were investigated [69, 96].

The third source of literature is the robustness analysis for a class of polytopic linear uncertain systems. When a linear nominal system

$$\dot{x}(t) = A_0x(t) + B_0u(t)$$

is perturbed by a time-varying uncertainty with a polyhedral bound

$$d(t) \in \mathrm{co}\big\{A_1x(t), \ldots, A_mx(t)\big\},$$

the perturbed system can be described by

$$\dot{x}(t) = A_0x(t) + B_0u(t) + d(t) = A\big(\omega(t)\big) + B_0u(t),$$

where $\omega(t) \in \{w \in \mathbf{R}^m : w_i \geq 0, \sum_{i=1}^m w_i = 1\}$ and $A(w) = A_0 + \sum_{i=1}^m w_i A_i$. A more general description is

$$\dot{x}(t) = A\big(\omega(t)\big) + B\big(\omega(t)\big)u(t),$$

where the input gain matrix is also perturbed. For this class of systems, the main problems are of various (gain-scheduling, robust) stabilizing design and robustness analysis. Blanchini and his coworkers [30, 32, 34] developed a nonquadratic Lyapunov scheme for stabilizing design of polytopic linear uncertain systems. By developing a Brayton–Tong-like recursive procedure, it was possible to evaluate both the transient performance and the asymptotic behavior of the linear uncertain systems [33].

The fourth source of literature is the stability of differential inclusions. Differential inclusions provide a unified representation of a wide class of dynamical systems. For a differential inclusion

$$\dot{x}(t) \in F\big(x(t)\big)$$

and its relaxed convex system

$$\dot{x}(t) \in \operatorname{co} F\big(x(t)\big),$$

the solution sets for both systems admit a common closure (w.r.t. an appropriate normed functional space) [13, 83]. This implies the fact that the two systems share same stability properties. This observation bridges the stability theories for the switched linear system, the polytopic linear uncertain system (without control input), and the linear convex differential inclusion, as the last one can be seen as the relaxed system for the former two. On the other hand, it has long been established [135, 188] that an asymptotically stable differential inclusion admits a strictly convex and homogeneous Lyapunov function. This paves the way for finding more universal sets of Lyapunov functions. Indeed, as a strictly convex level set can be arbitrarily approximated via a polyhedral set or an intersection of a set of ellipsoids, both sets of piecewise linear and piecewise quadratic functions are universal.

Finally, the literature on switched and hybrid systems has grown rapidly since the 1990s. The common Lyapunov function approach was proposed based on the fact that, if all the subsystems share a common Lyapunov function, then the switched system is stable under arbitrary switching. Much effort was paid to find criteria for the existence of common quadratic Lyapunov functions for switched linear systems. Narendra and Balakrishnan [182] found that, if all linear subsystems are stable with commuting A-matrices, then they share a common quadratic Lyapunov function. The commutation condition in fact implies the simultaneous triangularizability, and for this, the commutation condition can be further relaxed [1, 164]. The commuting criterion was recently extended to switched nonlinear systems [256]. For planar switched linear systems, complete criteria for common quadratic stability were established [53, 204]. There were a few works reporting various converse Lyapunov theorems [64, 159, 168], which in fact can be seen as special cases of the earlier results in different contexts [34, 152, 173].

It should be noticed that, while many early studies focused on one-system framework, more and more researchers took advantage of the tight connections among the schemes [146, 149, 173, 234]. A notable example is that most researchers from various backgrounds realized the limitation of quadratic Lyapunov functions in tackling the stability and robustness problems for linear and quasi-linear systems, which leads to the active research into the nonquadratic Lyapunov approach. Another example is that the constructive criterion for the stability of planar switched linear systems [112] also provides a solution for the absolute stability of planar Lur'e systems [163].

While the above review provides a brief survey on the relevant literature, it only mentioned a small fraction of the existing results, methods, and literature. The reader is referred to [31, 62, 63, 146, 148] and references therein for more details.

In this chapter, we tried to integrate novel ideas, fresh methods, and rigorous results from various schemes into a systematic framework. The richness of the relevant material enables us to highlight the most notable progress within a unified framework.

The notational preliminaries in Sect. 2.1 were adapted from the books [146, 234]. The common Lyapunov function approach presented in Sect. 2.2 is a combination of several works including [31, 152, 224]. In particular, the notion of the common strong Lyapunov function is a mixture of the smooth version [159] and the locally Lipschitz version [30], and the notion of common weak Lyapunov function was taken from [224].

In Sect. 2.3, Lemma 2.11 can be found in [117], and Proposition 2.13 was adapted from [221]. As for Theorem 2.15, the fact of the equivalence between asymptotic stability and exponential stability was reported in [82, 139] for differential inclusions. The equivalence between local attractivity and global exponential stability was established in [8] for switched systems. The context of universal Lyapunov functions in Sect. 2.3.2 was mainly adapted from [173]. The continuous-time version of Propositions 2.21, 2.33, and 2.35 and Lemma 2.34 were adapted from [224], where their discrete-counterparts can be found in [22, 25, 33, 69], respectively. A simplified proof was presented in [156] for Proposition 2.35 in discrete time. Proposition 2.25 was reported in [69] for discrete-time systems and in [223] for continuous-time systems. The important algebraic criterion, Theorem 2.31, was adapted from [32, 173]. The context in Sect. 2.3.5 on triangular systems was adapted from [239]. While conceptually simple, the research on simultaneous triangularization has been quite active, which impacts greatly on the common Lyapunov function approach.

The computational issues presented in Sect. 2.4 highlight some progress in calculating the spectral radius of a matrix set and verifying the stability of the corresponding switched system. It has been revealed that the computation of the joint spectral radius is NP-hard, and the stability verification "$\rho(\mathbf{A}) \leq 1$" is undecidable [41, 251]. This reveals that effective approximating of the joint spectral radius is difficult in general. Nevertheless, much effort has been paid in investigating the approximation issues, and the reader is referred to [7, 37, 95, 158, 190, 249] and the references therein. Proposition 2.46 and Example 2.48 were taken from the thesis [249]. The accuracy of the ellipsoidal norm approximation, Proposition 2.49, could be found in [7, 38]. The approximation by means sum-of-squares homogeneous polynomial Lyapunov functions, which forms the main content of Sect. 2.4.1, was largely borrowed from [190]. The recursive procedure for calculating the largest invariant set contained in a polyhedron, which forms the main content of Sect. 2.4.2, was adapted from [33]. It is also possible to investigate the stability through the largest invariant set containing a polyhedron [47, 48, 193].

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