Solutions to Problems of Unified Signal Theory

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Abstract This report collects some of the solutions to the problems proposed at the end of each chapter of the book *Unified Signal Theory* by G. Cariolaro. Springer–Verlag, May 2011 [3].
The enumeration of figures and formulas are preceeded by the letter "S" in order to distinguish them from the enumeration in the book.

Chapter 2	problems 39
Chapter 3	problems 17
Chapter 4	problems 20
Chapter 5	problems 45
Chapter 6	problems 35
Chapter 7	problems 9
Chapter 8	problems 28
Chapter 9	problems 25
Chapter 10	problems 5
Chapter 11	problems 12
Chapter 12	problems 6
Chapter 13	problems 5
Chapter 14	problems 22
Chapter 15	problems 9
Chapter 16	problems 12
Chapter 17	problems 13

- global number of problems 299

The author will appreciate comments and error detection by the readers.
cariolar@dei.unipd.it

Problems of Chapter 2

2.1. $\star$ [Sect. 2.1] Assuming that a continuous–time signal $s(t)$ is the mathematical model of an electrical voltage, find the physical dimensions of the following quantities: area, mean value, (specific) energy and (specific) power.

Denoting a physical dimension by $[\cdot]$, one gets

$$[s(t)] = \mathrm{V}\ ,\ [\text{area}\,(s)] = \mathrm{V}\cdot\mathrm{s}\ ,\ [m_s] = \mathrm{V}\ ,\ [E_s] = \mathrm{V}^2\cdot\mathrm{s}\ ,\ [P_s] = \mathrm{V}^2\ .$$

2.2. $\star$ [Sect. 2.2] Show that the *area over a period* of a periodic signal defined by (2.17a) is independent of t_0.

It is sufficient to prove that

$$I = \int_{t_0}^{t_0+T_p} s(t)\ \mathrm{d}t = \int_0^{T_p} s(t)\ \mathrm{d}t\ .$$

To this end we divide the interval $[t_0, t_0+T_p)$ into the intervals $[t_0, 0)$, $[0, T_p)$ and $[T_p, t_0+T_p)$. Then we obtain the three contributions

$$\begin{aligned}
I_1 &= \int_{t_0}^{0} s(t)\ \mathrm{d}t = -\int_0^{t_0} s(t)\ \mathrm{d}t \\
I_2 &= \int_0^{T_p} s(t)\ \mathrm{d}t \\
I_3 &= \int_{T_p}^{t_0+T_p} s(t)\ \mathrm{d}t = \int_0^{t_0} s(t'+T_p)\ \mathrm{d}t' = \int_0^{t_0} s(t')\ \mathrm{d}t' = -I_1
\end{aligned}$$

where we have introduced the change of variable $t' = t - T_p$ and used the periodicity $s(t'+T_p) = s(t')$. In conclusion $I = I_2$.

2.3. $\star\star$ [Sect. 2.2] Show that the *mean value over a period* for a periodic signal, defined by (2.17b), is equal to the *mean value* defined in general by (2.10).

We prove that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\,\mathrm{d}t = \frac{1}{T_p} \int_0^{T_p} s(t)\,\mathrm{d}t\ . \tag{S2.1}$$

We use the fact that the limit as $T \to \infty$ is equal to the limit of nT_p as $n \to \infty$

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\,\mathrm{d}t &= \lim_{n\to\infty} \frac{1}{2nT_p} \int_{-nT_p}^{nT_p} s(t)\,\mathrm{d}t \\ &= \lim_{n\to\infty} \frac{1}{2nT_p} \cdot 2n \int_0^{T_p} s(t)\,\mathrm{d}t = \frac{1}{T_p} \int_0^{T_p} s(t)\,\mathrm{d}t\,.\end{aligned}$$

2.4. $\star$ [Sect. 2.3] Using the functions $1(x)$ and $\mathrm{rect}\,(x)$ write a concise expression for the signal

$$s(t) = 3 \quad \text{for } t \in (-5, 1)\,, \qquad s(t) = t \quad \text{for } t \in (2, 4)\,, \qquad s(t) = 0 \quad \text{otherwise}\ .$$

The signal, shown in Fig.S2.1, has the following expressions

$$\begin{aligned} s(t) &= 3\,[1(t+5) - 1(t-1)] + t\,[1(t-2) - 1(t-4)] \\ &= 3\ \mathrm{rect}\left(\frac{t+2}{6}\right) + t\ \mathrm{rect}\left(\frac{t-3}{2}\right)\ . \end{aligned}$$

where we have used the identity $\mathrm{rect}(t/T) = 1(t+T/2) - 1(t-T/2)$.

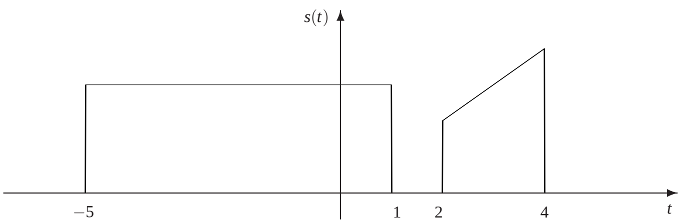

Fig. S2.1 Representation of the signal of the problem

2.5. $\star$ [Sect. 2.3] Find the extension, duration, area and energy of the signal of Problem 2.4.

It results

$$e(s) = (-5,1) \cup (2,4) \quad , \qquad D(s) = 6+2 = 8$$

$$\text{area}\,(s) = 18+6 = 24 \quad , \qquad E_s = 54 + \frac{56}{3} = \frac{218}{3} \; .$$

The smalled interval containing $e(s)$ is $(-5,4)$ and the corresponding duration is 9.

2.6. $\star$ [Sect. 2.3] Find the energy of the causal exponential with $p_0 = 2 + \mathrm{i}2\pi 5$.

More generally, we note that for the signal

$$s(t) = 1(t)\; \mathrm{e}^{pt}$$

it results

$$|s(t)|^2 = 1(t)\; \mathrm{e}^{\alpha t}\,, \qquad \alpha = 2\,\Re\, p\; .$$

If $\alpha < 0$ the signal has the finite energy $E_s = -1/\alpha$, otherwise $E_s = +\infty$.

2.7. $\star$ [Sect. 2.3] Write a mathematical expression of a triangular pulse $u(t)$ determined by the following conditions: $u(t)$ is even, has duration 2 and energy 10.

Considering in general a real amplitude $A_0 > 0$, the expression is

$$u(t) = A_0\; (1-|t|)\; \text{rect}\left(\frac{t}{2}\right) = A_0\, \text{triang}(t)$$

where $\text{triang}(t)$ is the function defined by (2.29). The corresponding energy is $E_u = A_0^2\,\frac{2}{3} = 10$. Then $A_0 = \sqrt{15}$.

2.8. $\star$ [Sect. 2.3] An even–symmetric triangular pulse $u(t)$ of duration 4 and amplitude 2 is periodically repeated according to (2.16). Draw the periodic repetition in the following cases: $T_p = 8$, $T_p = 4$ and $T_p = 2$.

Fig.S2.2 shows the pulse $u(t)$ and its periodic repetition

$$\text{rep}_{T_p}\, u(t) = \text{rep}_{T_p}\; \text{triang}\left(\frac{1}{2}\right) = \sum_{k=-\infty}^{+\infty} u(t-kT_p)$$

with $T_p = 2$, $T_p = 4$ e $T_p = 8$. Note that for $T_p \geq D(u) = 4$ the repetition terms do not overlap, while for $T_p < 4$ they overlap. In particular, for $T_p = 2$ the repetition gives a constant signal of amplitude 2.

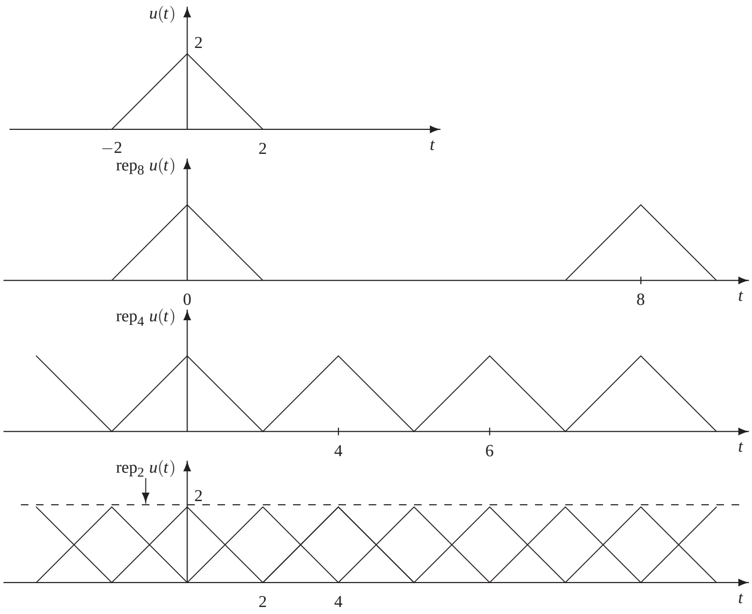

Fig. S2.2 The signals of the problem

2.9. ⋆⋆ [Sect. 2.3] Write the derivative $r'(t)$ of the rectangular pulse $r(t)$ defined by (2.26). Verify that the integral of $r'(t)$ from $-\infty$ to t recovers $r(t)$.

The impulse is

$$r(t) = A_0 \; \mathrm{rect}\left(\frac{t-t_0}{D}\right) = A_0 \; 1(t-t_1) - A_0 \; 1(t-t_2)$$

where $t_1 = t_0 - D/2$ and $t_2 = t_0 + D/2$. Considering that (see (2.34)) $\mathrm{d}1(t)/\mathrm{d}t = \delta(t)$, we obtain

$$r'(t) = A_0 \; \delta(t-t_1) - A_0 \; \delta(t-t_2) \, .$$

Next, we note that the delta function has unit area and

$$\int_{t_1-\varepsilon}^{t_1+\varepsilon} \delta(t-t_1) \, \mathrm{d}t = 1 \, , \qquad \varepsilon > 0 \, .$$

This is the key to prove that the integral of $r'(t)$ gives $r(t)$.

2.10. ⋆⋆[Sect. 2.3] Write the first and second derivatives of the rectified sinusoidal signal

$$s(t) = A_0\,|\cos\omega_0 t|\;.$$

We note that the signal $s(t)$ has period $T_0 = \pi/\omega_0$, that is half of the period of the non rectified signal (Fig.S2.3). The ordinary derivative of $s(t)$ is

$$\begin{aligned} s'(t) &= -A_0\omega_0\,\frac{|\cos\omega_0 t|}{\cos\omega_0 t}\,\sin\omega_0 t \\ &= \begin{cases} -A_0\omega_0\,\sin\omega_0 t\,, & -\dfrac{\pi}{2}+2k\pi < \omega_0 t < \dfrac{\pi}{2}+2k\pi \\ A_0\omega_0\,\sin\omega_0 t\,, & \dfrac{\pi}{2}+2k\pi < \omega_0 t < \dfrac{3}{2}\pi+2k\pi \end{cases} \end{aligned}$$

and is continuous almost everywhere, except in

$$t_k = t_0 + kT_0 \quad \text{with} \quad t_0 = \tfrac{1}{2}T_0$$

where it has an amplitude jump of $2A_0\,\omega_0$. Hence, in the second derivative we have to add impulses in t_k of area $2A_0\omega_0$, that is,

$$s''(t) = -A_0\,\omega_0^2\,|\cos\omega_0 t| + 2A_0\,\omega_0\sum_{k=-\infty}^{+\infty}\delta(t-t_k)\;.$$

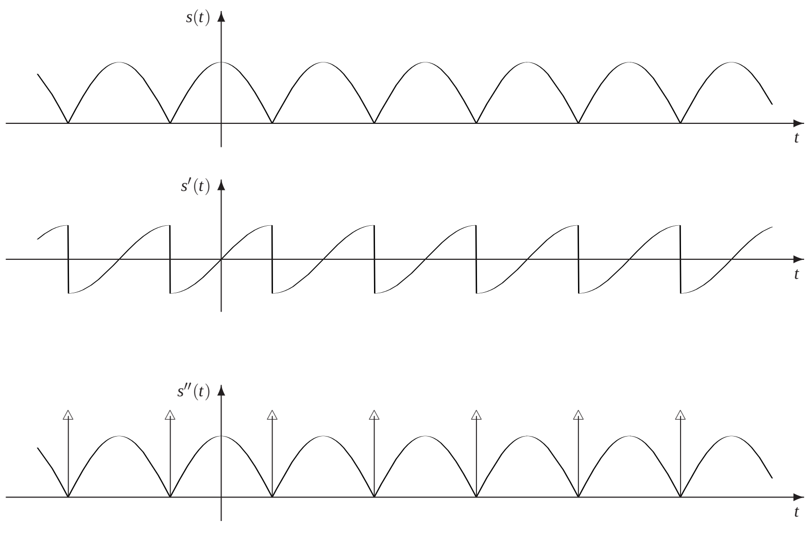

Fig. S2.3 The rectified sinusoidal signals and its first and second derivatives

2.11. ⋆⋆[Sect. 2.3] Find the (minimum) period of the signal

$$s(t) = 2\cos\frac{2}{3}\,\omega_0 t + 3\sin\frac{4}{5}\,\omega_0 t\ .$$

We let

$$s(t) = 2\ \cos\frac{2}{3}\,\omega_0 t + 3\ \sin\frac{4}{5}\,\omega_0 t \overset{\Delta}{=} s_1(t) + s_2(t)$$

and first we find before the periods of the two terms. With $T_0 = 2\pi/\omega_0$ we find that $s_1(t)$ has period $\frac{3}{2}T_0$ and $s_2(t)$ has period $\frac{5}{4}T_0$. The common period is therefore $T_p = \frac{15}{2}T_0$. This technique will be seen in detail in Sect. 3.6.

2.12. ⋆⋆[Sect. 2.4] Show that the (acyclic) convolution of an arbitrary signal $x(t)$ with a sinusoidal signal $y(t) = A_0\ \cos(\omega_0 t + \phi_0)$ is a sinusoidal signal with the same period as $y(t)$.

The convolution is

$$s(t) = y * x(t) = \int_{-\infty}^{+\infty} A_0 \cos\left[\omega_0(t-u) + \phi_0\right]\, x(u)\,\mathrm{d}u\,,$$

where we can use trigonometric formulas to obtain

$$\begin{aligned} s(t) = &\ \ A_0\ \cos(\omega_0 t + \phi_0)\int_{-\infty}^{+\infty} \cos\omega_0 u\, x(u)\,\mathrm{d}u \\ &- A_0\ \sin(\omega_0 t + \phi_0)\int_{-\infty}^{+\infty} \sin\omega_0 u\, x(u)\,\mathrm{d}u\,. \end{aligned}$$

Therefore, the convolution has the following structure

$$A_1\ \cos(\omega_0 t + \phi_0) + A_2\ \sin(\omega_0 t + \phi_0) = B_0\ \cos(\omega_0 t + \phi_0 + \beta_0) \quad ,$$

where A_1, A_2, B_0 and β_0 are constants.

2.13. ⋆[Sect. 2.4] Show that the derivative of the convolution $s(t)$ of two differentiable signals $x(t)$ and $y(t)$ is given by $s' = x' * y = x * y'$.

We write the convolution expression

$$s(t) = \int_{-\infty}^{+\infty} x(t-u)\,y(u)\ \mathrm{d}u\ .$$

Then, taking the derivative with respect to t and, assuming that order of integration and derivative can be interchanged, we find

$$s'(t) = \int_{-\infty}^{+\infty} x'(t-u)\,y(u)\,\mathrm{d}u\,.$$

2.14. $\star\star\star$ [Sect. 2.4] Evaluate the convolution of the following pulses:

$$x(t) = A_1\ \mathrm{rect}\,(t/2D)\,, \qquad y(t) = A_2\ \exp\left(-t^2/D^2\right)\,.$$

Hint. Express the result in terms of the normalized Gaussian distribution

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}y\,.$$

The convolution is

$$s(t) = \int_{-\infty}^{+\infty} x(t-u)\,y(u)\,\mathrm{d}u = A_1 A_2 \int_{t-D}^{t+D} \mathrm{e}^{-u^2/D^2}\,\mathrm{d}u \qquad u = \frac{D}{\sqrt{2}}y$$

$$= \frac{D}{\sqrt{2}} A_1 A_2 \int_{\frac{\sqrt{2}}{D}(t-D)}^{\frac{\sqrt{2}}{D}(t+D)} \mathrm{e}^{-y^2/2}\,\mathrm{d}y\,.$$

Considering that $\int_a^b = \int_{-\infty}^b + \int_a^{-\infty} = \int_{-\infty}^b - \int_{-\infty}^a$ we finally obtain

$$s(t) = \sqrt{2\pi}\,\frac{A_1 A_2 D}{\sqrt{2}}\left[\Phi\left(\frac{\sqrt{2}(t+D)}{D}\right) - \Phi\left(\frac{\sqrt{2}(t-D)}{D}\right)\right]\,.$$

2.15. $\star$ [Sect. 2.4] Evaluate the convolution of the signals

$$x(t) = A_1\ \mathrm{sinc}\,(t/D)\,, \qquad y(t) = A_2\ \delta(t) + A_3\ \delta(t-2D)\,.$$

In general, if one of two signals is given by impulses

$$y(t) = \sum_{n=1}^{N} A_i\ \delta(t-t_i)$$

we use the property (2.41), that is, $\delta(\cdot - t_i) * s(t) = s(t-t_i)$. Hence

$$x * y(t) = \sum_{n=1}^{N} A_i\,x(t-t_i)\,.$$

2.16. ⋆⋆⋆[Sect. 2.4] Evaluate the (cyclic) convolution of the signal

$$x(t) = \mathrm{rep}_{T_p}\ \mathrm{rect}\,(t/T)\ ,$$

with $x(t)$ itself (autoconvolution). Assume $T_p = 4T$.

We apply definition (2.44) with $t_0 = -2T = -T_p/2$

$$s(t) = \int_{-T_p/2}^{T_p/2} x(u)\, x(t-u)\, \mathrm{d}u$$

In this integral $u \in (-\frac{1}{2}T_p, \frac{1}{2}T_p)$ and in this interval the periodic repetition is present only with the zeroth term, that is $x(u) = \mathrm{rect}(t/T)$. If $t \in \{-\frac{1}{2}T_p, \frac{1}{2}T_p\}$ also $x(t-u) = \mathrm{rect}((t-u)/T)$ and then

$$\begin{aligned} s(t) &= \int_{-T_p/2}^{T_p/2} \mathrm{rect}\left(\frac{u}{T}\right)\ \mathrm{rect}\left(\frac{t-u}{T}\right)\ \mathrm{d}u \\ &= \int_{-T/2}^{T/2} \mathrm{rect}\left(\frac{t-u}{T}\right)\ \mathrm{d}u\,, \qquad t \in \left(-\tfrac{1}{2}T_p, \tfrac{1}{2}T_p\right) \end{aligned}$$

Continuing the assumption that $t \in (-\frac{1}{2}T_p, \frac{1}{2}T_p)$ we find

$$s(t) = \begin{cases} t + \frac{1}{2}T & -\frac{1}{2}T < t < 0 \\ -t + \frac{1}{2}T & 0 < t < \frac{1}{2}T \\ 0 & \text{elsewhere}\ . \end{cases} \quad \text{and} \quad -\tfrac{1}{2}T_p < t < \tfrac{1}{2}T_p$$

In conclusion, we have a triangular shape in a period, which we express in the form

$$s(t) = T\,\mathrm{triang}\left(\frac{t}{T}\right)\,, \qquad -\tfrac{1}{2}T_p < t < \tfrac{1}{2}T_p\,.$$

But, the convolution of periodic signals is periodic with the same period. Hence, to have the *expression for any* t, we have to write

$$s(t) = \mathrm{rep}_{T_p}\ T\ \ \mathrm{triang}\left(\frac{t}{T}\right)\ .$$

2.17. ⋆[Sect. 2.5] Show that the Fourier coefficients have the same physical *dimensions* as the signal. In particular, if $s(t)$ is a voltage (volt), also S_n must be expressed in volts.

If $[s(t)] = \mathrm{V}$ it results: $[S_n] = \mathrm{s}^{-1} \cdot \mathrm{V} \cdot \mathrm{s} = \mathrm{V}$.

2.18. ⋆[Sect. 2.5] Starting from the exponential form of the Fourier series and assuming a *real* signal, prove (2.49a) and (2.49b). Note that in this case S_0 is real.

If $s(t)$ is real, the real and the imaginary parts of S_n we have the symmetry

$$R_n = R_{-n}\,, \qquad X_n = -X_{-n}\,.$$

Then

$$\begin{aligned} s(t) &= R_0 + \sum_{n=1}^{\infty} S_n\,\mathrm{e}^{\mathrm{i}2\pi nFt} + \sum_{n=-\infty}^{-1} S_n\,\mathrm{e}^{\mathrm{i}2\pi nFt} \\ &= R_0 + \sum_{n=1}^{\infty} \left[R_n\left(\mathrm{e}^{\mathrm{i}2\pi nFt} + \mathrm{e}^{-\mathrm{i}2\pi nFt}\right) + \mathrm{i}\,X_n\left(\mathrm{e}^{\mathrm{i}2\pi nFt} - \mathrm{e}^{-\mathrm{i}2\pi nFt}\right)\right] \end{aligned}$$

and (2.49a) follows from Euler's identities. To prove (2.49b) we let $S_n = |S_n|\mathrm{e}^{\mathrm{i}\arg S_n}$. Then

$$\begin{aligned} s(t) &= S_0 + \sum_{n=1}^{\infty}\left[S_n\,\mathrm{e}^{\mathrm{i}2\pi nFt} + S_n^*\,\mathrm{e}^{-\mathrm{i}2\pi nFt}\right] \\ &= S_0 + 2\Re\sum_{n=1}^{\infty}|S_n|\mathrm{e}^{\mathrm{i}(2\pi nFt+\arg S_n)} \\ &= S_0 + 2\sum_{n=1}^{\infty}|S_n|\cos(2\pi nFt + \arg S_n)\,. \end{aligned}$$

Note that $S_0 = R_0$ is real.

2.19. ⋆[Sect. 2.5] Show that if $s(t)$ is real and even, then its sine–cosine expansion (2.49a) becomes an *only cosine expansion*.

We prove the statement by absurd. If in (2.49a) $X_k \neq 0$, the signal is not even for the presence of the terms $X_k\ \sin(2\pi kFt)$.

2.20. ⋆⋆⋆[Sect. 2.5] Assume that a periodic signal has the following symmetry:

$$s(t) = -s\left(t - T_p/2\right)\,.$$

Then, show that the Fourier coefficients S_n are zero for n even, i.e. the *even harmonics* disappear. *Hint*: use (2.50).

Consider the signal $x(t) \stackrel{\Delta}{=} s\left(t - \frac{1}{2}T_p\right)$, and use (2.50). Then

$$X_{2n} = S_{2n}\,\mathrm{e}^{-\mathrm{i}2\pi 2nF\,\frac{1}{2}T_p} = S_{2n}\,\mathrm{e}^{-\mathrm{i}2\pi n} = S_{2n}\,.$$

But by the symmetry hypothesis we also obtain $X_{2n} = -S_{2n}$, and we conclude that $S_{2n} = -S_{2n} = 0$.

2.21. ⋆⋆[Sect. 2.5] Evaluate the mean value, the root mean square value and the Fourier coefficients of the periodic signal

$$s(t) = \mathrm{rep}_{T_p}\left[\mathrm{rect}\left(\frac{t}{T_0}\right)A_0\left(1 - \frac{|t|}{T_0}\right)\right]$$

in the cases $T_p = 2T_0$ and $T_p = T_0$.

We use the definitions for periodic signals, (2.17b) and (2.18b). Considering that with $T_p = 2T_0$ and $T_p = T_0$, we have $T_p \geq \frac{1}{2}T_0$, so that the terms of the repetition do not overlap and

$$s(t) = \mathrm{rect}\left(\frac{t}{T_0}\right)A_0\left(1 - \frac{|t|}{T_0}\right), \qquad -\tfrac{1}{2}T_p < t < \tfrac{1}{2}T_p\,.$$

The mean value results

$$m_s(T_p) = \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} s(t)\,\mathrm{d}t = \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} A_0\left(1 - \frac{|t|}{T_0}\right)\mathrm{d}t = \frac{3}{4}A_0\,\frac{T_0}{T_p}\,.$$

The rms value is given by

$$\sqrt{P_s(T_p)} = \left\{\frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} |s(t)|^2\,\mathrm{d}t\right\}^{\frac{1}{2}} = \left\{\frac{A_0^2 T_0}{T_p}\,\frac{7}{12}\right\}^{\frac{1}{2}}.$$

The Fourier coefficients are given by

$$S_0 = \frac{3}{4}A_0\,\frac{T_0}{T_p}$$
$$S_n = \frac{A_0}{T_p}\left[\frac{\sin(\pi nFT_0)}{2\pi nF} + \frac{1 - \cos(\pi nFT_0)}{2\pi^2 n^2 F^2 T_0}\right], \qquad n \neq 0\,.$$

For $T_p < \frac{1}{2}T_p$ the terms of the repetition overlap and the evaluation is more complicated. In any case the preliminary step is the signal evaluation in a period.

2.22. ⋆[Sect. 2.5] Check Parseval's Theorem (2.51a) for a sinusoidal signal (see Example 2.1).

We must verify that

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p}|s(t)|^2\,\mathrm{d}t=\sum_{n=-\infty}^{+\infty}|S_n|^2$$

for the signal $s(t)=A_0\,\cos(2\pi f_0 t+\varphi_0)$. The direct evaluation in the time domain gives

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p}|s(t)|^2\,\mathrm{d}t=f_0\int_0^{1/f_0}A_0^2\,\cos^2(2\pi f_0 t+\varphi_0)\,\mathrm{d}t=\frac{1}{2}A_0^2\,.$$

On the other hand, considering that

$$s(t)=\frac{1}{2}A_0\,\mathrm{e}^{\mathrm{i}\varphi_0}\,\mathrm{e}^{\mathrm{i}2\pi f_0 t}+\frac{1}{2}A_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}\,\mathrm{e}^{-\mathrm{i}2\pi f_0 t}\,,$$

the Fourier coefficients are

$$S_1=\frac{1}{2}A_0\,\mathrm{e}^{\mathrm{i}\varphi_0}\,,\qquad S_{-1}=\frac{1}{2}A_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}\,,\qquad S_n=0\quad n\neq\pm 1\,.$$

Hence

$$\sum_{n=-\infty}^{+\infty}|S_n|^2=|S_1|^2+|S_{-1}|^2=\frac{1}{2}A_0^2\quad.$$

2.23. ⋆[Sect. 2.5] Evaluate the Fourier coefficients of the signal

$$s(t)=\mathrm{rep}_{T_p}\left[\delta\left(t-\frac{1}{4}T_p\right)-\delta\left(t-\frac{3}{4}T_p\right)\right]$$

and find symmetries (if any).

The Fourier coefficients are given by

$$S_n=\frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p}\left[\delta\left(t-\frac{1}{4}T_p\right)-\delta\left(t-\frac{3}{4}T_p\right)\right]\mathrm{e}^{-\mathrm{i}2\pi nFt}\,\mathrm{d}t$$

where we can use sifting property (2.41). We get

$$\begin{aligned}
S_n &= \frac{1}{T_p}\left[\mathrm{e}^{-\mathrm{i}2\pi nF\frac{1}{4}T_p} - \mathrm{e}^{-\mathrm{i}2\pi nF\frac{3}{4}T_p}\right] \\
&= \frac{1}{T_p}\left[\mathrm{e}^{-\mathrm{i}2\pi n/4} - \mathrm{e}^{-\mathrm{i}2\pi n/4}\right] \\
&= \frac{1}{T_p}\left[(-\mathrm{i})^n - (+\mathrm{i})^n\right] .
\end{aligned}$$

In conclusion

$$S_{2m} = 0\,, \qquad S_{2m+1} = \mathrm{i}\,\frac{2}{T_p}\,(-1)^{m+1}\,, \qquad m = \in \mathbb{Z}$$

The symmetry is the absence of even harmonics and in fact the signal verifies the symmetry of Problem 2.20.

2.24. $\star$ [Sect. 2.6] Find the physical dimension of the Fourier transform $S(f)$ when the signal is an electric voltage.

In (2.55b) the exponential is adimensional and then the integral has the dimensions of the signal, $[s] = \mathrm{V}$, multiplied by a time $[\mathrm{d}t] = \mathrm{s}$. Hence $[S(F)] = \mathrm{V}\cdot\mathrm{s}$.

2.25. $\star\star$ [Sect. 2.6] Show that if $s(t)$ is real, then $S(f)$ has the Hermitian symmetry. *Hint*: use (2.55).

Consider the FT

$$S(f) = \int_{-\infty}^{+\infty} s(t)\,\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}t$$

where $s(t) = s^*(t)$. Hence

$$S(f) = \int_{-\infty}^{+\infty} s^*(t)\,\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}t = \left\{\int_{-\infty}^{+\infty} s(t)\,\mathrm{e}^{\mathrm{i}2\pi ft}\,\mathrm{d}t\right\}^* = S^*(-f)$$

which states the Hermitian symmetry.

2.26. $\star\star$ [Sect. 2.6] Prove rule (2.60a) on the product of two signals.

Let $s(t) = x * y(t)$. Then, writing the convolution and the FT in an explicit form, we obtain

$$S(f)=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}x(u)\,y(t-u)\,\mathrm{d}u\right]\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}t\;.$$

Now, in the exponent we write $t=t-u+u$

$$\begin{aligned}S(f)&=\int_{-\infty}^{+\infty}x(u)\,\mathrm{e}^{-\mathrm{i}2\pi fu}\left\{\int_{-\infty}^{+\infty}y(t-u)\,\mathrm{e}^{-\mathrm{i}2\pi f(t-u)}\mathrm{d}t\right\}\mathrm{d}u\\&=\int_{-\infty}^{+\infty}x(u)\,\mathrm{e}^{-\mathrm{i}2\pi fu}\left\{\int_{-\infty}^{+\infty}y(t')\,\mathrm{e}^{-\mathrm{i}2\pi ft'}\mathrm{d}t'\right\}\mathrm{d}u \qquad (t'=t-u)\\&=\int_{-\infty}^{+\infty}x(u)\,\mathrm{e}^{-\mathrm{i}2\pi fu}\,Y(f)\,\mathrm{d}u\\&=X(f)\,Y(f)\end{aligned}$$

where we have used the variable change $t-u=t'$.

2.27. ⋆⋆[Sect. 2.6] Prove that the product $s(t)=x(t)\,y(t)$ of two strictly band–limited signal is strictly band–limited with

$$B(s)=B(x)+B(y)\;.$$

Hence, in particular the band of $x^2(t)$ is $2B(x)$.

Proceeding in the frequency domain the product becomes a convolution, that is $S(f)=X*Y(f)$. If the extension of the FTs are given respectively by the interval $[-B_x,B_x]$ and $[-B_y,B_y]$, the extension of their convolution $S(f)$ is $[-(B_x+B_y),B_x+B_y]$. Here, we have used the rule on the extension of convolution proved in Sect. 2.4 in the time domain (see (2.39)).

2.28. ⋆[Sect. 2.7] Evaluate the Fourier transform of the causal signal

$$s(t)=1(t)\,\mathrm{e}^{-t/T}\,,\qquad T>0$$

and then check that it verifies the Hermitian symmetry.

We obtain

$$S(f)=\int_0^{\infty}\mathrm{e}^{-t/T}\,\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}t=\frac{T}{1+\mathrm{i}2\pi fT}\;.$$

We immediately prove that $S^*(f)=S(-f)$. In fact

$$S^*(f)=\frac{T}{1-\mathrm{i}2\pi fT}=S(-f)\,.$$

2.29. $\star$ [Sect. 2.7] Prove the relationship

$$s(t)\,\cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{2}S(f-f_0)+\frac{1}{2}S(f+f_0)\,, \tag{2.2}$$

called *modulation rule*.

Applying the Euler formulas, we obtain:

$$s(t)\,\cos(2\pi f_0 t)=\frac{1}{2}s(t)\,e^{i2\pi f_0 t}+\frac{1}{2}s(t)\,e^{-i2\pi f_0 t} \quad .$$

Hence, the desired relation is obtained using the frequency shifting rule (2.59b).

2.30. $\star$ [Sect. 2.7] Using (2.112) evaluate the Fourier transform of the signal

$$s(t)=\mathrm{rect}(t/T)\,\cos 2\pi f_0 t\,.$$

Then, draw graphically $S(f)$ for $f_0T=4$, checking that it is an even real function.

Considering the FT pair (2.64) and using the *modulation rule* of the previous problem, we obtain

$$\begin{aligned}S(f)&=\frac{1}{2}T\,[\mathrm{sinc}((f-f_0)T)+\mathrm{sinc}((f+f_0)T)]\\&=\frac{1}{2}T\,[\mathrm{sinc}(fT-4)+\mathrm{sinc}(fT+4)]\,.\end{aligned}$$

By inspection we see that $S(f)$ is a real and even function.

2.31. $\star\star$ [Sect. 2.7] Using the rule on the product, prove the relationship

$$1(t)\,\cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{4}\left[\delta(f-f_0)+\delta(f+f_0)+\frac{1}{i\pi(f-f_0)}+\frac{1}{i\pi(f+f_0)}\right]\,.$$

The product in the time domain becomes the convolution in the frequency domain. Keeping in mind that

$$\begin{aligned}a(t)=\cos(2\pi f_0 t) &\quad\xrightarrow{\mathcal{F}}\quad A(f)=\frac{1}{2}\,\delta(f-f_0)+\frac{1}{2}\,\delta(f+f_0)\\ b(t)=1(t) &\quad\xrightarrow{\mathcal{F}}\quad B(f)=\frac{1}{2}\,\delta(f)+\frac{1}{i2\pi f} \quad ,\end{aligned}$$

Then, $s(t) = a(t)\,b(t)$ has FT

$$\begin{aligned} S(f) &= \int_{-\infty}^{+\infty} A(f-\lambda)\,B(\lambda)\mathrm{d}\lambda \\ &= \frac{1}{4}\int_{-\infty}^{+\infty} [\delta(f-\lambda-f_0)\,\delta(\lambda)+\delta(f-\lambda+f_0)\,\delta(\lambda)]\,\mathrm{d}\lambda \\ &+ \frac{1}{2}\int_{-\infty}^{+\infty} [\delta(f-\lambda-f_0)\frac{1}{\mathrm{i}2\pi\lambda}+\delta(f-\lambda+f_0)\frac{1}{\mathrm{i}2\pi\lambda}]\,\mathrm{d}\lambda \end{aligned}$$

Finally, we use four times the sifting property $\int_{-\infty}^{+\infty} \delta(u-\lambda)\,g(\lambda)\,\mathrm{d}\lambda = g(u)$, which gives

$$S(f) = \tfrac{1}{2}\left[\frac{1}{\mathrm{i}2\pi(f-f_0)} + \frac{1}{\mathrm{i}2\pi(f+f_0)}\right] .$$

2.32. ⋆⋆∇ [Sect. 2.7] The *scale change* (see Sect. 6.5) has the following rule

$$s(at) \xrightarrow{\mathcal{F}} (1/|a|)\,S(f/a) \qquad a \neq 0 . \tag{2.3}$$

Then, giving as known the pair $\mathrm{e}^{-\pi t^2} \xrightarrow{\mathcal{F}} \mathrm{e}^{-\pi f^2}$, evaluate the Fourier transform of the *Gaussian pulse*

$$u(t) = \frac{A_0}{\sqrt{2\pi}\,\sigma}\,\exp\left[-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2\right] .$$

We first evaluate the scale factor a

$$\mathrm{e}^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} = \mathrm{e}^{-\pi\left(\frac{t}{\sqrt{2\pi}\,\sigma}\right)^2} = \mathrm{e}^{-\pi\left(\frac{t}{a}\right)^2} \Rightarrow a = \sqrt{2\pi}\,\sigma .$$

Applying the scale change rule, we get the Fourier pair

$$\frac{A_0}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} \quad \xrightarrow{\mathcal{F}} \quad A_0\,\mathrm{e}^{-2\pi^2\sigma^2 f^2} .$$

2.33. ⋆⋆ [Sect. 2.7] Evaluate the Fourier transform of the periodic signal

$$s(t) = \mathrm{rep}_{T_p}\ \mathrm{rect}\left(\frac{t}{D}\right) .$$

The evaluation of the Fourier coefficients

$$S_n = \frac{1}{T_p} \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} s(t)\, e^{-i2\pi nFt}\, dt$$
$$= \frac{1}{T_p} \int_{-D/2}^{D/2} e^{-i2\pi nFt}\, dt$$

gives

$$S_n = F\, D \operatorname{sinc}(nFD)\,.$$

Applying the rule (2.67) for a periodic signal rule, we obtain

$$\sum_{n=-\infty}^{+\infty} F\, D \operatorname{sinc}(nFD)\, \delta(f - nF)\,.$$

2.34. ⋆⋆ [Sect. 2.7] Prove the Fourier pair

$$\operatorname{triang}\left(\frac{t}{D}\right) = \operatorname{rect}\left(\frac{t}{2D}\right)\left(1 - \frac{|t|}{D}\right) \xrightarrow{\mathcal{F}} D \operatorname{sinc}^2(fD)$$

where the signal is the *triangular pulse* of duration 2D.

We can apply FT definition and integrate by parts, or we can recognize that $s(t)$ is the convolution of two rectangular pulses. We develop this second possibility, noting that $s(t) = 1/Dx * x(t)$, where

$$x(t) = \operatorname{rect}\left(\frac{t}{D}\right)\,.$$

Then, recalling that $X(f) = D \operatorname{sinc}(fD)$ we obtain $S(f) = D \operatorname{sinc}^2(fD)$.

2.35. ⋆⋆⋆ [Sect. 2.7] Consider the decomposition of a *real* signal in an even and an odd components

$$s(t) = s_e(t) + s_o(t)\,.$$

Then, prove the relationship

$$s_e(t) \xrightarrow{\mathcal{F}} \Re\, S(f)\,, \qquad s_o(t) \xrightarrow{\mathcal{F}} j\,\Im\, S(f)\,.$$

For the even part of $s(t)$ we obtain:

$$s_p(t) = \frac{1}{2}\left[s(t) + s(-t)\right] \xrightarrow{\mathcal{F}} \frac{1}{2}\left[S(f) + S(-f)\right].$$

If $s(t)$ is real, we have the Hermitian symmetry $S(-f) = S^*(f)$. Then

$$s_p(t) \xrightarrow{\mathcal{F}} \frac{1}{2}\left[S(f)+S^*(f)\right] = \Re S(f)\,.$$

For the odd part of $s(t)$ we proceed analogously.

2.36. $\star$[Sect. 2.12] Evaluate the Fourier transforms of the discrete signals

$$s_1(nT) = \begin{cases} A_0 & n=\pm 1 \\ 0 & \text{otherwise} \end{cases}, \qquad s_2(nT) = \begin{cases} A_0 & n=-1,0,1 \\ 0 & \text{otherwise} \end{cases}$$

and check that $S_1(f)$ and $S_2(f)$ are a) periodic with period $F_p = 1/T$, b) real and c) even.

Using definition (2.91b) we get

$$S_1(f) = \sum_{n=-\infty}^{+\infty} T\, s_1(nT)\, \mathrm{e}^{-\mathrm{i}2\pi f nT} = T\,[A_0\, \mathrm{e}^{-\mathrm{i}2\pi fT} + A_0\, \mathrm{e}^{\mathrm{i}2\pi f_0 T}] = 2T\, A_0\, \cos(2\pi fT)$$

$$S_2(f) = \sum_{n=-\infty}^{+\infty} T\, s_2(nT)\, \mathrm{e}^{-\mathrm{i}2\pi f nT} = T\, A_0 + 2T\, A_0\, \cos(2\pi fT)\,.$$

We can see that $S_1(f)$ and $S_2(f)$ have period $F_p = 1/T$, and are real (as usual we suppose A_0 real) and even.

2.37. $\star$[Sect. 2.12] With the signals of the previous problem check the Parseval theorem (2.97).

We have to prove that

$$E_{s1} = \sum_{n=-\infty}^{+\infty} T\, |s_1(nT)|^2 = \int_{f_0}^{f_0+F_p} |S_1(f)|^2\, \mathrm{d}f$$

where f_0 is arbitrary, and we choose $f_0 = 0$. In the time domain

$$E_{s1} = \sum_{n=-\infty}^{+\infty} T\, |s_1(nT)|^2 = 2T\, A_0^2\,.$$

In the frequency domain, using the result of the previous problem, we get

$$\int_0^{F_p} |S_1(f)|^2\, \mathrm{d}f = \int_0^{F_p} 4T^2\, A_0^2\, \cos^2(2\pi fT)\, \mathrm{d}f = 2T\, A_0^2\,.$$

Analogously, we can proceed for the second signal.

2.38. $\star$[Sect. 2.12] Show the relationship

$$\operatorname{sinc}(nF_0T) \xrightarrow{\mathcal{F}} \frac{1}{F_0}\ \operatorname{rep}_{F_p}\ \operatorname{rect}\left(\frac{f}{F_0}\right),$$

illustrated in Fig.2.41 for $F_0T = \frac{1}{2}$. *Hint: show that the inverse Fourier transform of $S(f)$ is $s(nT)$.*

The inverse transform is

$$s(nT) = \int_{-\frac{1}{2}F_p}^{\frac{1}{2}F_p} S(f)\,\mathrm{e}^{\mathrm{i}2\pi fnT}\,\mathrm{d}f = \int_{-\frac{1}{2}F_0}^{\frac{1}{2}F_0} \frac{1}{F_0}\,\mathrm{e}^{\mathrm{i}2\pi fnT}\,\mathrm{d}f = \operatorname{sinc}(nTF_0)\,.$$

2.39. $\star\star$[Sect. 2.15] Apply the Sampling Theorem to the signal

$$s(t) = \operatorname{sinc}^3(Ft)\,, \qquad t \in \mathbb{R}$$

with $F = 4$ kHz.

We have to evaluate the spectral extension of $s(t)$, that is, the extension of $S(f)$. Recall the rule that $y(t) = x^2(t) \to Y(f) = X * X(f)$ and consequently the extensions are relate as $(-B_y, B_y) = (-2B_x, 2B_x)$. Extending this rule to their power, $y(t) = x^3(t)$, we have $(-B_y, B_y) = (-3B_x, 3B_x)$. In our case $s(t) = \operatorname{sinc}(Ft)$ has spectral extension $(-\frac{1}{2}F, \frac{1}{2}F)$ and hence the extension of $S(f)$ is $(-\frac{3}{2}F, \frac{3}{2}F)$. The Sampling Theorem states that we can sample $s(t)$ (correctly for the reconstruction) with $F_c = 3F = 12$k samples per second.

Problems of Chapter 3

3.1. $\star$[Sect. 3.2] Check that the additive set of complex numbers, $\mathbb{C}$, is an Abelian group.

It is immediate to verify that $(\mathbb{C},+)$ satisfies to properties required by general definition of group; particularly the identity element is the complex number $0+\mathrm{i}0$.

3.2. $\star$[Sect. 3.2] Prove the relations

$$\mathbb{Z}(2)+\mathbb{Z}(4)=\mathbb{Z}(2)\,, \qquad \mathbb{Z}(3)+\mathbb{R}=\mathbb{R}\,.$$

The two relations are particular cases of a general statement , that is, if $J\subset K$, then always $J+K=K$. In fact, if $j\in J$, then $j\in K$ and $j+K=K$, from the group properties.

3.3. $\star\star\star$[Sect. 3.2] Prove the relations

$$\mathbb{Z}(3)+\mathbb{Z}(5)=\mathbb{Z}(1)\,, \qquad \mathbb{Z}(6)+\mathbb{Z}(9)=\mathbb{Z}(3)\,.$$

We want to prove the relation $\mathbb{Z}(3)+\mathbb{Z}(5)=\mathbb{Z}(1)$. From the definition of the sum we obtain:

$$\mathbb{Z}(3)+\mathbb{Z}(5)=\{3n+5m \mid n,m\in\mathbb{Z}\}\,.$$

This set coincides with set $\mathbb{Z}(1)=\mathbb{Z}$. In fact, every integer k can always be expressed in the form $3n+5m$, with n and m opportune integers. To proof this, we divide the integer k by 3. In this way we have to consider three different cases:

$$\begin{aligned}
k &= 3n = 3n+5\cdot 0\\
k &= 3n+1 = 3(n-3)+5\cdot 2\\
k &= 3n+2 = 3(n+4)+5\cdot(-2)\,.
\end{aligned}$$

Analogously, to prove $\mathbb{Z}(6)+\mathbb{Z}(9)=\mathbb{Z}(3)$, we write:

$$\mathbb{Z}(6)+\mathbb{Z}(9)=\{6n+9m \mid n,m\in\mathbb{Z}\}=\{3(2n+3m) \mid n,m\in\mathbb{Z}\}\,.$$

This set coincides with set $\mathbb{Z}(3)$. In fact, every integer k can always be expressed in the form $2n+3m$ (by expression $2n+3m$), with n and m opportune integers.

A general result on the sum $\mathbb{Z}(T_1)+\mathbb{Z}(T_2)$ is given by Theorem 3.7.

3.4. $\star$[Sect. 3.2] Prove the relations

$$[0,2)+\mathbb{Z}(2)=\mathbb{R}\,,\qquad [0,3)+\mathbb{Z}(2)=\mathbb{R}\,.$$

The proof of $[0,2)+\mathbb{Z}(2)=\mathbb{R}$ is immediate. In fact, the sets $[0,2)+m$, with $m\in\mathbb{Z}(2)$ from a partition of $\mathbb{R}$. Analogous is the proof of the second relation.

3.5. $\star$[Sect. 3.2] Verify that $\mathbb{C}^*$ is an Abelian group, where the operation is the ordinary multiplication between complex numbers.

If a and b are complex numbers also $a\cdot b$ is a complex number. The identity in $\mathbb{C}^*$ is the complex number $1=1+\mathrm{i}0$. In fact, $1\cdot s=a$. Finally, for every $a\neq 0$, we can find a complex number $1/a$, such that $a\cdot 1/a=1$.

3.6. $\star\star$[Sect. 3.2] Verify that the 2D set $\mathbb{Z}_2^1$ consisting of the integer pairs (m,n), with *m, n both even or both odd*, is a subgroup of $\mathbb{R}^2$.

We limit ourselves to verify the closing property with respect to the addition, that is, for all a and b belonging to $\mathbb{Z}_2^1(d_1,d_2)$, $a+b$ and belongs to $\mathbb{Z}_2^1(d_1,d_2)\in\mathbb{Z}_2^1(d_2,d_1)$. In fact, let $a=(m_ad_1,\,n_ad_2)$, $b=(m_bd_1,\,n_bd_2)\in\mathbb{Z}_2^1(d_1,d_2)$, then $a+b=((m_a+m_b)\,d_1,\,(n_a+n_b)\,d_2)\in\mathbb{Z}_2^1(d_1,d_2)$. Now for the pairs (m_a,n_a) and (m_b,n_b) we find four possibilities: (m_a,n_a) even, (m_b,n_b) even, etc.. Examining these possibilities we realize that in all the four cases the integers m_a+m_b, n_a+n_b are both even or both odd; hence $a+b\in\mathbb{Z}_2^1(d_1,d_2)$.

3.7. $\star\star\star$[Sect. 3.3] With reference to representation (3.27), find the corresponding upper–triangular representation. *Hint: Use Proposition 3.1.*

The normalized lower–triangular and upper–triangular bases have respectively the form

$$\mathbf{G}_l=\begin{bmatrix}1 & 0\\ b & i\end{bmatrix},\qquad \mathbf{G}_u=\begin{bmatrix}i & c\\ 0 & 1\end{bmatrix}$$

where $0<b<i$ and $0<c<i$. According to Proposition 3.1, the two matrices must be related by

$$\mathbf{G}_u=\mathbf{G}_l\,\mathbf{E}=\mathbf{G}_l\begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix}\tag{S3.1}$$

where $\mathbf{E}$ is an integer matrix with $\det \mathbf{E} = \pm 1$. Since $\det \mathbf{G}_l = \det \mathbf{G}_u = i$, in this case $\det \mathbf{E} = 1$.

Explicitly writing (S3.1), we find the conditions

$$e_{11} = i \qquad e_{12} = c \qquad i(b + e_{21}) = 0 \,.$$

Hence

$$\mathbf{E} = \begin{bmatrix} i & c \\ -b & e_{22} \end{bmatrix}$$

and $\det \mathbf{E} = 1$ gives the condition

$$\boxed{ie_{22} + bc = 1 \,.} \tag{S3.2}$$

In conclusion, for given i, b, the integer c is found as the solution of the *integer equation* (S3.2) under the condition $0 < c < i$.

For instance, with $i = 5$, $b = 2$ the equation is

$$5e_{22} + 2c = 1$$

which has the solution $c = 3$, $e_{22} = -1$.

3.8. $\star$ [Sect. 3.5] Check that the set $A = [0,1) \cup [6,7) \cup [12,15)$ is a cell of $\mathbb{R}$ modulo $\mathbb{Z}(5)$.

First we note that the sets

$$A_k = [0,1) + 5k \quad , \quad B_k = [6,7) + 5k \quad , \quad C_k = [12,15) + 5k$$

respectively with $k = 0$, $k = -1$, $k = -2$, give a partition of the interval $[0,5)$, that is,

$$A_0 \cup B_{-1} \cup C_{-1} = [0,5) \,.$$

Then, $A = A_0 \cup B_0 \cup C_0$ and $A + 5k$, $k \in \mathbb{Z}$ give a partition of $\mathbb{R}$. In fact

$$\begin{aligned} \bigcup_{k=-\infty}^{+\infty} (A + 5k) &= \bigcup_{k=-\infty}^{+\infty} (A_k \cup B_k \cup C_k) \\ &= \bigcup_{k=-\infty}^{+\infty} (A_k \cup B_{k-1} \cup C_{k-2}) = \bigcup_{k=-\infty}^{+\infty} [5k,\, 5 + 5k) = \mathbb{R} \,. \end{aligned}$$

3.9. $\star\star$ [Sect. 3.5] Verify the relationship $[I_0/P_0) + [P_0/P) + P = I_0$ for $I_0 = \mathbb{R}$, $P_0 = \mathbb{Z}(2)$ and $P = \mathbb{Z}(10)$.

From (3.33) one gets

$$[\mathbb{R}/\mathbb{Z}(2)) + \mathbb{Z}(2) = \mathbb{R}\,, \qquad [\mathbb{Z}(2)/\mathbb{Z}(10)) + \mathbb{Z}(10) = \mathbb{Z}(2)\,.$$

Combining the two relations, one obtains

$$[\mathbb{R}/\mathbb{Z}(2)) + \{[\mathbb{Z}(2)/\mathbb{Z}(10) + \mathbb{Z}(10))\} = [\mathbb{R}/\mathbb{Z}(2)) + [\mathbb{Z}(2)) = \mathbb{R}\,.$$

3.10. $\star$ [Sect. 3.6] Find the periodicity of the continuous signal

$$s(t) = A_0\, \cos 2\pi f_1 t + B_0\, \sin 2\pi f_2 t\,, \qquad t \in \mathbb{R}$$

for $f_1/f_2 = 3/5$ and $f_1/f_2 = \sqrt{2}/5$.

The signal is

$$s(t) = A_0\, \cos 2\pi f_1 t + B_0\, \sin 2\pi f_2 t \stackrel{\Delta}{=} s_1(t) + s_2(t)\,, \qquad t \in \mathbb{R}$$

where minimum periods of $s_1(t)$ and $s_2(t)$ are, respectively

$$T_{p1} = \frac{1}{f_1}\,, \qquad T_{p2} = \frac{1}{f_2}\,.$$

The possible periods are $k\,T_{p1}$ and $h\,T_{p2}$ for all $k,h \in \mathbb{Z}$. For $f_1/f_2 = 3/5$ the minimum common period is

$$3\,T_{p1} = 5\,T_{p2}$$

and the *periodicity* is $\mathbb{Z}(3T_p) = \mathbb{Z}(5T_p)$. For $f_1/f_2 = \sqrt{2}/5$ we have $T_{p1}/T_{p2} = 2/\sqrt{5}$ and $s_1(t)$ and $s_2(t)$ have not a common period and the periodicity becomes $\mathbb{O}$.

3.11. $\star\star\star$ [Sect. 3.6] Find the periodicity of the discrete signal

$$s(t) = A_0\, \cos 2\pi f_1 t + B_0\, \sin 2\pi f_2 t\,, \qquad t \in \mathbb{Z}(2)$$

for $f_1 = 1/7$ and $f_2 = 1/4$.

Let

$$s(t) = A_0 \cos 2\pi f_1 t + B_0 \sin 2\pi f_2 t \stackrel{\Delta}{=} s_1(t) + s_2(t)\,, \qquad t \in \mathbb{Z}(2)\,.$$

The period of $s_1(t)$, interpreted as continuous signal, would be 7, but the period of a signal defined on $\mathbb{Z}(2)$ must be a multiple of 2 and so the minimum period is 14. The period of $s_2(t)$, interpreted as continuous signal, is 4, which is admissible also for a signal defined on $\mathbb{Z}(2)$. Hence the period of sum $s(t)$ is 28 and the periodicity is $\mathbb{Z}(28)$.

3.12. $\star$[Sect. 3.6] Find the minimum period of the discrete signal

$$s(t) = s_1(t)\, s_2^3(t)\,, \qquad t \in \mathbb{Z}(3)$$

where $s_1(t)$ has period $T_{p1} = 9$ and $s_2(t)$ has period $T_{p2} = 12$.

It is immediate to check that $y(t) = s_2^3(t)$ has the same period of $s_2(t)$, that is, $T_{p2} = 12$. The period of a product $s(t)\,y(t)$ is the common period of $s_1(t)$ and $y(t)$. Hence, the period of $s(t)$ is given by the *least common multiple* of 9 and 12, that is, 36.

3.13. $\star\star$[Sect. 3.8] Verify that any logarithmic function, $\log_b$, is an isomorphism from $(\mathbb{R}_p, \cdot)$ onto $(\mathbb{R}, +)$.

According to Definition 3.3 it is sufficient to note that

$$\alpha(t) = \log_b(t) : \mathbb{R}_p \to \mathbb{R}$$

is an *invertible map*. In fact, the inverse map is

$$\alpha^{-1}(u) = b^u : \mathbb{R} \to \mathbb{R}_p$$

for every *positive real* b.

3.14. $\star\star\star$[Sect. 3.9] Prove that if G_1 e G_2 are both subgroups of a group G, the *sum* $G_1 + G_2$ and the *intersection* $G_1 \cap G_2$ are subgroups of G.

The union $G_1 \cup G_2$ is not a group, in general, as we can check for the pair $G_1 = \mathbb{Z}(5)$ and $G_2 = \mathbb{Z}(3)$.

We limit ourself to verify, in both cases, the closing properties with respect to the group operation $+$.

Let $D = G_1 + G_2 = \{a + b \,|\, a \in G_1\,,\ b \in G_2\}$. Considered $a_1 + b_1, a_2 + b_2 \in D$, with $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$. Then

$$(a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)\in D$$

since $a_1+a_2 \in G_1$ e $b_1+b_2 \in G_2$.

Let $G_1 \cap G_2 = \{a \,|\, a \in G_1\ ,\ a \in G_2\}$ and let $a_1, a_2 \in G_1 \cap G_2$, with $a_1, a_2 \in G_1$ and $a_1, a_2 \in G_2$, we obtain

$$a_1+a_2 \in G_1\,, \qquad a_1+a_2 \in G_2$$

and hence $a_1+a_2 \in G_1 \cap G_2$.

3.15. ⋆⋆ [Sect. 3.9] Evaluate

$$\mathbb{Z}(T_1)\cap\mathbb{Z}(T_2)\cap\mathbb{Z}(T_3) \qquad \text{and} \qquad \mathbb{Z}(T_1)+\mathbb{Z}(T_2)+\mathbb{Z}(T_3)$$

for $T_1 = 0.018,\ T_2 = 0.039,\ T_3 = 0.045$.

We first find a common submultiple of T_1, T_2, T_3, which is given by $T_0 = 0.001$. Then

$$\mathbb{Z}(T_1)=\mathbb{Z}(18\,T_0)\,, \qquad \mathbb{Z}(T_2)=\mathbb{Z}(39\,T_0)\,, \qquad \mathbb{Z}(T_3)=\mathbb{Z}(45\,T_0)\ .$$

From (3.72a) we obtain $\mathbb{Z}(T_1)\cap\mathbb{Z}(T_2)=\mathbb{Z}(234\,T_0)$ and

$$(\mathbb{Z}(T_1)\cap\mathbb{Z}(T_2))\cap\mathbb{Z}(T_3)=\mathbb{Z}(1170\,T_0)=\mathbb{Z}(1.17)\ .$$

From (3.72b) we obtain $\mathbb{Z}(T_1)+\mathbb{Z}(T_2)=\mathbb{Z}(3\,T_0)$ and

$$(\mathbb{Z}(T_1)+\mathbb{Z}(T_2))+\mathbb{Z}(T_3)=\mathbb{Z}(3\,T_0)=\mathbb{Z}(0.003)\ .$$

3.16. ⋆⋆ [Sect. 3.9] Reconsider Problem 3.13 and Problem 3.14 using Theorem 3.7.

The application of Theorem 3.7 to Problem 3.13 states that the period T_p is obtained by solving the equation

$$\mathbb{Z}(T_p)=\mathbb{Z}(T_{p1})\cap\mathbb{Z}(T_{p2})\ .$$

In the specific case, with $T_0 = 1/f_1$, for $f_1/f_2 = 3/5$, the equation is

$$\mathbb{Z}(T_p)=\mathbb{Z}(T_0)\cap\mathbb{Z}\left(\frac{3}{5}\,T_0\right)=\mathbb{Z}(3T_0)\ .$$

For $f_1/f_2 = \sqrt{2}/5$ we find

$$\mathbb{Z}(T_p) = \mathbb{Z}(T_0) \cap \mathbb{Z}\left(\frac{\sqrt{2}}{5}\,T_0\right) = \{0\}\ .$$

Analogously in Problem 3.14 we obtain

$$\mathbb{Z}(T_p) = \mathbb{Z}(2) \cap \mathbb{Z}(7) \cap \mathbb{Z}(4) = \mathbb{Z}(28)$$

where $\mathbb{Z}(2)$ is inserted to impose that the periodicity of a signal on $\mathbb{Z}(2)$ must be a subgroup of $\mathbb{Z}(2)$.

3.17. $\star\star\star$ [Sect. 3.9] Find the periodicity of the discrete sinusoid

$$s(t) = A_0\ \cos(2\pi f_0 t + \varphi_0)\,, \qquad t \in \mathbb{Z}(T)$$

considering f_0 as a parameter.

Let $T_0 = 1/f_0$ be the period of the corresponding continuous signal. Then the periodicity of the discrete signal is given by

$$\mathbb{Z}(T_p) = \mathbb{Z}(T_0) \cap \mathbb{Z}(T)\ .$$

So, if $T/T_0 = T f_0$ is rational, we let $T/T_0 = N_1/N_2$, with N_1 and N_2 coprime. Then we obtain $\mathbb{Z}(T_p) = \mathbb{Z}(N_2 T) = \mathbb{Z}(N_1 T_0)$. If T/T_0 is irrational, then $\mathbb{Z}(T_p) = \{0\}$ and the signal is aperiodic.

Problems of Chapter 4

4.1. ⋆[Sect. 4.1] Explicitly write (4.12a) with $I_0 = \mathbb{R}$ and $P = \mathbb{Z}(T_p)$ and (4.12b) with $I_0 = \mathbb{R}$, $P = \mathbb{Z}(T_p)$ and $P_0 = \mathbb{Z}(\frac{1}{3}T_p)$. Then, combine these formulas.

Equation

$$\int_{I_0} \mathrm{d}t\, s(t) = \int_{I_0/P} \mathrm{d}u \sum_{p \in P} s(u-p)$$

with $I_0 = \mathbb{R}$ and $P = \mathbb{Z}(T_p)$ becomes

$$\int_{-\infty}^{+\infty} s(t)\,\mathrm{d}t = \int_0^{T_p} \sum_{k=-\infty}^{+\infty} s(u-kT_p)\,\mathrm{d}t\ .$$

The latter equation can be proved by partitioning $\mathbb{R}$ into the intervals $[kT_p, (k+1)T_p)$, $k \in \mathbb{Z}$.

Equation

$$\int_{I_0/P} \mathrm{d}t\, s(t) = \int_{I_0/P_0} \mathrm{d}u \sum_{p \in [P_0/P)} s(u-p)$$

with $I_0 = \mathbb{R}$ and $P = \mathbb{Z}(T_p)$ and $P_0 = \mathbb{Z}(\frac{1}{3}T_p)$ becomes

$$\begin{aligned}\int_0^{T_p} s(t)\,\mathrm{d}t &= \int_0^{T_p/3} \sum_{p \in \{0, \frac{1}{3}T_p, \frac{2}{3}T_p\}} s(u-p)\,\mathrm{d}u \\ &= \int_0^{T_p/3} \left[s(u) + s(u - \tfrac{1}{3}T_p) + s(u - \tfrac{2}{3}T_p)\right]\,\mathrm{d}t\end{aligned} \tag{S4.1}$$

where $[P_0/P) = \{0, \frac{1}{3}T_p, \frac{2}{3}T_p\}$. In this case we assume that $s(t)$ has period T_p and we partition the interval $[0, T_p)$ into $[0, \frac{1}{3}T_p)$, $[\frac{1}{3}T_p, \frac{2}{3}T_p)$ and $[\frac{2}{3}T_p, T_p)$. Hence

$$\int_0^{T_p} s(t)\,\mathrm{d}t = \int_0^{\frac{1}{3}T_p} s(t)\,\mathrm{d}t + \int_{\frac{1}{3}T_p}^{\frac{2}{3}T_p} s(t)\,\mathrm{d}t + \int_{\frac{2}{3}T_p}^{T_p} s(t)\,\mathrm{d}t$$

and with a change of variable we obtain (S4.1).

Note that the signal $s(u) + s(u - \frac{1}{3}T_p) + s(u - \frac{2}{3}T_p)$ becomes periodic of period $\frac{1}{3}T_p$ and then its formulation on $I_0/P_0 = \mathbb{R}/\mathbb{Z}(\frac{1}{3}T_p)$ is correct.

4.2. ⋆⋆[Sect. 4.1] Explicitly write the multirate identity (4.13) with $I_0 = \mathbb{Z}$ and $P = \mathbb{Z}(5)$. Then, prove the identity in the general case starting from (4.12).

With $I_0 = \mathbb{Z}$ and $P = \mathbb{Z}(5)$, the cell $[I_0/P) = \{0,1,2,3,4\}$ has cardinality $N = 5$. Then the multirate identity

$$\int_{I_0} \mathrm{d}t\, s(t) = \frac{1}{N} \sum_{p\in[I_0/P)} \int_P \mathrm{d}u\, s(u+p) \tag{S4.2}$$

becomes

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} s(n) &= \frac{1}{5} \sum_{p=0}^{4} \int_{\mathbb{Z}(5)} \mathrm{d}u\, s(u+p) \\ &= \frac{1}{5} \sum_{p=0}^{4} \sum_{u\in\mathbb{Z}(5)} 5\, s(u+p) \\ &= \sum_{p=0}^{4} \sum_{k=-\infty}^{+\infty} s(5k+p)\ .\end{aligned}$$

In the latter summation, the signal values are summed separately over the sets $\mathbb{Z}(5), \mathbb{Z}(5)+1, \ldots, \mathbb{Z}(5)+4$ and then the 5 sums are combined.

To prove relation (S4.2) in general, we note that I_0 and P are lattices. Then, (4.12b)ecomes

$$\begin{aligned}\int_{I_0} \mathrm{d}t\, s(t) &= \sum_{u\in[I_0/P)} \mathrm{d}(I_0) \sum_{p\in P} s(u-p) \\ &= \sum_{u\in[I_0/P)} \mathrm{d}(I_0) \left[\frac{1}{\mathrm{d}(P)} \int_P \mathrm{d}p\, s(u-p)\right]\end{aligned}$$

where $\mathrm{d}(I_0)/\mathrm{d}(P) = 1/N$. Then the multirate identity follows, considering that the integrals of $s(u)$ and of $s(u-p)$ coincides for the property of the Haar integral (see (4.2)).

4.3. $\star$[Sect. 4.2] Show that the ordinary integral over $\mathbb{R}$ verifies the general properties of the Haar integral.

The check is immediate. For instance, the property

$$\int_{-\infty}^{+\infty} s(t+t_0)\, \mathrm{d}t = \int_{-\infty}^{+\infty} s(t')\, \mathrm{d}t'$$

is obtained with the change of variable $t' = t + t_0$.

4.4. ⋆ [Sect. 4.2] Show that the Haar integral over $\mathbb{Z}(T)$ verifies the general properties of the Haar integral.

The Haar integral over $\mathbb{Z}(T)$ is given by

$$\int_{\mathbb{Z}(T)} \mathrm{d}t\, s(t) = \sum_{n=-\infty}^{+\infty} T\, s(nT)$$

and clearly it is a linear functional. Moreover the other properties

$$\sum_{n=-\infty}^{+\infty} T\, s(-nT) = \sum_{m=-\infty}^{+\infty} T\, s(mT)$$
$$\sum_{n=-\infty}^{+\infty} T\, s(nT-kT) = \sum_{m=-\infty}^{+\infty} T\, s(mT) \quad ,$$

can be proved by an index change.

4.5. ⋆⋆⋆ [Sect. 4.3] Prove the identity

$$\int_{-\infty}^{+\infty} s(t)\, \mathrm{d}t = \int_0^{T_p} \sum_{n=-\infty}^{+\infty} s(t-nT_p)\, \mathrm{d}t\,,$$

which is a particular case of (4.12a) for $I_0 = \mathbb{R}$ and $P = \mathbb{Z}(T_p)$.

It results

$$\int_0^{T_p} \sum_{n=-\infty}^{+\infty} s(t-nT_p)\, \mathrm{d}t = \sum_{n=-\infty}^{+\infty} \int_0^{T_p} s(t-nT_p)\, \mathrm{d}t\ .$$

With the change of variable $t - nT = u$, one gets

$$\sum_{n=-\infty}^{+\infty} \int_0^{T_p} s(t-nT_p)\, \mathrm{d}t = \sum_{n=-\infty}^{+\infty} \int_{-nT_p}^{-nT_p+T_p} s(u)\, \mathrm{d}u = \int_{-\infty}^{+\infty} s(t)\, \mathrm{d}t$$

where we have considered that the sequence of intervals $[-nT_P\ ,\ -nT_P + T_P)$, $n \in \mathbb{Z}$, forms a *partition* of $\mathbb{R}$.

4.6. ⋆⋆ [Sect. 4.3] Using (4.6), explicitly write the integral of a signal $(t_1,t_2), \in \mathbb{R} \times \mathbb{Z}(d)$. Then, evaluate the integral of the signal $s(t_1,t_2) = \mathrm{e}^{-(t_1+t_2)}$ for $t_1,t_2 \geq 0$ and $s(t_1,t_2) = 0$ elsewhere.

It results

$$I = \int_{\mathbb{R}} \mathrm{d}t_1 \int_{\mathbb{Z}(d)} \mathrm{d}t_2\, s(t_1,t_2) = \int_{-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} d\, s(t_1, nd) \right] \mathrm{d}t_1 \,.$$

The signal is *separable* as $s(t_1,t_2) = s_1(t_1)\,s_2(t_2)$ and therefore

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} s_1(t_1)\,\mathrm{d}t_1\, d \sum_{n=-\infty}^{+\infty} s(nd) \\ &= \int_0^{\infty} \mathrm{e}^{-t_1}\,\mathrm{d}t_1\, d \sum_{n=0}^{\infty} \mathrm{e}^{-nd} = I_1 I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{\infty} \mathrm{e}^{-t_1}\,\mathrm{d}t_1 = 1 \\ I_2 &= \sum_{n=0}^{\infty} \mathrm{e}^{-nd} = \sum_{n=0}^{\infty} (\mathrm{e}^{-d})^n = \frac{1}{1-\mathrm{e}^{-d}} \,. \end{aligned}$$

4.7. $\star\star\star$ [Sect. 4.5] Prove that the inner product of an even real signal and an odd real signal on $\mathbb{Z}(T)/\mathbb{Z}(NT)$ is zero. *Hint*: consider the cases N even and N odd separately.

Let $x(t), y(t), t \in \mathbb{Z}(T)/\mathbb{Z}(NT)$, where $x(nT)$ is real and even and $y(nT)$ is real and odd. Then their inner product is given by

$$\langle x,y \rangle = \int_{\mathbb{Z}(T)/\mathbb{Z}(N)} \mathrm{d}t\, x(t)y(t) \,.$$

For $N = 2M+1$ odd, we can choose as cell of integration $\{-MT, \dots, 0, \dots, MT\}$. Hence

$$\langle x,y \rangle = T \sum_{n=-M}^{M} x(nT)\,y(nT) = T \left\{ x(0)y(0) + \sum_{n=-M}^{M} [x(nT)y(nT) + x(-nT)y(-nT)] \right\}$$

where $y(0) = 0$, $x(-nT)y(-nT) = -x(nT)y(nT)$. Hence, $\langle x,y \rangle = 0$.

For $N = 2M$ we choose the cell $\{-(M-1)T, \dots, 0, \dots, MT\}$. Note that $y(0) = 0$, but also $y(MT) = 0$. In fact, for the periodicity $y(MT) = y(MT - 2MT) = y(-MT) = -y(MT)$. Then proceeding as in the case of N odd, we prove that $\langle x,y \rangle = 0$.

4.8. ⋆⋆ [Sect. 4.5] The abstract definition of the *adjoint* of an operator $\mathcal{L}$ is formulated through the inner product as an operator $\mathcal{L}^*$ such that

$$\langle \mathcal{L}[x], y\rangle = \langle x, \mathcal{L}^*[y]\rangle \,, \qquad \forall x, y \in L_2(I) \,. \tag{S4.3}$$

It can be shown that this condition uniquely define $\mathcal{L}^*$ from $\mathcal{L}$ [7].

Prove condition (S4.3) through the kernels, where the kernel of $\mathcal{L}^*$ is given by (4.47).

Let $h(t,u)$ be the kernel of $\mathcal{L}$. Then, the first inner product is explicitly

$$\langle \mathcal{L}[x], y\rangle = \int_I \mathrm{d}t \left\{ \int_I \mathrm{d}u\, h(t,u)\, x(u) \right\} y^*(t) \,.$$

The second inner product is

$$\langle x, \mathcal{L}^*[y]\rangle = \int_I \mathrm{d}t\, x(t) \left\{ \int_I \mathrm{d}u\, h^*(t,u)\, y(t) \right\}^* \,.$$

The equality follows after use of (4.47) and interchanging t with u in the second inner product.

4.9. ⋆⋆⋆ [Sect. 4.5] Prove that the operators $\mathcal{L}_E = \frac{1}{2}(\mathcal{I} + \mathcal{I}_-)$ and $\mathcal{L}_O = \frac{1}{2}(\mathcal{I} - \mathcal{I}_-)$ are idempotent and orthogonal to each other.

The square of $\mathcal{L}_E$ is given by

$$\mathcal{L}_E^2 = \frac{1}{4}(\mathcal{I}^2 + \mathcal{I}_-^2 + \mathcal{I}\mathcal{I}_- + \mathcal{I}\mathcal{I}_-)$$

where, considering the meaning, $\mathcal{I}^2 = \mathcal{I}$, $\mathcal{I}_-^2 = \mathcal{I}$, and $\mathcal{I}\mathcal{I}_- = \mathcal{I}_-$. Hence, $\mathcal{L}_E^2 = \mathcal{L}_E$ and analogously $\mathcal{L}_O^2 = \mathcal{L}_O$. The proof of the orthogonality, $\mathcal{L}_E\,\mathcal{L}_O = 0$, is similar.

4.10. ⋆⋆ [Sect. 4.5] Prove the identity of the inner product in $L_2(I)$

$$\langle \mathcal{L}[x], \mathcal{K}[x]\rangle = \langle x, \mathcal{L}^*\mathcal{K}[x]\rangle$$

where $\mathcal{L}$ and $\mathcal{K}$ are arbitrary operators on $L_2(I)$ and $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. *Hint*: use the abstract definition of the adjoint reported in Problem 4.8.

The abstract definition of adjoint is (see Problem 4.8)

$$\langle \mathcal{L}[x], y\rangle = \langle x, \mathcal{L}^*[y]\rangle \,.$$

With $y = \mathcal{K}[x]$ we obtain the identity.

4.11. $\star$ [Sect. 4.6] Show that class (4.55) consists of orthogonal functions.

The check is based on property of Nth root of the unity and explicitly

$$\sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi mk/N}\,\mathrm{e}^{-\mathrm{i}2\pi nk/N} = \sum_{k=0}^{N-1} \left\{\mathrm{e}^{\mathrm{i}2\pi(m-n)/N}\right\}^k = \frac{1-\mathrm{e}^{\mathrm{i}2\pi(m-n)}}{1-\mathrm{e}^{\mathrm{i}\,2\pi\,(m-n)/N}} = 0\,, \qquad m \neq n\,.$$

4.12. $\star\star\nabla$ [Sect. 4.6] Show the orthogonality of the cardinal functions (4.54).

The check in time domain is difficult. In the frequency domain we can use the Parseval Theorem (see Sect. 5.7), which gives

$$\int_I \mathrm{d}t\; \gamma_m(t)\; \gamma_n^*(t) = \int_{\widehat{I}} \mathrm{d}f\; \Gamma_m(f)\; \Gamma_n^*(f)$$

where $\Gamma_m(f)$ is the Fourier transform of $\gamma_m(t)$

$$\gamma_m(t) \xrightarrow{\mathcal{F}} \Gamma_m(f) = \frac{1}{F}\;\mathrm{rect}\left(\frac{f}{F}\right)\;\mathrm{e}^{-\mathrm{i}2\pi mf/F}\,, \qquad f \in \mathbb{R}\,.$$

By evaluating the second integral we obtain

$$\frac{1}{F}\;\mathrm{sinc}\,(m-n) = \frac{1}{F}\;\delta_{mn}$$

and this proves the orthogonality.

4.13. $\star\star$ [Sect. 4.6] Show that cross–energies verify the inequality

$$0 \leq E_{xy}\,E_{yx} \leq E_x E_y\,.$$

Let $E_{xy} = \langle x, y\rangle$ be the inner product of $x(t)$ and $y(t)$, $t \in I$. Then

$$E_{xy}\,E_{yx} = |E_{xy}|^2 \geq 0\,.$$

Next, from the Schwartz inequality (4.41) one gets

$$E_{xy}\,E_{yx} = |(x,y)|^2 \leq \|x\|^2\;\|y\|^2 = E_x\,E_y$$

where the equality holds if and only if the signals x and y are proportional.

4.14. ⋆⋆⋆ [Sect. 4.6] Using the inequality for complex numbers

$$|z+z^*| \le 2|z| \tag{S4.4}$$

prove the Schwartz–Gabor inequality (4.42). Note that in (S4.4) the equality holds if z is real.

Using the norm and inner–product notations, we have to prove that

$$|\langle x,y\rangle + \langle y,x\rangle|^2 \le 4\|x\|^2\|y\|^2 \;. \tag{S4.5}$$

In fact, using (S4.4)

$$|\langle x,y\rangle + \langle y,x\rangle| \le 2\langle x,y\rangle \;. \tag{S4.6}$$

Hence, using the Schwartz inequality in (S4.6)

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2 \tag{S4.7}$$

we obtain (S4.5). The equality holds in (S4.6) if (x,y) is real and in (4.7) if $y(t) = K\,x(t)$. Combination of these two conditions state that K is real.

4.15. ⋆ [Sect. 4.6] Formulate a basis on a finite group K/P starting from the impulse $\delta_{K/P}$.

If $I = K/P$ is a finite group, the impulse is given by

$$\delta_{K/P}(t) = \begin{cases} 1/\mathrm{d}(K) & t \in P \\ 0 & t \notin P \end{cases} \tag{S4.8}$$

and has periodicity P. Then using (S4.8) we see by inspection that the functions

$$\delta_{K/P}(t-u)\,, \qquad u \in [K/P) \tag{S4.9}$$

form an orthogonal basis for the class $L_2(K/P)$.

4.16. ⋆⋆ [Sect. 4.8] Find the extension and duration of the signal

$$x(t) = \sum_{n=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t-nT_p}{dT_p}\right)\,, \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)$$

where d is a positive real number. Discuss the result as a function of d.

The extension of the nth term is

$$e_n = (-\tfrac{1}{2}dT_p - nT_p, \tfrac{1}{2}dT_p + nT_p) = e_0 + nT_p$$

where $e_0 = (-\frac{1}{2}dT_p, \frac{1}{2}dT_p)$. Then, the extension of $x(t)$ is the periodic interval

$$e(x) = e_0 + \mathbb{Z}(T_p)\ .$$

For $d < 1$ the intervals are disjoint. For $d \geq 1$ the intervals overlap and $e(x) = \mathbb{R}$.

4.17. $\star$ [Sect. 4.9] Prove the following relations for the minimal extension of the product and the sum of two signals

$$e_0(xy) = e_0(x) \cap e_0(y)\,,$$
$$e_0(x+y) \subset e_0(x) \cup e_0(y)\ .$$

We recall that the *minimal extension* of a signal $s(t)$, $t \in I$ is given by the *support* of $s(t)$, that is, the set $e_0(s) = \{t \in I \mid s(t) \neq 0\}$. Then, the first relation follows from the fact that $s(t) = x(t)y(t) \neq 0$, if and only if both $x(t) \neq 0$ and $y(t) \neq 0$. The second is proved by noting that if $x(t) = 0$ *and* $y(t) = 0$ for a given t, then $s(t) = x(t) + y(t) = 0$. Then, for the complement $\bar{e}_0$ of the minimal extension we have

$$\bar{e}_0(x+y) \supset \bar{e}_0(x) \cup \bar{e}_0(y)\ .$$

4.18. $\star\star$ [Sect. 4.9] The signal $x(t)$ and $y(t)$, defined on $\mathbb{R}/\mathbb{Z}(10)$, have the following extensions

$$e(x) = [0,1) + \mathbb{Z}(10)\,, \qquad e(y) = [0,2) + \mathbb{Z}(10)\ .$$

Find the extension of their convolution.

From (4.72) it results

$$e(x * y) = [0,3) + \mathbb{Z}(10) \quad .$$

4.19. $\star\star$ [Sect. 4.9] Consider the *self–convolution* $s(t) = x * x(t)$, $t \in \mathbb{R}/\mathbb{Z}(T_p)$ of the signal

$$x(t) = \sum_{n=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t - nT_p}{dT_p}\right), \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)\ .$$

Find the extension as a function of the parameter d.

We have found that the extension of $e(x)$ is given by $e_0+\mathbb{Z}(T_p)$, where $E_0 = (-\frac{1}{2}dT_p, \frac{1}{2}dT_p)$. Then, applying the rule on the extension of convolution

$$\begin{aligned}e(s) =& e(x)+e(x) = [e_0+\mathbb{Z}(T_p)]+[e_0+\mathbb{Z}(T_p)]\\ =&(e_0+e_0)+\mathbb{Z}(T_p)\end{aligned}$$

where $e_0+e_0 = (-dT_p, dT_p)$. Hence,

$$e(s) = [-dT_p, dT_p]+\mathbb{Z}(T_p) \quad .$$

Clearly, for $d \geq \frac{1}{2}$ it results $e(s) = \mathbb{R}$.

4.20. $\star\star$ [Sect. 4.13] Prove that (4.122), where $\mathcal{P}_i$ form a system of M orthogonal projectors, defines an M–ary reflector, that is an operator with the property $\mathcal{B}^M = \mathcal{I}$. *Hint*: first evaluate $\mathcal{B}^2$ using the orthogonality of the $\mathcal{P}_i$, then evaluate $\mathcal{B}^3 = \mathcal{B}^2\,\mathcal{B}$, etc.

We first evaluate $\mathcal{B}^2$, which is given by

$$\mathcal{B}^2 = \sum_{i,j=0}^{M-1} \mathcal{P}_i\,\mathcal{P}_j\,W_M^{-(i+j)} = \sum_{i=0}^{M-1} \mathcal{P}_i\,W_M^{-2i}$$

where we have used the properties $\mathcal{P}_i\,\mathcal{P}_j = \delta_{ij}\,\mathcal{P}_i^2 = \delta_{ij}\,\mathcal{P}_i$. Analogously

$$\mathcal{B}^3 = \mathcal{B}^2\,\mathcal{B} = \sum_{i,j=0}^{M-1} \mathcal{P}_i\,\mathcal{P}_j\,W_M^{-(2i+j)} = \sum_{i=0}^{M-1} \mathcal{P}_i\,W_M^{-3i} \,.$$

Proceeding in this way we obtain

$$\mathcal{B}^M = \sum_{i=0}^{M-1} \mathcal{P}_i\,W_M^{-Mi} = \sum_{i=0}^{M-1} \mathcal{P}_i = \mathcal{I}$$

where $W_M^{-Mi} = 1$ and we have used the second of (4.118).

Problems of Chapter 5

5.1. $\star$[Sect. 5.2] Write and prove the orthogonality conditions (5.11) for $I = \mathbb{Z}(T)/\mathbb{Z}(NT)$.

Considering that

$$I = \mathbb{Z}(T)/\mathbb{Z}(T_p)\ ,\ \widehat{I} = \mathbb{Z}(F)/\mathbb{Z}(F_p) \quad \text{with} \quad T_p = NT\ ,\ F = 1/T_p\ ,\ F_p = NF$$

and letting $t = nT$ and $f = kF$, the orthogonality conditions (5.11) become

$$\sum_{n=0}^{N-1} T\, \mathrm{e}^{\mathrm{i}2\pi kn/N} = \delta_{\mathbb{Z}(F)/\mathbb{Z}(NF)}(kF)$$
$$\sum_{k=0}^{N-1} F\, \mathrm{e}^{\mathrm{i}2\pi kn/N} = \delta_{\mathbb{Z}(T)/\mathbb{Z}(NT)}(nT)\ .$$

Next, recalling that

$$\delta_{\mathbb{Z}(F)/\mathbb{Z}(NF)}(kF) = \begin{cases} NT & k \in \mathbb{Z}(N) \\ 0 & k \notin \mathbb{Z}(N) \end{cases}$$

it is sufficient to prove the relation

$$\sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi kn/N} = \begin{cases} N & k \in \mathbb{Z}(N) \\ 0 & k \notin \mathbb{Z}(N)\ . \end{cases}$$

Now, if k is multiple of N, all the summation terms are unitary and their sum gives N. If k is not a multiple of N, we can use the identity

$$\sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z}\ ,$$

which gives 0 for $z = \mathrm{e}^{\mathrm{i}2\pi kn/N}$. The proof of the second relation is similar.

5.2. $\star$[Sect. 5.3] Show that the 1D LCA groups of $\mathbb{R}$, i.e., $\mathbb{R}$, $\mathbb{Z}(T)$ and $\mathbb{O}$, verify relation (5.21).

The check is an immediate consequence of (5.30). For instance, considering that $\mathbb{Z}(T) \subset \mathbb{R}$, where $\mathbb{Z}(T)^\star = \mathbb{Z}(\frac{1}{T})$ and $\mathbb{R}^\star = \mathbb{O}$, it follows that $\mathbb{Z}(T)^\star \supset \mathbb{R}^\star$.

5.3. $\star\star$ [Sect. 5.3] Starting from the general definition (5.18) of reciprocal, prove relation (5.21).

We have to prove that, if $K \subset J$, then $K^\star \supset J^\star$ with

$$\begin{aligned} K^\star &= \{f \mid ft \in \mathbb{Z}\,,\ \forall t \in K\} \\ J^\star &= \{f \mid ft \in \mathbb{Z}\,,\ \forall t \in J\} = \{f \mid ft \in \mathbb{Z}\,,\ \forall t \in K \text{ and } \forall t \in (J \backslash K)\}\,, \end{aligned}$$

where $J \backslash K$ denotes the set of the points J that do not belong to K. Hence

$$\begin{aligned} f_0 \in J^\star \quad &\Longleftrightarrow \quad f_0 t \in \mathbb{Z}\,,\ \forall t \in K \ \text{ and } \ \forall t \in (J \backslash K) \\ &\Longrightarrow \quad f_0 t \in \mathbb{Z}\,,\ \forall t \in K \quad \Longleftrightarrow \quad f_0 \in K^\star\,. \end{aligned}$$

5.4. $\star\nabla$ [Sect. 5.3] Find the reciprocals of the following groups

$$\mathbb{R} + \mathbb{Z}(2)\,, \qquad \mathbb{Z}(6) + \mathbb{Z}(15)\,, \qquad \mathbb{Z}(12) \cap \mathbb{Z}(40)\,, \qquad \mathbb{O} \cap \mathbb{Z}(3)\,.$$

From (3.72) and (5.30) one gets

$$\begin{aligned} &\mathbb{R} + \mathbb{Z}(2) = \mathbb{R} && \xrightarrow{\star} \mathbb{O} \\ &\mathbb{Z}(6) + \mathbb{Z}(15) = \mathbb{Z}(3) && \xrightarrow{\star} \mathbb{Z}(1/3) \\ &\mathbb{Z}(12) \cap \mathbb{Z}(40) = \mathbb{Z}(120) && \xrightarrow{\star} \mathbb{Z}(1/120) \\ &\mathbb{O} \cap \mathbb{Z}(3) = \mathbb{O} && \xrightarrow{\star} \mathbb{R}\,. \end{aligned}$$

5.5. $\star\star\star$ [Sect. 5.3] Prove that two *rationally comparable* groups J and K satisfy the relations

$$(J + K)^\star = J^\star \cap K^\star\,, \qquad (J \cap K)^\star = J^\star + K^\star\,.$$

In the class of the groups of $\mathbb{R}$ the reciprocal is given by (5.30). Then, for the relation $(J+K)^\star = J^\star \cap K^\star$ we find

$$\begin{aligned} f_0 \in (J+K)^\star \quad &\Longleftrightarrow \quad f_0 t \in \mathbb{Z}\,,\ \forall t \in (J+K) \\ &\Longleftrightarrow \quad f_0 t_1 + f_0 t_2 \in \mathbb{Z}\,,\ \forall t_1 \in J\,,\ \forall t_2 \in K \\ &\Longleftrightarrow \quad f_0 t_1 \in \mathbb{Z}\,,\ \forall t_1 \in J \ \text{ and } \ f_0 t_2 \in \mathbb{Z}\,,\ \forall t_2 \in K \\ &\Longleftrightarrow \quad f_0 \in (J^\star \cap K^\star)\,. \end{aligned}$$

The second relation is obtained by applying the first relation to the pair $J^\star$ e $K^\star$. In fact

$$(J^\star + K^\star)^\star = (J^\star)^\star \cap (K^\star)^\star = J \cap K$$

hence

$$(J \cap K)^\star = J^\star + K^\star \,.$$

For the multidimensional groups of $\mathbb{R}^m$ the proof is similar. Recall that in the mD case the product ft means $f_1 t_1 + \cdots + f_m t_m$.

5.6. ⋆⋆ [Sect. 5.4] For any $n \in \mathbb{Z}$, find the result of the application of the operator $\mathcal{F}^n$ on a signal $s(t)$, $t \in I$ ($\mathcal{F}^n$ denotes n applications of the operator $\mathcal{F}$).

From the graph (5.36) we have

$$\mathcal{F}^1 = \mathcal{F}\,, \qquad \mathcal{F}^2 = \mathcal{I}_-\,, \qquad \mathcal{F}^3 = \mathcal{F}_-\,, \qquad \mathcal{F}^4 = \mathcal{I}$$

where

- $\mathcal{I}$ is the *identity* operator,
- $\mathcal{I}_-$ is the *reflector* operator, giving the reverse signal $s(-t)$,
- $\mathcal{F}_-$ gives the reverse FT $S(-f)$.

For $\mathcal{F}^5$ we find $\mathcal{F}^4\mathcal{F} = \mathcal{I}\mathcal{F} = \mathcal{F}$, etc. For a generic $n \in \mathbb{Z}$ we can use the periodicity of period 4, e.g., $\mathcal{F}^7 = \mathcal{F}^3 = \mathcal{F}$, $\mathcal{F}^{-7} = \mathcal{F}^{-8}\mathcal{F} = \mathcal{F}$.

5.7. ⋆ [Sect. 5.5] Prove Rule 11a of Tab. 5.2 using Rule 10a.

Assume as known the rule on convolution (Rule 6a of Tab. 5.2). Then, Rule 11a

$$x * y_-^*(t) \xrightarrow{\mathcal{F}} X(f)\,\mathcal{F}\{y^*(-t)\}$$

can be proved as follows. The FT of $y^*(-t)$ is given by (see Rule 4b)

$$\mathcal{F}\{y^*(-t)\} = \int_I \mathrm{d}t\, y^*(-t)\,\mathrm{e}^{-\mathrm{i}2\pi f t} = \left(\int_I \mathrm{d}t\, y(-t)\,\mathrm{e}^{\mathrm{i}2\pi f t}\right)^* = Y^*(f)\,.$$

Hence Rule 11a

$$x * y_-^*(t) \xrightarrow{\mathcal{F}} X(f)\,Y^*(f)$$

is proved.

5.8. [Sect. 5.5] Write the Poisson summation formula with $I = \mathbb{Z}(T_0)$, $U = \mathbb{Z}(NT_0)$ and $P = \mathbb{O}$.

The general formula (5.46) requires to specify the groups I_0, U_0 and P and the corresponding reciprocals. In the specific case we have

$$I_0 = \mathbb{Z}(T_0)\,, \qquad U_0 = \mathbb{Z}(NT_0)\,, \qquad P = \mathbb{Z}(\infty) = \mathbb{O}$$

and

$$I_0^\star = \mathbb{Z}(F_0)\,, \qquad U_0^\star = \mathbb{Z}\left(\frac{1}{N}F_0\right)\,, \qquad P^\star = \mathbb{R} \quad (F_0 = 1/T_0)\ .$$

Therefore the arguments of the summations are

$$u = nNT_0 \in \mathbb{Z}(NT_0)\,, \qquad \lambda = kF \in \mathbb{Z}(F)/\mathbb{Z}(F_0) \quad \text{with } F = F_0/N$$

and considering that $\mathrm{d}(U_0) = NT_0$, we obtain

$$NT_0 \sum_{n=-\infty}^{+\infty} s(nNT_0) = \sum_{k=0}^{N-1} S(kF)$$

where the signal $s(t)$ is defined in $\mathbb{Z}(T_0)$ and the transform $S(f)$ on $\mathbb{R}/\mathbb{Z}(F_0)$.

5.9. $\star\star$[Sect. 5.5] Evaluate the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{1+an^2}$$

using Poisson's formula (5.46). *Hint*: consider $s(t) = \exp(-\alpha\,|t|)$, $t \in \mathbb{R}$ as the signal.

Relation (5.46) states the equality between the sum of two series. Then, if we are not able to calculate the first one, we try to calculate the second one, or vice versa. Now, using the Fourier pair

$$s(t) = \mathrm{e}^{-\alpha|t|} \xrightarrow{\mathcal{F}} S(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

the Poisson formula gives

$$\sum_{n=-\infty}^{+\infty} T\,\mathrm{e}^{-\alpha|n|T} = \sum_{k=-\infty}^{+\infty} \frac{2\alpha}{\alpha^2 + (2\pi k F_p)^2} = \frac{2}{\alpha} \sum_{k=-\infty}^{+\infty} \frac{1}{1+b^2k^2}$$

where $b = 2\pi F_p/\alpha$.

The last summation is of very difficult evaluation, while the first is easy because it is related to the geometric series. We get

$$\alpha T\left(2\sum_{n=0}^{\infty}\mathrm{e}^{-\alpha Tn}-1\right)=2\left(2\sum_{k=0}^{\infty}\frac{1}{1+b^2k^2}-1\right)$$

where $\alpha T=2\pi/b$. Hence, recalling that $\sum_{n=0}^{\infty}A^n=1/(1-A)$, for $|A|<1$, one obtains

$$\sum_{k=0}^{\infty}\frac{1}{1+b^2k^2}=\frac{\pi}{4b}\left(\frac{2}{1-\exp\left(-2\pi/b\right)}-1\right)+\frac{1}{2}\,.$$

5.10. $\star$[Sect. 5.5] Evaluate the Fourier transform of a periodic signal defined on $\mathbb{R}$, instead of $\mathbb{R}/\mathbb{Z}(T_p)$, starting from its Fourier series expansion.

See Chap. 2, eq. (2.67). In alternative, observe that an $\mathbb{R}\to\mathbb{R}/\mathbb{Z}(T_p)$ down–periodization is involved and apply the Duality Theorem of Chap.6.

5.11. $\star$[Sect. 5.6] Find the symmetries of the signal

$$s(t)=\mathrm{i}2\pi t\;1(t)\;\mathrm{e}^{-\alpha t}\,,\qquad t\in\mathbb{R}\,.$$

Considering the presence of step function, even, odd, Hermitian and anti–Hermitian symmetry are excluded. Then, we consider real and imaginary symmetries. Letting $\alpha=a+\mathrm{i}b$, the signal is written in the form

$$\begin{aligned}s(t)&=\mathrm{i}2\pi t\;1(t)\mathrm{e}^{at}[\cos(bt)+\mathrm{i}\sin(bt)]\\&=-2\pi t\;1(t)\mathrm{e}^{at}\sin(bt)+\mathrm{i}2\pi t\;1(t)\mathrm{e}^{at}\cos(bt)\end{aligned}$$

and we see that the signal become *imaginary* when $b=0$.

5.12. $\star\star$[Sect. 5.6] Decompose the signal of the previous problem into symmetric components, according to Symmetries 1, 2 and 3.

Symmetries 1

$$\begin{aligned}s_e(t)&=\frac{1}{2}\Big[s(t)+s(-t)\Big]=\mathrm{i}\pi t\;\operatorname{sgn}t\;\mathrm{e}^{-\alpha|t|}\\s_o(t)&=\frac{1}{2}\Big[s(t)-s(-t)\Big]=\mathrm{i}\pi t\;\mathrm{e}^{-\alpha|t|}\end{aligned}$$

where sgn$(\cdot)$ is the signum function. Symmetries 2 Letting $\alpha = a + \mathrm{i}b$

$$\Re\left[s(t)\right] = \frac{1}{2}\Big[s(t) + s^*(t)\Big] = 2\pi t\ 1(t)\ \mathrm{e}^{-at}\ \sin\Big(bt\Big)$$
$$\mathrm{i}\Im\left[s(t)\right] = \frac{1}{2}\Big[s(t) - s^*(t)\Big] = \mathrm{i}2\pi t\ 1(t)\ \mathrm{e}^{-at}\ \cos\Big(bt\Big)\,.$$

Symmetries 3

$$s_h(t) = \frac{1}{2}\Big[s(t) + s^*(-t)\Big] = \mathrm{i}\pi t\ \mathrm{e}^{-a|t|-\mathrm{i}bt}$$
$$s_{ah}(t) = \frac{1}{2}\Big[s(t) - s^*(-t)\Big] = \mathrm{i}\pi t\ \mathrm{e}^{-a|t|-\mathrm{i}bt}\ \mathrm{sgn}\,(t)\,.$$

5.13. ⋆⋆⋆ [Sect. 5.6] Decompose the signal

$$s(t) = \mathrm{sinc}_N\left(\frac{t-t_0}{T_p}\right), \qquad t \in \mathbb{R}/\mathbb{Z}(5T_p)\,,\ N = 5$$

into even and odd components, $s_e(t)$ and $s_o(t)$.

We get

$$s_e(t) = \frac{1}{2}\Big[s(t) + s(-t)\Big] = \frac{1}{2}\left[\mathrm{sinc}_5\left(\frac{t-t_0}{T_p}\right) + \mathrm{sinc}_5\left(\frac{t+t_0}{T_p}\right)\right]$$
$$s_o(t) = \frac{1}{2}\Big[s(t) - s(-t)\Big] = \frac{1}{2}\left[\mathrm{sinc}_5\left(\frac{t-t_0}{T_p}\right) - \mathrm{sinc}_5\left(\frac{t+t_0}{T_p}\right)\right] \quad .$$

5.14. ⋆⋆⋆∇ [Sect. 5.6] Calculate the Fourier transforms of the signals $s(t)$, $s_e(t)$ and $s_o(t)$ of the previous problem. Then, check that they verify the corresponding symmetries, e.g., $S_e(f)$ must be real and even.

Assume as known the pair

$$\mathrm{sinc}_5\left(\frac{t}{T_p}\right),\ t \in \mathbb{R}/\mathbb{Z}(5T_p) \quad \xrightarrow{\mathcal{F}} \quad T_p\ \mathrm{rect}\,(T_p f)\,,\ f \in \mathbb{Z}(F)$$

where $5FT_p = 1$. Then we get

$$s(t) = \mathrm{sinc}_5\left(\frac{t-t_0}{T_p}\right) \xrightarrow{\mathcal{F}} S(f) = T_p\ \mathrm{rect}\,(fT_p)\ \mathrm{e}^{-\mathrm{i}2\pi f t_0}\,,\ f \in \mathbb{Z}(F)$$

where $S(f)$ has the Hermitian symmetry. Moreover

$$\begin{aligned} s_e(t) \quad \xrightarrow{\mathcal{F}} \quad S_e(f) &= \frac{1}{2} T_p \ \mathrm{rect}\,(T_p f)\ [\mathrm{e}^{-\mathrm{i}2\pi f t_0} + \mathrm{e}^{\mathrm{i}2\pi f t_0}] \\ &= T_p \ \mathrm{rect}\,(T_p f)\ \cos\,(2\pi f t_0)\ ,\ f \in \mathbb{Z}(F) \end{aligned}$$

where $S_e(f)$ is real and even. Analogously

$$s_o(t) \quad \xrightarrow{\mathcal{F}} \quad S_o(f) = \mathrm{i}T_p \ \mathrm{rect}\,(T_p f)\ \sin\,(2\pi f t_0)\ ,\ f \in \mathbb{Z}(F)$$

where $S_o(f)$ is imaginary and odd.

5.15. ⋆[Sect. 5.7] Prove the following rule on the extension of the correlation

$$e(c_{xy}) = e(x) + [-e(y)]\ .$$

We recall that the correlation $c_{xy}(t)$ is equal to the convolution of x with y_-^*, where y_- is the reverse version of y, that is, $y_-(t) = y(-t)$ (see (5.56). Then, the rule follows from the rule on the extension of the convolution.

5.16. ⋆⋆[Sect. 5.7] Compute the correlation of the signals

$$x(t) = A_0 \ \mathrm{rect}\,((t/T))\ , \qquad y(t) = B_0 \ \exp\,(-|t|/T)\ , \qquad t \in \mathbb{R}\ .$$

Using definition (5.56) one gets

$$c_{xy}(\tau) = \int_{-\infty}^{+\infty} x(u-\tau)\, y^*(u)$$

where $y(t)$ is real and even. Then

$$c_{xy}(\tau) = \int_{-\infty}^{+\infty} B_0 \exp\left(\frac{-|u|}{T}\right) A_0 \,\mathrm{rect}\left(\frac{\tau-u}{T}\right)\ \mathrm{d}u\ .$$

Subdividing the integral appropriately, one gets

$$c_{xy}(\tau) = \begin{cases} \int_{\tau-metT}^{\tau+\frac{1}{2}T} A_0 B_0 \exp\left(\frac{u}{T}\right)\ \mathrm{d}u & \tau \le -\frac{1}{2}T \\ \int_{\tau-\frac{1}{2}T}^{0} A_0 B_0 \exp\left(\frac{u}{T}\right)\ \mathrm{d}u + \int_0^{\tau+\frac{1}{2}T} A_0 B_0 \exp\left(-\frac{u}{T}\right)\ \mathrm{d}u & -\frac{1}{2}T < \tau < \frac{1}{2}T \\ \int_{\tau-\frac{1}{2}T}^{0} A_0 B_0 \exp\left(-\frac{u}{T}\right)\ \mathrm{d}u & \tau > \frac{1}{2}T\ . \end{cases}$$

Finally, the explicit evaluation gives

$$c_{xy}(\tau) = \begin{cases} 2\exp\left(-\frac{|\tau|}{T}\right) A_0 B_0 T\left(\frac{1}{2}\right) & |\tau| \geq \frac{1}{2}T \\ 2A_0 B_0 T\left[1 - \exp\left(-\frac{1}{2}\right)\left(\frac{\tau}{T}\right)\right] & |\tau| < \frac{1}{2}T \ . \end{cases}$$

5.17. ⋆⋆⋆[Sect. 5.7] Calculate the energy spectral density of the two signals of the previous problem and verify their symmetries.

The Fourier transforms of the signals are respectively

$$X(f) = A_0 T \ \mathrm{sinc}(fT) , \qquad Y(f) = \frac{2B_0 T}{1 + 4\pi^2 f^2 T^2} \ .$$

Then, using (5.59), one gets

$$C_{xy}(f) = X(f)Y^*(f) = \frac{2A_0 B_0 T^2}{1 + (2\pi f T)^2} \,\mathrm{sinc}(fT) \ .$$

Considering the symmetry rules of Tab. 5.3, we find that the cross spectral density $C_{xy}(f)$ is *real* and *even*.

5.18. ⋆[Sect. 5.8] Show that by substituting the expression of S_k given by the first of (5.70) in in the second, we actually obtain s_n.

To avoid a conflict in symbolism, we rewrite the inverse FT (5.70b) as

$$\tilde{s}_m = \frac{1}{N} \sum_{k=0}^{N-1} S_k W_N^{km} \ .$$

Then, replacing the equation of S_k given by (5.70a), one gets

$$\tilde{s}_m = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} s_n W_N^{-kn} W_N^{km} = \frac{1}{N} \sum_{n=0}^{N-1} s_n \sum_{k=0}^{N-1} W_N^{-k(n-m)} ,$$

where the last summation gives N for $n = m$ and 0 for $n \neq m$, that is, $N \delta_{nm}$. Hence

$$\tilde{s}_m = \sum_{n=0}^{N-1} s_n \delta_{nm} = s_m \ .$$

5.19. ⋆⋆ [Sect. 5.8] Starting from (5.65) we can "prove" that *all continuous–time signals are constant valued*. Indeed

$$e^{i2\pi ft} = \left(e^{i2\pi}\right)^{ft} = 1^{ft} = 1 \,.$$

Therefore

$$s(t) = \int_{-\infty}^{+\infty} S(f)\,df = \text{area}(S) \quad !!!$$

Try to explain this paradox.

The mistake is in the first passage. In fact, $e^{i2\pi a} = 1$ if and only if $a \in \mathbb{Z}$, whereas in the exponential $ft \in \mathbb{R}$.

5.20. ⋆ [Sect. 5.8] Calculate the Fourier transform of the signal

$$t\, e^{-\alpha t}\, 1(t)\,, \qquad t \in \mathbb{R}\,,\ \alpha > 0$$

and then apply the Symmetry Rule.

Integrating by parts one gets

$$S(f) = \int_0^{\infty} t\, e^{-(\alpha + i2\pi f)t}\, dt = \frac{1}{(\alpha + i2\pi f)^2}\,, \qquad f \in \mathbb{R}\,.$$

Then the application of Symmetry Rule (5.37) yields

$$s_1(t) = \frac{1}{(\alpha + i2\pi t)^2}\,,\ t \in \mathbb{R} \xrightarrow{\mathcal{F}} S_1(f) = -f\, e^{af}\, 1(-f)\,,\ f \in \mathbb{R}\,.$$

5.21. ⋆⋆ [Sect. 5.8] Calculate the Fourier transform of the discrete signal

$$e^{-\alpha|t|}\,, \qquad t \in \mathbb{Z}(T)\,,\ \alpha > 0$$

and then apply the Symmetry Rule.

We apply the definition of FT on $I = \mathbb{Z}(T)$, which is given by a summation. Then, we subdivide the summation in such a way that the obsolete value becomes explicit, we get

$$S(f) = \sum_{n=0}^{+\infty} T\, e^{-\alpha nT}\, e^{-i2\pi fnT} + \sum_{n=-\infty}^{-1} T\, e^{\alpha nT}\, e^{-i2\pi fnT}\,.$$

Then, the substitutions $z_1 = \mathrm{e}^{-\alpha T - \mathrm{i}2\pi f T}$ and $z_2 = \mathrm{e}^{\alpha T - \mathrm{i}2\pi f T}$ give

$$S(f) = \sum_{n=0}^{+\infty} T\, z_1^n + \sum_{n=0}^{\infty} T\, z_2^{-n} - T\ .$$

Considering the sum of the geometric series, one obtain

$$S(f) = \frac{T}{1-z_1} + \frac{T}{1-z_2^{-1}} - T = T\ \frac{1-\mathrm{e}^{-2\alpha T}}{1-2\mathrm{e}^{-\alpha T}\cos 2\pi f T + \mathrm{e}^{-2\alpha T}}\ , \qquad f \in \mathbb{R}/\mathbb{Z}(F_p)\,, \qquad F_p = 1/T\ . \tag{S5.1}$$

Note that the two series are convergent because $|z_1|, |z_2| < 1$. Hence, we have obtained the Fourier pair

$$s(t) = \mathrm{e}^{-\alpha|t|}\ ,\ t \in \mathbb{Z}(T) \quad \xrightarrow{\mathcal{F}} \quad S(f)\ ,\ f \in \mathbb{R}/\mathbb{Z}(F_p)$$

with $S(f)$ given by (5.1). The symmetric pair is $S(t)$, $t \in \mathbb{R}/\mathbb{Z}(F_p)$

$$S(t)\ ,\ t \in \mathbb{R}/\mathbb{Z}(F_p) \quad \xrightarrow{\mathcal{F}} \quad s(-f)\ ,\ f \in \mathbb{Z}(T)$$

with $S(t)$ given by (S5.1) and $s(-f) = \mathrm{e}^{-\alpha|f|}$.

Remark. In general, after the application of the Symmetry Rule it is convenient to change some symbols. In the specific case, in the symmetric pair the *signal domain* turns out to be $\mathbb{R}/\mathbb{Z}(F_p)$, which is a symbol consolidated for the frequency domain and it will be convenient to replace F_p with T_p.

5.22. ⋆⋆ [Sect. 5.8] Referring to 3. of Tab. 5.4, define the *compatibility conditions* of a sinusoidal signal on the domains $\mathbb{Z}(T)$ and $\mathbb{Z}(T)/\mathbb{Z}(NT)$. In particular, determine in which domains the frequency $f_0 = \frac{7}{39}\,\frac{1}{T}$ is compatible.

The discrete signal $s(t) = \cos(2\pi f_0 t)$, $t \in \mathbb{Z}(T)$ is periodic if $f_0\, T = T/T_0$ is rational, $T_0 = 1/f_0$ being the period of the continuous sinusoids. Only in this case the signal can be represented on the proper quotient group $\mathbb{Z}(T)/\mathbb{Z}(NT)$. If $f_0 T$ is irrational, the representation must be done on $\mathbb{Z}(T)$.

When T/T_0 is rational, the minimum period T_p is given by the *least common multiple* between T_0 and T, as stated by

$$\mathbb{Z}(T_p) = \mathbb{Z}(T_0) \cap \mathbb{Z}(T)\ . \tag{S5.2}$$

In the specific case where $f_0 = \frac{7}{39}\,\frac{1}{T}$ one finds

$$\mathbb{Z}(T_p) = Z\left(\tfrac{39}{7}\,T\right) \cap \mathbb{Z}(T)$$

whose solution is $T_p = 39T$.

In conclusion, no compatibility is required for the representation on $\mathbb{Z}(T)$, while on $\mathbb{Z}(T)/\mathbb{Z}(T_p)$ the period T_p must be chosen according to (S5.2).

5.23. $\star$[Sect. 5.8] Starting from the Fourier pair (5.73), evaluate the Fourier transform of the even and odd parts of $s(t)$. Note in particular that $s_e(t) = \frac{1}{2}\exp(-\alpha|t|)$.

The pair is

$$s(t) = 1(t)\,\mathrm{e}^{-\alpha t}\,,\ t \in \mathbb{R} \xrightarrow{\mathcal{F}} S(f) = \frac{1}{\alpha + \mathrm{i}2\pi f}\,,\ f \in \mathbb{R}\,.$$

Considering the rule $s(-t) \xrightarrow{\mathcal{F}} S(-f)$, for the even part we find

$$s_e(t) = \frac{1}{2}[s(t) + s(-t)] \xrightarrow{\mathcal{F}} S_e(f) = \frac{1}{2}[S(f) + S(-f)]$$

and hence

$$S_e(f) = \frac{1}{2}[S(f) + S(-f)] = \frac{1}{2}\left[\frac{1}{\alpha + \mathrm{i}2\pi f} + \frac{1}{\alpha - \mathrm{i}2\pi f}\right] = \frac{\alpha}{\alpha^2 + (2\pi f)^2}\,.$$

Analogously for the odd part we find

$$S_o(f) = \frac{1}{2}[S(f) - S(-f)] = \frac{1}{2}\left[\frac{1}{\alpha + \mathrm{i}2\pi f} - \frac{1}{\alpha - \mathrm{i}2\pi f}\right] = -\frac{\mathrm{i}2\pi f}{\alpha^2 + (2\pi f)^2}\,.$$

5.24. $\star$[Sect. 5.8] Evaluate the area of the signal

$$s(t) = A_0\ \mathrm{sinc}\,(F_0 t)\,, \qquad t \in \mathbb{R}\,.$$

The evaluation in the time domain requires to use tables of integrals, whereas in the frequency domain it is immediate using the rule area$\,(s) = S(0)$

$$\text{area}\,(s) = S(0) = \frac{A_0}{F_0}\ \mathrm{rect}\left(\frac{f}{F_0}\right)\bigg|_{f=0} = \frac{A_0}{F_0}\,.$$

5.25. $\star$[Sect. 5.8] Evaluate the Fourier transform of the *even part* and *odd part* of the signal

$$s(t) = 1(t)\,t\,\mathrm{e}^{-\alpha t}\,, \qquad t \in \mathbb{R}\,,\ \alpha > 0\,.$$

We start from the Fourier pair

$$s(t) = 1(t)\, t\, \mathrm{e}^{-at}\,, \quad t \in \mathbb{R} \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{a + \mathrm{i}2\pi f}\,, \quad f \in \mathbb{R}\,.$$

Then, from (5.53) one gets

$$s_e(t) \xrightarrow{\mathcal{F}} \frac{a}{a^2 + 4\pi^2 f^2}\,, \qquad s_o(t) \xrightarrow{\mathcal{F}} \frac{-\mathrm{i}2\pi f}{a^2 + 4\pi^2 f^2}\,.$$

Remark. This problem is identical to Problem 5.23, but the solution follows a different approach.

5.26. $\star$[Sect. 5.8] Evaluate amplitude and phase of the Fourier transform of the signals

$$s_1(t) = A_0 \,\mathrm{rect}\left(\frac{t - t_0}{T}\right), \; t \in \mathbb{R}, \qquad s_2(t) = A_0 \,\mathrm{rep}_{10}\left[\mathrm{rect}\left(\frac{t-2}{3}\right)\right], \; t \in \mathbb{R}/\mathbb{Z}(10)\,.$$

The Fourier pairs are

$$\begin{aligned} s_1(t) = A_0 \,\mathrm{rect}\left(\frac{t-t_0}{T}\right) &\xrightarrow{\mathcal{F}} S_1(f) = \frac{A_0}{F} \,\mathrm{sinc}\left(\frac{f}{F}\right) \mathrm{e}^{-\mathrm{i}2\pi f t_0} \\ t \in \mathbb{R} \qquad & \qquad F = \frac{1}{T}\,,\; f \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} s_2(t) = A_0 \,\mathrm{rep}_{10}\left[\mathrm{rect}\left(\frac{t-2}{3}\right)\right] &\xrightarrow{\mathcal{F}} S_2(f) = 3A_0 \,\mathrm{sinc}(3f)\, \mathrm{e}^{-\mathrm{i}4\pi f} \\ t \in \mathbb{R}/\mathbb{Z}(10) \qquad & \qquad f \in Z(1/10)\,. \end{aligned}$$

The amplitude and the phase of the FTs are given by

$$\begin{aligned} |S_1(f)| &= (A_0/F)\, |\mathrm{sinc}\,(f/F)|\,, & \arg[S_1(f)] &= -2\pi f t_0 + \alpha(f) \\ |S_2(f)| &= 3A_0\, |\mathrm{sinc}\,(3f)|\,, & \arg[S_2(f)] &= -4\pi f + \beta(f) \end{aligned}$$

where $\alpha(f) = 0$ for $\mathrm{sinc}(f/F) > 0$ and $\alpha(f) = \pi$ for $\mathrm{sinc}(f/F) < 0$. Hence

$$\alpha(f) = \begin{cases} 0 & f \in \{(0,F) + \mathbb{Z}_0^+(2F)\} \cup \{(-F,0) + \mathbb{Z}_0^-(2F)\} \\ \pi & \text{otherwise} \end{cases}$$

where $\mathbb{Z}_0^+(2F) = \{0, 2F, 4F, \ldots\}$ and $\mathbb{Z}_0^-(2F) = \{\ldots, -4F, -2F, 0\}$. Analogously, one gets the phase $\beta(f)$ (given by $\alpha(f)$ for $F = \frac{1}{3}$).

5.27. ⋆⋆ [Sect. 5.8] Evaluate the Fourier transform of the signal

$$s(t) = \mathrm{rect}\,(t/T)\ \sin 2\pi f_0 t\,, \qquad t \in \mathbb{R}$$

in two different ways: 1) using Rule 6b and 2) using the Euler formulas and then Rule 5b.

The signal $s(t)$ is given by the product of the signals

$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right)\,, \qquad y(t) = \sin\left(2\pi f_0 t\right)$$

whose FTs are respectively

$$X(f) = T\ \mathrm{sinc}\,(Tf) \qquad \text{and} \qquad Y(f) = \frac{1}{2\mathrm{i}}\left[\delta(f-f_0) - \delta(f+f_0)\right]\,.$$

From the rule on convolution in the frequency domain (6b of Tab. 5.2) one gets

$$S(f) = X * Y(f) = \int_I d\lambda\ X(\lambda)\,Y(f-\lambda)\,.$$

After a single substitution, the use of *sifting property* of the impulse gives

$$S(f) = \frac{T}{2\mathrm{i}}\left\{\mathrm{sinc}\,[T(f-f_0)] - \mathrm{sinc}\,[T(f+f_0)]\right\}\,.$$

The generalization of this result is

$$s(t) = x(t)\ \sin\left(2\pi f_0 t\right) \xrightarrow{\mathcal{F}} S(f) = \frac{1}{2\mathrm{i}}\left[X(f-f_0) - X(f+f_0)\right]\,.$$

However, the evaluation through Eulero's formulas is simpler, using the rule

$$s(t)\,\mathrm{e}^{\mathrm{i}2\pi f_0 t} \xrightarrow{\mathcal{F}} S(f-f_0)\,.$$

5.28. ⋆ [Sect. 5.8] Write Parseval's theorem in the case $I = \mathbb{Z}(T)/\mathbb{Z}(10T)$.

Considering that

$$I = \mathbb{Z}(T)/\mathbb{Z}(10T) \xrightarrow{\text{duals}} \widehat{I} = \mathbb{Z}(F)/\mathbb{Z}(10F)\,, \qquad F = 1/(10T)\,.$$

Parseval's theorem (5.43)

$$\int_I \mathrm{d}t\ |s(t)|^2 = \int_{\widehat{I}} \mathrm{d}f\ |S(f)|^2\,,$$

becomes

$$\sum_{n=0}^{9} T\ |s(nT)|^2 = \sum_{k=0}^{9} F\ |S(kF)|^2\ .$$

5.29. ⋆[Sect. 5.8] Write decomposition (5.77) for the causal exponential signal

$$s(t) = 1(t)\ \mathrm{e}^{-3t}\,, \qquad t \in \mathbb{R}\ .$$

First, we write the Fourier pair

$$s(t) = 1(t)\ \mathrm{e}^{-3t}\ ,\ t \in \mathbb{R} \quad \xrightarrow{\mathcal{F}} \quad S(f) = \frac{1}{3+\mathrm{i}2\pi f}\ ,\ f \in \mathbb{R}\ .$$

Then, note that a singularity at the origin is not present and the decomposition gives

$$|S(f)| = A_S(f) = \frac{1}{\sqrt{9+4\pi^2 f^2}}\,, \qquad \arg[S(f)] = \beta_S(f) = -\arctan\frac{2}{3}\pi f$$

$$S_0 = \int_{\{0\}} S(f)\ \mathrm{d}f = 0\ .$$

Hence, (5.77) becomes

$$s(t) = \int_{0^+}^{+\infty} \frac{2}{\sqrt{9+4\pi^2 f^2}}\ \cos\left[2\pi f t - \arctan\frac{2}{3}\pi f\right] \mathrm{d}f\,, \qquad t \in \mathbb{R}\ .$$

5.30. ⋆⋆[Sect. 5.8] Write decomposition (5.77) for the signal

$$s(t) = 5 + 1(t)\ \mathrm{e}^{-3t}\,, \qquad t \in \mathbb{R}\ .$$

The problem is very similar to the previous one, but in this case the FT has an impulse at the origin

$$s(t) = 5 + 1(t)\ \mathrm{e}^{-3t}\ ,\ t \in \mathbb{R} \quad \xrightarrow{\mathcal{F}} \quad S(f) = 5\ \delta(f) + \frac{1}{3+\mathrm{i}2\pi f}\ ,\ f \in \mathbb{R}\ .$$

Then

$$A_S(f) = 5\delta(f) + \frac{1}{\sqrt{9+4\pi^2 f^2}}\,, \qquad \beta_S(f) = -\arctan\frac{2}{3}\pi f$$

$$S_0 = \int_{\{0\}} S(f)\,\mathrm{d}f = 5\,,$$

and decomposition (5.77) gives

$$s(t) = 5 + \int_{0+}^{+\infty} \frac{2}{\sqrt{9+4\pi^2 f^2}}\,\cos\left[2\pi f t - \arctan\frac{2}{3}\pi f\right]\mathrm{d}f\,, \qquad t \in \mathbb{R}\,.$$

5.31. ⋆⋆[Sect. 5.8] Write decomposition (5.77) for the discrete signal

$$s(nT) = 2 + (1/3)^{|n|}\,, \qquad nT \in \mathbb{Z}(T)\,.$$

The domains are

$$I = \mathbb{Z}(T)\,, \qquad \widehat{I} = \mathbb{R}/\mathbb{Z}(F_p)\,, \qquad F_p = \frac{1}{T}\,.$$

Now, on the quotient group $\widehat{I} = \mathbb{R}/\mathbb{Z}(T_p)$ the decomposition (5.74) requires a choice of a cell P. With the choice $P = \left[-\frac{1}{2}F_p,\,\frac{1}{2}F_p\right)$ one gets

$$\widehat{I}_z = \{0\}\,, \qquad \widehat{I}_+ = \left(0,\,\tfrac{1}{2}F_p\right)\,, \qquad \widehat{I}_- = \left(-\tfrac{1}{2}F_p,\,0\right)\,.$$

The corresponding decomposition, given by (5.77), results

$$s(t) = S_0 + \int_{0+}^{\frac{1}{2}F_p} 2A_S(f)\,\cos\left[2\pi f t + \beta_S(f)\right]\mathrm{d}f\,, \qquad t \in \mathbb{Z}(T)$$

where

$$S_0 = \int_{\{0\}} S(f)\,\mathrm{d}f$$

takes into account a possible impulse at $f = 0$. For the evaluation of the FT it is required a preliminary decomposition of the signal in the form $s(t) = s_0(t) + s_1(t)$, where $s_0(t) = 5 \to S_0(f) = 5\,\delta_{\mathbb{R}/\mathbb{Z}(F_p)}(f)$, and

$$\begin{aligned} S_1(f) &= \sum_{n=-\infty}^{+\infty} T\left(\frac{1}{3}\right)^{|n|} \mathrm{e}^{-\mathrm{i}2\pi f n T}\,, \qquad z \stackrel{\Delta}{=} \mathrm{e}^{\mathrm{i}2\pi f T} \\ &= T\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{-n} z^n + T\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n} z^{-n} - T = T\left[\frac{1}{1-3z} + \frac{1}{1-\frac{1}{3}z^{-1}} - 1\right]. \end{aligned}$$

Hence $S_0 = 5$, while for $f \in (0, +\infty)$ one gets

$$A_S(f) = |S(f)| = |S_1(f)|, \qquad \beta_S(f) = \arg S(f) = \arg S_1(f) .$$

The evaluation of the amplitude and phase is long and tedious.

5.32. $\star\star$ [Sect. 5.8] Show that on the discrete domain $\mathbb{Z}(T)$ the signal $x_0(nT) = z^n$, with z a complex constant, is a filter eigenfunction.

The input–output relation of filters on $\mathbb{Z}(T)$ is

$$y(nT) = \sum_{m=-\infty}^{+\infty} T\, g(nT - mT)\, x(mT) = \sum_{k=-\infty}^{+\infty} T\, g(kT)\, x(nT - kT) .$$

The input signal $x_0(nT) = z^n$ gives

$$y(nT) = \sum_{k=-\infty}^{+\infty} T\, g(kT)\, z^{n-k} = z^n\, \tilde{G}(z) ,$$

where

$$\tilde{G}(z) = \sum_{k=-\infty}^{+\infty} T\, g(kT)\, z^{-k} . \tag{S5.3}$$

In conclusion, one gets the response $y(nT) = \tilde{G}(z)\, x_0(nT)$ and therefore $x_0(nT) = z^n$ is an eigenfunction with eigenvalue $\tilde{G}(z)$. As we shall see in Chap.11, the function $G(z)$ is the *z–transform* of the impulse response.

5.33. $\star$ [Sect. 5.8] Explain why sinusoids are not filter eigenfunctions, although the response to a sinusoid is still a sinusoid.

The response of *real* filter with frequency response $G(f)$ to the signal $x(t) = A_0 \cos(2\pi f_0 t + \varphi_0)$ is (see Sect. 2.8)

$$y(t) = A_0\, |G(f_0)| \cos\Big(2\pi f_0 t + \varphi_0 + \arg G(f_0)\Big) .$$

Hence, the condition of proportionality $y(t) = \lambda\, x(t)$ is not verified. Ultimately, this is due to the fact that a sine signal $s(t)$ is not *separable*, that is, $\lambda s(t-u) \neq s(t)\, s(-u)$.

5.34. $\star\star\star$ [Sect. 5.8] Show that in the case of a continuous signal $s(t)$, $t \in \mathbb{R}$, the constant term in the composition (5.77) is given by the so called *continuous component* (see Sect. 2.1)

$$S_0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\, \mathrm{d}t\,,$$

provided that the limit exists and is finite.

Suppose that the limit

$$K = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\, \mathrm{d}t$$

exists with K finite. Then, $s(t)$ can be expressed as

$$s(t) = K + v(t)\,,\ t \in \mathbb{R} \qquad \text{where} \qquad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t)\, \mathrm{d}t = 0\,.$$

Hence, in the frequency domain we have $S(f) = K\,\delta(f) + V(f)$, with $f \in \mathbb{R}$, where $V(f)$ does not contain an impulse at $f = 0$. Then

$$S_0 = \int_{\{0\}} S(f)\, \mathrm{d}f = K\,.$$

5.35. $\star$ [Sect. 5.8] Write decomposition (5.74) in the case $I = \mathbb{R}/\mathbb{Z}(T_p)$ and then write the signal decomposition (5.77).

For signals defined on $\mathbb{R}/\mathbb{Z}(T_p)$ the frequency domain is $\widehat{I} = \mathbb{Z}(F)$ with $F = 1/T_p$. Then, decomposition (5.74) gives

$$\widehat{I}_z = \{0\}\,, \qquad I_+ = \{F, 2F, 3F \ldots\}$$

and (5.77) becomes

$$s(t) = S_0 + \sum_{n=1}^{\infty} 2F\, A_S(nF)\, \cos\left[2\pi nFt + \beta_S(nF)\right] \quad , \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)$$

with

$$S_0 = \int_{\{0\}} \mathrm{d}f\, S(f) = F\, S(0)\,.$$

5.36. $\star$ [Sect. 5.8] Write decompositions (5.74) and (5.77) in the case $I = \mathbb{Z}(T)/\mathbb{Z}(T_p)$.

The frequency domain is

$$\widehat{I} = \mathbb{Z}(F)/\mathbb{Z}(F_p) \quad \text{with} \quad F = 1/T_p \,,\; F_p = NF \;.$$

Considering $N = 2M + 1$ odd, a cell of $\mathbb{Z}(F)$ is given by $\mathbb{Z}(F_p)$ è

$$P = \{-MF, \dots, -F, 0, F, \dots, MF\} \;.$$

Then, decomposition (5.74) holds for

$$\widehat{I}_z = \{0\}\,, \qquad \widehat{I}_+ = \{F, 2F, \dots, MF\}$$

and (5.77) gives

$$s(t) = S_0 + \sum_{n=1}^{M-1} 2F\,A_S(nF)\,\cos\left[2\pi nFt + \beta_S(nF)\right], \qquad t \in \mathbb{Z}(T)/\mathbb{Z}(T_p)$$

with $S_0 = F\,S(0)$.

5.37. $\star\star\star$[Sect. 5.8] The decomposition into "positive" and "negative" frequencies (5.74) is not unique. Prove that, for $I = \mathbb{Z}(T)$ and $\widehat{I} = \mathbb{R}/\mathbb{Z}(F_p)$, in place of the decomposition indicated in (5.74) we can consider the alternative decomposition

$$\widehat{I}_+ = \left(0, \frac{1}{2}F_p\right), \qquad \widehat{I}_- = \left(\frac{1}{2}F_p, F_p\right) \;.$$

The alternative is explained by the fact that the frequencies in the domain $\mathbb{R}/\mathbb{Z}(F_p)$ are modulo $\mathbb{Z}(F_p)$. Thus, e.g., the negative frequency $-\frac{1}{4}F_p$ is equivalent to the positive frequency $-\frac{1}{4}F_p + F_p = \frac{3}{4}F_p$.

5.38. $\star$[Sect. 5.8] Let $s(t)$, $t \in \mathbb{R}$, be a signal with the limited spectral extension $e(S) = (-B, B)$. Find the spectral extension of $s^2(t)$, $t \in \mathbb{R}$.

In the frequency domain the product $s(t) \cdot s(t)$, $t \in \mathbb{R}$ becomes the convolution $S * S(f)$, $f \in \mathbb{R}$. Considering the rule on the extension of convolution, given by (4.73), one gets

$$e(S) + e(S) = (-B, B) + (-B, B) = (-2B, 2B) \;.$$

5.39. ⋆⋆ [Sect. 5.8] Find the spectral extension of the signal

$$s(t) = A_0 \ \mathrm{sinc}\,(t/T_0)\,, \qquad t \in \mathbb{Z}(T)\,,\ T_0 = T/10\,.$$

The FT of $s(t)$ is given by

$$S(f) = 10TA_0 \ \mathrm{rep}_{F_p} \ \mathrm{rect}\,(10Tf)\,, \qquad F_p = \frac{1}{T}\,.$$

The extension of $\mathrm{rect}(10fT)$ is the interval $e_0 = \left(-\frac{1}{20}F_p, \frac{1}{20}F_p\right)$. Since the repetition terms do not overlap, the spectral extension is given by the repetition of the interval e_0. Hence

$$e(S) = e_0 + \mathbb{Z}(F_p) = \left(-\tfrac{1}{20}F_p, \tfrac{1}{20}F_p\right) + \mathbb{Z}(F_p)\,.$$

5.40. ⋆ [Sect. 5.8] Find the spectral extension of the signal

$$s(t) = A_0 \ \sin^3(2\pi f_0 t) \ \cos\,(4\pi f_0 t)\,, \qquad t \in \mathbb{R}/\mathbb{Z}(T_0)\,,\ T_0 = 1/f_0\,.$$

It is convenient to use Eulero's formulas, which gives the following decomposition

$$\begin{aligned} s(t) &= A_0 \left(\frac{\mathrm{e}^{\mathrm{i}2\pi f_0 t} - \mathrm{e}^{-\mathrm{i}2\pi f_0 t}}{2\mathrm{i}}\right)^3 \frac{\mathrm{e}^{\mathrm{i}4\pi f_0 t} + \mathrm{e}^{-\mathrm{i}4\pi f_0 t}}{2} \\ &= \frac{A_0}{\mathrm{i}16}\left(\mathrm{e}^{\mathrm{i}2\pi 5 f_0 t} - 3\mathrm{e}^{\mathrm{i}2\pi 3 f_0 t} + 4\mathrm{e}^{\mathrm{i}2\pi f_0 t} - 4\mathrm{e}^{-\mathrm{i}2\pi f_0 t} + 3\mathrm{e}^{-\mathrm{i}2\pi 3 f_0 t} - \mathrm{e}^{-\mathrm{i}2\pi 5 f_0 t}\right)\,. \end{aligned}$$

Hence, considering that

$$\mathrm{e}^{\mathrm{i}2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta_{\mathbb{Z}(f_0)}(f - nf_0) \tag{S5.4}$$

one gets the Fourier transform

$$\begin{aligned} S(f) =& \frac{A_0}{\mathrm{i}16}\Big[\delta_{\mathbb{Z}(F)}(f - 5f_0) - 3\delta_{\mathbb{Z}(F)}(f - 3f_0) + 4\delta_{\mathbb{Z}(F)}(f - f_0) \\ & - 4\delta_{\mathbb{Z}(F)}(f + f_0) + 3\delta_{\mathbb{Z}(F)}(f + 3f_0) - \delta_{\mathbb{Z}(F)}(f + 5f_0)\Big]\,. \end{aligned}$$

Finally, considering that

$$e(\delta_{\mathbb{Z}(f_0)}(f - nf_0)) = \{nf_0\} \tag{S5.5}$$

the spectral extension is given by

$$\mathcal{E}_0(s) = \Big\{ -5f_0, -3f_0, -f_0, f_0, 3f_0, 5f_0 \Big\} .$$

5.41. ⋆⋆[Sect. 5.8] Prove that the spectral extension of the previous signal does not change if the domain/periodicity $\mathbb{R}/\mathbb{Z}(T_0)$ is replaced by $\mathbb{R}/\mathbb{Z}(2T_0)$.

First we note that the signal has *minimum period* $T_0 = 1/f_0$, but every multiple of T_0 is an *admissible period* of the signal, in particular $2T_0$. In the new representation $s(t)$, $t \in \mathbb{R}/\mathbb{Z}(2T_0)$ the previous relations hold with (S5.4) replaced by

$$\mathrm{e}^{\mathrm{i}2\pi n f_0 t} \xrightarrow{\mathcal{F}} \delta_{\mathbb{Z}(\frac{1}{2}f_0)}(f - nf_0)$$

but (S5.5) still hold and therefore the spectral extension does not change.

In the theory of *elementary transformations* of Chap. 6 the change $\mathbb{R}/\mathbb{Z}(T_0) \to \mathbb{R}/\mathbb{Z}(2T_0)$ is called a *down–periodization* and the corresponding transformation in the frequency domain of the form $\mathbb{Z}(f_0) \to \mathbb{Z}(\frac{1}{2}f_0)$ is called *up–sampling*. Now, the $\mathbb{Z}(f_0) \to \mathbb{Z}(\frac{1}{2}f_0)$ up–sampling multiplies the values of the FT $S(f)$ at the frequencies $f \in \mathbb{Z}(f_0)$ and *fills with zero values* the FT $S(f)$ at the frequencies $\frac{1}{2}f_0 \notin \mathbb{Z}(f_0)$. Hence, the spectral extension *after a down–periodization does not change* and this is a general statement.

5.42. ⋆[Sect. 5.9] Show that in 1D case the rule on the coordinate change becomes

$$s(at) \xrightarrow{\mathcal{F}} (1/|a|)S(f(a))$$

where a is an arbitrary non zero real number.

In the general mD case the rule on coordinate change is

$$s(\mathbf{a}\mathbf{t}) \xrightarrow{\mathcal{F}} \frac{1}{\mathrm{d}(\mathbf{a})} S(\mathbf{a}^\star f)$$

where $\mathbf{a}$ is a regular $m \times m$ real matrix and $\mathbf{a}^\star$ is the inverse transpose of $\mathbf{a}$. In the 1D case $\mathbf{a}$ becomes a scalar a and $\mathbf{a}^\star$ becomes $1/a$, where $a \neq 0$ because the matrix $\mathbf{a}$ is supposed to be regular. Moreover, $\mathrm{d}(\mathbf{a})$ is the absolute value of the determinant of $\mathbf{a}$, which becomes $|a|$ in the 1D case.

5.43. ⋆⋆⋆[Sect. 5.10] Prove that (5.97), when $\mathcal{B}$ is hexagonal, yields (5.101).

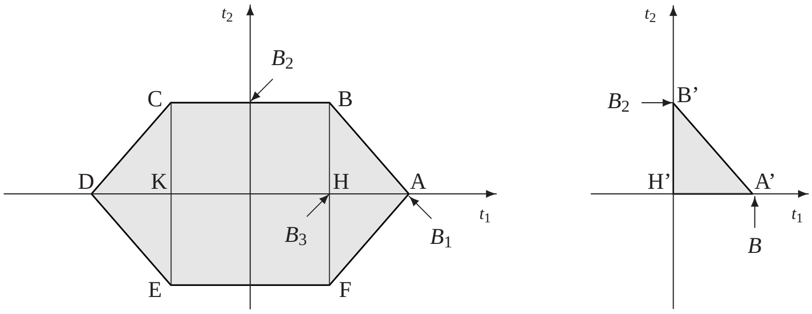

Fig. S5.1 Decomposition of the hexagon

The hexagon of Fig. 5.17 can be seen as the union of the rectangle BFEC and four rectangular triangles (Fig. S5.1). The rectangle is explicitly $[-B_3, B_3] \times [-B_2, B_2]$ where $2B_3$ is the length of the horizontal edges. The corresponding indicator function is

$$G(f_1, f_2) = \text{rect}(f_1/2B_3)\,\text{rect}(f_2/2B_2)\;.$$

Hence (see (5.98))

$$g_{\text{rect}}(t_1, t_2) = 2B_3\,\sin(2B_3 T_1)\;2B_2\,\text{sinc}(2B_2 t_2)\;.$$

As regards the triangles, it is convenient to start from the triangle HAB shifted to origin and denoted by H'A'B' in Fig.5.1. The lengths of the basis and of the altitude are respectively $B = B_1 - B_3$. The indicator function of H'A'B' is

$$Q(f) = \begin{cases} 1 & 0 < f_1 < B, 0 < F_2 < B_2(1 - f/B) \\ 0 & \text{elsewhere}\,. \end{cases}$$

Then, the indicator functions of the four triangles are given by (listed clockwise)

$$\begin{aligned} Q_{++}(f_1, f_2) &= Q(f_1 - B_3, f_2) \\ Q_{+-}(f_1, f_2) &= Q_{++}(f_1, -f_2) = Q(f_1 - B_3, -f_2) \\ Q_{--}(f_1, f_2) &= Q_{++}(-f_1, -f_2) = Q(-f_1 - B_3, -f_2) \\ Q_{-+}(f_1, f_2) &= Q_{++}(-f_1, f_2) = Q(-f_1 - B_3, f_2)\;. \end{aligned} \tag{S5.6}$$

Hence, it is sufficient the evaluation of the inverse FT of $Q(f)$, which is given by

$$\begin{aligned} q(t_1, t_2) &= \int_0^B \mathrm{d}f_1 \left\{ \int_0^{B_2(1-f_1/B)} \mathrm{d}f_2\; e^{i2\pi f_2 t_2} \right\} e^{i2\pi f_1 t_1} \\ &= \int_0^B \mathrm{d}f_1\, \frac{1}{i2\pi t_2} \left[e^{i2\pi B_2(1-f_1/B)t_2} - 1 \right] e^{i2\pi f_1 t_1} \\ &= \frac{1}{i2\pi t_2} \left\{ e^{i2\pi B_2 t_2}\, \frac{B}{i2\pi(Bt_1 - B_2 t_2)} \left[e^{i2\pi(Bt_1 - B_2 t_2)} - 1 \right] - \frac{1}{i2\pi t_1} \left[e^{i2\pi B t_1} - 1 \right] \right\} \end{aligned}$$

which can be written in the form

$$q(t_1,t_2) = \frac{B}{\mathrm{i}2\pi t_2}\left[\operatorname{sinc}(Bt_1 - B_2t_2)\,\mathrm{e}^{\mathrm{i}\pi(Bt_1+B_2t_2)} - \operatorname{sinc}(Bt_1)\,\mathrm{e}^{\mathrm{i}\pi Bt_1}\right]. \tag{S5.7}$$

Next, using (S5.6) we find

$$\begin{aligned} q_{++}(t_1,t_2) &= q(t_1,t_2)\,\mathrm{e}^{\mathrm{i}2\pi B_3 t_1} \\ q_{+-}(t_1,t_2) &= q(t_1,-t_2)\,\mathrm{e}^{\mathrm{i}2\pi B_3 t_1}\,. \end{aligned}$$

Considering that $Q(f_1,f_2)$ is *real* and

$$\begin{aligned} Q_{--}(f_1,f_2) &= Q_{++}(-f_1,-f_2) = Q^*_{++}(-f_1,-f_2) \\ Q_{-+}(f_1,f_2) &= Q_{+-}(-f_1,-f_2) = Q^*_{+-}(-f_1,-f_2)\,. \end{aligned}$$

From the rule $s^*(t) \xrightarrow{\mathcal{F}} S^*(-f)$ we get

$$\begin{aligned} q_{++}(t_1,t_2) + q_{--}(t_1,t_2) &= 2\Re q_{++}(t_1,t_2) \\ q_{+-}(t_1,t_2) + q_{-+}(t_1,t_2) &= 2\Re q_{+-}(t_1,t_2) \end{aligned}$$

and the contribution of the four triangles is given by

$$\begin{aligned} g_{\mathrm{tr}}(t_1,t_2) =& 2\Re\{q_{++}(t_1,t_2) + q_{+-}(t_1,t_2)\} \\ & 2\Re\left\{[q(t_1,t_2) + q(t_1,-t_2)]\,\mathrm{e}^{\mathrm{i}\pi B_3 t_1}\right\}\,. \end{aligned}$$

Using (S5.7) we find

$$\begin{aligned} &\Re\left\{[q(t_1,t_2) + q(t_1,-t_2)]\,\mathrm{e}^{\mathrm{i}\pi B_3 t_1}\right\} \\ &\quad = \frac{B}{\mathrm{i}2\pi t_2}\left[\operatorname{sinc}(B_4t_1 - B_2t_2)\,\mathrm{e}^{\mathrm{i}\pi(B_4t_1+B_2t_2)} - \operatorname{sinc}(B_4t_1 + B_2t_2)\,\mathrm{e}^{\mathrm{i}\pi(B_4t_1-B_2t_2)}\right] \end{aligned}$$

where

$$B_4 = B + 2B_3 = B_1 + B_3\,.$$

Hence

$$\begin{aligned} g_{\mathrm{tr}}(t_1,t_2) = \frac{B}{2\pi t_2}\Big\{&\operatorname{sinc}(B_4t_1 - B_2t_2)\ \sin[\pi(B_4t_1 + B_2t_2)] - \\ & - \operatorname{sinc}(B_4t_1 + B_2t_2)\ \sin[\pi(B_4t_1 - B_2t_2)]\Big\} \end{aligned}$$

We can check this result by applying the rule

$$g_{\mathrm{tr}}(0,0) = \operatorname{area}(G_{\mathrm{tr}})$$

where the area of the four triangles is (see Fig.5.1) $2BB_2$. In fact, to solve the indeterminacy at $(t_1,t_2) = (0,0)$ we note that $\operatorname{sinc}(0) = 1$ and $\sin(x) = x + \mathrm{O}(x^3)$. Hence

$$g_{\text{tr}}(t_1,t_2) \longrightarrow \frac{2B}{2\pi t_2}[\pi(B_4t_1 - B_2t_2) - \pi(B_4t_1 - B_2t_2)] = 2BB_2\ .$$

We recall the symbols

- B_1 and B_2 as in the figure,
- B_3 half of horizontal edges,
- $B = B_1 - B_3$,
- $B_4 = B + 2B_3 = B_1 + B_3$ and write the final result

$$\begin{aligned} g_{\text{hexagon}}(t_1,t_2) =& 2B_3\ \sin(2B_3T_1)\ 2B_2\ \text{sinc}(2B_2t_2) + \\ &+ \frac{B}{2\pi t_2}\Big\{\text{sinc}\,(B_4t_1 - B_2t_2)\ \sin[\pi(B_4t_1 + B_2t_2)] - \\ &\qquad - \text{sinc}\,(B_4t_1 + B_2t_2)\ \sin[\pi(B_4t_1 - B_2t_2)]\Big\} \end{aligned}$$

where B_1 and B_2 are defined as in the figure, B_3 is half of the horizontal edges, $B = B_1 - B_3$ and $B_4 = B_1 + B_3$.

5.44. ⋆⋆⋆[Sect. 5.10] Prove that (5.97), when $\mathcal{B}$ is circular, yields (5.102).

Hint: use the Bessel function identity [1]: $\int_0^1 xJ_0(cx)\,\mathrm{d}x = \frac{1}{c}J_1(c)$.

We apply the theory of signals and FTs with a *circular symmetry*, which states that if the signal has the form

$$s(t_1,t_2) = g\left(\sqrt{t_1^2 + t_2}\right) \tag{S5.8}$$

also its FT has the form

$$S(f_1,f_2) = G\left(\sqrt{f_1^2 + f_2}\right)\ . \tag{S5.9}$$

The functions $g(a)$ and $G(b)$ are related by the Hankel transform (see (5.87)). In the present case, the FT is given by (S5.9) with

$$G(b) = \begin{cases} 1 & 0 < b < B \\ 0 & b > B\ . \end{cases}$$

The corresponding inverse Hankel transform is

$$\begin{aligned} g(a) &= 2\pi \int_0^\infty \mathrm{d}b\, b\, G(b)\, J_0(2\pi ab) \\ &= 2\pi \int_0^B \mathrm{d}b\, b\, J_0(2\pi ab)\ . \end{aligned} \tag{S5.10}$$

The suggested identity gives

$$g(a) = 2\pi B^2 \int_0^1 x\, J_0(2\pi a x)\, \mathrm{d}x = \frac{B}{a}\, J_1(2\pi a B)$$

and (5.102) follows.

5.45. ⋆⋆ [Sect. 5.10] Consider the Fourier transform of the pyramidal signal given by (5.103). Evaluate the value at the origin $\mathrm{PYR}(0,0)$. *Hint*: to evaluate the indeterminacy $0/0$ use the expansion $\sin(x) = x - x^3/6 + O(x^4)$.

Expression (5.103) is indeterminate for $(f_1, f_2) = (0,0)$ and therefore we use the suggested expansion, which gives

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x} = 1 - \frac{1}{6}(\pi x)^2 + \mathrm{O}(x^3)\ .$$

Then

$$\mathrm{PIR}(f_1, f_2) \;\to\; -\frac{1}{6}\,\frac{4\pi^2(f_1 - f_2)^2 - 4\pi^2(f_1 + f_2)}{2\pi^2 f_1 f_2} \;\to\; \frac{4}{3}\ .$$

Problems of Chapter 6

6.1. ⋆⋆[Sect. 6.2] Consider the $\mathbb{R} \to \mathbb{R}$ tf with following input–output relation

$$y(t) = \mathrm{e}^{\mathrm{i}2\pi f_0 t} x^2(t), \qquad t \in \mathbb{R}$$

where $f_0 > 0$ is a constant frequency.

Check if the tf 1) is *real*, 2) is *shift–invariant*, 3) is *invertible*.

This tf *is not a real tf*. In fact, if $x(t)$ is a real valued signal, $y(t)$ is complex. This tf *is not periodically invariant*. In fact, if we shift the input signal, we obtain the signal

$$y_1(t) = \mathrm{e}^{\mathrm{i}2\pi f_0 t} x^2(t - t_0)$$

which is not the shifted version $y(t - t_0)$. But, if we restrict t_0 to $\mathbb{Z}(T_0)$, where $T_0 = 1/f_0$, we can write

$$y_1(t) = \mathrm{e}^{\mathrm{i}2\pi f_0 (t-t_0)} x^2(t - t_0) = y(t - t_0), \qquad \forall t_0 \in \mathbb{Z}(1/f_0).$$

Hence, the proposed tf is *periodically invariant* with periodicity $\mathbb{Z}(1/f_0)$. Finally, it is clear that the above tf *is not invertible* owing to the presence of a quadratic term, which makes it not possible to recover the input signal.

6.2. ⋆⋆[Sect. 6.2] Consider the $\mathbb{R} \to \mathbb{R}$ tf with input–output relationship

$$y(t) = \mathrm{e}^{-\alpha|t|} x^2(t), \qquad t \in \mathbb{R}$$

where $\alpha > 0$. Determine when the tf is *conditionally invertible* and express the inverse tf.

We first consider the relation

$$y_0(t) = x^2(t)$$

where the possibility of the recovery of $x(t)$ from $y_0(t)$, without ambiguity, requires that $x(t)$ is real and nonnegative. If this is the case, the recovery is obtained as $x(t) = \sqrt{y_0(t)}$.

Next, considering that $\mathrm{e}^{-\alpha|t|} > 0$ for every $t \in \mathbb{R}$, the condition is still given by $x(t) \geq 0$ and the recovery is obtained as

$$x(t) = \sqrt{y(t)\,\mathrm{e}^{\alpha|t|}} = \mathrm{e}^{\frac{1}{2}\alpha|t|} \sqrt{y(t)}. \qquad \text{(S6.1)}$$

In conclusion, the tf is conditionally invertible for the class of *real nonnegative input signals*. The inverse ft is $\mathbb{R} \to \mathbb{R}$ with input–output relation given by (S6.1).

6.3. ⋆⋆ [Sect. 6.2] Prove that the set of shift invariance defined by (6.8) of Definition 6.3 is always an Abelian group.

Let $y = \mathcal{T}[x]$ be the input–output relation and

$$\Pi = \{p \mid y_p = \mathcal{T}[x_p]\}$$

the set of shift–invariance. We have to prove that 1) $p = 0 \in \Pi$ (this is evident), 2) if $p \in \Pi$ then also $-p \in \Pi$, 3) if $p, q \in \Pi$, then also $p+q \in \Pi$.

These properties follows from the fact that the input and output domains I ans U are Abelian groups. The input–output relation is explicitly

$$y(t) = \mathcal{T}[x(\cdot) \mid t] \qquad \forall x \in \mathcal{I}, \qquad \forall t \in U \tag{S6.2}$$

where $\mathcal{I}$ is the class of the input signals. Now, if $p \in \Pi$, then $y_p(t) = \mathcal{T}[x_p]$ and more explicitly

$$y(t-p) = \mathcal{T}[x(\cdot - p) \mid t-p] \, . \tag{S6.3}$$

Now, to prove that also $-p \in \Pi$, we use the fact that (S6.2) and (S6.3) holds simultaneously, and we let (Fig.S6.1)

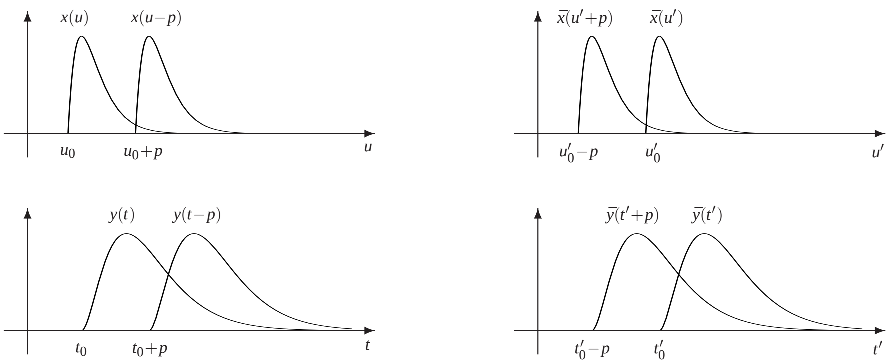

Fig. S6.1 Illustration of symbols for the proof in the problem

$$\begin{aligned} \overline{x}(u) &= x(u-p) = x_p(u)\,, \qquad u' = u - p \\ \overline{y}(t) &= y(t-p) = y_p(u)\,, \qquad t' = t - p\,. \end{aligned}$$

Then

$$\begin{aligned} x(u') &= \overline{x}(u'+p) = \overline{x}_{-p}(u') \\ y(t') &= \overline{t}(t'+p) = \overline{y}_{-p}(t')\,. \end{aligned}$$

With these notations (S6.3) becomes

$$\overline{y}(t') = \mathcal{T}[\overline{x}(\cdot) \mid t']$$

and (S6.2) gives

$$\overline{y}(t'+p) = \mathcal{T}[\overline{x}(\cdot + p) \mid t'+p] \ .$$

The last two relations state that also $-p \in \Pi$.

In a similar way we can prove that, if $p, q \in \Pi$, also $p+q \in \Pi$.

6.4. ⋆[Sect. 6.4] Find the kernel of the $\mathbb{R} \to \mathbb{R}$ linear tf with input–output relation

$$y(t) = \int_{-\infty}^{t} x(u)\, \mathrm{d}u\,, \qquad t \in \mathbb{R}\ .$$

The kernel is given by

$$h(t,u) = 1(t-u)\,, \qquad t,u \in \mathbb{R}$$

where $1(x)$ is the unit step function. In fact

$$y(t) = \int_{\mathbb{R}} \mathrm{d}u\, h(t,u)\, x(u) = \int_{-\infty}^{+\infty} 1(t-u)\, x(u)\, \mathrm{d}u$$

where

$$1(t-u) = \begin{cases} 1 & u < t \\ 0 & u > t \end{cases} \ .$$

Hence

$$y(t) = \int_{-\infty}^{t} x(t)\, \mathrm{d}u\ .$$

6.5. ⋆⋆[Sect. 6.4] Find the kernel of the $\mathbb{R} \to \mathbb{Z}(T)$ linear tf with input–output relation

$$y(t) = \int_{-\infty}^{t} x(u)\, \mathrm{d}u\,, \qquad t \in \mathbb{Z}(T)\ .$$

We can write

$$y(t) = \int_{-\infty}^{t} x(u)\, \mathrm{d}u = \int_{-\infty}^{+\infty} 1(t-u)\, x(u)\, \mathrm{d}u\,, \qquad t \in \mathbb{Z}(T) \quad ,$$

so that the kernel results $h(t,u) = 1(t-u)$ with $t \in \mathbb{Z}(T)$, $u \in \mathbb{R}$. the only difference with respect to the previous exercise is in that the kernel is defined over $\mathbb{Z}(T) \times \mathbb{R}$ instead over $\mathbb{R}^2$.

6.6. $\star$ [Sect. 6.4] Explain why a transformation with input–output relation

$$y(t) = A\,x(t) + B\,,$$

where A and B are constants and $B \neq 0$, is not linear.

The transformation does not satisfy the *condition of additivity*.

6.7. $\star\star$ [Sect. 6.4] Explain why the transformation that gives the conjugate of a signal is not linear, although if it satisfies additivity: $(x_1 + x_2)^* = x_1^* + x_2^*$.

The transformation does not satisfy the *condition of homogeneity*.

6.8. $\star\star$ [Sect. 6.4] Find the impulse response of the $I \to I$ linear tf with input–output relation

$$y(t) = x(t)\ \cos\,\omega_0 t + x(t - t_0)\ \sin\,\omega_0 t\ .$$

The term $x(t)\ \cos\,\omega_0 t$ corresponds to a window with kernel $\cos\,\omega_0 t\ \delta_I(t-u)$, while the term $x(t - T_0)\ \sin\,\omega_0 t$ corresponds to the cascade of a delay of T_0 with a window with kernel $\sin\,\omega_0 t\ \delta_I(t-u)$. Hence, the overall kernel is:

$$h(t,u) = \cos(\omega_0 t)\ \delta_I(t-u) + \sin(\omega_0 t)\ \delta_I(t - T_0 - u)\,, \qquad t\ ,\ u \in I\ .$$

6.9. $\star\star$ [Sect. 6.4] Find the model of the operation that, starting from a signal $x(t)$, $t \in I$, gives its even and odd parts.

The tf has one input and two outputs and therefore the kernel $\mathbf{h}(t,u)$ is a 2×1 matrix and the the input–output relationship results Fig. S6.2

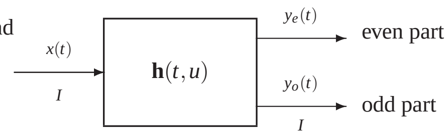

Fig. S6.2 Transformation that gives the odd and even part of a signal

$$\begin{bmatrix} y_e(t) \\ y_o(t) \end{bmatrix} = \int_I \mathrm{d}u \begin{bmatrix} h_e(t,u) \\ h_o(t,u) \end{bmatrix} x(u)\,,$$

that is,

$$y_e(t) = \int_I \mathrm{d}u\, h_e(t,u)\, x(u) = \frac{1}{2}\Big[x(t) + x(-t)\Big]$$
$$y_o(t) = \int_I \mathrm{d}u\, h_o(t,u)\, x(u) = \frac{1}{2}\Big[x(t) - x(-t)\Big]\,.$$

taking into account that, in order to obtain $x(t)$ from $x(u)$, we must let $\delta_I(t-u)$ in the integrand and, to obtain $x(-t)$ from $x(t)$ we must let $\delta_I(t+u)$. The kernel of the tf results:

$$h(t,u) = \begin{bmatrix} h_e(t,u) \\ h_o(t,u) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \delta_I(t-u) + \delta_I(t+u) \\ \delta_I(t-u) - \delta_I(t+u) \end{bmatrix}\,.$$

6.10. ⋆⋆⋆[Sect. 6.4] Show that necessary and sufficient conditions for a linear tf to be real is that its kernel is real.

The proof of the sufficient condition is trivial: if $x(u)$ and $h(t,u)$ are real also $y(t)$ is real.

To prove the necessary condition we must apply on the input the *real* signal $x(u) = \delta_I(u-u_0)$, which gives the output $y(t) = h(t,u_0)$. From the definition of *real* tf the output of a real input signal must be real. Hence $y(t) = h(t,u_0)$ must be real $\forall u_0$.

6.11. ⋆[Sect. 6.4] Find the equivalent tf of the cascade of a filter on $\mathbb{Z}(T)$ followed by a delay of $t_0 = 5T$.

To identify the kernel we can apply the impulse at the input $x(u) = \delta_{\mathbb{Z}(T)}(u-u_0)$, $u_0 \in \mathbb{Z}(T)$. After the filter we obtain the signal $g(t-u_0)$ and hence, finally $g(t-t_0-u_0)$. Thus, the kernel results:

$$h(t,u) = g(t-u-t_0)\,, \qquad t\,,\ u \in \mathbb{Z}(T)\,.$$

6.12. ⋆[Sect. 6.5] Prove the rule on the scale change in the frequency domain, given by (6.31) in the case $I = \mathbb{Z}(T)$.

Letting $y(t) = x(at)$, $t \in \mathbb{Z}(T/a)$, we obtain:

$$Y(f) = \sum_{n=-\infty}^{+\infty} \frac{T}{a}\, y\left(n\frac{T}{a}\right) \mathrm{e}^{-\mathrm{i}2\pi f n \frac{T}{a}} \,, \qquad f \in \mathbb{R}/\mathbb{Z}(aF_p)\,,\, F_p = \frac{1}{T}\,.$$

Therefore it results:

$$Y(f) = \sum_{n=-\infty}^{+\infty} \frac{T}{a}\, x(nT)\, \mathrm{e}^{-\mathrm{i}2\pi f n \frac{T}{a}} = \frac{1}{a} X\left(\frac{f}{a}\right), \qquad f \in \mathbb{R}/\mathbb{Z}(aF_p)\,.$$

6.13. $\star$[Sect. 6.5] Find the time domain I_a and the frequency domain $\widehat{I}_a$ after a scale change with $a = \frac{2}{5}$, in the cases: $I = \mathbb{R}/\mathbb{Z}(10)$ and $I = \mathbb{Z}(3)/\mathbb{Z}(12)$.

Since $a = \frac{2}{5} < 1$, a time–scale occurs and thus, from $I = \mathbb{R}/\mathbb{Z}(10)$ we obtain $I_a = \mathbb{R}/\mathbb{Z}(25)$. On the other hand, in the frequency domain $\widehat{I} = Z\left(\frac{1}{10}\right)$ we obtain $\widehat{I}_a = Z\left(\frac{1}{25}\right)$ and therefore a frequency compression occurs.

Similar considerations can be made in the second case, obtaining

$$\begin{aligned} I = \mathbb{Z}(3)/\mathbb{Z}(12) &\quad \xrightarrow{\text{dual}} \quad \widehat{I} = \mathbb{Z}(\tfrac{1}{12})/\mathbb{Z}(\tfrac{1}{3}) \\ I_a = \mathbb{Z}(\tfrac{15}{2})/\mathbb{Z}(30) &\quad \xrightarrow{\text{dual}} \quad \widehat{I}_a = \mathbb{Z}(\tfrac{1}{30})/\mathbb{Z}(\tfrac{2}{15})\,. \end{aligned}$$

6.14. $\star$[Sect. 6.6] Classify the $\mathbb{R} \to \mathbb{R}$ tfs with input–output relations

$$y_1(t) = 1(x(t))\,, \qquad y_2(t) = x(t)\; 1(x(t) - A_0) \qquad \text{with} \qquad A_0 > 0\,.$$

The transformations are *nonlinear* since neither homogeneity nor additivity conditions are satisfied (see Sect. 6.2). Also, the tf are *memoryless*

We rewrite the input–output relationships in a more explicit form

$$\begin{aligned} y_1(t) &= 1(x(t)) = \begin{cases} 1 \quad , & x(t) > 0 \\ 0 \quad , & \text{elsewhere} \end{cases} \\ y_2(t) &= x(t)\; 1\Big(x(t) - A_0\Big) = \begin{cases} x(t) \quad , & x(t) > A_0 \\ 0 \quad , & \text{elsewhere}\;. \end{cases} \end{aligned}$$

From the former relation we obtain on the output a unitary signal on those intervals where $x(t) > 0$ and a zero signal on those intervals where $x(t) < 0$; hence, $y(t)$ is a binary signal. From the latter relation we obtain on the output the input signal when it is greater than A_0, and zero elsewhere.

Notice that in both cases we assume that $x(t)$ is a real signal.

6.15. ⋆⋆ [Sect. 6.6] Show that the dual of a Volterra tf is still a Volterra tf.

We can easily obtain (assuming to operate over groups of $\mathbb{R}$)

$$\begin{aligned}
Y_1(f) &= \int_{\widehat{I}} \mathrm{d}\lambda \left[\int_U \mathrm{d}t \int_I \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}2\pi f t}\, h_1(t,u)\, \mathrm{e}^{\mathrm{i}2\pi\lambda u} \right] X(\lambda) \\
&= \int_{\widehat{I}} \mathrm{d}\lambda\, \hat{h}_1(f,\lambda)\, X(\lambda) \\
Y_2(f) &= \int_{\widehat{I}} \mathrm{d}\lambda_1 \int_{\widehat{I}} \mathrm{d}\lambda_2 \left[\int_u \mathrm{d}t \int_I \mathrm{d}u_1 \int_I \mathrm{d}u_2\, \mathrm{e}^{-\mathrm{i}2\pi f t} \right. \\
&\qquad \left. h_2(t,u_1,u_2)\, \mathrm{e}^{\mathrm{i}2\pi\lambda_1 u_1}\, \mathrm{e}^{\mathrm{i}2\pi\lambda_2 u_2} \right] X(\lambda_1)\, X(\lambda_2) \\
&= \int_{\widehat{I}} \mathrm{d}\lambda_1 \int_{\widehat{I}} \mathrm{d}\lambda_2\, \hat{h}_2(f,\lambda_1,\lambda_2)\, X(\lambda_1)\, X(\lambda_2)\,, \qquad f \in \widehat{U}\,.
\end{aligned}$$

We proceed in the same way for $Y_3(f)$, $Y_4(f)$, ... Thus, we deduce that the dual transformation is still a Volterra transformation with kernels $\hat{h}_1, \hat{h}_2, \ldots$ which are obtained as Fourier transforms of the corresponding kernels, except the sign of λ arguments, which must be changed (see (6.75a)).

6.16. ⋆ [Sect. 6.8] Calculate the response of the QIL linear $\mathbb{Z}(T) \to \mathbb{R}$ tf, with the impulse response

$$g(t) = \mathrm{rect}\,(t/T - 1)$$

to the signal $x(t) = \exp\,(-2\,|t|/T)$.

Using the input–output relationship for the QIL tf (see (6.44)) we obtain:

$$\begin{aligned}
y(t) &= \int_{\mathbb{Z}(T)} \mathrm{d}u\, g(t-u)\, x(u)\,, \qquad t \in \mathbb{R} \\
&= \sum_{n=-\infty}^{+\infty} T\ \mathrm{rect}\left(\frac{t-nT}{T} - 1\right) \exp\left(-\left|2\frac{nT}{T}\right|\right)
\end{aligned}$$

The final result can be expressed as the following step function

$$y(t) = T\, \mathrm{e}^{-2|k|}\,, \qquad t \in (kT + \tfrac{1}{2}T\,,\ kT + \tfrac{3}{2}T)\,,\ k \in \mathbb{Z}\,.$$

6.17. ⋆ [Sect. 6.8] Find the domain of the kernel of an $I \to U$ QIL linear tf, with $I = \mathbb{Z}(6)/\mathbb{Z}(30)$ and $U = \mathbb{Z}(9)/\mathbb{Z}(90)$.

The sum of two quotient groups is by definition (see (3.79))

$$D = I + U = I_0/S_1 + U_0/S_2 \stackrel{\Delta}{=} (I_0 + U_0)/(S_1 + S_2)\ .$$

Thus with $I = \mathbb{Z}(6)/\mathbb{Z}(30)$ and $U = \mathbb{Z}(9)/\mathbb{Z}(90)$ we find

$$D = (\mathbb{Z}(6) + \mathbb{Z}(9))\,/\,(\mathbb{Z}(30) + \mathbb{Z}(90)) = \mathbb{Z}(3)/\mathbb{Z}(30)\ .$$

6.18. ⋆⋆ [Sect. 6.8] Find the kernel of the linear tfs given by the cascade of two QIL linear tf: $\mathbb{Z}(T) \to \mathbb{Z}(3T)$ with kernel $g_1(t)$ and $\mathbb{Z}(3T) \to \mathbb{Z}(6T)$ with kernel $g_2(t)$. Is the result a QIL linear tf?

Applying (6.18a), which gives the kernel of the cascade of two linear tfs, over the domains $I_1 = \mathbb{Z}(T)$, $I_2 = \mathbb{Z}(3T)$, $I_3 = \mathbb{Z}(3T)$ we obtain

$$\begin{aligned} h(6mT, nT) &= \sum_{k=-\infty}^{+\infty} 3T\ g_2(6mT - 3kT)\ g_1(3kT - nT) \\ &= \sum_{k=-\infty}^{+\infty} 3T\ g_2(6mT - nT - mT)\ g_1(mT). \end{aligned}$$

The transformation is still QIL, because the kernel depends only on the difference $t_3 - t_1$, $t_3 \in \mathbb{Z}(6T)$, $t_1 \in \mathbb{Z}(T)$.

The positive answer to the equivalence of the cascade to a tf QIL could be obtained directly from the Principle of Composition (see Sect. 7.2), since the domains of the cascade satisfy

$$I_1 \cap I_2 \cap I_3 = I_1 \cap I_3\ . \tag{6.4}$$

6.19. ⋆⋆⋆ [Sect. 6.8] Repeat the previous problem with $\mathbb{R} \to \mathbb{Z}(T)$ and $\mathbb{Z}(T) \to \mathbb{R}$.

The kernel results:

$$h(t,u) = \sum_{k=-\infty}^{+\infty} T\ g_2(t - kT)\ g_1(kT - u)\,, \qquad t, u \in \mathbb{R}\,,$$

and cannot be expressed in the form $h(t,u) = g(t-u)$. Hence, the tf is not QIL. This is in agreement with the Principle of Composition (see Sect. 7.2) which for the equivalence to hold requires that condition (6.4) which is not satisfied in this case.

6.20. $\star$ [Sect. 6.8] Find the *domain complexity* of a QIL $\mathbb{Z}(10)/\mathbb{Z}(30) \to \mathbb{R}/\mathbb{Z}(45)$ tf.

Taking into account that $\mathbb{Z}(10)+\mathbb{R}=\mathbb{R}$ and $\mathbb{Z}(30)+\mathbb{Z}(45)=\mathbb{Z}(15)$ the domain of the impulse response results $\mathbb{R}/\mathbb{Z}(15)$. Hence, referring to (6.48) we obtain

$$\mathbb{Z}(10) \neq \mathbb{R} = \mathbb{R} \quad , \qquad \mathbb{Z}(30) \neq \mathbb{Z}(15) \neq \mathbb{Z}(45)$$

so that the domain complexity is $c = 3$.

6.21. $\star\star\star\nabla$ [Sect. 6.8] Find the *domain complexity* of a $\mathbb{Z}_2^1(d_1,d_2) \to \mathbb{Z}_3^2(2d_1,d_2)$ QIL tf. (for the evaluation of the sum of two lattices, see Chap.16).

The lattices $J = \mathbb{Z}_2^1(d_1,d_2)$ and $K = \mathbb{Z}_3^2(2d_1,d_2)$ are shown in Fig.S6.3 and we can

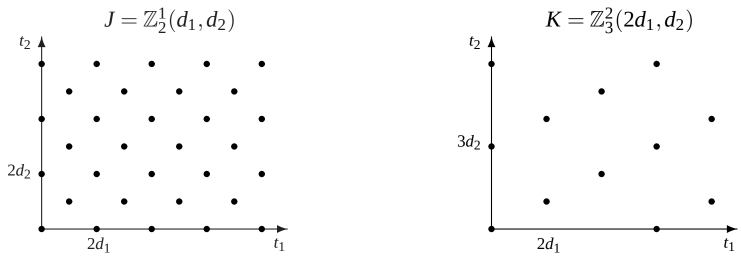

Fig. S6.3 The lattices of the problem

see by inspection that they are *not ordered*, that is $J \not\supset K$ and $K \not\supset J$. For instance, the point (d_1,d_2) belongs to J but not to K and the point $(0,3d_2)$ belongs to K but not to J. This is sufficient to establish that the *domain complexity* is 2.

However, we can explicit the relation

$$J \neq J+K \neq K$$

which requires to evaluate the sum. The sum $J+K$ (and also the intersection) can be found with the Mathematica© procedure`demo4.m` [4], which gives:

Sum and intersection of two lattices

Given **J** first lattice basis and **K** second lattice basis, the procedure `demo3` finds sum lattice basis **C** and intersection lattice basis **D**.

matrices 2×2.

$$\mathbf{J} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \mathsf{GCLD}(\mathbf{J},\mathbf{K}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \mathsf{lcrm}(\mathbf{J},\mathbf{K}) = \begin{bmatrix} 6 & 4 \\ 0 & 2 \end{bmatrix}$$

Determinant check

$$\mathrm{d}(\mathbf{J})\,\mathrm{d}(\mathbf{K}) = \mathrm{d}(\mathbf{C})\,\mathrm{d}(\mathbf{D}) \qquad 2 \cdot 6 = 1 \cdot 12 \quad (12 = 12)$$

In conclusion, the sum of the two lattices is $J + K = \mathbb{Z}^2$ and is different from both J and K.

6.22. $\star$[Sect. 6.9] Show that the cascade of two down–samplers is a down–sampler. Consider the case $\mathbb{R} \to \mathbb{Z}(T) \to \mathbb{Z}(5T)$ as an example.

Let us consider in general two samplers on $I_1 \to I_2 \to I_3$, where the sampling condition imposes $I_1 \supset I_2 \supset I_3$. The first sampling has kernel $\delta_{I_1}(t_2 - t_1)$ and the second has kernel $\delta_{I_2}(t_3 - t_2)$. Hence, from the cascade formula, we obtain the kernel

$$\begin{aligned} h(t_3, t_1) &= \int_{I_2} \mathrm{d}t_2\, \delta_{I_2}(t_3 - t_2)\, \delta_{I_1}(t_2 - t_1) \\ &= \quad \delta_{I_1}(t_3 - t_1)\,, \qquad t_1 \in I_1\,,\ t_3 \in I_3 \end{aligned}$$

so that, the equivalent tf on $I_1 \to I_3$ is ideal and since $I_1 \supset I_3$ it is a sampler.

6.23. $\star\star$[Sect. 6.9] The cascade of two impulse tfs with domains $I_1 \to I_2 \to I_3$ is not, in general, an impulse. For instance, the domains $\mathbb{R} \to \mathbb{Z}(T) \to \mathbb{R}$ do not lead to an impulse tf. Explain why.

The input output relationship has again the form (6.51). Thus we need to find the domain $D = I + U$. Considering that $\mathbb{Z}(30) + \mathbb{Z}(45) = \mathbb{Z}(15)$, it results $D = \mathbb{R}/\mathbb{Z}(15)$. Therefore

$$y(t) = \int_{\mathbb{R}/\mathbb{Z}(30)} \mathrm{d}u\, \delta_{\mathbb{R}/\mathbb{Z}(15)}(t - u)\, x(u)\,, \qquad t \in \mathbb{R}/\mathbb{Z}(45)\ .$$

A this point we could write the explicit form of the integral, but this is not convenient. In fact, it is convenient to observe that the impulse on $\mathbb{R}/\mathbb{Z}(15)$ can be decomposed into two impulses on $\mathbb{R}/\mathbb{Z}(30)$ (see (4.85))

$$\delta_{\mathbb{R}/\mathbb{Z}(15)}(t) = \delta_{\mathbb{R}/\mathbb{Z}(30)}(t) + \delta_{\mathbb{R}/\mathbb{Z}(30)}(t-15) \ .$$

Then, we can use the sifting property of the impulse, which let us simplify the input–output relationship into the form

$$y(t) = x(t) + x(t-15) \ .$$

In general, the kernel of the cascade is given by

$$h(t_3,t_2) = \int_{I_2} \mathrm{d}t \ \delta_{I_2+I_3}(t_3-t_2)\,\delta_{I_1+I_2}(t_2-t_1)$$

whereas the kernel of the $I_1 \to I_3$ impulse tf is

$$h'(t_1,t_3) = \delta_{I_1+I_3}(t_3-t_1) \ .$$

The two kernels are equal only in particular cases.

6.24. $\star\star\star$ [Sect. 6.10] Prove that the inverse of an $\mathbb{R}/\mathbb{Z}(T_p) \to \mathbb{R}$ down–periodization is given by the cascade of a window on $\mathbb{R}$ whose shape is the indicator function of the cell $[0,T_p)$, followed by an $\mathbb{R} \to /\mathbb{R}/\mathbb{Z}(T_p)$ up–periodization.

The down–periodization $\mathbb{R}/\mathbb{Z}(T_p) \to \mathbb{R}$ has input–output relation $y(t) = x(t)$, $t \in \mathbb{R}$, where $x(t)$ is a periodic signal with periodicity T_p. We observe that the inverse tf is not given directly by the up–periodization $\mathbb{R} \to \mathbb{R}/\mathbb{Z}(T_p)$, because $y(t)$, even down–periodized, is equal to $x(t)$ and hence it is periodic (its periodicity is simply ignored on the quotient group) and the direct application of the periodization could not be a convergent signal.

The multiplication performed by the window gives the signal

$$u(t) = w(t)\,x(t) = \begin{cases} x(t) & t \in [0,T_p) \\ 0 & t \notin [0,T_p) \end{cases} \ ,$$

which turns out to be actually periodic. The successive application of the periodization $\mathbb{R} \to \mathbb{R}/\mathbb{Z}(T_p)$ produces the signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} u(t-kT_p)$$

where the terms of the periodization do not overlap and thus in $[0,T_p)$ we find

$$\tilde{x}(t) = u(t) = x(t) \quad , \qquad t \in [0, T_p) \quad .$$

Since $\tilde{x}(t)$ is by construction periodic with periodicity T_p, the relationship obtained guarantees the equality of $\tilde{x}(t)$ to the original signal over $\mathbb{R}$.

6.25. $\star\star\nabla$ [Sect. 6.10] Prove that the inverse of a $\mathbb{Z}(T) \to \mathbb{R}$ up–sampling is given by the cascade of a filter on $\mathbb{R}$ with frequency response $G(f) = \mathrm{rect}(fT)$, followed by an $\mathbb{R} \to \mathbb{Z}(T)$ down–sampling.

Consider the scheme of Fig.S6.4. The $\mathbb{Z}(T) \to \mathbb{R}$ up–sampling gives

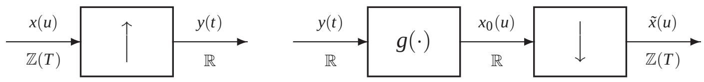

Fig. S6.4 The $\mathbb{Z}(T) \to \mathbb{R}$ up–sampling and its inverse transformation

$$y(t) = \sum_{n \in \mathbb{Z}} T\, x(nT)\, \delta(t - nT) \; .$$

The filter followed by the $\mathbb{R} \to \mathbb{Z}(T)$ down–sampling gives

$$x_0(t) = g * y(t) = \sum_{n \in \mathbb{Z}} T\, x(nT)\, g(t - nT)$$
$$\tilde{x}(mT) = x_0(mT) = \sum_{n \in \mathbb{Z}} T\, x(nT)\, g(mT - nT) \; .$$

The impulse response of the filter is

$$g(t) = (1/T)\,\mathrm{sinc}(t/T) \; .$$

Hence

$$T\, g(mT - nT) = \mathrm{sinc}(m - n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

and $\tilde{x}(mT) = x(nT)$. Then we have recovered the original signal.

6.26. $\star\star\nabla$ [Sect. 6.10] The down–sampling of a periodic signal $s(t), t \in \mathbb{R}/\mathbb{Z}(10)$ with sampling period $T_0 = 3$ cannot be formulated as an $\mathbb{R}/\mathbb{Z}(10) \to \mathbb{Z}(3)/\mathbb{Z}(10)$ down–sampling because $\mathbb{Z}(3) \not\supset \mathbb{Z}(10)$. Nevertheless a sampling, with sampling period $T_0 = 3$, is possible. Formulated this operation.

The down–sampling of the type $\mathbb{R}/\mathbb{Z}(10) \to \mathbb{Z}(T)/\mathbb{Z}(10)$ implies the condition that $\mathbb{Z}(10)$ is a subgroup of $\mathbb{Z}(T)$. The trick to find a correct down–sampling, where the period is $T_0 = 3$, lies in the introduction of an $\mathbb{R}/\mathbb{Z}(10) \to \mathbb{R}/\mathbb{Z}(30)$ down–periodization (which is only conceptual without a signal modification). Then we have the cascade of

1) $\mathbb{R}/\mathbb{Z}(10) \to \mathbb{R}/\mathbb{Z}(30)$ down–periodization
2) $\mathbb{R}/\mathbb{Z}(30) \to \mathbb{Z}(3)/\mathbb{Z}(30)$ down–sampling.

A check can be made in the frequency domain, where the wrong $\mathbb{R}/\mathbb{Z}(10) \to \mathbb{Z}(3)/\mathbb{Z}(10)$ down–sampling would give the $\mathbb{Z}(1/10) \to \mathbb{Z}(1/10)/\mathbb{Z}(1/3)$ periodization with the wrong relation

$$Y\left(m\tfrac{1}{10}\right) = \sum_{k=-\infty}^{+\infty} X\left(m\tfrac{1}{10} - k\tfrac{1}{3}\right) .$$

Instead, the cascade 1), 2) gives a $\mathbb{Z}(1/10) \to \mathbb{Z}(1/30)$ interpolator with relation

$$X_0\left(k\tfrac{1}{30}\right) = \begin{cases} 3X\left(k\tfrac{1}{30}\right)) & k \in \mathbb{Z}(3) \\ 0 & k \notin \mathbb{Z}(3) \end{cases}$$

and a $\mathbb{Z}(1/30) \to \mathbb{Z}(1/30)/\mathbb{Z}(1/3)$ periodization with relation

$$Y\left(k\tfrac{1}{30}\right) = \sum_{m=-\infty}^{+\infty} X_0\left(k\tfrac{1}{30} - m\tfrac{1}{3}\right) .$$

6.27. $\star$[Sect. 6.13] Find the dual of a $\mathbb{Z}(T) \to \mathbb{Z}(NT)$ down–sampling and write the corresponding input–output relation.

From $I = \mathbb{Z}(T_0) \to U = \mathbb{Z}(NT_0)$ it follows

$$\widehat{I} = \mathbb{R}/\mathbb{Z}(NF_c) \to \widehat{U} = \mathbb{R}/\mathbb{Z}(F_c) \quad , \qquad F_c = \frac{1}{NT_0}$$

therefore the dual of the down–sampling $\mathbb{Z}(T_0) \to \mathbb{Z}(NT_0)$ is the up–periodization $\mathbb{R}/\mathbb{Z}(NF_c) \to \mathbb{R}/\mathbb{Z}(F_c)$. From the general relation (6.80), that is,

$$Y(f) = \sum_{p \in [U_0^\star/I_0^\star)} X(f-p) \quad ,$$

taking into account that

$$p \in [U_0^\star/I_0^\star) = [\mathbb{Z}(F_c)/\mathbb{Z}(NF_c)) = \{0, F_c, \ldots, (N-1)F_c\}$$

we obtain

$$Y(f) = \sum_{k=0}^{N-1} X(f - kF_c), \qquad f \in \mathbb{R}/\mathbb{Z}(F_c) .$$

In this relationship $X(f)$ has periodicity NF_c, while $Y(f)$ has periodicity F_c, that is, N times smaller.

6.28. $\star\star$ [Sect. 6.13] Consider the $\mathbb{R}/\mathbb{Z}(20) \to \mathbb{R}/\mathbb{Z}(60)$ down–periodization. Find a) the impulse response, b) the impulse response of the dual tf, and c) the input–output relation of the dual tf.

The impulse response of an impulse tf $I \to U$ is given by the impulse on $I+U$. Since it is

$$\mathbb{R}/\mathbb{Z}(20) + \mathbb{R}/\mathbb{Z}(60) = \mathbb{R}/\mathbb{Z}(20)$$

the impulse response is $\delta_{\mathbb{R}/\mathbb{Z}(20)}(v)$. Taking into account that

$$\widehat{I} = \mathbb{Z}(3F_0), \qquad \widehat{U} = \mathbb{Z}(F_0) \qquad F_0 = \tfrac{1}{20}$$

and also that $\widehat{I} + \widehat{U} = \mathbb{Z}(F_0)$, the impulse response of the dual tf is $\delta_{\mathbb{Z}(F_0)}(f)$.

Since we are dealing with $\mathbb{Z}(F_0) \to \mathbb{Z}(\frac{1}{3}F_0)$ up–sampling, the relation results explicitly (see (6.57)):

$$Y(kF_0) = \begin{cases} 3X(kF_0) & k \in \mathbb{Z}(3) \\ 0 & k \notin \mathbb{Z}(3) . \end{cases}$$

6.29. $\star$ [Sect. 6.14] Find the dual tf of a $\mathbb{Z}(3T) \to \mathbb{Z}(5T)$ interpolator/decimator.

The filter can be decomposed into the cascade of: 1) a $\mathbb{Z}(3T) \to \mathbb{Z}(T)$ interpolator; 2) a filter on $\mathbb{Z}(T)$; 3) a $\mathbb{Z}(T) \to \mathbb{Z}(5T)$ down–sampler. Hence, the dual tf is the cascade of:

1) a $\mathbb{R}/\mathbb{Z}\left(\frac{1}{3T}\right) \to \mathbb{R}/\mathbb{Z}\left(\frac{1}{T}\right)$ down–periodization,
2) a window on $\mathbb{R}/\mathbb{Z}\left(\frac{1}{T}\right)$,
3) a $\mathbb{R}/\mathbb{Z}\left(\frac{1}{T}\right) \to \mathbb{R}/\mathbb{Z}\left(\frac{1}{5T}\right)$ up–periodization.

6.30. $\star$ [Sect. 6.14] Find the Fourier transform of the output of a window on $\mathbb{R}$ with shape and input signal given by, respectively:

$$w(t) = \mathrm{rect}(t/T) , \quad x(t) = 1(t)\, \mathrm{e}^{-\alpha t} \qquad \text{with} \qquad \alpha = 1/T .$$

The output signal results:

$$y(t) = x(t)\,w(t) = \mathrm{e}^{-at}\,1(t)\,1\left(\tfrac{1}{2}T - t\right) = \begin{cases} \mathrm{e}^{-\alpha t} & 0 < t < \frac{1}{2}T \\ 0 & \text{elsewhere} \end{cases}, \qquad t \in \mathbb{R}\,.$$

Hence, applying the definition of a transform on $\mathbb{R}$, given by (5.65a), we obtain:

$$Y(f) = T\,\frac{1 - \mathrm{e}^{-\left(\frac{1}{2}+\mathrm{i}\pi f T\right)}}{1 + \mathrm{i}2\pi f T}\,, \qquad f \in \mathbb{R}\,.$$

6.31. $\star$[Sect. 6.15] Restate the axiomatic derivation of (6.88) for a discrete filter on $\mathbb{Z}(T)$.

On the discrete time–domain the integral (6.51) becomes a summation, namely

$$y(t) = \sum_{n=-\infty}^{+\infty} T\,g(t - nT)\,x(nT)\,. \tag{S6.5}$$

For the proof we decompose the input signal into impulses

$$x(t) = \sum_{n=-\infty}^{+\infty} T\,x(nT)\,\delta_{\mathbb{Z}(T)}(t - nT) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} x_n(t)\,, \qquad t \in \mathbb{Z}(T)\,.$$

and we take into account the response of the filter to the impulse $\delta_{\mathbb{Z}(T)}(t)$ is given, by definition, by the impulse response $g(t)$. Then, the response to the nth impulse results

$$x_n(t) = T\,x(nT)\,\delta_{\mathbb{Z}(T)}(t - nT) \xrightarrow{\text{filter}} T\,x(nT)\,g(t - nT)\,.$$

The overall response is obtained from the superposition of each contribution. Then, (S6.5) follows.

6.32. $\star$[Sect. 6.15] Prove the decomposition into sinusoidal components (6.90b) for a *real* filter with a *real* input.

Since $g(t)$ and $x(u)$ are real, $G(f)$ and $X(f)$ have the Hermitian symmetry

$$G(-f) = G^*(f)\,, \qquad X(-f) = X^*(f)\,.$$

On the other hand, the output signal results

$$y(t) = \int_{-\infty}^{+\infty} G(f)X(f)\,\mathrm{e}^{\mathrm{i}2\pi ft}\,\mathrm{d}f$$
$$= \int_{0}^{\infty} [G(f)X(f)\mathrm{e}^{\mathrm{i}2\pi ft} + G(-f)X(-f)\,\mathrm{e}^{-\mathrm{i}2\pi ft}]\,\mathrm{d}f$$

where the two terms are complex conjugate, so that

$$y(t) = 2\Re\int_{0}^{\infty} G(f)X(f)\,\mathrm{e}^{\mathrm{i}2\pi ft}\,\mathrm{d}f\ .$$

Hence, letting

$$G(f) = A_G(f)\,\mathrm{e}^{\mathrm{i}\varphi_G(f)}, \qquad X(f) = A_X(f)\,\mathrm{e}^{\mathrm{i}\varphi_X(f)}$$

we get

$$y(t) = \int_{0}^{\infty} 2A_G(f)\,A_X(f)\,\cos\Big(2\pi ft + \varphi_G(f) + \varphi_x(f)\Big)\,\mathrm{d}f\ .$$

This result holds if the input signal has no continuous component (see Sect. 2.1). In the presence of a continuous component X_0, which gives as Fourier transform $X_0\,\delta(f)$, we need to add the term $Y_0\,A_G(0)$, which yields the continuous component of the output signal.

6.33. $\star$[Sect. 6.15] Find the impulse response of a discrete ideal filter on $\mathbb{Z}(T)$ with pass–band

$$e(G) = (-B,B) + \mathbb{Z}(F_p)\ , \quad F_p = \frac{1}{T}\ ,\ B < \frac{1}{2}F_p\ .$$

The frequency response can be expressed in the form:

$$G(f) = \mathrm{rep}_{F_p}\ \mathrm{rect}\left(\frac{f}{2B}\right), \qquad f \in \mathbb{R}/\mathbb{Z}(F_p)\ .$$

Hence, taking the inverse FT according to (6.31b), we obtain:

$$g(t) = 2B\ \mathrm{sinc}\,(2Bt), \qquad t \in \mathbb{Z}(T)\ ,\ T = \frac{1}{F_p}\ .$$

6.34. [Sect. 6.15] Calculate the output of a discrete time filter with impulse response

$$g(nT) = 1_0(n)\,a^n\ , \quad a \text{ real}$$

when the input is a generic sinusoid $A_0\ \cos(2\pi f_0 t + \varphi_0)$.

We evaluate the frequency response $G(f)$ and hence we take into account that filters eigenfunctions are the signals $x_0(t) = \mathrm{e}^{\mathrm{i}2\pi f t}$ with eigenvalues $G(f)$

$$x_0(t) = \mathrm{e}^{\mathrm{i}2\pi f t} \xrightarrow{\text{filter}} y_0(t) = G(f)\,\mathrm{e}^{\mathrm{i}2\pi f t} \,.$$

Decomposing the input signal in the form

$$x(t) = A_0 \cos 2\pi f_0 t = \tfrac{1}{2} A_0\,\mathrm{e}^{\mathrm{i}2\pi f t} + \tfrac{1}{2} A_0\,\mathrm{e}^{-\mathrm{i}2\pi f t}$$

we find

$$\begin{aligned} y(t) &= \tfrac{1}{2} A_0\,\mathrm{e}^{\mathrm{i}\varphi_0}\, G(f_0)\,\mathrm{e}^{\mathrm{i}2\pi f t} + \tfrac{1}{2} A_0\,\mathrm{e}^{-\mathrm{i}\varphi_0}\, G(-f_0)\,\mathrm{e}^{-\mathrm{i}2\pi f t} \\ &= \Re A_0\,\mathrm{e}^{\mathrm{i}\varphi_0}\, G(f_0)\,\mathrm{e}^{\mathrm{i}2\pi f t} = A_0\,|G(f_0)| \cos(2\pi f_0 t + \varphi_0 + \beta_0) \end{aligned}$$

where $\beta_0 = \arg G(f_0)$ and we recalled that the filter is *real*, so that the frequency response has Hermitian symmetry $G(-f_0) = G^*(f_0)$.

In this specific case it results

$$G(f) = \sum_{n=0}^{\infty} T\, a^n\, \mathrm{e}^{-\mathrm{i}2\pi f t} = T\, \frac{1}{1 - a\exp(-\mathrm{i}2\pi f T)} \,.$$

Hence

$$\begin{aligned} |G(f)| &= \frac{1}{|1 - a\,\exp(-\mathrm{i}2\pi f t)|} = \frac{1}{1 + a^2 - 2a\cos 2\pi f T} \\ \arg G(f) &= -\arg\left(1 - a\,\exp(-\mathrm{i}2\pi f T)\right) = -\tan^{-1} \frac{a \sin 2\pi f T}{1 - a\cos 2\pi f T} \end{aligned}$$

from which, letting $f = f_0$, we can obtain the expression of $y(t)$. With $fT = \frac{1}{3}$ and $a = \frac{1}{2}$ one gets

$$|G(f_0)| = 0.571\,, \qquad \beta_0 = -0.33 \text{ rad } = -19.1^{\text{o}} \,.$$

6.35. $\star\star$ [Sect. 6.15] Prove that the impulse response of the ideal band–pass filter with the rhomboidal frequency response of Fig.6.49 is given by (see (5.100))

$$g(t_1,t_2) = B_1 B_2 \operatorname{sinc}(B_1 t_1 - B_2 t_2) \operatorname{sinc}(B_1 t_1 + B_2 t_2) \cos\left(2\pi(f_{01} t_1 + f_{02} t_2)\right) \,,$$

where $f_0 = (f_{01}, f_{02}) = (3B_1, 3B_2)$.

See the solution of Problem 8.20

Problems of Chapter 7

7.1. $\star\star$[Sect. 7.1] Show that an exponential modulator on a lattice J with a frequency $\lambda \in P^\star$ is a PIL tf with periodicity P (P is a sublattice of J and $P^\star$ is the reciprocal of P).

The input–output relation of an exponential modulator is

$$y(t) = \mathrm{e}^{\mathrm{i}2\pi\lambda t}\, x(t)\,, \qquad t \in J\,. \tag{S7.1}$$

The condition for a PIL with periodicity P is

$$y(t+p) = y(t)\,, \qquad p \in P$$

where $y(t)$ is given by (S7.1) and

$$y(t+p) = \mathrm{e}^{\mathrm{i}2\pi\lambda(t+p)}\, x(t+p)\,.$$

If we apply a shift of $p \in P$ at the input, the response is the signal

$$y'(t) = \mathrm{e}^{\mathrm{i}2\pi\lambda t}\, x(t-p)\,.$$

For the periodicity this response must be the p–shifted version of $y(t)$, that is,

$$y(t-p) = \mathrm{e}^{\mathrm{i}2\pi\lambda(t-p)}\, x(t-p)\,.$$

Now, the condition $y'(t) = y(t-p)$ implies that

$$\mathrm{e}^{\mathrm{i}2\pi(\lambda p)t} = \mathrm{e}^{\mathrm{i}2\pi\lambda t} \Rightarrow \mathrm{e}^{-\mathrm{i}2\pi\lambda p} = 1\,.$$

On the other hand, the reciprocal of $P^\star$ is defined by

$$J^\star = \{\lambda \mid \mathrm{e}^{\mathrm{i}2\pi\lambda p} = 1\,, \qquad p \in J\}\,.$$

Hence, the conclusion.

7.2. $\star\star$[Sect. 7.3] Apply Theorem 7.3 to the case $\mathbb{R}/\mathbb{Z}(3) \to \mathbb{R}/\mathbb{Z}(5)$ and discuss the result by considering the signals in the cascades.

Considering that

$$J = \mathbb{R}/\mathbb{Z}(3)\,, \qquad K = \mathbb{R}/\mathbb{Z}(5)\,, \qquad J+K = \mathbb{R}/\mathbb{Z}(1)\,, \qquad J \cap K = \mathbb{R}/\mathbb{Z}(15)$$

we find the decomposition of Fig.S7.1.

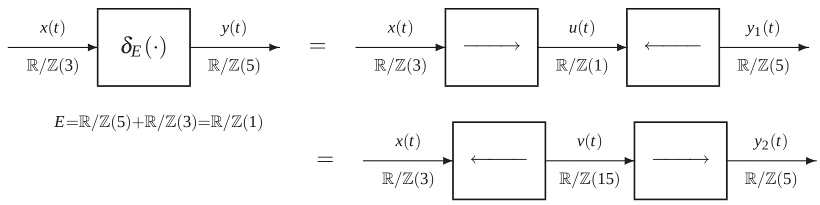

Fig. S7.1 Application of Theorem 7.3

Then, the $\mathbb{R}/\mathbb{Z}(3) \to \mathbb{R}/\mathbb{Z}(5)$ impulse tf can be decomposed in the two equivalent cascades

1) an $\mathbb{R}/\mathbb{Z}(3) \to \mathbb{R}(1)$ up–periodization, followed by an $\mathbb{R}/\mathbb{Z}(1) \to \mathbb{R}/\mathbb{Z}(5)$ down–periodization
2) an $\mathbb{R}/\mathbb{Z}(3) \to \mathbb{R}/\mathbb{Z}(15)$ down–periodization, followed by an $\mathbb{R}/\mathbb{Z}(15) \to \mathbb{R}/\mathbb{Z}(5)$ up–periodization.

Considering that the down–periodizations are irrelevant for the signals, we find respectively the relations

$$y_1(t) = \sum_{k=0}^{2} x(t-k) = x(t) + x(t-1) + x(t-2) \,,$$
$$y_2(t) = \sum_{k=0}^{2} v(t-5k) = \sum_{k=0}^{2} x(t-5k) = x(t) + x(t-5) + x(t-10) \,.$$

Now, $x(t)$ has period 3 and therefore

$$x(t-5) = x(t-5+3) = x(t-2)\,, \qquad x(t-10) = x(t-10+9) = x(t-1)$$

which states that $y_2(t) = y_1(t)$, that is, the two decompositions have the same input–output relations.

The signals in the decompositions are illustrated in Fig.S7.2.

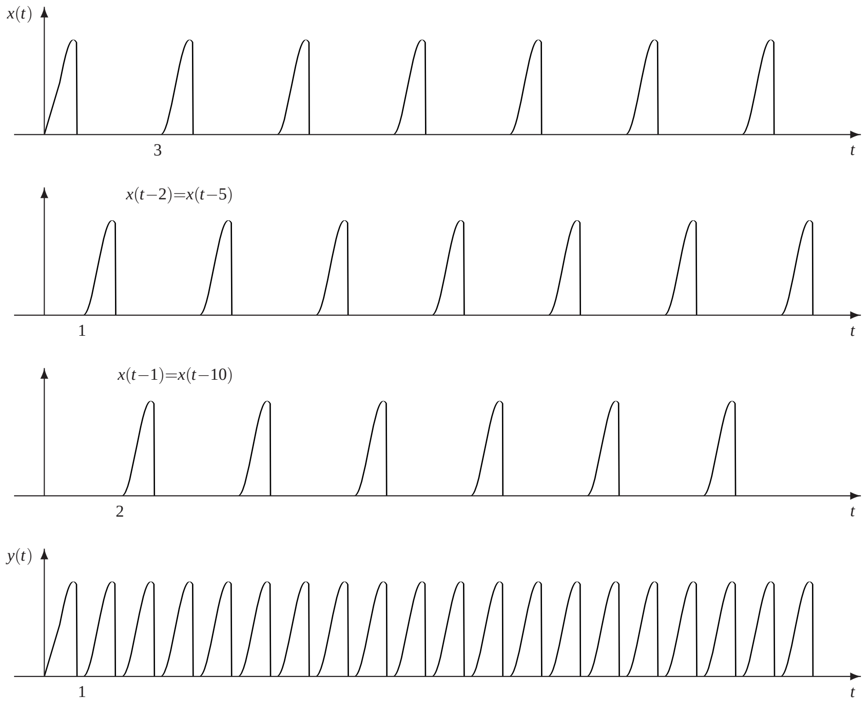

Fig. S7.2 The signals $x(t-2)$ and $x(t-5)$ coincide because of the periodicity. The same is for the signals $x(t-1)$ and $x(t-10)$

7.3. ⋆⋆[Sect. 7.4] Consider a $\mathbb{Z} \to \mathbb{Z}(T_0)$ interpolator with impulse response $g(t) = (1/10T_0)\,\mathrm{sinc}(t/10T_0)$, $t \in \mathbb{Z}(T_0)$ followed by an EM with frequency $\lambda = 1/(5T_0)$. Apply Noble Identity NI6.

The Noble Identity NI6 is shown in Fig.S7.3 in the specific case of the problem. Let

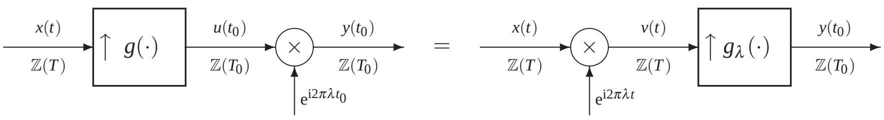

Fig. S7.3 Application of Noble Identity N16

$T = NT_0$. Then the relations in the first cascade are

$$
\begin{aligned}
u(nT_0) &= T \sum_{k\in\mathbb{Z}} g(nT_0 - kT)\,x(kT) \\
y(nT_0) &= e^{i2\pi\lambda nT_0}\,u(nT_0) = e^{i2\pi n/5}\,u(nT_0)\ .
\end{aligned}
$$

In the second cascade

$$
\begin{aligned}
v(nT) &= e^{i2\pi\lambda nT}\,x(nT) = e^{i2\pi nN/5}\,x(nT) \\
y(nT_0) &= T \sum_{k\in\mathbb{Z}} g_\lambda(nT_0 - kT)\,v(kT)
\end{aligned}
$$

where

$$
\begin{aligned}
g_\lambda(nT_0) &= g(nT_0)\,e^{-i2\pi\lambda nT_0} = g(nT_0)\,e^{-i2\pi n/5} \\
&= \frac{1}{10T_0}\,\mathrm{sinc}\left(\frac{n}{10}\right) e^{-i2\pi n/5}\ .
\end{aligned}
$$

Note that if N is a multiple of 5 the exponential modulator in the second cascade becomes irrelevant and we have a *carrierless* modulation.

7.4. ⋆⋆⋆[Sect. 7.4] Consider a $J \to K$ down–sampler followed by a $K \to J$ up–sampler. Prove that the cascade is equivalent to a modulator on J with the carrier $\gamma(t)$, $t \in J$ given by the indicator function of K multiplied by the amplification $A_0 = \mathrm{d}(K)/\mathrm{d}(J)$ of the up–sampler.

We have to prove the cascade of Fig.S7.4 is equivalent to a modulator with relation

$$
y(t) = \gamma(t)\,x(t) \tag{S7.2}
$$

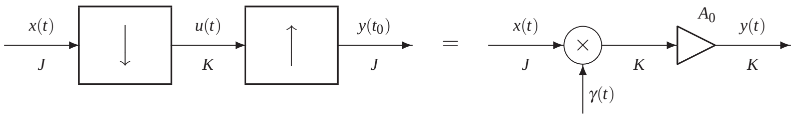

Fig. S7.4 Illustration of the statement of the problem

where $\gamma(t) = A_0\, \eta_K(t)$, $t \in J$, with $\eta(t) = 1$ for $t \in K$ and $\eta_K(t)$ for $t \notin K$.

Let us consider a preliminary case: $J = \mathbb{Z}$, $K = \mathbb{Z}(3)$, where the down–sampling relation is

$$u(3n) = x(3n)\,, \qquad 3n \in \mathbb{Z}$$

and the up–sampling relation is

$$y(k) = \begin{cases} 3u(k) & k \in \mathbb{Z}(3) \\ 0 & k \notin \mathbb{Z}(3) \end{cases} \qquad k \in \mathbb{Z}\,.$$

Then, the global relation is

$$y(k) = \begin{cases} 3x(k) & k \in \mathbb{Z}(3) \\ 0 & k \notin \mathbb{Z}(3) \end{cases}$$

which can be written in the form (S7.2).

In the general case the relations are

$$u(t) = x(t)\,, \qquad t \in K \qquad\qquad y(t) = \begin{cases} A_0\, u(t) & t \in K \\ 0 & t \notin K \end{cases}$$

and globally

$$y(t) = \begin{cases} A_0\, x(t) & t \in K \\ 0 & t \notin K \end{cases} \tag{S7.3}$$

where $A_0 = \mathrm{d}(J)/\mathrm{d}(K)$ is the amplification of the interpolator. Relation (S7.3) is the explicit form of (S7.2).

7.5. [Sect. 7.5] 1 Study the S/P and P/S conversions on $\mathbb{Z}(T_0)/P \to \mathbb{Z}(T)/P$ with $T = 5T_0$ and $P = \mathbb{Z}(15T_0)$. Write all the cells involved and the frequency domain relationships.

The S/P conversion converts a discrete periodic signal $x(nT_0)$, with period $T_p = 15T_0$, into 5 discrete periodic signals $x_k(n5T_0)$, with period $T_p = 15T_0$. The relation is the same as in the aperiodic case, that is,

$$x_i(n5T_0) = x(iT_0 + n5T_0)\,, \qquad i = 0, 1, 2, 3, 4\,.$$

The cell in this decomposition is $A = \mathbb{Z}_5(T_0) = \{0, T_0, 2T_0, 3T_0, 4T_0\}$. The reciprocal cell is

$$\begin{aligned}A^\star &= [J_0^\star/I_0^\star) = [\mathbb{Z}(1/(5T_0))/\mathbb{Z}(1/T_0))\\ &= [\mathbb{Z}(F)/\mathbb{Z}(5F)) = \{0, F, 2F, 3F, 4F, 5F\}\end{aligned}$$

where $F = 1/(5T_0)$. The domain/periodicities in the frequency domain are

$$\widehat{I} = P^\star/I_0^\star = \mathbb{Z}(F_0)/\mathbb{Z}(15F_0)\,, \qquad \widehat{J} = P^\star/J_0^\star = \mathbb{Z}(F_0)/\mathbb{Z}(3F_0)$$

where $F_0 = 1/(15T_0)$.

The frequency–domain relation of the S/P is still given by (7.31) with the cell $[J^\star/I^\star)$ replaced by $A^\star = [J_0^\star/I_0^\star) = \mathbb{Z}_N(F)$, that is,

$$\begin{aligned}X_i(f) &= \sum_{\lambda\in A^\star} \mathrm{e}^{\mathrm{i}2\pi(f-\lambda)iT_0} X(f-\lambda)\\ &= \sum_{s=0}^{4} \mathrm{e}^{\mathrm{i}2\pi(f-sF)iT_0} X(f-sF)\end{aligned}$$

where $f \in \widehat{J} = \mathbb{Z}(F_0)/\mathbb{Z}(3F_0)$.

The relations of the P/S conversion can be obtained in a similar way.

7.6. ⋆⋆ [Sect. 7.7] Explicitly write the parallel decomposition of an $I = \mathbb{Z}(2) \to U = \mathbb{Z}(5)$ QIL tf, choosing as inner domains $J = \mathbb{Z}(6)$ and $K = \mathbb{Z}(10)$.

The parallel architecture consists of

1) a $\mathbb{Z}(2) \to \mathbb{Z}(6)$ S/P converter with generator $A = [I/J) = \{0, 1, 3\}$,
2) a $\mathbb{Z}(6) \to \mathbb{Z}(10)$ QIL with 3 inputs and 2 outputs, and
3) a $\mathbb{Z}(10) \to \mathbb{Z}(5)$ P/S converter with generator $B = [U/K) = \{0, 5\}$.

The impulse response $g(v_0)$ is defined on $I + U = \mathbb{Z}(2) + \mathbb{Z}(5) = \mathbb{Z}$ and in the parallel architecture becomes the matrix

$$\mathbf{g}(v) = \begin{bmatrix} g_{00}(v) & g_{01}(v) & g_{02}(v)\\ g_{10}(v) & g_{11}(v) & g_{12}(v)\end{bmatrix} = \frac{1}{3}\begin{bmatrix} g(v) & g(v-1) & g(v-2)\\ g(v+5) & g(v+4) & g(v+3)\end{bmatrix}\,. \tag{S7.4}$$

where $v \in \mathbb{Z}(2)$.

7.7. ⋆⋆ [Sect. 7.7] In the previous problem suppose that the impulse response of the $\mathbb{Z}(2) \to \mathbb{Z}(5)$ QIL tf is given by

$$g(n) = A_0 \operatorname{sinc}(n/10)\,, \qquad n \in \mathbb{Z}\,.$$

Find the frequency response of the corresponding 5–input 2–output $\mathbb{Z}(10) \to \mathbb{Z}(5)$ QIL parallel architecture.

We rewrite the impulse response in the form

$$g(n) = A_0 \operatorname{sinc}(nB), \qquad n \in \mathbb{Z}, \ B = 1/10.$$

The corresponding frequency response is (see Tab.11.1))

$$G(f) = A_0 \operatorname{rep}_{F0} \operatorname{rect}(f/B), \qquad f \in \mathbb{R}/\mathbb{Z}(1)$$

The elements $g_{ba}(v), v \in \mathbb{Z}(2)$ are obtained from $g(v), v \in \mathbb{Z}$ by shifts of $a-b$ followed by a $\mathbb{Z}(1) \to \mathbb{Z}(2)$ downsampling. The shifts give $\tilde{G}_{ba}(f) = G(f)\mathrm{e}^{-i2\pi f(a-b)}$ and the $\mathbb{R}/\mathbb{Z}(1) \to \mathbb{R}/\mathbb{Z}(\frac{1}{2})$ periodization gives

$$G_{ba}(f) = \tilde{G}_{ba}(f) + \tilde{G}_{ba}(f - \tfrac{1}{2}) = G(f)\mathrm{e}^{-i2\pi f(a-b)} + G(f - \tfrac{1}{2})\mathrm{e}^{-i2\pi(f-\frac{1}{2})(a-b)}$$

with $a - b = 0, 1, 2, -5, -4, -3$.

7.8. $\star\star$ [Sect. 7.7] Consider a $\mathbb{Z}(3) \to \mathbb{Z}(5)$ QIL tf and its parallel decomposition obtained with $J = \mathbb{Z}(15)$, $K = \mathbb{Z}(25)$. Explicitly write the matrix $\mathbf{g}(v)$ of decomposition (7.42) and show that its elements are a circulant replica of the elements of the first row.

The parallel decomposition is illustrated in Fig. 7.24. The impulse responses in the decomposition are given by (7.42), that is,

$$g_{ba}(v) = (1/M)\, g(v + b - a), \qquad v = J + K, \qquad a \in A, \qquad b \in B$$

where

$$\begin{aligned} &J + K = \mathbb{Z}(15) + \mathbb{Z}(25) = \mathbb{Z}(5), \qquad M = \mathrm{d}(J)/\mathrm{d}(I) = 5 \\ &A = [I/J) = [\mathbb{Z}(3)/\mathbb{Z}(15)) = \{0, 3, 6, 9, 12\} \\ &B = [U/K) = [\mathbb{Z}(5)/\mathbb{Z}(25)) = \{0, 5, 10, 15, 20\}\,. \end{aligned}$$

The elements of the matrix $\mathbf{g}(v)$ are

$$g_{ba}(v), \qquad a \in A, \qquad b \in B\,.$$

To write the matrix in a standard form it is convenient to let $a = 3i$ and $b = 5j$, with $i, j = 0, 1, 2, 3, 4$. Hence

$$g_{ba}(v) = \frac{1}{5} g(v + b - a) \to g_{ji}(v) = \frac{1}{5} g(v + 5j - 3i)$$

where j is the row index and i the column index. We have

$$\mathbf{g}(v) = \frac{1}{5}\begin{bmatrix} g_{00}(v) & g_{10}(v) & g_{20}(v) & g_{30}(v) & g_{40}(v) \\ g_{01}(v) & g_{11}(v) & g_{21}(v) & g_{31}(v) & g_{41}(v) \\ g_{02}(v) & g_{12}(v) & g_{22}(v) & g_{32}(v) & g_{42}(v) \\ g_{03}(v) & g_{13}(v) & g_{23}(v) & g_{33}(v) & g_{43}(v) \\ g_{04}(v) & g_{14}(v) & g_{24}(v) & g_{34}(v) & g_{44}(v) \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} g(v) & g(v+5) & g(v+10) & g(v+15) & g(v+20) \\ g(v-3) & g(v+2) & g(v+7) & g(v+12) & g(v+17) \\ g(v-6) & g(v-1) & g(v+4) & g(v+9) & g(v+14) \\ g(v-9) & g(v-4) & g(v+1) & g(v+6) & g(v+11) \\ g(v-12) & g(v-7) & g(v-2) & g(v+3) & g(v+8) \end{bmatrix}.$$

Now, the original impulse response $g(v_0)$ is defined on $I+U = \mathbb{Z}(3)+\mathbb{Z}(5) = \mathbb{Z}(1)$ and the elements $g_{ji}(v)$ are defined on $\mathbb{Z}(5)$. All the $5\times 5 = 25$ impulse responses $g_{ij}(v) = \frac{1}{5}g(v+5j-3i)$ can be obtained from polyphase decomposition of $g(v_0)$, $v_0 \in \mathbb{Z}(1)$ given by $q_k(v) = g(v+k)$, $k = 0,1,2,3,4$ and appropriate delays of $\mathbb{Z}(5)$. For instance $g(v-6) = g(v-10+4) = g_4(v-10)$ and $g(v+17) = g(v+15+2) = g_2(v+15)$.

7.9. $\star\star\star$[Sect. 7.10] Prove the orthogonality condition of the perfect time–limited OFDM.

We can prove the orthogonality condition using the efficient architecture of Fig.7.42. We have to prove that the cascade modulator/demodulator is equivalent to the $M\times M$ identity on $\mathbb{Z}(T)$. Now, in this case the P/S converter followed by the S/P converter is equivalent to the identity and, in the proof, can be dropped.

It remains the cascade of the IDFT followed by the DFT, which are the inverse of each other and therefore are equivalent to the identity.

Problems of Chapter 8

8.1. $\star$[Sect. 8.1] Explicit the Nyquist criterion for $I = \mathbb{Z}(T)$ and $U = \mathbb{Z}(T_0)$ with $T = NT_0$.

The Nyquist criterion is expressed by (8.14) for a pair of domains I, U with $I \subset U$. Now, the dual of

$$I = \mathbb{Z}(T)\,, \qquad U = \mathbb{Z}(T_0)\,, \qquad \text{with} \quad T = NT_0 \tag{S8.1}$$

are respectively

$$\widehat{I} = \mathbb{R}/\mathbb{Z}(F_c)\,, \qquad \widehat{U} = \mathbb{R}/\mathbb{Z}(F_0) \quad \text{with} \quad F_c = 1/T\ ,\ F_0 = 1/T_0 = NF_c\ .$$

We also have $I^\star/U_0^\star = \mathbb{Z}(F_c)/\mathbb{Z}(F_0)$. Therefore, the Nyquist criterion on the domains (S8.1) becomes

$$\sum_{k=0}^{N-1} G(f - kF_c) = 1\ . \tag{S8.2}$$

The frequency response $G(f)$ has period F_0 and the periodic repetition $\mathbb{R}/\mathbb{Z}(F_0) \to \mathbb{R}/\mathbb{Z}(F_c)$ leads to a period F_c, which is N times smaller than F_0. For the interpolating function $g_0(t_0), t_0 \in \mathbb{Z}(T_0)$ the condition is

$$g_0(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0\ . \end{cases} \tag{S8.3}$$

8.2. $\star$[Sect. 8.1] Verify that an interpolating filter, whose frequency response $G(f)$, $f \in \mathbb{R}$ has an *isosceles triangle shape* over $(-F_c, F_c)$, satisfies the correct interpolation condition.

The check in the frequency domain must be done according to (8.17), that is

$$\sum_{k=-\infty}^{+\infty} G(f - kF_c) = 1\ . \tag{S8.4}$$

If the frequency response $G(f)$ is isosceles triangle shaped between $-F_c$ and F_c, then the graphical check of (S8.4) is immediate (the height of the triangle must equal 1).

The check in the time domain follows from (S8.3), that is

$$g_0(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \,. \end{cases}$$

The inverse transform of $G(f)$ is

$$g(t) = T\,\mathrm{sinc}^2(F_c t)\,, \qquad t \in \mathbb{R}\,, \quad F_c = 1/T \tag{S8.5}$$

and then the interpolating function $g_0(t) = g(t)/T$ actually verifies eq. (S8.4), recalling that $\mathrm{sinc}(0) = 1$ and $\mathrm{sinc}(k) = 0$ for $k \neq 0$

$$\sum_{k=-\infty}^{+\infty} G(f - kF_c) = 1\,, \qquad f \in \mathbb{R}/\mathbb{Z}(F_c)\,,\ F_c = \frac{1}{T}\,. \tag{S8.6}$$

8.3. ⋆⋆⋆[Sect. 8.1] Consider the frequency response $G(f)$, $f \in \mathbb{R}$, defined for $f > 0$, as follows

$$G(f) = \begin{cases} 1 & 0 < f < \frac{1}{2}F_c(1-r) \\ \alpha(f) & \frac{1}{2}F_c(1-r) < f < \frac{1}{2}F_c(1+r) \\ 0 & f > \frac{1}{2}F_c(1+r) \end{cases}\,, \qquad 0 \le r \le 1$$

and extended by the even symmetry for $f < 0$.

Find the conditions on the function $\alpha(f)$ such that $G(f)$ verifies the Nyquist criterion (S8.6).

The Nyquist criterion requires that

$$\tilde{G}(f) \stackrel{\Delta}{=} \sum_{k=-\infty}^{+\infty} G(f - kF_c) = 1\,.$$

Considering the periodicity of $\tilde{G}(f)$, we only need to verify that $\tilde{G}(f) = 1$ in a period, as $-\frac{1}{2}F_c < f < \frac{1}{2}F_c$. Furthermore, recalling that $G(f)$ is even and, consequently, $\tilde{G}(f)$ is even too, the check can be limited to half a period, that is

$$\tilde{G}(f) = G(f) = 1\,, \qquad 0 < f < \tfrac{1}{2}F_c(1-r)\,.$$

The extension of $G(f)$ is

$$e(G) = \left(-\tfrac{1}{2}F_c(1+r)\,,\ \tfrac{1}{2}F_c(1+r)\right) \subset (-F_c, F_c)$$

and hence in the considered interval $\left[0, \frac{1}{2}F_c\right]$ the summation includes only two terms (Fig.S8.1)

$$\tilde{G}(f) = G(f) + G(f - F_c)\,.$$

For $0 < f < \frac{1}{2}F_c(1-r)$, the term centered in F_c gives no contribution and then

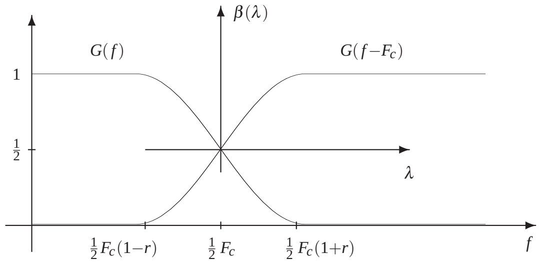

Fig. S8.1 Example of *even* roll–off with respect to the coordinate pair $(\frac{1}{2}F_c, \frac{1}{2})$.

$$\tilde{G}(f) = G(f) = 1\,, \qquad 0 < f < \tfrac{1}{2}F_c(1-r)\,.$$

For $\frac{1}{2}F_c(1-r) < f < \frac{1}{2}F_c$, both terms produce a roll-off and precisely

$$\tilde{G}(f) = \alpha(f) + \alpha(f - F_c) = \alpha(f) + \alpha(F_c - f)\,,$$

where we have used the fact that $\alpha(f)$ is even. Hence the following condition on the junction must be imposed

$$\boxed{\alpha(f) + \alpha(F_c - f) = 1\,.}$$

To improve the interpretation of this condition, we introduce the function

$$\beta(\lambda) = \alpha(\lambda + \tfrac{1}{2}F_c) - \tfrac{1}{2}\,, \tag{S8.7}$$

which has to verify the condition $\beta(\lambda) = -\beta(\lambda)$, that is the function $\beta(\lambda)$ must be odd.

From eq. (S8.7) we can conclude that *every roll–off that is an odd function with respect to the point of coordinates* $(\frac{1}{2}F_c, \frac{1}{2})$ leads a characteristic which verifies the Nyquist criterion.

8.4. $\star\star$ [Sect. 8.4] In Fig.8.13 the transform

$$S(f) = A_0\, \exp(-|f|\,T_0)\,, \qquad f \in \mathbb{R}\,.$$

is drawn. Calculate its periodic repetition $S_c(f),\ f \in \mathbb{R}/\mathbb{Z}(F_c)$. Recall that it is sufficient to perform the evaluation over a period, as $[0, F_c)$.

We need to compute

$$S_c(f) = \sum_{k=-\infty}^{+\infty} S(f - kF_c) = \sum_{k=-\infty}^{+\infty} A_0\, \mathrm{e}^{-|f-kF_c|T_0}\,.$$

Restricting the computation to the cell $[0,F_c)$ and partitioning the summation, we find

$$\begin{aligned} S_c(f) &= \sum_{k=1}^{\infty} A_0\, e^{-|f-kF_c|T_0} + \sum_{k=0}^{\infty} A_0\, e^{-|f+kF_c|T_0} \\ &= \sum_{k=1}^{\infty} A_0\, e^{(f-kF_c)T_0} + \sum_{k=0}^{\infty} A_0\, e^{-(f+kF_c)T_0} \end{aligned}$$

where, keeping in mind that $f \in [0,F_c)$ in the first summation, it results $f - kF_c < 0$, whereas in the second it results $f + kF_c > 0$. Then, using the identities:

$$\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}\,, \qquad \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}\,,$$

we find:

$$S_c(f) = A_0\, e^{fT_0}\, \frac{e^{-F_cT_0}}{1-e^{-F_cT_0}} + A_0\, e^{-fT_0}\, \frac{1}{1-e^{-F_cT_0}}\,, \qquad f \in [0,F_c)\,.$$

8.5. $\star$ [Sect. 8.4] Find the minimum sampling frequency (for a perfect reconstruction) of the signal:

$$s(t) = A_0\, \mathrm{sinc}^2(F_0 t)\,, \qquad t \in \mathbb{R}\,,$$

with $F_0 = 2$ MHz.

From the notable transform

$$A_0\, \mathrm{sinc}^2(t) \quad \xrightarrow{\mathcal{F}} \quad A_0\, \mathrm{triang}(f)\,,$$

by applying a scale change, we have:

$$s(t) = A_0\, \mathrm{sinc}^2(F_0 t) \quad \xrightarrow{\mathcal{F}} \quad S(f) = \frac{A_0}{F_0}\, \mathrm{triang}\left(\frac{f}{F_0}\right)\,.$$

Hence, the spectral extension is $e(S) = (-F_0, F_0)$ and the band $B = F_0$. The condition $F_c \geq 2B$ stated that the minimum sampling frequency is $F_c = 2F_0 = 4\,\mathrm{Ms/s}$ (Ms/s=megasamples per second).

8.6. $\star\star$ [Sect. 8.4] Find the alias–free condition (8.34) for the signal

$$s(t) = A_0\, \mathrm{sinc}(F_0 t)\, \cos(2\pi f_0 t)\,,$$

with $F_0 = 2$ kHz and $f_0 = 1$ MHz.

The Fourier transform of the signal $s(t)$ is

$$S(f) = \frac{A_0}{2F_0}\left[\mathrm{rect}\left(\frac{f-f_0}{F_0}\right) + \mathrm{rect}\left(\frac{f+f_0}{F_0}\right)\right],$$

and then the spectral extension is given from

$$e(S) = (-f_0 - \tfrac{1}{2}F_0, -f_0 + \tfrac{1}{2}F_0) \bigcup (f_0 - \tfrac{1}{2}F_0, f_0 + \tfrac{1}{2}F_0)$$

which is included in the cell $C_0 = [-f_0 - \frac{1}{2}F_0, f_0 + \frac{1}{2}F_0)$. Then the Fundamental Sampling Theorem can be applied with

$$F_c \geq 2f_0 + F_0 = 2002\,\mathrm{ks/s}\,.$$

However this form of sampling is poorly efficient since the measure of the spectral extension is $\mathrm{mis}\,e(S) = 2F_0 = 4\,\mathrm{kHz}$. Considering the schemes of up/down–sampling of Sect. 8.7 we can confine the sampling frequency to $2F_0 = 4$ ks/s.

8.7. ⋆⋆⋆ [Sect. 8.4] Show that, if the hypotheses of the Fundamental Sampling Theorem are verified, the energy of samples equals that of the signal:

$$\sum_{n=-\infty}^{+\infty} T\,|s(nT)|^2 = \int_{-\infty}^{+\infty} |s(t)|^2\,\mathrm{d}t\,.$$

Observing that the signal has limited extension $e(S) \subset (-\frac{1}{2}F_c, \frac{1}{2}F_c) \stackrel{\Delta}{=} C_0$, from Parseval's theorem we obtain:

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2\,\mathrm{d}t = \int_{-\infty}^{+\infty} |S(f)|^2\,\mathrm{d}f = \int_{C_0} |S(f)|^2\,\mathrm{d}f\,.$$

Still, applying Parseval's Theorem on the domains $I = \mathbb{Z}(T)$, $\widehat{I} = \mathbb{R}/\mathbb{Z}(F_c)$, the energy of the down–sampled signal results:

$$E_{s_c} = \int_{C_0} |S_c(f)|^2\,\mathrm{d}f = \sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\int_{C_0} S(f-nF_c)\,S^\star(f-kF_c)\,\mathrm{d}f\,.$$

If the alias–free conditions are verified, we obtain

$$S(f-nF_c)\,S^\star(f-kF_c) = 0\,, \qquad n \neq k,$$

and hence

$$E_{s_c} = \sum_{n=-\infty}^{+\infty}\int_{C_0} |S(f-nF_c)|^2\,\mathrm{d}f\,.$$

But, inside the cell, we have: $S(f-nF_c) = 0$, $n \neq 0$. Thus

$$E_{s_c} = \int_{C_0} |S(f)|^2 \, df = E_s \, .$$

8.8. $\star$ [Sect. 8.7] Find the spectral extension of the signal

$$s(t) = A_0 \, \mathrm{sinc}^2(F_0 t) \, \cos(2\pi f_0 t) \, ,$$

with $f_0 = 10F_0$. Then express the efficiency η_{si} that can be achieved with a *direct* down–sampling.

The Fourier transform of the signal results:

$$S(f) = \frac{1}{2}\frac{A_0}{F_0}\left[\left(1 - \frac{|f-f_0|}{F_0}\right)\mathrm{rect}\left(\frac{f-f_0}{2F_0}\right) + \left(1 - \frac{|f+f_0|}{F_0}\right)\mathrm{rect}\left(\frac{f+f_0}{2F_0}\right)\right]$$

and hence the spectral extension is

$$e(S) = (-f_0 - F_0, -f_0 + F_0) \bigcup (f_0 - F_0, f_0 + F_0) \, , \qquad f_0 = 10F_0 \, .$$

Considering the cell (see Problem 8.6)

$$C_0 = \left(-f_0 - \tfrac{11}{10}F_0, -f_0 + \tfrac{11}{10}F_0\right) \bigcup \left(f_0 - \tfrac{11}{10}F_0, f_0 + \tfrac{11}{10}F_0\right) \, ,$$

and down–sampling with $F_c = \mathrm{mis}\, C_0 = \frac{22}{5}F_0 > 2B = 4F_0$, the efficiency results

$$\eta_{si} = \frac{2B}{\mathrm{mis}\, C_0} = \frac{10}{11} \, .$$

8.9. $\star$ [Sect. 8.7] Referring to the sampling of Fig.8.22, find the interpolating function.

From Fig.8.22, the frequency response of the interpolating filter results

$$Q(f) = \mathrm{rect}\left(\frac{f-f_0}{F}\right) + \mathrm{rect}\left(\frac{f+f_0}{F}\right) \, , \qquad F = \frac{1}{2}F_c \, , \quad f_0 = \frac{7}{4}F_c \, ,$$

where F_c is the sampling frequency. The corresponding impulse response is

$$\begin{aligned} q(t) &= F \, \mathrm{sinc}(Ft) \, e^{i2\pi f_0 t} + F \, \mathrm{sinc}(Ft) \, e^{-i2\pi f_0 t} \\ &= 2F \, \mathrm{sinc}(Ft) \, \cos(2\pi f_0 t) \, , \end{aligned}$$

where $2F = 1/T$ is the sampling frequency. Hence the interpolating function results:

$$q_0(t) = T\,q(t) = \text{sinc}\left(\frac{t}{2T}\right)\cos(2\pi f_0 t)\ .$$

8.10. ⋆⋆ [Sect. 8.7] Find the smallest cell of $\mathbb{R}$ modulo $\mathbb{Z}(F_c)$ containing the extension

$$e(S) = (-23 \text{ kHz}, -19 \text{ kHz}) \cup (19 \text{ kHz}, 23 \text{ kHz})$$

and then calculate the efficiency η_{si}.

Bimodal cell of $\mathbb{R}$ modulo $\mathbb{Z}(2F)$ have the following general structure

$$C_0^{(m)} = -C_m \cup C_m \quad \text{with} \quad C_m = (mF\ ,\ (m+1)F)\,, \qquad m \in \mathbb{N} \tag{S8.8}$$

and the alias–free condition can be limited to positive frequencies:

$$e(S_+) \subset C_m = (mF\ ,\ (m+1)F)\ .$$

Hence we have to choose m and F such that

$$(19\text{kHz}, 23\text{kHz}) \subset (mF\ ,\ (m+1)F) \tag{S8.9}$$

from which we get the constraint $F \geq 4$ (kHz). With such constraint, feasible values for mF are

$$F\ ,\ 2F\ ,\ 3F\ ,\ 4F \leq 19\ ,$$

among which we choose $4F$. From (S8.9) we obtain $4F \leq 19 < 23 \leq 5F$, which has the minimal solution $F_0 = \frac{23}{5} = 4.6$kHz.

Hence, the minimal sampling frequency is $F_c = 2F_0 = 9.2$ ks/s; and, being the bandwidth $B = 8$ kHz, efficiency results $\eta_{si} = 8/9.2 = 86.9\%$.

8.11. ⋆⋆⋆ [Sect. 8.7] Consider a bimodal symmetric spectrum with the extension indicated in (8.70). Evaluate the smallest cell C_0 containing such extension, for any value of the ratio f_0/B.

As in the previous problem, bimodal cells have the structure in (S8.8), so it is sufficient to impose the alias–free condition on the positive frequency mode, that is

$$(f_0\ ,\ f_0 + B) \subset C_m = (mF\ ,\ (m+1)F)\,, \tag{S8.10}$$

where B represents the band and F is half the sampling frequency, so that efficiency results

$$\eta_{si} = \frac{2B}{2F} = \frac{B}{F}\ .$$

Minimizing cells means finding the minimum F which is consistent with (S8.10).

It is evident that if $f_0 = mF$, then we must consider the cell $C_m = (mF\ ,\ (m+1)F)$ with $F = B$ and, in this case, we obtain unitary efficiency. For a generic f_0 it is better to set m in (S8.10) and examine the result, which leads to the conditions

$$mF \le f_0\,, \qquad (m+1)F \ge f_0 + B\,. \tag{S8.10a}$$

On the f_0, F plane, these two conditions with the identity sign set two lines (Fig.S8.2), which meet in the point (mB, B). Then, we realize that equations (S8.10)

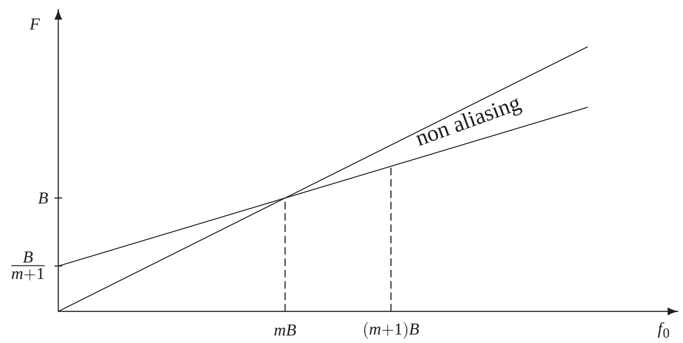

Fig. S8.2 Lines that determine the possible choices of the sampling frequency $2F$

are verified in the points of the angular sector given from these two lines and having abscissa $f \ge mB$. The minimum sampling semi frequency in this sector is

$$F_0 = \frac{f_0 + B}{m+1}\,, \qquad f_0 \ge mB\,.$$

Considering that when f_0 grows, efficiency decreases, it will be better to use the fixed value of m until we will achieve the value $(m+1)F$, and then pass to $m+1$.

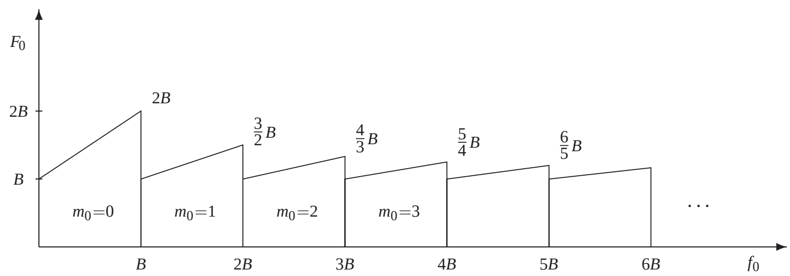

Fig. S8.3 Minimun sampling frequency of a signal with a bimodal spectrum

From a practical point of view, we can proceed like this:

1) given f_0 and B, we compute the least value of m verifying $f_0 \geq B$, that is

$$m_0 = \text{integer part of } f_0/B\,,$$

2) we compute the minimum sampling frequency as

$$F_0 = \frac{f_0 + B}{m_0 + 1}\,.$$

In this way, on the (f_0, F_0) plane we obtain a piecewise behavior between $m_0 B$ and $(m_0+1)B$ with F_0 varying between B and $B(m_0+2)/(m_0+1)$ (Fig.S8.3).

Note that if $f_0 \gg B$ then, the frequency F_0 doesn't moves aside from B very much and efficiency becomes nearly unitary.

8.12. ⋆⋆⋆ [Sect. 8.7] Referring to Fig.8.23, suppose that the signal $s(t)$, $t \in \mathbb{R}$, is real. Show that the signals in the lower branch of the block diagram are the conjugate of those in the upper branch and in particular $\tilde{s}_2(t) = \tilde{s}_1^*(t)$.

The scheme is shown in Fig.S8.4. The impulse responses of the filters in the scheme

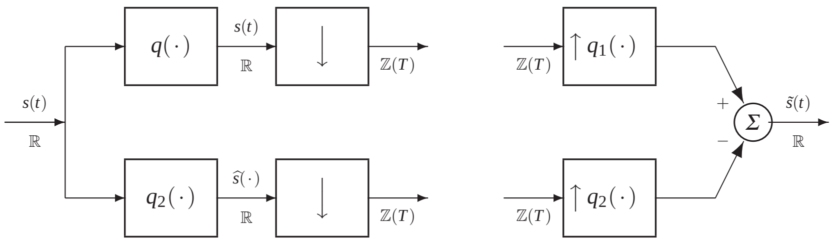

Fig. S8.4 Sampling/interpolation for *real* band–pass signals implemented with *complex*filters

are

$$q_i(t) = B_i \operatorname{sinc}(B_i t)\, e^{i2\pi f_i t}\,, \qquad i = 1, 2.$$

If the input signal is *real*, then its spectral extension is symmetric with respect to the origin, so that we can choose $B_1 = B_2$ and $f_2 = -f_1$. With this choice $q_1(t)$ and $q_2(t)$ become complex conjugate

$$q_2(t) = q_1^\star(t)\,.$$

Now, writing down the input–output relationship, we deduce that, with $s(t)$ real, it results $s_2(t) = s_1^*(t)$, and then, in the two branches, all the corresponding signals are conjugate among them. Then, in the end, signal recovery can be done from the superior branch only, according to

$$\tilde{s}(t) = \tilde{s}_1(t) + \tilde{s}_2(t) = 2\Re s_1(t)\ .$$

We thus obtain the simplified scheme of Fig. S8.5.

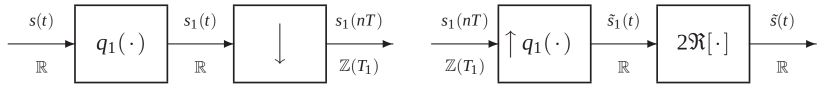

Fig. S8.5 Simplification of the scheme of the previous figure

We observe that in these schemes the input and output signals are *real*, whereas intermediate signals are *complex* and the two filters are *complex*. Applying the methodology of the Modulation Theory, the scheme can be transformed into a scheme with both *real* components and signals (Fig. S8.6).

We recognise that $\tilde{s}_1(t)$ is the positive frequency component of $s(t)$, denoted in Modulation Theory as $s_+(t)$; this signal can be expressed as (see analytic signal, in Sect. 9.10)

$$z_s(t) = 2\,s_+(t) = s(t) + \mathrm{i}\,\widehat{s}(t)$$

where $\widehat{s}(t)$ is the Hilbert transform in the signal bandwidth. In the scheme of Fig. S8.6 the signals and the component are *real*.

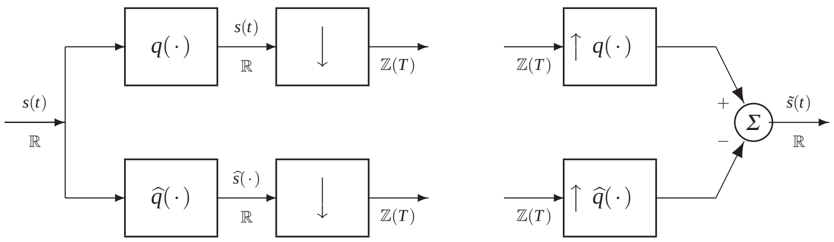

Fig. S8.6 Sampling/interpolation for *real* band–pass signals implemented with *real* components ; $q(t)$ is an ideal band–pass filter, with the same band as the signal $s(t)$, and therefore irrelevant, and $\widehat{q}(t)$ is the filter that gives the Hilbert transform $\widehat{s}(t)$ of $s(t)$

8.13. ⋆⋆ [Sect. 8.8] Consider a discrete signal $s(t)$, $t \in \mathbb{Z}(T_0)$, with extension $e(S) = (-B,B) + \mathbb{Z}(F_0)$ and $B = \frac{1}{7}F_0$. Find the minimum sampling frequency.

We can propose the down–sampling $\mathbb{Z}(T_0) \to \mathbb{Z}(2T_0)$ or, at most, $\mathbb{Z}(T_0) \to \mathbb{Z}(3T_0)$. With the first down–sampling the spectral extension of the sampled signal results

$$e(S_c) = e(S) + \mathbb{Z}(\tfrac{1}{2}F_0) = (-B,B) + \mathbb{Z}(F_0) + \mathbb{Z}(\tfrac{1}{2}F_0) = (-B,B) + \mathbb{Z}(\tfrac{1}{2}F_0)\ .$$

The extension of the side term is

$$(-B,B)+\tfrac{1}{2}F_0=(-B+\tfrac{1}{2}F_0,B+\tfrac{1}{2}F_0)=(\tfrac{5}{14}F_0,\tfrac{9}{14}F_0)\,,$$

which is disjointed from the central term $(-\frac{1}{7}F_0,\frac{1}{7}F_0)$. Considering the symmetry, this is enough to prove that the $\mathbb{Z}(T_0)\to\mathbb{Z}(2T_0)$ sampling, the side term results

$$(-B,B)+\tfrac{1}{3}F_0=\left(\tfrac{4}{21}F_0,\tfrac{10}{21}F_0\right)$$

which does not overlap the central one $(-\frac{1}{7}F_0,\frac{1}{7}F_0)$.

On the contrary with the $\mathbb{Z}(T_0)\to\mathbb{Z}(4T_0)$ down–sampling there is overlapping. In conclusion, the minimum sampling frequency is $F_c=\frac{1}{3}F_0$, with which we achieve the efficiency $\eta_{si}=2B/F_c=6/7=85.9\%$.

A direct solution could be obtained from (8.79).

8.14. $\star\star\star$[Sect. 8.8] Consider a discrete signal with extension

$$e(S)=(-B,B)+\mathbb{Z}(F_0)\,. \tag{S8.11}$$

Find the minimum sampling frequency with a *direct* down–sampling at the varying of B/F_0.

We can directly use (8.79), that is

$$F_c\ge 2B\,,\qquad F_c=\frac{1}{N}F_0 \tag{S8.12}$$

where N is a natural number. Hence the minimum sampling frequency can be obtained finding the *largest natural* N_0 verifying (S8.12), that is $\frac{1}{N_0}F_0\ge 2B$. Through the function "integer part", we find

$$N_0=\operatorname{int}\left(\frac{F_0}{2B}\right)\,,\qquad N_0\ge 1$$

and the minimum sampling frequency results $F_{\mathrm{c,min}}=F_0/N_0$. We find, in particular:

$$\begin{array}{lll}
2B\le F_0<4B & N_0=1 & F_{\mathrm{c,min}}=F_0\\
4B\le F_0<6B & N_0=2 & F_{\mathrm{c,min}}=\frac{1}{2}F_0\\
6B\le F_0<8B & N_0=3 & F_{\mathrm{c,min}}=\frac{1}{3}F_0\\
8B\le F_0<10B & N_0=4 & F_{\mathrm{c,min}}=\frac{1}{4}F_0\,.
\end{array}$$

8.15. $\star\star\star$[Sect. 8.8] Consider the down–sampling of a discrete–time signal with extension (S8.11) with $B=\frac{3}{11}F_0$. Find a scheme that, using a pre–filtering allows for down–sampling with $2B$ samples/s.

We observe that, in order to achieve unitary efficiency with direct sampling, $2B$ must be a submultiple of the period F_0' of the Fourier transform of the signal to be sampled, that is

$$2B = \frac{1}{M} F_0', \qquad M \in \mathbb{N}\,. \tag{S8.13}$$

As $2B = \frac{6}{11}F_0$, we need to modify the signal $s(t)$, $t \in \mathbb{Z}(T_0)$ into the signal $s'(t)$, $t \in \mathbb{Z}(T_0')$, verifying (S8.13). We have to carry out a transformation as $\mathbb{Z}(T_0) \to \mathbb{Z}(T_0')$, where $T_0' = 1/F_0'$ is determined by

$$\frac{1}{M} F_0' = \frac{6}{11} F_0$$

and we can choose $M = 11$, resulting $F_0' = 6F_0$.

Hence we need to apply a $\mathbb{Z}(T_0) \to \mathbb{Z}(\frac{1}{6}T_0)$, transformation, followed by a $\mathbb{Z}(\frac{1}{6}T_0) \to \mathbb{Z}(\frac{11}{6}T_0)$ down–sampling. The scheme is illustrated in Fig. S8.7, where

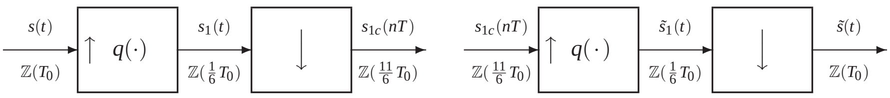

Fig. S8.7 Sampling/interpolation scheme of discrete signals to achieve unitary efficiency

we chose a $\mathbb{Z}(T_0) \to \mathbb{Z}(\frac{1}{6}T_0)$ interpolator and the signal recovery is done using a $\mathbb{Z}(\frac{11}{6}T_0) \to \mathbb{Z}(\frac{1}{6}T_0)$ interpolator, followed by a $\mathbb{Z}(\frac{1}{6}T_0) \to \mathbb{Z}(T_0)$ down–sampler.

The analysis of this scheme can be subdivided into two parts concerning

1) the recovery of the signal $s_1(t)$ from its samples;
2) the recovery of $s(t)$ from $s_1(t)$.

Point 1) must be solved using the Sampling Theorem and requires that the alias–free condition is verified. Such condition however is imposed from (S8.13) and gives unitary efficiency. The $\mathbb{Z}(T_0) \to \mathbb{Z}(\frac{1}{6}T_0)$ interpolating filter will be chosen so that to verify this condition.

Now, it becomes necessary to proceed in the frequency domain, as depicted in Fig. S8.8. The transform $S(f)$ has period F_0 and extension $(-B,B) + \mathbb{Z}(F_0)$ with $B = \frac{3}{11}F_0$. The frequency response $Q(f)$ has period $6F_0$ and must be chosen in order to preserve the shape of $S(f)$ in $(-B,B)$ canceling the five side terms. Therefore $Q(f)$ is given from the indicating function of the extension $(-B,B) + \mathbb{Z}(6F_0)$. At the output of this interpolating filter the transform has the same extension as the filter, hence

$$e(S_1) = (-B,B) + \mathbb{Z}(6F_0)\,.$$

Being

$$2B = \frac{6}{11}F_0 = \frac{1}{11}F_0' \quad \text{with} \quad F_0' = 6F_0\,,$$

the periodic repetition, due to the $\mathbb{Z}(T_0') \to \mathbb{Z}(11T_0')$ down–sampling, involves juxtaposition of the side terms (which confirms unitary efficiency).

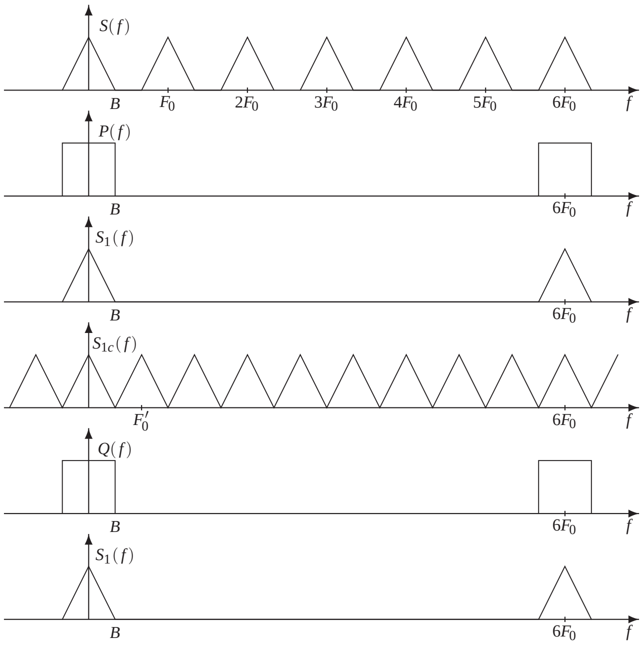

Fig. S8.8 Fourier analysis of the sampling scheme of Fig.8.7

The frequency response of the interpolating filter $Q(f)$, indicated by the Sampling Theorem, coincides with $P(f)$. This filter allows to exactly recover $S_1(f)$, eventually, in order to recover $S(f)$, we just have to apply a $\mathbb{Z}(T_0') \to \mathbb{Z}(6T_0')$ down–sampling where $6T_0' = T_0$.

8.16. $\star$ [Sect. 8.9] The signal

$$s(t) = \mathrm{rep}_{F_p}\ \mathrm{rect}\,(t/d\,T_p)\,, \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)\,,$$

with $T_p = 1$ ms and $d = 20\%$, is filtered by an ideal low–pass filter with band $B_0 = 3.5$ kHz and then down–sampled with an $\mathbb{R}/\mathbb{Z}(T_p) \to \mathbb{Z}(T)/\mathbb{Z}(T_p)$ sampling.

Find the minimum number of samples per period and write the expression of the recovered signal.

The Fourier transform of the signal $s(t)$, $t \in \mathbb{R}/\mathbb{Z}(T_p)$ can be computed using the Duality Theorem. From the Fourier pair

$$x(t) = \mathrm{rect}\left(\frac{t}{dT_p}\right) \quad \xrightarrow{\mathcal{F}} \quad X(f) = dT_p\ \mathrm{sinc}(f dT_p)\,,$$

since $s(t)$ is the periodic repetition of $x(t)$, then $S(f)$ is the sampled version of $X(f)$, that is

$$S(f) = dT_p \operatorname{sinc}(f dT_p), \qquad f \in \mathbb{Z}(F_p),\ F_p = 1/T_p\ . \tag{S8.14}$$

Filtering $s(t)$ with an ideal unitary filter with extension $(-B_0, B_0)$ we obtain the signal $x(t)$, $t \in \mathbb{R}/\mathbb{Z}(T_p)$ with transform

$$X(f) = \begin{cases} S(f) & f \in (-B_0, B_0) \\ 0 & f \notin (-B_0, B_0) \end{cases} . \tag{S8.15}$$

Hence, the spectral extension of $x(t)$ is, by construction,

$$\begin{aligned} e(x) &= (-B_0, B_0) \cap \mathbb{Z}(F) = (-3.5, 3.5) \cap \mathbb{Z}(1) \\ &= \{-3, -2, -1, 0, 1, 2, 3\} \qquad \text{kHz}\ . \end{aligned}$$

The Sampling Theorem for periodic signals guarantees that the recovery of $x(t)$ from the samples can be done considering 7 samples per period, that is with a sampling period $T_c = \frac{1}{7} T_p = \frac{1}{7}$ ms. The expression of the reconstructed signal can be achieved from (8.82), that is

$$x(t) = \sum_{n=0}^{N-1} x(nT) \operatorname{sinc}_N(F_c t), \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)$$

where $N = 7$, $T = \frac{1}{7} T_p$, $F_c = \frac{1}{T}$. This expression requires knowledge of the samples $x(nT)$ and, in short of $x(t)$. The expression of $x(t)$ can be achieved through the inverse transform of (S8.15), as

$$\begin{aligned} x(t) &= \sum_{k=-3}^{3} F\, X(kF)\, \mathrm{e}^{\mathrm{i}2\pi kFt} = \sum_{k=-3}^{3} F\, S(kF)\, \mathrm{e}^{j2\pi kFt} \\ &= d \sum_{k=-3}^{3} \operatorname{sinc}(kd)\, \mathrm{e}^{\mathrm{i}2\pi kFt} = d + 2d \sum_{k=1}^{3} \operatorname{sinc}(kd) \cos 2\pi kFt\ . \end{aligned}$$

8.17. ⋆⋆ [Sect. 8.9] Explicit the Unified Sampling Theorem for discrete periodic signals, that is, with $\mathbb{Z}(T_0)/\mathbb{Z}(T_p) \to \mathbb{Z}(T)/\mathbb{Z}(T_p)$.

The $\mathbb{Z}(T_0)/\mathbb{Z}(T_p) \to \mathbb{Z}(T)/\mathbb{Z}(T_p)$ down–sampling must verify the compatibility condition $\mathbb{Z}(T_0) \subset \mathbb{Z}(T) \subset \mathbb{Z}(T_p)$, which requires that T is a multiple of T_0 and a submultiple of T_p. Hence

$$T = NT_0, \qquad T = \frac{1}{M} T_p, \qquad T_0 = \frac{1}{MN} T_p\ .$$

Afterwards, we pass from MN points per period to M points per period.

In frequency we obtain the $\mathbb{Z}(F)/\mathbb{Z}(F_0) \to \mathbb{Z}(F)/\mathbb{Z}(F_c)$ periodic repetition with

$$F = \frac{1}{T_p}\,,\; F_0 = \frac{1}{T_0} = MNF\,,\; F_c = \frac{1}{T} = \frac{1}{N}F_0 = MF$$

and with the relationship

$$S_c(f) = \sum_{k=0}^{N-1} S(f - kF_c) = \sum_{p \in P} S(f-p)$$

where $P = [\mathbb{Z}(F_c)/\mathbb{Z}(F_0)) = \{0, F_c, \ldots, (N-1)F_c\}$.

The alias–free condition is that the spectral extension of the signal is contained in a cell of $\mathbb{Z}(F)$ modulo P. A reference cell can be

$$C_0 = \{0, F, \ldots, (M-1)F\} + \mathbb{Z}(F_0)$$

leading to the following interpolating function

$$q_0(t) = T \int_{C_0} \mathrm{d}f\, \mathrm{e}^{\mathrm{i}2\pi f t} = TF \sum_{m=0}^{M-1} \mathrm{e}^{\mathrm{i}2\pi m F t}\,, \qquad t \in \mathbb{Z}(T_0)$$

where $TF = 1/M$ (this function can be expressed through the $\mathrm{sinc}_M(\cdot)$).

The reconstruction formula results

$$s(t) = \sum_{n=0}^{M-1} s(nT)\, q_0(t - nT)\,, \qquad t \in \mathbb{Z}(T_0)\,.$$

8.18. $\star\star$ [Sect. 8.10] Prove that if the reference cell C_0 in the 2D sampling verifies the symmetry condition

$$-C_0 \;=\; C_0\,,$$

then the interpolating filter is *real*, i.e. with a real impulse response.

If the reference cell C_0 verifies the symmetry condition $-C_0 = C_0$, the frequency response of the interpolator

$$Q(f_1, f_2) = \begin{cases} 1 & (f_1, f_2) \in C_0 \\ 0 & (f_1, f_2) \notin C_0 \end{cases}$$

has the Hermitian symmetry $Q(\mathbf{f}) = Q^\star(-\mathbf{f})$ and therefore the frequency response is real, that is, $q(\mathbf{t}) = q^\star(\mathbf{t})$, where

$$q(t_1, t_2) = \int_{C_0} \mathrm{d}f_1\, \mathrm{d}f_2\, \mathrm{e}^{\mathrm{i}2\pi(f_1 t_1 + f_2 t_2)}\,.$$

8.19. $\star\star$ [Sect. 8.10] Consider an $\mathbb{R}^2 \to \mathbb{Z}(d_1,d_2)$ sampling and assume that the reference cell C_0 is a parallelogram, instead of a rectangle.

Write the frequency response of the interpolator characterized by this cell.

In $\mathbb{R}^2 \to \mathbb{Z}(d_1,d_2)$ down–sampling a reference cell $\mathbb{R}^2$ modulo $\mathbb{Z}(F_1,F_2)$ may be the centered parallelepiped, given by

$$C_0 = \left\{ \alpha \begin{bmatrix} F_1 & 0 \\ F_2 & F_2 \end{bmatrix} \middle| \alpha \in [-\tfrac{1}{2}, \tfrac{1}{2})^2 \right\} .$$

The corresponding frequency response is given by

$$Q(f_1, f_2) =$$

$$\eta_{C_0}(f_1, f_2) = \begin{cases} 1 & f_1 \in \left[-\frac{1}{2}F_1, \frac{1}{2}F_1\right], f_2 \in \left[\frac{1}{2}F_2\left(1-\frac{2f_1}{F_1}\right), \frac{1}{2}F_2\left(1+\frac{2f_1}{F_1}\right)\right] \\ & \text{elsewhere.} \end{cases}$$

8.20. $\star\star\star$ [Sect. 8.10] Consider the sampling $\mathbb{R}^2 \to \mathbb{Z}_2^1(d_1,d_2)$ and assume the reference cell C_0 is a rhombus.

Determine the impulse response of the interpolator characterized by this cell.

The evaluation could be done using general formula (5.84), but after the integration it is difficult to combine the terms into a synthetic formula. It is more convenient to use a *coordinate change* $\boldsymbol{\lambda} = \mathbf{b}\,\mathbf{f}$ that map the rhombus into a rectangle, as shown in Fig.S8.9 and then we apply the corresponding FT relation.

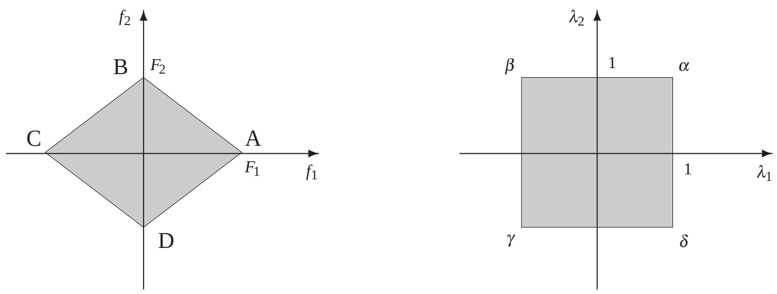

Fig. S8.9 Coordinate change to map a rhombus into a rectangle

$$Y(\mathbf{f}) = X(\mathbf{b}\,\mathbf{f}) \xrightarrow{\mathcal{F}^{-1}} y(\mathbf{t}) = d(\mathbf{b}^{\star})\, x(\mathbf{b}^{\star}\,\mathbf{t}) \,. \tag{S8.16}$$

The coordinate change is explicitly

$$\lambda_1 = b_{11} f_1 + b_{12} f_2, \qquad \lambda_2 = b_{21} f_1 + b_{22} f_2 \tag{S8.17}$$

where the coefficients b_{ij} can be determined by imposing that the centered rectangle of the (λ_1, λ_2) plane is mapped into the rhombus ABCD of the (f_1, f_2) plane. Hence imposing tha the vertex $\alpha = (1,1)$ becomes the vertex $A = (F_1, 0)$ and that the vertex $\beta = (-1,1)$ becomes the vertex $B = (0, F_2)$, from (S8.17) one gets

$$\begin{cases} 1 = b_{11} F_1 \\ 1 = b_{21} F_1 \end{cases} \qquad \begin{cases} -1 = b_{12} F_2 \\ \ \ 1 = b_{22} F_2 \, . \end{cases}$$

The solution is

$$\mathbf{b} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1/F_1 & -1/F_2 \\ 1/F_1 & 1/F_2 \end{bmatrix} \xrightarrow{\quad\star\quad} \mathbf{b}^\star = \begin{bmatrix} \frac{1}{2} F_1 & -\frac{1}{2} F_2 \\ \frac{1}{2} F_1 & \frac{1}{2} F_2 \end{bmatrix}$$

where $\mathrm{d}(\mathbf{b}^\star) = \frac{1}{2} F_1 F_2$. Now letting

$$X(\lambda_1, \lambda_2) = \mathrm{rect}(\frac{1}{2}\lambda_1)\,\mathrm{rect}(\frac{1}{2}\lambda_2)$$

which represents the indicator function of the rectangle, the relation $Y(\mathbf{f}) = X(\mathbf{b}\,\mathbf{f})$ gives the indicator function of the rhombus. Considering that

$$x(u_1, u_2) = 4\,\mathrm{sinc}(2u_1)\,\mathrm{sinc}(2u_2)$$

from (S8.16) one gets

$$y(t_1, t_2) = 2F_1 F_2\,\mathrm{sinc}(F_1 t_1 - F_2 t_2)\,\mathrm{sinc}(F_1 t_1 + F_2 t_2) = g(t_1, t_2)$$

which represents the impulse response of the interpolator. Then we obtain the interpolating function as $q_0(t_1, t_2) = \mathrm{d}(J) y(t_1, t_2)$, where $\mathrm{d}(J) = 2 d_1 d_2$ and $F_1 = 1/(2d_1)$, $F_2 = 1/(2d_2)$. The final result is

$$g_0(t_1, t_2) = \frac{1}{2}\,\mathrm{sinc}(F_1 t_1 - F_2 t_2)\,\mathrm{sinc}(F_1 t_1 + F_2 t_2) \, .$$

8.21. ⋆⋆ [Sect. 8.11] Consider the sampling $\mathbb{R} \to \mathbb{Z}(T)$ of the signal

$$s(t) = 1(t)\,\mathrm{e}^{-\alpha t}, \qquad t \in \mathbb{R} \, .$$

Find the sampling frequency F_c that ensures $\Lambda_{\min} = 48$ dB, using a pre–filter and assuming $F_0 = \alpha/2\pi = 1\mathrm{MHz}$.

The energy of the signal $s(t)$ is

$$E_s = \int_0^{\infty} |\mathrm{e}^{-\alpha t}|^2 \,\mathrm{d}t = \frac{1}{2\alpha}\,, \qquad \alpha > 0\,.$$

The Fourier transform results

$$S(f) = \frac{1}{\alpha + \mathrm{i}2\pi f}$$

and hence the signal has not limited bandwidth.

We can compute the conventional bandwidth according to the criterion of negligible energy (see Sect. 13.11). We have

$$2\int_{B_c}^{+\infty} \frac{1}{\alpha^2 + (2\pi f)^2}\,\mathrm{d}f = \varepsilon\,\frac{1}{2\alpha} = \varepsilon E_s\,.$$

Solving, we find B_c as a function of ε

$$B_c = F_0 \tan\frac{\pi}{2}(1-\varepsilon)\,, \qquad F_0 \stackrel{\Delta}{=} \frac{\alpha}{2\pi}\,. \tag{S8.18}$$

Recalling that $\Lambda_{min} = 48$ dB=63095, we obtain $\varepsilon = 1/\Lambda = 1.584 \cdot 10^{-5}$ and, from (S8.18), the conventional bandwidth results $B_c = 40\,168 F_0 = 40\,168$ MHz. Hence, in order to guarantee $\Lambda_{min} = 48$ dB we have to down–sample with frequency

$$F_c = 2B_c = 80\,336\ \mathrm{Ms/s}.$$

8.22. ⋆⋆ [Sect. 8.11] As in the previous problem, but without the pre–filter (assume $\mu = 1$).

Assuming $\mu = 1$ in (8.95), we find

$$\varepsilon_{\mathrm{dB}} = \Lambda_{\mathrm{dB}} + 3\ \mathrm{dB} = 51\ \mathrm{dB} = 125\,893\,.$$

Hence we need to impose a value ε, which is half than in the previous case, giving a conventional bandwidth $B_c = 80\,146 F_0 = 80\,146$ MHz. As a consequence, the sampling frequency, which without a prefix guarantees SNR equal to 48 dB, becomes

$$F_c = 160\,292\ \mathrm{Ms/s}\,.$$

8.23. $\star\star\star$[Sect. 8.11] Evaluate the ratio μ defined in (8.95) for the signal:

$$s(t) = \frac{A_0}{1+(F_0 t)^2} \,.$$

The signal $s(t)$ transform is

$$S(f) = \frac{\pi A_0}{F_0}\,\exp\left(-2\pi\frac{|f|}{F_0}\right) \,.$$

The energy of the error outside bandwidth, that is $f \notin (-B_c, B_c)$ with $F_c = 2B$, can be computed considering (8.93) and it results

$$\begin{aligned} E_{e_1} &= \int_{-\infty}^{-B_c} |S(f)|^2 \,\mathrm{d}f + \int_{B_c}^{+\infty} |S(f)|^2 \,\mathrm{d}f = \frac{A_0^2 \pi}{2F_0}\,\exp(-4\pi\,\frac{B_c}{F_0}) \\ &= E_s \exp\left(-2\pi\frac{F_c}{F_0}\right) \,, \end{aligned}$$

where

$$E_s = \frac{A_0^2\pi}{2F_0} = \text{signal energy} \,.$$

The Fourier transform of the error signal inside bandwidth is given by

$$\begin{aligned} E_{e_2} &= \sum_{p\neq 0} S(f-p)\,G(f) \\ &= \frac{\pi A_0}{F_0}\left[\sum_{n=1}^{\infty} \mathrm{e}^{2\pi(f-nF_c)/F_0} + \sum_{n=0}^{\infty} \mathrm{e}^{-2\pi(f+nF_c)/F_0} - \mathrm{e}^{-2\pi f/F_0}\right] G(f) \\ &= \frac{\pi A_0}{F_0}\,\frac{\mathrm{e}^{-2\pi F_c/F_0}}{1-\mathrm{e}^{-2\pi F_c/F_0}}\,\left(\mathrm{e}^{2\pi f/F_0} + \mathrm{e}^{-2\pi f/F_0}\right) \,, \qquad F_c = 2B_c \,, \end{aligned}$$

from which the energy can be obtained, as

$$E_{e_2} = \int_{-B_c}^{B_c} |E_2(f)|^2 \,\mathrm{d}f \,.$$

After few passages, we obtain

$$E_{e_2} = 2E_s\,\frac{1}{\left(\exp\left(2\pi\,\frac{F_c}{F_0}\right)-1\right)^2}\,\left[\sinh\left(2\pi\,\frac{F_c}{F_0}\right) + 2\pi\frac{F_c}{F_0}\right] \,.$$

Hence

$$\mu = \frac{E_{e_2}}{E_{e_1}} = \frac{\exp\left(-2\pi\frac{F_c}{F_0}\right)}{\left(\exp\left(2\pi\frac{F_c}{F_0}\right)-1\right)^2}\left[2\sinh\left(2\pi\frac{F_c}{F_0}\right)+4\pi\frac{F_c}{F_0}\right] \stackrel{\Delta}{=} \mu\left(\frac{F_c}{F_0}\right) .$$

The shaping of μ as a function of F_c/F_0 is depicted in Fig. S8.10 and it asymptotically tends to 1. For $F_c/F_0 = 1$, the ratio μ varies very little from 1, namely 3%.

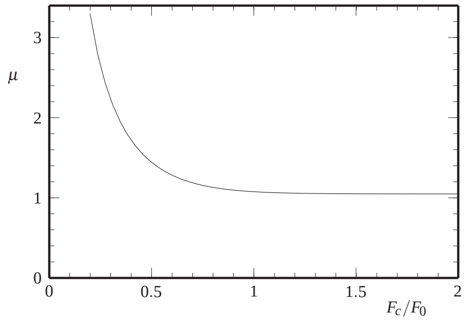

Fig. S8.10 Ratio in–bound energy/out–bound energy as a function of F_c/F_0.

8.24. ⋆⋆ [Sect. 8.11] Consider the down–sampling $\mathbb{R} \to \mathbb{Z}(T)$ with a pre–filter (see Fig.8.30), with

$$D(f) = \mathrm{e}^{-|f|/F_0}\ \mathrm{rect}(f/F_c)\ , \qquad G(f) = \mathrm{rect}(f/F_c)\ .$$

Show that this scheme is equivalent to a filter on $\mathbb{R}$ and find the equivalent filter.

From the frequency analysis of the scheme of Fig.8.30 we obtain

$$\begin{aligned} S_d(f) &= D(f)\,S(f) \\ S_c(f) &= \sum_{k=-\infty}^{+\infty} D(f-kF_c)\,S(f-kF_c)\,, \\ \tilde{S}(f) &= \sum_{k=-\infty}^{+\infty} D(f-kF_c)\,S(f-kF_c)\,G(f)\,. \end{aligned}$$

Recalling that the spectral extension of the two filters is $\left(-\frac{1}{2}F_c, \frac{1}{2}F_c\right)$, we have $D(f-kF_c)\,G(f) = 0$ for $k \neq 0$. Hence

$$\tilde{S}(f) = D(f)\,G(f)\,S(f)\,,$$

which shows that the scheme is equivalent to a filter having frequency response $Q(f) = D(f)\,G(f)$. In this specific case it results

$$Q(f) = \exp\left(-\frac{|f|}{F_0}\right) \operatorname{rect}\left(\frac{f}{F_c}\right), \qquad f \in \mathbb{R}.$$

8.25. $\star$ [Sect. 8.11] In the $\mathbb{R} \to \mathbb{Z}(T)$ sampling verify that, if $e(S) \subset (-F_c, F_c)$, the in–band energy $\mathcal{E}_{\text{in}}$ equals the out–band energy $\mathcal{E}_{\text{out}}$.

The out–band error energy is

$$\mathcal{E}_{\text{out}} = \int_{-\frac{3}{2}F_c}^{-\frac{1}{2}F_c} |S(f)|^2 \,\mathrm{d}f + \int_{\frac{1}{2}F_c}^{\frac{3}{2}F_c} |S(f)|^2 \,\mathrm{d}f .$$

The in–band error is

$$\mathcal{E}_{\text{in}}(f) = S(f+F_c) + S(f-F_c), \qquad f \in \left(-\tfrac{1}{2}F_c, \tfrac{1}{2}F_c\right)$$

and the two terms do not overlap as soon as $e(S) \subset (-F_c, F_c)$; then $S(f+F_c)\,S^*(f-F_c) = 0$ and

$$\mathcal{E}_{\text{in}} = \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} |S(f+F_c)+S(f-F_c)|^2 \,\mathrm{d}f = \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} |S(f+F_c)|^2 \,\mathrm{d}f + \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} |S(f-F_c)|^2 \,\mathrm{d}f ,$$

So, we find that $\mathcal{E}_{\text{in}} = \mathcal{E}_{\text{out}}$.

8.26. $\star$ [Sect. 8.12] Consider the *sampling and hold* with a fundamental pulse

$$p_0(t) = \cos 2\pi \frac{t}{T_0} \operatorname{rect}\left(\frac{t}{dT}\right),$$

where $T_0 = 2dT$ and $d = 20\%$. Find the frequency response of the filter that allows the perfect reconstruction of the signal.

In general, the filter frequency response is given by (8.102), that is

$$G(f) = \frac{Q(f)}{P(f)} = \frac{T\,Q(f)}{P_0(f)} = \frac{T \operatorname{rect}(f/F_c)}{P_0(f)} \tag{S8.19}$$

where $P_0(f)$ is the Fourier transform of the fundamental pulse. In the reference case, the fundamental pulse is rectangular, whereas in the problem we have

$$p_0(t) = \cos 2\pi \frac{t}{T_0} \operatorname{rect}\left(\frac{t}{dT}\right), \qquad T_0 = 2dT\ ,\ d = 20 \quad .$$

Applying the rule on modulation (see (2.112)), we obtain

$$P_0(f) = \frac{1}{2}\, dT \,\mathrm{sinc}\left[\left(f - \frac{1}{T_0}\right) dT\right] + \frac{1}{2}\, dT \,\mathrm{sinc}\left[\left(f + \frac{1}{T_0}\right) dT\right]$$

which, substituted in (S8.19) gives the desired frequency response.

8.27. ⋆⋆[Sect. 8.12] A real signal $s(t)$, $t \in \mathbb{R}$, with bandwidth $B = 4$ kHz is sampled and held with $F_c = 2B$ and then filtered with a real pass–band filter with band–pass $(3F_c - B, 3F_c + B)$ and unitary frequency response over the band.

Find the signal expression at the filter output.

The Fourier transform of the signal obtained with a *sampling and hold* is given by (8.106), that is

$$S_c(f) = \sum_{k=-\infty}^{+\infty} V_k\, S(f - kF_c) \quad . \tag{S8.20}$$

The subsequent filter has frequency response

$$G(f) = \mathrm{rect}\left(\frac{f + 3F_c}{2F_c}\right) + \mathrm{rect}\left(\frac{f - 3F_c}{2F_c}\right) \quad .$$

Examining the spectral extension we can state that the filter takes out the terms of the weighted repetition (S8.20) for $k = \pm 3$. Hence, we achieve

$$Y(f) = V_{-3}\, S(f + 3F_c) + V_3\, S(f - 3F_c) \tag{S8.21}$$

where (see (8.105a))

$$V_3 = d\,\mathrm{sinc}(3d)\,\mathrm{e}^{-\mathrm{i}\pi 3d}\,, \qquad V_{-3} = V_3^{\star} \quad .$$

In order to find the signal expression, we compute the inverse transform of (S8.21) and achieve

$$\begin{aligned} y(t) &= V_{-3}\, s(t)\,\mathrm{e}^{-\mathrm{i}6\pi F_c t} + V_3\, s(t)\,\mathrm{e}^{\mathrm{i}6\pi F_c t} \\ &= 2d\,\mathrm{sinc}(3d)\, s(t)\, \cos(6\pi F_c t - 3\pi d) \quad . \end{aligned}$$

Problems of Chapter 9

9.1. ⋆⋆ [Sect. 9.1] Consider the signal

$$s(t) = A_0 \operatorname{rect}_+(t/D) \cos 2\pi t/T_0$$

and find the decompositions into a continuous and a discontinuous part in the cases: a) $D = 3T_0$ and b) $D = 3.25T_0$.

Case $D = 3T_0$

In this case the signal has two discontinuities at $t = 0$ and $t = D$ (Fig. S9.1) with discontinuity sizes $d_1 = A_0$ at $t_1 = 0$ and $d_2 = -A_0$ at $t_2 = D$. The dicontinuities are balanced $(d_1 + d_2 = 0)$, and the two decompositions coincide

$$s_{d1}(t) = A_0 1(t) - A_0 1(t - D) = s_{d2}(t) = \tfrac{1}{2} A_0 \operatorname{sgn}(t) - \tfrac{1}{2} A_0 \operatorname{sgn}(t - D)\,.$$

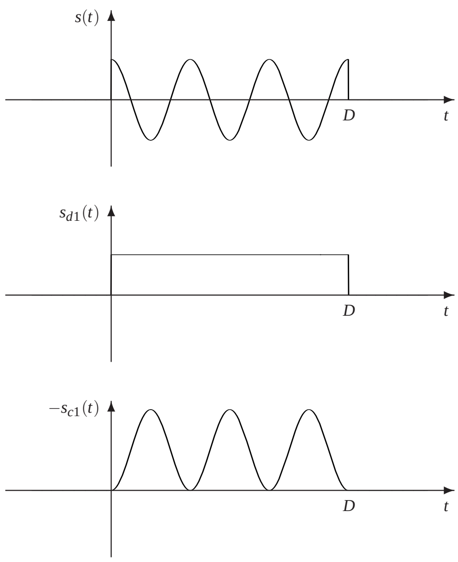

Fig. S9.1 The signal of the problem and its decomposition for $D = 3T_0$. The discontinuities are balanced and the two types of decompositions coincide: $s_{d1}(t) = s_{d2}(t)$

Case $D = 3.25T_0$

In this case the signal has only one discontinuity at $t = 0$ of size $d_1 = A_0$ (Fig.S9.2) and there is no balancement. Therefore the two decompositions are different

$$s_{d1}(t) = A_0 1(t) \neq s_{d2}(t) = \tfrac{1}{2} A_0 \,\mathrm{sgn}(t)\ .$$

Note the presence of a dc component in $s_{d1}(t)$, whereas $s_{d2}(t)$ is free of a dc component.

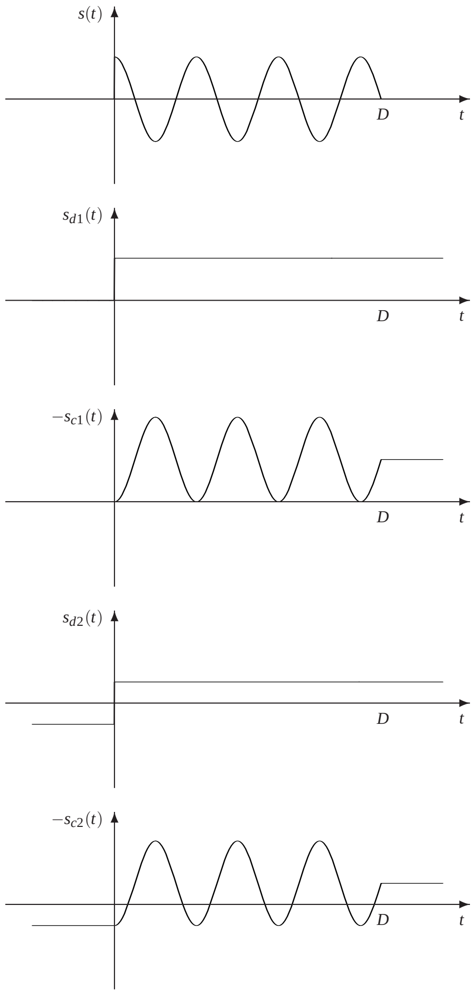

Fig. S9.2 The signal of the problem and its decomposition for $D = 3.25T_0$. The discontinuities are not balanced and the two types of decompositions are different: $s_{d1}(t) \neq s_{d2}(t)$

9.2. ⋆[Sect. 9.4] Starting from the pair (9.19), find the Fourier transform of the signal

$$s_1(t) = A_0\, \mathrm{e}^{-t^2/T^2} .$$

We start from the known Fourier pair (see Tab. 9.2, 24)

$$\mathrm{e}^{-\pi t^2} \quad \xrightarrow{\mathcal{F}} \quad \mathrm{e}^{-\pi f^2}$$

and then apply the scale change relationship (5.93) that is

$$s(at) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{|a|} S\left(\frac{f}{a}\right) \quad \text{with} \quad a = \frac{1}{T\sqrt{\pi}} .$$

We thus obtain

$$s_1(t) = A_0\, \mathrm{e}^{-t^2/T^2} \quad \xrightarrow{\mathcal{F}} \quad S_1(f) = A_0\, T\sqrt{\pi}\, \mathrm{e}^{-\pi^2 T^2 f^2} . \qquad \text{(S9.1)}$$

9.3. ⋆⋆[Sect. 9.4] Find the Fourier transform of the signal

$$s_2(t) = A_0 \int_{-\infty}^{t} \mathrm{e}^{-u^2/T^2}\, \mathrm{d}u .$$

We preliminarly note that the signal $s_2(t)$ is the integral of $s_1(t)$. Thus, from the “integral in time” rule of Tab. 9.1, we obtain

$$S_2(f) = \frac{1}{\mathrm{i}2\pi f}\, S_1(f) + \frac{1}{2} S_1(0)\, \delta(f)$$

where $S_1(f)$ was derived in the previous exercise. To conclude with, we have

$$S_2(f) = \frac{1}{\mathrm{i}2\pi f}\, A_0\, T\sqrt{\pi}\, \mathrm{e}^{-\pi^2 T^2 f^2} + \frac{1}{2} A_0\, T\sqrt{\pi}\, \delta(f) .$$

9.4. ⋆⋆[Sect. 9.4] Find the Fourier transform of the signal

$$s_3(t) = t\, \mathrm{e}^{-t^2/T^2} .$$

From the second rule on “differentiation in frequency” of Tab. 9.1 we obtain

$$s_3(t) = t\, s(t) \quad \xrightarrow{\mathcal{F}} \quad S_3(f) = -\frac{1}{\mathrm{i}2\pi}\,\frac{\mathrm{d}S(f)}{\mathrm{d}f}$$

where (see (S9.1))

$$s(t) = \mathrm{e}^{-\frac{t^2}{T^2}} \quad \xrightarrow{\mathcal{F}} \quad T\sqrt{\pi}\,\mathrm{e}^{-\pi^2 f^2 T^2}\ .$$

So, we have

$$S_3(f) = -\mathrm{i}\pi f \sqrt{\pi}\, T^3\, \mathrm{e}^{-\pi^2 f^2 T^2}\ .$$

9.5. $\star$ [Sect. 9.4] Find the Fourier transform of the signal

$$s_4(t) = 1(t)\, t^2\, \mathrm{e}^{-t/T}\ \sin\omega_0 t\ .$$

First it is convenient to get the Fourier transform of $s(t) = 1(t)\, t^2\, \mathrm{e}^{-t/T}$ and then apply the general rule on modulation (9.14b), that is

$$s_4(t) = s(t)\sin\omega_0 t \quad \xrightarrow{\mathcal{F}} \quad S_4(f) = \frac{1}{2\mathrm{i}}\,[S(f-f_0) - S(f+F_0)]\ .$$

Thus we evaluate

$$S(f) = \int_0^\infty t^2\, \mathrm{e}^{-\alpha t}\, \mathrm{e}^{-\mathrm{i}2\pi f t}\, \mathrm{d}t\ .$$

Integrating by parts one gets

$$S(f) = \frac{2}{(\mathrm{i}2\pi f - \alpha)^3}\,, \qquad \alpha = \frac{1}{T}\ .$$

In alternative one can evaluate $S(f)$ trough the derivative rule in the frequency (Tab. 9.1), which must be applied twice and gives

$$(-\mathrm{i}2\pi t)^2\, s(t) \quad \xrightarrow{\mathcal{F}} \quad \frac{\mathrm{d}^2 S(f)}{\mathrm{d}f^2}\ .$$

In any case, we have

$$S_4(f) = \frac{\mathrm{i}}{[\mathrm{i}2\pi(f+f_0) - \alpha]^3} - \frac{\mathrm{i}}{[\mathrm{i}2\pi(f-f_0) - \alpha]^3}\,, \qquad \alpha = \frac{1}{T}\ .$$

9.6. $\star$ [Sect. 9.4] Find the Fourier transform of the signal

$$s_5(t) = A_0 \ \mathrm{sinc}(f_1 t)\ \cos 2\pi f_2 t\ .$$

By applying the modulation rule (9.14a) we immediately obtain

$$S_5(f) = \frac{A_0}{2f_1}\left[\mathrm{rect}\left(\frac{f-f_2}{f_1}\right)+\mathrm{rect}\left(\frac{f+f_2}{f_1}\right)\right]\ .$$

9.7. $\star\star$ [Sect. 9.4] The derivative of a triangular pulse is given by the sum of two rectangular pulses. Use this remark to find its Fourier transform from pairs 17) of Tab. 9.2.

Fig. S9.3 shows the triangular pulse $s(t)$ and its derivative. Since the derivative is the sum of two triangles, and in particular we have

$$x(t) = \frac{d\,s(t)}{\mathrm{d}t} = \frac{1}{b}\ \mathrm{rect}\left(\frac{t+\frac{1}{2}b}{b}\right) - \frac{1}{a}\ \mathrm{rect}\left(\frac{t-\frac{1}{2}a}{a}\right)\ .$$

We straightforwardly obtain

$$X(f) = \mathrm{sinc}(fb)\,\mathrm{e}^{\mathrm{i}\pi bf} - \mathrm{sinc}(fa)\,\mathrm{e}^{-\mathrm{i}\pi af}\ .$$

Finally we evaluate the Fourier transform of $s(t)$, considering that

$$s(t) = \int_{-\infty}^{t} x(u)\,\mathrm{d}u\,,$$

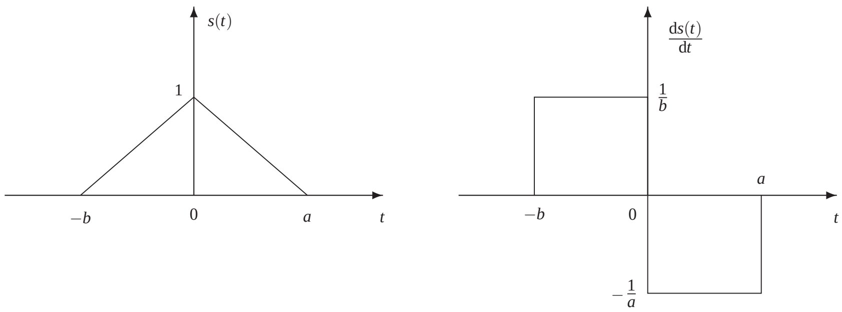

Fig. S9.3 The triangular pulse and its derivative and then we apply the integration rule in the time domain of Tab. 9.1 by recalling that $X(0) = \mathrm{area}(x) = 0$. We obtain

$$S(f) = \frac{1}{\mathrm{i}2\pi f} X(f) = \frac{1}{\mathrm{i}2\pi f} \left[\mathrm{sinc}\,(fb)\,\mathrm{e}^{\mathrm{i}\pi fb} - \mathrm{sinc}\,(fa)\,\mathrm{e}^{-\mathrm{i}\pi fa} \right] .$$

9.8. ⋆⋆[Sect. 9.4] Prove Fourier pair 18) using the technique suggested in the previous problem.

Fig.S9.4 shows the trapetium pulse and its derivative. The derivative can be written as

$$x(t) = \frac{d\,s(t)}{\mathrm{d}t} = \frac{1}{b-a}\ \mathrm{rect}\left[\frac{t+c}{b-a}\right] - \frac{1}{b-a}\ \mathrm{rect}\left[\frac{t-c}{b-a}\right], \qquad c = \tfrac{1}{2}(a+b)$$

and its Fourier transform is

$$\begin{aligned} X(f) &= \mathrm{sinc}[f(b-a)]\,\mathrm{e}^{\mathrm{i}2\pi fc} - \mathrm{sinc}[f(b-a)]\,\mathrm{e}^{S}R26{-}\mathrm{i}2\pi fc \\ &= 2\mathrm{i}\ \mathrm{sinc}[f(b-a)]\ \sin 2\pi fc\,. \end{aligned}$$

With considerations analogous to the ones of Problem 9.7, we obtain

$$\begin{aligned} S(f) &= \frac{1}{\mathrm{i}2\pi f} X(f) \\ &= (a+b)\ \mathrm{sinc}\left[f(b-a)\right]\ \mathrm{sinc}\left[f(b+a)\right] . \end{aligned}$$

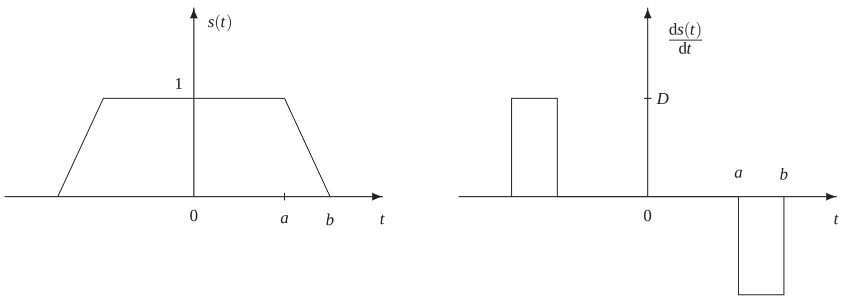

Fig. S9.4 The trapezoidal pulse and its derivative

9.9. ⋆⋆[Sect. 9.5] Check that for a Gaussian pulse (9.37) holds with equality sign.

Let $p(t)$ be the Gaussian pulse and let $P(f)$ be its Fourier transform. We have to prove that

$$(2\pi)^2 B_q^2 D_q^2 = \frac{E_{P'} E_{p'}}{E_p^2} \tag{S9.2}$$

where $p'(t)$ is the time derivate of $p(t)$ and $P'(f)$ is the frequency derivate of $P(f)$

$$D_q^2 = \int_{-\infty}^{+\infty} t^2 |p(t)|^2 \, \mathrm{d}t \big/ E_p$$
$$B_q^2 = \int_{-\infty}^{+\infty} f^2 |P(f)|^2 \, \mathrm{d}f \big/ E_P$$

with

$$E_p = \int_{-\infty}^{+\infty} |p(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{+\infty} |P(f)|^2 \, \mathrm{d}f = E_p \,.$$

As we see, the proof of the statement lies on the evaluation of integrals. To this end it is convenient to consider the normalized Gaussian pulse and its FT

$$p_0(t) = \mathrm{e}^{-\pi t^2} \xrightarrow{\mathcal{F}} \mathrm{e}^{-\pi f^2} = P_0(f)$$

and then we will prove that the introduction of an amplitude and of a scale change does not modify the result.

We start from the integrals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}x^2} = 1$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 \, \mathrm{e}^{-\frac{1}{2}x^2} = 1$$

which is known from Probability Theory (the first gives the normalization condition of a normalized Gaussian probability density and the second gives the variance). Then, with a change of variable we get

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-Ay^2} \, \mathrm{d}y = \sqrt{\pi/A}$$
$$\int_{-\infty}^{+\infty} y^2 \, \mathrm{e}^{-Ay^2} \, \mathrm{d}y = \frac{\sqrt{\pi}}{2A^{3/2}} \,.$$

Hence, considering that $p_0(x) = P_0(x)$, we obtain

$$
\begin{aligned}
&E_{p_0} = 1/\sqrt{2}\\
&D_q^2 = B_q^2 = 1/(4\pi)\\
&p_0'(t) = -2\pi t\,\mathrm{e}^{-\pi t^2} = -2\pi t\,p_0(t)\\
&E_{p_0'} = (2\pi)^2 \int_{-\infty}^{+\infty} t^2\,p_0^2(t)\,\mathrm{d}t = (2\pi)^2 D_q^2 E_{p_0} = E_{P_0'}\,.
\end{aligned}
$$

Substituting in (S9.2) gives

$$
\begin{aligned}
&(2\pi)^2 B_q^2 D_q^2 = (2\pi)^2 \frac{1}{(4\pi)^2} = \frac{1}{4}\\
&\frac{E_{P_0'} E_{p_0'}}{E_{p_0}^2} = \frac{[(2\pi)^2\ 1/(4\pi)^2\ (1/\sqrt{2})]^2}{(1/\sqrt{2})^2} = \frac{1}{4}\,.
\end{aligned}
$$

In conclusion, the statement of the problem is proved for the normalized Gaussian pulse. To prove the general case of the pulse

$$
p(t) = A_0\,\mathrm{e}^{-\alpha t^2}
$$

we remark that the amplitude A_0 is compensated in the ratio. But, also a scale factor, which gives

$$
p(t) = A_0\,p_0(\sqrt{\alpha/\pi}\,t)
$$

is compensated in the ratios recalling that the FT is given by

$$
P(f) = A_0 \sqrt{\frac{\pi}{\alpha}}\,P_0\left(\sqrt{\frac{\pi}{\alpha}}\,f\right)\,.
$$

9.10. ⋆⋆ [Sect. 9.5] Evaluate the product $B_q D_q$ for the triangular pulse

$$
p(t) = A_0\,\mathrm{triang}(t/D)\,.
$$

The solution requires the following integrals

$$
\begin{aligned}
&\int (1-y)^2\,\mathrm{d}y = -\frac{1}{3}(1-y)^3\\
&\int (1-y)^2 y^2\,\mathrm{d}y = \frac{1}{3}y^3 - \frac{1}{2}y^4 + \frac{1}{5}y^5\\
&\int_0^\infty x^2\,\mathrm{sinc}^4(x)\,\mathrm{d}x = \frac{1}{4\pi^2}\,.
\end{aligned}
$$

We can also assume $A_0 = 1$ since the amplitude is compensated in the product $B_q D_q$. Considering that $\mathrm{triang}(x)$ is even and $\mathrm{triang}(x) = 1 - x$, $0 \le x < 1$, the energy of

$p(t)$ is given by

$$E_p = 2\int_0^D (1-t/D)^2\,\mathrm{d}t = 2D\int_0^1 (1-x)^2\,\mathrm{d}x = \frac{2}{3}D\,.$$

Analogously

$$I_1 = \int_{-\infty}^{+\infty} t^2\,p^2(t)\,\mathrm{d}t = 2\int_0^D t^2(1-t/D)^2\,\mathrm{d}t = 2D^3\int_0^1 x^2(1-x)^2\,\mathrm{d}x = \frac{1}{15}D^3\,.$$

Hence

$$D_q^2 = I_1/E_p = \frac{1}{10}D^2\,.$$

The FT of $p(t)$ is

$$P(f) = D\,\mathrm{sinc}^2(fD)\,.$$

Hence

$$I_2 = \int_{-\infty}^{+\infty} f^2 P^2(f)\,\mathrm{d}f = 2D^2\int_0^{\infty} f^2\,\mathrm{sinc}^4(fD)\,\mathrm{d}f$$
$$\frac{2}{D}\int_0^{\infty} x^2\,\mathrm{sinc}^4(x)\,\mathrm{d}x = \frac{2}{D}\,\frac{1}{4\pi^2} = \frac{1}{2\pi^2 D}$$

and

$$B_q^2 = I_2/E_p = \frac{1}{D^2}\,\frac{3}{4\pi^2}\,.$$

The product $B_q D_q$ is therefore

$$B_q D_q = \sqrt{\frac{3}{40}}\,\frac{1}{\pi}\,.$$

Note that $B_q D_q = 0.0872 < 1/(4\pi) < 0.0796$, where $1/(4\pi)$ refers to a Gaussian pulse (which gives the minimum value of $B_q D_q$).

9.11. ⋆⋆[Sect. 9.5] Check bounds (9.33) and (9.37) for the signal

$$s(t) = 1(t)\,t\,\mathrm{e}^{-t/T}\,, \qquad T > 0\,.$$

Let us recall the following integral, which can be calculated recursively using integration by parts

$$\int_0^{+\infty} x^n\,\mathrm{e}^{-\alpha x}\,\mathrm{d}x = \left(\frac{1}{\alpha}\right)^{n+1} n!\,, \qquad \alpha > 0\,. \tag{S9.3}$$

Then, the energy of the signal is given by

$$E_s = \int_0^{+\infty} t^2 \mathrm{e}^{-2t/T}\,\mathrm{d}t = \left(\frac{T}{2}\right)^3 2! = \frac{T^3}{4}\ .$$

The root mean square (rms) duration D_q of the signal is obtained applying (S9.3):

$$D_q^2 = \frac{1}{E_s}\int_0^{+\infty} t^4\,\mathrm{e}^{-2t/T}\,\mathrm{d}t = \frac{1}{E_s}\left(\frac{T}{2}\right)^5 4! = 3T^2\ .$$

Hence $D_q = \sqrt{3}T$. To calculate the rms bandwidth, the Fourier transform of the signal is required

$$S(f) = \frac{T^2}{(1+\mathrm{i}2\pi T f)^2} \quad \Longrightarrow \quad |S(f)|^2 = \frac{T^4}{\left(1+(2\pi T)^2 f^2\right)^2}\ .$$

Then

$$B_q^2 = \frac{1}{E_S}\int_{-\infty}^{+\infty} f^2 |S(f)|^2\,\mathrm{d}f = \frac{1}{E_s}\int_{-\infty}^{+\infty} \frac{T^4 f^2}{(1+(2\pi T)^2 f^2)^2}\,\mathrm{d}f$$

where, with the variable change $x = 2\pi T f$, we obtain

$$B_q^2 = \frac{1}{E_s}\frac{T}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{x^2}{(1+x^2)^2}\,\mathrm{d}x$$

and using the integral

$$\int \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = -\frac{x}{2(1+x^2)} + \frac{1}{2}\arctan x$$

we obtain

$$B_q^2 = \frac{1}{2T^2\pi^3}\left[-\frac{x}{2(1+x^2)} + \frac{1}{2}\arctan x\right]_{-\infty}^{+\infty} = \frac{1}{4T^2\pi^2}\ .$$

Hence $B_q = 1/(2T\pi)$. Now, we see that the inequality (9.33) is satisfyed. In fact

$$D_q B_q = \sqrt{3}T\frac{1}{2T\pi} = \frac{\sqrt{3}}{2}\frac{1}{\pi} > \frac{1}{4\pi}\ .$$

The centroid abscissa of $|s(t)|^2$ is obtained from (S9.3):

$$t_c = \frac{1}{E_s}\int_0^{+\infty} t^3 \mathrm{e}^{-2t/T}\,\mathrm{d}t = \frac{4}{T^3}\left(\frac{T}{2}\right)^4 3! = \frac{3}{2}T\ .$$

So that

$$\Delta D_q = \sqrt{D_q^2 - t_c^2} = \sqrt{3T^2 - \tfrac{9}{4}T^2} = \frac{\sqrt{3}T}{2}$$

The centroid abscissa of $|S(f)|^2$ is easily obtained and is

$$f_c = \frac{1}{E_S} \int_{-\infty}^{+\infty} f |S(f)|^2 \, \mathrm{d}f = 0$$

because $|S(f)|^2$ is an even function and $|S(f)|^2 = \mathrm{O}(t^{-3})$ is an integrable odd function. Then, $\Delta B_q = B_q = 1/(2\pi T)$. Inequality (9.37) is verified:

$$B_q D_q = \frac{\sqrt{3}}{2} \frac{1}{\pi} > \Delta B_q \Delta D_q = \frac{\sqrt{3}}{4} \frac{1}{\pi} > \frac{1}{4\pi} \, .$$

The abscissas t_c anf f_c can also be obtained by (9.35), but we need the derivative of $S(f)$ and $s(t)$, given by

$$S'(f) = -\mathrm{i}4\pi T^3 \frac{1 + \mathrm{i}2\pi T f}{(1 + \mathrm{i}2\pi T f)^3} \, , \qquad s'(t) = \left(1 - \frac{t}{T}\right) \mathrm{e}^{-t/T} \, 1(t) \, .$$

Then

$$E_{S'S} = \int_{-\infty}^{+\infty} S'(f) S^*(f) \, \mathrm{d}f = -\mathrm{i}4\pi T^5 \int_{-\infty}^{+\infty} \frac{1 - \mathrm{i}2\pi T f}{(1 + (2\pi T)^2 f^2)^3} \, \mathrm{d}f$$

noticing that the imaginary part of the function is odd and hence it gives a zero and, performing the change $x = 2\pi T f$, we have

$$\begin{aligned} E_{S'S} &= -\mathrm{i}2T^4 \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^3} \, \mathrm{d}x = -\mathrm{i}2T^4 \left[\frac{x}{4(1+x^2)^2} + \frac{3x}{8(1+x^2)} + \frac{3}{8} \arctan x \right]_{-\infty}^{+\infty} \\ &= -\mathrm{i}2T^4 \frac{3}{8} \pi = -\mathrm{i} \frac{3}{4} T^4 \pi \, . \end{aligned}$$

From (9.35)

$$t_c = \frac{\mathrm{i}}{2\pi} \frac{E_{S'S}}{E_S} = \frac{3}{2} T$$

as previously found. For f_c we have from (S9.3)

$$E_{s's} = \int_0^{+\infty} s'(t) s^*(t) \, \mathrm{d}t = \int_0^{+\infty} \left(t - \frac{t^2}{2}\right) \mathrm{e}^{-2t/T} \, \mathrm{d}t = \left(\frac{T}{2}\right)^2 - \frac{2}{T} \left(\frac{T}{2}\right)^3 = 0$$

as previously found.

9.12. ⋆[Sect. 9.6] Find the first three derivatives of the function (9.45) and then establish that the damping of the correspondent pulse is of the type $1/t^3$.

It is sufficient to recall the normalized expression (9.21a). Let

$$R_0(f) = \mathrm{rcos}(f, \alpha)$$

whose first derivative is (see (9.71))

$$R_0'(f) = P(f + \tfrac{1}{2}) - P(f - \tfrac{1}{2})$$

where

$$P(f) = \frac{\pi}{2\alpha} \cos \frac{\pi}{\alpha} f \operatorname{rect}\left(\frac{f}{\alpha}\right) .$$

Thus we have

$$R_0''(f) = P'(f + \tfrac{1}{2}) - P'(f - \tfrac{1}{2})$$

where

$$P'(f) = -\frac{\pi^2}{2\alpha^2} \sin \frac{\pi}{\alpha} f \operatorname{rect}\left(\frac{f}{\alpha}\right) .$$

The behaviors are illustrated in Fig.S9.5, which clearly shows that $R_0(f)$ is continuous with a continuous first derivative, but discontinuous second derivative. Thus, we can apply Theorem 9.3 with $n = 3$, and we have that the inverse FT has a $1/t^3$ damping.

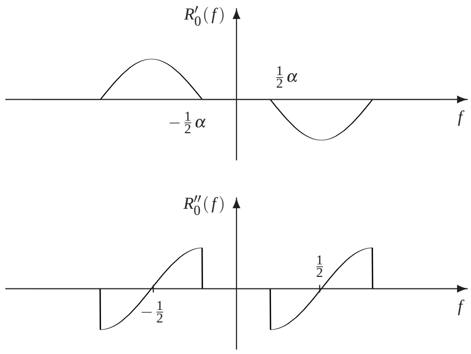

Fig. 9.5 The first two derivatives of the *raised–cosine* function for $\alpha = 0.6$

9.13. ⋆⋆ [Sect. 9.6] Find the damping of a pulse whose FT is given by the convolution of the raised cosine transform (9.45) with $\operatorname{rect}(f/(2F_0))$.

The raised cosine function in frequency corresponds to a pulse in time with a $1/t^3$ damping, while the rectangular function corresponds to a pulse with a $1/t$ damping. Since convolution in frequency gives a product in time, the resulting function shows a $1/t^4$ damping.

9.14. $\star$ [Sect. 9.8] Find the Laplace transform of the signal

$$s_1(t) = 1(t)\, t^2\, \mathrm{e}^{-t/T_0}\,, \qquad t \in \mathbb{R}\,.$$

The definition gives

$$S_L(p) = \int_0^{+\infty} t^2\, \mathrm{e}^{-t/T_0}\, \mathrm{e}^{-pt}\,.$$

Integrating by parts one gets

$$S_L(p) = \frac{2}{(a_0+p)^3}\,, \qquad a_0 = \frac{1}{T_0}\,.$$

Letting $p = \sigma + \mathrm{i}\omega$, it must be $a_0 + \sigma \geq 0$ and thus the convergence region is

$$\Gamma = C\,(-a_0, +\infty)\,.$$

9.15. $\star\star$ [Sect. 9.8] Find the Laplace transform of the signal

$$s_2(t) = 1(t)\, A_0\, \cos \omega_0 t\,, \qquad t \in \mathbb{R}\,.$$

The definition gives

$$S_L(p) = \int_0^{+\infty} A_0\, \cos \omega_0 t\, \mathrm{e}^{-pt}\, \mathrm{d}t\,.$$

By applying Euler's formulas we obtain

$$S_L(p) = A_0\, \frac{p}{\omega_0^2 + p^2}\,, \qquad \Gamma = C(0, +\infty)\,.$$

9.16. $\star\star$ [Sect. 9.8] Find the inverse Laplace transform of

$$S_L(p) = \frac{(p+1)}{p^2+p+1}\,, \qquad p \in \mathbb{C}\left(-\frac{1}{2}, +\infty\right)\,.$$

We can write

$$S_L(p) = \frac{A}{p-p_1} + \frac{B}{p-p_2} \quad \text{with} \quad p_1 = -\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\,,\; p_2 = -\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}\,, \qquad (9.4)$$

from which we can obtain A and B, namely

$$A = \frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{6}\,, \qquad B = \frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{6}\ .$$

Now, recalling the Fourier pairs of Tab. 9.4 and considering that $\Re p_1 = \Re p_2 = \frac{1}{2}$, we obtain

$$s(t) = A\,\mathrm{e}^{p_1 t}\,1(t) + B\,\mathrm{e}^{p_2 t}\,1(t)\ .$$

9.17. ⋆⋆⋆ [Sect. 9.8] As in the previous problem, but with convergence region given by $\mathbb{C}\left(-\infty, -\frac{1}{2}\right)$.

The decomposition (9.26) holds in this case as well, but the convergence region $C\left(-\infty, -\frac{1}{2}\right)$ provides a non–causal signal. In fact, in Sect. 9.7 we have found

$$1(-t)\,\mathrm{e}^{p_0 t} \quad \xrightarrow{\mathcal{L}} \quad -\frac{1}{p-p_0}\,, \qquad p \in C(-\infty, \sigma_0)$$

where $\sigma_0 = \Re\, p_0$. Thus, from decomposition (9.26) we conclude that

$$s(t) = -A\,1(-t)\,\mathrm{e}^{p_1 t} - B\,1(-t)\,\mathrm{e}^{p_2 t}\,, \qquad t \in \mathbb{R}\ .$$

9.18. ⋆⋆⋆ [Sect. 9.9] Find the frequency response $G(f)$ of a *real causal filter*, such that

$$\Re\, G(f) = \mathrm{rect}\,(f/2B)\ .$$

For a real and causal filter, real and imaginary parts of the frequency response $G(f)$ are linked by the Hilbert transform (see Sect. 9.9), that is,

$$\Im\, G(f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\Re\, G(\lambda)}{f-\lambda}\, d\lambda\ .$$

Since $\Re G(f) = \mathrm{rect}\,(\frac{f}{2B})$, we have

$$
\begin{aligned}
\Im(G(f)) &= \frac{1}{\pi}\int_{-B}^{B}\frac{1}{f-\lambda}\,d\lambda = -\frac{1}{\pi}\int_{-B}^{B}\frac{1}{\lambda-f}\,d\lambda \\
&= -\left[\frac{1}{\pi}\left(\log|B-f| - \log|-B-f|\right)\right] = \frac{1}{\pi}\log\left|\frac{f+B}{f-B}\right| \ .
\end{aligned}
\tag{9.5}
$$

Thus the frequency response of the filter is given by

$$
G(f) = \mathrm{rect}\left(\frac{f}{2B}\right) + \frac{\mathrm{i}}{\pi}\log\left|\frac{f+B}{f-B}\right| \ .
$$

9.19. $\star$[Sect. 9.9] Explicitly write the impulse responses of the ideal filters whose frequency responses are shown in Fig.9.22.

For the low–pass filter we have

$$
G(f) = \mathrm{rect}\left(\frac{f}{2B}\right) \xrightarrow{\mathcal{F}^{-1}} g(t) = 2B\,\mathrm{sinc}\,(2Bt) \ .
$$

For the pass–band filter, letting f_0 the center band frequency and $2\Delta f$ the bandwidth, we have

$$
G(f) = \mathrm{rect}\left(\frac{f-f_0}{2\Delta f}\right) + \mathrm{rect}\left(\frac{f+f_0}{2\Delta f}\right)
$$

and thus

$$
\begin{aligned}
g(t) &= 2\Delta f\ \mathrm{sinc}\,(2\Delta f\,t)\,\mathrm{e}^{\mathrm{i}2\pi f_0 t} + 2\Delta f\ \mathrm{sinc}\,(2\Delta f\,t)\,\mathrm{e}^{-\mathrm{i}2\pi f_0 t} \\
&= 4\Delta f\ \mathrm{sinc}\,(2\Delta f t)\cos(2\pi f_0 t) \ .
\end{aligned}
$$

For the high–pass filter we have

$$
G(f) = 1 - \mathrm{rect}\left(\frac{f}{2B}\right) \quad \xrightarrow{\mathcal{F}^{-1}} \quad g(t) = \delta(t) - 2B\,\mathrm{sinc}\,(2Bt) \ .
$$

9.20. [Sect. 9.9] Find the responses of an ideal low–pass filter when the input is: 1) a unit step and 2) a rectangular pulse. *Hint:* Use the *sine integral* function

$$
\mathrm{Si}\,(x) \triangleq \int_0^x \frac{\sin(y)}{y}\,\mathrm{d}y \ .
$$

The response of a filter to the unitary step is in general given by

$$u(t) = \int_{-\infty}^{t} g(a)\,\mathrm{d}a\,, \qquad t \in \mathbb{R}\,.$$

Letting $g(t) = 2B\,\mathrm{sinc}\,(2Bt)$, we thus obtain

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} 2B\,\mathrm{sinc}\,(2Ba)\,\mathrm{d}a \\ &= \frac{1}{\pi}\int_{-\infty}^{2\pi Bt} \frac{\sin v}{v}\,\mathrm{d}v = \frac{1}{\pi}\left[\int_{0}^{2\pi Bt} \frac{\sin v}{v}\,\mathrm{d}v - \int_{0}^{-\infty} \frac{\sin v}{v}\,\mathrm{d}v\right] . \end{aligned}$$

By further use of the integral sine function (Fig.9.6)

$$\mathrm{Si}(x) \stackrel{\Delta}{=} \int_0^x \frac{\sin y}{y}\,\mathrm{d}y\,,$$

we have

$$u(t) = \frac{1}{\pi}\left[\mathrm{Si}(2\pi Bt) - \lim_{x\to\infty}\mathrm{Si}(x)\right] = \frac{1}{\pi}\,\mathrm{Si}(2\pi Bt) + \frac{1}{2}\,.$$

To get the response to the rectangular pulse $p(t)$ with unitary amplitude and extension from 0 to T, we express such pulse as a difference of two unitary steps, that is,

$$p(t) = 1(t) - 1(t-T)\,.$$

We then obtain

$$u(t) - u(t-T) = \frac{1}{\pi}\left[\mathrm{Si}(2\pi Bt) - \mathrm{Si}\,(2\pi B(t-T))\right]\,.$$

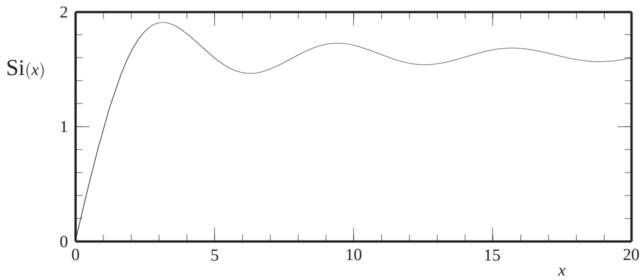

Fig. 9.6 The function *integral sine*

9.21. ⋆⋆⋆[Sect. 9.9] Show that the impulse response of the ideal *real* phase–shifter of β_0 is

$$g(t) = \delta(t)\,\cos\beta_0 - \frac{1}{\pi t}\,\sin\beta_0\,.$$

The frequency response of an ideal *real* β_0 phase–shifter can be written in the form (see (9.57b))

$$G(f) = e^{i\beta_0 \operatorname{sgn}(f)} = \cos\beta_0 + i\sin\beta_0 \operatorname{sgn}(f) .$$

By recalling that the inverse transform of $\operatorname{sgn}(f)$ is $i/(\pi t)$, we have

$$g(t) = \cos\beta_0\, \delta(t) - \sin\beta_0 \frac{1}{\pi t} .$$

9.22. ⋆⋆ [Sect. 9.9] Find the frequency response of a low–loss coaxial cable (see (9.56)).

The transfer function is given by (9.56), that is,

$$G_L(p) = \exp(-\sqrt{p/p_0} - pt_0)$$

where p_0 and t_0 are real. Letting $p = i2\pi f$ one gets

$$\sqrt{p} = \sqrt{i}\sqrt{2\pi f} = \frac{\sqrt{2}}{2}(1+i)\sqrt{2\pi f} = (1+i)\sqrt{\pi f}, \qquad f > 0 .$$

Hence

$$G(f) = G_L(i2\pi f) = \exp(-(1+i)\sqrt{\pi f/p_0} - i2\pi f t_0), \qquad f > 0$$

which is in agreement with 36 of Tab. 9.2.

We now discuss the Fourier pair 36. In Angot's book [2], p. 544, we find the Laplace pair

$$h(t) = \frac{\alpha}{2\sqrt{\pi t^3}} e^{-\alpha^2/(4t)}\, 1(t) \xrightarrow{\mathcal{L}} H(p) = e^{-\alpha\sqrt{p}}, \qquad \alpha > 0 .$$

If we let $\alpha = 1/\sqrt{p_0}$ we get

$$h(t) = \frac{1}{2\sqrt{p_0}\sqrt{\pi t^3}} e^{-1/(4p_0 t)}\, 1(t)$$

which gives the signal of pair 36 with $p_0 = \pi f_0$.

9.23. ⋆ [Sect. 9.10] Show that the following is a Hilbert pair

$$s(t) = \operatorname{sinc}(Ft), \qquad \widehat{s}(t) = \frac{1 - \cos 2\pi Ft}{\pi Ft} .$$

For the Hilbert transform evaluation it is convenient to choose an alternative (non–

direct) path based on the analytic signal. In fact from

$$v(t) = \mathrm{sinc}(Ft) \quad \xrightarrow{\mathcal{F}} \quad V(f) = \frac{1}{F}\,\mathrm{rect}\left(\frac{f}{F}\right)$$

the transform of the analytic signal is

$$Z(f) = 2 \cdot 1(f) \cdot V(f) = \frac{2}{F}\,\mathrm{rect}\left(\frac{f - \frac{1}{4}F}{\frac{1}{2}F}\right) .$$

By inverse transformation we obtain

$$z(t) = \mathrm{sinc}\left(\tfrac{1}{2}F\,t\right) \mathrm{e}^{\mathrm{i}2\pi\frac{1}{4}Ft} ,$$

from which the Hilbert transform is given by the imaginary part

$$\widehat{v}(t) = \Im\, z(t) = \mathrm{sinc}\left(\tfrac{1}{2}Ft\right) \sin(\tfrac{\pi}{2}Ft) = \frac{1 - \cos \pi F t}{\pi F t} .$$

9.24. ⋆⋆ [Sect. 9.10] Show that the following is a Hilbert pair

$$s(t) = \mathrm{rect}\left(\frac{t}{T}\right) , \qquad \widehat{s}(t) = \frac{1}{\pi}\,\log\left|\frac{2t+T}{2t-T}\right| .$$

By straightforwardly applying the definition (9.65) of Hilbert transform, we have

$$\begin{aligned}\widehat{v}(t) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{rect}(u/T)}{t-u}\,\mathrm{d}u = \frac{1}{\pi}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} \frac{1}{t-u}\,\mathrm{d}u \\ &= -\frac{1}{\pi}\left[\log|t - \tfrac{1}{2}T| - \log|t + \tfrac{1}{2}T|\right] = \frac{1}{\pi}\,\log\left|\frac{2t+T}{2t-T}\right| .\end{aligned}$$

9.25. ⋆⋆ [Sect. 9.10] Find the analytic signal associated to the signal

$$s(t) = \mathrm{sinc}^2\left(\frac{t}{T}\right)\,\cos 2\pi f_0 t \qquad \text{with} \qquad f_0 T > 1 .$$

It is convenient to evaluate the transform of the analytic signal from the relation $Z_v(t) = 2 \cdot\ 1(f)V(f)$ and then apply the inverse transformation. From

$$\mathrm{sinc}^2\left(\frac{t}{T}\right) \quad \xrightarrow{\mathcal{F}} \quad T\ \mathrm{triang}(fT)$$

where $\text{triang}(x)$ indicates the triangular function ranging from -1 to 1 and with unitary maximum amplitude, we obtain

$$V(f) = \tfrac{1}{2}\,\text{triang}((f - f_0)T) + \tfrac{1}{2}\,\text{triang}((f + f_0)T)\ .$$

Hence

$$\begin{aligned} V(f) &= 1(f)\,\text{triang}((f - f_0)T) + 1(f)\,\text{triang}((f - f_0)T) \\ &= \text{triang}((f - f_0)T) \end{aligned}$$

where we exploited the hypothesis $f_0 T > 1$ so that the first term has extension ranging on positive frequencies, and the second term has extension ranging on negative frequencies.

By inverse transformation, we obtain

$$z(t) = \text{sinc}^2\left(\frac{t}{T}\right)\,\mathrm{e}^{\mathrm{i}2\pi f_0 t}\ .$$

Problems of Chapter 10

10.1. ⋆⋆ [Sect. 10.1] In the previous chapter we have seen that a discontinuous signal on $\mathbb{R}$ can be decomposed into a continuous signal and a piecewise constant signal. Find the decomposition for a signal defined on $\mathbb{R}/\mathbb{Z}(T_p)$.

For an aperiodic signal $s(t)$, $t \in \mathbb{R}$ with discontinuities, we have obtained that the discontinuous part can be written in the form (9.2), that is,

$$s_d(t) = \sum_i d_i\, 1(t - t_i)\,, \qquad d_1 = s(t_i^+) - s(t_i^-)$$

or in the form (9.3), that is,

$$\tilde{s}_d(t) = \sum_i \tfrac{1}{2} d_i \operatorname{sgn}(t - t_i)\ .$$

These expressions hold also for a periodic signal $s(t)$, $t \in \mathbb{R}/\mathbb{Z}(T_p)$, with the remark that the periodicity implies that the discontinuities are periodic. Hence, if in the reference period $[0, T_p)$ the signal is discontinuous, say at t_1 and t_2 with amounts d_1 and d_2, respectively, we find discontinuities at the instants $t_1 + nT_p$ and $t_2 + nT_p$ with the same amounts. Hence, e.g., $s_d(t)$ takes the form

$$s_d(t) = d_1\, 1(t - t_1) + d_2\, 1(t - t_2)\,, \qquad 0 < t < T_p$$

and for any t we have to write

$$s_d(t) = \sum_{n=-\infty}^{+\infty} \left[d_1\, 1(t - t_1 - nT_p) + d_2\, 1(t - t_2 - nT_p)\right]\ .$$

10.2. ⋆ [Sect. 10.1] Find conditions on the signal

$$s(t) = A_1 \operatorname{rep}_{T_p} \operatorname{rect}\left(\frac{t}{d\,T_p}\right) + A_2 \operatorname{rep}_{T_p} \operatorname{rect}\left(\frac{t}{d\,T_p} - \frac{1}{2}\right), \qquad d = 20\%\,,$$

which assure that its integral $y(t)$, defined by (10.2), is still periodic. Then, evaluate $y(t)$ and its Fourier transform.

The signal can be written as

$$s(t) = A_1 \operatorname{rep}_{\mathrm{T_p}} \operatorname{rect}\left(\frac{t}{2T_0}\right) + A_2 \operatorname{rep}_{\mathrm{T_p}} \operatorname{rect}\left(\frac{t - T_0}{2T_0}\right) \qquad \text{(S10.1)}$$

where $T_0 = \frac{1}{2} d T_p = \frac{1}{10} T_p$ and is illustrated in Fig. S10.1 for $A_1 < 0$ and $A_2 = -A_1$. The integral $y(t)$ of $s(t)$, defined by (10.2), is

$$y(t) = \int_{t_0}^{t} s(u)\,\mathrm{d}u\,, \qquad t_0 \in \mathbb{R}\,.$$

The condition of periodicity for the integral is that the *mean value in a period* of $s(t)$, given by

$$m_s = \frac{1}{T_p} \int_0^{T_p} s(t)\,\mathrm{d}t$$

is zero. We find

$$m_1 = \frac{1}{T_p}(A_1 2T_0 + A_2 2T_0) = \frac{2T_0}{T_p}(A_1 + A_2)\,.$$

Then the periodicity condition is

$$A_1 = -A_2\,.$$

Under this assumption, the integral $y(t)$, obtained with $t_0 = 0$, is given by the periodic repetition of a trapezoidal pulse, as shown in Fig. S10.1.

The FT of $y(t)$, $t \in \mathbb{R}/\mathbb{Z}(T_p)$ can be calculated by the definition (10.4a), but it is more convenient the application of the integration rule, that is,

$$y(t) = \int_{t_0}^{t} s(u)\,\mathrm{d}u \quad \xrightarrow{\mathcal{F}} \quad Y(f) = \frac{1}{\mathrm{i}2\pi f} S(f)\,, \qquad f \neq 0\,. \tag{S10.2}$$

For the evaluation of $Y(f)$ for $f = 0$ we can use the rule

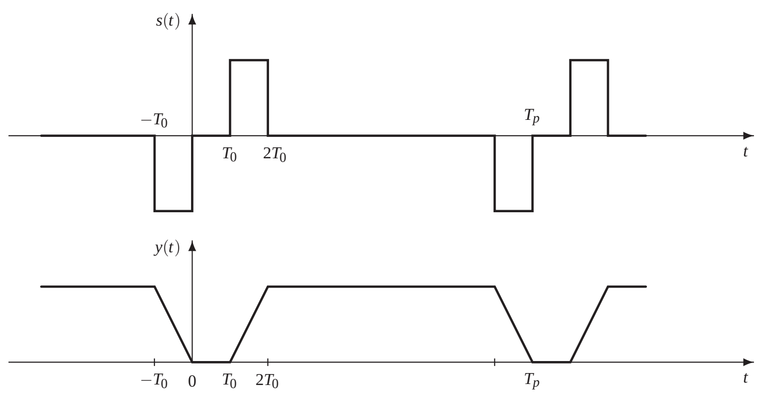

Fig. S10.1 Signals in Problem 10.2

$$Y(0) = \text{area}(y) = \int_0^{T_p} y(t)\,\mathrm{d}t = A_1\,T_0\,(T_p - 2T_0)$$

where the area is limited to a period.

On the other, for the evaluation of the FT $S(f)$ we can apply Proposition 10.1, which gives

$$\text{rep}_{\mathrm{T_p}}\,\text{rect}\left(\frac{t}{2T_0}\right) \quad \xrightarrow{\mathcal{F}} \quad 2T_0\,\text{sinc}(f2T_0)\,, \qquad f \in \mathbb{Z}(F)\,.$$

Hence, from (S10.1) one gets (with $A_1 = -A_2$)

$$\begin{aligned} S(f) &= -A_2\,2T_0\,\text{sinc}(f2T_0) + A_2\,2T_0\,\text{sinc}(f2T_0)\,e^{-j2\pi fT_0} \\ &= A_2\,2T_0\,\text{sinc}(f2T_0)\,\left(-1 + e^{-j2\pi f2T_0}\right)\,. \end{aligned}$$

Hence, (S10.2) yields

$$Y(f) = A_2\,2T_0\,\text{sinc}(f2T_0)\,\frac{-1 + \mathrm{e}^{-\mathrm{i}2\pi f2T_0}}{\mathrm{i}2\pi f}\,.$$

This result can be written in the form

$$Y(f) = \tfrac{1}{2}A_2\,(2T_0)^2\,\text{sinc}(f2T_0)\,\text{sinc}(fT_0)\,\mathrm{e}^{-\mathrm{i}\pi fT_0}$$

and is in agreement with the Fourier pair 9 of Tab. 10.1.

10.3. $\star$ [Sect. 10.3] Compute the Fourier coefficients of the "two waves" rectified sinusoid

$$s(t) = |\cos 2\pi F_0 t|\,, \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)\,.$$

The signal is shown in Fig. S10.2 and has minimum period $T_p = 1/(2F_0)$. Letting $F = 1/T_p = 2F_0$ the Fourier coefficients are given by

$$S_k = \frac{1}{T_p}\,S(kF) = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} s(t)\,\mathrm{e}^{-\mathrm{i}2\pi kFt}\,\mathrm{d}t\,.$$

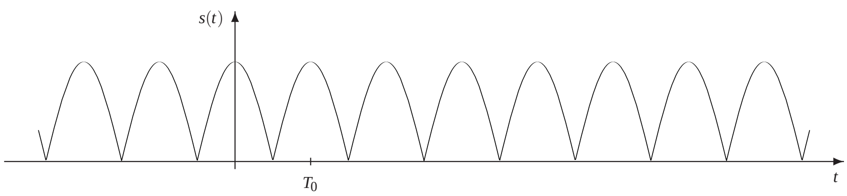

Fig. S10.2 The rectified sinusoidal signal

In this case we choose $t_0 = -\frac{1}{2}T_p$ which allows to remove the absolute value. Thus, we find

$$\begin{aligned} S_k &= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \cos(2\pi F_0 t)\,\mathrm{e}^{-\mathrm{i}2\pi kFt}\,\mathrm{d}t = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos x\,\mathrm{e}^{-\mathrm{i}2\pi kx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}\right)\mathrm{e}^{-\mathrm{i}2kx}\,\mathrm{d}x = \frac{1}{\pi}\left\{\frac{\sin(2k-1)\frac{1}{2}\pi}{(2k-1)} + \frac{\sin(2k+1)\frac{1}{2}\pi}{(2k+1)}\right\} \\ &= \operatorname{sinc}(k-\tfrac{1}{2}) + \operatorname{sinc}(k+\tfrac{1}{2}) \end{aligned}$$

where we have used the Euler formulas.

10.4. ⋆⋆ [Sect. 10.3] A signal with minimum period T_0 can be represented on $\mathbb{R}/\mathbb{Z}(T_0)$, but also on $\mathbb{R}/\mathbb{Z}(3T_0)$. Let $s_1(t)$ and $s_3(t)$ be the two representations, then find the relationship between $S_1(f)$ and $S_3(f)$.

The relations can be found by a separate evaluation of $S_1(f)$ and $S_3(f)$, given by

$$\begin{aligned} S_1(kF) &= \int_0^{T_0} s(t)\,\mathrm{e}^{-\mathrm{i}2\pi kF_0 t}\,\mathrm{d}t\,, \qquad F_0 = 1/T_0 \\ S_3(kF) &= \int_0^{3T_0} s(t)\,\mathrm{e}^{-\mathrm{i}2\pi kFt}\,\mathrm{d}t\,, \qquad F = 1/(3T_0) = F_0/3\,. \end{aligned}$$

However, a more elegant and general solution is obtained with the application of *elementary transformations*, developed in Chap. 6, and in particular of the Duality Theorem. Then, we realize that the signal $s_3(t)$, $t \in \mathbb{Z}(3T_0)$ is given by the *down–periodization* of $s_1(t), t \in \mathbb{Z}(T_0)$. The Duality Theorem states that the corresponding operation in the frequency domain is the $\mathbb{Z}(3F) \to \mathbb{Z}(F)$ up–sampling of $S_1(f)$, $f \in \mathbb{Z}(F_0)$. The up–sampling relation is

$$S_3(kF) = \begin{cases} 3\,S_1(kF) & k \in \mathbb{Z}(3) \\ 0 & k \notin \mathbb{Z}(3)\,. \end{cases}$$

10.5. ⋆⋆⋆ [Sect. 10.3] Using the Fourier series expansion of the signal (10.10a), prove that the modulated signal (10.11) can be written in the form

$$v(t) = \sum_{k=-\infty}^{+\infty} V_0\,J_k(A)\,\cos\left[2\pi\,(f_0 + kF)\,t\right]\,.$$

The signal (10.11) can be written in the form

$$v(t) = V_0\,\Re\left[\mathrm{e}^{\mathrm{i}2\pi f_0 t}\,\mathrm{e}^{\mathrm{i}A\sin 2\pi F t}\right]$$

where, for the second exponential, we can use the Fourier expansion (10.10c). Hence

$$v(t) = V_0\,\Re\left[\sum_{n=-\infty}^{+\infty} J_n(A)\,\mathrm{e}^{\mathrm{i}2\pi(f_0+nF)t}\right] .$$

The result is finally obtained considering that $J_n(A)$ is real.

Problems of Chapter 11

11.1. ⋆⋆ [Sect. 11.4] Evaluate the *running sum* $y(nT)$ of the causal exponential $1_0(nT)a^n$ and its Fourier transform.

The running sum is given by

$$\begin{aligned}y(nT) &= 1_0 * s(nT) = \sum_{k=-\infty}^{+\infty} T\,1_0(nT-kT)\,s(kT)\\ &= T\sum_{k=-\infty}^{n} s(kT) = T\sum_{k=-\infty}^{n} a^k\,.\end{aligned}$$

In order to apply the geometric series, we let $k = n - i$. Then

$$y(nT) = Ta^n\sum_{i=0}^{\infty} a^{-i} = Ta^n\,\frac{1}{1-1/a}$$

which requires that $|a| > 1$. For $|a| \le 1$, the series is not convergent. Therefore the current sum (for $|a| > 1$) is an exponential signal of the form $y(nT) = ka^n$ *whose FT does not exist*.

11.2. ⋆⋆ [Sect. 11.4] Prove the discrete modulation rule

$$s(nT)\,\cos(2\pi f_0 nT) \xrightarrow{\mathcal{F}} \frac{1}{2}S(f-f_0) + \frac{1}{2}S(f+f_0)$$

and apply it to the signal $s(nT) = 1_0(nT)$.

The rule can be obtained by using the Euler formulas

$$y(nT) = s(nT)\,\cos 2\pi f_0 T = \tfrac{1}{2}\,s(nT)\,e^{j2\pi f_0 nT} + \tfrac{1}{2}\,s(nT)\,e^{-j2\pi f_0 nT}$$

and by subsequently applying the shifting rule in the frequency domain.

With $s(nT) = 1_0(nT)$, it is sufficient to recall that the FT $U(f)$ of a discrete step is given by (11.17). Then

$$\begin{aligned}Y(f) &= \tfrac{1}{2}\,U_0(f-f_0) + \tfrac{1}{2}\,U(f-f_0)\\ &= \tfrac{1}{4}\,\delta_{\mathbb{R}/Z(F_p)}(f-f_0) + \tfrac{1}{4}\,\delta_{\mathbb{R}/Z(F_p)}(f+f_0)\\ &+ T\,\tfrac{1}{4\mathrm{i}}\,[\cot(\pi(f-f_0)T) + \cot(\pi(f+f_0)T] + \tfrac{1}{2}T\,.\end{aligned}$$

11.3. ⋆⋆ [Sect. 11.4] Evaluate the Fourier transform of a discrete triangular pulse (see pair 14 of Tab. 11.1).

The expression of discrete triangular pulse can be written in the form

$$s(nT) = \begin{cases} 1 - \frac{|n|}{N_0} & n = -N_0, \dots, 0, \dots, N_0 \\ 0 & \text{elsewhere} \end{cases}$$

From the definition of the Fourier transform one gets

$$\begin{aligned} S(f) &= \sum_{k=-\infty}^{+\infty} T\, s(nT)\, \mathrm{e}^{-\mathrm{i}2\pi f nT}\,, \qquad f \in \mathbb{R}/Z(1/T) \\ &= \sum_{n=-N_0}^{+N_0} T \left(1 - \frac{|n|}{N_0}\right) \mathrm{e}^{-\mathrm{i}2\pi f nT} \\ &= T + T \sum_{n=1}^{N_0} \left(1 - \frac{n}{N_0}\right) \left(\mathrm{e}^{\mathrm{i}2\pi f nT} + \mathrm{e}^{-\mathrm{i}2\pi f nT}\right)\,. \end{aligned}$$

Finally, considering Euler formulas, one obtains

$$S(f) = T + 2T \sum_{n=1}^{N_0} \left(1 - \frac{n}{N_0}\right) \cos(2\pi f nT)\,.$$

11.4. ⋆⋆ [Sect. 11.4] Find the Fourier transform of the signal

$$s(nT) = \begin{cases} 1 & n = 0, 3, 6, \dots \\ 0 & \text{elsewhere.} \end{cases}$$

Hint: use Proposition 11.2.

The evaluation of the FT can be handled as done for the FT of the discrete step signal $1_0(nT)$. An alternative evaluation can be done using the theory of *elementary tfs* developed in Chap. 6. In fact, $s(nT)$ may be regarded as the up–sampled version of the discrete step signal $\frac{1}{3}u_0(n3T)$ defined on $\mathbb{Z}(3T)$. The factor $\frac{1}{3}$ is due to the fact that the $\mathbb{Z}(3T) \to \mathbb{Z}(T)$ up–sampling introduces an amplification of 3 times.

In the frequency domain, the $\mathbb{Z}(3T) \to \mathbb{Z}(T)$ up–sampling becomes the $\mathbb{R}/\mathbb{Z}(F_0) \to \mathbb{R}/\mathbb{Z}(3F_0)$ down–periodization with $F_0 = 1/3T$. Therefore we obtain

$$S(f) = \tfrac{1}{3} U(f)\,, \qquad f \in \mathbb{R}/Z(F_p)$$

where $U(f)$ is the FT of the discrete step $1_0(n3T)$. This FT is obtained from (11.18) with the substitution $3T \to T$ and $F_0 \to F_p$. The final result is

$$S(f) = \tfrac{1}{3}\left[\tfrac{1}{2}\,\delta_{\mathbb{R}/Z(F_0)}(f) + 3T\,\tfrac{1}{2\mathrm{i}}\cot(\pi f 3T) + 3T\,\tfrac{1}{2}\right] .$$

11.5. ⋆[Sect. 11.6] Find the z–transform of the signal

$$s_1(nT) = 1_0(nT)\, n^2\, a^n .$$

We introduce the auxiliary signals

$$\begin{aligned} x(nT) &= 1_0(nT)\, a^n \\ y(nT) &= n\, x(nT) = 1_0(nT)\, n\, a^n . \end{aligned}$$

such that

$$s_1(nT) = 1_0(nT)\, n^2\, a^n = n\, y(nT) .$$

Now, the z–transform $x(nT)$, is given by

$$X(z) = T\,\frac{1}{1 - az^{-1}}\,, \qquad z \in \Gamma(|a|, \infty)$$

and we apply twice rule 11 of Tab. 11.2, i.e.,

$$n\, s(nT) \xrightarrow{\;\mathcal{Z}\;} -z\,\frac{d\,S(z)}{dz} \tag{S11.1}$$

Hence

$$\begin{aligned} Y(z) &= -z\,\frac{dX(z)}{dz} = T\,\frac{az^{-1}}{(1 - az^{-1})^2} \\ \Rightarrow \quad S_1(z) &= -z\,\frac{Y(z)}{dz} = T\,\frac{az^{-1}(1 + az^{-1})}{(1 - az^{-1})^3} \end{aligned}$$

The region of convergence is still $\Gamma(|a|, \infty)$.

11.6. ⋆⋆[Sect. 11.6] Find the z–transform of the signal

$$s_2(nT) = 1_0(nT)\, n\, \cos 2\pi f_0 nT .$$

It is convenient to start from the signal

$$x(nT) = 1_0(nT)\cos 2\pi f_0 T$$

and apply Rule 11 recalled in Problem 11.5.

The z–transform of $x(nT)$ is given by (see Tab. 11.3)

$$X(z) = T\,\frac{1-\beta z^{-1}}{1-2\beta z^{-1}+z^{-2}}\,, \qquad z \in \Gamma(1,\infty)$$

where

$$\beta = \cos 2\pi f_0 T\ .$$

Hence

$$\begin{aligned} S_2(z) &= -z\,\frac{dX(z)}{dz} \\ &= T\left[-z\,\frac{\beta z^{-2}(1-2\beta z^{-1}+z^{-2})-(1-\beta z^{-1})(2\beta z^{-2}-2z^{-3})}{(1-2\beta z^{-1}+z^{-2})^2}\right] \\ &= \frac{z^{-1}(\beta-2z^{-1}+\beta z^{-2})}{(1-2\beta z^{-1}+z^{-2})^2}\ . \end{aligned}$$

The region of convergence is $\Gamma(1,\infty)$.

11.7. $\star$ [Sect. 11.6] Find the z–transform of the signal

$$s_3(nT) = a^{|n|}\ .$$

The signal $s_3(nT)$ can be decomposed into causal and anticausal part in this form

$$s_3(nT) = 1_0(nT)\,a^n + 1_0(-nT)\,a^{-n} - 1 - \delta_{n0}$$

where the term δ_{n0} takes into account the fact that $1_0(0) = 1$. Then, recalling that (see Sect. 11.5)

$$\begin{array}{llll} 1_0(nT)\,a^n & \xrightarrow{\ z\ } & T\,\dfrac{1}{1-az^{-1}}\,, & z \in \Gamma(|a|,\infty) \\ 1_0(-nT)\,a^{-n} & \xrightarrow{\ z\ } & T\,\dfrac{1}{1-az}\,, & z \in \Gamma(0,1/|a|) \\ \delta_{n0} & \xrightarrow{\ z\ } & T\,, & z \in \mathbb{C}\ . \end{array}$$

The region of convergence is given by the intersection of the three regions and is not empty only if $|a| < 1$. In this hypothesis, it is given by $\Gamma(|a|,1/|a|)$. Hence

$$S_3(z) = T\left[\frac{1}{1-az^{-1}} + \frac{1}{1-az} - 1\right] = T\,\frac{1-a^2}{1+a^2-a(z+z^{-1})}\,, \qquad z \in \Gamma\left(|a|,1/|a|\right)\ .$$

11.8. $\star$ [Sect. 11.7] Find the impulse response of the discrete low–pass filter with frequency response (11.37).

We use Proposition 11.1 relating the FT on $\mathbb{Z}(T)$ to the FT on $\mathbb{R}$. The Fourier pair on $\mathbb{R}$ is

$$S_0(f) = \operatorname{rect}\left(\frac{f}{2B}\right), \qquad f \in \mathbb{R} \quad \xrightarrow{\mathcal{F}^{-1}} \quad s_0(t) = 2B \operatorname{sinc}(2Bt), \qquad t \in \mathbb{R}\,.$$

Hence

$$S(f) = \operatorname{rep}_{\mathrm{F_p}} \operatorname{rect}\left(\frac{f}{2B}\right), \qquad f \in \mathbb{R}/\mathbb{Z}(F_p) \quad \xrightarrow{\mathcal{F}^{-1}} \quad s(t) = 2B \operatorname{sinc}(2Bt), \qquad t \in \mathbb{Z}(T)\,.$$

Therefore the impulse response of the ideal band–pass filter on $\mathbb{Z}(T)$ is given by $g(nT) = 2B \operatorname{sinc}(2BnT)$.

11.9. $\star\star$ [Sect. 11.7] Show that the impulse response of the discrete Hilbert filter is given by (11.39b).

If we consider that in the period $(-\frac{1}{2}F_p, \frac{1}{2}F_p)$ the FT $G(f)$ of $g(t)$ is given by

$$G(f) = \begin{cases} \mathrm{i} & -\frac{1}{2} < f < F_p \\ -\mathrm{i} & 0 < f < \frac{1}{2}F_p\,, \end{cases}$$

we can express its inverse FT as

$$\begin{aligned} g(nT) &= \int_{-\frac{1}{2}F_p}^{0} \mathrm{i}\,\mathrm{e}^{\mathrm{i}2\pi fnT}\,\mathrm{d}f - \int_{0}^{\frac{1}{2}F_p} \mathrm{i}\,\mathrm{e}^{\mathrm{i}2\pi fnT}\,\mathrm{d}f \\ &= \mathrm{i}\int_{0}^{\frac{1}{2}F_p} (\mathrm{e}^{-\mathrm{i}2\pi fnT} - \mathrm{e}^{\mathrm{i}2\pi fnT})\,\mathrm{d}f = 2\int_{0}^{\frac{1}{2}F_p} \sin 2\pi fnT\,\mathrm{d}f\,. \end{aligned}$$

For $n \neq 0$ we obtain

$$g(nT) = \frac{1}{\pi nT}(1 - \cos \pi nT)$$

while $g(0) = 0$. Hence

$$g(nT) = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi nT} & n \text{ odd}\,. \end{cases}$$

11.10. ⋆[Sect. 11.7] Show that the impulse response of the filter whose frequency response is defined by (11.42), is given by

$$h_z(nT) = \frac{1}{T}\,\mathrm{sinc}\left(\frac{1}{2}n\right)\mathrm{i}^n\,.$$

The frequency response of the filter in the period $[-\frac{1}{2}F_p, \frac{1}{2}F_p]$ is

$$H_z(f) = \begin{cases} 0 & -\frac{1}{2}F_p < f < 0 \\ 2 & 0 < f < \frac{1}{2}F_p\,. \end{cases}$$

Hence

$$\begin{aligned} h_z(nT) &= 2\int_0^{F_p/2} \mathrm{e}^{\mathrm{i}2\pi f nT}\,\mathrm{d}f \\ &= 2\frac{1}{\mathrm{i}2\pi nT}\left(\mathrm{e}^{\mathrm{i}2\pi\frac{F_p}{2}nT} - 1\right), \qquad n \neq 0 \\ &= 2\frac{1}{\mathrm{i}2\pi nT}\mathrm{e}^{\mathrm{i}2\pi\frac{F_p}{4}nT}\left(\mathrm{e}^{\mathrm{i}2\pi\frac{F_p}{4}nT} - \mathrm{e}^{-\mathrm{i}2\pi\frac{F_p}{4}nT}\right) \\ &= \frac{1}{t}\mathrm{e}^{\mathrm{i}2\pi\frac{F_p}{4}nT}\,\mathrm{sinc}\left(\frac{F_p}{2}nT\right) \end{aligned}$$

where $F_pT = 1$.

11.11. ⋆⋆⋆[Sect. 11.7] Prove that the impulse response of the *discrete real* phase shifter of β_0 is

$$g(nT) = -\frac{\sin\beta_0}{\pi nT} + \frac{2\sin(\beta_0 + n\pi)}{\pi nT}\,, \qquad n \neq 0$$

while $g(0) = 0$.

Considering that for $|f| < \frac{1}{2}F_p$

$$G(f) = \mathrm{e}^{\mathrm{i}\beta_0\,\mathrm{sgn}\,f} = \begin{cases} \mathrm{e}^{-\mathrm{i}\beta_0} & -\frac{1}{2}F_p < f < 0 \\ \mathrm{e}^{\mathrm{i}\beta_0} & 0 < f < \frac{1}{2}F_p \end{cases}$$

the inverse FT is

$$g(nT) = \int_{-\frac{1}{2}F_p}^{0} \mathrm{e}^{-\mathrm{i}\beta_0}\,\mathrm{e}^{\mathrm{i}2\pi f nT}\,\mathrm{d}f + \int_0^{\frac{1}{2}F_p} \mathrm{e}^{\mathrm{i}\beta_0}\,\mathrm{e}^{\mathrm{i}2\pi f nT}\,\mathrm{d}f\,.$$

The integrals give

$$\begin{aligned} g(nT) &= \frac{e^{-i\beta_0}}{i2\pi nT} - \frac{e^{-i\beta_0}\, e^{-i2\pi\frac{1}{2}F_p nT}}{i2\pi nT} + \frac{e^{i\beta_0}\, e^{i2\pi\frac{1}{2}F_p nT}}{i2\pi nT} - \frac{e^{i\beta_0}}{i2\pi nT} \\ &= -\frac{1}{\pi nT}\,\frac{e^{i\beta_0} - e^{-i\beta_0}}{2i} + \frac{1}{\pi nT}\,\frac{e^{i(\pi n+\beta_0)} - e^{-i(\pi n+\beta_0)}}{2i} \\ &= -\frac{\sin\beta_0}{\pi nT} + \frac{\sin(\beta_0+\pi n)}{\pi nT} = \frac{\sin\beta_0}{\pi nT}\,[(1-(-1)^n)] \end{aligned}$$

and $g(0) = 0$. In confirmation of this result, for $\beta_0 = -\frac{1}{2}\pi$, we obtain (11.39b).

11.12. [Sect. 11.8] Show that if the input to a $\mathbb{Z}(T_0) \to \mathbb{Z}(T)$ down–sampler is causal with the rational z–transform

$$X(z) = \frac{Tz^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

the equality $\Gamma_y = \Gamma_x^N$ holds for the convergence regions.

We assume $p_0 \neq p_1$, and recall that, by causality, the input convergence region is $\Gamma_x = \Gamma\left(\frac{1}{3},\infty\right)$. Then, using the partial function expansion of $X_z(z)$, we find

$$X_z(z_0) = \frac{A}{1-\frac{1}{3}z_0^{-1}} + \frac{B}{1-\frac{1}{4}z_0^{-1}}\,, \qquad \Gamma_x = \Gamma\left(\tfrac{1}{3},\infty\right)$$

where $A = 12T$ and $B = -A$. Then

$$x(nT_0) = 1_0(n)\left[A\left(\frac{1}{3}\right)^n + B\left(\frac{1}{4}\right)^n\right] .$$

After the $\mathbb{Z}(T) \to \mathbb{Z}(T_0)$ down–periodization with $T = NT_0$, the signal is

$$y(nT) = x(nNT_0) = 1_0(n)\left[A\left(\frac{1}{3}\right)^{nN} + B\left(\frac{1}{4}\right)^{nN}\right] .$$

Hence

$$Y_z(z) = \frac{A}{1-\left(\frac{1}{3}\right)^N z_0^{-1}} + \frac{B}{1-\left(\frac{1}{4}\right)^N z_0^{-1}}\,, \qquad \Gamma_y = \left(\left(\tfrac{1}{3}\right)^N,\infty\right) .$$

In conclusion, after the down–sampling, the original poles $z_1 = 1/3$ and $z_2 = 1/4$ become z_1^N and z_2^N.

Problems of Chapter 12

12.1. ⋆⋆ [Sect. 12.3] Evaluate the DFT of the $\mathbb{Z}(T) \to \mathbb{Z}(T)/\mathbb{Z}(T_p)$ up–periodization of the unitary rectangular pulse on $[0,D) \cap \mathbb{Z}(T)$ with $D = MT \le T_p$.

For the DFT evaluation, we consider the cell

$$\mathcal{I} = [0, MT) \cap \mathbb{Z}(T) = \{0, T, \dots, (M-1)T\} \ .$$

Inside this cell, the signal is explicitly given by

$$w(nT) = \begin{cases} 1 & 0 \ge n < M-1 \\ 0 & \text{elsewhere} \ . \end{cases}$$

The application of the DFT gives

$$W(f) = \sum_{n=0}^{M-1} T\, \mathrm{e}^{-\mathrm{i}2\pi f nT} \,, \qquad f \in \mathbb{Z}(F)/\mathbb{Z}(F_p)$$

and, explicitly,

$$W(f) = \begin{cases} TM & fT \in \mathbb{Z} \\ T(1-\mathrm{e}^{-\mathrm{i}2\pi fMT})/(1-\mathrm{e}^{-\mathrm{i}2\pi fT}) & fT \notin \mathbb{Z} \end{cases} \qquad f \in \mathbb{Z}(F)/\mathbb{Z}(F_p) \ .$$

This result can be expressed by means of the *periodic sinc function* as

$$W(f) = D\,\mathrm{sinc}_M(fT)\,\mathrm{e}^{-\mathrm{i}2\pi f t_c} \,, \qquad f \in \mathbb{Z}(F)/\mathbb{Z}(F_p)$$

where

$$D = MT \,, \qquad t_c = \tfrac{1}{2}(M-1)T \ .$$

12.2. ⋆ [Sect. 12.3] The signal $s_0(t) = \cos 2\pi t/T_0$, $t \in \mathbb{Z}(T)/\mathbb{Z}(T_0)$, is truncated on the interval $[0, \alpha T_0)$ and repeated with period $T_p = \alpha T_0$, giving the signal $s(t)$, $t \in \mathbb{Z}(T)/\mathbb{Z}(T_p)$. Find the DFT of $s(t)$.

The signal $s(t)$ is obtained from $s_0(t) = \cos 2\pi t/T_0$, $t \in \mathbb{Z}(T)/\mathbb{Z}(T_0)$, by the following operations (Fig.S12.1)

1) $\mathbb{Z}(T)/\mathbb{Z}(T_0) \to \mathbb{Z}(T)$ down–periodization,

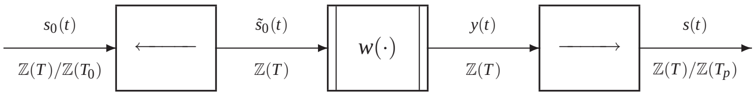

Fig. S12.1 Operations needed for the generation of the signal

2) multiplication by an indicator function $w(t), t \in \mathbb{Z}(T)$, of the "discrete interval" $[0, \alpha T_0) \cap \mathbb{Z}(T)$ (window with shape $w(t)$),
3) $\mathbb{Z}(T) \to /\mathbb{Z}(T)\mathbb{Z}(T_p)$ up–periodization.

Denoting the signal after the down–periodization with $\tilde{s}_0(t)$, after point 2) we obtain the signal $y(t) = w(t)\tilde{s}_0(t)$ and after point 3) we get

$$s(t) = \mathrm{rep}_{T_p} y(t) = \mathrm{rep}_{T_p}[w(t)\tilde{s}_0(t)], \qquad t \in \mathbb{Z}(T)/\mathbb{Z}(T_p) .$$

The Fourier transform of $\tilde{s}_0(t) = \cos 2\pi t/T_0$, $t \in \mathbb{Z}(T)$, is given by

$$\tilde{S}(f) = \tfrac{1}{2}\delta_{\mathbb{R}/\mathbb{Z}(F_p)}(f - f_0) + \tfrac{1}{2}\delta_{\mathbb{R}/\mathbb{Z}(F_p)}(f + f_0), \qquad f \in \mathbb{R}/\mathbb{Z}(F_p)$$

where $F_p = 1/T$ and $f_0 = 1/T_0$. The Fourier transform of $y(t)$ is obtained as the convolution

$$Y(f) = W * \tilde{S}(f) = \tfrac{1}{2}W(f - f_0) + \tfrac{1}{2}W(f + f_0), \qquad f \in \mathbb{R}/\mathbb{Z}(F_p)$$

and, finally, the Fourier transform $S(f)$ is obtained as the down–sampling $\mathbb{R}/\mathbb{Z}(F_p) \to \mathbb{Z}(F)/\mathbb{Z}(F_p)$ of $Y(f)$, that is,

$$S(f) = Y(f) = \tfrac{1}{2}W(f - f_0) + \tfrac{1}{2}W(f + f_0), \qquad f \in \mathbb{Z}(F)/\mathbb{Z}(F_p) \qquad \text{(S12.1)}$$

where

$$F = \frac{1}{T_p} = \frac{1}{\alpha T_0} = \frac{1}{\alpha} f_0 .$$

Finally, we have to calculate the FT $W(f)$, $f \in \mathbb{R}/\mathbb{Z}(F_p)$ of $w(t)$. We observe that, for the compatibility condition, αT_0 must be a multiple of T, say $\alpha T_0 = MT$. Then the FT can be expressed in the form

$$W(f) = (MT)\,\mathrm{sinc}_M(fT)\,\mathrm{e}^{-\mathrm{i}2\pi f t_c}, \qquad t_c = \tfrac{1}{2}(M-1)T \qquad \text{(S12.2)}$$

where $\mathrm{sinc}_M(x)$ is the periodic sinc function.

In conclusion, the DFT of $s(t)$ is given by

$$\begin{aligned} S(f) = {} & \tfrac{1}{2}MT\,\mathrm{sinc}_M((f - f_0)T)\,\mathrm{e}^{-\mathrm{i}2\pi(f-f_0)t_c} \\ & + \tfrac{1}{2}MT\,\mathrm{sinc}_M((f + f_0)T)\,\mathrm{e}^{-\mathrm{i}2\pi(f+f_0)t_c} . \end{aligned} \qquad \text{(S12.3)}$$

12.3. ⋆⋆[Sect. 12.3] In the previous problem suppose that α is a natural. Check that the DFT $S(f)$ consists of two impulses and explain why.

In (S12.2) and (S12.3) we let $f_0 = 1/T_0$ and $MT = \alpha T_0$. For the check, we must recall the definition of sinc_M and, in particular, the following property (valid if M is odd)

$$\mathrm{sinc}_M\left(\frac{m}{M}\right) = \begin{cases} 1 & m \in \mathbb{Z}(M) \\ 0 & m \notin \mathbb{Z}(M) \end{cases} \qquad m \in \mathbb{Z}\,.$$

Considering that

$$FT = \frac{1}{M}\,, \qquad f_0 T = \alpha FT = \frac{\alpha}{M}\,, \qquad \alpha \stackrel{\Delta}{=} n_0 \in \mathbb{N}$$

and indicating the frequency f by kF in (S12.3), we obtain

$$\mathrm{sinc}_M((f+f_0)T) = \mathrm{sinc}_M(kFT \pm n_0 FT) = \mathrm{sinc}_M\left(\frac{k \pm n_0}{M}\right)$$

so the transform is null except for $f = \pm f_0 + mF_p$. This proves that $S(f)$ is composed by two impulses $\delta_{\mathbb{Z}(F)/\mathbb{Z}(F_p)}(f \pm f_0)$ of appropriate area.

The obtained result agrees with fact that truncating a sinusoidal signal on an integer number of periods and repeating it periodically is equivalent to leave the signal unchanged. This observation holds for every periodic signal.

12.4. ⋆[Sect. 12.3] Show that the discrete chirp signal $s(nT) = W_{2N}^{n^2}$ has period NT for N even and period $2NT$ for N odd.

The following equality holds:

$$s((n+N)T) = W_{2N}^{(n+N)^2} = W_{2N}^{n^2} W_{2N}^{2nN} W_{2N}^{N^2}\,.$$

Considering that $W_{2N}^{mN} = W_2^m$, we have

$$W_{2N}^{2nN} = 1\,, \qquad W_{2N}^{N^2} = W_2^N\,.$$

Hence

$$s((n+N)T) = s(nT)\,W_2^N\,.$$

If N is even, $W_2^N = 1$ and the signal has period NT. Similarly, if N is odd, it is possible verify that $s((n+2NT)) = s(nT)$.

12.5. ⋆⋆[Sect. 12.5] Show that the sequence s_n that has a constant DCT (see Tab. 12.2) is given by

$$s_n = A_0 \frac{2N-1}{\sqrt{N}} \operatorname{sinc}_{(2N-1)}\left(\frac{(2N-1)(2n+1)}{2N}\right).$$

Take uniform weights and use identity (2.54).

For the sake of generality, we consider the DCT given by the second of (12.33). Now, by letting $S_n = A_0$ and $\lambda_k = \mu_k/\alpha_k$ in this relation, we obtain

$$s_n = \frac{A_0}{\sqrt{N}} \sum_{k=0}^{N-1} \lambda_k \cos 2\pi \frac{(2n+1)k}{4N} = \frac{A_0}{\sqrt{N}} [\lambda_0 + \tfrac{1}{2}\lambda_1 y_n]$$

where we let $\lambda_k = \lambda_1$ for $k \geq 1$ and

$$y_n = 2 \sum_{k01}^{N-1} \cos 2\pi \frac{(2n+1)k}{4N}.$$

Next, we use the identity

$$1 + 2\sum_{k=1}^{n_0} \cos \pi k x = M \operatorname{sinc}_M(Mx), \qquad M = 2n_0 + 1$$

and we let $n_0 = N - 1$ and $x = (2n+1)/(2N)$, thus obtaining

$$y_n = M \operatorname{sinc}_M\left(\frac{M(2n+1)}{2N}\right) - 1, \qquad M = 2N - 1$$

and hence

$$s_n = \frac{A_0}{\sqrt{N}}\left[\lambda_0 - \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\lambda_1 M \operatorname{sinc}_M\left(\frac{M(2n+1)}{2N}\right)\right].$$

Finally, with uniform weights ($\alpha_k = 1$), we have $\lambda_k = \mu_k$; therefore $\lambda_0 - \frac{1}{2}\lambda_1 = 0$ and $\frac{1}{2}\lambda_1 = 1$.

12.6. ⋆⋆[Sect. 12.5] Suppose that the DCT and the IDCT, given by (12.33), hold. Then, prove orthogonality conditions (12.35).

Relations (12.33) have the following structure

$$S_k = \sum_{n=0}^{N-1} A_{kn} s_n, \qquad s_n = \sum_{k=0}^{N-1} B_{nk} S_k$$

and can be rewritten in the matrix form (see Sect. 12.6–A)

$$\mathbf{S} = \mathbf{A}\mathbf{s}\,, \qquad \mathbf{s} = \mathbf{B}\mathbf{S}$$

where $\mathbf{A} = [A_{kn}]$ and $\mathbf{B} = [B_{nk}]$ are $N \times N$ matrices. Hence, combining these relations, we find that $\mathbf{S} = \mathbf{ABS}$ and $\mathbf{s} = \mathbf{BAs}$, and, consequently,

$$\mathbf{AB} = \mathbf{I} \quad \text{and} \quad \mathbf{BA} = \mathbf{I} \tag{S12.4}$$

where $\mathbf{I} = [\delta_{rs}]$ is the identity matrix. (This establishes, as obvious, that $\mathbf{A}$ and $\mathbf{B}$ are one the inverse of the other). By writing explicitly the (12.4), we have

$$\sum_{n=0}^{N-1} A_{hn} B_{nk} = \delta_{hk}\,, \qquad \sum_{k=0}^{N-1} B_{mk} A_{kn} = \delta_{mn}$$

which correspond to (12.35) as soon as we recall (12.33) with uniform weights $(\alpha_k = 1)$.

Problems of Chapter 13

13.1. ⋆[Sect. 13.4] Show that with $N = 2^m$ the general solution of recurrence (13.19) is

$$\mu(N) = \frac{1}{2}N\mu(2) + N(\log_2 N - 1) \qquad \text{(S13.1)}$$

with $\mu(2)$ the initial condition.

As done in the book, it is convenient to let $\mu_m = \mu(2^m)$, so that from the general formula of parallel computation one gets (see (13.20))

$$\mu_m = 2\mu_{m-1} + 2^m, \quad m \geq 2 .$$

This is a *difference equation*. For the first orders it gives

$$\begin{aligned} \mu_2 &= 2\mu_1 + 2^2 \\ \mu_3 &= 2\mu_2 + 2^3 = 2^2\mu_1 + 2 \cdot 2^3 \\ \mu_4 &= 2\mu_3 + 2^4 = 2^3\mu_1 + 3 \cdot 2^4 \end{aligned}$$

and *obviously* the general solution is

$$\mu_m = 2^{m-1}\mu_1 + (m-1)2^m . \qquad \text{(S13.2)}$$

However, this results can be formally proved *by induction*.

Finally, to get (S13.1) from (S13.2), it is sufficient to recall that $N = 2^m$, $m = \log_2 N$ and $\mu_1 = \mu(2)$.

13.2. ⋆⋆[Sect. 13.4] Prove (13.22) concerning the parallel computation of an a–point DFT.

With $N = a^m$, a decomposition $N = LM$ is given by $L = a^{m-1}$, $M = a$ and then the general relation (13.13) becomes

$$\mu(a^m) = a\,\mu(a^{m-1}) + (a-1)a^m .$$

Letting $\mu_m = \mu(a^m)$ one gets the following *difference equation*

$$\mu_m = a\,\mu_{m-1} + (a-1)a^m, \quad m \geq 2$$

where the initial condition is $\mu_1 = \mu(a)$.

Without entering in the theory of difference equations, we can proceed as in the solution of the previous problems. We write the solution for the first orders m and finally we get

$$\mu_m = a^{m-1}\,\mu_1 + (m-1)(a-1)a^m\,. \tag{S13.3}$$

Next, considering that $\mu_m = \mu(a^m)$, (S13.3) is equivalent to (13.22).

13.3. $\star$[Sect. 13.4] Gauss dedicated several years to compute the orbit of the asteroid Ceres. In particular he was engaged on a 12–point DFT, and found it convenient to use the decompositions $12 = 3\cdot 4$ and $12 = 3\cdot 2\cdot 2$ (Fig.13.20).

Discuss the advantage of such decompositions with respect to the direct 12–point DFT computation.

We consider only multiplications. The direct computation for $N = 12$ would require $12^2 = 144$ multiplications. From $\mu(N) = M\,\mu(L) + (M-1)N$ with the decomposition $M = 3$, $L = 4$, one gets

$$\mu(12) = 3\,\mu(4) + 2\cdot 12 = 3\cdot 16 + 24 = 72$$

although we assume $\mu(4) = 4^2 = 16$. With the decomposition $M = 3$, $L = 4 = 2\times 2$ one gets

$$\mu(12) = 3\mu(4) + 24, \quad \mu(4) = 2\mu(2) + 4$$

which, with $\mu(2) = 0$, gives $\mu(12) = 36$.

Hence, the reduction is from half to one fourth. A very relevant reduction for Gauss, who did not have a computer!

13.4. $\star\star$[Sect. 13.8] In the previous chapter (Sect. 12.4) we have introduced the *cosine* DFT. Organize its numerical computation and evaluate the number of operations.

Given the sequence $s_0, s_1, \ldots, s_{N-1}$, the *cosine* DFT is given by (12.23a), that is

$$S(kF) = T\sum_{n=0}^{N} \mu_n\, s(nT)\,\cos 2\pi\,\frac{nk}{2N}\,.$$

As seen in Sect. 12.4, the *cosine* DFT is given by the standard DFT on $2N$ points of an appropriate even sequence obtained from the given N–point sequence. Hence, the computation can be done by means of a $2N$–point FFT and the corresponding complexity is of $2N\log_2(2N)$ operations.

13.5. ⋆⋆ [Sect. 13.8] In the previous chapter (Sect. 12.4) we have introduced the *sine* DFT. Organize its numerical computation and evaluate the number of operations.

The sine DFT and IDFT on $N-1$ points are defined by (12.24) and can be obtained starting from $2N$–point DFT and IDFT, where the signal $s(nT)$ is *real and odd* and the DFT $S(kF)$ is *imaginary and odd*. Also, the signal has the constraints $s(0)=0$ and $s(NT)=0$ and the DFT has the same constraints $S(0)=0$ and $S(NF)=0$.

Hence, we start from a real sequence $s_n = s(nT)$, $n = 1,\dots,N-1$, of length $N-1$ and we construct a vector of $2N$ points using the above constraints and evaluate the $2N$–point DFT giving an imaginary vector from which we read the sine DFT at the frequencies $F, 2F, \dots, (N-1)F$.

The computational complexity is of $2N\log(2N)$ complex operations. Note the direct evaluation, based on (12.24a), would require $(N-1)^2$ real operations.

Problems of Chapter 14

14.1. ⋆⋆[Sect. 14.1] Given a generalized transform on $I \mapsto U$ with kernels $\theta(u,t)$ and $\varphi(t,u)$, prove that the kernels of the *dual* transform on $\widehat{I} \mapsto \widehat{U}$ are given by

$$\widehat{\theta}(\lambda, f) = \Theta(\lambda, -f)\,, \qquad \widehat{\varphi}(f, \lambda) = \Phi(f, -\lambda) \tag{S14.1}$$

where $\Theta(\lambda, f)$ and $\Phi(f, \lambda)$ are respectively the FTs of $\theta(u,t)$ and $\varphi(t,u)$. Here *dual* is not intended in the sense of (14.13), but as a frequency representation.

From a transform pair (s,S) we obtain the dual transform pair $(\widehat{s},\widehat{S})$

$$\widehat{\Theta}: \qquad \widehat{S}(\lambda) = \int_{\widehat{I}} \mathrm{d}f\, \widehat{\theta}(\lambda, f)\widehat{s}(f)\,, \qquad \lambda \in \widehat{U}$$

$$\widehat{\Phi}: \qquad \widehat{s}(f) = \int_{I} \mathrm{d}\lambda\, \widehat{\varphi}(f, \lambda)\widehat{S}(\lambda)\,, \qquad f \in \widehat{I}$$

where $\widehat{s}(f)$ and $\widehat{S}(\lambda)$ are the FTs of the signal $s(t)$ and of the transform $S(u)$, respectively.

The dual kernels can be obtained by applying of the FT, that is, more specifically (see Theorem 6.7),

$$\widehat{\theta}(\lambda, f) = \Theta(\lambda, -f)\,, \qquad \widehat{\varphi}(f, \lambda) = \Phi(f, -\lambda) \tag{S14.2}$$

where $\Theta(\lambda, f)$ and $\Phi(f, \lambda)$ are the FTs of the original kernels.

The dual transform is important in itself (as a new generalized transform), but also to establish properties of the given generalized transform.

14.2. ⋆⋆[Sect. 14.1] Prove that, if the kernels of a generalized transform $\theta(u,t)$ and $\varphi(t,u)$ are self–reciprocal, also the kernels of the dual transform $\hat{\theta}(\lambda, f)$ and $\hat{\varphi}(f, \lambda)$ are self–reciprocal.

The self–reciprocity condition is

$$\theta(u,t) = \varphi^*(t,u)\,. \tag{S14.3}$$

Then, the application of the conjugation rule of the FT, namely $s^*(t) \xrightarrow{\mathcal{F}} S^*(-f)$, gives in this case $\varphi^*(t,u) \xrightarrow{\mathcal{F}} \Phi^*(-t,-u)$. Then, in the frequency domain, (S14.3) becomes

$$\Theta(\lambda, f) = \Phi^*(-f, -\lambda)$$

where Θ and Φ are the FTs of θ and φ, respectively. On the other hand, the dual kernels found in the previous solution are given by (S14.2) and, therefore,

$$\tilde{\theta}(\lambda, f) = \tilde{\varphi}^*(f, \lambda)\,.$$

14.3. ⋆[Sect. 14.2] Check that the IDFT/DFT are a special case of (14.16a) and (14.16b).

The DFT and IDFT are respectively (see Sect. 5.8)

$$S(kF) = \sum_{n=0}^{N-1} T\, s(nT)\, W_N^{-kn}$$
$$s(nT) = \sum_{k=0}^{N-1} F\, S(kF)\, W_N^{kn}$$

which are a special case of (14.16) with

$$\begin{aligned} &I = \mathbb{Z}(T)/\mathbb{Z}(NT)\,, \qquad U = \mathbb{Z}(F)/\mathbb{Z}(NF) \\ &\varphi_k(n) = W_N^{-kn}\,, \qquad \theta_k(n) = W_N^{kn} \\ &S_k = \mathrm{d}(\mathbb{Z}(F))\, S(kF) = F\, S(kF)\,. \end{aligned}$$

In particular, (14.16b) becomes

$$s(nT) = \sum_{k=0}^{N-1} S_k\, W_N^{kn}\,.$$

14.4. ⋆⋆[Sect. 14.2] Show that for the Fourier series expansion both orthogonality conditions (14.6) and (14.8) hold.

The Fourier series expansion is (see Sect. 2.5)

$$s(t) = \sum_{n\in\mathbb{Z}} S_n\, \mathrm{e}^{\mathrm{i}2\pi nFt}\,, \qquad t \in \mathbb{R}/\mathbb{Z}(T_p)$$

where

$$S_n = \frac{1}{T_p}\int_0^{T_p} s(t)\, \mathrm{e}^{-\mathrm{i}2\pi nFt}\, \mathrm{d}t\,, \qquad n \in \mathbb{Z}\,.$$

This is a special case of signal expansion with $I = \mathbb{R}/\mathbb{Z}(T_p)$, $U = \mathbb{Z}(F)$, being $F = 1/T_p$. The basis functions are

$$\theta_n(t_0) = \mathrm{e}^{\mathrm{i}2\pi nFt_0}\,, \qquad \varphi_n(t_0) = \mathrm{e}^{-\mathrm{i}2\pi nFt_0}\,, \qquad t_0 \in \mathbb{R}/\mathbb{Z}(T_p)\,.$$

The FRC is

$$\sum_{n\in\mathbb{Z}} F\,\varphi_n(t_0')\,\theta_n(t_0) = F\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi nF(t_0'-t_0)} = \delta_{\mathbb{R}/\mathbb{Z}(T_p)}(t_0'-t_0)$$

where we have used the orthogonality condition of the FT on $\mathbb{R}/\mathbb{Z}(T_p)$ (see (5.11)). The IRC is given by

$$\begin{aligned}\int_{\mathbb{R}/\mathbb{Z}(T_p)} \mathrm{d}t_0\,\varphi_{n'}(t_0)\,\theta_n(t_0) &= \int_0^{T_p} \mathrm{e}^{-\mathrm{i}2\pi(n'-n)Ft_0}\,\mathrm{d}t_0 \\ &= T_p\,\delta_{n',n} = (1/F)\,\delta_{n',n}\end{aligned} \tag{S14.4}$$

where we have used the orthogonality of the exponential functions. The result of (S14.4) is in agreement with (14.18).

14.5. ⋆⋆ [Sect. 14.2] Apply the expansion/reconstruction (14.22) to the signal

$$s(t) = \mathrm{sinc}^2(Ft)\,, \qquad t \in \mathbb{R}$$

and show that, if the sampling frequency $F_c = 1/T < 2F$, the imperfect reconstruction gives the projection of $s(t)$ onto the class $H(B)$. *Hint*: consider that the projector defined by (14.23) is given by the cascade sampling/interpolation of the Fundamental Sampling Theorem and proceed in the frequency domain.

The FT of $s(t)$ is given by

$$S(f) = (1/F)\,\mathrm{triang}(f/F)$$

and therefore the spectral extension is $(-B, B)$ with $B = F$. If the sampling frequency is $F_c = 1/T \geq 2F$, the ideal interpolator with frequency response $Q(f) = \mathrm{rect}(f/F_c)$ gives the correct reconstruction.

If $F_c < 2B$, we find that the FT of the reconstructed signal $\tilde{s}(t)$ is given by

$$\tilde{S}(f) = Q(f)\sum_{k\in\mathbb{Z}} S(f - kF_c) \tag{S14.5}$$

and represents the projection of $S(f)$ onto the subspace of $L_2(\mathbb{R})$ of the FTs having extension $(-\frac{1}{2}F_c, \frac{1}{2}F_c)$.

Fig. S14.1 illustrates the FT $\tilde{S}(f)$ in the case $F_c = \frac{3}{2}F$, where (S14.5) has only two terms of aliasing

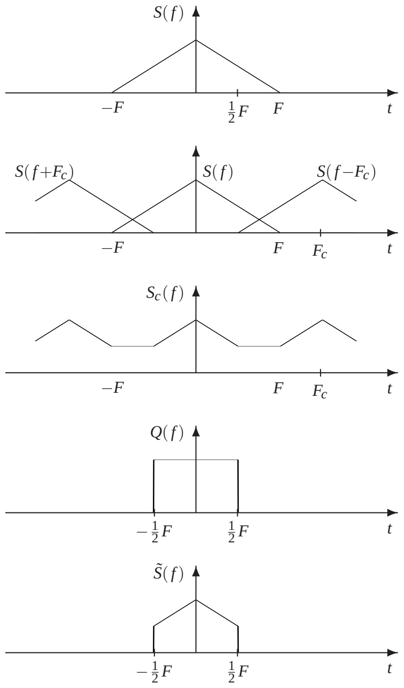

Fig. S14.1 The Fourier transforms in Problem 14.5 with $F_c = \frac{3}{2}F$

$$\tilde{S}(f) = Q(f)[S(f+F_c)+S(f)+S(f-F_c)] \ .$$

14.6. $\star\star$ [Sect. 14.3] Consider the Mercedes Benz frame defined in Example 14.3. Verify that it is a tight frame and find the redundancy.

Let $\mathbf{s} = (s_0, s_1) \in \mathbb{R}^2$. Then

$$\langle s, \varphi_0 \rangle = s_1 \ , \qquad \langle s, \varphi_1 \rangle = -\frac{\sqrt{3}}{2}s_0 - \frac{1}{2}s_1 \ , \qquad \langle s, \varphi_2 \rangle = \frac{\sqrt{3}}{2}s_0 - \frac{1}{2}s_1$$

and

$$\begin{aligned}\sum_{n=0}^{2} |\langle s, \varphi_n \rangle|^2 &= s_1^2 + \left(-\frac{\sqrt{3}}{2}s_0 - \frac{1}{2}s_1\right)^2 + \left(\frac{\sqrt{3}}{2}s_0 - \frac{1}{2}s_1\right)^2 \\ &= \frac{3}{2}s_0^2 + \frac{3}{2}s_1^2 = \frac{3}{2}||s||^2 \ .\end{aligned}$$

Then, the frames bounds are $A = B = \frac{3}{2}$. The redundancy is 50%.

14.7. ⋆⋆ [Sect. 14.3] Find a dual of the Mercedes Benz frame and discuss its multiplicity.

Given the 2×3 matrix of the Mercedes Benz frame

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{bmatrix}$$

we have to find a 3×2 matrix

$$\boldsymbol{\Theta} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

such that

$$\boldsymbol{\Phi}\boldsymbol{\Theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{S14.6}$$

This relation gives four linear equations is the six unknowns a_{ij}. Thus, we have infinitely many solutions.

Letting $a_{11} = x$, $a_{12} = y$ we solve the system (S14.6)

$$\begin{aligned} \frac{1}{2}\sqrt{3}a_{31} - \frac{1}{2}\sqrt{3}a_{21} &= 1 \\ \frac{1}{2}\sqrt{3}a_{32} - \frac{1}{2}\sqrt{3}a_{22} &= 0 \\ x - \frac{1}{2}a_{21} - \frac{1}{2}a_{31} &= 0 \\ y - \frac{1}{2}a_{22} - \frac{1}{2}a_{32} &= 1 \end{aligned}$$

with respect to x, y. The solution gives the matrix

$$\boldsymbol{\Theta} = \begin{bmatrix} x & y \\ x - 1/\sqrt{3} & y - 1 \\ x + 1/\sqrt{3} & y - 1 \end{bmatrix} .$$

For instance, with $x = 1$, $y = 0$ we get

$$\boldsymbol{\Theta} = \begin{bmatrix} 1 & 0 \\ 1 - 1/\sqrt{3} & -1 \\ 1 + 1/\sqrt{3} & -1 \end{bmatrix} .$$

This dual of the Mercedes Benz frame is shown in Fig.S14.2.

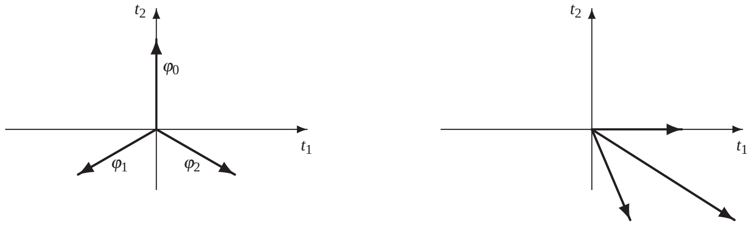

Fig. 14.2 The *Mercedes Benz* frame and one of its dual

> **14.8.** ⋆[Sect. 14.6] Formulate Proposition 14.2 in the case $I = \mathbb{R}$, $U = \mathbb{Z}(T)$ and $P = \mathbb{Z}$.

This problem wants to stress the generality of subband decomposition, which holds also in the decomposition of a continuous signal, with expansions obtained with $I = \mathbb{R}$ and $U = \mathbb{Z}$, namely

$$s(t_0) = \sum_{n\in\mathbb{Z}} S(nT)\,\varphi(t_0 - nT)\,, \qquad t \in \mathbb{R} \tag{S14.7}$$

where

$$S(nT) = \int_{-\infty}^{+\infty} \theta(t_0 - nT)\,s(t)\,\mathrm{d}t\,, \qquad n \in \mathbb{Z}\,. \tag{S14.8}$$

The above expansion is PI with periodicity $\mathbb{Z}(T)$. In fact, the basis functions are obtained from the functions $\varphi(t)$ and $\theta(t)$ with T–translations. We have

$$I = \mathbb{R}\,, \qquad U = \mathbb{Z}(T)\,, \qquad P = \mathbb{Z}(T)\,, \qquad B = [U/P) = \{0\}\,.$$

Since B degenerates, in the subband architecture of Fig. 14.15 the P/S and S/P disappear and we simply have: in the Analysis an $\mathbb{R} \to \mathbb{Z}(T)$ decimator with relation (S14.8) and with impulse response $q_0(t_0) = \theta(-t_0)$ and in the Synthesis a $\mathbb{Z}(T) \to \mathbb{R}$ interpolator with impulse response $g_0(t_0) = \varphi(t_0)$, as shown in Fig. S14.3.

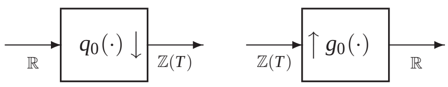

Fig. S14.3 Filter implementation of the sinc basis. The filters are ideal low–pass with $q_0(t_0) = g_0(t_0) = (1/T)\,\mathrm{sinc}(t_0/T)$

> **14.9.** ⋆[Sect. 14.6] Interpret the *Fundamental Sampling Theorem* (see Sect. 8.4) as a subband expansion on $I = \mathbb{R}$ and $U = \mathbb{Z}(T)$, where the Analysis performs the $\mathbb{R} \to \mathbb{Z}(T)$ sampling and the Synthesis gives the reconstruction of the signal from the samples.

This problem may be viewed as a reformulation of Example 14.2. The theorem claims that a band–limited signal $s(t)$ with spectral extension $(-B,B)$ can be reconstructed from its sample values $s(nT)$ with $T = 1/(2B)$, as

$$s(t) = \sum_{n\in\mathbb{Z}} s(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right), \qquad t \in \mathbb{R}.$$

Hence, we have (S14.7) with

$$\varphi_n(t) = \mathrm{sinc}\left(\frac{t-nT}{T}\right) \quad \text{and} \quad S_n = s(nT). \tag{S14.9}$$

This set of functions verifies the orthogonality condition (see Sect. 8.6). However, the set is not complete because the expansion holds only for band–limited signals of $L_2(\mathbb{R})$. Now, (S14.9) verifies the PI with $P = \mathbb{Z}(T)$.

The implementation according to the subband coding architecture is a special case of the previous problem. Note that the interpolator is irrelevant after the band–limitation of the signal and, therefore, the whole architecture degenerates into an $\mathbb{R} \to \mathbb{Z}(T)$ down–sampler.

14.10. $\star\star$ [Sect. 14.6] Formulate Proposition 14.2 in the case $I = \mathbb{Z}^2$, $U = \mathbb{Z}^2$ and $P = \mathbb{Z}_2^1(2,2)$, the quincunx lattice defined in Sect. 3.3.

The parameters are

$$I = \mathbb{Z}^2, \qquad U = \mathbb{Z}^2, \qquad P = \mathbb{Z}_1^2(2,2)\,.$$

To write explicitly the subband architecture we have to find a cell $B = [U/P)$. Considering the lattice P has basis

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix},$$

in order to find the cell we can use the fundamental parallelepiped P_0 determined by the basis vector, as shown in Fig.S14.4. The parallelepiped gives a cell $C_0 = [\mathbb{R}^2/P)$ and, according to Proposition 3.6, the desired cell B is obtained as the intersection of C_0 with the lattice $U = \mathbb{Z}^2$, that is,

$$B = \mathbb{Z}^2 \cap C_0\,.$$

Thus, we find that the cell B consists of 8 points given by

$$B = \{(0,0),(0,1),(0,2),(0,3),(1,1),(2,1),(3,1),(3,2)\}\,. \tag{S14.10}$$

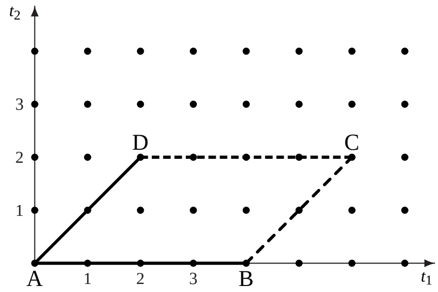

Fig. S14.4 Evaluation of the cell B: ABCD is the fundamental parallelepiped of the lattice $P = \mathbb{Z}_2^2(2,2)$; the points of the cell B are the points of $\mathbb{Z}^2$ contained in the parallelepiped

As a check, $\mathrm{d}(P) = 8$, $\mathrm{d}(U) = 1$ and then $N = (U : P) = \mathrm{d}(P)/\mathrm{d}(U) = 8$.

Note that $I = U$ and, then, $A = [I/P)$ has the same cardinality as B. In other words, the filter banks are critically sampled. The Analysis consists of $N = 8$ $\mathbb{Z}^2 \to \mathbb{Z}_2^1(2,2)$ decimators and the Synthesis of $N = 8$ $\mathbb{Z}_2^1(2,2) \to \mathbb{Z}^2$ interpolators.

14.11. $\star\star$ [Sect. 14.6] Formulate Proposition 14.2 in the case $I = \mathbb{Z}(2,2)$, $U = \mathbb{Z}^2$ and $P = \mathbb{Z}_2^1(2,2)$. Which is the main difference with respect to the previous problem?

We have to find the cells $B = [U/P)$ and $A = [I/P)$. In the previous solution we have found that the cell $B = [\mathbb{Z}^2/\mathbb{Z}_2^1(2,2))$ consists of $N = 8$ points. Now, using the same procedure, we find that the cell A is given as the intersection of the fundamental parallelepiped C_0 with the input domain $I = \mathbb{Z}(2,2)$. We find

$$A = \mathbb{Z}(2,2) \cap C_0 = \{(0,0),(0,2)\}$$

and the cardinality is $M = \mathrm{d}(P)/\mathrm{d}(I) = 8/4 = 2$.

The Analysis consists of $N = 8$ decimators and the Synthesis of $N = 8$ interpolators; the down–sampling and up–sampling ratios are given by $M = 2 < N = 8$. Then, the subband coding is *oversampled*.

14.12. $\star\star$ [Sect. 14.7] The direct connection of the transmultiplexer architecture consists of an N–input one–output $P \to U$ interpolator (transmitter side) and one–input N–output $U \to P$ decimator (receiver side). The $N \times N$ global impulse response is the matrix

$$||\tilde{q}_b * \tilde{g}_c(t)||, \qquad t \in P \quad b,c \in B\,.$$

Note that this is a convolution between the high rate signals $\tilde{q}_b(t_0)$ and $\tilde{g}_c(t_0), t_0 \in U$, subsequently evaluated at the low rate argument $t \in P$ (after the evaluation on $t_0 \in U$, there is a $U \to P$ downsampling). Evaluate the global frequency response (note that the connection transmitter–receiver is equivalent to a filter on P).

The *direct connection* of the transmultiplexer architecture (without the S/P and the

P/S) consists of an M–input one–output $P \to U$ interpolator followed by a one–input M–output $U \to P$ decimator. The global kernel is given by

$$h(t',t) = \int_U \mathrm{d}u_0 \, \tilde{\mathbf{q}}(t-u_0)\, \tilde{\mathbf{g}}(u_0 - t') \tag{S14.11}$$

where $\tilde{\mathbf{q}}$ is $M \times 1$ and $\tilde{\mathbf{g}}$ is $1 \times M$.

Since the domain ordering is $P \supset U$ and $U \subset P$ we can apply the recomposition.

14.13. $\star$[Sect. 14.9] Consider a subband decomposition with $I = \mathbb{Z}$, $U = \mathbb{Z}$ and $P = \mathbb{Z}(4)$. Explicitly write the distortion–free and the alias–free conditions.

The frequency domains are

$$\widehat{I} = \mathbb{R}/\mathbb{Z}, \qquad \widehat{U} = \mathbb{R}/\mathbb{Z}, \qquad \widehat{P} = \mathbb{R}/\mathbb{Z}(1/4)$$

and the cells are

$$\begin{aligned} A &= [I/P) = [\mathbb{Z}/\mathbb{Z}(4)) = \{0,1,2,3\}\,, \\ B &= [U/P) = A\,, \\ A^\star &= [P^\star/I^\star) = [\mathbb{Z}(1/4)/\mathbb{Z}) = \{0, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}\}\,. \end{aligned}$$

Hence $N = M = 4$ and there is a *critical sampling*.

The distortion–free and the alias–free conditions in the frequency domain are given by (14.108), namely

$$D(f) = \mathbf{G}(f)\,\mathbf{Q}(f) = 1\,, \qquad \mathbf{G}(f)\,\mathbf{Q}(f-\lambda)\,, \qquad \lambda \neq 0 \tag{S14.12}$$

where

$$\mathbf{G}(f) = [G_0(f), G_1(f), G_2(f), G_3(f)]\,, \qquad \mathbf{Q}(f) = \begin{bmatrix} Q_0(f) \\ Q_1(f) \\ Q_2(f) \\ Q_3(f) \end{bmatrix}\,.$$

Hence

$$\begin{aligned} D(f) &= \sum_{i=0}^{3} G_i(f)\,Q_i(f) \\ \mathbf{G}(f)\,\mathbf{Q}(f-\lambda) &= \sum_{i=0}^{3} G_i(f)\,Q_i(f-\lambda) \end{aligned}$$

where $\lambda \in A^\star$, $\lambda \neq 0$ and, therefore,

$$\lambda = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\,.$$

14.14. $\star\star$ [Sect. 14.9] Consider a subband decomposition with $I = \mathbb{Z}$, $U = \mathbb{Z}(2/3)$ and $P = \mathbb{Z}(4)$. Explicitly write the distortion–free and the alias–free conditions.

The frequency domains are

$$\widehat{I} = \mathbb{R}/\mathbb{Z}, \qquad \widehat{U} = \mathbb{R}/\mathbb{Z}(3/2), \qquad \widehat{P} = \mathbb{R}/\mathbb{Z}(1/4)$$

and the cells are

$$\begin{aligned} A &= [I/P) = [\mathbb{Z}/\mathbb{Z}(4)) = [\mathbb{Z}/\mathbb{Z}(4)) = \{0, 1, 2, 3\}, \\ B &= [U/P) = [\mathbb{Z}(2/3)/\mathbb{Z}(4)) = \{0, \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{6}{3}, \tfrac{8}{5}, \tfrac{10}{3}\}, \\ A^\star &= [P^\star/I^\star) = [\mathbb{Z}(1/4)/\mathbb{Z}) = \{0, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}\}. \end{aligned}$$

Hence, $M = |A| = 4$, $N = |B| = 6$ and there is an *oversampling*.

The distortion–free and alias–free conditions are the same as in the previous problem with the difference that now $G_i(f)$ have period 1 and $Q_i(f)$ have period $3/2$.

14.15. $\star\star$ [Sect. 14.9] Consider a 2D subband decomposition with (see Problem 14.10) $I = \mathbb{Z}^2$, $U = \mathbb{Z}^2$ and $P = \mathbb{Z}_2^1(2,2)$. Explicitly write the distortion–free and the alias–free conditions.

The frequency domains are

$$I = \mathbb{R}^2/\mathbb{Z}^2, \qquad U = \mathbb{R}^2/\mathbb{Z}^2, \qquad \widehat{P} = \mathbb{R}^2/\mathbb{Z}_2^1(F,F),$$

where $F = 14$. In fact, in Sect. 5.3 we have seen that the reciprocal of the quincunx lattice $\mathbb{Z}_2^1(d_1, d_2)$ is the quincunx lattice $\mathbb{Z}_2^1(F_1, F_2)$ with $F_1 = 1/(2d_2)$.

We have to evaluate the cells $A = B = [\mathbb{Z}^2/\mathbb{Z}_2^1(2,2))$ and $A^\star = [P^\star/I^\star) = [\mathbb{Z}_2^1(1/4, 1/4)/\mathbb{Z}^2)$. The cell $A = B$ has been evaluated in the solution of Problem 14.10 and is given by

$$A = B = \{(0,0), (0,1), (0,2), (0,3), (1,1), (2,1), (3,1), (3,2)\}. \tag{S14.13}$$

Considering that $|A| = |B| = 8$, we have a *critical sampling*. For the evaluation of the cell $A^\star$ we use the technique of Proposition 3.6. Then, we evaluate a cell of $[\mathbb{R}^2/\mathbb{Z}^2)$, which may be the square $C_0 = [0,1) \times [0,1)$ and, therefore, the intersection is

$$A^\star = C_0 \cap \mathbb{Z}_2^1(1/4, 1/4).$$

We find (see Fig.S14.5)

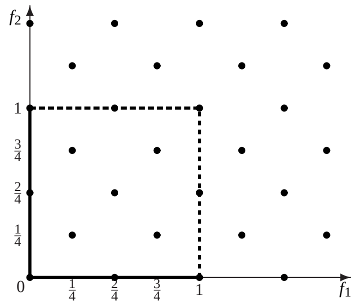

Fig. S14.5 Evaluation of the cell $A^\star$ as the intersection of the fundamental parallelepiped $C_0 = [0,1) \times [0,1)$ of the lattice $\mathbb{Z}^2$ and the lattice $\mathbb{Z}_2^1(\frac{1}{4}, \frac{1}{4})$

$$A^\star = \{(0,0), (\tfrac{2}{4}, 0), (\tfrac{1}{4}, \tfrac{1}{4}), (\tfrac{3}{4}, \tfrac{1}{4}), (0, \tfrac{2}{4}), (\tfrac{2}{4}, \tfrac{2}{4}), (\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{3}{4}, \tfrac{3}{4})\}$$

which consists of $M = N = 8$ 2D frequencies.

The distortion–free and alias–free conditions are given by the general formulas (14.108), that is,

$$D(f) = \mathbf{G}(f)\,\mathbf{Q}(f) = 1\,, \qquad \mathbf{G}(f)\,\mathbf{Q}(f-\lambda)\,, \qquad \lambda \neq 0$$

where $\mathbf{G}(f)$ is a row vector (1×8) and $\mathbf{Q}(f)$ is a column vector (8×1). The indexes of the entries are given by the cell A which, in (S14.13), is written with the lexicographical order. Then, for instance, we have

$$\mathbf{G} = [G_{0,0}, G_{0,1}, G_{0,2}, G_{0,3}, G_{1,1}, G_{2,1}, G_{3,1}, G_{3,2}]\,.$$

The distortion–free condition can be written as

$$D(f) = \sum_{(i,j) \in A} G_{i,j}(f)\,Q_{i,j}(f) = 1\,.$$

Analogously, we have the alias–free condition

$$\sum_{(i,j) \in A} G_{i,j}(f)\,Q_{i,j}(f-\lambda)$$

where $\lambda \in A^\star$ with $\lambda = (0,0)$ omitted.

14.16. $\star\star$ [Sect. 14.9] Write the IRC condition for the subband decomposition architecture obtained by imposing that the cascade of the $P \to I$ interpolator followed by the $I \to P$ decimator be equivalent to the N–input N–output identity on P (see Tab. 14.1).

The *inverse* subband architecture consists of the cascade of (Fig.S14.6)

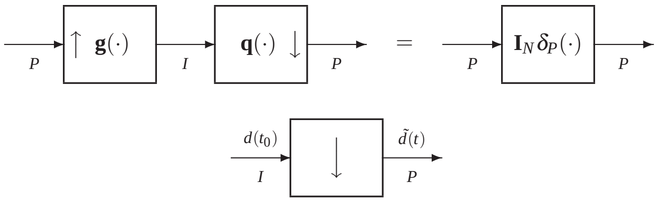

Fig. S14.6 Illustration of the IRC condition in subband architecture

1) an N–input one–output interpolator on $P \to I$ with impulse response (see (14.89))

$$\mathbf{g}(t_0) = [g_0(t_0), \ldots, g_{N-1}(t_0)] \,,$$

2) a one–input N–output decimator on $I \to P$ with impulse response

$$\mathbf{q}(t_0) = [q_0(t_0), \ldots, q_{N-1}(t_0)]' \,.$$

From the decomposition/recomposition of QIL tfs (see Sect. 6.11, Theorem 6.6) cascade of these two blocks is equivalent to a filter on P with impulse response given by the $I \to P$ *down–sampled version of the convolution*

$$\mathbf{d}(t_0) = \mathbf{q} * \mathbf{g}(t_0) = \int_P \mathrm{d}p\, \mathbf{q}(t_0 - p)\, \mathbf{g}(p) \,, \qquad t_0 \in I \tag{S14.14}$$

where $\mathbf{d}(t_0)$ is an $N \times N$ matrix.

The IRC condition is that the down–sampled version $\tilde{\mathbf{d}}(t)$, $t \in P$, of this matrix is the N–input N–output identity on P, that is,

$$\tilde{\mathbf{d}}(t) = \mathbf{q} * \mathbf{g}(t) = \mathbf{I}_N\, \delta_P(t) \,, \qquad t \in P \,. \tag{S14.15}$$

14.17. ⋆⋆ [Sect. 14.9] Write the IRC condition of the previous problem in the frequency domain (see Tab. 14.1).

In the frequency domain the overall response given by (S14.14) becomes

$$\mathbf{D}(f) = \mathbf{Q}(f)\, \mathbf{G}(f) \,, \qquad f \in \widehat{I} = \mathbb{R}^m / I^\star \,.$$

The $I \to P$ down–sampling of $\mathbf{d}(t_0)$ becomes the $\widehat{I} \to \widehat{P}$ up–periodization, with relation

$$\tilde{\mathbf{D}}(f) = \sum_{\lambda \in [P^\star/I^\star)} \mathbf{D}(f - \lambda) \,, \qquad f \in \widehat{P} = \mathbb{R}^m / P^\star \,.$$

The FT of the impulse $\delta_P(t)$ is 1. Hence, the IRC condition in the frequency domain is $\tilde{\mathbf{D}}(f) = 1$ and, more explicitly,

$$\sum_{\lambda \in [P^\star/I^\star)} \mathbf{Q}(f-\lambda)\mathbf{G}(f-\lambda) = \mathbf{I}_N \ .$$

14.18. $\star\star$ [Sect. 14.9] Consider a subband decomposition with $I = \mathbb{Z}$, $U = \mathbb{Z}(2/3)$ and $P = \mathbb{Z}(4)$. Explicitly write the polyphase matrix $\mathbf{g}_\pi(t)$.

Given the high–rate impulse responses $\mathbf{g}_i(t_0)$, $t_0 \in I$, the polyphase matrix is the $M \times N$ matrix defined by (14.127). In order to write this matrix, we have to evaluate the integers M, N and the shifts τ_i. These are obtained from the cells

$$A = [I/P) = \{0,1,2,3\}, \qquad B = [U/P) = \{0, \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{6}{3}, \tfrac{8}{3}, \tfrac{10}{3}\} \ .$$

Then, $M = |A| = 4$, $N = 6$ and $\{\tau_0, \tau_1, \tau_2, \tau_3\} = \{0,1,2,3\}$. Hence

$$\mathbf{g}_\pi(t) = \begin{bmatrix} g_0(t) & g_1(t) & g_2(t) & g_3(t) & g_4(t) & g_5(t) \\ g_0(t+1) & g_1(t+1) & g_2(t+1) & g_3(t+1) & g_4(t+1) & g_5(t+1) \\ g_0(t+2) & g_1(t+2) & g_2(t+2) & g_3(t+2) & g_4(t+2) & g_5(t+2) \\ g_0(t+3) & g_1(t+3) & g_2(t+3) & g_3(t+3) & g_4(t+3) & g_5(t+3) \end{bmatrix}$$

where $t \in \mathbb{Z}(4)$.

Note that the jth row of the polyphase matrix is obtained from the S/P conversion on $\mathbb{Z} \to \mathbb{Z}(4)$ of the jth impulse response $g_j(t_0)$, $t_0 \in \mathbb{Z}$.

14.19. $\star\star$ [Sect. 14.9] Consider the subband decomposition of the previous problem. Explicitly write the polyphase matrix $\mathbf{G}_\pi(f)$ in the frequency domain in terms of the original frequency responses.

We have remarked that the jth column of the polyphase matrix is obtained from the S/P conversion of the impulse response $g_j(t_0)$, $t_0 \in I = \mathbb{Z}$. Then, we have to apply the theory of S/P conversion in the frequency domain developed in Sect. 7.5 and, in particular, relation (7.31).

Now, in this conversion $g_j^{(k)}(t) = g_j(t+k)$, $k = 0,1,2,3$, are the polyphase components. This writing implies two operations

1) a shift of $-k$ on $g_j(t_0)$, giving $\tilde{g}_j^{(k)}(t_0) = g_j(t_0+k)$,
2) a $\mathbb{Z} \to \mathbb{Z}(4)$ down–sampling.

In the frequency domain 1) gives

$$\tilde{G}_j^{(k)}(f) = G_j(f)\,\mathrm{e}^{\mathrm{i}2\pi f k}$$

and 2) gives

$$G_j^{(k)} = \sum_{\lambda \in [P^\star/I^\star)} \tilde{G}_j^{(k)}(f-\lambda)$$

where $P^\star = \mathbb{Z}(1/4)$ and $I^\star = \mathbb{Z}$. Hence, $\lambda \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$ so that

$$G_j^{(k)}(f) = \sum_{m=0}^{3} \tilde{G}_j^{(k)}(f-m/4)\ .$$

In conclusion

$$G_j^{(k)}(f) = \sum_{m=0}^{3} G_j^{(k)}(f-m/4)\,\mathrm{e}^{\mathrm{i}2\pi(f-m/4)k}\ .$$

This relation, with $k = 0,1,2,3$, gives the four entries of the jth column of the 4×6 matrix $\mathbf{G}_\pi(f)$.

14.20. ⋆⋆[Sect. 14.10] Consider a 2D subband decomposition with (see Problem 14.10) $I = \mathbb{Z}^2$, $U = \mathbb{Z}^2$ and $P = \mathbb{Z}_2^1(2,2)$. Explicitly write the the polyphase matrix $\mathbf{g}_\pi(t)$. *Hint:* use the lexicographical order (see Sect. 7.6).

In order to solve this problem, we have to find the cells $A = [I/P)$ and $B = [U/P)$, which give the cardinalities $M = |A|$ and $N = |B|$ and the indexes. In (14.126) and in (14.127), the entries of the polyphase matrix $\mathbf{g}_\pi(t)$ are written as

$$g_m^{(j)}(t) = g_m(t+\tau_j) \tag{S14.16}$$

where m is the column index, given by the cell B, and j is the row index with τ_j given by the cell A.

In this specific case, the cells are equal and given by (see (S14.10))

$$\begin{array}{ccccccccc} A=B= & \{(0\,,0), & (0\,,1), & (0\,,2), & (0\,,3), & (1\,,1), & (2\,,1), & (3\,,1), & (3\,,2)\} \\ n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

where in the second line we have indicated a natural n which is in correspondence with the elements of the cell. This natural number can be used for both m and j in notation (S14.16). In such a way, we can handle $\mathbf{g}_\pi(t)$ as an ordinary 8×8. For instance, we have that

$$g_3^{(5)}(t) = g_3(t+\tau_5) = g_3(t+(2,1))\,, \qquad t \in \mathbb{Z}_2^1(2,2)$$

represents the 3rd elements of the 5th row of the matrix.

14.21. ⋆[Sect. 14.12] Prove that (14.147) $D(f)$ verifies the condition $D(f-F) = -D(f)$, which implies that in the impulse response $d(nT_0)$ is zero for n even.

Given that $2D(f) = Q_0^2(f) - Q_0^2(f-F)$, the function $Q_0(f)$ has period $F_0 = 2F$. Hence

$$\begin{aligned} 2D(f-F) &= Q_0^2(f-F) - Q_0^2(f-2F) \\ &= Q_0^2(f-F) - Q_0^2(f) = 2D(f) \ . \end{aligned}$$

In the time domain we have

$$(-1)^n \, \mathrm{d}(nT_0) = -\mathrm{d}(nT_0) \ .$$

Then, for n even $\mathrm{d}(nT_0) = -\mathrm{d}(nT_0) \to \mathrm{d}(nT_0) = 0$.

14.22. ⋆[Sect. 14.12] Prove relation (14.149) on the Synthesis filters of Choice II.

It is convenient the introduction of auxiliary signals and the application of simple FT rules. We let

$$\begin{aligned} a(nT_0) &= g_0(-nT_0) \\ b(nT_0) &= a((n-1)T_0) = a(nT_0 - T_0) \\ c(nT_0) &= b^*(nT_0) \\ d(nT_0) &= (-1)^n \, c(nT_0) = \mathrm{e}^{\mathrm{i}2\pi F nT_0} \, c(nT_0) = g_1(nT_0) \ . \end{aligned}$$

Then, the corresponding FTs are

$$\begin{aligned} A(f) &= G_0(-f) \\ B(f) &= A(f) \, \mathrm{e}^{-\mathrm{i}2\pi f T_0} = G_0(-f) \, \mathrm{e}^{\mathrm{i}2\pi f T_0} \\ C(f) &= B^*(-f) = G_0^*(f) \, \mathrm{e}^{-\mathrm{i}2\pi f T_0} \\ D(f) &= C(f-F) = G_0^*(f-F) \, \mathrm{e}^{-\mathrm{i}2\pi (f-F) T_0} \\ &= G_0^*(f-F) \, \mathrm{e}^{-\mathrm{i}2\pi (f-F) T_0} (-1) = G_1(f) \ . \end{aligned}$$

14.23. ⋆⋆[Sect. 14.16] Prove that with the impulse response given by (14.161), the kernel $\theta(u_0, t_0)$ of the corresponding PI transformation is specified by

$$h(b,a) = K_{ba},$$

where K_{ba} are the entries of $\mathbf{K}$. Prove also that the extension of h is given by

$$\theta(h) = \{B \times A + (t,t) | t \in P\}.$$

It is convenient to express the relation between the kernel $\theta(u_0, t_0)$ and the polyphase matrix $\mathbf{q}_\pi(t)$ in the general case. Recall that the polyphase matrix is the result of two polyphase decompositions applied to $\theta(u_0, t_0)$ with $u_0 = u + b$, $b \in B$, $u \in P$ and $t_0 = t + a$, $a \in A$, $t \in P$.

We can start from the first of (14.67), namely

$$q_b(t_0) = \theta_b(-t_0) = \theta(b, -t_0) ,$$

and note that $q_b(t_0)$ identifies $\theta(u_0, t_0)$ by means of PI. In fact, we have

$$\theta(b + p, t_0) = \theta(b, t_0 - p) = q_b(-(t_0 - p)) .$$

Next, we decompose t_0 as $t_0 = a + t$, in order to get

$$\theta(b + p, t + a) = q_b(-(t + a - p)) \stackrel{\Delta}{=} q_{ba}(-(t - p))$$

where $q_{ba}(t)$ is the (b, a) entry of the matrix $\mathbf{q}_\pi(t)$. In conclusion, the general relation is

$$\boxed{\theta(b + p, t + a) = q_{ba}(-(t - p)) .}$$

In the specific case of (14.161) we get

$$\theta(b + p, t + a) = M K_{ba}\, \delta_P(-(t - p))$$

which clearly shows that the kernel is identified by the matrix $\mathbf{K} = [K_{ba}]$.

The extension of the kernel is obtained by noting that $\delta_P(-(t - p)) \neq 0$ only for $t = p \in P$. Hence

$$\begin{aligned} e(\theta) &= \{(b + t, a + t) \mid a \in A, b \in B, t \in P\} \\ &= \{(b, a) + (t, t) \mid a \in A, b \in B, t \in P\} \\ &= \{B \times A + (t, t) \mid t \in P\} . \end{aligned}$$

Problems of Chapter 15

15.1. $\star$[Sect. 15.8] Prove the mirror symmetry in the frequency domain, as stated by (15.44).

The proof is not immediate. It is convenient to introduce some auxiliary signals, namely

$$\begin{aligned} a(n) &= g^*(-n)\ , \\ b(n) &= a(n-1) = g^*(-(n-1))\ , \\ c(n) &= (-1)^n b(n) = \mathrm{e}^{\mathrm{i}2\pi n/a}\, b(n) = g_1(n)\ . \end{aligned}$$

Then the application of elementary FT rules gives

$$\begin{aligned} A(f) &= G^*(f)\ , \\ B(f) &= \mathrm{e}^{-\mathrm{i}2\pi f} A(f) = \mathrm{e}^{-\mathrm{i}2\pi f} G^*(f)\ , \\ C(f) &= B(f+\tfrac{1}{2}) = \mathrm{e}^{-\mathrm{i}2\pi(f+\frac{1}{2})} G^*(f+\tfrac{1}{2})\ , \end{aligned}$$

where $\mathrm{e}^{\mathrm{i}2\pi f/2} = -1$.

15.2. $\star$[Sect. 15.9] Using the orthogonality $\mathbf{G}_0 \perp \mathbf{G}_1$ in Proposition 15.4, prove the orthogonality $\boldsymbol{\Phi}_0 \perp \boldsymbol{\Psi}_0$ in Proposition 15.5. In other words, prove that $\psi(t-k)$ and $\varphi(t-k')$, $k,k' \in \mathbb{Z}$, are orthogonal using (15.53).

In the inner product

$$\alpha_{kk'} \stackrel{\Delta}{=} \langle \psi(\cdot - k), \varphi^*(\cdot - k') \rangle = \int_{\mathbb{R}} \mathrm{d}t\, \psi(t-k)\, \varphi^*(t-k')$$

we use (15.46) and (15.47). Hence

$$\begin{aligned} \alpha_{kk'} &= \sum_{n,n'} g_1(n) g_0^*(n') \int_{\mathbb{R}} \mathrm{d}t\, \varphi_{n/2}^{(-1)}(t)\, \varphi_{n'/2}^{(-1)*}(t) \\ &= \sum_{n,n'} g_1(n) g_0^*(n') \int_{\mathbb{R}} \mathrm{d}t\, \varphi_{n/2}^{(-1)}(t-k)\, \varphi_{n'/2}^{(-1)*}(t-k) \end{aligned}$$

where $\varphi_{n/2}^{(-1)}(t) = \varphi_{n'/2}^{(-1)}(t - n/2 - k)$. Then, by substituting $m = n + 2k$ and $m' = n' + 2k$, we obtain

$$\begin{aligned}\alpha_{kk'} &= \sum_{m,m'} g_1(m-2k)g_0^*(m'-2k') \int_{\mathbb{R}} \mathrm{d}t\, \varphi^{(-1)}(t-m/2)\, \varphi^{(-1)*}(t-m'/2) \\ &= \sum_{m,m'} g_1(m-2k)g_0^*(m'-2k')\, \delta_{m.m'}\ , \\ &= \sum_{m,m'} g_1(m-2k)g_0^*(m'-2k') = 0 \qquad m' \neq m\ .\end{aligned}$$

15.3. ⋆⋆ [Sect. 15.9] Prove the recurrence

$$p_m(t,t') + r_m(t,t') = p_{m-1}(t,t')$$

of the projector kernels. *Hint*: prove the equivalent relation $\mathcal{P}_m + \mathcal{R}_m = \mathcal{P}_{m-1}$ and use the property $s = \mathcal{P}_{m-1}[s]$ for every $s \in V_{m-1}$.

For simplicity, we consider the case $m = 0$. If $s_{-1} \in V_{-1}$, from the idempotency of $\mathcal{P}_{-1}$ we have $s_{-1} = \mathcal{P}_{-1}[s_{-1}]$ ($\mathcal{P}_{-1}$ is the identity on V_{-1}). Considering that s_{-1} has the unique decomposition

$$\mathcal{P}_0[s_{-1}] + \mathcal{R}_0[s_{-1}] = s_{-1}\ ,$$

we can write

$$\mathcal{P}_0[s_{-1}] + \mathcal{R}_0[s_{-1}] = \mathcal{P}_{-1}[s_{-1}]$$

for every $s_{-1} \in V_{-1}$. This states the relation $\mathcal{P}_0 + \mathcal{R}_0 = \mathcal{P}_{-1}$.

15.4. ⋆ [Sect. 15.10] Prove the relations of Mallat filters in the frequency domain

$$G_0^{(m)}(f) = G_0^*(2^m f)\,, \qquad G_1^{(m)}(f) = G_1^*(2^m f)$$

that is, prove (15.61) starting from (15.60).

It is convenient to let $u = n2^m$ in the impulse responses (15.60), that is,

$$g^{(m)}(u) = 2^{-m} g^*(-u2^{-m})\,, \qquad u \in \mathbb{Z}_m\ .$$

Then, we apply the rule on scale change

$$s(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}\, S\left(\frac{f}{a}\right)$$

with $s(t) = 2^{-m} g(t)$ and $a = -2^{-m}$, $g^{(m)}(u) = s^*(at)$. We find

$$G^{(m)}(f) = G^*(2^m f)\ .$$

15.5. ⋆[Sect. 15.10] Evaluate Mallat filters in the case of the discrete sinc basis defined in Example 14.5.

We first deduce the discrete sinc basis (14.46) from the frequency reponses of ideal filters on $\mathbb{Z}(T)$.

We start from continuous–time ideal low–pass filter with response (Fig.S15.1)

$$q(t) = 2BA_0 \operatorname{sinc}(2Bt) \xrightarrow{\mathcal{F}} Q(f) = A_0 \operatorname{rect}\left(\frac{f}{2B}\right), \qquad t, f \in \mathbb{R} .$$

Then, we want to construct N frequency responses obtained by cosine modulations with frequencies f_k. Hence, we get the response

$$g_k(t) = q(t) \cos(2\pi f_k t) = 2BA_0 \operatorname{sinc}(2Bt) \cos(2\pi f_k)$$
$$G_k(f) = \tfrac{1}{2}Q(f - f_k) + \tfrac{1}{2}Q(f + f_k) = \tfrac{1}{2}A_0 \left[\operatorname{rect}\left(\frac{f - f_k}{2B}\right) + \operatorname{rect}\left(\frac{f + f_k}{2B}\right)\right] .$$

The conditions are

1) the bands of filters $G_k(f)$ are disjoint,
2) the frequencies f_k are equally spaced,
3) the N filters provide a partition of the intervals $(\frac{1}{2}, -\frac{1}{2})$.

Considering that the extension of $G_k(f)$ is

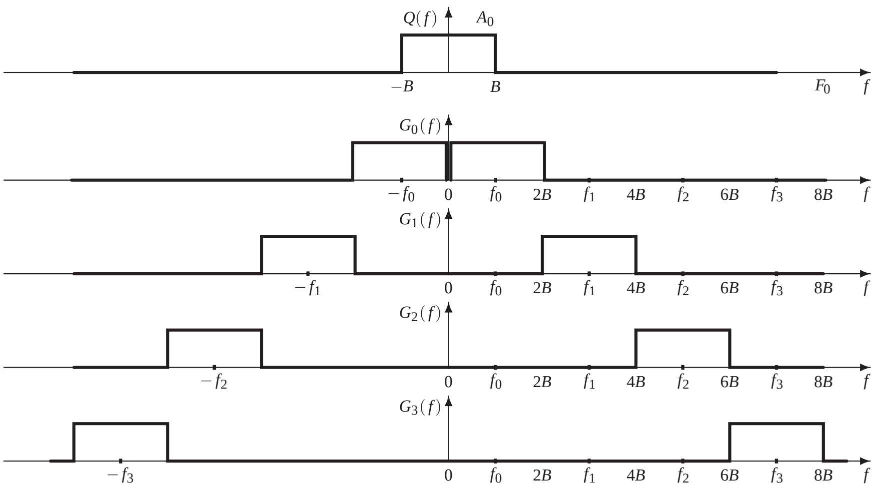

Fig. S15.1 For the deduction of discrete sinc filters with $N = 4$

$$(-f_k - B, -f_k + B) \cup (f_k - B, f_k + B)$$

the solution is (see Fig.S15.1 with $N = 4$)

$$B = \frac{1}{4N} \qquad f_k = B + 2Bk = \frac{1}{4N} + k\frac{1}{2N}, \qquad k = 0, 1, \ldots, N-1 .$$

We also want that the $G_k(f)$ are normalized, that is,

$$\int_{\mathbb{R}} \mathrm{d}f |G_k(f)|^2 = \tfrac{1}{4}A_0^2 4B = 1 \longrightarrow A_0 = 2\sqrt{N}$$

and the amplitudes of the $g_k(t)$ becomes

$$2BA_0 = \frac{1}{\sqrt{N}} .$$

Finally, the impulse response of the discrete filters are obtained from $g_k(t)$, $t \in \mathbb{R}$, by an $\mathbb{R} \to \mathbb{Z}$ down–sampling, that is,

$$g_k(n) = \frac{1}{\sqrt{N}} \operatorname{sinc}\left(\frac{n}{2N}\right) \cos(2\pi f_k n) ,$$

in agreement with (14.46).

Finally, we apply the above result to get the Mallat filters. For $N = 2$ the impulse responses are

$$g_k(n) = \frac{1}{\sqrt{2}} \operatorname{sinc}(n/4) \cos(2\pi f_k n), \qquad k = 0, 2$$

with $f_0 = 1/8$, $f_1 = 3/8$, and the frequency responses are depicted in Fig.S15.2

$$\begin{aligned} G_0(f) &= \sqrt{2}\, \mathrm{rep}_1 [\mathrm{rect}(4(f - f_0)) + \mathrm{rect}(4(f + f_0))] \\ &= \sqrt{2}\, \mathrm{rep}_1\, \mathrm{rect}(f) \\ G_1(f) &= \sqrt{2}\, \mathrm{rep}_1 [\mathrm{rect}(4(f - f_1)) + \mathrm{rect}(4(f - f_1))] . \end{aligned}$$

Then, we get the Mallat filters as

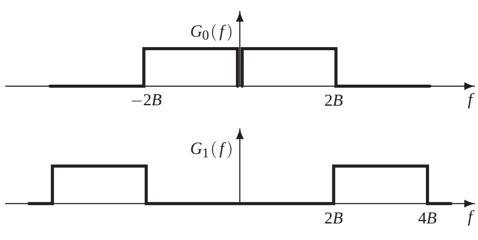

Fig. S15.2 The frequency responses of discrete sinc filters with $N = 2$

$$G_0^{(m)}(f) = G_0(2^m f)\,, \qquad G_1^{(m)}(f) = G_1(2^m f)\,.$$

15.6. $\star$ [Sect. 15.10] Prove Proposition 15.14 of Appendix A. *Hint*: write the global kernel of g_1 followed by g_2 and realize that it is a sampled version of a convolution.

The kernels of the interpolator $g_1(t)$ and of the decimator $g_2(t)$ are respectively

$$h_1(t_2,t_1) = g_1(t_2 - t_1)\,, \qquad h_2(t_3,t_2) = g_2(t_3 - t_2)$$

with $t_1 \in \mathbb{Z}_m$, $t_2 \in \mathbb{R}$, $t_3 \in \mathbb{Z}_{m+1}$. Hence, the application of the kernel–composition rule gives

$$h_{12}(t_3,t_1) = \int_{\mathbb{R}} \mathrm{d}t_2\, g_2(t_3 - t_2)\, g_1(t_2 - t_1)\,.$$

With the change $t_3 - t_2 = u$ one gets

$$h_{12}(t_3,t_1) = \int_{\mathbb{R}} \mathrm{d}u\, g_2(u)\, g_1(t_3 - t_1 - u)\,.$$

and we see that the global tf is LQI with impulse response

$$g(t) = \int_{\mathbb{R}} \mathrm{d}u\, g_2(u)\, g_1(t-u)\,, \qquad t \in \mathbb{Z}_m\,.$$

On the right–hand side we have the convolution of $g_2(t)$ and $g_1(t)$ on $\mathbb{R}$, but on the left–hand side the time t is restricted to $\mathbb{Z}_m$. This means that the convolution on $\mathbb{R}$ is $\mathbb{R} \to \mathbb{Z}_m$ down–sampled, as shown in Fig.15.38.

15.7. $\star\star\star$ [Sect. 15.11] Evaluate the mother wavelet $\psi(t)$ in the case of Example 15.2 with roll-off $\alpha = 1$.

With a roll–off $\alpha = 1$, we can take advantage of significant simplifications. The band is $B = 1$, the expression of $\Phi(f)$ is simple (Fig.15.3), i.e.,

$$\Phi(f) = \begin{cases} \cos\left(\frac{\pi}{2} f\right) & |f| < 1 \\ 0 & |f| < 1 \end{cases}$$

and the scaling function becomes

$$\begin{aligned}\varphi(t) &= \int_{-1}^{1} \cos\left(\frac{\pi}{2} f\right) \mathrm{e}^{\mathrm{i}2\pi f t}\, \mathrm{d}f = \int_{-1}^{1} \cos\left(\frac{\pi}{2} f\right) \cos(2\pi f t)\, \mathrm{d}f \\ &= \frac{4\cos(2\pi t)}{\pi(1-16t^2)}\,.\end{aligned}$$

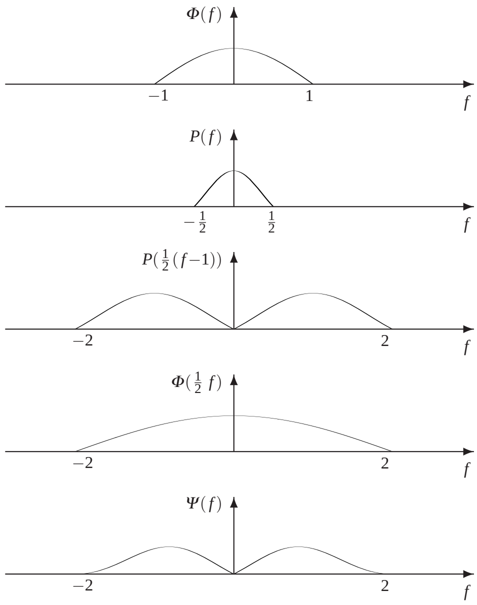

Fig. 15.3 Evaluation of the Fourier transform $\Psi(f)$ of the mother wavelet starting from the Fourier transform $\Phi(f)$ of the scaling function

The function $P(\frac{1}{2}f)$ is

$$P(\tfrac{1}{2}f) = \Phi(\tfrac{1}{2}f)\,\Phi(f)\ \cos\left(\tfrac{\pi}{4}f\right)\,\cos\left(\tfrac{\pi}{2}f\right)$$

and has extension. Furthermore, accoding to (15.70), we have

$$\psi(t) = -2\int_0^2 \Phi(\tfrac{1}{2}f)P(\tfrac{1}{2}(f-1))\cos\left[2\pi f(t-\tfrac{1}{2})\right]\mathrm{d}f\ .$$

The integration gives explicitly

$$\psi(t) = \frac{-64t^3+96t^2-38t+(6t-3)\cos(4\pi t)+\left[-16t^2+16t-3\right]\sin(4\pi t)+3}{16\sqrt{2}\pi t(4t-1)(2t-1)(t-1)(4t-3)}\ .$$

The scaling function and the mother wavelet are illustrated in Fig.15.4.

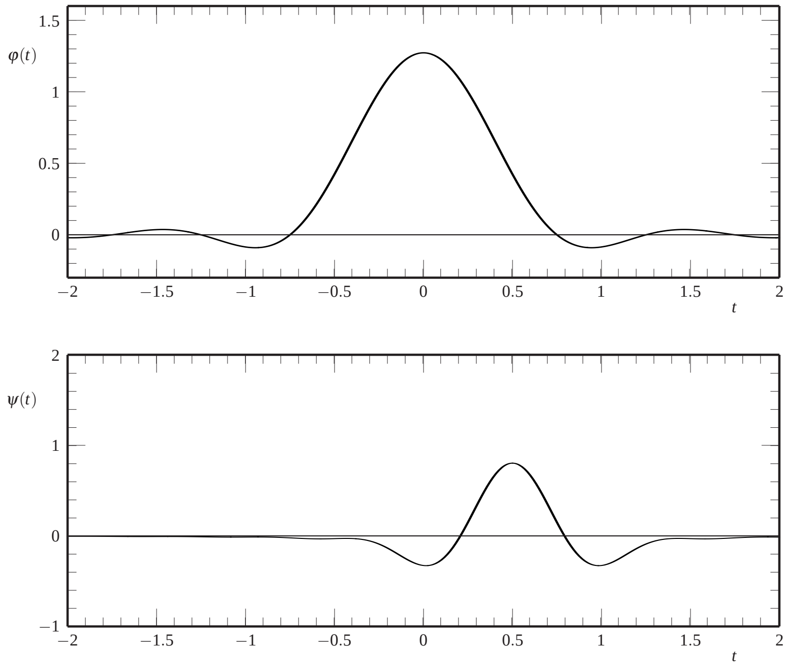

Fig. 15.4 The scaling function $\varphi(t)$ and the mother wavelet $\psi(t)$ of the problem

15.8. $\star\star$ [Sect. 15.14] Find the set of *admissible shifts* $D_m = \mathbf{D}^m \mathbb{Z}^N$, where the dilation matrix $\mathbf{D}$ is the first matrix in (15.90), that is,

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

The lattice $D = \mathbf{D}_1\mathbb{Z}^2$ is the quincunx lattice $\mathbb{Z}_2^1(1,1)$. In fact, with elementary operators on $\mathbf{D}_1$ we find

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \stackrel{\Delta}{=} \mathbf{D}$$

where the latter is the canonical basis of $\mathbb{Z}_2^1(1,1)$ (see (3.22)). The powers of $\mathbf{D}_1$ are

$$\mathbf{D}_1^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\mathbf{I}_2 , \qquad \mathbf{D}_1^3 = \mathbf{D}_1^2\mathbf{D}_1 = 2\mathbf{D}_1$$

and in general

$$\mathbf{D}_1^{2k} = 2^k\mathbf{I}_2 , \qquad \mathbf{D}_1^{2k+1} = 2^k\mathbf{D}_1 .$$

Then, $D_{2k} = \mathbf{D}^{2k}\mathbb{Z}^2$ is the separable lattice $\mathbb{Z}(2^k,2^k)$ and $\mathbf{D}_{2k} = \mathbf{D}^{2k}\mathbb{Z}^2$ is the quincunx lattice $\mathbb{Z}_2^1(2^k,2^k)$.

15.9. $\star\star$ [Sect. 15.14] Consider the dilation matrix given by (15.91). Prove that, if $\varphi(t)$ and $\psi(t)$ are respectively a 1D scaling function and a mother wavelet, then the 2D scaling function is $\varphi(t_1)\,\varphi(t_2)$, and the 2D mother wavelets are given by $\varphi(t_1)\,\psi(t_2)$, $\psi(t_1)\,\varphi(t_2)$ and $\psi(t_1)\,\psi(t_2)$.

We refer to Proposition 15.13, where in the 1D case the basis $\boldsymbol{\Gamma}$ is given by (15.89)

$$\boldsymbol{\Gamma} = \boldsymbol{\Phi}_0 \cup \boldsymbol{\Psi}_0$$

where (see (15.26)) $\boldsymbol{\Phi}_0 = \{\varphi(t-n) \mid n \in \mathbb{Z}\}$ and (see (15.49)) $\boldsymbol{\Psi}_0 = \{\psi(t-n) \mid n \in \mathbb{Z}\}$. Considering the separability, the 2D basis is given by

$$\begin{aligned} \boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma} \times \boldsymbol{\Gamma} &= [\boldsymbol{\Phi}_0 \cap \boldsymbol{\Psi}_0] \times [\boldsymbol{\Phi}_0 \cap \boldsymbol{\psi}_0] \\ &= (\boldsymbol{\Phi}_0 \times \boldsymbol{\Phi}_0) \cap (\boldsymbol{\Phi}_0 \times \boldsymbol{\Psi}_0) \cap (\boldsymbol{\Psi}_0 \times \boldsymbol{\Phi}_0) \cap (\boldsymbol{\Psi}_0 \times \boldsymbol{\Psi}_0) . \end{aligned}$$

This gives the decomposition of the basis $\boldsymbol{\Gamma}$ in the orthogonal subfamilies (15.88). The family $\boldsymbol{\Phi}_0 \times \boldsymbol{\Phi}_0$ is the basis of V_0 with scaling function $\varphi(t_1)\,\varphi(t_2)$, the family $\boldsymbol{\Phi}_0 \times \boldsymbol{\Psi}_0$ is the basis of $W_0^{(1)}$ with mother wavelet $\varphi(t)\,\psi(t)$, etc..

Problems of Chapter 16

16.1. $\star$[Sect. 16.3] Explicitly write (16.30) and (16.31) for the grating G_2 of Example 16.4.

The grating G_2 has the following *reduced* representation (see (16.14))

$$\mathbf{G}_2 = \mathbf{G}_r = \begin{bmatrix} 1 & 0 & 0 \\ a & d_2 & 0 \\ b & 0 & d_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{E} & \mathbf{F} \end{bmatrix}, \qquad \mathbb{R}\times\mathbb{Z}^2$$

where

$$\mathbf{E} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} d_2 & 0 \\ 0 & d_3 \end{bmatrix}. \tag{S16.1}$$

The reciprocal $G_r^\star$ has representation (see the relation below (16.25))

$$\mathbf{G}_r^\star = \begin{bmatrix} 1 & -\mathbf{E}'\mathbf{F}^\star \\ \mathbf{O} & \mathbf{F}^\star \end{bmatrix} = \begin{bmatrix} 1 & -aF_2 & -bF_3 \\ 0 & F_2 0 & \\ 0 & 0 & F_3 \end{bmatrix}, \qquad \mathbb{O}\times\mathbb{Z}^2$$

where $F_2 = 1/d_2$ and $F_3 = 1/d_3$. Hence, $G_r^\star$ is a 2D lattice in $\mathbb{R}^3$ with coordinates

$$f_1 = -amF_2 - bnF_3, \qquad f_2 = mF_2,\; f_3 = nF_3, \qquad m,n \in \mathbb{Z}.$$

The dual of G_r is $\widehat{G}_r = \mathbb{R}^3/G_r^\star$.

The reference *separable* grating is $G_0 = \mathbb{R}\times\mathbb{Z}(d_2,d_3)$ with reciprocal $G_0^\star = \mathbb{O}\times F^\star = \mathbb{O}\times\mathbb{Z}(F_2,F_3)$.

Using (S16.1) in (16.25) we get

$$\mathbf{a}_E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \qquad \mathbf{a}_E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}, \qquad \mathbf{a}_E' = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{a}_E^\star = \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The explicit form of coordinate changes is

$$\begin{aligned} s_0(v_1,v_2,v_3) &= s(v_1, av_1+v_2, v_1+bv_3) \\ s(t_1,t_2,t_3) &= s_0(t_1, -at_1+t_2, -bt_1+v_3) \\ S(f_1,f_2,f_3) &= S_0(f_1+af_2+af_3, f_2, f_3) \\ S_0(\lambda_1,\lambda_2,\lambda_3) &= S(\lambda_1 - a\lambda_2 - a\lambda_2, \lambda_1, \lambda_3) \end{aligned}$$

where $(v_1,v_2,v_3)\in\mathbb{R}\times F = G_0$ with $F=\mathbb{Z}(d_2,d_3)$, $(t_1,t_2,t_3)\in G_2$, $(\lambda_1,\lambda_2,\lambda_3)\in\widehat{G}_0 = \mathbb{R}\times\mathbb{R}/\mathbb{Z}(1/d_2)\times\mathbb{R}/\mathbb{Z}(1/d_3)$, $(f_1,f_2,f_3)\in\widehat{G}_2$.

16.2. ⋆⋆ [Sect. 16.4] Write a reduced basis $\mathbf{G}_r$ of a grating G with signature $\mathbb{R}^2 \times \mathbb{Z}^2$. Then, find all subgroups of G with signature $\mathbb{R} \times \mathbb{Z}^3$.

With reference to the general reduced basis given by (16.14a), in this specific case we have

$$\mathbf{G}_r = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{E} & \mathbf{F} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e_{11} & e_{12} & f_{11} & f_{12} \\ e_{21} & e_{22} & f_{21} & f_{22} \end{bmatrix}$$

where e_{ij} and f_{ij} are real numbers. In particular, the f_{ij} can be chosen such that (see (16.19)) $\det \mathbf{F} = f_{11} f_{22} - f_{12} f_{22} > 0$.

To find all the subgroups of G with signature $\mathbb{R} \times \mathbb{Z}^3$, we apply Theorem 16.1 with

$$p = 2\,, \qquad q = 2\,, \qquad a = 1\,.$$

Then the bases of these subgroups are obtained from (16.38), that is,

$$\mathbf{J} = \mathbf{G}_r \mathbf{K}$$

where the first two rows of $\mathbf{K}$ are real and the last two rows are integer. Hence

$$\mathbf{K} = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{bmatrix}$$

with r_i, s_i real and m_i, n_i integers.

16.3. ⋆⋆ [Sect. 16.4] Find all 1D subgroups of the grating $\mathbb{R}\mathbb{Z}(2,1)$ (see (3.26)).

The grating $\mathbb{R}\mathbb{Z}(2,1)$ has representation

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \qquad H = \mathbb{R} \times \mathbb{Z}$$

and consists of the points of $\mathbb{R}^2$

$$(t_1, t_2) = \{r, 2r + n \mid r \in \mathbb{R}, n \in \mathbb{Z}\}\,.$$

The subgroups may be found by inspection. We have:

1) the subgratings obtained by restricting n from $n \in \mathbb{Z}$ to $n \in \mathbb{Z}(M)$, which have representations

$$\mathbf{J} = \begin{bmatrix} 1 & 0 \\ 2 & M \end{bmatrix}, \qquad H = \mathbb{R} \times \mathbb{Z}$$

and can be symbolized as $\mathbb{R}\mathbb{Z}(2, M)$.

2) the sublattices obtained by restricting also r from $r \in \mathbb{R}$ to $r \in \mathbb{Z}(d)$, which have representation

$$\mathbf{L} = \begin{bmatrix} d & 0 \\ 2d & M \end{bmatrix}, \qquad H = \mathbb{Z}^2 .$$

with points

$$(t_1, t_2) = \{md, 2md + nM \mid m \in \mathbb{Z}, n \in \mathbb{Z}\} . \tag{S16.2}$$

3) the 1D group in $\mathbb{R}^2$ obtained by restricting n from $n \in \mathbb{Z}$ to $n \in \mathbb{O}$, that is, with points

$$(t_1, t_2) = \{r, 2r \mid r \in \mathbb{R}\} .$$

4) 1D groups in $\mathbb{R}$ (degenerate lattices) obtained by restricting in (S16.2) n from $n \in \mathbb{Z}$ to $n \in \mathbb{O}$, that is, with points

$$(t_1, t_2) = \{md, 2md \mid m \in \mathbb{Z}\} .$$

Of course, the trivial group $\mathbb{O}^2$ is a subgroup of the given grating.

16.4. [Sect. 16.6] Find the aligned bases of the 2D lattices starting from the bases

$$\mathbf{G} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 6 & 13 \\ 2 & 6 \end{bmatrix} .$$

We first evaluate

$$\mathbf{H} = \mathbf{G}^{-1}\mathbf{J} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} .$$

Since $\mathrm{d}(\mathbf{H}) = 2$ we find that the sublattice J is 2 times sparser than G. The Smith decomposition gives

$$\mathbf{H} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \mathbf{E}_1 \boldsymbol{\Delta} \mathbf{E}_2$$

and the aligned bases are

$$\mathbf{G}_0 = \mathbf{G}\mathbf{E}_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{J}_0 = \mathbf{J}\mathbf{E}_2^{-1} = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}, \qquad \mathbf{J}_0 = \mathbf{G}_0 \,\mathrm{diag}\,[2, 1] .$$

Consequentially: $\mathbf{j}_1 = 2\mathbf{g}_1$ and $\mathbf{j}_2 = 1\mathbf{g}_2$.

16.5. $\star$[Sect. 16.10] Check that the kernels of zero and integral–reductions are respectively

$$h(\mathbf{u};\mathbf{u}_0,\mathbf{v}_0) = \delta_V(\mathbf{v}_0)\delta_U(\mathbf{u}-\mathbf{u}_0)\,, \qquad h(\mathbf{u};\mathbf{u}_0,\mathbf{v}_0) = \delta_U(\mathbf{u}-\mathbf{u}_0)\,.$$

The zero–reduction is $U \times V \to U$ linear transformation with input–output relation

$$\begin{aligned} y(\mathbf{U}) &= \int_U \mathrm{d}\,\mathbf{u}_0 \int_V \mathrm{d}\,v_0\; h(\mathbf{u};\mathbf{u}_0,\mathbf{v}_0)\,x(\mathbf{u}_0,\mathbf{v}_0) \\ &= \int_U \mathrm{d}\,\mathbf{u}_0 \int_V \mathrm{d}\,\mathbf{v}_0\; \delta_U(u-u_0)\,\delta_V(\mathbf{v}_0)\,x(\mathbf{u}_0,v_0)\,. \end{aligned}$$

Then, by applying twice the *sifting properties*, we can drop the integrals by setting $\mathbf{u}_0 = \mathbf{u}$ and $\mathbf{v}_0 = \mathbf{O}$, thus getting

$$y(\mathbf{u}) = x(\mathbf{u},\mathbf{O})\,.$$

Analogously, the integral–reduction has input–output relation

$$\begin{aligned} y(\mathbf{u}) &= \int_U \mathrm{d}\,\mathbf{u}_0 \int_V \mathrm{d}\,\mathbf{v}_0\; h(\mathbf{u};u_0,v_0)\,x(u_0,v_0) \\ &= \int_U \mathrm{d}u_0 \int_V \mathrm{d}\,\mathbf{v}_0\; \delta_U(u-u_0)\,x(u_0,v_0) \end{aligned}$$

where we apply the sifting properties with respect to $\mathbf{u}_0$ to get

$$y(\mathbf{u}_0) = \int_V \mathrm{d}\,v_0\; x(\mathbf{u},\mathbf{v}_0)\,.$$

16.6. $\star$[Sect. 16.10] Check that the kernels of the hold and delta–increases are respectively

$$h(\mathbf{u},\mathbf{v};\mathbf{u}_0) = \delta_U(\mathbf{u}-\mathbf{u}_0)\,, \qquad h(\mathbf{u},\mathbf{v};\mathbf{u}_0) = \delta_U(\mathbf{u}-\mathbf{u}_0)\,\delta_V(\mathbf{u})\,.$$

These elementary dimensionality increases are $U \to U \times V$ linear transformations having the general input–output relation

$$y(\mathbf{u},\mathbf{v}) = \int_U \mathrm{d}\,\mathbf{u}_0\; h(\mathbf{u},\mathbf{v};\mathbf{u}_0)\,x(\mathbf{u}_0)\,.$$

With the first kernel we find

$$y(\mathbf{u},\mathbf{v}) = \int_U \mathrm{d}\,\mathbf{u}_0\; \delta_U(\mathbf{u}-\mathbf{u}_0)\,x(\mathbf{u}_0) = x(\mathbf{u})$$

and with the second kernel

$$y(\mathbf{u},\mathbf{v}) = \int_U \mathrm{d}\mathbf{u}_0\ \delta_U(\mathbf{u}-\mathbf{u}_0)\,\delta_V(\mathbf{v})\,x(\mathbf{u}_0)$$
$$= x(\mathbf{u})\,\delta_V(\mathbf{v})\ .$$

In both cases we have applied the sifting property.

16.7. $\star\star\star$[Sect. 16.12] Consider the $\mathbb{Z}_a^p(d_1,d_2) \to \mathbb{Z}(ad_1)$ reading of Proposition 16.14. Prove the following statements (recall that a and p are relatively prime):

1) the abscissa of the first pixel of line n is given by $m_n d_1$, where $m_n = \mu_a(nb)$, with $\mu_a(x)$ the remainder of the integer division of x by a;
2) the number of pixels of line n is given by

$$M_n = \rho_a(M-1-m_n)+1$$

where $\rho_a(x)$ denotes the integer part of x/a
3) m_n and M_n have period a;
4) the number of pixels in a period is M.

For instance, with $a=5$, $b=2$ and $M=13$ we find $m_0=0$, $m_1=2$, $m_2=4$, $m_3=1$, $m_4=3$ and $M_0=3$, $M_1=3$, $M_2=2$, $M_3=3$, $M_4=2$ and the sum of M_N is $M=13$.

1) The points of the normalized lattice $\mathbb{Z}_a^p$ are given by

$$x = am+nb\,, \qquad y=n \qquad m,n\in\mathbb{Z}$$

where n represents the line index (we consider $m,n \geq 0$). With n fixed, the abscissas of the pixels are spaced by a and the abscissa of the first pixel is given by the smallest nonnegative $x = am+nb$. This is expressed as $\mu_a(x) = \mu_a(nb)$.
2) In line n the pixels have abscissas

$$m_n, m_n+a, m_n+2a, \ldots, m_n+(M_n-1)a$$

where $m_n+(M_n-1)a \leq M-1$, that is,

$$\frac{M-1-m_n}{a} \geq M_n - 1\ .$$

Hence

$$M_n = \rho_a\,(M-1-m_n)+1\ . \tag{S16.3}$$

3) $m_n = \mu_a(nb)$ has period a because $\mu_a((n+a)b) = \mu(nb)$, and so is for M_n.
4) We first remark that $\mu_a(nb)$ in a single period has a distinct values: $0,1,\ldots,a-1$ (a and b, in fact, are relatively prime). Then we let

$$M = aM_0 - j \qquad \text{with} \qquad 0 \leq j \leq a-1$$

and we see from (S16.3) that M_n can assume in a period only the values M_0 and M_0-1 and, more specifically, $a-j$ values equal to M_0 and j values equal to M_0-1. Hence, the number of pixels in a period is

$$(a-j)M_0 + j(M_0-1) = aM_0 - j = M\ .$$

16.8. ⋆⋆[Sect. 16.12] Show that in the 3D→1D reading with lattice (16.93) and extension (16.94) the period of the fields Q_k is given by $L = ai$ (recall that a, p, q and i, b have no common factor). *Hint*: the period is given by the smallest integer $k > 0$ such that the lattice coordinates result $(x, y) = (0, 0)$, which represents the position of the first pixel of the field Q_0.

From (16.97) we find that condition $(x, y) = (0, 0)$ implies that

$$am + pn + qk = 0\,, \qquad in + bk = 0\ .$$

Since i and b are relatively primes, the second condition requires that k must be multiple of i, say $k = ik_0$. Then, $n = -bk_0$ and the first condition becomes

$$am + k_0(-pb + iq) = 0\ .$$

A solution is $k_0 = a$ and then $k_{min} = ia$. Note that, by assumption, b and q are relatively prime and p and q cannot have in common the factor a.

16.9. ⋆⋆⋆[Sect. 16.12] Continuing the previous problem, show that the number of lines in a period is Na and the number of pixels is MN.

The position of the lines c_{nk} in the field Q_k is determined by the vertical coordinate y according to $y_{nk} = (ni + kvd_2)$, where

$$0 \le ni + kb < N\ . \tag{16.4}$$

The lines are vertically spaced of id_2 and have a period i. Now, the fields in a period $Q_0, Q_1, \ldots, Q_{i-1}$ contain exactly N pixels. In fact, with the constraint (16.4), where i and b are relatively primes, y_{nk} spans without superposition the set $\{0, d_2, \ldots, (N-1)d_2\}$ of cardinality N (see Fig. 16.30). But the period of the fields is ia, where we find a times the periodic pattern of the lines in a period. Hence, the number of lines in a field period is Na.

For the evaluation of the number of pixels in a field period, it is convenient to consider the global projection of a field period on the t_1–axis, after the application of the composite shifts, and use conditions (16.95), which ensure that the projection points fall on $\mathbb{Z}(ad_1)$. In the projection, the room for a line is $D_1 = Md_1$ and for the aN lines in a field period the global room is $aNMd_1$, which contains exactly NM equally spaced of ad_1.

16.10. $\star$ [Sect. 16.13] Find the Fourier analysis of the $\mathbb{R} \times \mathbb{Z}(d_2, d_3) \to \mathbb{R}$ reading, where the composite shift matrix $\mathbf{A}$ is given by (16.85)

The dual domains are $\widehat{U} = \mathbb{R}$ and $\widehat{V} = \mathbb{Z}(F_2, F_3)$ with $F_2 = 1/d_2$, $F_3 = 1/d_3$. The reciprocal of matrix (16.85) is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & D_1/d_2 & MD_1/d_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\star} \mathbf{A}^\star = \begin{bmatrix} 1 & 0 & 0 \\ D_1/d_2 & 1 & 0 \\ MD_1/d_2 & 0 & 1 \end{bmatrix} .$$

Then, $\mathbf{M}' = \begin{bmatrix} D_1/d_2 \\ MD_1/d_2 \end{bmatrix}$ and (16.103), where $f \in \mathbb{R}$ is scalar, $\mathbf{M}' f = (D_1/d_2 f, MD_1/d_2 f)$ and $\mu(V) = \mu(\mathbb{Z}(d_1, d_2)) = F_1 F_2$, gives

$$Y_0(f) = F_1 F_2 \, X\left(f, \tfrac{D_1}{d_2} f, \tfrac{MD_1}{d_2} f\right), \qquad f \in \mathbb{R} .$$

16.11. $\star\star$ [Sect. 16.13] Find the Fourier analysis of the $\mathbb{Z}_2^1(d_1, d_2) \to \mathbb{Z}(2d_2)$ reading, where the composite shift matrix is given by (16.87) with M odd.

The dual groups are $J = \mathbb{Z}_2^1(F_1, F_2)$, $U = \mathbb{Z}(F_1)$, $V = \mathbb{Z}(F_2)$ with $F_1 = 1/(2d_1)$, $F_2 = 1/(2d_2)$ and $\mathbf{M}$ is a scalar given by Md_1/d_2. Then (16.103), where $\mathrm{d}(V) = F_2$, gives

$$Y_0(f) = F_2 \, X\left(f, \tfrac{Md_1}{d_2} f\right), \qquad f \in \mathbb{Z}(F_1) .$$

16.12. $\star\star$ [Sect. 16.13] Find the Fourier analysis of the writing operation of Fig.16.24.

The first operation is a hold increase $U \to U \times V$ with relation (see (16.74))

$$x_h(\mathbf{u}, \mathbf{v}) = y_0(\mathbf{u}), \qquad (\mathbf{u}, \mathbf{v}) \in U \times V$$

which becomes a delta increase with relation

$$X_h(\boldsymbol{\lambda},\boldsymbol{\mu}) = Y_0(\boldsymbol{\lambda})\,\delta_{\widehat{V}}(\boldsymbol{\mu})\,, \qquad (\boldsymbol{\lambda},\boldsymbol{\mu}) \in \widehat{U}\times\widehat{V}\ .$$

The second operation is a window with shape $\eta_0(\mathbf{u},\mathbf{v})$ given by the indicator function of the extension of the signal $x_0(\mathbf{u},\mathbf{v})$ (see Fig.16.24). The relation is

$$\tilde{x}_0(\mathbf{u},\mathbf{v}) = \eta_0(\mathbf{u},\mathbf{v})\,\mathbf{x}_h(\mathbf{u},\mathbf{v})$$

which becomes a convolution

$$\begin{aligned}
&\tilde{X}_0(\boldsymbol{\lambda},\boldsymbol{\mu}) = N_0 * X_h(\boldsymbol{\lambda},\boldsymbol{\mu}) \\
&= \int_{\widehat{U}} \mathrm{d}\boldsymbol{\lambda}' \int_{\widehat{V}} \mathrm{d}\mathbf{u}'\, N_0(\boldsymbol{\lambda}',\boldsymbol{\mu}')\, X_h(\boldsymbol{\lambda}-\boldsymbol{\lambda}',\boldsymbol{\mu}-\boldsymbol{\mu}') \\
&= \int_{\widehat{U}} \mathrm{d}\boldsymbol{\lambda}' \int_{\widehat{V}} \mathrm{d}\mathbf{u}'\, N_0(\boldsymbol{\lambda}',\boldsymbol{\mu}')\, Y_0(\boldsymbol{\lambda}-\boldsymbol{\lambda}')\ \delta_{\widehat{V}}(\boldsymbol{\mu}-\boldsymbol{\mu}') \\
&= \int_{\widehat{U}} \mathrm{d}\boldsymbol{\lambda}'\, N_0(\boldsymbol{\lambda}',\boldsymbol{\mu})\, Y_0(\boldsymbol{\lambda}-\boldsymbol{\lambda}')
\end{aligned}$$

where $N_0(\cdot)$ is the FT of $\eta_0(\cdot)$ and we have used the sifting property of the impulse. The final operation is a coordinate change with matrix $\mathbf{A}^{-1}$ with relation

$$\tilde{x}(\mathbf{t}) = \mathbf{x}_0(\mathbf{A}^{-1}(\mathbf{u},\mathbf{v}) \quad \text{with} \quad \mathbf{t} = \mathbf{A}^{-1}(\mathbf{u},\mathbf{v})$$

which becomes a coordinate change with matrix $(\mathbf{A}^{-1})^* = \mathbf{A}'$ and a multiplication by $\mu(\mathbf{A}^{-1}) = \det\mathbf{A}$ (see Sect. 6.5). Hence

$$\tilde{X}(\mathbf{f}) = \det\mathbf{A}\, X_0(\mathbf{A}'(\boldsymbol{\lambda},\boldsymbol{\mu})) \quad \text{with} \quad \mathbf{f} = \mathbf{A}'(\boldsymbol{\lambda},\boldsymbol{\mu})\ .$$

Problems of Chapter 17

17.1. $\star$[Sect. 17.2] Starting from the Fourier transform $L(f_x, f_y)$ of a source 2D image, write explicitly the Fourier transform $L_Q(f_x, f_y)$ of the framed image.

The relation is $\ell(x,y) = w_Q(x,y)\,\ell(x,y)$ and becomes a convolution in the frequency domain

$$L_Q(f_x, f_y) = \int_{\mathbb{R}^2} \mathrm{d}\lambda_x \mathrm{d}\lambda_y\, W_Q(f_x - \lambda_x, f_y - \lambda_y)\, L(\lambda_x, \lambda_y)\,, \qquad (f_x, f_y) \in \mathbb{R}^2 \quad (17.1)$$

where $W_Q(f_x, f_y)$ is the FT of the indicator function $w_Q(x,y)$ of the frame $Q = [0, D_x) \times [0, D_y)$. Considering that $w_Q(x,y) = \mathrm{rect}_+(x/D_x)\,\mathrm{rect}_+(y/D_y)$, one easily finds

$$W_Q(f_x, f_y) = W_1(f_x)\, W_2(f_y)\,, \qquad (17.2)$$

where

$$W_1(f_x) = D_x\ \mathrm{sinc}(f_x D_x)\ \exp(-\mathrm{i}\pi\, f_x D_x)\,, \qquad (17.2\mathrm{a})$$

$$W_2(f_y) = D_y\ \mathrm{sinc}(f_y D_y)\ \exp(-\mathrm{i}\pi\, f_y D_y)\,. \qquad (17.2\mathrm{b})$$

Note that the framing operation does not change the signal domain, which is still $\mathbb{R}^2$. So $L_Q(f_x, f_y)$ is a 2D continuous frequency function, as $L(f_x, f_y)$.

17.2. $\star$[Sect. 17.2] Consider the 2D continuous scanning where the framed image $\ell_Q(x,y)$, $(x,y) \in \mathbb{R}^2$, is down–sampled in the form $\mathbb{R}^2 \to \mathbb{R} \times \mathbb{Z}(d_y)$ to give $\ell_{QS}(x,y)$. Explicitly write the relationship between the Fourier transforms. In addition, write the expression of the Fourier transform of $\ell_{QS}(x,y)$ and its inverse.

The general relation is given by (17.22). With $I_S^\star = \mathbb{O} \times \mathbb{Z}(F_y)$, it gives

$$L_{QS}(f_x, f_y) = \sum_{k=-\infty}^{+\infty} L_Q(f_x, f_y - kF_y)\,, \qquad f_x \in \mathbb{R}\,,\ f_y \in \mathbb{R}/\mathbb{Z}(F_y)$$

Next, we write the expression of the FT of the grating image and its inverse. Since $I_S = \mathbb{R} \times \mathbb{Z}(d_y)$ and $\widehat{I}_S = \mathbb{R} \times \mathbb{R}/\mathbb{Z}(F_y)$, from Sect. 5.9 one obtains

$$\begin{aligned} L_{QS}(f_x, f_y) &= \int_{-\infty}^{+\infty} \left\{ \sum_{n=-\infty}^{+\infty} d_y\, \ell_{QS}(x, nd_y)\, \mathrm{e}^{-\mathrm{i}2\pi(f_x x + f_y n d_y)} \right\} \mathrm{d}x \\ \ell_{QS}(x, nd_y) &= \int_{-\infty}^{+\infty} \int_0^{F_y} \mathrm{d}f_x\, \mathrm{d}f_y\, L_{QS}(f_x, f_y)\, \mathrm{e}^{\mathrm{i}2\pi(f_x x + f_y n d_y)}\,. \end{aligned} \qquad (17.3)$$

17.3. ⋆⋆[Sect. 17.2] Consider the 2D discrete scanning where the framed image $\ell_Q(x,y)$, $(x,y) \in \mathbb{R}^2$ is down–sampled in the form $\mathbb{R}^2 \to \mathbb{Z}(d_x,d_y)$ to give $\ell_{QS}(x,y)$. Write explicitly the relationship between the Fourier transforms. In addition, write the expression of the Fourier transform of $\ell_{QS}(x,y)$ and its inverse.

From (17.22) with $I_S^\star = \mathbb{Z}(F_x,F_y)$ we obtain

$$L_{QS}(f_x, f_y) = \sum_{h=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} L_Q(f_x - hF_x, f_y - kF_y) \quad \text{with} \quad f_x \in \mathbb{R}/\mathbb{Z}(F_x)\,,\ f_y \in \mathbb{R}/\mathbb{Z}(F_y) \tag{17.4}$$

that is, we have a full periodic repetition along f_x and along f_y. Considering that the domain of $\ell_{QS}(x,y)$ is $I_S = \mathbb{Z}(d_x,d_y)$ and that the dual domain is $\widehat{I}_S = \mathbb{R}^2/\mathbb{Z}(F_x,F_y)$, the expression of the Fourier transform and of its inverse are respectively

$$L_{QS}(f_x, f_y) = \sum_{m=-\infty}^{+\infty} \mathrm{d}x \sum_{d_y\,=-\infty}^{+\infty} \ell_{QS}(md_x, nd_y)\, \mathrm{e}^{-\mathrm{i}2\pi(f_x md_x + f_y nd_y)}$$

$$\ell_{QS}(md_x, nd_y) = \int_{=}^{F_x} \mathrm{d}f_x \int_0^{F_y} \mathrm{d}f_y\, L_{QS}(f_x, f_y)\, \mathrm{e}^{\mathrm{i}2\pi(f_x md_x + f_y nd_y)}\ .$$

17.4. ⋆⋆[Sect. 17.2] Consider a general discrete scanning obtained with a general lattice $I_S = \mathbb{Z}_a^p(d_x,d_y)$. Write explicitly the Fourier transform of $\ell_{QS}(x,y)$ and its inverse.

Considering that I_S is a lattice and $\widehat{I}_S = \mathbb{R}^2/I_S^\star$, one gets

$$L_S(f_x, f_y) = \sum_{(x,y)\in I_S} \mathrm{d}(I_S)\, \ell_S(x,y)\, \mathrm{e}^{-\mathrm{i}2\pi(f_x x + f_y y)} \tag{17.5a}$$

$$\ell_S(x,y) = \int_{\mathbb{R}^2/I_S^\star} \mathrm{d}x\, \mathrm{d}y\, L_S(f_x, f_y)\, \mathrm{e}^{\mathrm{i}2\pi(f_x x + f_y y)} \tag{17.5b}$$

where the integral is over a cell of $\mathbb{R}^2$ modulo $I_S^\star$. This cell can be the fundamental parallelepiped of I_S^*, but the double integration can be split into two 1D integral if we use an *orthogonal* cell (see Sect. 16.9).

17.5. $\star$ [Sect. 17.2] Consider the discrete scanning of a still image. Write the reconstruction of the image starting from the video signal $u(mT_0)$.

The latticed image can be recovered from discrete video signal $u(mT_0)$ as

$$\ell_{QS}(x,y) = u\left(x/v_x + y/v_y\right)\; w_r(x)\,, \qquad (x,y) \in \mathbb{Z}(d_x,d_y)\;.$$

After this step, one needs to interpolate both horizontally and vertically, obtaining

$$\ell_i(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \ell_{QS}(md_x,nd_y)\;\mathrm{sinc}\left(\frac{x-md_x}{d_x}\right)\;\mathrm{sinc}\left(\frac{y-nd_y}{d_y}\right)\;.$$

This reconstruction refers to full discrete scanning. For a discrete scanning on a general lattice, we can proceed in a similar way.

17.6. $\star\star$ [Sect. 17.10] Consider the expression of the Radon transform given by (17.66). Prove that it can be written in the form (17.67).

Relation (17.66) gives the *projections* of $\ell(x,y)$, that is,

$$g(s_0,\theta) = \int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}y\;\ell(x,y)\,\delta_{\mathbb{R}}(x\cos\theta + y\cos\theta - s_0)\;.$$

With the change of coordinates

$$\begin{cases} x = s\cos\theta - u\sin\theta \\ y = s\sin\theta + u\cos\theta \end{cases} \qquad\qquad \begin{cases} s = x\cos\theta + y\sin\theta \\ u = x\sin\theta - y\cos\theta \end{cases}$$

whose Jacobian is 1, we obtain

$$g(s_0,\theta) = \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}u\;\ell(s\cos\theta - u\sin\theta, s\sin\theta + u\cos\theta)\,\delta_{\mathbb{R}}(s-s_0)\;.$$

Then, the application of the sifting property on $\delta_{\mathbb{R}}(s-s_0)$ allows dropping the integral with respect to s under the condition $s-s_0=0$. Hence

$$g(s_0,\theta) = \int_{\mathbb{R}} \mathrm{d}u\;\ell(s_0\cos\theta - u\sin\theta, s_0\sin\theta + u\cos\theta)\,\delta_{\mathbb{R}}(s-s_0)\;.$$

17.7. $\star\star$ [Sect. 17.10] Show that the image of Fig.17.35 with polar representation

$$\widetilde{\ell}(r,\varphi) = \mathrm{rect}_+(1-r^2)\cos 3\varphi$$

has Cartesian representation

$$\ell(x,y) = \left[(x^3-3xy^2)/|x+\mathrm{i}y|^3\right]\mathrm{rect}_+(1-x^2-y^2)\ .$$

Hint: Use identity $\mathrm{e}^{\mathrm{i}n\arg(x+\mathrm{i}y)} = (x+\mathrm{i}y)^n/|x+\mathrm{i}y|^n$.

The polar representation

$$\widetilde{\ell}(r,\varphi) = \mathrm{rect}_+(1-r^2)\cos 3\varphi$$

with $r^2 = x^2+y^2$, $\varphi = \arg(x+\mathrm{i}y)$, gives

$$\begin{aligned}\cos 3\varphi &= \Re[\arg \mathrm{e}^{\mathrm{i}3\arg(x+\mathrm{i}y)}]\\ &= \Re\frac{(x+\mathrm{i}y)^3}{|x+\mathrm{i}y|^2} = \frac{x^3-3xy^2}{|x+\mathrm{i}y|^2}\end{aligned}$$

and $\ell(x,y)$ takes the form indicated above.

17.8. $\star$ [Sect. 17.10] Write the polar representation $\widetilde{\ell}(r,\varphi)$ of the reference image (17.72).

In the polar representation $x = r\cos\varphi$, $y = r\sin\varphi$ and hence $x^2+y^2 = r^2$. Then (17.72) becomes

$$\widetilde{\ell}(r,\varphi) = 2^4\,4!\,\frac{J_4(r)}{r^2}\,\widetilde{p}(r,\varphi)$$

where

$$\widetilde{p}(r,\varphi) = 1 + Ar\cos\varphi + Br\sin\varphi + Cr^2\cos\varphi\sin\varphi\ .$$

17.9. $\star\star$ [Sect. 17.10] The reference image has the structure $\ell(x,y) = \ell_0(x,y)\,p(x,y)$, where

$$\ell_0(x,y) = J_4(\sqrt{x^2+y^2})\Big/(x^2+y^2)$$

has circular symmetry and its FT $L_0(f_x,f_y)$ can be calculated via Hankel transform (see Sect. 5.9). Considering that the Hankel transform of $J_4(r)/r^2$ is $\frac{\pi}{24}(1-(2\pi\lambda)^2)^3\,\mathrm{rect}(\pi\lambda)$, find the FT of $\ell(x,y)$. *Hint*: $p(x,y)$ is a polynomial and using the differentiation rule

$$(-\mathrm{i}2\pi x)^m(-\mathrm{i}2\pi y)^n\ell_0(x,y) \quad \xrightarrow{\mathcal{F}} \quad \frac{\partial^{m+n}L_0(f_x,f_y)}{\partial f_x^m\,\partial f_y^n}\ .$$

we can obtain the terms as $(-\mathrm{i}2\pi x)(-\mathrm{i}2\pi y)\ell_0(x,y) \xrightarrow{\mathcal{F}} \frac{\partial L_0(f_x,f_y)}{\partial f_x \partial f_y}$.

The image is given as the product

$$\ell(x,y) = \ell_0(x,y)\, p(x,y) \quad \text{with} \quad \ell_0(x,y) = c\,(|x+\mathrm{i}y|)$$

and $p(x,y) = (1 + Ax + By + Cxy)$ is a polynomial. We first calculate the Fourier transform $L_0(f_x, f_y)$ of the first factor and then use the differentiation rule. Since $\ell_0(x,y)$ has a circular symmetry, its Fourier transform has the same symmetry

$$L_0(f_x, f_y) = \Gamma(|f_x + \mathrm{i}\, f_y|)$$

where $\Gamma(\lambda)$ is the Hankel transform of $c(r)$, which is given by (see [6] p.145)

$$\Gamma(\lambda) = \frac{\pi}{24}\left[1-(2\pi\lambda)^2\right]^3 \operatorname{rect}(\pi\lambda)\ . \tag{S17.6}$$

The function $\Gamma(\lambda)$ and its two first derivatives are illustrated in Fig.S17.1.

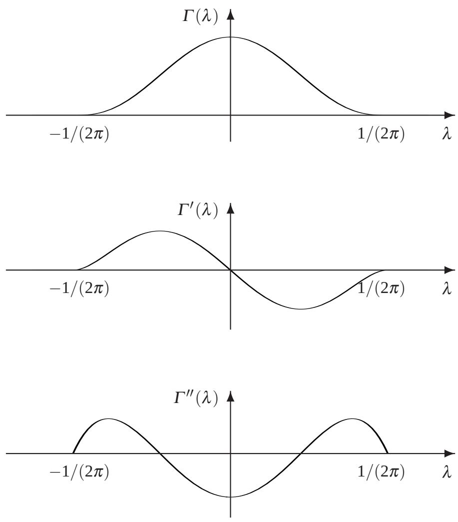

Fig. S17.1 The function $\Gamma(\lambda)$ and its two first derivatives

Now, we use the FT frequency differentiation rule, that is,

$$(-\mathrm{i}2\pi x)^m(-\mathrm{i}2\pi y)^n \ell_0(x,y) \quad \xrightarrow{\mathcal{F}} \quad \frac{\partial^{m+n} L_0(f_x, f_y)}{\partial f_x^m\, \partial f_y^n}\ .$$

Letting $P(\lambda) = 1 - (2\pi\lambda)^2$ and $\lambda^2 = f_x^2 + f_y^2$ we find

$$L(f_x, f_y) = \frac{\pi}{24} Q(f_x, f_y)\, \mathrm{rect}\,(\pi |f_x + \mathrm{i}\, f_y|) \tag{S17.7}$$

where

$$\begin{aligned} Q(f_x, f_y) =& P^3(\lambda) + 6P^2(\lambda)\,(-\mathrm{i}2\pi f_x A - \mathrm{i}2\pi f_y B) \\ &+ 384C\,(-\mathrm{i}2\pi f_x)\,(-\mathrm{i}2\pi f_y)\, P(\lambda) \end{aligned} \tag{S17.7a}$$

is a polynomial in f_x, f_y. Since $\mathrm{rect}(x) = 0$ for $|x| > \frac{1}{2}$, we find that $L(f_x, f_y) = 0$ for $|f_x + \mathrm{i} f_y| > 1/(2\pi)$, that is, $L(f_x, f_y)$ has a circular extension of radius $1/(2\pi)$.

Note that no singularity (delta function) is present in the boundary because the function $\Gamma(\lambda)$ is continuous at $\lambda = \pm 1/(2\pi)$ together with its first two derivatives, as shown in Fig. S17.1.

17.10. ⋆⋆ [Sect. 17.15] Explain why the lattice $\mathbb{Z}_3^1(\Delta s, \Delta\theta)$ cannot be used in the projection sampling.

The condition on the lattice J for the projection sampling are

1) J must contain the periodicity $P = \mathbb{O} \times \mathbb{R}(2\pi)$,
2) J must be symmetric with respect to the vertical axis.

The lattice $J = \mathbb{Z}_3^1(\Delta s, \Delta\theta)$, with basis

$$\mathbf{J} = \begin{bmatrix} \Delta s & 0 \\ 0 & \Delta\theta \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

is shown in Fig. S17.2. We realize that it is not symmetric with respect to s. For instance, the point $(\Delta s, \Delta\theta) \in J$, but the symmetric point $(-\Delta s, \Delta\theta) \notin J$.

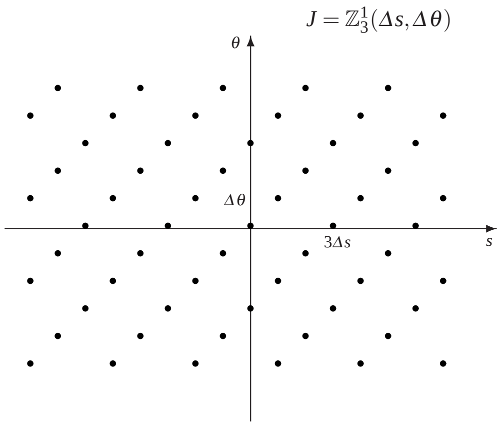

Fig. S17.2 The lattices of the problem

17.11. $\star\star$ [Sect. 17.15] Show that with the cell

$$C = (-B_s, B_s) \times (-N_a F_a, \cdots, -F_a, 0, F_a, \cdots, (N_a+1)F_a)$$

the interpolator impulse response is given by (17.130).

It is convenient to recall the several symbols:

- impulse response $h(s,\theta)$, $(s,\theta) \in I = \mathbb{R} \times \mathbb{R}/\mathbb{Z}(2\pi)$
- frequency response $H(f_s, f_\theta)$, $(f_s, f_\theta) \in \widehat{I} = \mathbb{R} \times \mathbb{Z}(F_a)$, $F_a = 1/(2\pi)$.

Given the cell $C \in \widehat{I}$, the frequency response is given by the indicator function of the cell C

$$H(f_s, f_\theta) = \eta_C(f_s, f_\theta), \qquad f_s \in \mathbb{R},\; f_\theta = nF_a$$

and the impulse response is the inverse FT of the frequency response, that is,

$$\begin{aligned} h(s,\theta) &= \int_{\widehat{I}} \mathrm{d}\mathbf{f}\, \eta_C(\mathbf{f})\, \mathrm{e}^{\mathrm{i}2\pi \mathbf{f}(s,\theta)} \\ &= \int_C \mathrm{d}f_s\, \mathrm{d}f_\theta \mathrm{e}^{\mathrm{i}2\pi(f_s s + f_\theta \theta)}\,. \end{aligned} \tag{S17.8}$$

Now, the cell C is separable in the form

$$C = (-B_s, B_s) \times \{-N_a F_a, \ldots, 0, (N_a+1)F_a\}$$

and therefore

$$h(s,\theta) = h_r(s)\, h_a(\theta)$$

where

$$h_r(s) = \int_{-B_s}^{B_s} \mathrm{d}f_s\, \mathrm{e}^{\mathrm{i}2\pi f_s s} = 2B_0\, \mathrm{rect}(sB_s)$$

and

$$h_a(\theta) = \sum_{n=-N_a}^{N_a+1} F_a\, \mathrm{e}^{\mathrm{i}2\pi n F_a \theta}\,.$$

Letting $X = \mathrm{e}^{\mathrm{i}2\pi F_a \theta} = \mathrm{e}^{\mathrm{i}\theta}$

$$h_a(\theta) = F_a \sum_{n=-N_a}^{N_a+1} X^n$$

and recalling the identity

$$\sum_{n=n_1}^{n_2} X^n = \frac{(1 - X^{n_2 - n_1 + 1}) X^{n_1}}{1 - X}$$

we get

$$h_a(\theta) = F_a \frac{1 - X^{2N_a+2}}{(1-X)X^{N_a}}$$
$$= F_a \frac{1 - \mathrm{e}^{\mathrm{i}(2N_a+2)\theta}}{(1-\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}N_a\theta}} \; .$$

17.12. ⋆⋆⋆ [Sect. 17.15] Write the frequency response of the rhomboidal cell of Fig.17.44 and prove that the corresponding impulse response is given by

$$h(s,\theta) = 2F_aB_s\,\mathrm{sinc}(2B_ss) + \sum_{n=1}^{N_a} 4F_aB_n\,\mathrm{sinc}(2B_ns)\cos n\theta$$

where $B_n = B_s\,(1 - n/(N_a+1))$.

We use the general formulation presented in the solution of the previous problem. The rhomboidal cell of Fig.17.44 consists of lines of the (f_s, f_θ) plane, whose projection on the f_s axis are given by

$$e_n = (-B_n, B_n) \qquad -N_a' \le n \le N_a', \qquad N_a' = N_a + 1$$

where $B_n = B_s(1 - |n|/N_a')$. The vertical abscissa is at the frequencies nF_a and, therefore, the cell is

$$C = \bigcup_{n=-N_a'}^{N_a'} e_n \times \{nF_a\} \; .$$

Consequently (S17.8), with $f_\theta = nF_a$, gives

$$h(s,\theta) = \sum_{n=-N_a}^{N_a'} F_a\,\mathrm{e}^{\mathrm{i}2\pi nF_a\theta} \int_{e_n} \mathrm{d}f_s\,\mathrm{e}^{\mathrm{i}2\pi f_s s}$$

where

$$\int_{e_n} \mathrm{d}f_s\,\mathrm{e}^{\mathrm{i}2\pi f_s s} = \int_{-B_n}^{B_n} \mathrm{d}f_s\,\mathrm{e}^{\mathrm{i}2\pi f_s s} = 2B_n\,\mathrm{sinc}(2B_ns) \; .$$

Hence

$$h(s,\theta) = 2B_sF_a \sum_{n=-N_a}^{N_a'} (1 - |n|/N_a')\mathrm{e}^{\mathrm{i}2\pi nF_a\theta}\,\mathrm{sinc}[2B_s(1 - |n|/N_a')s]$$

which can be written as in the statement of the problem.

Closed Form Results of the Reference Example

This part deals with the representations of an image in the context of *Radom transform*. It refers to Chapter 17, where the Radom transform is developed and a “Reference Example” is introduced to illustrate the sequel of ancillary functions related to the Radom transform. Here, all the expressions of the ancillary functions are reported and commented.

1) Image Cartesian representation

The Reference Example starts from the specific expression of a 2D image given by

$$\ell(x,y) = c\,(|x+\mathrm{i}y|)\,(1+Ax+By+Cxy) \tag{S17.9}$$

where

$$c(r) = \frac{J_4(r)}{r^4}\,.$$

Eq. (S17.9) gives the image in terms of the *Cartesian representation*.

2) Polar representation of $\ell(x,y)$

$$\widehat{\ell}\,(r,\varphi) = c(r)\left[1+r(A\cos\varphi+B\sin\varphi)+\tfrac{1}{2}Cr^2\sin 2\phi\right]\,.$$

3) Harmonic expansion of $\widehat{\ell}\,(r,\varphi)$

$$\widehat{\ell}\,(r,\varphi) = \sum_{n=-2}^{2}\widehat{\ell}_{\,n}(r)\,\mathrm{e}^{\mathrm{i}n\varphi}\,, \qquad \widehat{\ell}_{\,-n}(r) = \widehat{\ell}_{\,n}(-r) \tag{S17.10}$$

where

$$\begin{aligned} &\widehat{\ell}_{\,0}(r) = c(r)\,, \qquad \widehat{\ell}_{\,1}(r) = \tfrac{1}{2}r\,c(r)\,[A-\mathrm{i}B] \\ &\widehat{\ell}_{\,2}(r) = \tfrac{1}{2}r^2\,c(r) - \mathrm{i}\tfrac{1}{4}Cr^2\,c(r)\,. \end{aligned} \tag{S17.10a}$$

4) Angular Fourier transform of $\widehat{\ell}\,(r,\varphi)$

$$\mathcal{L}_a(r,nF_a) = 2\pi\,\ell_n(r)$$

where $\ell_n(r)$ is given by (S17.10a).

5) Fourier transform $L(f_x, f_y)$ of the image $\ell(x,y)$

See the solution of Problem 17.9.

6) Polar representation of $L(f_x, f_y)$

By letting $f_x = \lambda\cos\alpha,\ f_y = \lambda\sin\alpha$ in (S17.7), we obtain

$$\widehat{L}(\lambda,\alpha) = \frac{\pi}{24}\widehat{Q}(\lambda,\alpha)\,\mathrm{rect}(\pi\lambda) \tag{S17.11}$$

where

$$\begin{aligned}\widehat{Q}(\lambda,\alpha) = P^3(\lambda) - \mathrm{i}2\pi\lambda P^2(\lambda)\,(A\cos\alpha + B\sin\alpha)\\ + (-\mathrm{i}2\pi\lambda)^2 192CP(\lambda)\sin 2\alpha\,.\end{aligned} \tag{S17.11a}$$

From Projection Theorem, the radial FT of projections is given by

$$G(f_s,\theta) = \widehat{L}(f_s,\theta) = \frac{\pi}{24}\,\mathrm{rect}(\pi f_s)\,\widehat{Q}(f_s,\theta)\,. \tag{S17.12}$$

7) Harmonic expression of $\widehat{L}(\lambda,\alpha)$

$$\widehat{L}(\lambda,\alpha) = \sum_{n=-2}^{2}\widehat{L}_n(\lambda)\,\mathrm{e}^{\mathrm{i}n\alpha}\,, \qquad \widehat{L}_{-n}(\lambda) = \widehat{L}_n^*(-\lambda) \tag{S17.13}$$

where

$$\begin{aligned}\widehat{L}_0(\lambda) &= \frac{\pi}{24}\,\mathrm{rect}(\pi\lambda)P^3(\lambda)\\ \widehat{L}_1(\lambda) &= \frac{\pi}{24}\,\mathrm{rect}(\pi\lambda)\left[-\mathrm{i}2\pi\lambda P^2(\lambda)\,6(A-\mathrm{i}B)\right]P^2(\lambda)\\ \widehat{L}_2(\lambda) &= \frac{\pi}{24}\,\mathrm{rect}(\pi\lambda)\,(-\mathrm{i}2\pi\lambda)^2(-96\mathrm{i}C)\,P(\lambda)\,.\end{aligned} \tag{S17.13a}$$

8) Projections

The radial FT of projections is given by (S17.12), where $\widehat{Q}(f_s,\theta)$ is a polynomial in f_s of degree 6. With $z = -\mathrm{i}2\pi f_s$ it is given by

$$\widehat{Q}\,(f_s,\theta)=\frac{\pi}{24}\sum_{n=0}^{6}q_n(\theta)\,z^n$$

where

$$\begin{aligned}
&q_0(\theta)=1\,, \qquad q_1(\theta)=-6\,(A\cos\theta+B\sin\theta)\,, \qquad q_2(\theta)=192C\pi^4\sin 2\theta\\
&q_3(\theta)=-12\,(A\cos\theta+B\sin\theta)\,, \qquad q_4(\theta)=3+192C\pi^4\sin 2\theta\\
&q_5(\theta)=-6\,(A\cos\theta+B\sin\theta)
\end{aligned} \tag{S17.14}$$

Next, considering that

$$\begin{aligned}
&\mathrm{rect}(\pi f_s) \quad \xrightarrow{\mathcal{F}^{-1}} \quad \frac{\sin s}{\pi s} \stackrel{\Delta}{=} p(s)\,,\\
&(-\mathrm{i}2\pi f_s)^n\,\mathrm{rect}(\pi f_s) \quad \xrightarrow{\mathcal{F}^{-1}} \quad p^{(n)}(s)\,,
\end{aligned}$$

we find

$$g(s,\theta)=\frac{\pi}{24}\sum_{n=0}^{6}q_n(\theta)\,p^{(n)}(s)\,.$$

The derivatives $p^{(n)}(s)$ of $p(s)$ are

$$\begin{aligned}
p^{(1)}(s)&=\frac{\cos s}{\pi s}-\frac{\sin s}{\pi s^2}\,,\\
p^{(2)}(s)&=\frac{-2\cos s}{s^2}+\frac{\sin s}{\pi s}\left(\frac{2}{s^2}-1\right)\,,\\
p^{(3)}(s)&=\frac{\cos s}{\pi s}\left(\frac{6}{s^2}-1\right)+\frac{\sin s}{\pi s^2}\left(3-\frac{6}{s^2}\right)\,,\\
p^{(4)}(s)&=\frac{\cos s}{\pi s^2}\left(4-\frac{24}{s^2}\right)+\frac{\sin s}{\pi s}\left(1-\frac{12}{s^2}+\frac{24}{s^4}\right)\,,\\
p^{(5)}(s)&=\frac{\cos s}{\pi s}\left(1-\frac{2}{s^2}+\frac{120}{s^4}\right)+\frac{\sin s}{\pi s^2}\left(-5+\frac{12}{s^2}-\frac{24}{s^4}\right)\,,\\
p^{(6)}(s)&=\frac{\cos s}{\pi s^2}\left(-6+\frac{5!}{s^2}-\frac{6!}{s^4}\right)+\frac{\sin s}{\pi s}\left(-1+\frac{30}{s^2}-\frac{360}{s^4}+\frac{6!}{s^6}\right)\,.
\end{aligned} \tag{S17.15}$$

9) Hankel transform of harmonic expansion of $\widehat{\ell}\,(r,\varphi)$

The harmonic expansion was evaluated in step 3). Taking the Hankel transform of the harmonics gives the harmonics of the radial FT $G_r(f_s,\theta)$ of projections

$$\widehat{\ell}_n(r) \quad \xrightarrow{\mathcal{H}_n} \quad G_{rn}(f_s)\,.$$

From the expressions (S17.10a) of $\widehat{\ell}_n(r)$, we see that we have to evaluate the Hankel transforms

$$c(r) \xrightarrow{\mathcal{H}_0} \Gamma(\lambda)\,, \qquad rc(r) \xrightarrow{\mathcal{H}_1} \Gamma_1(\lambda)\,, \qquad r^2c(r) \xrightarrow{\mathcal{H}_2} \Gamma_2(\lambda)\,.$$

More specifically,

$$\begin{aligned}
\Gamma(\lambda) &= 2\pi \int_0^\infty rc(r)\, J_0(2\pi\lambda r)\, \mathrm{d}r\,,\\
\Gamma_1(\lambda) &= 2\pi \mathrm{i}^{-1} \int_0^\infty r^2 c(r)\, J_1(2\pi\lambda r)\, \mathrm{d}r\,,\\
\Gamma_2(\lambda) &= 2\pi \mathrm{i}^{-2} \int_0^\infty r^3 c(r)\, J_2(2\pi\lambda r)\, \mathrm{d}r\,.
\end{aligned} \tag{S17.16}$$

Then, the harmonic coefficients are given by (see (S17.10a))

$$G_{r0}(f_s) = \Gamma(f_s)\,, \quad G_{r1}(f_s) = \frac{1}{2}(A - \mathrm{i}B)\Gamma_1(f_s)\,, \quad G_{r2}(f_s) = -\mathrm{i}\frac{1}{4}C\Gamma_2(f_s)\,. \tag{S17.17}$$

The first one is an ordinary Hankel transform (of order zero) given by (see (S17.6))

$$\Gamma(\lambda) = \frac{\pi}{24} P^3(\lambda)\, \mathrm{rect}(\pi\lambda)\,. \tag{17.18}$$

For the evaluation of the second one, we note that in (S17.16) the derivative of $\Gamma(\lambda)$ is given by

$$\Gamma'(\lambda) = 2\pi \int_0^\infty r^2 c(r)\, J_0'(2\pi\lambda r)\, \mathrm{d}r = -2\pi \int_0^\infty r^2 c(r)\, J_1(2\pi\lambda r)\, \mathrm{d}r\,, \tag{17.19}$$

where we have used the identity $J_0'(z) = -J_1(z)$. Hence, we find that

$$\Gamma_1(\lambda) = \mathrm{i}\Gamma'(\lambda) = -\mathrm{i}\pi^3 \lambda P^2(\lambda)\,.$$

In order to evaluate the third transform, we use the identity $J_2(z) = (1/z)J_1(z) - J_1'(z)$. We get

$$\begin{aligned}
\Gamma_2(\lambda) &= \frac{1}{\lambda} \int_0^\infty r^2 c(r)\, J_1(2\pi\lambda)\, \mathrm{d}r + 2\pi - \int_0^\infty r^3 c(r)\, J_1'(2\pi)\, \mathrm{d}r\\
&= -\frac{1}{(2\pi)^2\lambda}\Gamma'(\lambda) - 2\pi \int_0^\infty r^3 c(r)\, J_1'(2\pi\lambda)\, \mathrm{d}r\,.
\end{aligned}$$

To calculate the last integral we differentiate (S17.19), thus getting

$$\Gamma''(\lambda) = -(2\pi)^3 \int_0^\infty r^3 c(r)\, J_1'(2\pi\lambda)\, \mathrm{d}r\,.$$

Hence

$$\Gamma_2(\lambda) = \frac{1}{(2\pi)^2}\left[\frac{\Gamma'(\lambda)}{\lambda} - \Gamma''(\lambda)\right] = -(2\pi\lambda)^2\pi\left[1-(2\pi\lambda)^2\right]$$
$$= -(2\pi\lambda)^2\pi P(\lambda)\ .$$

From (S17.17), we finally obtain

$$G_{r0}(\lambda) = \frac{\pi}{24}\,\mathrm{rect}\,(\pi\lambda)\,P^3(\lambda)$$
$$G_{r1}(\lambda) = \frac{\pi}{24}\,\mathrm{rect}\,(\pi\lambda)\,\left[-\mathrm{i}2\pi\lambda P^2(\lambda)(A-\mathrm{i}B)\right]\pi$$
$$G_{r2}(\lambda) = \frac{\pi}{24}\,\mathrm{rect}\,(\pi\lambda)\,24\mathrm{i}(2\pi\lambda)^2\,P(\lambda)\ .$$

10) Harmonic expansion of projections

$$g(s,\theta) = \sum_{n=-1}^{1} g_n(s)\,\mathrm{e}^{\mathrm{i}n\theta} \tag{S17.20}$$

where $g_n(s)$ is the inverse FT of $G_n(f_s)$.

Considering that

$$\mathrm{rect}\,(\pi f_s) \quad \xrightarrow{\mathcal{F}^{-1}} \quad \frac{1}{\pi}\,\mathrm{sinc}(s/\pi) = \frac{1}{\pi}\frac{\sin s}{s} \stackrel{\Delta}{=} p(s)\ ,$$

the inverse FT can be obtained by differentiation rule, namely

$$\widehat{L}_0(f_s) \quad \xrightarrow{\mathcal{F}^{-1}} \quad Q_0\left[p(s)+p''(s)\right] = g_0(s)\ ,$$
$$\widehat{L}_1(f_s) \quad \xrightarrow{\mathcal{F}^{-1}} \quad \tfrac{1}{2}\left[Q_{10}-\mathrm{i}Q_{01}\right]p'(s) = g_1(s)\ .$$

11) Projection

From (S17.20) one gets

$$g(s,\theta) = g_0(s) + 2\Re g_1(s)\,\mathrm{e}^{\mathrm{i}\theta}$$
$$= Q_0\left[p(s)+p''(s)\right] + Q_{10}p'(s)\cos\theta + Q_{01}p'(s)\sin\theta\ .$$

13) Fourier transform of the polar representation

Here we use the integrals (see [5])

$$\int_0^{+\infty} \frac{J_4(r)}{r^4} \cos(\omega r)\,\mathrm{d}r = \frac{1}{7!!}(1-\omega^2)^{7/2}\,\mathrm{rect}(\tfrac{1}{2}\omega)$$
$$\int_0^{+\infty} \frac{J_4(r)}{r^3} \sin(\omega r)\,\mathrm{d}r = \frac{1}{5!!}(1-\omega^2)^{5/2}\,\omega\,\mathrm{rect}(\tfrac{1}{2}\omega)$$
$$\int_0^{+\infty} \frac{J_4(r)}{r^4} \cos(\omega r)\,\mathrm{d}r = \frac{1}{5!!}(1-\omega^2)^{3/2}(1-6\omega^2)\,\mathrm{rect}(\tfrac{1}{2}\omega)$$

in order to obtain

$$\mathcal{L}_n(f_r) = \frac{2^8}{5}(1-(2\pi f_r)^2)^{3/2}\,\mathrm{rect}(\pi f_r) C_n(2\pi f_r)\,,$$

where

$$C_0(\omega) = \tfrac{1}{7} p_{00}(1-\omega^2)^2 + \tfrac{1}{2}(p_{20}+p_{02})(1-6\omega^2)\,,$$
$$C_1(\omega) = \tfrac{1}{2} p_{01}\omega(1-\omega^2) + \mathrm{i}\tfrac{1}{2} p_{10}\omega(1-\omega^2)\,,$$
$$C_2(\omega) = \tfrac{1}{4}(p_{20}-p_{02})(1-6\omega^2) - \mathrm{i}\tfrac{1}{4} p_{11}(1-6\omega^2)\,,$$
$$C_{-1}(\omega) = \tilde{C}_1^*(-\omega)\,, \qquad \tilde{C}_{-2}(\omega) = \tilde{C}_2^*(-\omega)\,.$$

13) Radial and angular FT of the polar representation

$$\mathcal{L}_a(r, nF_a) = 2\pi \widehat{\ell}_n(r)\,, \qquad 4\mathcal{L}_r(f_r, nF_a) = 2\pi \widehat{\mathcal{L}}_n(f_r)$$
$$\begin{aligned}\mathcal{L}_r(f_r,\varphi) &= \sum_{n=-\infty}^{+\infty} C_n(f_r)\mathrm{e}^{\mathrm{i}n\varphi} \\ &= \frac{2^8}{5}(1-(2\pi f_r)^2)^{3/2}\,\mathrm{rect}(\pi f_r)\bigg[\tfrac{1}{7} p_{00}(1-\omega^2)^2 \\ &\quad - \mathrm{i}2\pi f_r(1-(2\pi f_r)^2)(p_{10}\cos\varphi + p_{01}\sin\varphi) \\ &\quad + (1-6(2\pi f_r)^2)\left(\frac{p_{20}+p_{02}}{2} + \frac{p_{20}-p_{02}}{2}\cos 2\varphi + \frac{p_{11}}{2}\sin 2\varphi\right)\bigg]\,.\end{aligned} \tag{S17.21}$$

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