Chapter 2
Decentralized Control of Nonlinear Systems I

In this chapter, we examine decentralized control techniques for classes of nonlinear interconnected systems. We identify classes for the system structure along with the underlying assumptions and emphasize the information and design constraints. The subsequent sections focus on a class of large-scale interconnected minimum-phase nonlinear systems with parameter uncertainty and nonlinear interconnections. The uncertain parameters are allowed to be time-varying and enter the systems nonlinearly. The interconnections are bounded by nonlinear functions of states. The problem we address is to design a decentralized robust controller such that the closed-loop large-scale interconnected nonlinear system is globally asymptotically stable for all admissible uncertain parameters and interconnections. It is shown that decentralized global robust stabilization of the system can be achieved using a control law obtained by a recursive design method together with an appropriate Lyapunov function.

The problem of decentralized output-feedback tracking with disturbance attenuation is addressed for a new class of large-scale and minimum-phase nonlinear systems. Common assumptions like matching and growth conditions are not required for the underlying decentralized system with a diagonal structure. An observer-based decentralized controller design is presented. The proposed decentralized output-feedback laws achieve asymptotic tracking and internal Lagrange stability when the disturbance inputs disappear, and, guarantee external stability in the presence of disturbance inputs. These external stability properties include Sontag's ISS and iISS conditions and standard $\mathcal{L}_2$-gain property.

2.1 Classes of Nonlinear Interconnected Systems

In what follows, we summarize the classes of nonlinear interconnected systems (NIS) that will be treated in the subsequent sections. We focus on the features of each class before addressing the topics of stability analysis and decentralized output-feedback control design.

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2.1.1 Class I

In recent years, modern control methods have found their way into decentralized design of interconnected systems leading to a wide variety of new concepts and results. This includes, but not limited to, the framework of $\mathcal{H}_\infty/\mathcal{H}_2$ design and linear matrix inequalities (LMIs) [1] which has been shown [6, 44] to be very attractive particularly when coping with high dimensional systems. Applications having sophisticated theoretical generalizations of the underlying concepts have been in control of multi-agent systems, such as platoons of vehicles on highways and in the air, interconnected spatially-invariant systems, and large-scale power systems [5–7]. It turns out that, the decentralized control designs imply, either explicitly or implicitly, that the system, with local feedback loops closed around the subsystems, remains stable despite changes in its interconnection topology [4, 60, 66, 67].

2.1.1.1 System Description

According to this class, a nonlinear interconnected system **S** is considered to be composed of a finite number N of subsystems represented by

$$\begin{aligned} \mathbf{S}_j\colon\quad \dot{x}_j &= A_j x_j + B_j u_j + h_j(t,x), \\ y_j &= C_j x_j, \end{aligned} \tag{2.1}$$

where $x_j \in \Re^{n_j}$, $u_j \in \Re^{m_j}$ and $y_j \in \Re^{p_j}$ are the subsystem state, input and output vectors, respectively, $x = [x_1^t, \dots, x_N^t]^t$ is the global state vector with $\sum_{i=1}^N n_j = n$ and $h_j(t,x)\colon \Re^{n+1} \to \Re^{n_j}$ are piecewise continuous vector functions in both arguments, satisfying in their domains of continuity the following quadratic inequalities

$$h_j^t(t,x) h_j(t,x) \le \widetilde{\sigma}_j^{-2} x^t \widetilde{H}_j^t \widetilde{H}_j x, \tag{2.2}$$

where $\widetilde{\sigma}_j > 0$ are bounding parameters and $\widetilde{H}_j$ are constant $\alpha_j \times n$ matrices, $j = 1, \dots, N$.

The interconnected system can be represented as

$$\begin{aligned} \mathbf{S}\colon\quad \dot{x} &= Ax + Bu + h(t,x), \\ y &= Cx, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} u = [u_1^t, \dots, u_N^t]^t, \qquad y = [y_1^t, \dots, y_N^t]^t, \\ h(t,x) = [h_1(t,x)^t, \dots, h_N(t,x)^t]^t \end{aligned}$$

are the global input, output and interconnection vectors, respectively, with

$$\sum_{i=1}^{N} m_j = m, \qquad \sum_{i=1}^{N} p_j = p,$$

$$A = \mathrm{diag}[A_1, \ldots, A_N], \qquad B = \mathrm{diag}[B_1, \ldots, B_N], \qquad C = \mathrm{diag}[C_1, \ldots, C_N]$$

and $h(t,x)$ is the global interconnection function. Proceeding further, define $\bar{H}^t = [H_1^t \vdots \ldots \vdots H_N^t]$, where $\widetilde{H}_j$, $j = 1, \ldots, N$, are defined in (2.2), and

$$\widetilde{\Gamma} = \mathrm{diag}[\tilde{\gamma}_1 I_{\alpha 1}, \ldots, \tilde{\gamma}_N I_{\alpha N}], \quad \tilde{\gamma}_j = \bar{\alpha}_j^{-2}, \; I_{\alpha j} \in \Re^{\alpha j \times \alpha_j}$$

then, it is always possible to find matrices H, Γ such that

$$h(t,x)^t h(t,x) \le x^t \bar{H}^t \bar{\Gamma}^{-1} \bar{H}_x \le x^t H^t \Gamma^{-1} H x, \tag{2.4}$$

where

$$H = \mathrm{diag}[H_1, \ldots, H_N], \quad H_j \in \Re^{\alpha_j \times n_j}, \qquad \Gamma = \mathrm{diag}[\gamma_1 I_{\alpha 1}, \ldots, \gamma_N I_{\alpha N}],$$
$$j = 1, \ldots, N.$$

It is not difficult to show that matrices H and Γ satisfy

$$\lambda_M(\bar{H}^t \bar{H}) \min_i \bar{\gamma}_j \le \max_i \gamma_j \min_i \lambda_{\min}(H_j H_j^t)$$

represent a possible choice; different structures can be chosen in accordance with the problem under consideration, see [55, 58] for further elaboration.

Remark 2.1 The main feature of this class is its suitability to develop an LMI-based method for designing dynamic output feedback for robust decentralized stabilization of interconnected systems. This scheme is selected as a methodological basis for several reasons [55]. First, the method applies to systems composed of linear subsystems coupled by nonlinear interconnections. This type of model is attractive since, in most practical situations, local subsystem models are known with sufficient precision to make the linearization successful, while the interconnections are largely unknown: only their bounds are available for control design. Second, the scheme allows for maximization of interconnection bounds, and third, the resulting closed-loop system is connectively stable. Elaborations of the basic scheme in [55] presented in the literature have been related either to the state feedback [55], or to output feedback schemes containing an observer of Luenberger type [9, 46–54, 56–65, 67–89].

As will be shown later on that by assuming decentralized dynamic linear output feedback with a general structure, we apply the classical methodology of $\mathcal{H}_\infty$ controller design [6, 11, 24] to the basic scheme from [55]. As a result, a new two-step LMI-based design procedure is obtained, providing at the first step the block-diagonal Lyapunov matrix, together with the robustness degree vector, and at the second step the decentralized controller parameters.

2.1.2 Class II

Large-scale systems, frequently consisting of a set of small-interconnected subsystems, can be found in many applications such as electric power systems, industrial manipulators, computer networks, to name a few. On one hand, the centralized control of these systems is usually infeasible mainly due to the requirement of a formidable amount of information exchange. In this regard, decentralized control is often preferable [60] whereby a control law based only on local information is designed and implemented. In view of the interconnections among subsystems, the design of a decentralized control is in general more difficult than that of a centralized control. On the other hand, due to their complexity, exact modeling of large-scale systems is usually impossible. Therefore, it is of practical significance that decentralized control must reflect such design constraints by taking into account possible modeling uncertainties. Usually, the uncertainties for interconnected systems appear not only in local subsystems but also in interconnections.

From the literature, decentralized robust control for interconnected linear systems with uncertainties satisfying the so-called strict matching conditions was investigated in [3, 17, 56] and references cited therein. The interconnections among subsystems treated in these works are mostly bounded by first-order polynomials. It was pointed out in [13, 18, 38, 56] that interconnected systems with a decentralized control based on the first-order bounded interconnections may become unstable when the interconnections are of higher order. Decentralized robust stabilization was considered in [20] for systems with interconnections bounded by some nonlinear functions and uncertainties satisfying the so-called matching conditions. Decentralized adaptive control for a class of interconnected nonlinear systems was proposed in [22, 25] based on exact linearization by following the development of centralized control of nonlinear systems [23, 32, 39] and where the strict matching condition was relaxed and higher-order interconnections among subsystems were introduced.

2.1.2.1 System Description

The second class of systems considered in this chapter looks at a large-scale nonlinear system as comprised of N interconnected subsystems with time-varying unknown parameters and/or disturbances entering nonlinearly into the state equation. The jth subsystem is given as

$$\begin{aligned}
\dot{z}_j &= f_{i0}(z_j, x_{j1}) + g_{j0}(z_j, \bar{x}_{j0}, Z_j, Y_j; \theta)x_{j1},\\
\dot{x}_{j1} &= x_{j2} + g_{j1}(z_j, \bar{x}_{j1}, Z_j, Y_j; \theta),\\
\dot{x}_{j2} &= x_{j3} + g_{j2}(z_j, \bar{x}_{j2}, Z_j, Y_j; \theta),\\
&\ \ \vdots
\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
\dot{x}_{j,r_j-1} &= x_{j,r_j} + \phi_{j,r_j-1}(z_j, \bar{x}_{j,r_j-1}, Z_j, Y_j; \theta),\\
\dot{x}_{j,r_j} &= v_j + \phi_{j,r_j}(z_j, \bar{x}_{j,r_j}, Z_j, Y_j; \theta),\\
y_j &= x_{j1},
\end{aligned}
$$

where $\bar{x}_{j,k} = [x_{j1}\, x_{j2}\, \ldots\, x_{jk}]^t$ with $\bar{x}_{j0} = x_{j1}$, $x_j = \bar{x}_{jr_j}$, (z_j, x_j) is the state vector of the jth subsystem with

$$
\begin{aligned}
&z_j \in \Re^{n_j - r_j}, \quad Z_j = [z_1^t\; z_2^t\; \ldots\; z_{j-1}^t\; z_{j+1}^t\; \ldots\; z_N^t]^t,\\
&Y_j = [y_1\; y_2\; \ldots\; y_{j-1}\; y_{j+1}\; \ldots\; y_N]^t
\end{aligned}
$$

and $v_j \in \Re$ is the control input, $y_j \in \Re$ is the output, θ is a vector of unknown, time-varying piecewise continuous parameters and/or disturbances which belong to a known compact set Ω, the vector fields f_{j0} and ϕ_{jk} are smooth with $f_{j0}(0,0) = 0$ and $g_{jk}(0,0,0,0;\theta) = 0$, $\forall \theta \in \Omega$, $1 \le j \le N$, $0 \le j \le r_j$. Observe that the vector (g_{jk}), $k = 0, 1, 2, \ldots, r_j$, represents the interconnections of the ith subsystem with the other subsystems.

Remark 2.2 In what follows, we consider the decentralized robust control problem for a wider class of interconnected systems with partially feedback linearizable subsystems and nonlinear parameterization of time-varying parametric uncertainty. Observe from (2.5) that the interconnections involve the zero-dynamics and outputs of other subsystems. This is in contrast to [25] where an adaptive stabilization was considered for a class of interconnected nonlinear systems whose subsystems are exactly feedback linearizable and with linear parameterization of parameter uncertainty. Geometrical conditions on the isolated subsystems and interconnections such that the interconnected nonlinear systems are transformable into the so-called decentralized strict feedback form has been characterized in [25].

Remark 2.3 Similar to the centralized case discussed in [35, 40], the zero dynamics of each subsystem in (2.5) are independent of the uncertain parameter vector θ.

In the sequel, we assume that $n_j = n$, $r_j = r$, $1 \le j \le N$. Then, by considering $y_j = x_{j1}$, system (2.5) becomes

$$
\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + g_{j0}(z_j, \bar{x}_{i0}, Z_j, X_{j1}; \theta) x_{j1},\\
\dot{x}_{j1} &= x_{j2} + g_{j1}(z_j, \bar{x}_{j1}, Z_j, X_{j1}; \theta),\\
\dot{x}_{j2} &= x_{j3} + g_{j2}(z_j, \bar{x}_{j2}, Z_j, X_{j1}; \theta),\\
&\vdots\\
\dot{x}_{j,r-1} &= x_{j,r} + g_{j,r-1}(z_j, \bar{x}_{j,r-1}, Z_j, X_{j1}; \theta),\\
\dot{x}_{j,r} &= v_j + g_{j,r}(z_j, \bar{x}_{j,r}, Z_j, X_{j1}; \theta),
\end{aligned}
\tag{2.6}
$$

where $X_{j1} = Y_j = [x_{11}\; x_{21}\; \ldots x_{j-1,1}\; x_{j+1,1}\; \ldots\; x_{N1}]^t$.

The following assumptions are made for system (2.6).

Assumption 2.1 There exist some smooth real-valued functions

$$V_{j0}(z_j), \quad j = 1, 2, \ldots, N,$$

which are positive definite and proper (radially unbounded), such that

$$\frac{\partial V_{j0}}{\partial z_j} f_{j0}(z_j, 0) \leq -\nu_j \|z_j\|^2, \quad 1 \leq j \leq N, \tag{2.7}$$

for some positive real numbers $\nu_j > 0$.

Assumption 2.2 The nonlinear interconnections g_{jk} in (2.6) satisfy

$$\begin{aligned} &|g_{jk}(z_j, \bar{x}_{jk}, Z_j, X_{j1}; \theta) - \phi_{jk}(z_j, \bar{x}_{jk}, 0, 0, \theta)| \\ &\quad \leq \sum_{\ell=1}^{N} \eta_{jk\ell}(z_j, \bar{x}_{jk}) [\zeta^0_{jk\ell}(\|z_l\|)\|z_l\| + \zeta^1_{jk\ell}(z_\ell, x_{\ell 1})|x_{\ell 1}|] \\ &\quad \leq \sum_{\ell=1}^{N} \eta_{jk\ell}(z_j, \bar{x}_{jk}) \zeta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|), \end{aligned} \tag{2.8}$$

for any $\theta \in \Omega$, $\eta_{jk\ell}(\cdot)$, $\zeta^0_{jk\ell}(\cdot)$ and $\zeta^1_{jk\ell}(\cdot)$, $\ell = 1, 2, \ldots, N$, $0 \leq k \leq r$, $1 \leq j \leq N$ are nonnegative smooth functions with $\zeta^0_{jki}(\cdot) = \zeta^1_{jkj}(\cdot) \equiv 0$.

Remark 2.4 By the well-known converse Lyapunov theorem [29, 31], the zero dynamics of each subsystem are globally asymptotically stable if and only if there exists a positive definite and proper Lyapunov function V_{j0} such that $(\partial V_{j0}/\partial z_j) f_{j0}(z_j, 0) < 0$, $\forall z_j \neq 0$. Indeed, the requirement (2.7) is more restrictive than this. However, a globally exponentially minimum-phase nonlinear system (that is, the zero-dynamics of the system are globally exponentially stable) always satisfies condition (2.7).

Remark 2.5 The interconnections in Assumption 2.2 are very general, including many types of interconnections considered in existing literature as special cases, for example, interconnections bounded by linear (first-order) polynomials [3], and higher-order polynomials [56]. By contrast to the work in [3, 20, 27, 56], no matching conditions are imposed for system (2.6).

Later on, we will deal with the decentralized global robust stabilization problem for system (2.6) satisfying Assumptions 2.1 and 2.2. More precisely, we are concerned with the design of decentralized robust control laws $v_j = v_j(z_j, x_j)$, $j = 1, \ldots, N$, such that the overall closed-loop interconnected system (2.6) with the control laws is globally asymptotically stable for all admissible uncertainties and interconnections.

2.1.3 Class III

Recent years have seen steady progress in the field of decentralized control of both linear and nonlinear systems. Decentralized control issues naturally arise from controlling large complex systems found in the power industry, aerospace and chemical engineering applications, and telecommunication networks, to name only a few. Among the main characteristics of decentralized control are the dramatic reduction of computational complexity and the enhancement of robustness and reliability against interacting operation failures. Many researchers have made significant contributions to the development of decentralized control theory for large-scale, or interconnected, dynamic systems ([60] and a rather complete list of earlier references cited therein).

In Class III of this chapter, we study a broad class of large-scale nonlinear systems with output measurements. This problem, usually referred to as decentralized output-feedback control, is technically challenging because of the lack of a general theory for nonlinear observer design and the nonlinear version of the well-known "Separation Principle".

2.1.3.1 System Description

According to this Class III, a large-scale nonlinear system is viewed as comprised of N interconnected subsystems with time-varying unknown parameters and/or disturbances entering nonlinearly into the state equation. The jth subsystem is given as

$$\dot{x}_j = F_j(x_j) + G_j(x_j)u_j + \Pi_{j1}(y_1, \dots, y_N)x_j + \Pi_{j2}(y_1, \dots, y_N)w_j, \tag{2.9}$$

$$y_j = h_j(x_j), \tag{2.10}$$

where $1 \le j \le N$, $x_j \in \Re^{n_j}$, $u_j \in \Re$ and $y_j \in \Re$ represent the state, the single control input and the single output of the local jth subsystem, respectively, and $w_j \in \Re^{n_{w_j}}$ is the disturbance input. Also, $F_j, G_j, h_j, \Lambda_{j1}$ and Π_{j2} are sufficiently smooth functions. In the absence of the interacting terms Π_{j1} and Π_{j2}, the system (2.9)–(2.10) collapses to an isolated single-input single-output SISO system and has been extensively studied in the recent literature. Various constructive control algorithms have been developed for large classes of centralized nonlinear systems in special normal forms. Similar questions in the decentralized context should be addressed, that is, in the presence of strong interactions among local systems of the form (2.9)–(2.10). In the sequel, attention is focused on large-scale dynamic systems of type (2.9)–(2.10) transformable to

$$\begin{aligned}
\dot{z}_j &= Q_j z_j + f_{j0}(y_1, \dots, y_N) + p_{j0}(y_1, \dots, y_N)w_j, \\
\dot{x}_{j1} &= x_{i2} + f_{j1}(y_1, \dots, y_N) + g_{j1}(y_1, \dots, y_N)z_j + p_{j1}(y_1, \dots, y_N)w_j, \\
&\ \vdots
\end{aligned} \tag{2.11}$$

$$\dot{x}_{jn_j} = u_j + f_{jn_j}(y_1, \dots, y_N) + g_{jn_j}(y_1, \dots, y_N)z_j + p_{jn_j}(y_1, \dots, y_N)w_j,$$
$$y_j = x_{j1},$$

where for each $1 \le j \le N$, $z_j \in \Re^{n_{z_j}}$ and $x_j = (x_{i1}, \dots, x_{in_j}) \in \Re^{n_j}$ are the states of the ith transformed subsystem. For every j, Q_j is a constant matrix with appropriate dimension, f_{jk}, g_{jk} and p_{ij} are known and smooth functions.

In the sequel, the following minimum-phase condition is assumed.

Condition A For every $1 \le j \le N$, Q_j is a Hurwitz matrix.

The structure involved in (2.11) is commonly utilized in the past literature in both centralized and decentralized control, the reader is referred to [20, 23, 26, 32, 40, 48, 56, 80]. In view of the existing results on geometric nonlinear control [23, 29, 32, 40], necessary and sufficient geometric conditions can be easily derived under which a nonlinear system (2.9), (2.10) is transformed into (2.11), yielding the so-called "disturbed decentralized output-feedback form".

Remark 2.6 It is worth noting that the nonlinearities in (2.9) depend only on the output $y = (y_1, \dots, y_N)$ and that the unmeasured states $X_j[z_j, x_{j2}, \dots, x_{jn_j}]$ in (2.11) appear linearly. This feature is found appealing in recent studies in global output-feedback control for both centralized and decentralized nonlinear systems, in the framework of robust and/or adaptive control. As a matter of fact, simple counterexamples found in [43] reveal the fundamental limitation of global output-feedback control for systems with strong nonlinearities due to unmeasured states. For example, it has been shown in [43] that there is no continuous (static or dynamic) output-feedback control law that can globally asymptotically stabilize a nonlinear system $\dot{x}_1 = x_2, \dot{x}_2 = x_2^n + u$ with output $y = x_1$ whenever $n \ge 3$.

2.2 Dynamic Output Feedback: Class I

The objective of this section is to propose an approach to robust stabilization of systems which are composed of linear subsystems coupled by nonlinear time-varying interconnections satisfying quadratic constraints. The proposed algorithms, which are formulated within the convex optimization framework, employ linear dynamic feedback structure involving local Luenberger-type observers. It is also shown how the new methodology can produce improved results if interconnections have linear parts that are known a priori. Examples of output stabilization of inverted pendulums and decentralized control of a platoon of vehicles are used to illustrate the applicability of the design method.

With the emergence of the powerful convex optimization toolboxes involving linear matrix inequalities (LMIs), solving problems of controller design within the convex optimization framework became very attractive, see [1, 6, 10, 11, 14, 21, 24, 61]. Of wide-spread interest have been the control problems of formulating sufficient conditions for computing output feedback control laws using convex optimization methods due to the fact that the necessary and sufficient conditions are known to be non convex, in general. These problems become increasingly more difficult to solve when decentralized information structure constraints are imposed in the control design [2, 15, 16, 49, 59, 83, 85]. These information structures can be found in important applications, such as power systems [86], control of formations of unmanned vehicles [65] and control of large structures [34], to name few.

2.2.1 Observer-Based Control Design

In what follows, we propose novel sufficient conditions for the design of decentralized dynamic output controllers in the convex optimization context for stabilization of interconnected systems with linear subsystems and nonlinear time-varying interconnections. Controllers are designed to guarantee robust stability of the overall system and, in addition, maximize the bounds of unknown interconnection terms, starting from the methodology proposed in [55]. In what follows, we adopt here the controller structure containing local observers of Luenberger type. Several algorithms are proposed in the general case of full order observers, differing by complexity and the degree of interdependence between the observer and the feedback gains, where no additional constraints on the parameters of the system model are imposed [46, 58]. It is also shown how the proposed scheme can be used to build reduced-order observers. Particular attention is paid to the case when linear parts of interconnections are known a priori, and an algorithm is proposed which takes advantage of this knowledge to come up with improved results. To illustrate the application of the proposed schemes we include two examples, the first dealing with interconnected pendulums, and the second with the problem of platoons of vehicles in the case when the velocity and acceleration of the neighboring vehicles are not accessible.

Reference is made to model of Class I as described by (2.1)–(2.4). To proceed further, we consider that

1. *The dynamic controller* $\mathbf{F}$ *for* $\mathbf{S}$ *is linear*,
2. *It obeys the decentralized information structure constraint requiring that each subsystem is controlled using its own local output* and
3. *It is composed of an observer of Luenberger-type and a static observer state feedback.*

This motivates us to express controller $\mathbf{F}$ into the

$$\mathbf{F}: \quad \dot{w} = Aw + Bu + L(y - Cw), \qquad u = Kw, \tag{2.12}$$

where $w \in \Re^n$ is the observer state, with $w = [w_1^t, \dots, w_N^t]^t$, $w_j \in \Re^{n_j}$ and

$$K = \mathrm{diag}\{K_1, \dots, K_N\}, \qquad L = \mathrm{diag}\{L_1, \dots, L_N\}$$

represent the global controller parameter matrices while pairs (K_j, L_j) determine the local dynamic controllers.

The resulting closed-loop system $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ can be expressed as

$$\mathbf{S}_c\colon \quad \dot{z} = A_c z + h_c(t, z), \qquad y = C_c z, \tag{2.13}$$

where z is the state vector. Defining

$$z = [z_1^t, z_2^t]^t, \quad z_1 = w, \; z_2 = w - x$$

we obtain

$$\begin{aligned} A_c &= \begin{bmatrix} A + Bk & -LC \\ 0 & A - LC \end{bmatrix}, \qquad C_c = \begin{bmatrix} C \,\vdots\, -C \end{bmatrix}, \\ h_c(t, z) &= \begin{bmatrix} 0 \,\vdots\, -h(z_1 - z_2)^t \end{bmatrix}^t. \end{aligned} \tag{2.14}$$

In view of (2.4), we have now

$$h_c(t, z)^t h_c(t, z) \le z^t H_c^t \Gamma^{-1} H_c z, \tag{2.15}$$

where $H_c = [H \,\vdots\, - H]$.

We now address the key feature of dynamic controller $\mathbf{F}$, that is, it must robustly stabilizes $\mathbf{S}$. According to the results of [55, 58], it is shown that $\boldsymbol{S}$ is robustly stabilized with vector degree $\alpha = [\alpha_1, \ldots, \alpha_N]^t$ if the equilibrium $z = 0$ of the closed-loop system $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ is globally asymptotically stable for all $h_c(t, z)$ satisfying (2.15) for some H_c and Γ.

It turns out that the controller stabilizes the linear part of $\mathbf{S}$ and, at the same time, maximizes its tolerance to uncertain nonlinear interconnections and perturbations. This is nicely expressed by the following LMI-based formulation:

System $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ i*s robustly stable with vector degree* α *if the following problem is feasible*:

$$\begin{aligned} &\min \ \mathrm{Tr}\, \Gamma \\ &\text{subject to} \quad X_c > 0, \quad \begin{bmatrix} X_c A_c + A_c^t X_c & X_c & H_c^t \\ \bullet & -I & 0 \\ \bullet & \bullet & -\Gamma \end{bmatrix} < 0. \end{aligned} \tag{2.16}$$

It must be observed that, by and large, observer-based feedback design cannot be completed directly using (2.16). The main reason for this is that the second matrix inequality is not an LMI in both X_c and the feedback parameter matrix.

Remark 2.7 At this stage we should recall some basic results from [1, 16]. In the case of state-feedback the problem can be readily transformed into an LMI problem by a simple change of variables (convexification procedure). However, in the case of dynamic output feedback the problem becomes far more complex. A decoupled quadratic Lyapunov function with block-diagonal weighting matrix has been used in

[58] to determine the dynamic controller parameters. However, the proposed design procedure imposes additional constraints on the system model characteristics.

In what follows we will provide some modifications of problem (2.16) obtained by convexifying the constraints. Solutions to these problems will provide guaranteed feasible solutions to (2.16) and the upper bound of the objective function Tr Γ.

2.2.1.1 Full Order Observer

Introducing the following matrices

$$
\begin{aligned}
Q &= \mathrm{diag}\{Q_1, \ldots, Q_N\}, \quad Q_j \in \Re^{n_j \times n_j},\\
P &= \mathrm{diag}\{P_1, \ldots, P_N\}, \quad P_j \in \Re^{n_j \times n_j},\\
W &= \mathrm{diag}\{W_1, \ldots, W_N\}, \quad W_j \in \Re^{m_j \times n_j},\\
V &= \mathrm{diag}\{V_1, \ldots, V_N\}, \quad V_j \in \Re^{n_j \times p_j}.
\end{aligned}
$$

For the purpose of simplifying the subsequent analysis, we define the matrix function

$$
\Psi(S, L, M, \Gamma) = \begin{bmatrix} S & L & M \\ \bullet & -I & 0 \\ \bullet & \bullet & -\Gamma \end{bmatrix}, \tag{2.17}
$$

for some S, L, M, Γ matrices with appropriate dimensions.

Problem 2.1

$$
\begin{aligned}
&\min \ \mathrm{Tr}\,\Gamma\\
&\text{subject to} \quad Q > 0, \quad P > 0,\\
&\qquad\qquad \Psi(S_1, I, QH^t, \Gamma) < 0, \quad \Psi(S_2, P, -H^t, \Gamma) < 0,
\end{aligned}
\tag{2.18}
$$

where $S_1 = AQ + QA^t + BW + W^tB^t$ and $S_2 = PA + A^tP - VC - C^tV^t$.

We have the following result:

Theorem 2.1 *System **S** is robustly stabilized by the controller **F** if Problem 2.1 is feasible. The controller parameters are given by*

$$
K = WQ^{-1}, \qquad L = P^{-1}V. \tag{2.19}
$$

Proof In what follows it will be shown that there exists a real number $\lambda > 0$ such that the matrix $X_c = \mathrm{diag}\{\lambda^{-1}Q^{-1}, P\}$ satisfies LMIs (2.16) for some $\Gamma > 0$, where

P and Q are solutions of Problem 2.1. Substituting (2.14) and X_c into (2.16), we obtain

$$\begin{bmatrix} \lambda S_1 & -LC & I & 0 & \lambda QH^t \\ \bullet & S_2 & 0 & P & -H^t \\ \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Gamma \end{bmatrix} < 0. \tag{2.20}$$

By Schur complements, we obtain the following conditions equivalent to (2.20):

$$\begin{aligned} &\Xi_1 < 0, \qquad \lambda \Xi_3(\Gamma_\lambda) - \Xi_2 \Xi_1^{-1} \Xi_2^t + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0, \\ &\Xi_1 = \begin{bmatrix} -I & P \\ P & S_2 \end{bmatrix}, \qquad \Xi_2 = \begin{bmatrix} 0 & -LC \\ 0 & -H \end{bmatrix}, \qquad \Xi_3(X) = \begin{bmatrix} S_1 & QH^t \\ HQ & -X \end{bmatrix}, \\ &\Gamma_\lambda = \lambda^{-1}\Gamma, \qquad X \in \Re^{n \times n}. \end{aligned} \tag{2.21}$$

Now let $\Gamma_0 = \mathrm{diag}\{\gamma_1^0 I_{l_1}, \ldots, \gamma_N^0 I_{l_N}\}$ is the optimal Γ obtained by solving Problem 2.1 and define

$$\nu = \lambda_{\min}(\Xi_1^{-1}), \qquad a = \lambda_M(\Xi_2 \Xi_2^t), \qquad \mu = \lambda_M(\Xi_3(\Gamma_0)).$$

It is easy to see that Ξ_1 and $\Xi_3(\Gamma_0)$ represent principal minors of the matrices $\Psi(S_1, I, QH^t, \Gamma_0) < 0$ and $\Psi(S_2, P, -H^t, \Gamma_0) < 0$ and hence both eigenvalues μ and ν are negative.

Selecting $\Gamma = \lambda^* \Gamma_0, \lambda^* > |\theta|/|\mu|, \theta = -1 + a\nu$ and assuming that $0 < \lambda < \lambda^*$, it follows that

$$\lambda_M\{\Xi_3(\Gamma_\lambda)\} = \lambda_M\{\Xi_3((\lambda^*/\lambda)\Gamma_0)\} \le \lambda_M\{\Xi_3(\Gamma_0)\} = \mu$$

bearing in mind that $\lambda^*/\lambda > 1$. For this selection of Γ and λ, (2.21) is implied by

$$\mu\lambda - \theta < 0, \tag{2.22}$$

which holds true for $|\theta|/|\mu| < \lambda < \lambda^*$. Therefore, the desired λ exists and the proof is completed. □

Remark 2.8 The local robustness degrees defined by

$$\alpha_j = 1 \Big/ \sqrt{\gamma_j^0 |\theta|/|\mu|}, \quad j = 1, \ldots, N$$

guaranteed from Theorem 2.1 are generally conservative. More realistic values can be obtained by plugging the controller parameters obtained by (2.19) into (2.16) and by solving the corresponding minimization problem with variables X_c and Γ. This will be demonstrated in the numerical examples presented later on.

Remark 2.9 It is interesting to note that Problem 2.1 implements the separation principle. The constituent problems

$$Q > 0, \qquad \Psi(S_1, I, QH^t, \Gamma^1) < 0, \qquad P > 0, \qquad \Psi(S_2, P, -H^t, \Gamma^2) < 0$$

can be readily solved independently, the first providing K as in the state feedback design and the second L, robustly stabilizing the observer, so that

$$\Gamma = \mathrm{diag}\{\max(\gamma_1^1, \gamma_1^2) I_{\ell_1}, \ldots, \max(\gamma_N^1, \gamma_N^2) I_{\ell_N}\}.$$

Remark 2.10 An alternative procedure to simplify LMIs in (2.18) is as follows:

Problem 2.2

$$\begin{aligned} &\min\ \mathrm{Tr}\,\Gamma \\ &\text{subject to} \quad Q > 0, \quad P > 0, \quad \Xi_3(\Gamma) < 0, \quad \Xi_1 < 0 \end{aligned} \tag{2.23}$$

while the controller parameters are obtained by using (2.19).

Generally speaking, the achievable robustness degree is lower than the one obtained by solving Problem 2.1. Specifically, it is possible to show using the methodology of Theorem 2.1 that if Q_0, W_0 and Γ_0 are obtained by solving Problem 2.2, then there exist $\rho > 0$ and $\beta > 1$ such that $\Psi(\rho(AQ_0 + Q_0 A^t + BW_0 + W_0^t B^t), I, \rho Q_0 H^t, \beta \Gamma_0) < 0$.

By taking into consideration the interdependence between K and L in the LMIs (2.16), we will attempt to exploit the structure of (2.20) to construct improved algorithms with higher robustness degree.

Problem 2.3

$$\begin{aligned} &\min\ \mathrm{Tr}\,\Gamma \\ &\text{subject to} \quad P > 0, \quad \Psi(S_2, P, -H^t, \Gamma) < 0. \end{aligned} \tag{2.24}$$

1. Use the solutions $P, S_2, \Gamma, L = P^{-1} V$.
2.

$$\begin{aligned} &\min\ \mathrm{Tr}\,\Delta \\ &\text{subject to} \quad Q > 0, \\ &\qquad \begin{bmatrix} S_1 & I & -LC & 0 & QH^t \\ \bullet & -I & 0 & 0 & 0 \\ \bullet & \bullet & S_2 & P & -H^t \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Gamma\Delta \end{bmatrix} < 0, \end{aligned} \tag{2.25}$$

where $\Delta = \mathrm{diag}\{\delta_1 I_{l_1}, \ldots, \delta_N I_{l_N}\}$, $\delta_j > 0, \forall j$.

The following result stands-out:

Theorem 2.2 *System* $\boldsymbol{S}$ *is robustly stabilized by the controller* $\boldsymbol{F}$ *if Problem 2.3 is feasible. Controller parameters are given by (2.19). The robustness degree bounds are given by* $\alpha_j = 1/\sqrt{\gamma_j \delta_j}$.

Proof It is readily seen that the second inequality in (2.25) is identical to inequality (2.16) for $X_c = \text{diag}\{Q^{-1}, P\}$, with Γ replaced by $\Gamma\Delta$ and hence the desired result. □

Remark 2.11 It should be noted that Steps 1 and 2 have to be performed consecutively and not simultaneously, like in Problems 2.1 and 2.2. Alternative algorithms could be derived if one takes, for example,

$$z = [z_1^t, z_2^t]^t, \quad z_1 = x, \; z_2 = x - w,$$
$$A_c = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix}, \qquad C_c = \begin{bmatrix} C \,\vdots\, 0 \end{bmatrix},$$
$$h_c(t, z) = \begin{bmatrix} h^t(z_1) \,\vdots\, h^t(z_1) \end{bmatrix}^t$$

and arrives at a problem similar to Problem 2.3, in which K is determined in the first step, and L in the second step.

2.2.1.2 Reduced Order Observer

The results of the foregoing section can be directly extended to the design of controllers with decentralized reduced order observers. For this purpose, we assume that $C_j = [0_{(n_j - p_j)\times n_j} \,\vdots\, I_{Pi}]$, $p_j \le n_j$ if x_j is divided into

$$x_j = [(x_j^a)^t, (x_j^c)^t]^t, \quad x_j^a \in \Re^{n_j - p_j}, \; x_j^c \in \Re^{p_j}$$

then $y_j = x_j^c$ and the output $w_j \in \Re^{n_j - p_j}$ of the local reduced order observer is an estimate of x_j^a. Similar to [33], we assume that the local dynamic controllers F_j have the form:

$$\dot{w}_j = A_j^{11} w_j + A_j^{12} y_j + B_j^1 u_j + L_j[\dot{y}_j - A_j^{21} w_j - A_j^{22} y_j - B_j^2 u_j], \tag{2.26}$$
$$u_j = G_j w_j + J_j y_j = K_j \xi_j, \tag{2.27}$$

where

$$A_j = \begin{bmatrix} A_j^{11} & A_j^{12} \\ A_j^{21} & A_j^{22} \end{bmatrix}, \qquad B_j = \begin{bmatrix} B_j^1 \\ B_j^2 \end{bmatrix},$$
$$\xi_j = [w_j^t, y_j^t]^t = [w_j^t, (x_j^c)^t]^t.$$

Note that differentiation of y_j in (2.26) can be avoided by standard transformation of variables. Defining

$$\eta_j = w_j - x_j^a, \qquad \xi = [\xi_1^t, \dots, \xi_N^t]^t, \qquad \eta = [\eta_1^t, \dots, \eta_N^t]^t$$

we take $z = [\xi^t, \eta^t]^t$ as a new state vector for $S_c = (S, F)$, and obtain

$$\mathbf{S}_f\colon \quad \dot{z} = \begin{bmatrix} A + BK & \bar{L}A^{12} \\ 0 & A^{11} - LA^{21} \end{bmatrix} z + h_c(t, z), \tag{2.28}$$

where

$$\begin{aligned}
A^{11} &= \operatorname{diag}\{A_1^{11}, \dots, A_N^{11}\}, \qquad A^{12} = \operatorname{diag}\{A_1^{12}, \dots, A_N^{12}\}, \\
A^{21} &= \operatorname{diag}\{A_1^{21}, \dots, A_N^{21}\}, \qquad K = \operatorname{diag}\{K_1, \dots, K_N\}, \\
L &= \operatorname{diag}\{L_1, \dots, L_N\}, \qquad \bar{L} = \operatorname{diag}\{\bar{L}_1, \dots, \bar{L}_N\}, \\
\bar{L}_j &= \left[-L_j^t \; -I_{pi}\right]^t, \\
h_c(t, z) &= \left[\left[0_{n_1-p_1}^t \,\vdots\, h_1^c(x)^t\right], \dots, \left[0_{n_N-p_N}^t \,\vdots\, h_N^c(x)^t\right], -h_1^a(x)^t, \dots, -h_N^a(x)^t\right]^t
\end{aligned}$$

where the decomposition $h_j(x) = (h_j^a(x)^t, h_j^c(x)^t)^t$ is induced by the decomposition of x_j into x_j^a and x_j^c. This leads to

$$h_c(t, z)^t h_c(t, z) \le \alpha^2 z^t \bar{H}_c^t \bar{H}_c z, \tag{2.29}$$

where $\bar{H}_c = [H \,\vdots\, -\bar{H}]$, $\bar{H}^t = [\bar{H}_1^t \,\vdots\, \dots \,\vdots\, \bar{H}_N^t]$, while $\bar{H}_j$ is an $l_j \times (n_j - p_j)$ matrix containing the first $n_j - p_j$ columns of H_j, having in mind that $H_j x = H_j\xi - \bar{H}_j\eta$.

The structure of the closed-loop model (2.28) shows that controller design can be entirely based on the methodology developed earlier. Hence, Problem 2.1 and Theorem 2.1 yield

Corollary 2.1 *System* $\mathbf{S}$ *in which*

$$\begin{aligned}
&C = \operatorname{diag}\left\{\left[0_{(n_1-p_1)\times p_1} \,\vdots\, I_{p1}\right], \dots, \left[0_{(n_N-p_N)\times p_N} \,\vdots\, I_{p_N}\right]\right\}, \\
&\quad p_j \le j = 1, \dots, N
\end{aligned}$$

is robustly stabilized by the dynamic controller $\boldsymbol{F}$ *defined by* (2.26), (2.27) *if the following problem is feasible*:

$$\begin{aligned}
&\min \ \operatorname{Tr} \Gamma \\
&\text{subject to} \quad Q > 0, \quad \bar{P} > 0, \\
&\qquad\qquad\quad \Psi(S_1, I, QH^t, \Gamma) < 0, \\
&\qquad\qquad\quad \Psi(\bar{S}_2, \bar{P}, -H^t, \Gamma) < 0,
\end{aligned} \tag{2.30}$$

where

$$\bar{S}_2 = \bar{P}A^{11} + (A^{11})^t\bar{P} - \bar{V}A^{21} - (A^{21})^t\bar{V}^t \in \Re^{n_j - p_j \times n_j - p_j},$$
$$\bar{P} = \mathrm{diag}\{\bar{P}_1, \ldots, \bar{P}_N\} \in \Re^{n_j - p_j \times p_j}, \qquad \bar{V} = \mathrm{diag}\{\bar{V}_1, \ldots, \bar{V}_N\}.$$

The controller parameters are obtained by using (2.19).

2.2.1.3 Important Special Case

We now look at the special case where the interconnections between the subsystems $\mathbf{S}_j$ in $\mathbf{S}$ is known, linear and can be represented by a full matrix $A_s \in \Re^{n\times n}$ containing off diagonal interconnection blocks, so that $A + A_s$ becomes the new state matrix in the linear part of $\mathbf{S}$ in (2.2). The function $h(t, x)$ still represents the unknown part of interconnections.

The foregoing design methodology can be extended to this case while aiming to exploit the additional *a priori* information constraint. A point of caution must be entertained here. By replacing $\mathbf{A}$ by $A + A_s$ in the observer equation for $\mathbf{F}$ in (2.12) one violates the adopted information structure constraint, i.e. the dynamic controller ceases to be decentralized. Inserting $A + A_s$ only in the state model (2.3), we obtain

$$A_c = \begin{bmatrix} A + BK & \vdots & -LC \\ -A_\delta & \vdots & A + A_s - LC \end{bmatrix}.$$

This fact indicates that the design scheme could now be based on modifying the problems described in Sects. 2.2.1.1 and 2.2.1.2 by inserting the new information in the form of A_s at the corresponding places in the related LMIs. Robust stabilization is achievable however, when the interconnections are sufficiently weak. For example, Problem 2.1 turns to be:

Problem 2.4

$$\min \ \mathrm{Tr}\,\Gamma \tag{2.31}$$
$$\text{subject to} \quad P > 0, \quad Q > 0, \tag{2.32}$$
$$\Psi(S_1, I, QH^t, \Gamma) < 0, \tag{2.33}$$
$$\Psi(S_{2s}, P, -H^t, \Gamma) < 0, \tag{2.34}$$

where $S_{2s} = P(A + A_s) + (A + A_s)^t P - VC - C^tV^t$.

Theorem 2.3 *The system* $\mathbf{S}$ *with known linear interconnections* (*modeled by adding* A_δ *to* A *in* (2.3)) *is robustly stabilized by the decentralized dynamic controller* $\mathbf{F}$ *in* (2.12) *if Problem* 2.4 *is feasible and*

$$\delta < \frac{\mu^2}{8\theta_\delta \nu_\delta \lambda_p \lambda_Q}, \tag{2.35}$$

where $\delta = \lambda_M(A_s^t A_s)$, $\lambda_P = \lambda_M(P^2)$, $\lambda_Q = \lambda_M(Q^2)$, $\nu_\delta = \lambda_m(\Xi_{1\delta}^{-1})$, *matrix* Ξ_{1s} *is obtained from* Ξ_1 *in* (2.21) *by replacing* S_2 *with* S_{2s}, *and* $\theta_s = -1 + 2a\upsilon_s$.

Proof The proof is based on a line of thought similar to that applied in Theorem 2.1. Inserting

$$X_c = \mathrm{diag}\{\lambda^{-1}Q^{-1}, P\}, \qquad A_c = \begin{bmatrix} A + BK & -LC \\ -A_s & A + A_s - LC \end{bmatrix}$$

into (2.16) we obtain

$$\begin{bmatrix} \lambda S_1 & -L_\delta & I & 0 & \lambda Q H^t \\ \bullet & S_{2s} & 0 & P & -H^t \\ \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Gamma \end{bmatrix} < 0, \tag{2.36}$$

where $L_s = LC + \lambda Q A_s^t P$. The last inequality is equivalent to $\Xi_{1s} < 0$ and

$$\lambda \Xi_3(\Gamma_s) - (\Xi_2 + \lambda \Xi_{2s})\Xi_{1\delta}^{-1}(\Xi_2 + \lambda \Xi_{2\delta})^t + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0, \tag{2.37}$$

where

$$\Xi_{2s} = \begin{bmatrix} 0 & -Q A_s^t P \\ 0 & 0 \end{bmatrix}.$$

By similarity to Theorem 2.1, we let $P = \lambda^* \Gamma_0$ for some $\lambda^* > 0$, where Γ_0 is the optimal value obtained by solving Problem 2.4. Assume that $0 < \lambda < \lambda^*$. Then, (2.37) is implied by

$$-2\delta \nu_s \lambda_P \lambda_Q \lambda^2 + \mu\lambda - \theta_s < 0, \tag{2.38}$$

bearing in mind that $\lambda_M\{\Xi_3(\Gamma_\lambda)\}\lambda_M\{\Xi_3(\Gamma_0)\} = \mu$. Observe that $\nu_s < 0$ by assumption, as a consequence of the feasibility of Problem 2.4. The existence of $\lambda > 0$ satisfying (2.38) is guaranteed if (2.35) holds, since then we have $D = \mu^2 - 8\delta\theta_s \nu_s \lambda_P \lambda_Q > 0$. Consequently, we choose

$$\frac{-\mu - \sqrt{D}}{-4\delta \nu_s \lambda_P \lambda_Q} = \lambda_1 < \lambda^* \le \lambda_2 = \frac{-\mu + \sqrt{D}}{-4\delta \nu_s \lambda_P \lambda_Q},$$

where $0 < \lambda_1 < \lambda_2$ since $\mu < 0$ and $\sqrt{D} \le |\mu|$, so that λ can take any value in the interval $[\lambda_1, \lambda^*]$. The local guaranteed robustness degree bounds are now $\alpha_j = 1/\sqrt{\gamma_j^0 \lambda_1}$, $j = 1, \ldots, N$, which concludes the proof. $\square$

2.2.2 Simulation Example 2.1

Consider the motion of two inverted pendulums connected by a spring which can slide up and down the rods of the pendulums in jumps of unpredictable size and direction between the support and the height equal to 1 [55]. An appropriate linearized and normalized model is given by

$$
\begin{aligned}
\mathbf{S}\colon\quad \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + h(t,x), \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x, \\
h(t,x) &= e(t,x)Gx, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix},
\end{aligned}
\tag{2.39}
$$

where $e(t,x) : \Re^5 \to [0, 1]$ represents a normalized interconnection parameter.

It is required to compute a decentralized control law which would connectively stabilize the system for all values of $e(t,x) \in [0, 1]$.

A decentralized state-feedback is designed to provide $\alpha = 4.4950$ with the local controller gain matrix $K = [-725.9085 \; -40.4346]$ and the corresponding closed-loop poles $\{-20 \pm j17.8093\}$.

Computer simulation shows that the system is not stabilizable by static output feedback, since two coefficients of the characteristic equation remain fixed to zero irrespective of the controller parameters.

Turning to dynamic output feedback obtained by the proposed algorithms, Table 2.1 provides results on robustness degree α. In this table, Case A corresponds to the situation in which $H = G$ in the three algorithms from Sect. 2.2.1.1. Case B refers to $H = G$ with

$$
A_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & -0.5 & 0 \end{bmatrix}
$$

when the algorithms derived from Problems 2.1–2.3 in accordance with the methodology of Problem 2.4 and Theorem 2.3. Case C represents the situation with no a

Table 2.1 Robustness degree α for different algorithms

	Problem 2.1	Problem 2.2	Problem 2.3
Case A	5.6450	0.3813	21.7304
Case B	4.3840	0.5787	15.2214
Case C	0.6564	0.3191	0.7003

priori knowledge, when $H = I$ and $A_s = 0$, and the algorithms from Sect. 2.2.1.1 are applied.

The ensuing results lead to the conclusion that the best results are obtained by solving Problem 2.3; the worst case corresponds to Problem 2.2. This is quite expected. In view of the results of [55], we note that in Case C none of the algorithms ensures connective stability. For Problem 2.2, connective stability is achieved only in Case B, when the information about the interconnections is included. This corresponds in Case B to have, in fact, $e(t,x) = 0.5 + e^a(t,x)$, where $e^a(t,x) \in [-0.5, 0.5]$, so that any value of $\alpha > 0.5$ is sufficient for connective stability. All values of K and L and the corresponding modes are not presented because of the lack of space. For example, for Problem 2.1 and Case A we have $K_j = [-79.1666 \;\; -11.2883]$, $L_j^t = [27.7711 \;\; 15.7991]$, with local closed-loop poles $\{-27.2275, -0.5435, -0.5441 \pm j6.8052\}$.

2.2.3 Simulation Example 2.2

This example is concerned with the decentralized control of a platoon of vehicles. A feedback-linearized state space model of a platoon of N automotive vehicles is based, according to [65], on the following feedback linearized individual vehicle model:

$$\dot{d}_j = v_{j-1} - v_j, \qquad \dot{v}_j = a_j, \qquad \dot{a}_j = -\tau_j^{-1} a_j + \tau_j^{-1} u_j, \tag{2.40}$$

where $d_j = x_{j-1} - x_j$ is the distance between two consecutive vehicles, x_{j-1} and x_j being their positions, v_j and a_j are the velocity and acceleration of ith vehicle, respectively, u_j the input signal chosen to make the closed-loop system satisfy certain performance criteria, and τ_j the time constant of the engine. After obtaining the overall platoon state space model with the state

$$X = (d_1 - d_r, v_1 - v_r, a_1 - a_r, \dots, d_N - d_r, v_N - v_r, a_N - a_r)^t$$

and input

$$u = (u_1, u_2, \dots, u_N)^t,$$

where d_r, v_r, a_r are the reference values for inter-vehicle distance, velocity and acceleration, respectively, and applying the state and input expansion by using convenient full-rank linear transformations, the following model in the expanded space is obtained [65]:

$$\tilde{\mathbf{S}}\colon \quad \dot{\xi} = \tilde{A}\xi + \tilde{B}\zeta, \tag{2.41}$$

where

$$\xi = [\xi_1^t, \dots, \xi_N^t]^t, \quad \zeta = [\zeta_1^t, \dots, \zeta_N^t]^t,$$
$$\tilde{A} = \operatorname{diag}\{A_1, \dots, A_N\}, \quad \tilde{B} = \operatorname{diag}\{B_1, \dots, B_N\}$$

with vectors ξ_j and ζ_j and matrices A_j and B_j are defined within the formally defined subsystem models connected to each vehicle:

$$\mathbf{S}_j: \quad \dot{\xi}_j = A_j \xi_j + B_j \zeta_j = \begin{bmatrix} A_j^l & 0 \\ A_d & A_j^v \end{bmatrix} \xi_j + \begin{bmatrix} B_j^l & 0 \\ 0 & B_j^v \end{bmatrix} \zeta_j, \tag{2.42}$$

with $\xi_j = [v_{j-1} - v_r, a_{i-1} - a_r, d_j - d_r, v_j - v_r, a_j - a_r]^t$ being the state vector of jth subsystem, $\zeta_j = (u_{j-1}, u_j)^t$ represents its control vector, while

$$A_j^\ell = \begin{bmatrix} 0 & 1 \\ 0 & \tau_j^{-1} \end{bmatrix}, \qquad \bar{A}_d^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_j^\ell = \begin{bmatrix} 0 \\ \tau_j^{-1} \end{bmatrix},$$

$$A_j^v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau_j^{-1} \end{bmatrix}, \qquad B_j^v = \begin{bmatrix} 0 \\ 0 \\ \tau_j^{-1} \end{bmatrix}.$$

This model is treated in [65] where it is shown that a decentralized dynamic control law can be designed for the expanded system using the methodology from Sect. 2.2.1.2, supposing that only the subsystem states $d_j - d_r$, $v_j - v_r$ and $a_j - a_r$ are exactly known in jth vehicle (subsystem), that is, v_{j-1} and a_{j-1} are not accessible in ith vehicle. Applying the results of Sect. 2.2.1.2, the reduced-order Luenberger observer for $\xi_j^1 = (v_{j-1} - v_r, a_{j-1} - a_r)^t$ is given by

$$\dot{w}_j = A_j^l w_j + B_j^l u_{i-1} + L_j[\dot{\xi}_j^2 - \bar{A}_d w_j - A_j^v \xi_j^2], \tag{2.43}$$

where $\xi_j^2 = (d_j - d_r, v_j - v_r, a_j - a_r)^t$. The local control law has the following specific structure:

$$u_{j-1} = G_j^1 w_j, \qquad u_j = G_j^2 w_j + J_j^2 \xi_j^2, \tag{2.44}$$

having in mind that $(j-1)$th vehicle does not have any information about ith vehicle. Matrices

$$K_j = \begin{bmatrix} G_j^1 & 0 \\ G_j^2 & J_j^2 \end{bmatrix}, \quad L_j, \quad j = 1, \ldots, N$$

can now be obtained by using the algorithm from Corollary 2.1, exploiting the specific lower-block-triangular structure of K_j.

For $\tau_j = \tau = 0.1$, one obtains:

$$G_j = G = [-38.6940\ -2.1224], \qquad G_j^1 = G^1 = [-38.6940\ -2.1224],$$
$$G_j^2 = G^2 = [0.0095\ 0.0005],$$
$$J_j^2 = J^2 = [351.4028\ -319.3970\ -13.2356],$$
$$L_j = L = 10^4 \begin{bmatrix} 0.0001 & 0 & 0 \\ 3.2068 & 0 & 0 \end{bmatrix}, \qquad \alpha_j = \alpha = 1/4.080$$

generating the closed loop poles

$$10^2\{-1.1480, -0.0116, -0.1561 \pm j0.1197, -0.2640, -320.68, -0.00004\}.$$

Obviously, it is also possible to apply the alternative design schemes from Sect. 2.2.1.1. By using the expansion/contraction matrices as in [60] and [65], the obtained controller has to be finally contracted to the original space for implementation.

2.2.4 Simulation Example 2.3

The third example considered here is a linearized two-tank system modeled in the form (2.1) with data

$$A_1 = \begin{bmatrix} 0.703 & 0 & 0.395 & -0.320 \\ -0.052 & 0 & 0 & -0.137 \\ 0 & 0 & 0 & 0.619 \\ 0 & 1.028 & 1.752 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.402 & 0.978 \\ 0 & 0 \\ -0.263 & 0.159 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.423 & 0 & 0 & 0.317 \\ 0 & 0.137 & 0.576 & 0.340 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.695 & 0.013 & 0.315 & -0.414 \\ -0.193 & 0 & 0 & 0.258 \\ 0 & 0 & 0 & -0.834 \\ 0 & 0.879 & 0.978 & 0.015 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.375 & 0.888 \\ 0 & 0 \\ -0.249 & 0.147 \\ 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.462 & 0 & 0 & 0.351 \\ 0 & 0.098 & 0.685 & 0.742 \end{bmatrix},$$

$$h(t,x) = f(t,x)Mx, \quad M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $f(t,x) : \Re^4 \to [0, 1]$ represents a normalized coupling parameter. Exploring decentralized control design, we get state-feedback results with local gains as

$$K_1 = \begin{bmatrix} -6.222 & 18.345 & 28.367 & 17.793 \\ -1.893 & -8.148 & -13.479 & -8.542 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -13.028 & 34.718 & 52.797 & 33.092 \\ -5.766 & 12.867 & 18.465 & 11.302 \end{bmatrix},$$

which do not stabilize the two-tank system. On the other hand, the output feedback gains are given by

$$K_1 = \begin{bmatrix} -36.856 & 10.094 \\ 15.441 & 9.313 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -12.188 & 16.158 \\ 35.738 & 2.568 \end{bmatrix},$$

which stabilize the system with robustness degree $\alpha = 3.8436$.

2.2.5 Dynamic Control Design

Extending on the foregoing section, we now consider a general linear time-invariant dynamic controller **F** for **S** which obeys the decentralized information structure constraint. This entails that each subsystem is controlled using only its own local output. Therefore,

$$\mathbf{F}: \quad \dot{w} = F_c w + L_c y, \qquad u = K_c w + G_c y, \tag{2.45}$$

where $w \in \Re^s$ is the global observer state, and

$$w = \left[w_1 \ldots w_N \right]^t, \quad w_j \in \Re^{s_j}, \; s = \sum_{j=1}^{N} s_j,$$

$$F_c = \mathrm{diag}\{F_{c1}, \ldots, F_{cN}\}, \qquad L_c = \mathrm{diag}\{L_{c1}, \ldots, L_{cN}\},$$
$$K_c = \mathrm{diag}\{K_{c1}, \ldots, K_{cN}\}, \qquad G_c = \mathrm{diag}\{G_{c1}, \ldots, G_{cN}\}.$$

For simplicity in exposition, we denote

$$J_j = \begin{bmatrix} F_{cj} & L_{cj} \\ K_{cj} & G_{cj} \end{bmatrix} \in \Re^{s_j + m_j \times s_j + p_j}$$

the local controller parameter matrices, and by $J = \mathrm{diag}\{J_1, \ldots, J_N\}$ the global controller parameter matrix.

By standard algebraic manipulations, the resulting closed-loop system $\mathbf{S}_c = (\mathbf{S}, \mathbf{F})$ can be represented by

$$\mathbf{S}_c: \quad \dot{z} = A_c z + h_c(t, z), \qquad y = C_c z, \tag{2.46}$$

where

$$z = \left[x_1^t \; w_1^t \ldots x_N^t \; w_N^t \right]^t, \qquad A_c = \mathrm{diag}\{A_{c1}, \ldots, A_{cN}\},$$

$$C_c = \mathrm{diag}\{C_{c1}, \ldots, C_{cN}\}, \qquad h_c(t, z)^t = \left[h_{c_1}^t(t, z) \ldots h_{c_N}^t(t, z) \right]^t,$$

$$A_{cj} = \begin{bmatrix} A_j + B_j G_j C_j & B_j K_j \\ L_j C_j & F_j \end{bmatrix}, \qquad C_{cj} = \begin{bmatrix} C_j \vdots 0 \end{bmatrix},$$
$$h_{cj} = \begin{bmatrix} h_j^t(t,x) \vdots 0 \end{bmatrix}^t.$$

In view of the structural constraint (2.4), we have

$$h^t_{c_j}(t,z) h_{c_j}(t,z) \le z^t \bar{H}^t_c \Gamma^{-1} \bar{H}_c z \le z^t \tilde{H}^t \Gamma^{-1} \tilde{H} z, \tag{2.47}$$

where

$$\bar{H}^f = \begin{bmatrix} \bar{H}_1^{f^t} \vdots \ldots \vdots \bar{H}_j^{f^t} \end{bmatrix}^t,$$
$$\bar{H}_j^f = \begin{bmatrix} \bar{H}_j^1 \vdots 0 \vdots \bar{H}_j^2 \vdots 0 \vdots \ldots \vdots \bar{H}_j^N \vdots 0 \end{bmatrix}$$

in which $\nu_j \times n_j$ matrices $\bar{H}_j^t$ $(j = 1, \ldots, N)$ follow from the decomposition $\bar{H}_j = [\bar{H}_j^1 \vdots \ldots \vdots \bar{H}_j^N]$ while $\tilde{H} = \operatorname{diag}\{\tilde{H}_1, \ldots, \tilde{H}_N\}$ with $\tilde{H}_j = [H_j \vdots 0]$.

Our immediate objective is to design the dynamic controller $\mathbf{F}$ which robustly stabilizes $\mathbf{S}$. Following the results of [9, 46–54, 56–58], it follows that

System $\mathbf{S}$ *is robustly stabilized with vector degree*

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_1 \ldots \bar{\alpha}_N \end{bmatrix}^t = \begin{bmatrix} 1/\sqrt{\bar{\gamma}_1} \ldots 1/\sqrt{\bar{\gamma}_N} \end{bmatrix}^t$$

if the equilibrium $x = 0$ of the closed-loop system $\mathbf{S}_f = (\mathbf{S}, \mathbf{F})$ is globally asymptotically stable for all $h(t,z)$ satisfying (2.4) for some given $\bar{H}$ and $\bar{\alpha}$, according to the first inequalities in (2.4) and (2.47).

It turns out that maximizing $\bar{\alpha}$, the controller stabilizes the linear part of $\mathbf{S}$ and, at the same time, maximizes its tolerance to uncertain nonlinear interconnections and perturbations. In this regard, the nonlinear interconnections bound is represented by a full matrix $\bar{H}$. Bearing in mind that the system model sparsity implied by (2.1) and (2.3) and the developed controller structure in (2.45) designates the perfectly decentralized control [52, 53], the corresponding controller subspace is not quadratically invariant. This entails that the related optimization problem is not convex.

In order to convexify the problem under consideration, we invoke further decompositions by applying the second (right hand side) inequalities in (2.4) and (2.47), and formulate the following modified robust stabilization problem:

System $\mathbf{S}_f = (\mathbf{S}, \mathbf{F})$ *is robustly stable with vector degree* $\alpha = (\alpha_1, \ldots, \alpha_N)^t = (1/\sqrt{\gamma_1}, \ldots, 1/\sqrt{\gamma_N})^t$ *if the following problem is feasible*:

$$\begin{aligned} &\textit{Minimize} \quad \sum_{i=1}^{N} \gamma_j \\ &\textit{subject to} \quad \tilde{X} > 0, \quad \begin{bmatrix} \tilde{X} A^f + A^{f^t} \tilde{X} & \tilde{X} & \tilde{H}^t \\ \tilde{X} & -I & 0 \\ \tilde{H} & 0 & -\Gamma \end{bmatrix} < 0, \end{aligned} \tag{2.48}$$

where $\tilde{X}$ *is the global Lyapunov matrix.*

It must be noted that the matrix $\tilde{H}$ is block-diagonal in accordance with the assumed system sparsity, that is, with the subsystem dimensions. The second matrix inequality in (2.48) however is still not an LMI in both $\tilde{X}$ and the controller parameter matrix.

In the next section, we show that the above general robust stabilization problem can also be formulated as an LMI problem.

2.2.6 Robust Decentralized Design

Having in mind the availability of the system structure, together with the *a priori* knowledge about the interconnection bounds, it is quite natural to consider global Lyapunov matrices $\tilde{X}$ structurally adapted to $\mathbf{S}$ and $\mathbf{F}$:

Assumption 2.3 Matrix $\tilde{X}$ in (2.48) possesses the block-diagonal structure, that is, $\tilde{X} = \mathrm{diag}\{\tilde{X}_1, \dots, \tilde{H}_N\}$ where $\tilde{X}_j \in \Re^{n_j+s_j \times n_j+s_j}$, $j = 1, \dots, N$ are the local Lyapunov matrices.

It must be emphasized that this choice does not represent a significant restriction, giving the fact that the original problem has been already decomposed in (2.48) into N independent robust dynamic output feedback design problems.

Proceeding further, we let

$$\bar{A} = \mathrm{diag}\{\bar{A}_1, \dots, \bar{A}_N\}, \qquad \bar{B} = \mathrm{diag}\{\bar{B}_1, \dots, \bar{B}_N\}, \qquad \bar{C} = \mathrm{diag}\{\bar{C}_1, \dots, \bar{C}_N\},$$

$$\bar{A}_j = \begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{B}_j = \begin{bmatrix} 0 & B_j \\ I & 0 \end{bmatrix}, \qquad \bar{C}_j = \begin{bmatrix} 0 & I \\ C_j & 0 \end{bmatrix}$$

and then write $\widetilde{A} = \bar{A} + \bar{B} J \bar{C}$, where J is the global controller parameter matrix. Consequently, the second inequality in (2.48) can be written as

$$\widetilde{R} + \widehat{B} J \widetilde{C} + \widetilde{C}^t J^t \widehat{B}^t < 0, \tag{2.49}$$

where $\widetilde{R} = \mathrm{diag}\{\widetilde{R}_1, \dots, \widetilde{R}_N\}$, $\widetilde{B} = \mathrm{diag}\{\widetilde{B}_1, \dots, \widetilde{B}_N\}$, $\tilde{C} = \mathrm{diag}\{\widetilde{C}_1, \dots, \widetilde{C}_N\}$

$$\widetilde{R}_j = \begin{bmatrix} \tilde{X}_j \bar{A}_j + \bar{A}_j^t \tilde{X}_j & \tilde{X}_j & \tilde{H}_j^t \\ \bullet & -I & 0 \\ \bullet & \bullet & -\gamma_j I \end{bmatrix}, \qquad \widehat{B}_j = \begin{bmatrix} \tilde{X}_j \bar{B}_j \\ 0 \\ 0 \end{bmatrix}, \qquad \widetilde{C}_j^t = \begin{bmatrix} \bar{C}_j^t \\ 0 \\ 0 \end{bmatrix}.$$

It is interesting to note that the problem (2.49) resembles a compact formulation of a set of N local classical $\mathcal{H}_\infty$ problems for virtual subsystems defined as

$$\dot{x}_j = A_j x_j + B_j u_j + w_j, \qquad z_j = H_j x_j,$$

where the immediate objective is to compute local controllers that render the $\mathcal{H}_\infty$-norms of the transfer functions between w_j and z_j are less than γ_j.

An important observation arises here. The block matrix (2.48) contains the entire Lyapunov matrix $\tilde{X}$, and not of $\tilde{X}\,\mathrm{col}[I, ., 0]$ as it should be in the case of the classical $\mathcal{H}_\infty$ problems [6, 10, 11].

The following lemma provides a pertinent result:

Lemma 2.1 *Let Assumption 2.3 hold and let* $\tilde{X} > 0$. *Then*, (2.49) *holds if and only if*

$$\begin{aligned}
&\widetilde{B}^{\perp}\widetilde{T}\widetilde{B}^{\perp t} < 0, \qquad \tilde{C}^{t\perp}\tilde{R}\tilde{C}^{t\perp t} < 0,\\
&\widetilde{T} = \mathrm{diag}\{\widetilde{T}_1, \ldots, \widetilde{T}_N\},\\
&\widetilde{T}_j = \begin{bmatrix} \tilde{X}_j^{-1}\bar{A}_j^t + \bar{A}_j\tilde{X}_j^{-1} & I_j & \tilde{X}_j^{-1}\tilde{H}_j^t \\ \bullet & -I_j & 0 \\ \bullet & \bullet & -\gamma_j I_j \end{bmatrix},\\
&\widetilde{B}^{\perp} = \mathrm{diag}\{\widetilde{B}_1^{\perp}, \ldots, \widetilde{B}_N^{\perp}\},
\end{aligned} \tag{2.50}$$

where $\widetilde{B}_j^t = [\,\bar{B}_j^t\ 0\ 0]$.[1]

Proof The structure of $\tilde{X}$ and J implies that (2.49) decouples into N independent inequalities

$$\widetilde{R}_j + \widehat{B}_j J_j \widetilde{C}_j + \widetilde{C}_j^t J_j^t \widehat{B}_j^t < 0$$

with general $(s_j + m_j) \times (s_j + p_j)$ matrices J_j. According to the elimination lemma [1], the necessary and sufficient conditions for these inequalities are

$$\widehat{B}_j^{\perp}\tilde{R}_j\widehat{B}_j^{\perp t} < 0, \quad \tilde{C}_j^{t\perp}\tilde{R}_j\tilde{C}_j^{t\perp t} < 0, \quad j = 1, \ldots, N. \tag{2.51}$$

Note that $\widehat{B}_j^{\perp}\tilde{R}_j\widehat{B}_j^{\perp t} < 0$ holds if and only if $\widetilde{B}_j^{\perp}\widetilde{T}_j\widetilde{B}_j^{\perp t} < 0$.

Since $\widehat{B}_j = S_j[\bar{B}_j^t\ 0\ 0]^t$, $S_j = \mathrm{diag}\{\tilde{X}_j, I, I\}$, we have $\widehat{B}_j^{\perp} = \tilde{B}_j^{\perp}S_j^{-1}$, taking into consideration that $S_j^{-1}\tilde{R}_jS_j^{-1} = \tilde{T}_j$ and $\tilde{X} > 0$. This concludes the proof. □

Proceeding further, we follow the approach of [6] and introduce the decompositions:

$$\tilde{X}_j = \begin{bmatrix} X_j & X_{2j} \\ X_{2j}^t & X_{3j} \end{bmatrix}, \qquad \tilde{Y}_j = \tilde{X}_j^{-1} = \begin{bmatrix} Y_j & Y_{2j} \\ Y_{2j}^t & Y_{3j} \end{bmatrix}, \tag{2.52}$$

where $0 < X_j = X_j^t$ and $0 < Y_j = Y_j^t$ are $n_j \times n_j$ real matrices for $j = 1, \ldots, N$. The following result is established:

[1] $A^{\perp}$ denotes a matrix with the properties $\mathcal{N}(A^{\perp}) = \mathcal{R}(A)$ and $A^{\perp}A^{\perp t} > 0$, where $\mathcal{N}(.)$, $\mathcal{R}(.)$ denote the null space and the range space of an indicated matrix.

Lemma 2.2 *Let Assumption 2.3 hold, let* $\tilde{X} > 0$, *and let* X_j, Y_j *and* X_{2j} *be given by* (2.52), $j = 1, \ldots, N$. *Then inequalities* (2.50) *hold if and only if*

$$E^c V E^{ct} < 0, \qquad E^b W E^{bt} < 0, \tag{2.53}$$

where

$$\begin{aligned}
V &= \operatorname{diag}\{V_1, \ldots, V_N\}, \qquad W = \operatorname{diag}\{W_1, \ldots, W_N\}, \\
E^c &= \operatorname{diag}\{E_1^c, \ldots, E_N^c\}, \qquad E^b = \operatorname{diag}\{E_1^b, \ldots, E_N^b\}, \\
E_j^c &= \begin{bmatrix} C_j^{t\perp} & 0 \\ 0 & I \end{bmatrix}, \qquad E_j^b = \begin{bmatrix} B_j^{\perp} & 0 \\ 0 & I \end{bmatrix} \\
V_j &= \begin{bmatrix} X_j A_j + A_j^t X_j & X_j & X_{2j} & H_j^t \\ \bullet & -I & 0 & 0 \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -\gamma_j I \end{bmatrix}, \\
W_j &= \begin{bmatrix} Y_j A_j^t + A_j Y_j & I & Y_j H_j^t \\ \bullet & -I & 0 \\ \bullet & \bullet & -\gamma_j I \end{bmatrix}.
\end{aligned}$$

Proof By definition, we have

$$\tilde{R}_j = \begin{bmatrix} X_j A_j + A_j^t X_j & A_j^t X_{2j} & X_j & X_{2j} & H_j^t \\ \bullet & 0 & X_{2j}^t & X_{3j} & 0 \\ \bullet & \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j I_j \end{bmatrix}.$$

On the other hand, taking into consideration the structure of $\tilde{C}_j$ and $\bar{C}_j$, we have

$$\tilde{C}_j^{t\perp} = \begin{bmatrix} C_j^{t\perp} & 0 & 0 \\ 0 & 0 & I_j \end{bmatrix}.$$

As the second block-column in $\tilde{C}_j^{t\perp}$ contains only zero matrices, the second inequality in (2.50) gives the first inequality in (2.53).

Turning to the second inequality in (2.53), it is not difficult to show that it can be obtained analogously. From

$$\tilde{T}_j = \begin{bmatrix} A_j Y_j + Y_j A_j^t & A_j Y_{2j} & I & 0 & Y_j H_j^t \\ \bullet & 0 & 0 & I_j & Y_{2j}^t H_j^t \\ \bullet & \bullet & -I_j & 0 & 0 \\ \bullet & \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_j I_j \end{bmatrix}$$

and deleting the unnecessary block-rows and block-columns, we arrive at the desired result. □

It is readily seen that the matrices X_j, Y_j and X_{2j} are constrained by (2.53). This is in contrast to the standard $\mathcal{H}_\infty$ design [6, 10, 24]) where only the first diagonal blocks of the global Lyapunov matrix and its inverse are constrained by the corresponding LMIs.

Once X_j, Y_j and X_{2j} are determined, the next problem is to find $\tilde{X}_j > 0$ satisfying (2.52), $j = 1, \ldots, N$.

Lemma 2.3 *Assume that*:

(1) $s_j = n_j$,
(2) X_{2j} *in* (2.52) *is nonsingular, and*
(3) $Q_j = \left[\begin{smallmatrix} X_j & I \\ I & Y_j \end{smallmatrix}\right] > 0$. *Then*,

$$X_{3j} = X_{2j}^t (X_j - Y_1^{-1})^{-1} X_{2j} \quad \Longrightarrow \quad \tilde{X}_j > 0, \quad j = 1, \ldots, N. \tag{2.54}$$

Proof From (2.52), we obtain $Y_{2j}^t = X_{2j}^{-1}(I - X_j Y_j)$, yielding directly (2.54). Obviously, $\tilde{X}_j > 0$, since $X_j > 0$ and $X_j - X_{2j} X_{2j}^{-1} (X_j - Y_j^{-1})(X_{2j}^t)^{-1} X_{2j}^t = Y_j^{-1} > 0$, which completes the proof. □

By combining the foregoing results, we have the following theorem:

Theorem 2.4 *Under Assumption* 2.3, *system* $\boldsymbol{S}$ *in* (2.3) *is robustly stabilized by the dynamic controller* $\boldsymbol{F}$ *in* (2.45) *with* $s_j = n_j$ *if the following problem is feasible*:

$$\begin{aligned} &\textit{minimize} \quad \sum_{j=1}^{N} \gamma_j \\ &\textit{subject to} \quad X > 0, \quad Y > 0, \quad Q > 0, \quad Z > 0, \quad \bar{E}^c \bar{V} \bar{E}^{ct} < 0, \\ &\qquad\qquad E^b W E^{bt} < 0, \end{aligned} \tag{2.55}$$

where $X = \mathrm{diag}\{X_1, \ldots, X_N\}$, $Y = \mathrm{diag}\{Y_1, \ldots, Y_N\}$, $Q = \mathrm{diag}\{Q_1, \ldots, Q_N\}$, $Z = \mathrm{diag}\{Z_1, \ldots, Z_N\}$, $\bar{V} = \mathrm{diag}\{\bar{V}_1, \ldots, \bar{V}_N\}$,

$$\bar{V}_j = \begin{bmatrix} X_j A_j + A_j^t X_j + Z_j & X_j & H_j^t \\ \bullet & -I_j & 0 \\ \bullet & \bullet & -\gamma_j I_j \end{bmatrix}$$

while matrix $\bar{E}^c$ *is a matrix having the same structure as* E^c *in* (2.53), *but with the elements* $\bar{E}_j^c$ *obtained from* $E_j^c = \begin{bmatrix} C_j^{t\perp} & 0 \\ 0 & I \end{bmatrix}$ *in such a way that the dimension of the identity matrix ensures compatibility of the product with* $\bar{V}_j$ (*instead of* V_j), $j = 1, \ldots, N$.

Proof Notice that the inequality $\bar{E}^c \bar{V} \bar{E}^{ct} < 0$ from the problem (2.55) follows immediately from the first inequality in (2.53) in Lemma 2.2 after applying the Schur's complement formula and replacing $X_{2j} X_{2j}^t$ by Z_j in view of the expression for $\bar{V}_j$. Condition $Z > 0$ results from the requirement that the matrices X_{2j} are nonsingular, $j = 1, \ldots, N$. The inequality $E^b W E^{bt} < 0$ is identical to the second inequality in (2.55). This completes the proof. □

Remark 2.12 Solving (2.55), one gets $X > 0$, $Y > 0$ and $Z > 0$. Nonsingular matrices X_{2j} can always be constructed from any given $Z_j > 0$; one gets X_{3j} from (2.54), and, consequently, $\tilde{X}_j > 0$ from (2.52), $j = 1, \ldots, N$. Then, we come back to the original inequality (2.49), which represents then a system of N independent LMIs with unconstrained matrix variables J_j, $j = 1, \ldots, N$. Any solution to these LMIs gives the required block-diagonal parameter matrix $J = \mathrm{diag}\{J_1, \ldots, J_N\}$, that is, a robustly stabilizing decentralized dynamic controller **F** for **S**.

The underlying assumptions in Lemma 2.3 are important for the formulation of Theorem 2.1 in terms of LMIs. In general, in the case of reduced order observers (when $s_j < n_j$), one is faced with the problem of the existence of solutions for Y_{2j}, Y_{3j} and X_{3j} satisfying (2.52); notice that in the case of $\mathcal{H}_\infty$ design we have the rank condition in addition to the condition of the type $Q_j > 0$ [6]. The obtained estimates of the robustness degree α may appear to be too conservative. A better insight into the real robustness can be obtained by calculating A^f with the obtained parameter matrix J, replacing it in (2.48), and solving (2.48) for $\tilde{X}$ and Γ. An even more realistic and less conservative estimate can be obtained by using (2.48) with $\tilde{H}$ being replaced by $\bar{H}$ and Γ by $\bar{\Gamma}$, and by solving the corresponding LMI problem for $\tilde{X}$ and $\bar{\Gamma}$. By limiting the norm of the gain matrices J_j via the procedure of [55, 58] some benefits are anticipated.

Remark 2.13 In the case that the interconnection function in **S** is in the form $h(t,x) = h_L(t,x) + h_N(t,x)$, where $h_L(t,x) = A^h x$ is a known linear part in which A^h is a constant $N \times N$ block-matrix with blocks A^h_{jk}, $j, k = 1, \ldots, N$, and $h_N(t,x)$ is an unknown nonlinear part satisfying inequality (2.4). Taking $A^* = A + A^h$ as a new state matrix in (2.3), instead of (2.49) we have

$$\tilde{R}^* + \Delta\tilde{R} + \tilde{B}^x J \tilde{C}^t + \tilde{C}^t J^t \tilde{B}^{xt} < 0, \tag{2.56}$$

where $\Delta\tilde{R}$ is an $N \times N$ block-matrix with blocks

$$\Delta\tilde{R}_{ij} = \begin{bmatrix} \tilde{X}_j \bar{A}^h_{ij} + \bar{A}^{ht}_{ji} \tilde{X}_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}^h_{ij} = \begin{bmatrix} A^h_{jk} & 0 \\ 0 & 0 \end{bmatrix}, \quad j, k = 1, \ldots, N, \ j \neq k, \qquad \Delta\tilde{R}_{mm} = 0, \quad m = 1, \ldots, N$$

and $\tilde{R}^*$ is obtained from $\tilde{R}$ in (2.49) by replacing A_j by $A^*_{ii} = A_j + A^h_{ii}$. Bearing in mind that $\tilde{R}^* + \Delta\tilde{R}$ is not block-diagonal, Theorem 2.1 cannot be directly applied to (2.56). However, (2.56) can have a solution satisfying Assumption 2.3; it is reasonable to expect that the resulting controller provides better performance than the one obtained in the absence of the assumed a priori knowledge about linear interconnections.

2.2.7 Simulation Example 2.4

This examples uses the model of two inverted pendulums connected by a spring treated in the simulation Example 2.1.

From [55], the decentralized robust linear static state feedback provides $\alpha^* = \alpha_1 = \alpha_2 = 4.4950$, with the local gain matrix $K = [-725.909\ -40.435]$ and the local closed-loop poles $\{-20 \pm j17.8093\}$. It easy to see that the system is not stabilizable by any linear static output feedback.

The local dynamic output feedback controller parameters obtained on the basis of Theorem 2.1, with $H_j = I$ are

$$F_j = 10^4 \begin{bmatrix} -0.4670 & -1.4182 \\ -1.0131 & -3.1931 \end{bmatrix}, \quad L_j = 10^4 \begin{bmatrix} -3.3926 \\ 1.5118 \end{bmatrix},$$
$$K_j = \begin{bmatrix} 243.5166 & 767.0817 \end{bmatrix}, \quad G_j = -333.7029, \quad j = 1, 2$$

with the local closed-loop poles

$$\{-3.6543 \times 10^4, -0.0390 \times 10^4, -0.7455 \pm j0.5605\},$$

with $\alpha^* = 0.5670 < 1$—that is, the desired property is not achieved.

Assuming now that $e(t,x) = 0.5 + \acute{e}(t,x)$, where $\acute{e}(t,x) \in [-0.5, 0.5]$ one obtains the structure with known linear interconnections with

$$A^h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & -0.5 & 0 \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

In this case the LMI (2.56) is feasible and one gets a decentralized stabilizing controller with $\alpha^* = 1.3526$, ensuring stability for all spring positions. The local controller parameter matrices are in this case

$$F_j = 10^6 \begin{bmatrix} -1.4090 & -1.5774 \\ -1.0938 & -1.2571 \end{bmatrix}, \qquad L_j = 10^6 \begin{bmatrix} -5.0635 \\ 5.3116 \end{bmatrix},$$
$$K_j = 10^6 [0.5753\ 0.6573], \qquad G_j = 10^6 \times -1.69250,$$

and the local closed loop poles

$$\{-2.6487 \times 10^6, -1.7217 \times 10^4, -84.866, -1.8834\}.$$

A direct comparison with the results presented in relation with the same example in [67] shows that a better performance is obtained by using an observer of Luenberger type, incorporating the state matrix of the system model and leaving a smaller number of free parameters in the controller design procedure.

2.3 Robust Control Design: Class II

In this section, we investigate the problem of robust decentralized control for a wider class of large-scale nonlinear systems with parametric uncertainty and nonlinear interconnections. This class of systems was labeled in Sect. 2.1.2 as Class II. In this class, each subsystem of the interconnected system is assumed to be partially feedback linearizable and minimum phase. The uncertain parameters and/or disturbances are allowed to be time-varying and enter the system nonlinearly. The nonlinear interconnections are bounded by general nonlinear functions of the zero-dynamics and outputs of other subsystems. Inspired by the centralized nonlinear control results [9, 23, 35, 39, 51], we show in the sequel that decentralized global robust stabilization can be achieved for the uncertain interconnected systems by employing a Lyapunov-based recursive controller design method. Our result relies on a proper construction of Lyapunov function for the interconnected systems.

2.3.1 Construction Procedure

In what follows, we first present the following lemma which provides the first step of the induction in the construction of robust decentralized state feedback control laws of system (2.6).

Lemma 2.4 *Consider the first two state equations of system* (2.6):

$$\begin{aligned} \dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, x_{j1}, Z_j, X_{j1}; \theta)x_{j1}, \\ \dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}, \theta), \\ y_j &= x_{j1}, \end{aligned} \tag{2.57}$$

satisfying Assumptions 2.1 *and* 2.2. *Then, there exists a smooth function* $x_{i2}^*(z_j, x_{j1})$ *with* $x_{j2}^*(0,0) = 0$ *such that system* (2.12) *with the control* $x_{j2} = x_{j2}^*(z_j, x_{j1})$ *in the coordinates*

$$z_j = z_j, \qquad \tilde{x}_{j1} = x_{j1}$$

satisfies

$$\dot{V}_{j1} \le \frac{dW_j(V_{j0})}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{h00} - b_j(z_j, x_{j1}) x_{j1}^2$$
$$- r\tilde{x}_{j1}^2 + \|z_j\|^2 + \frac{1}{2} \sum_{l=1}^{N} \delta_{j1l}(\|(z_l, x_{l1})\|), \tag{2.58}$$

where

$$V_{j1} = W_j(V_{j0}) + \frac{1}{2}\tilde{x}_{j1}^2, \tag{2.59}$$

with V_{j0} given in Assumption 2.1, $W_j(\cdot)$ and $b_j(\cdot,\cdot)$ are, respectively, a smooth $\mathcal{K}_\infty$-function and a smooth function to be chosen; and

$$f_{j00}(z_j) = f_{j0}(z_j, 0), \tag{2.60}$$
$$\delta_{j1l}(\|(z_l, x_{l1})\|) = \beta_{j0l}^{-1}(\zeta_{j0l}(\|(z_l, x_{l1})\|))^2 + \beta_{j1l}^{-1}(\zeta_{j1l}(\|(z_l, x_{l1})\|))^2, \tag{2.61}$$

with β_{j0l} and β_{j1l} being positive scaling constants.

Proof First, since $f_{j0}(z_j, x_{j1})$ of (2.12) is a smooth vector with $f_{j0}(0,0) = 0$, there exists a smooth vector $f_{j1}(z_j, x_{j1})$ such that

$$f_{j0}(z_j, x_{j1}) = f_{j00}(z_j) + f_{j1}(z_j, x_{j1}) x_{j1},$$

where $f_{j00}(z_j)$ is as in (2.60). By virtue of Assumption 2.2 and along the state trajectory of system (2.57), we have

$$\dot{V}_{j1} = \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j0} + \phi_{j0} x_{j1}) + x_{j1}[x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}; \theta)]$$
$$= \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}(f_{j00} + f_{j1} x_{j1}) + x_{j1} x_{j2} + x_{j1} \sum_{j=0}^{1} \psi_{j1}^{1}(z_j) \phi_{il}(z_j, x_{j1}, 0, 0; \theta)$$
$$+ x_{j1} \sum_{j=0}^{1} \psi_{j1}^{1}(z_j) \phi_{jl}(z_j, x_{j1}, Z_j, X_{j1}; \theta) - \phi_{jl}(z_j, x_{j1}, 0, 0, \theta)), \tag{2.62}$$

where

$$\psi_{j1}^{0}(z_j) = \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j}, \qquad \psi_{j1}^{1}(z_j) = 1.$$

Since $\phi_{j0}(0,0,0,0;\theta) = \phi_{j1}(0,0,0,0;\theta) = 0, \forall\theta$, there exists some function $\alpha_{j1}(z_j, x_{j1})$ such that

$$\left| x_{j1} \sum_{\iota=0}^{1} \psi_{i1}^{\iota}(z_j) \phi_{i\iota}(z_j, x_{j1}, 0, 0; \theta) \right| \le |x_{j1}| \alpha_{j1}(z_j, x_{j1})(\|z_j\| + \|x_{j1}\|). \tag{2.63}$$

In view of Assumption 2.2, it follows from (2.62) with some algebraic manipulations that

$$
\begin{aligned}
\dot{V}_{j1} \leq\; & \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}(f_{j00}+f_{j1}x_{j1})+x_{j1}x_{j2} \\
& + |x_{j1}|\left|\frac{dW_j}{dV_{j0}}\right|\left\|\frac{\partial V_{j0}}{\partial z_j}\right\|\sum_{\ell=1}^{N}\eta_{j0\ell}(z_j,x_{j1})\zeta_{j0\ell}(\|(z_l,x_{\ell1})\|) \\
& + |x_{j1}|\sum_{\ell=1}^{N}\eta_{j1\ell}(z_j,x_{j\ell})\zeta_{j1\ell}(\|(z_j,x_{\ell1})\|) \\
& + |x_{j1}|\alpha_{j1}(z_j,x_{j1})(\|z_j\|+\|x_{j1}\|) \\
\leq\; & \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}(f_{j00}+f_{j1}x_{j1})+x_{j1}(x_{j2}+x_{j1}\alpha_{j1}(z_j,x_{j1})) \\
& + \frac{1}{2}x_{j1}^2\left|\frac{dW_j}{dV_{j0}}\right|^2\left\|\frac{\partial V_{j0}}{\partial z_j}\right\|^2\sum_{\ell=1}^{N}\beta_{j0\ell}\eta_{j0\ell}^2(z_j,x_{j1}) \\
& + \frac{1}{2}\sum_{\ell=1}^{N}\beta_{j0\ell}^{-1}(\zeta_{j0\ell}(\|(z_\ell,x_{\ell1})\|))^2 \\
& + \frac{1}{2}x_{i1}^2\sum_{l=1}^{N}\beta_{i1l}\eta_{i1l}^2(z_j,x_{i1})+\frac{1}{2}\sum_{l=1}^{N}\beta_{i1l}^{-1}(\zeta_{j1\ell}(\|(z_\ell,x_{\ell1})\|))^2 \\
& + \frac{1}{4}x_{j1}^2\alpha_{j1}^2(z_j,x_{j1})+\|z_j\|^2 \\
=\; & \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00}+x_{j1}(x_{j2}+M_{j1}(z_j,x_{j1})) \\
& + \|z_j\|^2+\frac{1}{2}\sum_{\ell=1}^{N}\delta_{j1\ell}(\|(z_\ell,x_{\ell1})\|),
\end{aligned}
\tag{2.64}
$$

where $\delta_{j1\ell}$ is given in (2.61) and

$$
\begin{aligned}
M_{j1}(z_j,x_{j1}) =\; & \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j1}+\frac{1}{2}x_{j1}\left|\frac{dW_j}{dV_{j0}}\right|^2\left\|\frac{\partial V_{j0}}{\partial z_j}\right\|^2 \\
& \times\sum_{\ell=1}^{N}\beta_{j0\ell}\eta_{j0\ell}^2(z_j,x_{j1})+\frac{1}{2}\sum_{\ell=1}^{N}\beta_{j1\ell}\eta_{j1\ell}^2(z_j,x_{j1}) \\
& + x_{j1}\alpha_{j1}(z_j,x_{j1})+\frac{1}{4}x_{j1}\alpha_{j1}^2(z_j,x_{j1}).
\end{aligned}
\tag{2.65}
$$

Now, select

$$x_{j2} = x_{j2}^* = -M_{j1} - b_j(z_j, x_{j1})x_{j1} - r x_{j1}, \tag{2.66}$$

where $b_j(\cdot,\cdot)$ is a smooth function to counteract the effect of the interconnections and is to be determined. Then, (2.58) is obtained and the proof of Lemma 2.4 is now completed. □

Remark 2.14 For the case when $r = 1$, that is, $x_{j2} = v_j$ in (2.57) is the actual control input, it can be shown, refer to the proof of Theorem 2.5, that the design functions $b_j(\cdot,\cdot)$ and $W_j(\cdot)$, $j = 1, 2, \ldots, N$ can be chosen such that the decentralized state feedback control $v_j = x_{j2}^*(z_j, x_{j1})$ solves the robust decentralized stabilization problem.

2.3.2 Recursive Design

Next, we proceed toward the systematic recursive design methodology for constructing robust decentralized control laws for the system (2.6) when $r \geq 2$. A preliminary result is provided.

Lemma 2.5 *Consider the first $\rho + 1$ state equations of system (2.6):*

$$\begin{aligned}
\dot{z}_j &= f_{j0}(z_j, x_{j1}) + \phi_{j0}(z_j, x_{j1}, Z_j, X_{j1}; \theta)x_{j1}, \\
\dot{x}_{j1} &= x_{j2} + \phi_{j1}(z_j, x_{j1}, Z_j, X_{j1}; \theta), \\
\dot{x}_{j2} &= x_{j3} + \phi_{j2}(z_j, \bar{x}_{j2}, Z_j, X_{j1}; \theta), \\
&\vdots \\
\dot{x}_{j,\rho-1} &= x_{j,\rho} + \phi_{j,\rho-1}(z_j, \bar{x}_{j,\rho-1}, Z_j, X_{j1}; \theta), \\
\dot{x}_{j,\rho} &= x_{j,\rho+1} + \phi_{j,\rho}(z_j, \bar{x}_{j\rho}, Z_j, X_{j1}; \theta),
\end{aligned} \tag{2.67}$$

satisfying Assumptions 2.1 and 2.2. Suppose that for any given index $\rho = m$ $(1 \leq m \leq r-1)$, there exist smooth functions

$$\begin{aligned}
&x_{j2}^*(z_j, x_{j1}), \quad x_{j3}^*(z_j, \bar{x}_{j2}), \quad \ldots, \quad x_{j,m+1}^*(z_j, \bar{x}_{jm}); \\
&x_{jk}^*(0,0) = 0, \quad 2 \leq k \leq m+1
\end{aligned}$$

such that system (2.67) with the control $x_{j,m+1} = x_{j,m+1}^*(z_j, \bar{x}_{j,m})$ in the new coordinates

$$\begin{aligned}
z_j &= z_j, \qquad \tilde{x}_{j1} = x_{j1}, \\
\tilde{x}_{j2} &= x_{j2} - x_{j2}^*(z_j, x_{j1}), \quad \ldots, \quad \tilde{x}_{jm} = x_{jm} - x_{j,m}^*(z_j, \bar{x}_{j,m-1}),
\end{aligned}$$

satisfies

$$
\begin{aligned}
\dot{V}_{jm} \le \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1})x_{j1}^2 - (r-m+1)\sum_{k=1}^{m}\tilde{x}_{jk}^2 + m\|z_j\|^2 \\
+ \frac{1}{2}\sum_{\ell=1}^{N}\delta_{jm\ ell}(\|(z_\ell, x_{\ell 1})\|),
\end{aligned}
\tag{2.68}
$$

where

$$
V_{im} = W_j(V_{j0}) + \frac{1}{2}\sum_{k=1}^{m}\tilde{x}_{jk}^2,
$$

with V_{j0} as given in Assumption 2.1 and

$$
\begin{aligned}
&\delta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|) \equiv 0, \\
&\delta_{jk\ell}(\|(z_\ell, x_{\ell 1})\|) = \delta_{j,k-1,\ell}(\|(z_\ell, x_{\ell 1})\|) \\
&\qquad\qquad + \sum_{\iota=0}^{k}\beta_{\ell\iota\ell}^{-1}(\zeta_{\ell\iota\ell}(\|(z_\ell, x_{\ell 1})\|))^2, \quad 1 \le k \le r.
\end{aligned}
\tag{2.69}
$$

Then for system (2.67) with $\rho = m+1$, there exists a smooth decentralized state feedback control law

$$
x_{j,m+2} = x_{j,m+2}^*(z_j, \bar{x}_{j,m+1}); \qquad x_{j,m+2}^*(0,0) = 0 \tag{2.70}
$$

such that system (2.67) with (2.70) in the new coordinates

$$
\begin{aligned}
&z_j = z_j, \quad \tilde{x}_{jk}, \quad 1 \le k \le m, \\
&\tilde{x}_{j,m+1} = x_{j,m+1} - x_{j,m+1}^*(z_j, \bar{x}_{j,m}),
\end{aligned}
$$

satisfies

$$
\begin{aligned}
\dot{V}_{j,m+1} \le \frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1})x_{j1}^2 - (r-m)\sum_{k=1}^{m+1}\tilde{x}_{jk}^2 \\
+ (m+1)\|z_j\|^2 + \frac{1}{2}\sum_{\ell=1}^{N}\delta_{\ell,m+1,\ell}(\|(z_\ell, x_{\ell 1})\|),
\end{aligned}
\tag{2.71}
$$

where

$$
V_{j,m+1} = V_{jm} + \frac{1}{2}\tilde{x}_{j,m+1}^2.
$$

Proof Initially, the derivative of $\tilde{x}_{j,m+1} = x_{j,m+1} - x^*_{j,m+1}$ is given by

$$\begin{aligned}\dot{\tilde{x}}_{j,m+1} &= x_{j,m+2} + a_{j,m+1}(z_j, \bar{x}_{j,m+1}) \\ &\quad + \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}(z_j, \bar{x}_{j,m})\phi_{j\iota}(z_j, \bar{x}_{j\iota}, Z_j, X_{j1}; \theta),\end{aligned}$$

where

$$\begin{aligned}a_{j,m+1}(z_j, \bar{x}_{j,m+1}) &= -\frac{\partial x^*_{j,m+1}}{\partial z_j} f_{j0}(z_j, x_{j1}) - \sum_{\iota=1}^{m} \frac{\partial x^*_{j,m+1}}{\partial x_{j,\iota}} x_{j,\iota+1}, \\ \psi^0_{j,m+1}(z_j, \bar{x}_{j,m}) &= -\frac{\partial x^*_{j,m+1}}{\partial z_j} x_{j1}, \\ \psi^{\iota}_{j,m+1}(z_j, \bar{x}_{j,m}) &= -\frac{\partial x^*_{j,m+1}}{\partial x_{j,\iota}}, \quad 1 \le \iota \le m, \\ \psi^{m+1}_{j,m+1}(z_j, \bar{x}_{j,m}) &= 1.\end{aligned}$$

The time derivative of $V_{j,m+1}$ is given by

$$\begin{aligned}\dot{V}_{j,m+1} &= \dot{V}_{j,m} + \tilde{x}_{j,m+1}\Bigg[x_{j,m+2} + a_{j,m+1} \\ &\quad + \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}(z_j, \bar{x}_{i,m})\phi_{j\iota}(z_j, \bar{x}_{j\iota}, Z_j, X_{j1}; \theta)\Bigg] \\ &= \dot{V}_{jm} + \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) + \tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}\phi_{i\iota}(z_j, \bar{x}_{i\iota}, 0, 0; \theta) \\ &\quad + \tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \psi^{\iota}_{j,m+1}[\phi_{j\iota}(z_j, \bar{x}_{j\iota}, Z_j, X_{j1}; \theta) \\ &\quad - \phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta)].\end{aligned} \tag{2.72}$$

Define

$$\begin{aligned}\tilde{\phi}_{j\iota}(z_j, \bar{\tilde{x}}_{j\iota}; \theta) &= \phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta) \\ &= \phi_{j\iota}(z_j, \bar{\tilde{x}}_{j\iota} + \bar{x}^*_{j\iota}, 0, 0; \theta), \quad 2 \le \iota \le m+1\end{aligned} \tag{2.73}$$

where $\bar{\tilde{x}}_{j\iota} = (\tilde{x}_{j1}, \ldots, \tilde{x}_{j\iota})$ and $\bar{x}^*_{j\iota} = (x^*_{j1}, x^*_{j2}, \ldots, x^*_{j\iota})$ with $\bar{\tilde{x}}_{j0} = \tilde{x}_{j1}$ and $\bar{x}^*_{j0} = x^*_{j1}$.

Now since $\phi_{j\iota}(0, 0, 0, 0; \theta) = 0, \forall \theta \in \Omega, 0 \le \iota \le m+1$, it is easy to verify that $\tilde{\phi}_{j\iota}(0, 0; \theta) = 0, \forall \theta \in \Omega$. Thus, there exist smooth bounding functions $\alpha_{j\iota}(z_j, \bar{\tilde{x}}_{j,\iota})$,

$\iota = 0, 1, \ldots, m+1$ such that

$$|\phi_{j0}(z_j, x_{j1}, 0, 0; \theta)| = |\tilde{\phi}_{j0}(z_j, \tilde{x}_{j1}; \theta) \leq \alpha_{j0}(z_j, \tilde{x}_{j1})(\|z_j\| + \|\tilde{x}_{j1}\|),$$
$$|\phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta)| = |\tilde{\phi}_{i\iota}(z_j, \bar{\tilde{x}}_{j\iota}; \theta) \leq \alpha_{j\iota}(z_j, \bar{\tilde{x}}_{j,\ell})\left[\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}|\right], \quad 1 \leq \ell \leq m+1. \tag{2.74}$$

Hence, the second last term of (2.72) satisfies

$$\begin{aligned}
&\tilde{x}_{j,m+1} \sum_{\ell=0}^{m+1} \psi_{j,m+1}^{j\iota} \phi_{j\iota}(z_j, \bar{x}_{j\iota}, 0, 0; \theta) \\
&\leq |\tilde{x}_{j,m+1}| \left[\psi_{j,m+1}^{0} |\alpha_{j0}(\|z_j\| + |\tilde{x}_{j1}|) + \sum_{\ell=1}^{m+1} |\psi_{j,m+1}^{\ell}| \alpha_{j\iota} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right) \right] \\
&= |\tilde{x}_{j,m+1}| \left[|\psi_{j,m+1}^{0}| \alpha_{j0}(\|z_j\| + |\tilde{x}_{j1}|) + \sum_{\ell=1}^{m} |\psi_{j,m+1}^{\ell}| \alpha_{i\iota} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right) \right] \\
&\quad + |\tilde{x}_{j,m+1}| \alpha_{j,m+1} \left(\|z_j\| + \sum_{k=1}^{m} |\tilde{x}_{jk}| \right) + \alpha_{j,m+1} \tilde{x}_{j,m+1}^2 \\
&\leq \tilde{x}_{ij,m+1}^2 \sum_{\ell=0}^{m} (\psi_{j,m+1}^{\ell})^2 \alpha_{j\iota}^2 (m+1)(\ell+1) \\
&\quad + \frac{1}{4(m+1)} \left[(\|z_j\| + |\tilde{x}_{j1}|)^2 + \sum_{\ell=1}^{m} \frac{1}{(\ell+1)} \left(\|z_j\| + \sum_{k=1}^{\ell} |\tilde{x}_{jk}| \right)^2 \right] \\
&\quad + \frac{1}{2}(m+1) \tilde{x}_{j,m+1}^2 \alpha_{j,m+1}^2 + \frac{1}{2(m+1)} \left(\|z_j\| + \sum_{k=1}^{m} |\tilde{x}_{jk}| \right)^2 + \alpha_{j,m+1} \tilde{x}_{j,m+1}^2 \\
&\leq \tilde{x}_{j,m+1}^2 \sum_{\ell=0}^{m} (\psi_{j,m+1}^{\ell})^2 \alpha_{i\iota}^2 (m+1)(\ell+1) + \frac{1}{2} \|z_j\|^2 + \frac{1}{2} \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
&\quad + \frac{1}{2}(m+1) \tilde{x}_{j,m+1}^2 \alpha_{j,m+1}^2 + \frac{1}{2} \left(\|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \right) + \alpha_{j,m+1} \tilde{x}_{j,m+1}^{*} \\
&= \left[\sum_{\ell=0}^{m} (\psi_{j,m+1}^{\ell})^2 \alpha_{i\iota}^2 (m+1)(\ell+1) + \frac{1}{2}(m+1) \alpha_{j,m+1}^2 + \alpha_{j,m+1} \right] \tilde{x}_{j,m+1}^2 \\
&\quad + \|z_{jk}\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
&\leq \tilde{x}_{j,m+1}^2 E_{j,m+1}(z_j, \bar{\tilde{x}}_{j,m+1}) + \|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2.
\end{aligned} \tag{2.75}$$

Invoking Assumption 2.2 and (2.75), it follows that (2.72) can be written as

$$
\begin{aligned}
\dot{V}_{j,m+1} \leq{}& \dot{V}_{jm} + \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) \\
&+ |\tilde{x}_{j,m+1}| \sum_{\ell=0}^{m+1} |\psi_{j,m+1}^{\ell}| \sum_{\ell=1}^{N} \eta_{j\iota\ell}(z_j, \bar{x}_{j\iota}) \zeta_{j\ell\ell}(\|(z_\ell, x_{\ell 1})\|) \\
&+ \tilde{x}_{j,m+1}^2 E_{j,m+1} + \|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
\leq{}& \dot{V}_{jm} + \tilde{x}_{j,m+1}(x_{j,m+2} + a_{j,m+1}) + \tilde{x}_{j,m+1}^2 E_{j,m+1} \\
&+ \|z_j\|^2 + \sum_{k=1}^{m} |\tilde{x}_{jk}|^2 \\
&+ \frac{1}{2}\tilde{x}_{j,m+1}^2 \sum_{\ell=0}^{m+1} \sum_{\ell=1}^{N} (\psi_{j,m+1}^{\ell})^2 (\eta_{j\iota\ell}(z_j, \bar{x}_{j\iota}))^2 \beta_{j\iota\ell} \\
&+ \frac{1}{2} \sum_{\ell=0}^{m+1} \sum_{\ell=1}^{N} (\zeta_{j\ell\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \beta_{j\iota\ell}^{-1} \\
\leq{}& \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{j00} - b_j(z_j, x_{j1}) x_{j1}^2 - (r - m + 1) \sum_{k=1}^{m} \tilde{x}_{jk}^2 \\
&+ m\|z_j\|^2 + \frac{1}{2} \sum_{\ell=1}^{N} \delta_{jm\ell}(\|(z_\ell, x_{\ell 1})\|) + \tilde{x}_{jm} \tilde{x}_{j,m+1} \\
&+ \tilde{x}_{j,m+1}(x_{j,m+2} + M_{j,m+1}) + \|z_j\|^2 + \sum_{k=1}^{m} \tilde{x}_{jk}^2 \\
&+ \frac{1}{2} \sum_{\iota=0}^{m+1} \sum_{\ell=1}^{N} (\zeta_{j\iota\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \beta_{\ell\ell l}^{-1},
\end{aligned}
\tag{2.76}
$$

where

$$
\begin{aligned}
M_{i,m+1}(z_j, \bar{\tilde{x}}_{j,m+1}) ={}& a_{j,m+1} + \tilde{x}_{j,m+1} E_{j,m+1} \\
&+ \frac{1}{2}\tilde{x}_{j,m+1} \sum_{\iota=0}^{m+1} \sum_{\ell=1}^{N} (\psi_{j,m+1}^{\ell})^2 (\eta_{j\iota\ell}(z_j, \bar{x}_{j\iota}))^2 \beta_{j\iota\ell}.
\end{aligned}
\tag{2.77}
$$

Select

$$
x_{j,m+2} = x_{j,m+2}^*(z_j, x_{j1}, \ldots, x_{j,m+1}) = -M_{j,m+1} - \tilde{x}_{jm} - (r - m)\tilde{x}_{j,m+1}. \tag{2.78}
$$

This makes (2.71) in Lemma 2.5 is valid, which completes the proof. $\square$

By combining Lemmas 2.4 and 2.5 the construction of robust decentralized control law stabilizing the uncertain interconnected nonlinear systems (2.6) can be completed. This is demonstrated below.

Theorem 2.5 *Consider the uncertain interconnected system* (2.6) *satisfying Assumptions* 2.1 *and* 2.2. *Then there exists a decentralized control law,* $v_j = v_j(z_j, x_j)$, $j = 1, 2, \ldots, N$, *such that the overall system with the decentralized controller is globally asymptotically stable for all admissible uncertainties and interconnections. Indeed*; *a suitable decentralized controller is given by*

$$v_j := x^*_{j,r+1}(z_j, \bar{x}_{j,r}) = -M_{jr} - \tilde{x}_{j,r-1} - \tilde{x}_{jr}, \tag{2.79}$$

where M_{jr} *is given in* (2.35) *with* $m + 1 = r$.

Proof By Lemma 2.4, it is not difficult to show that the induction hypotheses of Lemma 2.5 is satisfied. This motivates us to build a Lyapunov-based recursive decentralized control law by applying Lemma 2.5 repeatedly until the rth step. Therefore, we can construct $x^*_{j2}(z_j, x_{j1}), \ldots, x^*_{j,r+1}(z_j, \bar{x}_{jr})$ such that under the new coordinates

$$z_j, \quad \tilde{x}_{j1} = x_{j1}, \quad \tilde{x}_{j2} = x_{j2} - x^*_{j2}(z_j, x_{j1}), \quad \ldots, \quad \tilde{x}_{jr} = x_{jr} - x^*_{j,r}(z_j, \bar{x}_{j,r-1})$$

system (2.2) with control law (2.79) satisfies

$$\begin{aligned}\dot{V}_{jr} \le{}& \frac{dW_j}{dV_{j0}} \frac{\partial V_{j0}}{\partial z_j} f_{ij00} - b_j(z_j, x_{j1}) x_{j1}^2 - \sum_{k=1}^{r} \tilde{x}_{jk}^2 + r\|z_j\|^2 \\ &+ \frac{1}{2} \sum_{\ell=1}^{N} \delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|),\end{aligned} \tag{2.80}$$

where $V_{jr} = W_j(V_{j0}) + \frac{1}{2}\sum_{k=1}^{r} \tilde{x}_{jk}^2$ and

$$\begin{aligned}\delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|) ={}& r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}(\|(z_\ell, x_{\ell 1})\|))^2 \\ &+ \sum_{\iota=1}^{r} (r - \iota + 1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}(\|(z_\ell, x_{\ell 1})\|))^2.\end{aligned} \tag{2.81}$$

By Assumption 2.2, we have

$$\begin{aligned}\delta_{jr\ell}(\|(z_\ell, x_{\ell 1})\|) ={}& r\beta_{j0\ell}^{-1}(\zeta^0_{j0\ell}(\|z_\ell\|)\|z_\ell\| + \zeta^1_{j0\ell}(z_\ell, x_{\ell 1})|x_{\ell 1}|)^2 \\ &+ \sum_{\ell=1}^{r} (r - \ell + 1)\beta_{j\iota\ell}^{-1}(\zeta^0_{j\iota\ell}(\|z_\ell\|)\|z_\ell\| + \zeta^1_{j\ell\ell}(z_\ell, x_{\ell 1})|x_{\ell 1}|)^2 \\ \le{}& 2r\beta_{j0\ell}^{-1}((\zeta^0_{j0\ell}(\|z_\ell\|))^2\|z_\ell\|^2 + (\zeta^1_{j0\ell}(z_\ell, x_{\ell 1}))^2 x_{\ell 1}^2)\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{\ell=1}^{r}(r-\ell+1)\beta_{j\iota\ell}^{-1} \\
&\times((\zeta_{j\ell\ell}^{0}(\|z_\ell\|))^2\|z_\ell\|^2+(\zeta_{j\iota\ell}^{1}(z_\ell,x_{\ell 1}))^2x_{\ell 1}^2) \\
\le\ &2\Delta_{j\ell}(\|z_\ell\|)\|z_\ell\|^2+2D_{j\ell}(z_\ell,x_{\ell 1})x_{\ell 1}^2,
\end{aligned}
\tag{2.82}
$$

where

$$
\Delta_{j\ell}(\|z_\ell\|)=r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^{0}(\|z_\ell\|))^2+\sum_{\iota=1}^{r}(r-\iota+1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}^{0}(\|z_\ell\|))^2, \tag{2.83}
$$

$$
D_{j\ell}(z_\ell,x_{\ell 1})=r\beta_{j0\ell}^{-1}(\zeta_{j0\ell}^{1}(z_\ell,x_{\ell 1}))^2+\sum_{\iota=1}^{r}(r-\iota+1)\beta_{j\iota\ell}^{-1}(\zeta_{j\iota\ell}^{1}(z_\ell,x_{\ell 1}))^2. \tag{2.84}
$$

Define

$$
V=\sum_{i=1}^{N}V_jr.
$$

Observing the interconnection structural constraint

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta_{j\ell}(\|z_\ell\|)\|z_\ell\|^2+D_{j\ell}(z_\ell,x_\ell)x_{\ell 1}^2] \\
&\quad=\sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta_{\ell j}(\|z_j\|)\|z_j\|^2+D_{\ell j}(z_j,x_j)x_{j1}^2]
\end{aligned}
$$

and Assumption 2.1 and by noting that $W_j(V_{j0})$ is a $\mathcal{K}_\infty$ function of V_{j0}, we have

$$
\begin{aligned}
\dot V_{jr}\le&\sum_{j=1}^{N}\left\{\frac{dW_j}{dV_{j0}}\frac{\partial V_{j0}}{\partial z_j}f_{j00}-b_j(z_j,x_{j1})x_{j1}^2-\sum_{k=1}^{r}\tilde x_{jk}^2+r\|z_j\|^2\right. \\
&\left.+\sum_{\ell=1}^{N}[\Delta_{\ell j}(\|z_j\|)\|z_j\|^2+D_{\ell j}(z_j,x_j)x_{j1}^2]\right\} \\
\le&\sum_{j=1}^{N}\left\{-\frac{dW_j}{dV_{j0}}\nu_j\|z_j\|^2+\left[r+\sum_{\ell=1}^{N}\Delta_{\ell j}(\|z_j\|)\right]\|z_j\|^2\right. \\
&\left.-\sum_{k=1}^{r}\tilde x_{jk}^2-\left[b_j(z_j,x_{j\ell})x_{j\ell}^2-\sum_{\ell=1}^{N}D_{\ell j}(z_j,x_{j\ell})\right]x_{j\ell}^2\right\}.
\end{aligned}
\tag{2.85}
$$

Since $V_{j0}(z_j)$ in Assumption 2.1 is radially unbounded and positive definite, there exists a $\mathcal{K}_\infty$ function $\kappa_{\ell j}$ such that

$$\Delta_{\ell j}(\|z_j\|) \le \Delta_{\ell j}(0) + \kappa_{\ell j}(V_{j0}). \tag{2.86}$$

Now select

$$b_j(z_j, x_{j1}) = \sum_{\ell=1}^{N} D_{\ell j}(z_j, x_{j1}) \tag{2.87}$$

and

$$\frac{dW_j}{dV_{j0}} = k_j + \frac{1}{\nu_j}\left[r + \sum_{\ell=1}^{N}(\Delta_{\ell j}(0) + \kappa_{\ell j}(V_{j0}))\right], \qquad W_j(0) = 0, \tag{2.88}$$

where $k_j > 0$ is a constant. It is obvious that $W_j(\cdot)$ is a smooth $\mathcal{K}_\infty$-function. Then it follows that

$$\dot{V} \le \sum_{j=1}^{N}\left\{\left(-k_j\nu_j\|z_j\|^2 - \sum_{k=1}^{r}\tilde{x}_{jk}^2\right)\right\}. \tag{2.89}$$

Therefore, due to the onto-relation between (z_j, x_j) and $(z_j, \tilde{x}_j)$, where $\tilde{x}_j = (\tilde{x}_{j1}, \dots, \tilde{x}_{ir})$, the closed-loop interconnected system of (2.2) with the decentralized controller (2.79) is globally asymptotically stable for all admissible uncertainties and interconnections. □

Remark 2.15 Observe from Theorem 2.5 that the functions $b_j(z_j, x_{i1})$ and $W_j(V_{i0})$, $i = 1, 2, \dots, N$, can be chosen before we start the recursive design of the robust decentralized stabilization controller.

Remark 2.16 Theorem 2.5 presents a decentralized global stabilization result for uncertain interconnected minimum-phase nonlinear systems with parametric uncertainty and interconnections bounded by general nonlinear functions. This result extends centralized results in [35, 39] to decentralized control of large-scale interconnected systems.

2.3.3 Simulation Example 2.5

Consider the following large-scale system which is composed of two subsystems:

$$\begin{aligned}
\textit{Subsystem 1:}\quad \dot{z}_1 &= -2z_1 + z_1x_{11},\\
\dot{x}_{11} &= x_{12} + x_{11}z_1\sin\theta_1 + x_{21}^2 z_2\cos\theta_1^2,\\
\dot{x}_{12} &= u_1 + x_{12}^2(x_{11}z_1 + z_1^2)\sin\theta_1 + x_{21}z_2\cos(\theta_1 z_1);
\end{aligned} \tag{2.90}$$

$$\begin{aligned}
\textit{Subsystem 2:}\quad \dot{z}_2 &= -z_2 + x_{21}^2,\\
\dot{x}_{21} &= x_{22} + (x_{11}^2 z_1 + x_{21}^2 z_2)\sin(z_2\theta_2), \\
\dot{x}_{22} &= u_2 + x_{22}^2(x_{11}z_1^2 + x_{21}z_2^2)\sin\theta_2 + x_{22}^2 z_2^3\cos(\theta_2^2 z_2^2),
\end{aligned}\tag{2.91}$$

where $\theta_1, \theta_2 \in [-2, 2]$.

It is easy to verify that the interconnections in the above interconnected system satisfy Assumption 2.2. Choose $\beta_{jkm} = 1$, $j, k, m = 1, 2$. It follows from (2.83) and (2.84) that

$$\Delta_{11} = \Delta_{12} = \Delta_{21} = \Delta_{22} = 0$$

$$D_{11} = 0, \qquad D_{12} = 2x_{21}^2 z_2^2 + z_2^2, \qquad D_{21} = 2x_{11}^2 z_1^2 + z_1^4, \qquad D_{22} = 0.$$

1. Let $V_{10} = \frac{1}{2}z_1^2$ and $V_{20} = \frac{1}{2}z_2^2$. Then,

$$\frac{\partial V_{10}}{\partial z_1} f_{10}(z_1, 0) = -2z_1^2; \qquad \frac{\partial V_{20}}{\partial z_2} f_{20}(z_2, 0) = z_2^2.$$

Obviously, Assumption 2.1 is satisfied with $\nu_1 = 2$ and $\nu_2 = 1$.

It also follows from (2.86) that

$$\kappa_{11}(V_{10}) = \kappa_{21}(V_{10}) = \kappa_{12}(V_{20}) = \kappa_{22}(V_{20}) = 0.$$

By choosing $k_1 = k_2 = 3$, according to (2.87) and (2.88), we have

$$\frac{dW_1}{dV_{10}} = 4, \qquad \frac{dW_2}{dV_{20}} = 5$$

and

$$b_1 = D_{11} + D_{21}, \qquad b_2 = D_{12} + D_{22}.$$

It follows from (2.18) and (2.20) that

$$\alpha_{11} = x_{11}^2 + 0:25, \qquad \alpha_{21} = x_{21}^2$$

and

$$\begin{aligned}
M_{11} &= \frac{dW_1}{dV_{10}} z_1^2 + 0.5x_{11} + x_{11}\alpha_{11} + 0.25\alpha_{11}^2,\\
M_{21} &= \frac{dW_2}{dV_{20}} z_2 x_{21} + 0.5x_{21} + x_{21}\alpha_{21} + 0.25\alpha_{21}^2.
\end{aligned}$$

Hence, we can compute the virtual control

$$\begin{aligned}
x_{12}^* &= -M_{11} - b_1 x_{11} - 2x_{11},\\
x_{22}^* &= -M_{21} - b_2 x_{21} - 2x_{21}.
\end{aligned}$$

2. Letting $\tilde{x}_{i2} = x_{i2} - x_{i2}^*$, $i = 1, 2$, we have

$$\psi_{12}^0 = -\frac{\partial x_{12}^*}{\partial z_1} x_{11}, \qquad \psi_{22}^0 = -\frac{\partial x_{22}^*}{\partial z_2} x_{21},$$

$$\psi_{12}^1 = -\frac{\partial x_{12}^*}{\partial x_{11}}, \qquad \psi_{22}^1 = -\frac{\partial x_{22}^*}{\partial x_{21}}, \qquad \psi_{21}^2 = \psi_{22}^2 = 1,$$

$$a_{12} = -\frac{\partial x_{12}^*}{\partial z_1}(-2z_1 + x_{11}z_1) - \frac{\partial x_{12}^*}{\partial x_{11}} x_{12},$$

$$a_{22} = -\frac{\partial x_{22}^*}{\partial z_2}(-z_2 + x_{21}^2) - \frac{\partial x_{22}^*}{\partial x_{21}} x_{22}.$$

According to (2.74), we can choose

$$\alpha_{12} = x_{12}^2(z_1^2 + 0.25), \qquad \alpha_{22} = x_{22}^2 z_2^2.$$

Hence, it follows from (2.77) that

$$M_{12} = a_{12} + \tilde{x}_{12}(4(\psi_{12}^1)^2\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{12}) + 0.5\tilde{x}_{12}((\psi_{12}^1)^2 + (\psi_{12}^2)^2),$$

$$M_{22} = a_{22} + \tilde{x}_{22}(4(\psi_{22}^1)^2\alpha_{21}^2 + \alpha_{22}^2 + \alpha_{22}) + 0.5\tilde{x}_{22}((\psi_{22}^1)^2 + (\psi_{22}^2)^2 x_{22}^4).$$

The control law can be obtained from (2.78) as follows:

$$u_1 = -x_{11} - M_{12} - \tilde{x}_{12}, \tag{2.92}$$

$$u_2 = -x_{21} - M_{22} - \tilde{x}_{22}. \tag{2.93}$$

Systems (2.90)–(2.91) were simulated with the controller (2.92) and (2.93) to demonstrate the effectiveness of the decentralized robust control design procedure. The initial conditions are set to be

$$z_1 = 1.0, \qquad x_{11} = -1.0, \qquad x_{12} = 1.5,$$

$$z_2 = 1.0, \qquad x_{21} = -1.0, \qquad x_{22} = 1.5$$

and the uncertainties θ_1 and θ_2 are given by $\theta_1 = 2\sin t$ and $\theta_2 = 2\cos t^2$. Obviously, the uncertainties are time-varying ones and belong to the set $[-2, 2]$. The closed-loop responses for the two subsystems are plotted in Figs. 2.1 and 2.2 from which the stability is clearly seen.

2.4 Decentralized Tracking: Class III

In this section, we attend to the problem of class III that was presented in Sect. 2.1.3. In the problem description there, attention was given to a class of large-scale nonlinear systems which is comprised of N interconnected subsystems with time-varying

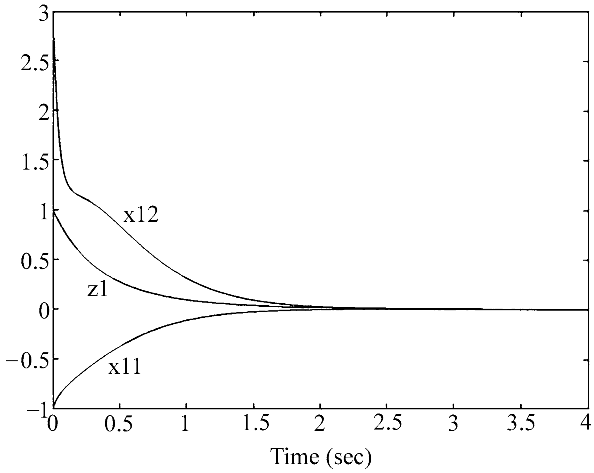

Fig. 2.1 Closed-loop responses of subsystem 1

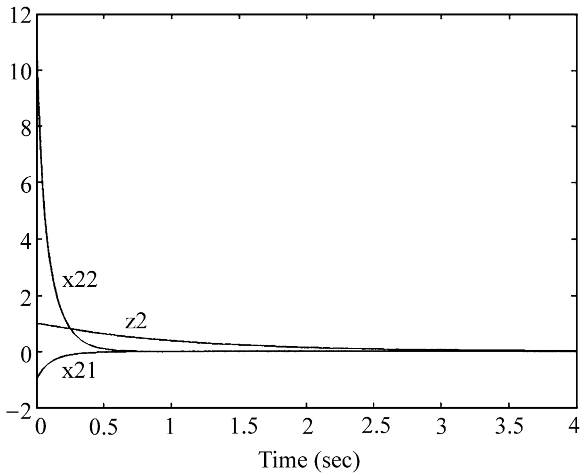

Fig. 2.2 Closed-loop responses of subsystem 2

unknown parameters and/or disturbances entering nonlinearly into the state equation as modeled by (2.9) and (2.10).

In what follows, we focus on studying the problem of decentralized output-feedback tracking with disturbance attenuation. Thus, with reference to the model (2.9) and (2.10), for every $1 \leq j \leq N$ and a given time-varying signal $y_{ir}(t)$ whose derivatives up to order n_j are bounded over $[0, \infty)$, our objective hereafter is to design a smooth, decentralized, dynamic, output-feedback controller of the form

$$\dot{x}_j = \nu_j(x_j, y_j, t), \quad u_j = \mu_j(x_j, y_j, t), \quad x_j \in \Re^{\bar{n}_j} \tag{2.94}$$

such that the following properties hold for the resulting closed-loop large-scale nonlinear system (2.11), (2.94):

1. *When the signal* $w_j \equiv 0$ *for all* $1 \le j \le N$, *the tracking error signal* $y_j - y_{ir}$ *goes to zero asymptotically and all other closed-loop signals remain bounded over* $[0, \infty)$.
2. *When* $w_j \neq 0$ *for all* $1 \le j \le N$, *the closed-loop system is bounded-input bounded-state* **BIBS** *stable and, in appropriate coordinates, is integral-input-to-state stable* **iISS** *with respect to the disturbance input* w [63]. *In particular, there exists a class-*$\mathcal{K}$ *function* γ_d (*that is,* γ_d *is continuous, strictly increasing and vanishes at the origin*) *such that, for any* $\rho > 0$, *the controller* (2.94) *can be tuned to satisfy the inequality*

$$\int_{t_0}^{t} |y(\tau) - y_r(t)|^2 d\tau \le \rho \int_{0}^{t} \gamma_d(|w(\tau)|) d\tau + \eta_0(z(0), x(0), x(0))$$
$$\forall t \ge 0, \tag{2.95}$$

where η_0 is a nonnegative C^0 function, and

$$z(0) = [z_1^t(0), \dots, z_N^t(0)]^t, \quad x(0) = [x_1^t(0), \dots, x_N^t(0)]^t,$$
$$x(0) = [x_1^t(0), \dots, x_N^t(0)]^t.$$

Remark 2.17 Property (1) above means that decentralized asymptotic tracking is achieved for each local jth subsystem (2.11) in the absence of disturbance inputs. Property (2) with (2.95) in implies that, in the presence of disturbances, the decentralized output-feedback controller (2.94) has the ability to attenuate the effect of the disturbances on the tracking error arbitrarily for a fixed class-K gain-function γ_d. As we shall see later, $\gamma_d(s) = s^2 + s^4 + s^8$ in our case.

In the sequel, sufficient conditions are provided to yield the standard $\mathcal{L}_2$-gain disturbance rejection property—that is, $\gamma_d(s) = s^2$ in (2.95). It is interesting to note that a similar problem has been studied in [41] in the framework of centralized output-feedback tracking with almost disturbance decoupling.

The control problem formulated above will be solved in two steps demonstrated in the following sections. We first introduce a (partially) decentralized observer in order to obtain an augmented decentralized system with partial-state information. Then, we base the decentralized controller design on this enlarged dynamic system.

2.4.1 Partially Decentralized Observer

Owing to the structure in every local system of (2.11), for each $1 \le j \le N$, we introduce the following state estimator for the (z_j, x_j)-subsystem:

$$\begin{aligned}
\dot{\hat{z}}_j &= Q_j \hat{z}_j + f_{j0}(y_{1r}, \dots, y_{Nr}), \\
\dot{\hat{x}}_{j1} &= \hat{x}_{j2} + L_{j1}(y_j - x_{j1}) + f_{j1}(y_{1r}, \dots, y_{Nr}) \\
&\quad + g_{j1}(y_{1r}, \dots, y_{Nr})\hat{z}_j, \\
&\vdots \\
\hat{x}_{jn_j} &= u_j + L_{jn_j}(y_j - \hat{x}_{j1}) + f_{jn_j}(y_{1r}, \dots, y_{Nr}) \\
&\quad + g_{jn_j}(y_{1r}, \dots, y_{Nr})\hat{z}_j,
\end{aligned} \tag{2.96}$$

$$A_j = \begin{bmatrix} -L_{j1} & \\ -L_{j2} & I_{n_j-1} \\ \vdots & \\ -L_{jn_j} & 0 \dots 0 \end{bmatrix}. \tag{2.97}$$

Notice that the eigenvalues of A_j can be assigned to any desired location in the open left-half plane via the choice of appropriate constants $\{L_{jm}\}_{m=1}^{n_m}$, provided complex conjugate eigenvalues appear in pair. In (2.97), I_{n_j-1} is the unit matrix of order $n_j - 1$.

Introducing the new variables

$$\tilde{z}_j = z_j - \hat{z}_j, \quad \tilde{x}_{jk} = x_{jk} - \hat{x}_{jk}, \quad 1 \le k \le n_j, \ 1 \le j \le N. \tag{2.98}$$

Then from (2.11) and (2.96), it follows that:

$$\begin{aligned}
\dot{\tilde{z}}_j &= Q_j \tilde{z}_j + f_{j0}(y_1, \dots, y_N) - f_{j0}(y_{1r}, \dots, y_{Nr}) \\
&\quad + p_{j0}(y_1, \dots, y_N) w_j,
\end{aligned} \tag{2.99}$$

$$\begin{aligned}
\dot{\tilde{x}}_j &= A_j \tilde{x}_j + f_j(y_1, \dots, y_N) - f_j(y_{1r}, \dots, y_{Nr}) \\
&\quad + g_j(y_1, \dots, y_N) z_j - g_j(y_{1r}, \dots, y_{Nr})\hat{z}_j \\
&\quad + p_j(y_1, \dots, y_N) w_j,
\end{aligned} \tag{2.100}$$

where

$$\begin{aligned}
\tilde{x}_j &= (\tilde{x}_{j1}, \dots, \tilde{x}_{jn_j})^t, & f_j &= (f_{j1}, \dots, f_{jn_j})^t, \\
g_j &= (g_{i1}, \dots, g_{jn_j})^t, & p_j &= (p_{j1}, \dots, p_{jn_j})^t.
\end{aligned}$$

Since every f_{jk} is a smooth function and every y_{jr} is a bounded signal, there exist a finite number of nonnegative smooth functions $\{\varphi_{j0k}\}_{k=1}^N$, $\{\varphi_{jk}\}_{k=1}^N$ such that

$$|f_{j0}(y_1, \dots, y_N) - f_{j0}(y_{1r}, \dots, y_{Nr})| \le \sum_{k=1}^N |\tilde{x}_{k1}| \varphi_{j0k}(\tilde{x}_{k1}), \tag{2.101}$$

$$|f_j(y_1, \dots, y_N) - f_j(y_{1r}, \dots, y_{Nr})| \le \sum_{k=1}^N |\tilde{x}_{k1}| \varphi_{jk}(\tilde{x}_{k1}). \tag{2.102}$$

In a similar way, we can obtain a functional bound for $g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j$. Indeed, we have

$$\begin{aligned}
&g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j\\
&\quad = g_j(y_1,\dots,y_N)\tilde{z}_j + (g_j(y_1,\dots,y_N) - g_j(y_{1r},\dots,y_{Nr}))\hat{z}_j. \quad (2.103)
\end{aligned}$$

Using the Mean-Value Theorem [29], there exist nonnegative smooth functions ϕ_{ik} $(1 \le k \le N)$ such that

$$\begin{aligned}
&|g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j|\\
&\quad \le |g_j(y_1,\dots,y_N)||\tilde{z}_j| + \sum_{k=1}^{N}|\tilde{x}_{k1}|\phi_{ik}(\tilde{x}_{k1})|\hat{z}_j|. \qquad (2.104)
\end{aligned}$$

It must be noted that, by means of these inequalities (2.101)–(2.104), it is easy to show that, in the absence of disturbance inputs, the solutions $(\tilde{z}_j(t), \tilde{x}_j(t))$ of the cascade system (2.99)–(2.100) go to zero, if $y_j(t) - y_{ir}(t) \to 0$ for all $1 \le j \le N$. The latter property will be guaranteed with the help of the decentralized controller to be designed next.

Remark 2.18 It should be mentioned that the observer (2.96) is not asymptotic and is totally decentralized only if the reference signals $y_{jr} = 0$ for all $1 \le j \le N$. Proceeding further, we select a partially decentralized observer so that; in appropriate coordinates; the system (2.105) has an equilibrium point and therefore there is a solution to decentralized asymptotic tracking. In general, when $y_{jr}(t)$ are general time-varying signals, the system augmented with a totally decentralized observer does not have a fixed equilibrium. Thus, only practical tracking can be achieved by means of high-gain feedback [60].

2.4.2 Design Procedure

From the forgoing development of partially decentralized observers, we derive the following controller-observer combined system for the purpose of feedback design:

$$\begin{aligned}
\dot{\tilde{z}}_j &= Q_j\tilde{z}_j + f_{j0}(y_1,\dots,y_N) - f_{i0}(y_{1r},\dots,y_{Nr})\\
&\quad + p_{j0}(y_1,\dots,y_N)w_j,\\
\dot{\tilde{x}}_j &= A_j\tilde{x}_j + f_j(y_1,\dots,y_N) - f_j(y_{1r},\dots,y_{Nr})\\
&\quad + g_j(y_1,\dots,y_N)z_j - g_j(y_{1r},\dots,y_{Nr})\hat{z}_j\\
&\quad + p_j(y_1,\dots,y_N)w_j,\\
\dot{y}_j &= \hat{x}_{j2} + \tilde{x}_{i2} + f_{j1}(y_1,\dots,y_N) + g_{j1}(y_1,\dots,y_N)z_j
\end{aligned} \qquad (2.105)$$

$$
\begin{aligned}
&+ p_{i1}(y_1, \ldots, y_N) w_j, \\
\dot{\hat{x}}_{j2} &= \hat{x}_{j3} + L_{i2}(y_j - \hat{x}_{i1}) + f_{j2}(y_{1r}, \ldots, y_{Nr}) \\
&+ g_{j2}(y_{1r}, \ldots, y_{Nr}) \hat{z}_j, \\
&\vdots \\
\dot{\hat{x}}_{in_j} &= u_j + L_{jn_j}(y_j - \hat{x}_{i1}) + f_{jn_j}(y_{1r}, \ldots, y_{Nr}) \\
&+ g_{jn_j}(y_{1r}, \ldots, y_{Nr}) \hat{z}_j.
\end{aligned}
$$

Notice that the state variables $(y_j, \hat{x}_{j1}, \hat{x}_{j2}, \ldots, \hat{x}_{jn_j})$, and then $\tilde{x}_{j1}$, are available for feedback design. Also note that the states $(\tilde{z}_j, \tilde{x}_j)$ are unmeasured and that the outputs y_j, with $j \neq i$, of other subsystems are unavailable for the design of the regional input u_j.

We now direct attention to the jth local system (2.105) with u_j as the control input. For the sake of clarity, the arguments of a function are often omitted in case no possible confusion arises. For notational simplicity, denote

$$
\tilde{f}_{j0} = f_{j0}(y_1, \ldots, y_N) - f_{j0}(y_{1r}, \ldots, y_{Nr}), \tag{2.106}
$$

$$
\tilde{f}_j = f_j(y_1, \ldots, y_N) - f_j(y_{1r}, \ldots, y_{Nr}), \tag{2.107}
$$

$$
\tilde{g}_j = g_j(y_1, \ldots, y_N) z_j - g_j(y_{1r}, \ldots, y_{Nr}) \hat{z}_j. \tag{2.108}
$$

A step-by-step constructive controller design procedure is now developed, leading to an improved solution to the decentralized problem under consideration with desired tracking controllers.

Step J.1: Starting with the first $(\tilde{z}_j, \tilde{x}_j, y_j)$-subsystem of (2.105). Introduce the new variable $\xi_{j1} = y_j - y_{jr}$ $(= \tilde{x}_{j1})$ and consider the Lyapunov function

$$
V_{j1} = \lambda_{j1} \tilde{z}_j^t P_{j1} \tilde{z}_j + \lambda_{j2} (\tilde{z}_j^t P_{j1} \tilde{z}_{j1})^2 + \tilde{x}_j^t P_{j2} \tilde{x}_j + \frac{1}{2} \xi_{i1}^2, \tag{2.109}
$$

where $\lambda_{j1}, \lambda_{j2} > 0$ are design parameters, $P_{j1} = P_{j1}^t > 0$ and $P_{i2} = P_{i2}^t > 0$ satisfy

$$
P_{i1} Q_j + Q_j^t P_{i1} = -2 I_{n_{z_j}}, \tag{2.110}
$$

$$
P_{i2} A_j + A_j^t P_{i2} = -2 I_{n_j}. \tag{2.111}
$$

This guarantees that $V_{j1} > 0$. Then by evaluating the time derivative of V_{i1} along the solutions of (2.105), we obtain

$$
\begin{aligned}
\dot{V}_{j1} = {} & (\lambda_{j1} + 2\lambda_{j2} \tilde{z}_j^t P_{j1} \tilde{z}_j)(-2|\tilde{z}_j|^2 + 2\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0} w_j)) \\
& - 2|\tilde{x}_j|^2 + 2\tilde{x}_j^t P_{j2}(\tilde{f}_j + \tilde{g}_j + p_j w_j) + \xi_{j1}(\hat{x}_{j2} + \tilde{x}_{j2} \\
& + f_{j1}(y_1, \ldots, y_N) + g_{j1}(y_1, \ldots, y_N) z_j \\
& + p_{j1}(y_1, \ldots, y_N) w_j - \dot{y}_{ir}).
\end{aligned} \tag{2.112}
$$

We first examine the term $2\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{i0} w_j)$. Using (2.106) and (2.101), with the help of Young's inequality (see Chap. 9) and some algebraic manipulations, it follows that:

$$\begin{aligned}
&2(\lambda_{j1} + 2\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)\tilde{z}_j^t P_{j1}(\tilde{f}_{j0} + p_{j0} w_j) \\
&\quad \le \lambda_{j1}|\tilde{z}_j|^2 + \frac{3\lambda_{i2}}{\lambda_{\max}(P_{i1})}(\tilde{z}_j^t P_{j1}\tilde{z}_j)^2 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{ik1}(\xi_{k1}) \\
&\qquad + c_{j2}|w_j|^2 + c_{j3}|w_j|^4 + |w_j|^8,
\end{aligned} \tag{2.113}$$

where $c_{j1}, c_{j2}, c_{j3} > 0$ and ψ_{jk1} is a nonnegative smooth function.

In a similar way, there exist positive constants κ_{j1}, c_{j4} and a nonnegative smooth function ψ_{jk2} such that

$$\begin{aligned}
&2\tilde{x}_j^t P_{j2}(\tilde{f}_j + \tilde{g}_j + p_j w_j) \\
&\quad \le |\tilde{x}_j|^2 + \kappa_{j1}|\tilde{z}_j|^2 + |\tilde{z}_j|^4 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{jk2}(\xi_{k1}) + c_{j4}|w_j|^2 + |w_j|^4,
\end{aligned} \tag{2.114}$$

where we have used the fact that $\hat{z}_j$ is bounded.

By substituting (2.113) and (2.114) into (2.112), we readily obtain

$$\begin{aligned}
\dot{V}_{i1} \le& -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j)|\tilde{z}_j|^2 - |\tilde{x}_j|^2 + \sum_{k=1}^{N} \xi_{k1}^2(\psi_{ik1} + \psi_{jk2}) \\
&+ \kappa_{j1}|\tilde{z}_j|^2 + |\tilde{z}_j|^4 + (c_{j2} + c_{j4})|w_j|^2 + (c_{j3} + 1)|w_j|^4 \\
&+ |w_j|^8 + \xi_{j1}(\hat{x}_{i2} + \tilde{x}_{j2} + f_{j1}(y_1, \ldots, y_N) \\
&+ g_{j1}(y_1, \ldots, y_N) z_j + p_{j1}(y_1, \ldots, y_N) w_j - \dot{y}_{jr}).
\end{aligned} \tag{2.115}$$

It is significant to note that κ_{j1} does not depend on λ_{j1} and λ_{j2} while c_{jk}'s may depend on λ_{j1} and λ_{j2}.

Proceeding further, using (2.102) and (2.104), we have

$$\begin{aligned}
&\xi_{j1}(\tilde{x}_{j2} + \tilde{f}_{j1} + \tilde{g}_{j1} + p_{j1} w_j) \\
&\quad \le \frac{1}{2}|\tilde{x}_j|^2 + \sum_{k=1}^{N} \xi_{k1}^2 \psi_{jk3}(\xi_{k1}) + |\tilde{z}_j|^2 + |w_j|^2,
\end{aligned} \tag{2.116}$$

where ψ_{jk3} is a nonnegative smooth function.

Taking into consideration the decomposition in (2.107) and (2.108) and letting $\hat{\psi}_{jk1} = \psi_{jk1} + \psi_{jk2} + \psi_{jk3}$, the following holds true:

$$\begin{aligned}\dot{V}_{j1} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 + (c_{j3} + 1)|w_j|^4 \\ & + |w_j|^8 + \xi_{j1}(\hat{x}_{j2} + f_{j1}(y_{1r}, \ldots, y_{Nr}) \\ & + g_{j1}(y_{1r}, \ldots, y_{Nr})\hat{z}_j - \dot{y}_{jr}) + \sum_{k=1}^{N} \xi_{k1}^2 \hat{\psi}_{ik1}. \end{aligned} \tag{2.117}$$

This motivates us to choose a control function $\xi_{j1}^\star$ and a new variable ξ_{j2} in the form

$$\begin{aligned}\xi_{j1}^\star = & -k_{j1}\xi_{j1} - \xi_{j1}K_j(\xi_{j1}) - f_{j1}(y_{1r}, \ldots, y_{Nr}) \\ & - g_{j1}(y_{1r}, \ldots, y_{N}r)\hat{z}_j + \dot{y}_{jr},\end{aligned} \tag{2.118}$$

$$\xi_{j2} = \hat{x}_{j2} - \xi_{j1}^\star(y_j, y_{1r}, \ldots, y_{Nr}, \dot{y}_{jr}, \hat{z}_j), \tag{2.119}$$

where $k_{j1} > 0$ is a design parameter and K_j is a nonnegative, smooth function such that

$$K_{j1}(\xi_{j1}) \geq \sum_{k=1}^{N} \hat{\psi}_{kj1}(\xi_{j1}). \tag{2.120}$$

This leads us to

$$\begin{aligned}\dot{V}_{j1} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 \\ & + (c_{j3} + 1)|w_j|^4 + |w_j|^8 - k_{i1}\xi_{j1}^2 - \xi_{j1}^2 K_j(\xi_{i1}) \\ & + \sum_{k=1}^{N} \xi_{k1}^2 \hat{\psi}_{jk1}(\xi_{k1}) + \xi_{j1}\xi_{i2}. \end{aligned} \tag{2.121}$$

Step J.*k* ($2 \leq k \leq n_k$): Consider the $(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{i2}, \ldots, \hat{x}_{jk})$-subsystem of (2.105) with $\hat{x}_{j,jk+1}$ as the virtual control. For notational simplicity, we define $\hat{x}_{j,n_k+1} := u_k$.

Rolling over from Step J.1 to Step J.$k-1$, we assume that we have designed intermediate control functions $\{\xi_{j\ell}^\star\}_{\ell=1}^{k-1}$, and that we have introduced new variables

$$\begin{aligned}& \xi_{j,\ell+1} = \hat{x}_{j,\ell+1} - \xi_{j\ell}^\star(y_j, \hat{x}_{j2}, \ldots, \hat{x}_{j\ell}, y_{\ell r}, \ldots, y_{Nr}, \dot{y}_{jr}, \ldots, y_{jr}^{(\ell)}, \hat{z}_j) \\ & \forall 1 \leq \ell \leq k-1 \end{aligned} \tag{2.122}$$

and a positive-definite and proper function

$$V_{j,k-1}(\tilde{z}_j, \tilde{x}_j, \xi_{j\ell}, \ldots, \xi_{j,k-1}) = V_{j\ell}(\tilde{z}_j, \tilde{x}_j, \xi_{j\ell} + \sum_{\ell=2}^{k-1} \frac{1}{2}\xi_{j\ell}^2. \tag{2.123}$$

It is further assumed that the time derivative of $V_{j,k-1}$ along the solutions of (2.105) satisfies

$$\begin{aligned}\dot{V}_{j,k-1} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - k + 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2^{k-1}}|\tilde{x}_j|^2 + (k - 1 + c_{k2} + c_{k4})|w_j|^2 \\ & + (c_{j3} + 1)|w_j|^4 + |w_j|^8 - \sum_{\ell=1}^{k-1} k_{j\ell}\xi_{j\ell}^2 - \xi_{j\ell}^2 K_j(\xi_{j\ell}) \\ & + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jm(k-1)}(\xi_{m1}) + \xi_{j,k-1}\xi_{jk}\end{aligned} \tag{2.124}$$

with $k_{j\ell}$ $(1 \leq \ell \leq k-1)$ positive design parameters and $\hat{\psi}_{jm(k-1)}$ a nonnegative smooth function being independent of K_j.

The objective is to prove that a similar property to the above also holds for the subsystem

$$(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jk})$$

of (2.105) when $\hat{x}_{j,k+1}$ is considered as the (virtual) input.

Toward this end, consider the positive-definite and proper function

$$V_{jk} = V_{j,k-1}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \ldots, \xi_{j,k-1}) + \frac{1}{2}\xi_{jk}^2. \tag{2.125}$$

Evaluating the time-derivative of V_{jk} along the solutions of (2.105) yields

$$\begin{aligned}\dot{V}_{jk} = \dot{V}_{j,k-1}\xi_{jk}\Bigg[& \hat{x}_{j,k+1} + L_{jk}(y_j - \hat{x}_{j1}) \\ & + f_{jk}(y_{1r}, \ldots, y_{Nr}) + g_{jk}(y_{1r}, \ldots, y_N r)\hat{z}_j \\ & - \sum_{m=2}^{k-1} \frac{\partial \xi_{j,k-1}^\star}{\partial \hat{x}_{jm}}(\hat{x}_{j,m+1} + L_{jm}(y_j - \hat{x}_{j1}) \\ & + f_{jm}(y_{1r}, \ldots, y_{Nr}) + g_{jm}(y_{1r}, \ldots, y_{Nr})\hat{z}_j) \\ & - \sum_{m=1}^{N} \frac{\partial \xi_{j,k-1}^\star}{\partial y_{mr}}\dot{y}_{mr} - \sum_{m=1}^{k-1} \frac{\partial \xi_{j,k-1}^\star}{\partial y_{jr}^{(m+1)}} y_{jr}^{(m+1)} \\ & - \frac{\partial \xi_{j,k-1}^\star}{\partial \hat{z}_j}(Q_j\hat{z}_j + f_{k0}(y_{1r}, \ldots, y_{Nr})) \\ & - \frac{\partial \xi_{j,k-1}^\star}{\partial y_j}(\hat{x}_{i2} + \hat{x}_{k2} + f_{k1} + g_{k1}z_j + p_{k1}w_j)\Bigg].\end{aligned} \tag{2.126}$$

Adopting similar arguments to Step J.1, after algebraic routine manipulations, it follows the existence of nonnegative smooth functions $\{\psi_{jmk}\}_{m=1}^{N}$ and κ_{jk} such that:

$$
\begin{aligned}
&-\xi_{jk}\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_j}(\tilde{x}_{j2}+\tilde{f}_{j1}+\tilde{g}_{j1}+p_{j1}w_j)\\
&\quad\le\frac{1}{2^j}\tilde{x}_j^2+\xi_{jk}^2\kappa_{jk}+\sum_{m=1}^{N}\xi_{m1}^2\psi_{jmk}(\xi_{m1})+|\tilde{z}_j|^2+|w_j|^2.
\end{aligned}
\tag{2.127}
$$

Observe that κ_{jk} is a function of

$$
(y_j,\hat{x}_{j2},\ldots,\hat{x}_{jk},y_{1r},\ldots,y_{Nr},\dot{y}_{jr},\ldots,y_{jr}^{(\ell)},\hat{z}_j)
$$

and that every ψ_{jmk} does not depend on K_j.

This motivates us to select the following control function:

$$
\begin{aligned}
\xi^{\star}_{jk}&=-k_{jk}\xi_{jk}-\xi_{j,k-1}-\xi_{jk}\kappa_{jk}-L_{jk}(y_j-\hat{x}_{j1})\\
&\quad-f_{jk}(y_{1r},\ldots,y_{Nr})-g_{jk}(y_{1r},\ldots,y_{Nr})\hat{z}_j\\
&\quad+\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_j}(\hat{x}_{j2}+f_{j1}(y_{1r},\ldots,y_{Nr})+g_{j1}(y_{1r},\ldots,y_{Nr})\hat{z}_j)\\
&\quad+\sum_{m=2}^{k-1}\frac{\partial\xi^{\star}_{j,k-1}}{\partial\hat{x}_{jm}}(\hat{x}_{j,m+1}+L_{jm}(y_j-\hat{x}_{j1})\\
&\quad+f_{jm}(y_{1r},\ldots,y_{Nr})+g_{jm}(y_{1r},\ldots,y_{Nr})\hat{z}_j)\\
&\quad+\sum_{m=1}^{N}\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_{mr}}\dot{y}_{mr}+\sum_{m=1}^{j-1}\frac{\partial\xi^{\star}_{j,k-1}}{\partial y_{jr}^{(m)}}y_{jr}^{(m+1)}\\
&\quad+\frac{\partial\xi^{\star}_{j,k-1}}{\partial\hat{z}_j}(Q_j\hat{z}_j+f_{j0}(y_{1r},\ldots,y_{Nr})),
\end{aligned}
\tag{2.128}
$$

where $k_{jk}>0$ is a design parameter.

Denoting $\xi_{j,k+1}=\hat{x}_{j,k+1}-\xi^{\star}_{jk}$ and combining (2.124) with (2.126)–(2.128), we obtain

$$
\begin{aligned}
\dot{V}_{jk}&\le-(\lambda_{j1}+\lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j-\kappa_{j1}-j-|\tilde{z}_j|^2)|\tilde{z}_j|^2\\
&\quad-\frac{1}{2^j}|\tilde{x}_j|^2+(j+c_{j2}+c_{j4})|w_j|^2+(c_{j3}+1)|w_j|^4\\
&\quad+|w_j|^8-\sum_{\ell=1}^{j}k_{jl}\xi_{jl}^2-\xi_{j1}^2K_j(\xi_{j1})\\
&\quad+\sum_{m=1}^{N}\xi_{m1}^2(\hat{\psi}_{jm(k-1)}(\xi_{m1})+\psi_{jmk}(\xi_{m1}))+\xi_{jk}\xi_{j,k+1}.
\end{aligned}
\tag{2.129}
$$

That is, property (2.124) holds for the $(\tilde{z}_j, \tilde{x}_j, y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jk})$-subsystem with

$$\hat{\psi}_{jmk} = \hat{\psi}_{ik(j-1)} + \psi_{ikj}.$$

By induction, at Step n_j, setting the control law

$$u_j = \xi^{\star}_{jn_j}(y_j, \hat{x}_{j2}, \ldots, \hat{x}_{jn_j}, y_{1r}, \ldots, y_{Nr}, \dot{y}_{ir}, \ldots, y^{(n_j)}_{jr}, \hat{z}_j) \tag{2.130}$$

leads us to

$$\begin{aligned}\dot{V}_{jn_j} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}^t_j P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2^{n_j}}|\tilde{x}_j|^2 + (n_j + c_{i2} + c_{i4})|w_j|^2 \\ & + (c_{j3} + 1)|w_j|^4 + |w_j|^8 - \sum_{\ell=1}^{n_j} k_{j\ell}\xi^2_{j\ell} - \xi^2_{j1}K_j(\xi_{j1}) \\ & + \sum_{m=1}^{N} \xi^2_{m1}\hat{\psi}_{jmn_j}(\xi_{m1}),\end{aligned} \tag{2.131}$$

where by construction, $\hat{\psi}_{jmn_j}$ is independent of the design function K_j.

Consider now the positive-definite and proper Lyapunov function for the entire closed-loop interconnected system

$$V(\tilde{z}, \tilde{x}, \xi) = \sum_{j=1}^{N} V_{jn_j}(\tilde{z}_j, \tilde{x}_j, \xi_{j1}, \ldots, \xi_{jn_j}), \tag{2.132}$$

where

$$\tilde{z} = (\tilde{z}^t_1, \ldots, \tilde{z}^t_N)^t, \qquad \tilde{x} = (\tilde{x}^t_1, \ldots, \tilde{x}^t_N)^t, \qquad \xi = \xi^t_1, \ldots, \xi^t_N)^t.$$

Notice that the positive definiteness and properness of V in (2.132) follows from the foregoing recursive construction.

To eliminate the positive sum of the last term of (2.131), which also appears in the time derivative of V, we pick a set of appropriate smooth functions $\{K_j\}^N_{j=1}$ to check on the inequalities $(1 \leq j \leq N)$

$$K_j(\xi_{j1}) \geq \sum_{m=1}^{N} \hat{\psi}_{mjn_m}\xi_{j1}. \tag{2.133}$$

Obviously, such a design function K_j always exists.

2.4.3 Design Results

When applying the above-described control design to the uncertain large-scale system (2.11), we establish the following result.

Theorem 2.6 *The problem of decentralized output-feedback tracking with disturbance attenuation is solvable for the minimum-phase large-scale system* (2.11) *subject to Condition A.*

Proof By differentiating V defined by (2.132), along the solutions of the closed-loop system (2.11) and (2.130), it yields

$$\dot{V} \leq -\sum_{j=1}^{N}(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2$$
$$-\sum_{j=1}^{N}\left(\frac{1}{2^{n_j}}|\tilde{x}_j|^2 + \sum_{\ell=1}^{n_j} k_{j\ell}\xi_{j\ell}^2\right)$$
$$+\sum_{j=1}^{N}[(n_j + c_{j2} + c_{j4})|w_j|^2 + (c_{j3}+1)|w_j|^4 + |w_j|^8]. \quad (2.134)$$

By selecting sufficiently large design parameters λ_1 and λ_2 such that

$$(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - n_j - |\tilde{z}_j|^2)|\tilde{z}_j|^2$$
$$\geq \frac{\lambda_{j1}}{2}\tilde{z}_j P_{j1}\tilde{z}_j + \frac{\lambda_{j2}}{2}(\tilde{z}_j P_{j1}\tilde{z}_j)^2 \quad (2.135)$$

it follows from (2.134) and (2.132) that

$$\dot{V} \leq -\lambda V + \sum_{j=1}^{N}[(n_j + c_{j2} + c_{j4})|w_j|^2$$
$$+ (c_{j3}+1)|w_j|^4 + |w_j|^8], \quad (2.136)$$

where

$$\lambda = \min\left\{\frac{1}{2}, 1/2^{n_j}\lambda_{\max}(P_{j2}),\ k_{j\ell} \mid 1 \leq j \leq N,\ 1 \leq \ell \leq n_j\right\}.$$

The BIBS and iISS property (2) follows readily for the (transformed) closed-loop system (2.11), (2.130) by either applying the technique in [64] or the Gronwall-Bellman lemma [32] to (2.136). When $w_j = 0$ for all $1 \leq j \leq N$, the null solution is uniformly globally asymptotically stable (UGAS), leading to the asymptotic convergence of the tracking error $y - y_r$ because $\xi_1 = y - y_r$.

Now from (2.134), for any pair of instants $0 \leq t_0 \leq t$, we obtain

$$\int_{t_0}^{t}|\xi_1(\tau)|^2 d\tau \leq V(z(t_0), x(t_0), \xi(t_0)) + \rho\int_{t_0}^{t}(|w(\tau)|^2$$
$$+ |w(\tau)|^4 + |w(\tau)|^8)d\tau \quad (2.137)$$

where $\rho > 0$ is defined by

$$\rho = \max\left\{\frac{\max\{n_j + c_{j2} + c_{j3} | 1 \le j \le N\}}{\min\{k_{j1} | 1 \le j \le N\}}, \frac{\max\{c_{j3} + 1 | 1 \le j \le N\}}{\min\{k_{j1} | 1 \le i \le N\}}, \frac{1}{\min\{k_{j1} | 1 \le j \le N\}}\right\}.$$

It must be noted that ρ can be made as small as possible by selecting sufficiently large values of the constants k_{j1}. In the present case, (2.95) is met with $\gamma_d(s) = s^2 + s^4 + s^8$. The proof of Theorem 2.6 is now completed. □

Remark 2.19 It is of interest to observe that, in the absence of disturbance inputs w, (2.136) yields that V converges to zero at an exponential rate and; therefore; the tracking error $y(t) - y_r(t)$ goes to zero exponentially.

Remark 2.20 By similarity to the centralized output-feedback tracking with almost disturbance decoupling [41], Condition A can be weakened and the z_j-system in (2.11) can be broadened as follows:

$$\dot{z}_j = \Gamma_j(y_1, \dots, y_N) z_j + f_{i0}(y_1, \dots, y_N) + p_{i0}(y_1, \dots, y_N) w_j. \tag{2.138}$$

Assume that, for each $1 \le j \le N$, there are a pair of constant matrices $(0 < P_j = P_j^t, 0 < M_j = M_j^t)$ such that

$$\Gamma_j^t(y_1, \dots, y_N) P_j + P_j \Gamma_j(y_1, \dots, y_N) \le -M_j. \tag{2.139}$$

Under this hypothesis, the $\hat{z}_j$-system in the decentralized observer (2.96) is replaced by

$$\dot{\hat{z}}_j = \Gamma_j(y_{1r}, \dots, y_{Nr}) \hat{z}_j + f_{j0}(y_{1r}, \dots, y_{Nr}). \tag{2.140}$$

Using the same techniques as in Sect. 2.4.2, Theorem 2.6 can be extended to this situation.

To proceed further, we examine the situation when the developed controller design procedure yields a decentralized output-feedback law guaranteeing the standard $\mathcal{L}_2$-gain disturbance attenuation property (2.95) holds with $\gamma_d(s) = s^2$. The following additional sufficient condition is recalled.

Condition B For all $1 \le j \le N$ and $1 \le k \le n_k$, the function p_{jk} is bounded by a constant. Furthermore, $p_{j0} = 0$ for each $1 \le j \le N$.

The following lemma provides the desired result:

Lemma 2.6 *Under Condition A and Condition B, the problem of decentralized output-feedback tracking with $\mathcal{L}_2$-gain disturbance attenuation is solvable for the class of minimum-phase large-scale systems* (2.11).

Proof We initially note that the only place where $|w_j|^4$ and $|w_j|^8$ occur is Step J.1 during the controller development in Sect. 2.4.2. More precisely, they are brought up in the inequalities (2.113) and (2.114). Under Condition B, the function V_{j1} satisfies the following inequality, instead of (2.121):

$$\begin{aligned}\dot{V}_{j1} \leq & -(\lambda_{j1} + \lambda_{j2}\tilde{z}_j^t P_{j1}\tilde{z}_j - \kappa_{j1} - 1 - |\tilde{z}_j|^2)|\tilde{z}_j|^2 \\ & - \frac{1}{2}|\tilde{x}_j|^2 + (c_{j2} + c_{j4} + 1)|w_j|^2 - k_{j1})\xi_{j1}^2 \\ & - \xi_{i1}^2 K_j(\xi_{i1}) + \sum_{m=1}^{N} \xi_{m1}^2 \hat{\psi}_{jm1}(\xi_{m1}) + \xi_{j1}\xi_{j2}. \end{aligned} \tag{2.141}$$

The above Lyapunov function V satisfies

$$\dot{V} \leq -\lambda V + \sum_{j=1}^{N} [(n_j + c_{j2} + c_{j4})|w_j|^2]. \tag{2.142}$$

From (2.142), the standard $\mathcal{L}_2$-gain property from w to $\xi_1 = y - y_r$ follows readily. The proof of Lemma 2.6 is thus completed. □

Remark 2.21 As an immediate corollary of Theorem 2.6, the standard $\mathcal{L}_2$-gain property from w to $\xi_1 = y - y_r$ can also be established when all functions f_{jk}, g_{jk} in the decentralized system (2.11) are bounded by linear functions and the functions p_{jk} $(1 \leq j \leq N, 0 \leq k \leq n_k)$ are bounded by some constants (in this case, $p_{j0} \neq 0$). The derived decentralized output-feedback controllers are linear.

Remark 2.22 The main features are four-fold:

(i) identifying a wide class of large-scale nonlinear systems in disturbed decentralized output-feedback form;
(ii) proposing an effective systematic output-feedback controller design procedure for decentralized systems in the presence of strong nonlinearities appearing in the subsystems and interactions and
(iii) guaranteeing decentralized asymptotic tracking when the disturbance inputs disappear and achieving desirable external stability properties when the disturbance inputs are present;
(iv) extending further the earlier results of [23, 29, 32, 40] to uncertain large complex systems.

2.5 Decentralized Guaranteed Cost Control

In recent years, the problem of the decentralized robust control of large-scale systems with parameter uncertainties has been widely studied. Although there have been numerous studies on the decentralized robust control of large-scale uncertain systems, much effort has been made toward finding a controller that guarantees robust stability. However, when controlling such systems, it is also desirable to design control systems that guarantee not only robust stability but also an adequate level of performance. One approach to this problem is the so-called guaranteed cost control approach [47]. This approach has the advantage of providing an upper bound on a given performance index.

Recent advances in the LMI theory have allowed a revisiting of the guaranteed cost control approach [82]. In [82], the guaranteed cost control technique for interconnected systems by means of the LMI approach has been discussed. In the literature, the guaranteed cost control for nonlinear uncertain large-scale systems under gain perturbations has been considered. However, the time delays have not been considered in those reports. If the system does not have delays, the theoretical behavior would usually be more tractable. However, if delays are present, they may result in instability or serious deterioration in the performance of the resulting control systems. Therefore, the study of the control, considering these time delays on the guaranteed cost stability, is very important.

In what follows, the guaranteed cost control problem of the decentralized robust control for uncertain nonlinear large-scale systems that have delay in both state and control input is considered. It should be noted that although the robust control design method for parameter uncertain ordinary dynamic systems that have delay in both state and control input has been considered, the guaranteed cost control for nonlinear uncertain large-scale systems that have delay in both state and control input has never been discussed. A sufficient condition for the existence of the decentralized robust feedback controllers is derived in terms of the LMI. The main result shows that the guaranteed cost controllers can be constructed by solving the LMI. The crucial difference between the existing results [82] and that of the present study is that the controller that guarantees the stability and the adequate level of performance of the large-scale delay systems is given. Thus, the applicability of the resulting controllers can be extended to more practical large-scale systems. Moreover, since the construction of the guaranteed cost controller consists of an LMI-based control design, the proposed method is computationally attractive and useful.

2.5.1 Analysis of Robust Performance

To demonstrate ideas, we consider in the sequel a class of continuous-time autonomous uncertain nonlinear large-scale interconnected delay systems, which consist of N subsystems of the form:

$$\begin{aligned} \dot{x}_j(t) &= [\bar{A}_j + \Delta\bar{A}_j(t)]x_j(t) + [A_j^d + \Delta A_j^d(t)]xi(t-\tau_j) \\ &\quad + [H_j^d + \Delta H_j^d(t)]x_j(t-h_j) \\ &\quad + \sum_{j=1, j\neq k}^{N} [G_{kj} + \Delta G_{kj}(t)]g_{kj}(x_j, x_k), \end{aligned} \tag{2.143}$$

$$\begin{aligned} x_j(t) &= \phi_j(t), \quad t \in [-d_j, 0], \\ d_j &= \max\{\tau_j, h_j\}, \quad j = 1, \ldots, N, \end{aligned} \tag{2.144}$$

where $x_j(t) \in \Re^{n_j}$ are the states. $\tau_j > 0$ and $h_j > 0$ are the delay constants, and $\phi_j(t)$ are the given continuous vector valued initial functions. $\bar{A}_j$, A_j^d, and H_j^d are the constant matrices of appropriate dimensions. $G_{ij} \in \Re^{n_j \times l_j}$ are the interconnection matrices between the ith subsystems and other subsystems. $g_{kj}(x_j, x_k) \in \Re^{\ell_j}$ are unknown nonlinear vector functions that represent nonlinearity. The parameter uncertainties considered here are assumed to be of the following form:

$$[\Delta\bar{A}_j(t)\Delta A_j^d(t)\Delta H_j^d(t)] = D_j F_j(t)[\bar{E}_j^1 E_j^{1d} \bar{E}_j^{d_h}], \tag{2.145}$$

$$\Delta G_{jk}(t) = D_{jk} F_{jk}(t) E_{jk}, \tag{2.146}$$

where D_j, $\bar{E}_j^1$, E_j^{1d}, $\bar{E}_j^{d_h}$, D_{ij}, and E_{ij} are known constant real matrices of appropriate dimensions. $F_j(t) \in \Re^{p_j \times q_j}$ and $F_{ij}(t) \in \Re^{r_{ij} \times s_{jk}}$ are unknown matrix functions with Lebesgue measurable elements and satisfy

$$F_j^t(t)F_j(t) \leq I_{qi}, \qquad F_{ij}^t(t)F_{ij}(t) \leq I_{sij}. \tag{2.147}$$

We make the following assumptions concerning the unknown nonlinear vector functions.

(A1) *There exist known constant matrices V_j and W_{jk} such that for all $j, k, t \geq 0$, $x_j \in \Re^{n_j}$* and $x_j \in \Re^{n_j}$

$$\|g_{jk}(x_j, x_k)\| \leq \|V_j x_j\| + \|W_{jk} x_j\|.$$

(A2) *For all j, k*

$$U_j := 2 \sum_{j=1, j\neq k}^{N} (V_j^t V_j + W_{jk}^t W_{jk}) > 0.$$

The cost function of the associated system (2.143) is given as

$$J = \sum_{j=1}^{N} \int_0^\infty x_j^t(t)\bar{Q}_j x_j(t)dt, \quad 0 < \bar{Q}_j = \bar{Q}_j^t. \tag{2.148}$$

The following definition of the cost matrix for the uncertain large-scale interconnected delay systems is given in [47]:

Definition 2.1 The set of matrices $0 < P_j = P_j^t$ is said to be the quadratic cost matrix for the uncertain nonlinear large-scale interconnected delay systems (2.143) if the following inequality holds

$$\sum_{i=1}^{N} \left(\frac{d}{dt} x_j^t(t) P_j x_j(t) + x_j^t(t) \bar{Q}_j x_j(t) \right) < 0, \tag{2.149}$$

for all nonzero $x_j \in \Re^{n_j}$ and all uncertainties (2.145).

Theorem 2.7 *Under assumptions* (A1) *and* (A2), *suppose there exist matrices* $0 < P_j = P_j^t \in \Re^{n_j \times n_j}, 0 < S_j = S_j^t \in \Re^{n_j \times n_j}, 0 < T_j = T_j^t \in \Re^{n_j \times n_j}$ *such that for all admissible uncertainties satisfying* (2.145) *the following matrix inequality holds*:

$$\Lambda_j = \begin{bmatrix} \Xi_j & P_j \tilde{A}_j^d & P_j \tilde{H}_j^d & P_j \tilde{G}_{j1} & \dots & P_j \tilde{G}_{jN} \\ \bullet & -S_j & 0 & 0 & \dots & 0 \\ \bullet & \bullet & -T_j & 0 & \dots & 0 \\ \bullet & \bullet & \bullet & -I_{l_1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \dots & -I_{l_N} \end{bmatrix} < 0, \tag{2.150}$$

where

$$\begin{aligned}
&\Lambda_j \in \Re^{\bar{N} \times \bar{N}}, \quad \bar{N} = 3n_j + \sum_{m=1, j \neq m}^{N} \ell_m, \\
&\Xi_j := \tilde{A}_j^t P_j + P_j \tilde{A}_j + U_j + \bar{Q}_j + S_j + T_j, \quad \tilde{A}_j := \bar{A}_j + \Delta A_j(t), \\
&\tilde{A}_j^d := A_j^d + \Delta A_j^d(t), \qquad \tilde{H}_j^d := H_j^d + \Delta H_j^d(t), \\
&\tilde{G}_{jk} := G_{jk} + \Delta G_{jk}(t).
\end{aligned}$$

Then the free uncertain nonlinear large-scale interconnected systems (2.143) *are quadratically stable, and the corresponding value of the cost function* (2.148) *satis-*

fies the following inequality:

$$J < \sum_{i=1}^{N} \left[\phi_j^t(0) P_j \phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s) S_j \phi_j(s) ds + \int_{-h_j}^{0} \phi_j^t(s) T_j \phi_j(s) ds \right]. \tag{2.151}$$

Proof Based on the definitions $\tilde{A}_j$, $\tilde{A}_j^d$, $\tilde{H}_j^d$ and $\tilde{G}_{jk}$, we can change the form (2.143) to

$$\dot{x}_j(t) = \tilde{A}_j x_j(t) + \tilde{A}_j^d x_j(t-\tau_j) + \tilde{H}_j^d x_j(t-h_j) + \sum_{k=1, j\neq k}^{N} \tilde{G}_{jk} g_{jk}(x_j, x_k). \tag{2.152}$$

There exist matrices $0 < P_j = P_j^t \in \Re^{n_j \times n_j}$, $0 < S_j = S_j^t \in \Re^{n_j \times n_j}$, $0 < T_j = T_j^t \in \Re^{n_j \times n_j}$, $j = 1, \dots, N$ such that the matrix inequality (2.150) holds for all admissible uncertainties (2.145). To prove the asymptotic stability of the interconnected delay systems (2.152), we introduce the following Lyapunov function candidate

$$V\big(x(t)\big) = \sum_{i=1}^{N} \left[x_j^t(t) P_j x_j(t) + \int_{t-\tau_j}^{t} x_j^t(s) S_j x_j(s) ds + \int_{t-h_j}^{t} x_j^t(s) T_j x_j(s) ds \right], \tag{2.153}$$

where $x(t) = [x_1^t(t) \ \dots \ x_N^t(t)]^t$. Note by default that $V(x(t)) > 0$ whenever $x(t) \neq 0$. The time derivative of $V(x(t))$ along any trajectory of the interconnected delay systems (2.152) is given by

$$\frac{d}{dt} V(x(t)) = \sum_{i=1}^{N} z_j^t(t) \Lambda_j z_j(t) - \sum_{i=1}^{N} x_j^t(t) \bar{Q}_j x_j(t) - \sum_{i=1}^{N} \sum_{k=1, j\neq k}^{N} (2x_j^t V_j^t V_j x_j + 2x_j^t W_{jk}^t W_{jk} x_j - g_{jk}^t g_{jk}),$$

where

$$z_j = [x_j^t(t) \ x_j^t(t-\tau_j) \ x_j^t(t-h_j) \ g_{j1}^t \ \dots \ g_{jN}^t]^t \in \Re^{\bar{N}}$$

and Ξ_j and Λ_j are given in (2.151).

Under assumption (A1), it is easy to verify that the following inequality holds

$$2x_j^t V_j^t V_j x_j + 2x_j^t W_{jk}^t W_{jk} x_j \geq g_{jk}^t g_{jk}. \tag{2.154}$$

With inequalities (2.150) and (2.154) hold, it immediately follows that

$$\frac{d}{dt}V(x(t)) < -\sum_{j=1}^{N} x_j^t(t)\bar{Q}_j x_j(t) < 0, \tag{2.155}$$

which assures that $V(x(t))$ is a Lyapunov function for the interconnected delay system (2.152). Therefore, system (2.152) is asymptotically stable. Furthermore, by integrating both sides of the inequality (2.155) from 0 to T and using the initial conditions, we obtain

$$V(x(T)) - V(x(0)) < -\sum_{j=1}^{N} \int_0^t x_j^t(t)\bar{Q}_j x_j(t)dt. \tag{2.156}$$

Since system (2.152) is asymptotically stable, that is, $x(T) \to 0$ when $T \to \infty$, we obtain $V(x(T)) \to 0$. Thus we obtain

$$\begin{aligned} J &= \sum_{j=1}^{N} \int_0^t x_m^t(t)\bar{Q}_j x_j(t)dt < V(x(0)) \\ &= \sum_{j=1}^{N} \left[\phi_j^t(0)P_j\phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s)S_j\phi_j(s)ds + \int_{-h_j}^{0} \phi_j^t(s)T_j\phi_j(s)ds \right]. \end{aligned}$$

This completes the proof of Theorem 2.7. □

2.5.2 Including Input Delays

In what follows, we consider the problem of decentralized guaranteed cost control via the state feedback to the class of nonlinear uncertain interconnected systems with input delays. The class of system under consideration is described by

$$\begin{aligned} \dot{x}_j(t) &= [A_j + \Delta A_j(t)]x_j(t) + [B_j + \Delta B_j(t)]u_j(t) \\ &\quad + [A_{dj} + \Delta A_{dj}(t)]x_j(t-\tau_j) + [B_{dj} + \Delta B_{dj}(t)]u_j(t-h_j) \\ &\quad + \sum_{k=1, j\neq k}^{N} [G_{jk} + \Delta G_{jk}]g_{jk}(x_j, x_k), \end{aligned} \tag{2.157}$$

$$x_j(t) = \phi_j(t), \quad t \in [-d_j, 0], \quad d_j = \max\{\tau_j, h_j\}, \quad j = 1, \ldots, N, \tag{2.158}$$

where $u_j(t) \in \Re^{m_j}$ are the control inputs of the jth subsystems. The parameter uncertainties satisfy

$$[\Delta A_j(t)\ \Delta B_j(t)\ \Delta A_{dj}(t)\ \Delta B_{dj}(t)] = D_j F_j(t)[E_{1j}\ E_{2j}\ E_{1dj}\ E_{2dj}]. \tag{2.159}$$

$A_j, B_j, E_{1j}, E_{2j}, E_{d1j}, E_{d2j}$ are constant matrices of appropriate dimensions. The remaining constant real matrices and parameter uncertainties are the same as those in system (2.143). Moreover, it is assumed that Assumptions (A1) and (A2) hold for the unknown nonlinear vector functions $g_{jk}(x_j, x_k) \in \Re^{\ell_j}$.

Associated with system (2.157) is the cost function

$$J = \sum_{j=1}^{N} \int_0^{\infty} [x_j^t(t) Q_j x_j(t) + u_j^t(t) R_j u_j(t)] dt,$$
$$0 < Q_j = Q_j^t, \quad 0 < R_j = R_j^t. \tag{2.160}$$

In view of the results of [47], the definition of the guaranteed cost control for the class of uncertain interconnected systems (2.157) is now provided:

Definition 2.2 A decentralized control law $u_j(t) = K_j x_j(t)$ is said to be a quadratic guaranteed cost control related to the set of matrices $0 < P_j = P_j^t$ for the uncertain interconnected system (2.157) and cost function (2.160) if the closed-loop system is quadratically stable and the closed-loop value of the cost function (2.160) satisfies the bound $J \le J^*$ for all admissible uncertainties, that is,

$$\sum_{j=1}^{N} \left(\frac{d}{dt} x_j^t(t) P_j x_j(t) + x_j^t(t) [Q_j + K_j^t R_j K_j] x_j(t) \right) < 0, \tag{2.161}$$

for all nonzero $x_j \in \Re^{n_j}$.

The objective now is to design a decentralized guaranteed cost controller

$$u_j(t) = K_j x_j(t), \quad j = 1, \dots, N,$$

for the uncertain large-scale interconnected delay system (2.157).

2.5.3 Decentralized Design Results

We now present the LMI design approach to the construction of a guaranteed cost controller.

Theorem 2.8 *Under assumptions* (A1) *and* (A2), *suppose there exist scalar parameters* $\mu_j > 0, \varepsilon_j > 0$ *and matrices* $0 < X_j = X_j^t \in \Re^{n_j \times n_j}, 0 < \bar{S}_j = \bar{S}_j^t \in \Re^{n_j \times n_j}, 0 < X_j = X_j^t \in \Re^{n_j \times n_j}, Y_j = \in \Re^{m_j \times n_j}$, *such that for all* $j = 1, \dots, N$ *the* *following LMI*

$$
\left[\begin{array}{ccccccc}
\Phi_j & A_{dj}\bar{S}_j & B_{dj}Y_j & (E_{1j}X_j + E_{2j}Y_j)^t & G_{j1} & 0 & \dots \\
\bullet & -\bar{S}_j & 0 & \bar{S}_j E^t_{1dj} & 0 & 0 & \dots \\
\bullet & \bullet & -Z_j & Y^t_j E^t_{2dj} & 0 & 0 & \dots \\
\bullet & \bullet & \bullet & -\mu_j I_{qj} & 0 & 0 & \dots \\
\bullet & \bullet & \bullet & \bullet & -I_{\ell_1} & E^t_{1j} & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_j I_{s_{j\ell}} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
G_{jN} & 0 & X_j & Y^t_j & X_j & X_j \\
0 & 0 & 0 & 0 & 0 & 0 \\
Y^t_j B^t_{dj} & 0 & -Z_j & Y^t_j E^t_{2dj} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-I_{\ell_N} & E^t_{jN} & 0 & 0 & 0 & 0 \\
E_{jN} & -\varepsilon_j I_{s_{jN}} & 0 & 0 & 0 & 0 \\
\bullet & \bullet & -Q^{-1}_j & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -R^{-1}_j & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\bar{S}_j & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -U^{-1}_j
\end{array}\right] < 0, \qquad (2.162)
$$

has a feasible solution, *where*

$$
\Phi_j := A_j X_j + B_j Y_j + (A_j X_j + B_j Y_j)^t + Z_j + \mu_j D_j D^t_j + H_j,
$$

$$
H_j := \sum_{j=1,\, j\neq k}^{N} D_{jk} D^t_{jk}.
$$

Moreover, *the decentralized linear state feedback control laws*

$$
u_j(t) = K_j x_j(t) = Y_j X^{-1}_j x_j(t), \quad j = 1, \dots, N \qquad (2.163)
$$

are the guaranteed cost controllers and

$$J < \sum_{i=1}^{N} \Bigg[\phi_j^t(0) X_j^{-1} \phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s) \bar{S}_j^{-1}(s) ds + \int_{-h_j}^{0} \phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s) ds \Bigg] \tag{2.164}$$

is the associated guaranteed cost.

Proof Introducing the matrices $X_j := P_j^{-1}, Y_j := K_j P_j^{-1}, \bar{S}_j := S_j^{-1}$ and $Z_j := P_j^{-1} T_j P_j^{-1}$. Pre-and post-multiplying both sides of the inequality (2.162) by

$$\mathrm{blockdiag}[P_j \; S_j \; P_j \; I_{qj} \; I_{l1} \; I_{si1} \; \ldots \; I_{l_N} \; I_{siN} \; I_{n_j} \; I_{m_j} \; I_{n_j} \; I_{n_j}]$$

yields

$$\begin{bmatrix} \Psi_j & P_j A_{dj} & P_j B_{dj} K_j & \bar{E}_j^t & P_j G_{j1} & 0 & P_j G_{jN} & 0 & I_{n_j} & K_j^t & I_{n_j} & I_{n_j} \\ \bullet & -S_j & 0 & E_{1dj}^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -T_j & K_j^t E_{2dj}^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mu_j I_{qi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I_{\ell_1} & E_{j1}^t & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_j I_{s_{i1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & -I_{\ell_N} & E_{jN}^t & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & E_{jN} & -\varepsilon_j I_{s_{jN}} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & -Q_j^{-1} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & 0 & -R_j^{-1} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & 0 & 0 & -S_j^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & 0 & 0 & 0 & 0 & 0 & -U_j^{-1} \end{bmatrix} < 0, \tag{2.165}$$

where

$$\Psi_j := \bar{A}_j^t P_j + P_j \bar{A}_j + T_j + \mu_j P_j D_j D_j^t P_j + P_j H_j P_j,$$
$$\bar{A}_j := A_j + B_j K_j, \bar{E}_j := E_j^1 + E_{2j} K_j.$$

Using Schur complement, the matrix inequality (2.165) holds if and only if, the following inequality holds:

$$F_j := \begin{bmatrix} \Gamma_j & P_j A_{dj} + \mu_j^{-1} \bar{E}_j^t E_{1dj} & P_j B_{dj} K_j + \mu_j^{-1} \bar{E}_j^t E_{2dj} K_j & P_j G_{j1} & \ldots & P_j G_{jN} \\ \bullet & \mu_j^{-1} E_{1dj}^t E_{1dj} - S_j & \mu_j^{-1} E_{1dj}^t E^{2dj} K_j & 0 & \ldots & 0 \\ \bullet & \bullet & \mu_j^{-1} K_j^t E_{2dj}^t E_{2dj} K_j - T_j & 0 & \ldots & 0 \\ \bullet & \bullet & 0 & \Theta_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & 0 & 0 & \ldots & \Theta_N \end{bmatrix} < 0, \tag{2.166}$$

where

$$\begin{aligned}\Gamma_j &:= \bar{A}_j^t P_j + P_j \bar{A}_j + U_j + \bar{R}_j + S_j + T_j + \mu_j P_j D_j D_j^t P_j + P_j H_j P_j \\ &\quad + \mu_j^{-1} \bar{E}_j^t \bar{E}_j, \\ \bar{R}_j &:= Q_j + K_j^t R_j K_j, \qquad \Theta_j := \varepsilon_j^{-1} E_{jk}^t E_{jk} - I_{\ell_j}.\end{aligned}$$

Using a standard matrix inequality [30] for all admissible uncertainties (2.145) and (2.159), the following matrix inequality holds:

$$\begin{aligned}0 &> F_j \\ &\geq \begin{bmatrix} \bar{A}_j^t P_j + P_j \bar{A}_j + U_j + \bar{R}_j + S_j + T_j & P_j A_{dj} & P_j B_{dj} K_j & P_j G_{j1} & \dots & P_j G_{jN} \\ \bullet & -S_j & 0 & 0 & \dots & 0 \\ \bullet & \bullet & -T_j & 0 & \dots & 0 \\ \bullet & \bullet & \bullet & -I_{\ell_1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \dots & -I_{\ell_N} \end{bmatrix} \\ &\quad + \begin{bmatrix} P_j D_j \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} F_j(t) \begin{bmatrix} \bar{E}_j^t \\ E_{1dj}^t \\ K_j^t E_{2dj}^t \\ 0 \\ \vdots \\ 0 \end{bmatrix}^t + \begin{bmatrix} \bar{E}_j^t \\ E_{1dt}^t \\ K_j^t E_{2dj}^t \\ 0 \\ \vdots \\ 0 \end{bmatrix} F_j^t(t) \begin{bmatrix} P_j D_j \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^t \\ &\quad + \begin{bmatrix} 0 & 0 & 0 & P_j D_{j1} & \dots & P_j D_{jN} \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & F_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & F_{jN} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & E_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_{jN} \end{bmatrix}\end{aligned}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & E_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_{jN} \end{bmatrix}^t \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & F_{j1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & F_{jN} \end{bmatrix}^t$$

$$\times \begin{bmatrix} 0 & 0 & 0 & P_j D_{j1} & \dots & P_j D_{jN} \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}^t = L_j. \tag{2.167}$$

Taking into consideration

$$\begin{aligned} \bar{A}_{dj} &= A_{dj} + D_j F_j(t) E_{1dj}, \qquad \tilde{G}_{jk} = G_{jk} + D_{jk} F_{jk}(t) E_{jk}, \\ \tilde{A}_j &= \bar{A}_j + D_j F_j(t) \bar{E}_j = \bar{A}_j + \Delta \bar{A}_j(t), \\ B_{dj} K_j &= H_{dj}, \qquad \Delta B_{dj}(t) K_j = \Delta H_{dj}(t), \\ Q_j + K_j^t R_j K_j &= \bar{R}_j = \bar{Q}_j \end{aligned}$$

we readily obtain $L_j = \Lambda_j$. Hence, the individual closed-loop systems are asymptotically stable under Theorem 2.8. The results of the cost bound (2.164) can be proved by using similar arguments for the proof of Theorem 2.7. □

Remark 2.23 Since LMI (2.162) consists of a solution set of $(\mu_j, \varepsilon_j X_j, Y_j, \bar{S}_j, Z_j)$, various efficient convex optimization algorithms can be applied. Moreover, its solutions represent the set of guaranteed cost controllers. This parameterized representation can be exploited to design the guaranteed cost controllers, which minimizes the value of the guaranteed cost for the closed-loop uncertain interconnected delay systems.

Consequently, to determine the optimal cost bound we solve the following optimization problem:

$$\begin{aligned} D_0\colon \quad & \min_{X_j} \sum_{i=1}^{N} \bar{J}_j = J^*, \\ & \bar{J}_j := \alpha_j + \operatorname{Tr} M_j + c_j^2 \|N_j N_j^t\|_2 \operatorname{Tr} Z_j, \\ & X_j \in (\mu_j, \varepsilon_j X_j, Y_j, \bar{S}_j, Z_j, \alpha_j, M_j), \end{aligned} \tag{2.168}$$

such that (2.162) is satisfied and

$$\begin{bmatrix} -\alpha_j & \phi_j^t(0) \\ \bullet & -X_j \end{bmatrix} < 0, \tag{2.169}$$

$$\begin{bmatrix} -M_j & M_j^t \\ \bullet & -\bar{S}_j \end{bmatrix} < 0, \tag{2.170}$$

$$\begin{bmatrix} -c_j I_{n_j} & I_{n_j} \\ \bullet & -X_j \end{bmatrix} < 0, \tag{2.171}$$

where $c_j > 0$ are prescribed constants and

$$M_j M_j^t := \int_{-\tau_j}^0 \phi_j(s)\phi_j^t(s)ds, \qquad N_j N_j^t := \int_{-h_j}^0 \phi_j(s)\phi_j^t(s)ds.$$

The main design result is summarized by the following theorem:

Theorem 2.9 *If the foregoing optimization problem has the solution*

$$\mu_j, \varepsilon_j, X_j, Y_j, \bar{S}_j, Z_j, , \alpha_j, M_j,$$

then the control laws of the form (2.163) *are the decentralized linear state feedback control laws, which ensure the minimization of the guaranteed cost* (2.164) *for the uncertain interconnected delay systems.*

Proof By Theorem 2.8, the control laws (2.163) constructed from the feasible solutions

$$\mu_j, \varepsilon_j, X_j, Y_j, \bar{S}_j, Z_j, \alpha_j, M_j$$

are the guaranteed cost controllers of the uncertain interconnected delay systems (2.157). Applying the Schur complement to the LMI (2.169) and using the following inequality [12]:

$$\mathrm{Tr}\, XY \leq \|X\|_2 \,\mathrm{Tr}\, Y, \quad Y = Y^t \geq 0, \quad X = X^t,$$

we have the following

1.

$$\phi_j^t(0) X_j^{-1} \phi_j(0) < \alpha_j,$$

2.

$$\begin{aligned} \int_{-\tau_j}^0 \phi_j^t(s)\bar{S}_j^{-1}\phi_j(s)ds &= \int_{\tau_j}^0 \mathrm{Tr}[\phi_j^t(s)\bar{S}_j^{-1}\phi_j(s)]ds \\ &= \mathrm{Tr}[M_j^t \bar{S}_j^{-1} M_j] < \mathrm{Tr}[M_j], \end{aligned}$$

3.

$$
\begin{aligned}
&\int_{-h_j}^{0} \phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s) ds \\
&\quad = \int_{-h_j}^{0} \mathrm{Tr}[\phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s)] ds \\
&\quad = \mathrm{Tr}[N_j^t X_j^{-1} Z_j X_j^{-1} N_j] \le \|N_j N_j^t\|_2 \|X_j^{-1}\|_2^2 \,\mathrm{Tr}\, Z_j \\
&\quad < c_j^2 \|N_j N_j^t\|_2 \,\mathrm{Tr}\, Z_j.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
J &< \sum_{j=1}^{N} \biggl[\phi_j^t(0) X_j^{-1} \phi_j(0) + \int_{-\tau_j}^{0} \phi_j^t(s) \bar{S}_j^{-1} \phi_j(s) ds \\
&\quad + \int_{-h_j}^{0} \phi_j^t(s) X_j^{-1} Z_j X_j^{-1} \phi_j(s) ds \biggr] \\
&< \sum_{i=1}^{N} (\alpha_j + \mathrm{Tr}[M_j] + c_j^2 \|N_j N_j^t\|_2 \cdot \mathrm{Tr}[Z_j]) \\
&\le \min_{X_j} \sum_{j=1}^{N} \bar{J}_j = J^*.
\end{aligned}
\tag{2.172}
$$

Thus, the minimization of $\sum_{i=1}^{N} \bar{J}_j$ implies the minimum value J^* of the guaranteed cost for the interconnected uncertain delay systems (2.157). The optimality of the solution of the optimization problem follows from the convexity of the objective function under the LMI constraints. This is the required result. □

Remark 2.24 It must be noted that the original optimization problem for the guaranteed cost (2.168) can be appropriately decomposed into the following reduced optimization problems (2.173) since each optimization problem (2.173) is independent of each other. Hence, we only have to solve the optimization problems (2.173) for each independent subsystem:

$$
\begin{aligned}
&\min_{X_j} \sum_{j=1}^{N} \bar{J}_j = \sum_{j=1}^{N} \min_{X_j} \bar{J}_j, \\
&\quad X_j \in (\mu_j, \varepsilon_j X_j, Y_j, \bar{S}_j, Z_j, \alpha_j, M_j), \quad D_j\colon \quad \min_{X_j} \bar{J}_j, \quad j = 1, \ldots, N, \\
&\quad \bar{J}_j := \alpha_j + \mathrm{Tr}[M_j] + c_j^2 \|N_j N_j^t\|_2 \cdot \mathrm{Tr}[Z_j].
\end{aligned}
\tag{2.173}
$$

Remark 2.25 The constant parameter c_j, which is included in the inequality (2.171), needs to be optimized as the LMI constraints. In this case, it is hard to obtain the optimum guaranteed cost, because the resulting problem is nonconvex optimization problem. As an alternative, the above suboptimal guaranteed cost control is solved instead of solving the non convex optimization problem. Consequently, the decentralized robust suboptimal guaranteed cost controller, which minimizes the value of the guaranteed cost for the closed-loop uncertain delay systems, can be easily solved by using the LMI. The selected constant parameter c_j needs to be as small as since the matrix X_j is constrained by the inequality (2.169).

2.6 Global Robust Stabilization

2.6.1 Introduction

The decentralized control schemes, different from the classical centralized information structures, have been considered with significant interests for the control of interconnected systems in recent years. The main objectives of decentralized control are to find some feedback laws for adapting the interactions from the other subsystems where no state information is transferred. The advantage of decentralized control design is to reduce complexity and this therefore allows the control implementation to be more feasible.

Unlike centralized control design, decentralized control cannot have access to the entire state information. Therefore, interconnections between subsystems need to be analyzed, so that their influence on the system performance can be properly addressed by the control. As far as asymptotic stability of interconnected systems is concerned, there are two main approaches for the treatment of the interconnections in the literature. The first is to assume that the interconnections satisfy the matching conditions bounded by first-order polynomials of states [3] or higher-order polynomials [38, 56]. The second is to require that the interconnections meet a triangular structure bounded by first-order polynomials of states [79] or higher-order polynomials [25]. The matching condition guarantees that Lyapunov redesign is applicable, which begins with Lyapunov functions for nominal subsystems and then attempts to use these Lyapunov functions to design decentralized feedback laws. Most of the work in the literature falls into this category. On the other hand, the triangular structure makes it possible to apply backstepping technique to design the decentralized controllers. The backstepping design idea, which was initially introduced in [28] for nonlinear adaptive control and in [8] for nonlinear robust control, was applied to construct decentralized robust controllers in [79] and used in decentralized adaptive control by [25]. In the latter, we note that decentralized adaptive control design is addressed for a class of large-scale interconnected nonlinear systems with decentralized strict feedback form and single input subsystems. In the literature, the interconnections are assumed to be bounded by higher order polynomials of the states in the first integrator of every subsystem, whose coefficients admit a lower triangular structure.

One of the important problems in decentralized control is to relax restrictions on the interconnections and uncertainties. There exist two kinds of restrictions, such as matching conditions and strict feedback conditions in the literature. Many physical systems, such as power systems in [62], do not satisfy these conditions, so the study of relaxing these restrictions is of theoretical and practical importance.

Hereafter, the main objective is to investigate the problem of decentralized robust stabilization for a class of large-scale nonlinear systems with parameter uncertainties and nonlinear interconnections. Each system of the interconnected system is assumed to be controlled by multiple inputs and to be in a nested structure, which was first introduced by [37]. The uncertain parameters and/or disturbances are allowed to be time-varying and enter the system nonlinearly. The nonlinear interconnections are bounded by higher-order polynomials in the decentralized strict feedback form. Inspired by the recent work of centralized nonlinear control [36], it is proved that the global decentralized robust asymptotic stabilization problem can be solved for the uncertain interconnected nonlinear systems by applying a recursive design procedure.

2.6.2 Problem Formulation and Assumptions

Consider a large-scale nonlinear system composed of N interconnected subsystems with m inputs. The ith subsystem is given as

$$
\begin{aligned}
\dot{x}^i &= f^i(x^i, \xi^i_{11}) + \sum_{n=1}^{m} \Phi^{in}_{n0}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_n, \theta)\xi^i_{n1},\\
\xi^i_{j1} &= \xi^i_{j2} + \Psi^i_{j1}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_j, \theta)\\
&\quad + \sum_{n=j+1}^{m} \Phi^{in}_{j1}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_n, \theta)\xi^i_{n1},\\
&\;\;\vdots\\
\xi^i_{j,r_j-1} &= \xi^i_{jr_j} + \Psi^i_{j,r_j-1}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_j, \theta),\\
&\quad + \sum_{n=j+1}^{m} \Phi^{in}_{j,r_j-1}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_n, \theta)\xi^i_{n1},\\
\xi^i_{jr_j} &= u^i_j + \Psi^i_{jr_j}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_j, \theta)\\
&\quad + \sum_{n=j+1}^{m} \Phi^{in}_{jr_j}(\bar{x}^N, \bar{\xi}^N_1, \ldots, \bar{\xi}^N_n, \theta)\xi^i_{n1},
\end{aligned}
\tag{2.174}
$$

where

$$x^j \in \Re^{n_j}, \quad \bar{x}^N = [(x^1)^t, \dots, (x^N)^t]^t, \quad \bar{\xi}^j_{jd} = [\xi^j_{j1}, \dots, \xi^j_{jd}]^t,$$
$$\bar{\xi}^N_j = [(\bar{\xi}^1_{jr_j})', \dots, (\bar{\xi}^N_{jr_j})^t]^t, \quad i = 1, \dots, N, \ j = 1, \dots, m, \ d = 1, \dots, r_j.$$

The vector $\theta \in \Re^q$ is a time-varying uncertain parameters. All functions are smooth and vanishing at the origin for any θ.

Remark 2.26 Every subsystem in (2.174) possesses a nested structure, that is, the $(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{jr_j})$-blocks are nested in the $\bar{\xi}^j_{j+1,r_{j+1}}$-block through feedback connections between these blocks. Moreover, each block has a strict feedback structure with unmatched interconnections. Such a structure can be easily seen from (5).

Our objective is to design decentralized robust controllers

$$u^j_1 = u^j_1(x^j, \bar{\xi}^j_{1r_1}), \quad \dots, \quad u^j_m = u^j_m(x^j, \bar{\xi}^j_{1r_1}, \dots, \xi^j_{mr_m}), \quad j = 1, \dots, N$$

such that the origin of the corresponding closed-loop system is globally asymptotically stable for any θ. The recursive design technique, that is, back stepping with the aid of augmentation, developed in [36], will be applied to construct decentralized robust controllers for the system (2.174).

To this end, we impose the following assumptions:

Assumption 2.4 There exist positive definite and proper smooth functions

$$V^j(x^j), \quad j = 1, \dots, N, \quad p^{jt}_0 > 0$$

such that

$$\sum_{j=1}^{N} \frac{\partial V^j}{\partial x^j} f^j(x^j, 0) \le -\sum_{j=1}^{N} \sum_{t=1}^{\rho} p^{jt}_0 \|x^j\|^{2t}. \tag{2.175}$$

Assumption 2.5 There exist a series of non-negative smooth functions

$$\Psi^{ikt}_{jd0}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd}), \quad \Psi^{iit}_{jdls}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd}),$$
$$\Psi^{iit}_{jdjs}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd}), \quad \Psi^{ikt}_{jdl1}(x^j, \bar{\xi}^j_{1r_1}, \dots, \bar{\xi}^j_{j-1,r_{j-1}}, \bar{\xi}^j_{jd})$$

such that

$$\begin{aligned}
&\|\Psi^j_{jd}(\bar{x}^N, \bar{\xi}^N_1, \dots, \bar{\xi}^N_j, \theta)\| \\
&\quad \le \sum_{k=1}^{N} \sum_{t=1}^{\rho} \Psi^{ikt}_{jd0} \|x^k\|^t + \sum_{l=1}^{j-1} \sum_{s=2}^{r_1} \sum_{t=1}^{\rho} \Psi^{iit}_{jdls} |\xi^j_{ls}|^t \\
&\qquad + \sum_{s=2}^{d} \sum_{t=1}^{\rho} \Psi^{iit}_{jdjs} |\xi^j_{js}|^t + \sum_{k=1}^{N} \sum_{l=1}^{j} \sum_{t=1}^{\rho} \Psi^{ikt}_{jdl1} |\xi^k_{l1}|^t
\end{aligned} \tag{2.176}$$

for $j = 1, \dots, N$, $k = 1, \dots, m$ and $d = 1, \dots, r_j$.

Assumption 2.6 There exist a series of non-negative smooth functions

$$\Phi_{jd0}^{inkt}(x^j, \bar{\xi}_{1r_1}^j, \ldots, \bar{\xi}_{n-1,r_{n-1}}^j), \quad \Phi_{jdls}^{init}(x^j, \bar{\xi}_{1r_1}^j, \ldots, \bar{\xi}_{n-1,r_{n-1}}^j),$$
$$\Phi_{jdl1}^{inkt}(x^j, \bar{\xi}_{1r_1}^j, \ldots, \bar{\xi}_{n-1,r_{n-1}}^j),$$

such that

$$\begin{aligned}
&\|\Phi_{jd}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\| \\
&\quad \le \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Phi_{jd0}^{inkt}\|x^k\|^t + \sum_{l=1}^{n-1}\sum_{s=2}^{r_1}\sum_{t=1}^{\rho} \Phi_{jdls}^{init}|\xi_{ls}^j|^t \\
&\qquad + \sum_{k=1}^{N}\sum_{l=1}^{n}\sum_{t=1}^{\rho} \Phi_{jdl1}^{inkt}|\xi_{l1}^k|^t
\end{aligned} \tag{2.177}$$

for $j = 1, \ldots, N$, $k = 1, \ldots, m$, $n = j+1, \ldots, m$ and $d = 0, \ldots, r_j$.

Remark 2.27 It must be noted that Assumptions 2.5 and 2.6 imply that the interconnections are bounded by polynomial-type nonlinearities with the decentralized strict feedback form. In particular, the interconnections in the ith subsystem are bounded by polynomial-type nonlinearities which are composed of two parts: higher-order polynomials of its own states, i.e. the second and the third terms on the right-hand side of (2.176) and the second terms on the right-hand side of (2.177); higher-order polynomials of the states from other subsystems, that is. the first terms on the right-hand side of (2.176) and (2.177) which are comprised of all the zero-dynamic considered in [25], the last terms in (2.176) and (2.177) which are comprised of the first states of each subsystem.

Remark 2.28 Note also that the restrictions on the interconnections imposed in Assumptions 2.5 and 2.6 are very general which include many types of interconnections considered in the existing literature as special cases, for example, the interconnections bounded by first-order polynomials [3], higher-order polynomials [25, 38]. Compared with the work in [3, 56], no matching conditions are imposed in Assumptions 2.5 and 2.6. Furthermore, the kth subsystem's state variables x^k are allowed to appear in the higher-order polynomials in Assumptions 2.5 and 2.6.

Remark 2.29 In the literature, the decentralized robust stabilization problem has been addressed for a class of large-scale nonlinear systems of the form (2.178). In what follows, we consider the same problem for a wider class of large-scale systems with more than one input and less restrictions on interconnections.

2.6.3 Robust Control Design

We now look for designing decentralized robust controllers for the large-scale system (2.174). The design will be carried out step by step.

1. Consider system (2.174) with $m = 1$, that is

$$\begin{aligned}
\dot{x}^j &= f^j(x^j, \xi_{11}^j) + \Psi_{10}^{i1}(\bar{x}^N, \bar{\xi}_1^N, \theta)\xi_{11}^j, \\
\xi_{11}^j &= \xi_{12}^j + \Psi_{11}^j(\bar{x}^N, \bar{\xi}_1^N, \theta), \\
&\vdots \\
\xi_{1,r_1-1}^j &= \xi_{1r_1}^j + \Psi_{1,r_1-1}^j(\bar{x}^N, \bar{\xi}_1^N, \theta), \\
\xi_{1r_1}^j &= u_1^j + \Psi_{1r_1}^j(\bar{x}^N, \bar{\xi}_1^N, \theta),
\end{aligned} \tag{2.178}$$

where Φ_{10}^{i1} and Ψ_{1d}^j satisfy the following conditions:

$$\begin{aligned}
\|\Phi_{10}^{i1}(\bar{x}^N, \bar{\xi}_1^N, \theta)\| \le{}& \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Phi_{100}^{i1kt}(x^j, \xi_{11}^j)\|x^k\|^t \\
&+ \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Phi_{1011}^{i1kt}(x^j, \xi_{11}^j)|\xi_{11}^k|^t, \qquad (2.179) \\
\|\Psi_{1d}^j(\bar{x}^N, \bar{\xi}_1^N, \theta)\| \le{}& \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Psi_{1d0}^{ikt}(x^j, \bar{\xi}_{1d}^j)\|x^k\|^t \\
&+ \sum_{s=2}^{d}\sum_{t=1}^{\rho} \Psi_{1d1s}^{iit}(x^j, \bar{\xi}_{1d}^j)|\xi_{1s}^j|^t \\
&+ \sum_{k=1}^{N}\sum_{t=1}^{\rho} \Psi_{1d11}^{ikt}(x^j, \bar{\xi}_{1d}^j)|\xi_{11}^k|^t, \qquad (2.180)
\end{aligned}$$

which follows from Assumptions 2.5 and 2.6. It is readily seen that system (2.178) is quite general. Furthermore, conditions (2.179) and (2.180) are less restrictive due to the presence of the higher polynomial terms $|\xi_{1s}^j|^t$ in (2.180) and the interconnection terms $\|x^k\|^t$ in (2.179) and (2.180). With Assumption 2.4, (2.179) and (2.180), an appropriate design procedure can be applied to system (2.178), the result can be summarized by the following lemma:

Lemma 2.7 *Consider system* (2.178) *with Assumption* 2.4 *and* (2.179) *and* (2.180). *There exist a change of coordinates* $z_{1d}^j = \xi_{1d}^j - \alpha_{1,d-1}^j(x^j, \bar{\xi}_{1,d-1}^j)$ *with* $\alpha_{10}^j = 0$

and decentralized feedback laws $u_1^j = u_1^j(x^j, \bar{\xi}_{1r_1}^j)$ *such that the Lyapunov function*

$$W_1 = \sum_{i=1}^{N} V^j + \sum_{i=1}^{N} \sum_{d=1}^{r_1} \frac{1}{2} (z_{1d}^j)^2 \tag{2.181}$$

satisfies

$$\begin{aligned} \dot{W}_1 &\le -\sum_{i=1}^{N} \sum_{t=1}^{\rho} p_1^{it} \|x^j\|^{2t} - \sum_{i=1}^{N} \sum_{d=1}^{r_1} \sum_{t=1}^{\rho} c_{1d1}^{it} (z_{1d}^j)^{2t}, \\ p_1^{it} &> 0, \quad c_{1d2}^{it} > 0 \end{aligned} \tag{2.182}$$

along the solutions to system (2.178) *with* $u_1^j = u_1^j(x^j, \bar{\xi}_{1r_1}^j)$.

Remark 2.30 Note that Lemma 2.7 is an extension of the results given in the literature. The proof presented there can be modified to verify Lemma 2.7. However, a major modification should be made, that is, the terms like $|\xi_{1s}^j|$ should be expressed in terms of x^j and z_{1d}^j for $d = 1, \ldots, s$. Observe that $\alpha_{1,s-1}^j$ can be put into the form

$$\alpha_{1,s-1}^j = \bar{\alpha}_{1,s-1,0}^j(x^j) x^j + \sum_{d=1}^{s-1} \bar{\alpha}_{1,s-1,d}^j (x^j, \xi_{11}^j, \ldots, \xi_{1d}) z_{1d}^j$$

due to the smoothness of $\alpha_{1,s-1}^j$ and $\alpha_{1,s-1}^j(0) = 0$. It follows from

$$\xi_{1s}^j = z_{1s}^j + \alpha_{1,s-1}^j (x^j, \xi_{11}^j, \ldots, \xi_{1,s-1}^j)$$

that

$$\xi_{1s}^j = z_{1s}^j + \bar{\alpha}_{1,s-1,0}^j(x^j) x^j + \sum_{d=1}^{s-1} \bar{\alpha}_{1,s-1,d}^j (x^j, \xi_{11}^j, \ldots, \xi_{1d}) z_{1d}^j$$

which implies, according to Lemma 2.8 in Sect. 2.6.5, that

$$\begin{aligned} |\xi_{1s}^j|^t &\le (s+1)^{t-1} [|z_{1s}^j|^t + \|\bar{\alpha}_{1,s-1,0}^j(x^j)\|^t \|x^j\|^t] \\ &\quad + (s+1)^{t-1} \sum_{d=1}^{s-1} |\bar{\alpha}_{1,s-1,d}^j (x^j, \xi_{11}^j, \ldots, \xi_{1d})|^t |z_{1d}^j|^t. \end{aligned}$$

Step *T*: Consider system (2.174) with $m = T$, $T \ge 2$, that is,

$$\begin{aligned} \dot{x}^j &= f^j(x^j, \xi_{11}^j) + \sum_{n=1}^{t} \Phi_{n0}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta) \xi_{n1}^j, \\ \xi_{j1}^j &= \xi_{j2}^j + \Psi_{j1}^j(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_j^N, \theta) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{n=j+1}^{t} \Phi_{j1}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\xi_{n1}^j, \\
&\vdots \\
\xi_{j,r_j-1}^j &= \xi_{jr_j}^j + \Psi_{j,r_j-1}^j(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_j^N, \theta) \\
&\quad + \sum_{n=j+1}^{t} \Phi_{j,r_j-1}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\xi_{n1}^j, \\
\xi_{jr_j}^j &= u_j^j + \Psi_{jr_j}^j(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_j^N, \theta) \\
&\quad + \sum_{n=j+1}^{t} \Phi_{jr_j}^{in}(\bar{x}^N, \bar{\xi}_1^N, \ldots, \bar{\xi}_n^N, \theta)\xi_{n1}^j,
\end{aligned}
\tag{2.183}
$$

where $u_1^j = u_1^j(x^j, \bar{\xi}_{1r_1}^j),\ u_2^j = u_2^j(X_1^j, \bar{\xi}_{2r_2}^j),\ \ldots,\ u_{T-1}^j = u_{T-1}^j(X_{T-2}^j, \bar{\xi}_{T-1,r_{T-1}}^j)$ are determined in the first $T-1$ steps with

$$
\begin{aligned}
X_1^j &= [(x^j)', (\bar{\xi}_{1r_1}^j)']', \\
X_1^N &= [(X_1^1)', \ldots, (X_1^N)']', \\
F_1^j &= [(f_0^j + \Phi_{10}^j \xi_{11}^j)', \xi_{12}^j + \Psi_{11}^j, \ldots, \xi_{1r_1}^j \\
&\quad + \Psi_{1,r_1-1}^j, u_1^j(X_1^j) + \Psi_{1r_1}^j]', \\
\bar{\Phi}_1^{i2} &= [(\Phi_{20}^{i2})', \Phi_{11}^{i2}, \ldots, \Phi_{1,r_1-1}^{i2}, \Phi_{1r_1}^{i2}]', \\
&\vdots \\
X_{T-2}^j &= [(X_{T-3}^j)', \xi_{T-2,1}^j, \ldots, \xi_{T-2,r_{T-2}}^j]', \\
X_{T-2}^N &= [(X_{T-2}^1)', \ldots, (X_{T-2}^N)']', \\
F_{T-2}^j &= [(F_{T-3}^j + \bar{\Phi}_{T-3}^{i,T-2}\xi_{T-2,1}^j)', \xi_{T-2,2}^j + \Psi_{T-2,1}^j, \ldots, \\
&\quad \xi_{T-2,r_{T-2}}^j + \Psi_{T-2,r_{T-2}-1}^j, u_{T-1}^j(X_{T-2}^j) + \Psi_{T-2,r_{T-2}}^j]', \\
\bar{\Phi}_{T-2}^{i,T-1} &= [(\Phi_{T-1,0}^{i,T-1})', \Phi_{11}^{i,T-1}, \ldots, \Phi_{1r_1}^{i,T-1}, \ldots, \\
&\quad \Phi_{T-1,1}^{i,T-1}, \ldots, \Phi_{T-2,r_{T-2}-1}^{i,T-1}, \Phi_{T-2,r_{T-2}}^{i,T-1}]'.
\end{aligned}
$$

Such a system can be alternatively put into the following form:

$$
\begin{aligned}
\dot{X}_{T-1}^j &= F_{T-1}^j(\bar{X}_{T-1}^N, \theta) + \bar{\Phi}_{T-1}^{iT}(\bar{X}_{T-1}^N, \bar{\xi}_T^N, \theta)\xi_{T1}^j, \\
\xi_{T1}^j &= \xi_{T2}^j + \Psi_{T1}^j(\bar{X}_{T-1}^N, \bar{\xi}_T^N, \theta),
\end{aligned}
$$

$$\begin{aligned}
&\vdots \\
\xi^j_{T2} &= \xi^j_{T3} + \Psi^j_{T2}(\bar{X}^N_{T-1}, \bar{\xi}^N_T, \theta), \\
\xi^j_{Tr_T} &= u^j_T + \Psi^j_{Tr_T}(\bar{X}^N_{T-1}, \bar{\xi}^N_T, \theta),
\end{aligned} \tag{2.184}$$

where

$$\begin{aligned}
X^j_{T-1} &= [(X^j_{T-2})', \xi^j_{T-1,1}, \ldots, \xi^j_{T-1,r_{T-1}}]', \\
X^N_{T-1} &= [(X^1_{T-1})', \ldots, (X^N_{T-1})']', \\
F^j_{T-1} &= [(F^j_{T-2} + \bar{\Phi}^{i,T-1}_{T-2}\xi^j_{T-1,1})', \xi^j_{T-1,2} + \Psi^j_{T-1,1}, \ldots, \\
&\qquad \xi^j_{T-1,r_{T-1}} + \Psi^j_{T-1,r_{T-1}}, u^j_{T-1}(X^j_{T-1}) + \Psi^j_{T-1,r_{T-1}}]', \\
\bar{\Phi}^{iT}_{T-1} &= [(\Phi^{iT}_{T0})', \Phi^{iT}_{11}, \ldots, \Phi^{iT}_{1r_1}, \ldots, \Phi^{iT}_{T-1,1}, \ldots, \\
&\qquad \Phi^{iT}_{T-1,r_{T-1}-1}, \Phi^{iT}_{T-1,r_{T-1}}]'.
\end{aligned}$$

According to Step $T-1$, F^j_{T-1} satisfies the following inequality:

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{\partial W_{T-1}}{\partial X^j_{T-1}} F^j_{T-1}(X^N_{T-1}, \theta) \\
&\quad \leq -\sum_{i=1}^{N}\sum_{t=1}^{\rho} p^{it}_{T-1}\|x^j\|^{2t} - \sum_{i=1}^{N}\sum_{j=1}^{T-1}\sum_{d=1}^{r_j}\sum_{t=1}^{\rho} c^{it}_{jd,T-1}(z^j_{jd})^2 t\cdot
\end{aligned} \tag{2.185}$$

It follows from Assumptions 2.5 and 2.6 that Φ^{iT}_{T-1} and Ψ^j_{Td} satisfy the following inequalities:

$$\begin{aligned}
\|\Phi^{iT}_{T-1}(\bar{X}^N_{T-1}, \bar{\xi}^N_T, \theta)\| &\leq \|\Phi^{iT}_{T0}\| + \sum_{j=1}^{T-1}\sum_{d=1}^{r_j}\|\Phi^{iT}_{jd}\| \\
&\leq \sum_{k=1}^{N}\sum_{t=1}^{\rho} \bar{\Phi}^{iTkt}_{T-1,0}(X^j_{T-1}, \xi^j_{T1})\|x^k\|^t \\
&\quad + \sum_{k=1}^{N}\sum_{l=1}^{T-1}\sum_{s=2}^{r_j}\sum_{t=1}^{\rho} \bar{\Phi}^{iTit}_{T-1,ls}(X^j_{T-1}, \xi^j_{T1})|\xi^j_{ls}|^t \\
&\quad + \sum_{k=1}^{N}\sum_{l=1}^{T-1}\sum_{t=1}^{\rho} \bar{\Phi}^{iTkt}_{T-1,l1}(X^j_{T-1}, \xi^j_{T1})|\xi^j_{l1}|^t,
\end{aligned} \tag{2.186}$$

$$\begin{aligned}
\|\Psi^j_{Td}(\bar{X}^N_{k-1},\bar{\xi}^N_T,\theta)\| \le & \sum_{k=1}^{N}\sum_{t=1}^{\rho}\Psi^{ikt}_{Td0}(X^j_{T-1},\bar{\xi}^j_{Td})\|x^k\|^t \\
& + \sum_{l=1}^{T-1}\sum_{s=2}^{r_1}\sum_{t=1}^{\rho}\Psi^{iit}_{Tdls}(X^j_{T-1},\bar{\xi}^j_{Td})|\xi^j_{ls}|^t \\
& + \sum_{s=2}^{d}\sum_{t=1}^{\rho}\Psi^{iit}_{TdTs}(X^j_{T-1},\bar{\xi}^j_{Td})|\xi^j_{Ts}|^t \\
& + \sum_{k=1}^{N}\sum_{l=1}^{T-1}\sum_{t=1}^{\rho}\Psi^{ikt}_{Tdl1}(X^j_{T-1},\bar{\xi}^j_{Td})|\xi^k_{l1}|^t. \qquad (2.187)
\end{aligned}$$

With (2.185)–(2.187), it follows from Lemma 2.7 that there exits a change of coordinates $z^j_{Td}=\xi^j_{Td}-\alpha^j_{T,d-1}(X^j_{T-1},\bar{\xi}^j_{T,d-1})$ with $\alpha^j_{T0}=0$ and decentralized feedback laws $u^j_T=u^j_k(X^j_{T-1},\bar{\xi}^j_{Tr_T})$ so that the Lyapunov function

$$W_T = W_{T-1}+\sum_{i=1}^{N}\sum_{d=1}^{r_T}\frac{1}{2}(z^j_{Td})^{2t} \qquad (2.188)$$

satisfies

$$\dot{W}_T \le -\sum_{i=1}^{N}\sum_{t=1}^{\rho}p^{it}_T\|x^j\| - \sum_{i=1}^{N}\sum_{j=1}^{t}\sum_{d=1}^{r_j}\sum_{t=1}^{\rho}c^{it}_{jdT}(z^j_{jd})^{2t} \qquad (2.189)$$

along the solution of (2.183) with

$$u^j_1=u^j_1(x^j,\bar{\xi}^j_{1r_1}),\quad u^j_2=u^j_2(X^j_1,\bar{\xi}^j_{2r_2}),\quad \ldots,\quad u^j_T=u^j_2(X^j_{T-1},\bar{\xi}^j_{Tr_T}).$$

From the foregoing analysis, we have the following result for system (2.174):

Theorem 2.10 *Suppose that Assumptions 2.4–2.6 are satisfied. Then, system (2.174) can be globally asymptotically stabilized by decentralized robust control laws* $u^j_1=u_1(x^j,\bar{\xi}^j_{1r_1}),\ldots,u^j_m=u_m(x^j,\bar{\xi}^j_{1r_1},\ldots,\bar{\xi}^j_{mr_m})$.

2.6.4 Simulation Example 2.7

To illustrate the theoretical developments, we consider the large-scale nonlinear system

$$\begin{aligned}
\dot{x}^j &= -x^1-(x^1)^3+\xi^1_{11}(x^1)^2\theta\sin t \\
&\quad + \frac{1}{\Delta}\xi^1_{21}[(\xi^2_{11})^2+(\xi^2_{21})^2],
\end{aligned}$$

$$
\begin{aligned}
\xi_{11}^1 &= \xi_{12}^1 + \xi_{21}^1(\xi_{11}^2)^2\theta\cos t,\\
\xi_{12}^1 &= u_1^1,\\
\xi_{21}^1 &= \xi_{22}^1 + \frac{1}{\Delta}[x^1\sin t + (\xi_{21}^2)^2\theta\cos t],\\
\xi_{22}^1 &= u_2^1,\\
\xi_{11}^2 &= \xi_{12}^2 + \xi_{21}^2\xi_{11}^1\xi_{11}^2\theta\cos t,\\
\xi_{12}^2 &= u_1^2,\\
\xi_{21}^2 &= \xi_{22}^2 + \frac{1}{\Delta}[x^1\sin t + (\xi_{21}^1)^2\theta\cos t],\\
\xi_{22}^2 &= u_2^2,
\end{aligned}
\tag{2.190}
$$

where $\Delta = 1 + (x^1)^2 + \sum_{i=1}^2\sum_{j=1}^2(\xi_{1j}^j)^2 + \sum_{i=1}^2\sum_{j=1}^2(\xi_{2j}^j)^2$ and $|\theta| < 1$.

For this purpose we choose $V = \frac{1}{2}p(x^1)^2$, $p > 0$. Then, a simple calculation shows that

$$\frac{\partial V}{\partial x^1}[-x^1 - (x^1)^3] \le -p(x^1)^2 - \frac{p}{2}(x^1)^4$$

which implies that Assumption 2.4 is satisfied. In addition, it is not difficult to prove that Assumptions 2.5 and 2.6 are satisfied as well. Therefore, the design procedure developed in Sect. 2.6.3 is applicable. Note that the approach in [81] cannot be used to solve the problem for (2.190) because there exist interconnected terms, that is, the last terms in the first equation, the second equation, the fourth equation, the sixth equation, and the eighth equation of system (2.190).

First, consider the following system:

$$
\begin{aligned}
\dot{x}^1 &= -x^1 - (x^1)^3 + \xi_{11}^1(x^1)^2\theta\sin t,\\
\xi_{11}^1 &= \xi_{12}^1,\\
\xi_{12}^1 &= u_1^1,\\
\xi_{11}^2 &= \xi_{12}^2,\\
\xi_{12}^2 &= u_1^2.
\end{aligned}
\tag{2.191}
$$

It follows from Step 1 in Sect. 2.6.3 that the following controllers can be constructed:

$$
\begin{aligned}
u_1^1 &= -c_{121}^{11}z_{12}^1 - \xi_{11}^1 - \frac{\partial\alpha_{11}^1}{\partial x^1}[x^1 + (x^1)^3]\\
&\quad - \frac{1}{2}z_{12}^1\left(\frac{\partial\alpha_{11}^1}{\partial x^1}z_{11}^1\right)^2 + \frac{\partial\alpha_{11}^1}{\partial z_{11}^1}(z_{12}^1 + \alpha_{11}^1),\\
u_1^2 &= -c_{121}^{21}z_{12}^2 - z_{11}^2 + \frac{\partial\alpha_{11}^2}{\partial z_{11}^2}(z_{12}^2 + \alpha_{11}^2),
\end{aligned}
$$

so that the Lyapunov function

$$W_1 = V + \sum_{i=1}^{2}\sum_{d=1}^{2}(z_{1d}^{j})^2$$

satisfies

$$\begin{aligned}\dot{W}_1 \le {} & -p(x^1)^2\left(\frac{p}{2}-\frac{1}{2}\right)(x^1)^4 - c_{111}^{11}(z_{11}^1)^2\\ & - c_{111}^{21}(z_{11}^2)^2 - c_{111}^{22}(z_{11}^2)^4 - c_{121}^{11}(z_{12}^1)^2 - c_{121}^{21}(z_{12}^2)^2,\end{aligned}$$

where $z_{11}^1 = \xi_{11}^1$, $z_{11}^2 = \xi_{11}^2$, $z_{12}^1 = \xi_{12}^1 - \alpha_{11}^1$, $z_{12}^2 = \xi_{12}^2 - \alpha_{11}^2$, $\alpha_{11}^1 = -c_{111}^{11}\xi_{11}^1 - \frac{1}{2}p\xi_{11}^1(x^1)^2$ and $\alpha_{11}^2 = -c_{111}^{21}\xi_{11}^2 - c_{111}^{22}(\xi_{11}^2)^3$.

Second, consider (2.190) and carry out Step 2 in Sect. 2.6.3. We obtain the following controllers:

$$\begin{aligned}u_2^1 &= -z_{22}^1 - z_{21}^1 - \psi_{22}^1 - \delta_{22}^1 z_{22}^1,\\ u_2^2 &= -z_{22}^2 - z_{21}^2 - \psi_{22}^2 - \delta_{22}^2 z_{22}^2,\end{aligned}$$

where $z_{21}^1 = \xi_{21}^1$, $z_{21}^2 = \xi_{21}^2$, $z_{22}^1 = \xi_{22}^1 - \alpha_{21}^1$, $z_{22}^2 = \xi_{22}^2 - \alpha_{21}^2$, and

$$\begin{aligned}\alpha_{21}^1 &= -2z_{21}^1 - (z_{21}^1)^3 - z_{21}^1\left[\left(px^1 - z_{12}^1\frac{\partial\alpha_{11}^1}{\partial x^1}\right)^2 + \frac{1}{2}\left(z_{11}^1 - z_{12}^1\frac{\partial\alpha_{11}^1}{\partial z_{11}^1}\right)^2\right],\\ \alpha_{21}^2 &= -2z_{21}^2 - 2(z_{21}^2)^3 - z_{21}^2\left[\frac{1}{2}\left(z_{11}^2 - \frac{\partial\alpha_{11}^2}{\partial z_{11}^2}\right)^2(z_{11}^2)^2 + \frac{1}{2}(z_{21}^2)^2\right],\\ \psi_{22}^1 &= \frac{\partial\alpha_{21}^1}{\partial x^1}[x^1 + (x^1)^3] - \frac{\partial\alpha_{21}^1}{\partial z_{11}^1}(z_{12}^1 + \alpha_{11}^1)\\ &\quad - \frac{\partial\alpha_{21}^1}{\partial z_{12}^1}u_1^1 - \frac{\partial\alpha_{21}^1}{\partial z_{21}^1}(z_{22}^1 + \alpha_{21}^1),\\ \delta_{22}^1 &= \frac{1}{2}\left(\frac{\partial\alpha_{21}^1}{\partial x^1}z_{11}^1\right)^2 + \left(\frac{\partial\alpha_{21}^1}{\partial x^1}z_{21}^1\right)^2 + \frac{1}{2}\left(\frac{\partial\alpha_{21}^1}{\partial z_{11}^1}z_{21}^1\right)^2 + \left(\frac{\partial\alpha_{21}^1}{\partial z_{21}^1}\right)^2,\\ \psi_{22}^2 &= \frac{\partial\alpha_{21}^2}{\partial z_{11}^2}(z_{12}^2 + \alpha_{11}^2) - \frac{\partial\alpha_{21}^2}{\partial z_{12}^2}u_1^2 - \frac{\partial\alpha_{21}^2}{\partial z_{21}^2}(z_{22}^2 + \alpha_{21}^2),\\ \delta_{22}^2 &= \frac{1}{2}\left(\frac{\partial\alpha_{21}^2}{\partial z_{21}^2}z_{21}^2 z_{11}^2\right)^2 + \frac{1}{2}\left(\frac{\partial\alpha_{12}^2}{\partial\xi_{21}^2}\right)^2 + \left(\frac{\partial\alpha_{21}^2}{\partial z_{21}^2}\right)^2.\end{aligned}$$

The derived controllers stabilize the system (2.190) because they render the Lyapunov function

$$W_2 = W_1 + \sum_{i=1}^{2}\sum_{d=1}^{2}(z_{2d}^{j})^2$$

satisfy

$$\dot{W}_2 \leq -(p-2)(x^1)^2 - \left(\frac{p}{2}-1\right)(x^1)^4 - (c_{111}^{11}-1)(z_{11}^1)^2 - c_{111}^{21}(z_{11}^2)^2$$
$$- (c_{111}^{22}-2)(z_{11}^2)^4 - c_{121}^{11}(z_{12}^1)^2 - c_{121}^{21}(z_{12}^2)^2 - \sum_{i=1}^{2}\sum_{j=1}^{2} c_{2j2}^{i1}(z_{2j}^{j})^2.$$

For the purpose of demonstration, simulation is carried out for the initial conditions $x^1 = 0.9$, $\xi_{11}^1 = -0.9$, $\xi_{11}^2 = 0.5$, $\xi_{12}^1 = 0.5$, $\xi_{12}^2 = -0.7$, $\xi_{21}^1 = 0.7$, $\xi_{21}^2 = 0.8$, $\xi_{22}^1 = -0.8$, $\xi_{22}^2 = 0.9$ and the parameters $p = 3$, $c_{111}^{11} = 2$, $c_{111}^{21} = 1$, $c_{111}^{22} = 2$, $c_{121}^{11} = 1$, $c_{121}^{21} = 1$, $c_{2j2}^{i1} = 1$ for $i, j = 1, 2$. The responses for the closed-loop system are plotted in Fig. 2.3.

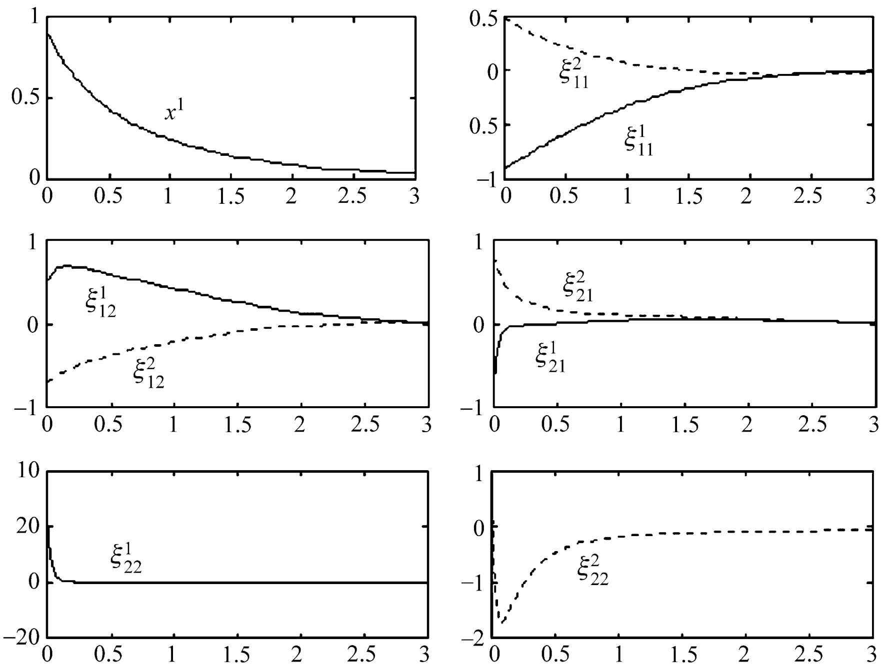

Fig. 2.3 Trajectories of the closed-loop system

2.6.5 *Proof of Lemma 2.8*

Lemma 2.8

$$(a_1 + \cdots + a_n)^t \le n^{t-1}(|a_1|^t + \cdots + |a_n|^t).$$

Proof It is obvious that

$$(a_1 + \cdots + a_n)^t \le (|a_1| + \cdots + |a_n|)^t.$$

Set $\bar{a} = (|a_1| + \cdots + |a_n|)/n$ and $f(x) = x^t$ for $t \ge 1$ and $x \ge 0$. Because $f(x)$ is c^∞ function, by Taylor expansion, there exists a real value ξ between x and $\bar{a}$, satisfying

$$f(x) = f(\bar{a}) + \dot{f}(\bar{a})(x - \bar{a}) + \frac{1}{2}\ddot{f}(\xi)(x - \bar{a})^2$$

which implies that

$$f(x) \ge f(\bar{a}) + \dot{f}(\bar{a})(x - \bar{a})$$

because $\ddot{f}(\xi)(x - \bar{a})^2 \ge 0$. Therefore

$$\begin{aligned} f(|a_n|) &\ge f(\bar{a}) + \dot{f}(\bar{a})(|a_1| - \bar{a}), \\ &\vdots \\ f(|a_n|) &\ge f(\bar{a}) + \dot{f}(\bar{a})(|a_n| - \bar{a}). \end{aligned}$$

Adding all these equations together gives

$$|a_1|^t + \cdots + |a_n|^t \ge nf(\bar{a}) = n(\bar{a})^t = \frac{(|a_1| + \cdots + |a_n|)^t}{n^{t-1}}$$

which implies that

$$\begin{aligned} (a_1 + \cdots + a_n)^t &\le (|a_1|^t + \cdots + |a_n|)^t \\ &\le n^{t-1}(|a_1| + \cdots + |a_n|^t). \end{aligned}$$

□

2.7 Notes and References

This chapter provided a critical overview of decentralized control techniques for classes of nonlinear interconnected continuous-time systems. The area of nonlinear control is so wide to accommodate new and research directions along the productive ideas [9, 19, 22, 23, 29, 40, 42, 43, 45, 46]. In particular, the topic of nonlinear interconnected discrete-time systems has not been fully investigated in the literature.

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