

Chapter 2

Classical Signal Theory

2.1 Continuous Signal Definitions

We begin with a formal definition:

Definition 2.1 A continuous-time signal is a complex function of a real variable $s : \mathbb{R} \rightarrow \mathbb{C}$, where the domain \mathbb{R} is the set of real numbers and the codomain \mathbb{C} is the set of complex numbers.

The signal will be denoted by $s(t)$, $t \in \mathbb{R}$, or simply by $s(t)$. The independent variable t is typically interpreted as *time*. From the historical viewpoint, *continuous-time* signals (more briefly, *continuous* signals) represent the most important class, and are the subject of several textbooks [1, 6, 8–25].

An important subclass of continuous signals is given by *real* signals, which can be defined by the relationship

$$s(t) = s^*(t), \quad (2.1)$$

where the asterisk denotes “complex conjugation”.

Another important subclass is given by *periodic* signals, characterized by the relationship

$$s(t + T_p) = s(t), \quad (2.2)$$

where the constant time $T_p > 0$ represents the *period*. Signals that do not satisfy Condition (2.2) are called *aperiodic*.

2.1.1 Signal Symmetries

A signal $s(t)$ is *even* (Fig. 2.1) if for any t

$$s(-t) = s(t), \quad (2.3a)$$

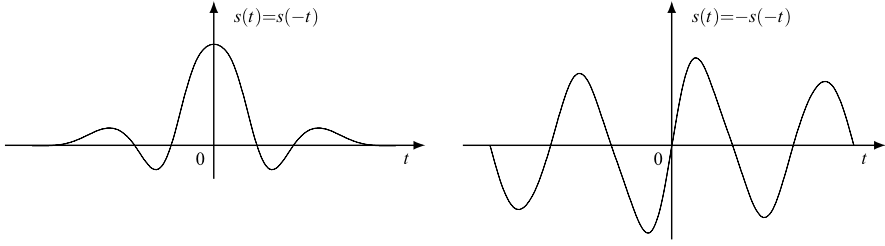


Fig. 2.1 Examples of even and odd signals

it is *odd* (Fig. 2.1) if

$$s(-t) = -s(t). \quad (2.3b)$$

An arbitrary signal can be always decomposed into the sum of an *even* component $s_e(t)$ and an *odd* component $s_o(t)$

$$s(t) = s_e(t) + s_o(t), \quad (2.4)$$

where

$$s_e(t) = \frac{1}{2}[s(t) + s(-t)], \quad s_o(t) = \frac{1}{2}[s(t) - s(-t)]. \quad (2.4a)$$

A signal is *causal* (Fig. 2.2) if it is zero for negative t ,

$$s(t) = 0 \quad \text{for } t < 0. \quad (2.5)$$

A causal signal is neither even nor odd, but can be decomposed into an even and an odd component, according to (2.4) to give $s_e(t) = s_o(t) = \frac{1}{2}s(t)$ for $t > 0$ and $s_e(t) = -s_o(t) = \frac{1}{2}s(-t)$ for $t < 0$.¹ We can link the even and odd components of a causal signal by the relationships

$$s_o(t) = \text{sgn}(t)s_e(t), \quad s_e(t) = \text{sgn}(t)s_o(t), \quad (2.6)$$

where $\text{sgn}(x)$ is the “signum” function

$$\text{sgn}(x) = \begin{cases} -1, & \text{for } x < 0; \\ 0, & \text{for } x = 0; \\ +1, & \text{for } x > 0. \end{cases} \quad (2.7)$$

¹The above relations hold for $t \neq 0$. For $t = 0$ we may have a discontinuity, as shown in Fig. 2.2. The problem of the signal value at discontinuities will be discussed below (see (2.19)).

Fig. 2.2 Decomposition of a causal signal $s(t)$ into the even part $s_e(t)$ and odd part $s_o(t)$

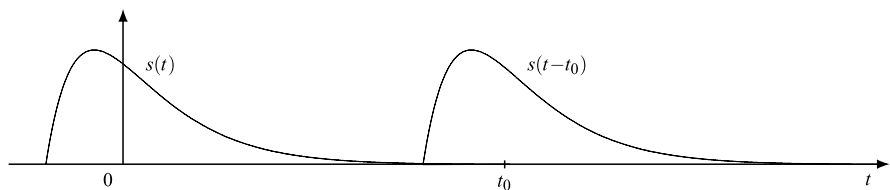
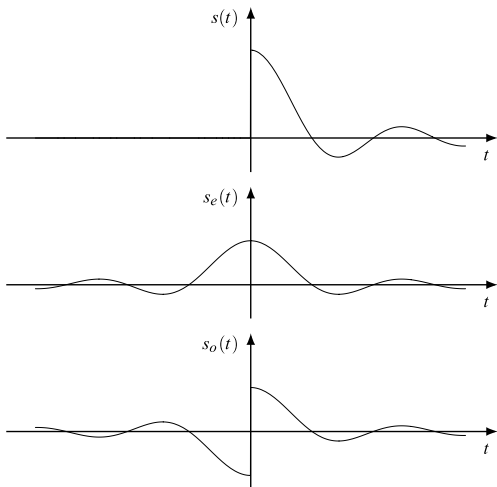


Fig. 2.3 Illustration of a t_0 -shift of a signal

2.1.2 Time-Shift

Given a signal $s(t)$ and a time value t_0 , the signal

$$s_{t_0}(t) = s(t - t_0) \quad (2.8)$$

represents a shifted version of $s(t)$, by the amount t_0 . If $t_0 > 0$ the time-shift is called a *delay* (Fig. 2.3), if $t_0 < 0$ it is called an *advance*, that is, a negative delay.

It is worth noting that to introduce a delay, e.g., of 5 units, we have to write $s(t - 5)$ and not $s(t + 5)$.

2.1.3 Area and Mean Value

The integral of a signal $s(t)$, $t \in \mathbb{R}$, extended over the whole domain \mathbb{R} is called the *area* of the signal

$$\text{area}(s) = \int_{-\infty}^{+\infty} s(t) dt. \quad (2.9)$$

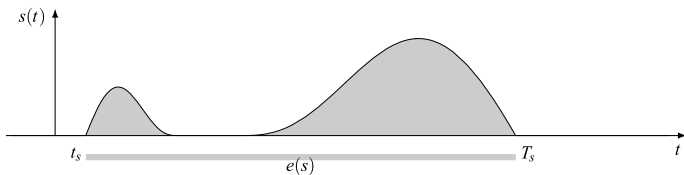


Fig. 2.4 Signal with an extension limited to the interval $[t_s, T_s]$

The limit

$$m_s = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T s(t) dt \quad (2.10)$$

is called the *mean value*. In the context of electrical circuits, m_s is called *direct current* component.

2.1.4 Energy and Power

The *specific energy*, or simply *energy*, is defined by

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt, \quad (2.11)$$

and the (specific) *power* by the limit

$$P_s = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |s(t)|^2 dt. \quad (2.12)$$

This terminology derives from the fact that, if $s(t)$ represents a voltage or a current applied to a unitary resistance, E_s equals the physical energy (in joules), while P_s equals the physical power (in watts) dissipated by the resistance.

If $0 < E_s < \infty$, then $s(t)$ is a *finite-energy* signal, and if $0 < P_s < \infty$ then $s(t)$ is a *finite-power* signal. Note that a finite-energy signal has $P_s = 0$ and a finite-power signal has $E_s = \infty$.

Typically, periodic signals have finite power and aperiodic signals have finite energy. However, some aperiodic signals, such as the step function, turn out to be finite-power signals.

2.1.5 Duration and Extension

A signal $s(t)$ that is zero-valued outside of a finite interval $[t_s, T_s]$ is called of *limited duration* and the measure of the interval is the *duration* of the signal. The interval $[t_s, T_s]$ is the *extension* of the signal and gives more information than the duration, because it indicates *where* the signal is significant (Fig. 2.4).

Definition 2.2 A set $e(s)$ such that

$$s(t) = 0, \quad t \notin e(s) \quad (2.13)$$

is the *extension* of $s(t)$ and its measure $D(s)$ is the *duration* of $s(t)$.

The above definition provides a basis for an obvious signal classification. If $e(s) = [t_s, T_s]$ is a finite interval, the signal has a *strictly-limited extension* or is *strictly time-limited*; if $e(s) = (-\infty, T_s]$ with T_s finite, the signal is *upper time-limited*, etc. In particular, the extension of *periodic* signals is always unlimited, $e(s) = (-\infty, +\infty) = \mathbb{R}$, and the extension of *causal* signals is lower time-limited with $e(s) = [0, +\infty)$.

Note that the above definitions are not stringent in the sense that duration and extension are not unique; in general, it is convenient to refer to as the smallest extension and duration (see Chap. 4).

2.1.6 Discontinuous Signals

The class of continuous-time signals includes discontinuous functions. The unit step function is a first example. In function theory, a function may be undefined at points of discontinuity, but in Signal Theory it is customary to assign a precise value at such a point. Specifically, if $s(t)$ has a discontinuity point at $t = t_0$, we assign the average value (*semi-value*)

$$s(t_0) = \frac{1}{2} [s(t_{0-}) + s(t_{0+})], \quad (2.14)$$

where $s(t_{0-})$ and $s(t_{0+})$ are the limits of $s(t)$ when t_0 is approached from the left and the right, respectively.

The reason of this convention is that, at discontinuities, the inverse Fourier transform converges to the semi-value.

2.2 Continuous Periodic Signals

Some of the general definitions given above for continuous-time signals hold for the subclass of periodic signals. This is the case for even and odd symmetries. Other definitions must be suitably modified.

It is worth stressing that in the *condition for periodicity*

$$s(t + T_p) = s(t), \quad t \in \mathbb{R}, \quad (2.15)$$

the period T_p is not unique. In fact, if T_p satisfies the condition (2.15), then also kT_p with k integer, satisfies the same condition. The smallest positive value of T_p

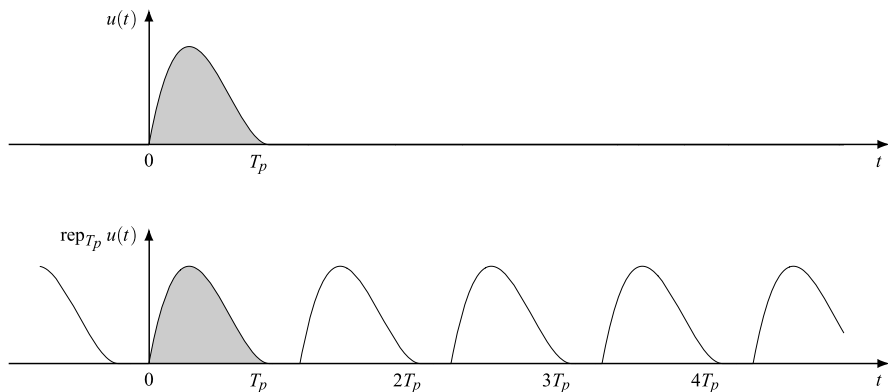


Fig. 2.5 Periodic repetition of an aperiodic signal with period T_p

will be called the *minimum period*, and a general positive value of T_p is a *period* of the signal. We also note that a periodic signal is fully identified by its behavior in a single period $[t_0, t_0 + T_p)$, where t_0 is arbitrary, since outside the period its behavior can be derived from the periodicity condition.

Note that “period” is used in two senses, as the positive real quantity T_p as well as an interval $[t_0, t_0 + T_p)$.

Periodic Repetition Sometimes a periodic signal is expressed as the *periodic repetition* of an aperiodic signal $u(t)$, $t \in \mathbb{R}$, namely (Fig. 2.5)

$$s(t) = \sum_{n=-\infty}^{+\infty} u(t - nT_p) \triangleq \text{rep}_{T_p} u(t), \quad (2.16)$$

where T_p is the *repetition period*.

Periodic repetition does not require that the signal $u(t)$ be of limited-duration as in Fig. 2.5. In general, for every $t \in \mathbb{R}$, a periodic repetition is given as a sum of a bilateral series (see Problem 2.8 for a periodic repetition in which the terms overlap, and see also Sect. 6.10).

2.2.1 Area, Mean Value, Energy and Power Over a Period

For periodic signals, the area definition given in (2.9) is not useful and is replaced by the *area over a period*

$$\text{area}(s) = \int_{t_0}^{t_0+T_p} s(t) dt. \quad (2.17a)$$

The *mean value over a period* is defined by

$$m_s(T_p) = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} s(t) dt. \quad (2.17b)$$

It can be shown that the mean value over a period equals the mean value defined as a limit by (2.10). Moreover, the periodicity property assures that both definitions (2.17a) and (2.17b) are independent of t_0 .

The definition of energy (2.11) is replaced by that of *energy over a period*

$$E_s(T_p) = \int_{t_0}^{t_0+T_p} |s(t)|^2 dt. \quad (2.18a)$$

The *mean power over a period* is defined by

$$P_s(T_p) = \frac{1}{T_p} E_s(T_p) = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} |s(t)|^2 dt. \quad (2.18b)$$

Note that the square root of $P_s(T_p)$ is known as the *root mean square* (rms) value.

2.3 Examples of Continuous Signals

We introduce the first examples of continuous signals, mainly to illustrate the usage of some functions, such as the step function and the Delta function.

2.3.1 Constant Signals

A *constant signal* has the form

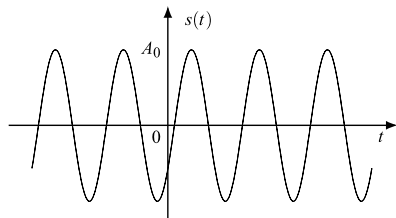
$$s(t) = A,$$

where A is a complex constant. It is even, with finite power, $P_s = |A|^2$, and with mean value A . Constant signals may be regarded as a limit case of periodic signals with an arbitrary period.

2.3.2 Sinusoidal and Exponential Signals

A *sinusoidal signal* (Fig. 2.6)

$$s(t) = A_0 \cos(\omega_0 t + \phi_0) = A_0 \cos(2\pi f_0 t + \phi_0) = A_0 \cos\left(2\pi \frac{t}{T_0} + \phi_0\right) \quad (2.19)$$

Fig. 2.6 Sinusoidal signal

is characterized by its amplitude A_0 , angular frequency ω_0 and phase ϕ_0 . Without loss of generality, we can always assume A_0 and ω_0 positive. The *angular* frequency ω_0 is related to the frequency f_0 by the relation $\omega_0 = 2\pi f_0$. Sinusoidal signals are periodic, with (minimum) period $T_0 = 1/f_0$, finite power, $P_s = \frac{1}{2}A_0^2$, and zero mean value. The signal (2.19) can be expressed as

$$s(t) = A_0 \cos \phi_0 \cos \omega_0 t - A_0 \sin \phi_0 \sin \omega_0 t,$$

which represents the decomposition into even and odd parts. By means of the very important Euler's formulas,

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i},} \quad (2.20)$$

a sinusoidal signal can also be decomposed into a sum of two *exponential signals*

$$s(t) = A_0 \cos(\omega_0 t + \phi_0) = \frac{1}{2} A_0 e^{i(\omega_0 t + \phi_0)} + \frac{1}{2} A_0 e^{-i(\omega_0 t + \phi_0)}. \quad (2.21)$$

Furthermore, it can be written as the real part of an exponential signal

$$s(t) = \Re A e^{i\omega_0 t}, \quad A = A_0 e^{i\phi_0}.$$

The *exponential signal* has the general form Ae^{pt} , where p is a complex constant. A particular relevance has the exponential signal with p imaginary, that is,

$$s(t) = A e^{i\omega_0 t} = A e^{i2\pi f_0 t}.$$

This signal is illustrated in Fig. 2.7. It has finite power $P_s = |A|^2$ and (minimum) period $1/|f_0|$. While for sinusoidal signals the frequency is commonly assumed to be positive, for exponential signals the frequencies may be negative, as well as positive.

Notation As a rule, a *real* amplitude will be denoted by A_0 , and a *complex* amplitude by A . In general, we suppose $A_0 > 0$.

Fig. 2.7 The exponential signal and its sine and cosine projections

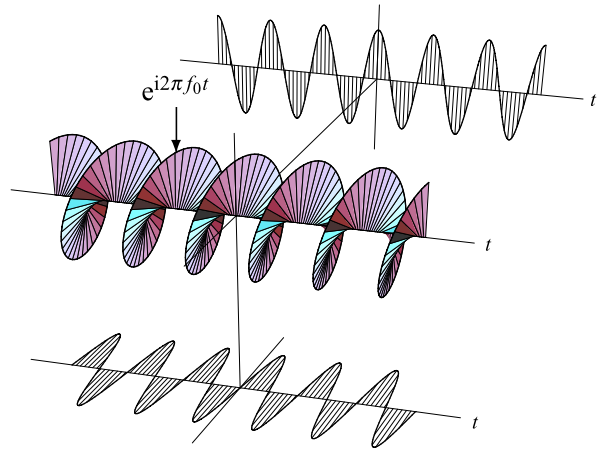
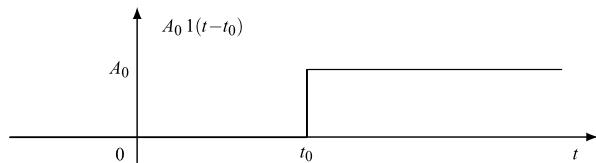


Fig. 2.8 Step signal of amplitude A_0 applied at the instant t_0



2.3.3 Step Signals

A *step signal* has the form (Fig. 2.8)

$$s(t) = A_0 1(t - t_0),$$

where $1(x)$ denotes the *unit step function*

$$1(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases} \quad (2.22)$$

It is aperiodic, with finite power $\frac{1}{2}A_0^2$ and mean value $\frac{1}{2}A_0$. Note that, by the conventions on discontinuities, $1(0) = \frac{1}{2}$ and $s(t_0) = \frac{1}{2}A_0$.

The following decomposition

$$A_0 1(t - t_0) = \frac{1}{2}A_0 + \frac{1}{2}A_0 \operatorname{sgn}(t - t_0), \quad (2.23)$$

is worth noticing, where $\operatorname{sgn}(x)$ is the *signum function*, $\frac{1}{2}A_0$ is the mean value and the last term has zero mean value.

The *unit step function* allows writing the *causal version* of a given signal $s(t)$ as

$$s_c(t) = 1(t)s(t),$$

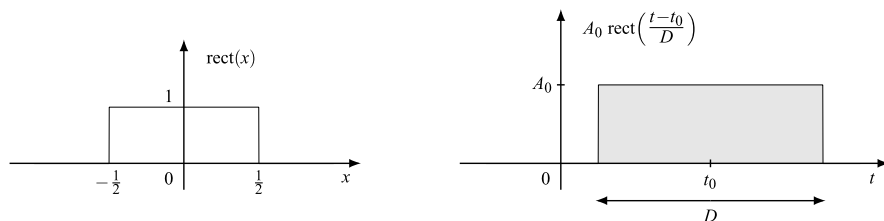


Fig. 2.9 The $\text{rect}(x)$ function and the rectangular impulse of duration D , amplitude A_0 and central instant t_0

which coincides with $s(t)$ for $t > 0$ and is zero for $t < 0$. For instance, the causal version of the linear signal $s(t) = \beta t$, with slope β , is the *ramp* signal $s_c(t) = 1(t)\beta t$.

A notable example of a causal signal is the *causal exponential*

$$s_c(t) = 1(t)e^{p_0 t} \quad (2.24)$$

where $p_0 = \sigma_0 + i\omega_0$ is a complex constant. If $\Re p_0 = \sigma_0 < 0$, this signal approaches zero as $t \rightarrow +\infty$, and has energy $1/|2\sigma_0|$; if $\sigma_0 > 0$ the signal diverges and has infinite energy.

2.3.4 Rectangular and Triangular Pulses

Using the definition

$$\text{rect}(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2}, \\ 0, & \text{for } |x| > \frac{1}{2}, \end{cases} \quad (2.25)$$

the pulse² centered at t_0 with duration D and amplitude A_0 (Fig. 2.9) can be written in the form

$$r(t) = A_0 \text{rect}\left(\frac{t - t_0}{D}\right). \quad (2.26)$$

Alternatively, we can express the pulse (2.26) as the difference between two step signals, namely

$$A_0 \text{rect}\left(\frac{t - t_0}{D}\right) = A_0 1(t - t_1) - A_0 1(t - t_2), \quad (2.27)$$

where $t_1 = t_0 - \frac{1}{2}D$ and $t_2 = t_0 + \frac{1}{2}D$. The pulse (2.26) has finite extension, $e(r) = [t_1, t_2]$, finite energy, $E_r = A_0^2 D$, and finite area, $\text{area}(r) = A_0 D$.

²Strictly speaking, a *pulse* denotes a signal of “short” duration, but more generally this term is synonymous with *aperiodic signal*.

Sometimes we shall use the *causal rect function*

$$\text{rect}_+(x) = \text{rect}\left(x - \frac{1}{2}\right) = \begin{cases} 1, & \text{for } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.28)$$

The rect functions are useful for writing concisely the *truncated versions* of a given signal $s(t)$ as $s(t) \text{rect}[(t - t_0)D]$ or $s(t) \text{rect}_+[(t - t_0)/D]$, which have extensions $(t_0 - \frac{1}{2}D, t_0 + \frac{1}{2}D)$ and $(t_0, t_0 + D)$, respectively.

A *triangular pulse* is introduced by the function

$$\text{triang}(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1; \\ 0 & \text{for } |x| > 1. \end{cases} \quad (2.29)$$

Note that $\text{triang}(x) = \text{rect}(x/2)(1 - |x|)$. The pulse $A_0 \text{triang}(t/D)$ has extension $(-D, D)$ and amplitude A_0 .

2.3.5 Impulses

Among the continuous signals, a fundamental role is played by the *delta function* or *Dirac function* $\delta(t)$. From a rigorously mathematical point of view, $\delta(t)$ is not an *ordinary function* and should be introduced as a *generalized function* in the framework of *distribution theory* [4] or of the *measure theory* [3].

On the other hand, for all practical purposes, a simple heuristic definition is adequate. Namely, $\delta(t)$ is assumed to vanish for $t \neq 0$ and satisfy the *sifting property*

$$\int_{-\infty}^{\infty} \delta(t) s(t) dt = s(0).$$

In particular, since

$$\int_{-\infty}^{\infty} \delta(t) dt = 1,$$

$\delta(t)$ may be interpreted as a signal with zero duration and unit area.

Intuitively, the Dirac function may be interpreted as a limit of a sequence of suitably chosen ordinary functions. For instance,

$$r_D(t) = \frac{1}{D} \text{rect}\left(\frac{t}{D}\right), \quad (2.30)$$

with $D > 0$, is a signal having unit area and duration D . As D tends to 0, the duration of r_D vanishes while the area maintains the unit value. Even though the limit diverges for $t = 0$, we find it useful to set

$$\delta(t) = \lim_{D \rightarrow 0} r_D(t).$$

Note that

$$\lim_{D \rightarrow 0} \int_{-\infty}^{\infty} r_D(t) s(t) dt = \lim_{D \rightarrow 0} \frac{1}{D} \int_{-D/2}^{D/2} s(t) dt = s(0),$$

so that the value of $s(t)$ at the origin is *sifted*. In conclusion, the sifting property applied to a signal $s(t)$ may be interpreted as a convenient shorthand for the following operations: (i) integrating the signal multiplied by $r_D(t)$, and (ii) evaluating the limit of the integral as $D \rightarrow 0$. Note that these limit considerations imply that

$$\delta(t) = 0 \quad \text{for } t \neq 0. \quad (2.31)$$

The choice of a rectangular pulse in the heuristic derivation is a mere mathematical convenience. Alternatively [1], we could choose a unitary area pulse $r(t)$, e.g., a triangular pulse or a Gaussian pulse, define $r_D(t) = (1/D)r(t/D)$ and apply the above operations.

In practice, we handle the delta function as an ordinary function, and indeed, it is called the *delta function* or *Dirac function*. For instance, we get

$$\boxed{\int_{-\infty}^{\infty} s(t) \delta(t - t_0) dt = \int_{-\infty}^{\infty} s(t + t_0) \delta(t) dt = s(t_0).} \quad (2.32)$$

Moreover,

$$\int_{-\infty}^{\infty} \delta(-t) s(t) dt = \int_{-\infty}^{\infty} \delta(t) s(-t) dt = s(0) = \int_{-\infty}^{\infty} \delta(t) s(t) dt,$$

and $\delta(t)$ is considered an even function. Then, (2.32) can be written in the alternative form

$$\boxed{s(t) = \int_{-\infty}^{+\infty} s(u) \delta(t - u) du.} \quad (2.33)$$

Of course, the delta function has singular properties. For instance, it allows writing a signal of zero duration and with finite area α as

$$\alpha \delta(t - t_0),$$

where t_0 is the *application instant*. In fact, from (2.31) it follows that

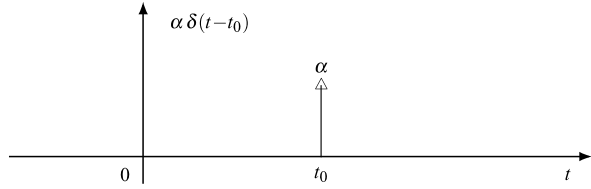
$$\alpha \delta(t - t_0) = 0 \quad \text{for } t \neq t_0,$$

so that the duration is zero and the area is

$$\int_{-\infty}^{+\infty} \alpha \delta(t - t_0) dt = \alpha \int_{-\infty}^{+\infty} \delta(t - t_0) dt = \alpha.$$

We shall use the graphical convention to represent $\alpha \delta(t - t_0)$ by a vertical arrow of length α applied at $t = t_0$ (Fig. 2.10), where the length of the arrow *does not*

Fig. 2.10 Graphical representation of the impulse of area α applied at the instant t_0



represent the amplitude of the impulse but its area. In the Unified Theory, the delta function will be called an *impulse*.

We note that the square of the delta function is undefined (even in distribution theory), so that it makes no sense to talk of energy and power of the delta function. Finally, we note that the formalism of the delta function allows writing the derivative of a discontinuous signal, for example,

$$\frac{dl(t)}{dt} = \delta(t). \quad (2.34)$$

More generally, for a discontinuity at t_0 the derivative of a signal has an impulse of area $s(t_{0+}) - s(t_{0-})$ at t_0 .

In the framework of distribution theory, also derivatives of any order of the delta function are defined, with useful applications in Signal and Control Theory. We confine us to the first derivative, in symbols

$$\delta'(t) = \frac{d\delta(t)}{dt}.$$

Formally, applying the integration by parts, we obtain the *derivative sifting property*

$$\int_{-\infty}^{\infty} \delta'(t)s(t) dt = \delta(t)s(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)s'(t) dt = -s'(0).$$

We may give a heuristic interpretation also to $\delta'(t)$ as

$$\delta'(t) = \lim_{D \rightarrow 0} u_D(t)$$

with

$$u_D(t) = \frac{1}{D^2} \left[\text{rect}\left(\frac{t + D/2}{D}\right) - \text{rect}\left(\frac{t - D/2}{D}\right) \right].$$

Indeed, it can be shown that, under mild conditions, $\lim_{D \rightarrow 0} \int_{-\infty}^{\infty} u_D(t)s(t) dt = s'(0)$.

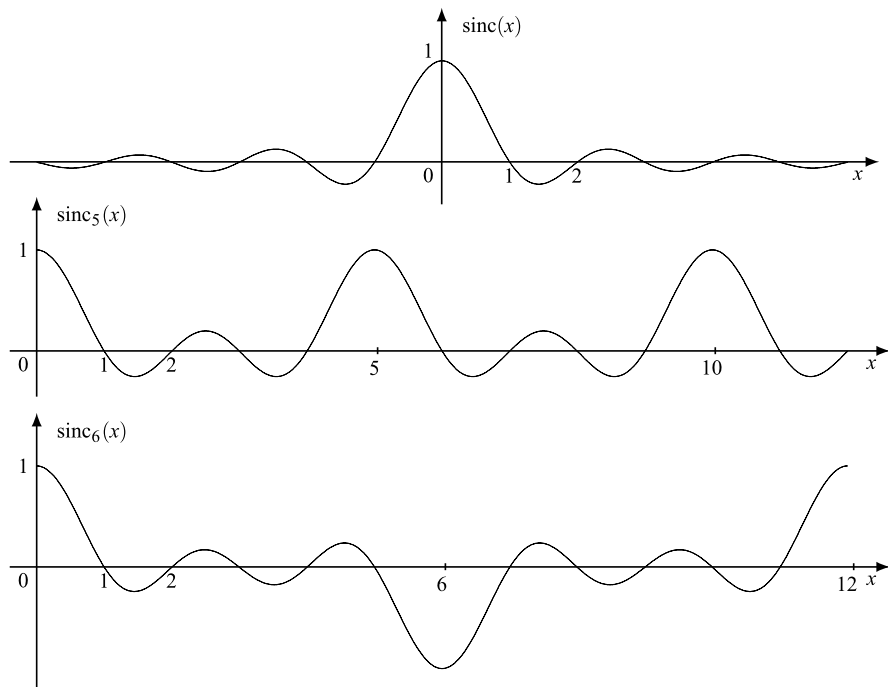


Fig. 2.11 The $\text{sinc}(x)$ function and its periodic version $\text{sinc}_N(x)$ shown for $N = 5$ and $N = 6$

2.3.6 Sinc Pulses

Sinc pulses have the form

$$A_0 \text{sinc}\left(\frac{t - t_0}{T}\right), \quad (2.35)$$

where (Fig. 2.11)

$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x} \quad (2.36)$$

and the value at $x = 0$ is $\text{sinc}(0) = 1$. The pulse (2.35) has a maximum value A_0 at $t = t_0$, it is zero at $t_0 + nT$, with $n = \pm 1, \pm 2, \dots$. It is even-symmetric about t_0 with finite energy $A_0^2 T$ and finite area $A_0 T$.

The sinc function has the *periodic* version (Fig. 2.11)

$$\text{sinc}_N(x) = \frac{1}{N} \frac{\sin \pi x}{\sin \frac{\pi x}{N}}, \quad (2.37)$$

where N is a natural number. This function has period N for N odd and period $2N$ for N even. Hence, the signal

$$s(t) = A_0 \operatorname{sinc}_N\left(\frac{t - t_0}{T}\right) \quad (2.37a)$$

has period NT for N odd and $2NT$ for N even, and similarly to the aperiodic sinc pulse (2.35) has equally spaced zeros, at intervals of length T .

Historical Note The functions sinc and rect were introduced by Woodward [7], who also introduced the symbol rep for periodic repetition. The definition (2.37) of the periodic sinc is new.

2.4 Convolution for Continuous Signals

Convolution is one of the most important operations of Signal and System Theory. It is now introduced for continuous aperiodic signals, and later for periodic signals.

2.4.1 Definition and Interpretation

Given two continuous signals $x(t)$ and $y(t)$, their convolution defines a new signal $s(t)$ according to

$$s(t) = \int_{-\infty}^{+\infty} x(u)y(t-u) du. \quad (2.38)$$

This is concisely denoted by $s = x * y$ or, more explicitly, by $s(t) = x * y(t)$ to indicate the convolution evaluated at time t . The signals $x(t)$ and $y(t)$ are called the *factors* of the convolution.

The interpretation of convolution is depicted in Fig. 2.12. We start with the two signals $x(u)$ and $y(u)$, expressed as functions of the time u . The second signal is then reversed to become $z(u) = y(-u)$, and finally shifted by a *chosen* time t to yield

$$z_t(u) = z(u-t) = y(-(u-t)) = y(t-u),$$

so that (2.38) becomes

$$s(t) = \int_{-\infty}^{+\infty} x(u)z_t(u) du. \quad (2.38a)$$

In conclusion, to evaluate the convolution *at the chosen time* t , we multiply $x(u)$ by $z_t(u)$ and integrate the product.

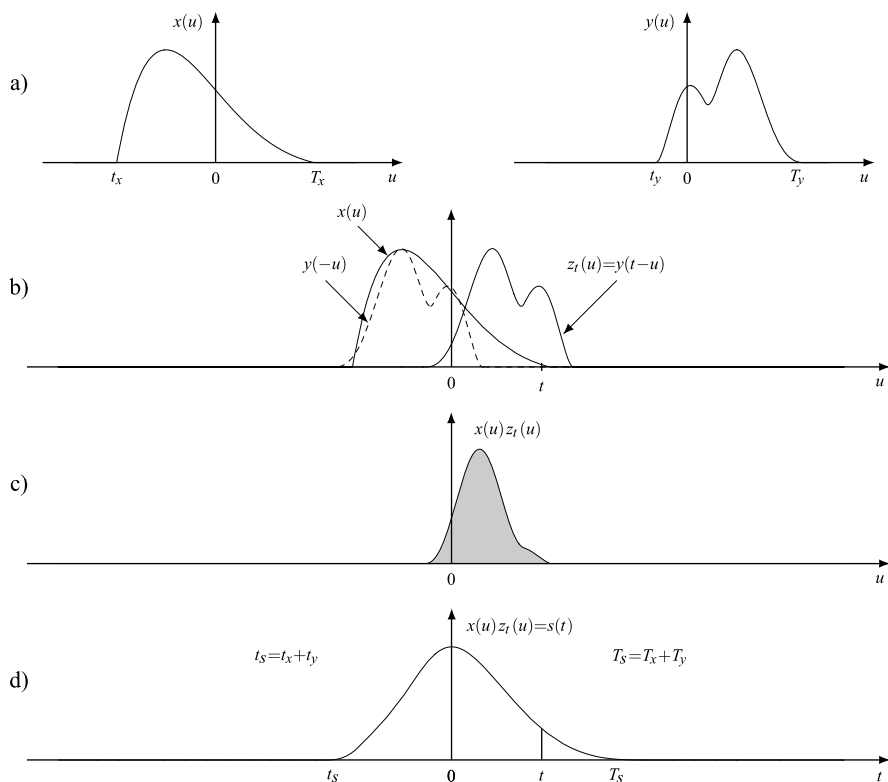


Fig. 2.12 Convolution interpretation: **(a)** signals to be convolved, **(b)** signals $x(u)$ and $z_t(u) = y(t - u)$, **(c)** product $x(u)z_t(u)$ for t fixed, **(d)** result of convolution

In this interpretation, based on (2.38), we hold the first signal while inverting and shifting the second. However, with a change of variable $v = t - u$, we obtain the alternative form

$$s(t) = \int_{-\infty}^{+\infty} x(t - u)y(u) du, \quad (2.38b)$$

in which we hold the second signal and manipulate the first to reach the same result.

Notation In the notation $x * y(t)$, the argument t represents the instant at which the convolution is evaluated; it does not represent the argument of the original signals. The notation $[x * y](t)$, used by some authors [5], is clearer, though a little clumsy, while the notation $x(t) * y(t)$ used by other authors [2] may be misleading, since it suggests interpreting the result of convolution at t as depending only on the values of the two signals at t .

Extension and Duration of the Convolution From the preceding interpretations, it follows that if both convolution *factors* are time-limited, also the convolution it-

self is time-limited. In fact, assuming that the factors have as extensions the finite intervals

$$e(x) = [t_x, T_x], \quad e(y) = [t_y, T_y],$$

then, the extension of $z(u) = y(-u)$ is $e(z) = [-T_y, -t_y]$ and after the t -shifting

$$e(z_t) = [t - T_y, t - t_y].$$

The extension of the integrand is given by the intersection of $e(x)$ and $e(z_t)$, so that (2.38a) can be rewritten in the more specific form

$$s(t) = \int_{e(x) \cap e(z_t)} x(u) z_t(u) du \quad (2.38c)$$

where the t -dependence also appears in the integration interval. If the intersection is empty, the integral is zero and $s(t) = 0$. This occurs whenever the intervals $e(x) = [t_x, T_x]$ and $e(z_t) = [t - T_y, t - t_y]$ are disjoint, and it happens for $t - t_y < t_x$ or $t - T_y > T_x$, i.e., for $t < t_x + t_y$, or $t > T_x + T_y$. Then, the convolution extension is given by the interval

$$e(x * y) = [t_x + t_y, T_x + T_y]. \quad (2.39)$$

In words, the infimum (supremum) of the convolution extension is the sum of the infima (suprema) of the factor extensions. The above rule yields for the durations

$$D(x * y) = D(x) + D(y) \quad (2.39a)$$

so that the convolution duration is given by the sum of the durations of the two factors.

Rule (2.39) is very useful in the convolution evaluation since it allows the knowledge of the extension in advance. It holds even in the limit cases; for instance, if $t_x = -\infty$, it establishes that the convolution is lower time-unlimited.

2.4.2 Convolution Properties

Commutativity We have seen that convolution operation is commutative

$$x * y(t) = y * x(t). \quad (2.40a)$$

Area If we integrate with respect to t in definition (2.38), we find

$$\int_{-\infty}^{+\infty} s(t) dt = \int_{-\infty}^{+\infty} x(t) dt \int_{-\infty}^{+\infty} y(t) dt. \quad (2.40b)$$

Recalling that the integral from $-\infty$ to $+\infty$ is the *area*, we get

$$\text{area}(x * y) = \text{area}(x) \text{area}(y). \quad (2.40c)$$

Time-Shifting By an appropriate variable changes, we can find that the convolution of the shifted signals $x(t - t_{0x})$ and $y(t - t_{0y})$ is given by

$$s(t - t_{0s}) \quad \text{with } t_{0s} = t_{0x} + t_{0y}, \quad (2.40d)$$

that is, the convolution is shifted by the sum of shifts on the factors.

Impulse The impulse has a central role in convolution. In fact, reconsidering (2.33)

$$s(t) = \int_{-\infty}^{+\infty} s(u)\delta(t - u) du \quad (2.41a)$$

and comparing it with definition (2.38), we find that the convolution of an arbitrary signal with the impulse $\delta(t)$ yields the signal itself

$$s(t) = s * \delta(t) = \delta * s(t). \quad (2.41b)$$

As we shall see better in Chap. 4, this result states that the impulse is the *unitary element* of the algebra of convolution.

2.4.3 Evaluation of Convolution and Examples

The explicit evaluation of a convolution may not be easy and must be appropriately organized. The first step is a choice between the two alternatives

$$s(t) = \int_{-\infty}^{+\infty} x(u)y(t - u) du = \int_{-\infty}^{+\infty} y(u)x(t - u) du$$

and, whenever convenient, we can use the rules stated above. In particular, the rule on the extension can be written more specifically in the forms (see (2.38c))

$$s(t) = \int_{e_t} x(u)y(t - u) du, \quad e_t = [t_x, T_x] \cap [t - T_y, t - t_y], \quad (2.42a)$$

$$s(t) = \int_{e_t} x(t - u)y(u) du, \quad e_t = [t - T_x, t - t_x] \cap [t_y, T_y]. \quad (2.42b)$$

Example 2.1 We want to evaluate the convolution of the rectangular pulses (Fig. 2.13)

$$x(t) = A_1 \operatorname{rect}\left(\frac{t}{4D}\right), \quad y(t) = A_2 \operatorname{rect}\left(\frac{t}{2D}\right).$$

Since $e(x) = (-2D, 2D)$ and $e(y) = (-D, D)$ we know in advance that

$$e(s) = (-3D, 3D)$$

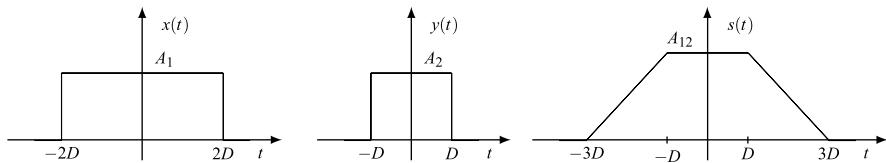


Fig. 2.13 Convolution $s(t) = x * y(t)$ of two rectangular pulses of duration $4D$ and $2D$; the trapezium amplitude is $A_{12} = 2DA_1A_2$

so we limit the evaluation to this interval.

Since the duration of the second pulse is less than the duration of the first one, it is convenient to hold the first while operating on the second. Using (2.42a) and considering that both the pulses are constant within their extensions, we find

$$s(t) = \int_{e_t} A_1 A_2 du = A_1 A_2 \text{meas } e_t$$

where $e_t = (-2D, 2D) \cap (t - D, t + D)$. Then, we have to find the intersection e_t for any t and the corresponding measure. The result is

$$e_t = \begin{cases} \emptyset, & \text{if } t < -3D \text{ or } t > 3D; \\ (-3D, t), & \text{if } -3D < t < -D; \\ (t - D, t + D), & \text{if } -D < t < D; \\ (t, 3D), & \text{if } D < t < 3D; \end{cases}$$

and then

$$s(t) = \begin{cases} 0, & \text{if } t < -3D \text{ or } t > 3D; \\ A_1 A_2 (t + 3D), & \text{if } -3D < t < -D; \\ A_1 A_2 2D, & \text{if } -D < t < D; \\ A_1 A_2 (3D - t), & \text{if } D < t < 3D. \end{cases} \quad (2.43)$$

In conclusion, the convolution of the rectangular pulses has an isosceles trapezoidal form, as illustrated in Fig. 2.13.

In (2.43), we have not specified the convolution values at the connection instants $t = \pm D$ and $t = \pm 3D$. Reconsidering the evaluation details, we find that in the four lines of (2.43) the open intervals can be replaced by closed intervals. Hence, the convolution $s(t)$ turns out to be a continuous function.

Example 2.2 We evaluate the convolution of the signals (Fig. 2.14)

$$x(t) = A_0 \text{rect}\left(\frac{t}{2D}\right), \quad y(t) = 1(t).$$

Since $e(x) = (-D, D)$ and $e(y) = (0, +\infty)$, it follows that $e(x * y) = (-D, +\infty)$. We note that in general the convolution of an arbitrary signal $x(t)$

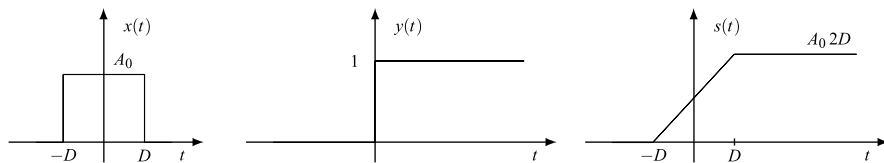


Fig. 2.14 Convolution $s(t) = x * y(t)$ of a rectangular pulse with the step signal

with the unit step signal $1(t)$ is given by the integral of $x(t)$ from $-\infty$ to t ,

$$s(t) = \int_{-\infty}^{+\infty} x(u)1(t-u) du = \int_{-\infty}^t x(u) du,$$

as soon as we take into account that $1(t-u) = 0$ for $u < t$.

In our specific case, we find

$$s(t) = \begin{cases} 0, & \text{if } t < -D; \\ A_0(t+D), & \text{if } -D < t < D; \\ A_0 2D, & \text{if } t > D, \end{cases}$$

which is similar to the step signal, but with a linear roll-off from $-D$ to D .

Example 2.3 We evaluate the convolution of the signals

$$x(t) = A_1 \operatorname{rect}\left(\frac{t}{2D}\right), \quad y(t) = A_2 \cos \omega_0 t.$$

Since $e(y) = (-\infty, +\infty)$, from the rule on extension it follows that also the convolution has the infinite extension $(-\infty, +\infty)$. Holding the second signal, we find

$$\begin{aligned} s(t) &= \int_{-\infty}^{+\infty} y(u)x(t-u) du = \int_{t-D}^{t+D} A_1 A_2 \cos \omega_0 u du \\ &= \frac{A_1 A_2}{\omega_0} [\sin \omega_0(t+D) - \sin \omega_0(t-D)] \\ &= 2 \frac{A_1 A_2}{\omega_0} \sin \omega_0 D \cos \omega_0 t. \end{aligned}$$

Hence, the convolution is a sinusoidal signal with the same frequency as $y(t)$.

2.4.4 Convolution for Periodic Signals

The convolution defined by (2.38) is typically used for *aperiodic* signals, but one of the signals to be convolved may be periodic. If this is the case, the convolution turns

out to be periodic with the same period as the periodic factor. When both signals are periodic, the integral in (2.38) may not exist and a specific definition must be issued.

The convolution of two periodic signals $x(t)$ and $y(t)$ with the same period T_p is then defined as

$$x * y(t) \triangleq \int_{t_0}^{t_0+T_p} x(u)y(t-u) du. \quad (2.44)$$

where the integral is over an arbitrary period $(t_0, t_0 + T_p)$. This form is sometimes called the *cyclic convolution* and then the previous form the *acyclic convolution*.

We can easily check that the periodic signal $s(t) = x * y(t)$ is independent of t_0 and has the same period T_p as the two factors. Moreover, the cyclic convolution has the same properties as the acyclic convolution, *provided that the results are interpreted within the class of periodic signals*. For instance, the area rule (2.40c) still holds provided that areas are interpreted as the integrals over a period (see (2.17a), (2.17b)).

2.5 The Fourier Series

In this section, continuous-time signals are examined in the *frequency domain*. The tool is given by the *Fourier series* for periodic signals and the *Fourier integral* for aperiodic signals.

We recall that in 1822 Joseph Fourier proved that an arbitrary (real) function of a real variable $s(t)$, $t \in \mathbb{R}$, having period T_p , can be expressed as the sum of a series of sine and cosine functions with frequencies multiple of the *fundamental frequency* $F = 1/T_p$, namely

$$s(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos 2\pi k F t + B_k \sin 2\pi k F t]. \quad (2.45)$$

This is the *Fourier series expansion*, which represents a periodic function by means of the coefficients A_k and B_k . In modern Signal Theory, the popular form of the Fourier series is the expansion into *exponentials*, equivalent to the sine–cosine expansion, but more compact and tractable.

2.5.1 The Exponential Form of Fourier Series

A continuous signal $s(t)$, $t \in \mathbb{R}$, with period T_p , can be represented by the *Fourier series*

$$s(t) = \sum_{n=-\infty}^{\infty} S_n e^{i2\pi n F t}, \quad F = \frac{1}{T_p}, \quad (2.46a)$$

where the *Fourier coefficients* S_n are given by

$$S_n = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} s(t) e^{-i2\pi n F t} dt, \quad n \in \mathbb{Z}. \quad (2.46b)$$

These relationships follow from the orthogonality of exponential functions, namely

$$\frac{1}{T_p} \int_{t_0}^{t_0+T_p} e^{i2\pi n F t} e^{-i2\pi m F t} dt = \delta_{mn} \quad (2.47)$$

where δ_{mn} is the Kronecker symbol ($\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ for $m \neq n$). Hence, (2.46a) is an orthogonal function expansion of the given signal in an arbitrary period $(t_0, t_0 + T_p)$. It represents the signal $s(t)$ as a sum of exponential components with frequencies being multiples of the fundamental frequency

$$f_n = nF, \quad n = 0, \pm 1, \pm 2, \dots$$

In the general case of a complex signal $s(t)$, the coefficients S_n have no symmetries. When the signal $s(t)$ is *real*, the coefficients have the *Hermitian symmetry*, namely

$$S_{-n} = S_n^* \quad (2.48)$$

and the signal identification can be limited to the Fourier coefficients S_n with $n \geq 0$.

If we let $S_n = R_n + iX_n$, the Hermitian symmetry (2.48) yields the two conditions

$$R_{-n} = R_n, \quad X_{-n} = -X_n,$$

which state that the real part is an even function (of the integer variable n) and the imaginary part is an odd function. These symmetries are illustrated in Fig. 2.15. The same symmetries hold respectively for the modulus and for the argument of the Fourier coefficients of a real signal.

Continuing with real signals, from the exponential form (2.48) the Hermitian symmetry allows obtaining the sine–cosine form (2.45) (where a real signal is tacitly assumed)

$$s(t) = R_0 + 2 \sum_{n=1}^{\infty} [R_n \cos 2\pi n F t - X_n \sin 2\pi n F t]. \quad (2.49a)$$

We can also obtain a form with only cosine terms but with appropriate phases in their arguments, namely

$$s(t) = S_0 + 2 \sum_{n=1}^{\infty} |S_n| \cos(2\pi n F t + \arg S_n). \quad (2.49b)$$

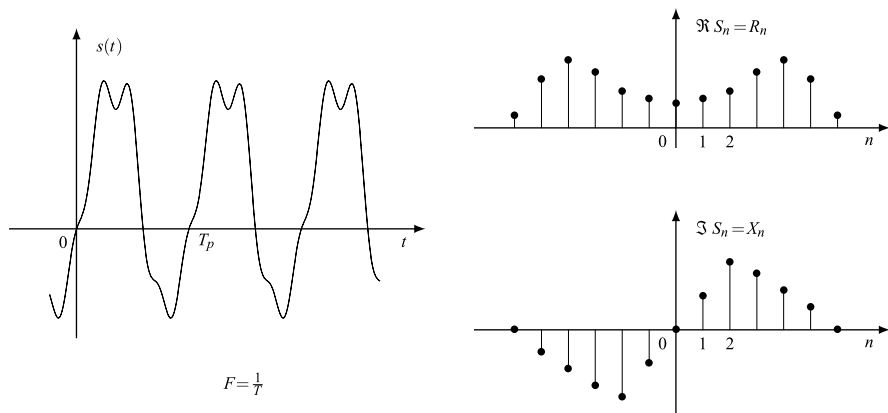


Fig. 2.15 Representation of Fourier coefficients of a *real* periodic signal illustrated by real and imaginary parts

Presence of Negative Frequencies In the exponential form, we find terms with *negative frequencies*. It is worth explaining this assertion clearly. To be concrete, let us assume that the periodic signal under consideration be the model of an electrical voltage $v(t)$. Since this signal is real, we can apply series expansion (2.49b), i.e.,

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(2\pi n F t + \varphi_n)$$

where all terms have *positive frequencies* (the constant V_0 can be regarded as a term with zero frequency). These terms, with positive frequencies nF , have a direct connection with the physical world and, indeed, they can be separated and measured by a filter-bank.

The presence of negative frequencies, related to exponentials, is merely a mathematical artifact provided by Euler's formulas (2.19), which yields

$$V_n \cos(2\pi n F t + \varphi_n) = \frac{1}{2} V_n e^{i\varphi_n} e^{i2\pi n F t} + \frac{1}{2} V_n e^{-i\varphi_n} e^{-i2\pi n F t}.$$

2.5.2 Properties of the Fourier Series

Fourier series has several properties (or rules) which represent so many theorems and will be considered systematically in Chap. 5 with the unified Fourier transform (which gives the Fourier series as a particularization). Here, we consider only a few of them.

- Let $s(t)$ be a periodic signal and $x(t) = s(t - t_0)$ a shifted version. Then, the relationship between the Fourier coefficients is

$$X_n = S_n e^{-i2\pi n F t_0}. \quad (2.50)$$

As a check, when t_0 is a multiple of the period $T_p = 1/F$, we find $x(t) = s(t)$, and indeed (2.50) yields $X_n = S_n$.

- The mean value in a period is the zeroth coefficient

$$m_s(T_p) = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} s(t) dt = S_0.$$

- The power given by (2.12) can be obtained from the Fourier coefficients as follows (Parseval's theorem)

$$P_s = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} |s(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |S_n|^2. \quad (2.51)$$

In particular, for a real signal, considering the Hermitian symmetry (2.48), Parseval's theorem becomes

$$P_s = S_0^2 + 2 \sum_{n=1}^{\infty} |S_n|^2. \quad (2.51a)$$

2.5.3 Examples of Fourier Series Expansion

We consider a few examples. The related problems are:

1. Given a periodic signal $s(t)$, evaluate its Fourier coefficients S_n , i.e., evaluate the integral (2.46b) for any n ;
2. Given the Fourier coefficients S_n , evaluate the sum of series (2.46a), to find $s(t)$.

Problem 1 is often trivial, whereas the inverse problem 2 may be difficult.

Example 2.4 Let

$$s(t) = A_0 \cos(2\pi f_0 t + \varphi_0)$$

with A_0 and f_0 positive. Letting $F = f_0$ and using Euler's formulas, we get

$$s(t) = \frac{1}{2} A_0 e^{i\varphi_0} e^{i2\pi F t} + \frac{1}{2} A_0 e^{-i\varphi_0} e^{-i2\pi F t}.$$

Then, comparison with (2.46a) (by the uniqueness of Fourier coefficients) yields:

$$S_1 = \frac{1}{2} A_0 e^{i\varphi_0}, \quad S_{-1} = \frac{1}{2} A_0 e^{-i\varphi_0}, \quad S_n = 0 \quad \text{for } |n| \neq 1.$$

Example 2.5 A periodic signal consisting of equally-spaced rectangular pulses can be written in the form

$$s(t) = \sum_{n=-\infty}^{+\infty} A_0 \operatorname{rect}\left(\frac{t - nT_p}{dT_p}\right) = A_0 \operatorname{rep}_{T_p} \operatorname{rect}\left(\frac{t}{dT_p}\right), \quad 0 < d \leq 1$$

where d is the pulse duration normalized to the period (d is called the *duty cycle*). Considering that in the interval $(-\frac{1}{2}T_p, \frac{1}{2}T_p)$ the signal $s(t)$ is given by the zeroth term of the periodic repetition,

$$s(t) = A_0 \operatorname{rect}\left(\frac{t}{dT_p}\right), \quad -\frac{1}{2}T_p < t < \frac{1}{2}T_p,$$

we get

$$S_n = \frac{1}{T_p} \int_{-\frac{1}{2}dT_p}^{\frac{1}{2}dT_p} A_0 e^{-i2\pi n F t} dt.$$

This integral can be expressed by the sinc function (2.36), namely

$$S_n = S_0 \operatorname{sinc}(nd), \quad S_0 = A_0 d. \quad (2.52)$$

As a check, for $d = 1$ all the Fourier coefficients are zero for $n \neq 0$, and indeed $s(t)$ becomes a constant signal.

As an opposite limit case, suppose that the duty cycle d tends to zero, but holding the mean value at the fixed value $S_0 = A_0 d$. Then, at the limit each rectangular pulse becomes a delta function of area $S_0 T_p$, that is,

$$s(t) = \sum_{n=-\infty}^{+\infty} T_p S_0 \delta(t - nT_p) = T_p S_0 \operatorname{rep}_{T_p} \delta(t).$$

Then, all the Fourier coefficients S_n are equal to S_0 . The interpretation is that a “train” of delta functions has all the “harmonics” with the same amplitude S_0 . From this result, follows the remarkable identity

$$\sum_{n=-\infty}^{+\infty} e^{i2\pi n F t} = T_p \sum_{n=-\infty}^{+\infty} \delta(t - nT_p), \quad F = \frac{1}{T_p}. \quad (2.53)$$

Example 2.6 We want to find the signal $s(t)$ whose Fourier coefficients are given by

$$S_n = \begin{cases} A_0 & \text{for } |n| \leq n_0; \\ 0 & \text{for } |n| > n_0, \end{cases}$$

i.e., the signal that has only the first n_0 harmonics with the same amplitude.

From (2.46a) we get

$$s(t) = A_0 + A_0 \sum_{n=1}^{n_0} (e^{i2\pi n F t} + e^{-i2\pi n F t}) = A_0 + 2A_0 \sum_{n=1}^{n_0} \cos 2\pi n F t.$$

An alternative expression is obtained by letting $z = e^{i2\pi Ft}$ and noticing that

$$\sum_{n=1}^{n_0} e^{i2\pi n Ft} = \sum_{n=1}^{n_0} z^n = \frac{z(1 - z^{n_0})}{1 - z}.$$

Hence

$$\begin{aligned} s(t) &= A_0 \left[1 + \frac{z(1 - z^{n_0})}{1 - z} + \frac{z^{-1}(1 - z^{-n_0})}{1 - z^{-1}} \right] \\ &= A_0 \frac{z^{n_0 + \frac{1}{2}} - z^{-(n_0 + \frac{1}{2})}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} = A_0 \frac{\sin 2\pi(n_0 + \frac{1}{2})Ft}{\sin 2\pi \frac{1}{2}Ft}. \end{aligned}$$

The last term compared with definition (2.37) of the periodic sinc can be written in the form:

$$s(t) = A_0 N \operatorname{sinc}_N(NFt), \quad N = 2n_0 + 1.$$

Thus, we have stated the following identity

$$1 + 2 \sum_{n=1}^{n_0} \cos 2\pi n Ft = N \operatorname{sinc}_N(NFt), \quad N = 2n_0 + 1. \quad (2.54)$$

2.6 The Fourier Transform

An aperiodic signal $s(t)$, $t \in \mathbb{R}$, can be represented by the *Fourier integral*

$$s(t) = \int_{-\infty}^{+\infty} S(f) e^{i2\pi ft} df, \quad t \in \mathbb{R}, \quad (2.55a)$$

where the function $S(f)$ is evaluated from the signal as

$$S(f) = \int_{-\infty}^{+\infty} s(t) e^{-i2\pi ft} dt, \quad f \in \mathbb{R}. \quad (2.55b)$$

These relationships allow the passage from the time domain to the frequency domain, and vice versa. The function $S(f)$ is the *Fourier transform* (FT) of the signal $s(t)$, and the signal $s(t)$, when written in the form (2.55a), is the *inverse Fourier transform* of $S(f)$. Concisely, we write $S(f) = \mathcal{F}[s \mid f]$ and $s(t) = \mathcal{F}^{-1}[S \mid t]$ where \mathcal{F} and \mathcal{F}^{-1} are the operators defined by (2.55a, 2.55b). We also use the notation

$$s(t) \xrightarrow{\mathcal{F}} S(f), \quad S(f) \xrightarrow{\mathcal{F}^{-1}} s(t).$$

The above relationships can be established heuristically with a limit consideration from the Fourier series. With reference to (2.46a), (2.46b), we limit the given

aperiodic signal to the interval $(-\frac{1}{2}T_p, \frac{1}{2}T_p)$ and we repeat it periodically outside, then we take the limit with $T_p \rightarrow \infty$. From a mathematical viewpoint, the conditions on the existence of the Fourier transform and its inverse are formulated in several ways, often having no easy interpretation [2, 6]. A sufficient condition is that the signal be absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} |s(t)| dt < \infty,$$

but this condition is too much stringent for Signal Theory where a very broad class of signals is involved, including “singular” signals as impulses, constant signals and periodic signals.

2.6.1 Interpretation

In the Fourier series, a *continuous-time* periodic signal is represented by a *discrete-frequency* function $S_n = S(nF)$. In the FT, this is no more true and we find a symmetry between the time domain and the frequency domain, which are both continuous. In (2.55a), a signal is represented as the sum of infinitely many exponential functions of the form

$$[S(f) df] e^{i2\pi f t}, \quad f \in \mathbb{R} \quad (2.56)$$

with frequency f and infinitesimal amplitude $S(f) df$.

In general, for a complex signal $s(t)$ the FT $S(f)$ has no peculiar symmetries. For a *real* signal, similarly to (2.48), we find that the Fourier transform has the *Hermitian symmetry*

$$S(-f) = S^*(f), \quad (2.57)$$

so that the portion of $S(f)$ for $f \geq 0$ completely specifies the signal. Letting

$$S(f) = R(f) + iX(f) = A_S(f)e^{i\beta_S(f)},$$

from (2.57) we find

$$R(f) = R(-f), \quad X(f) = -X(-f), \quad (2.57a)$$

which states that the real part of the FT is *even* and the imaginary part is *odd*. Analogously, we find for the modulus and the argument

$$A_S(f) = A_S(-f), \quad \beta_S(f) = -\beta_S(-f). \quad (2.57b)$$

These symmetries are illustrated in Fig. 2.16.

Continuing with the assumption of a *real signal*, the decomposition (2.55a) with both positive and negative frequencies can be set into a form with cosinusoidal terms

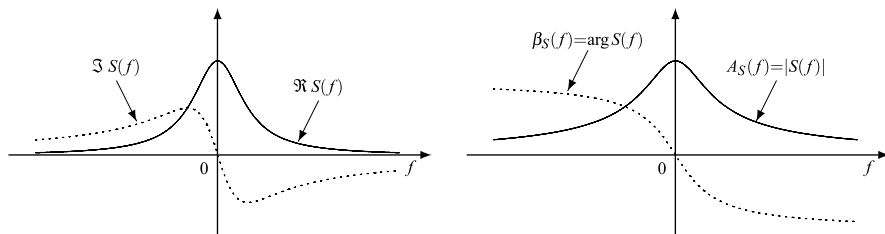


Fig. 2.16 Representation of the Fourier transform of a *real* signal $s(t)$, $t \in \mathbb{R}$, by real and imaginary parts, and by modulus and argument

with positive frequencies. In fact, by pairing the exponential terms (2.56) at frequency f with the terms at frequency $-f$, we get

$$\begin{aligned}
 & [S(f) df] e^{i2\pi ft} + [S(-f) df] e^{-i2\pi ft} \\
 &= [S(f) f] e^{i2\pi ft} + [S^*(f) df] e^{-i2\pi ft} = 2\Re\{[S(f) df] e^{i2\pi ft}\} \\
 &= 2[A_S(f) df] \cos(2\pi ft + \beta_S(f)), \quad f > 0.
 \end{aligned}$$

Hence, (2.55a) becomes

$$s(t) = \int_0^\infty 2A_S(f) \cos(2\pi ft + \beta_S(f)) df. \quad (2.58)$$

2.6.2 Properties of the Fourier Transform

The properties (or rules) of the FT will be seen in a unified form in Chap. 5 and in a specific form for continuous-time signals in Chap. 9. Here, we see the main rules. The formulation is simpler than with the Fourier series for the perfect symmetry between time and frequency domains.

- The *time-shifted* version $s(t - t_0)$ of a signal $s(t)$ gives the following FT pair

$$s(t - t_0) \xrightarrow{\mathcal{F}} S(f) e^{-i2\pi f t_0}. \quad (2.59a)$$

Symmetrically, the inverse FT of the *frequency-shifted* version $S(f - f_0)$ of $S(f)$ gives

$$S(f - f_0) \xrightarrow{\mathcal{F}^{-1}} s(t) e^{i2\pi f_0 t}. \quad (2.59b)$$

- The *convolution* $x * y(t)$ becomes the *product* for the FTs

$$x * y(t) \xrightarrow{\mathcal{F}} X(f) Y(f). \quad (2.60a)$$

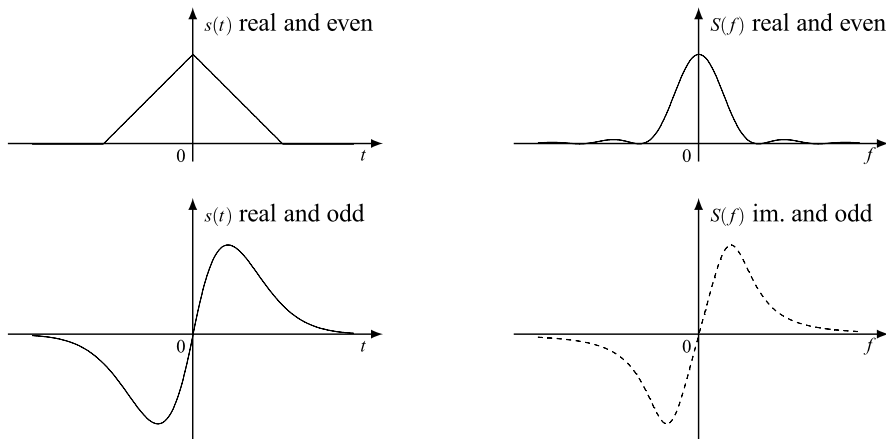


Fig. 2.17 Symmetric signals and the corresponding Fourier transforms

Symmetrically, the *products* $s(t) = x(t)y(t)$ becomes the *convolution*

$$S(f) = X * Y(f) = \int_{-\infty}^{+\infty} X(\lambda)Y(f - \lambda) d\lambda, \quad (2.60b)$$

where the operation is interpreted according to definition (2.38) for aperiodic continuous-argument functions, since $X(f)$ and $Y(f)$ belong to this class.

- Letting $t = 0$ and $f = 0$ in definitions (2.55a) and (2.55b), respectively, we get

$$s(0) = \int_{-\infty}^{+\infty} S(f) df = \text{area}(S), \quad S(0) = \int_{-\infty}^{+\infty} s(t) dt = \text{area}(s). \quad (2.61)$$

Hence, the signal area equals the FT evaluated at $f = 0$.

- The energy E_s of a signal $s(t)$, defined by (2.11), can be evaluated from the FT $S(f)$ as follows (Parseval Theorem):

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt = \int_{-\infty}^{+\infty} |S(f)|^2 df. \quad (2.62)$$

For a real signal, $|S(f)|$ is an even function of f , and the energy evaluation can be limited to positive frequencies, namely

$$E_s = \int_{-\infty}^{+\infty} s(t)^2 dt = 2 \int_0^{\infty} |S(f)|^2 df.$$

However, we note the perfect symmetry of (2.62), which emphasizes the opportunity to deal with complex signals.

- As seen above, the symmetry $s(t) = s^*(t)$ (real signal) yields the Hermitian symmetry, $S(f) = S^*(-f)$. Moreover, see Fig. 2.17,

1. If the signal is *real* and *even*, the FT is *real* and *even*;
2. If the signal is *real* and *odd*, the FT is *imaginary* and *odd*.

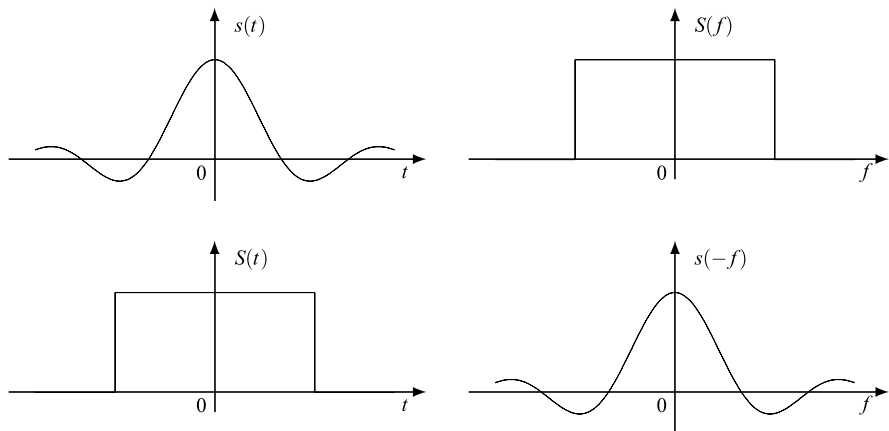


Fig. 2.18 Symmetry rule. If signal $s(t)$ has Fourier transform $S(f)$, then the signal $S(t)$ has transform $s(-f)$. In this specific case, $s(t)$ and $S(f)$ are even and $s(f) = s(-f)$

2.6.3 Symmetry Rule

Formulas (2.55a) and (2.55b), which express the signal and the FT, have a symmetrical structure, apart from a sign change in the exponential. This leads to the *symmetry rule*: If the FT of a signal $s(t)$ is $S(f)$, then, interpreting the FT as a signal $S(t)$, one obtains that the FT is $s(-f)$ (Fig. 2.18).

The symmetry rule is very useful in the evaluation, since, starting from the Fourier pair $(s(t), S(f))$, we get that also $(S(t), s(-f))$ is a consistent Fourier pair. The symmetry rule explains also the *symmetries* between the rules of the FT.

2.6.4 Band and Bandwidth of a Signal

In the time-domain, we have introduced the *extension* $e(s)$ and the *duration* $D(s) = \text{meas } e(s)$ of a signal. Symmetrically, in the frequency domain, we introduce the *spectral extension* $\mathcal{E}(s) = e(S)$, defined as the extension of the FT, and the *bandwidth*, defined as the measure of $\mathcal{E}(s)$:

$$B(s) = \text{meas } \mathcal{E}(s) = \text{meas } e(S). \quad (2.63)$$

Then, the property of the spectral extension is

$$\boxed{S(f) = 0, \quad f \notin \mathcal{E}(s).} \quad (2.63a)$$

For *real* signals, the Hermitian symmetry, $S(f) = S^*(-f)$, implies that the minimal extension \mathcal{E}_0 is symmetric with respect to the frequency origin and it will be

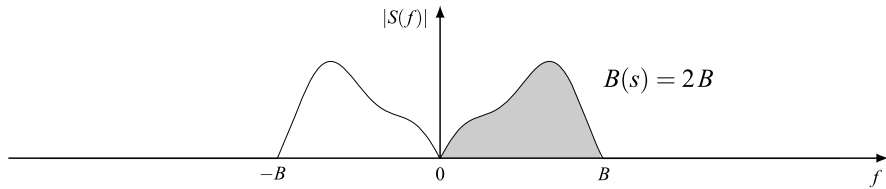


Fig. 2.19 Examples of limited spectral extension of a real signal; B represents the *band*

convenient to make such a choice also for an arbitrary extension $\mathcal{E}(s)$. Then, for a real *band-limited* signal we indicate the spectral extension in the form (Fig. 2.19):

$$e(S) = [-B, B]$$

for a finite frequency B , which is called the *band*³ of $s(t)$.

The first consequence of band limitation relies on the decomposition of a real signal into *sinusoidal* components (see (2.58)), i.e., $|S(f) df| \cos(2\pi ft + \arg S(f))$, $f > 0$, where $S(f) = 0$ for $f > B$, that is, the signal does not contain components with frequencies f greater than B . The second consequence will be seen with the *Sampling Theorem* at the end of the chapter.

2.7 Examples of Fourier Transforms

We develop a few examples of FTs. Note that the FT of some “singular” signals, as step signals and sinusoidal signals, can be written using the delta function, and should be interpreted in the framework of *distribution theory*.

2.7.1 Rectangular and Sinc Pulses

The FT of the rectangular pulse can be calculated directly from definition (2.55b), which yields

$$S(f) = A_0 \int_{-\frac{1}{2}D}^{\frac{1}{2}D} e^{-i2\pi ft} dt = \frac{A_0}{-i2\pi f} (e^{-i\pi f D} - e^{i\pi f D}) = A_0 \frac{\sin \pi f D}{\pi f}.$$

Then, using the sinc function,

$$A_0 \text{rect}(t/D) \xrightarrow{\mathcal{F}} A_0 D \text{sinc}(f D). \quad (2.64a)$$

³For real signals, it is customary to call as the *band* the half of the spectral extension measure.

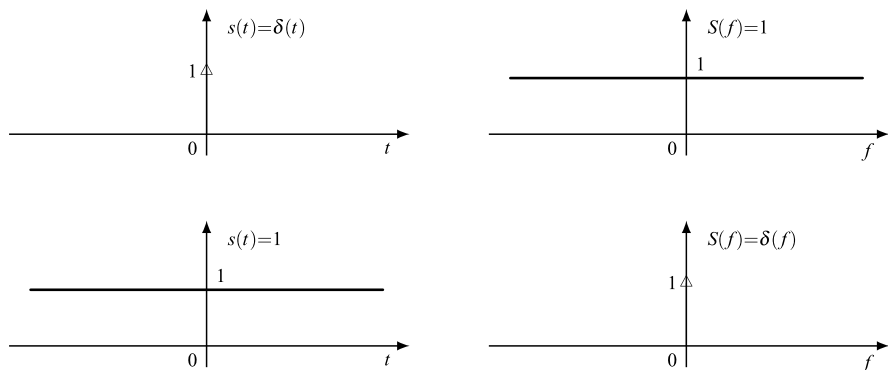


Fig. 2.20 Fourier transforms of the impulse and of the unit signal

In the direct evaluation of the FT of the sinc pulse, we encounter a difficult integral, instead we can apply the symmetry rule to the pair $(s(t), S(f))$ just evaluated. We get

$$S(t) = A_0 D \operatorname{sinc}(t D) \xrightarrow{\mathcal{F}} s(-f) = A_0 \operatorname{rect}(-f/D),$$

which is more conveniently written using the evenness of the rect function and making the substitutions $D \rightarrow 1/T$ and $A_0 D \rightarrow A_0$. Hence

$$A_0 \operatorname{sinc}(t/T) \xrightarrow{\mathcal{F}} A_0 T \operatorname{rect}(fT). \quad (2.64b)$$

This FT pair has been illustrated in Fig. 2.18 in connection with the symmetry rule.

2.7.2 Impulses and Constant Signals

The technique for the FT evaluation of the impulse $s(t) = \delta(t - t_0)$ is the usage of the sifting property (2.32) in definition (2.55b), namely

$$S(f) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-i2\pi f t} dt = e^{-i2\pi f t_0}.$$

Hence

$$\delta(t - t_0) \xrightarrow{\mathcal{F}} e^{-i2\pi f t_0} \quad (2.65)$$

and particularly for $t_0 = 0$

$$\delta(t) \xrightarrow{\mathcal{F}} 1, \quad (2.65a)$$

that is, the FT of the impulse centered at the origin is unitary (Fig. 2.20).

Note that the sifting property (2.32) holds also in the frequency domain, namely

$$\int_{-\infty}^{+\infty} X(f) \delta(f - f_0) df = X(f_0),$$

where $X(f)$ is an arbitrary frequency function. Then, with $X(f) = \exp(i2\pi ft)$ we find

$$\int_{-\infty}^{+\infty} \delta(f - f_0) e^{i2\pi ft} df = e^{i2\pi f_0 t}.$$

Hence, considering the uniqueness of the Fourier transform,

$$e^{i2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f - f_0). \quad (2.66)$$

In particular, for $f_0 = 0$

$$1 \xrightarrow{\mathcal{F}} \delta(f) \quad (2.66a)$$

which states that the FT of the unit signal is an impulse centered at the frequency origin (Fig. 2.20). Note that (2.66) could be obtained from (2.65) by the symmetry rule.

2.7.3 Periodic Signals

The natural tool for periodic signals is the Fourier series which represents the signal by a discrete-frequency function $S_n = S(nF)$. We can also consider the Fourier transform, but we obtain a “singular” result, however, expressed in terms of delta functions.

A first example of FT of a periodic signal is given by (2.66), which states that the FT of an exponential with frequency f_0 is the impulse applied at the frequency f_0 . A second example is given by sinusoidal signals, which can be decomposed into exponentials (see (2.21)). We find

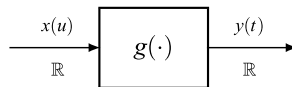
$$\begin{aligned} \cos 2\pi Ft &= \frac{1}{2} (e^{i2\pi Ft} + e^{-i2\pi Ft}) \xrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f - F) + \delta(f + F)], \\ \sin 2\pi Ft &= \frac{1}{2i} (e^{i2\pi Ft} - e^{-i2\pi Ft}) \xrightarrow{\mathcal{F}} \frac{1}{2i} [\delta(f - F) - \delta(f + F)]. \end{aligned}$$

More generally, for a periodic signal $s(t)$ that admits the Fourier series expansion, we find

$$s(t) = \sum_{n=-\infty}^{+\infty} S_n e^{i2\pi nFt} \xrightarrow{\mathcal{F}} \sum_{n=-\infty}^{+\infty} S_n \delta(f - nF). \quad (2.67)$$

Hence, the FT of a periodic signal consists of a train of delta functions at the frequencies $f = nF$ and with area given by the corresponding Fourier coefficients.

Fig. 2.21 Graphical representation of a continuous-time filter



2.7.4 Step Signals

First, it is convenient to consider the signum signal $\text{sgn}(t)$. In the appendix, we find that its FT is given by

$$\text{sgn}(t) \xrightarrow{\mathcal{F}} \frac{1}{i\pi f}.$$

This transform does not contain a delta function; anyway, it should be interpreted as a *distribution* [2].

For the FT of the unit step function, we use decomposition (2.22), which gives

$$1(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t) \xrightarrow{\mathcal{F}} \frac{1}{2} \delta(f) + \frac{1}{i2\pi f}.$$

Then, in the passage from the “signum” signal to the step signal, in the FT we have to add a delta function of area equal to half the step amplitude, that is, equal to the *continuous component* of the step signal.

2.8 Signal Filtering

Filtering is the most important operation used to modify some characteristics of signals. Historically, its original target was the “filtering” of sinusoidal components in the sense of passing some of them and eliminating the others. With the technology evolution, filtering has a broader and more articulated purpose.

2.8.1 Time-Domain Analysis

A filter (linear, invariant and continuous-time) may be introduced as the system characterized by the input–output relationship (Fig. 2.21)

$$y(t) = \int_{-\infty}^{+\infty} g(t-u)x(u) du = x * g(t), \quad (2.68)$$

where

- $x(t)$, $t \in \mathbb{R}$, is the *input signal*,
- $y(t)$, $t \in \mathbb{R}$, is the *output signal* or the *filter response*,
- $g(t)$, $t \in \mathbb{R}$, is the *impulse response*, which characterizes the filter.

The interpretation of the impulse response is obtained by applying an impulse to the input. Indeed, letting $x(t) = \delta(t)$ in (2.68) and considering property (2.41a, 2.41b), we get

$$y(t) = \delta * g(t) = g(t).$$

Then, the impulse response is the *filter response to the impulse applied at the origin*.

The filter model stated by (2.68) does not entail considerations of physical constraints. A constraint is the *causality condition* which states that the filter cannot “respond” before the application of the input signal (otherwise the filter would predict the future!). This condition implies that the impulse response must be a *causal* signal, i.e.,

$$g(t) = 0, \quad t < 0,$$

since it is the response to the impulse applied at $t = 0$ and cannot start at negative times. A filter with this property will be called *causal*, otherwise *anticipatory* (or *non-causal*). Physically implemented filters are surely causal, as correct models of “real” filters, but in Signal Theory we often encounter anticipatory filters, used in a simplified analysis (see below).

For causal filters the input–output relationship can be written in the more specific forms

$$y(t) = \int_{-\infty}^t x(u)g(t-u) du = \int_0^{+\infty} g(u)x(t-u) du,$$

whereas for anticipatory filters the general form (2.68) must be used.

2.8.2 Frequency-Domain Analysis

In the frequency-domain, input–output relationship (2.68) becomes

$$Y(f) = G(f)X(f) \tag{2.69}$$

where

- $X(f)$ is the FT of the input signal, $Y(f)$ is the FT of the output signal,
- $G(f)$ is the FT of the impulse response, which is called the *frequency response*.⁴

The frequency response $G(f)$ completely specifies a filter as well as the impulse response $g(x)$. When $g(t)$ is real, the frequency response has the *Hermitian symmetry* $G(f) = G^*(-f)$. Relationship (2.69) clearly states the advantage of dealing with the frequency-domain analysis, where the convolution becomes a *product*. This relationship, written as an inverse FT,

$$y(t) = \int_{-\infty}^{+\infty} Y(f)e^{i2\pi ft} df = \int_{-\infty}^{+\infty} G(f)X(f)e^{i2\pi ft} df,$$

⁴We prefer to reserve the term *transfer function* to the Laplace transform of the impulse response.

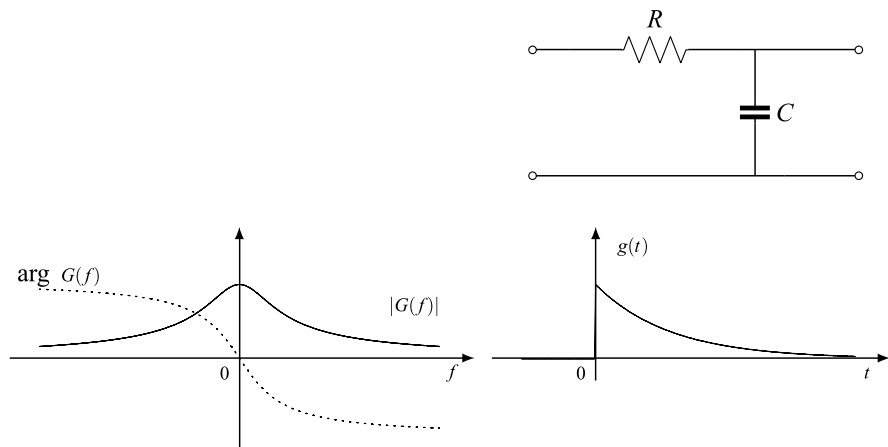


Fig. 2.22 The RC filter and the corresponding frequency and impulse responses

shows that each exponential component of the output signal is obtained from the corresponding component of the input signal as

$$[Y(f) df] e^{i2\pi ft} = G(f) [X(f) df] e^{i2\pi ft}, \quad f \in \mathbb{R}. \quad (2.70)$$

Hence, a filter modifies the complex amplitudes of the input signal components.

When both the input signal $x(t)$ and the impulse response $g(t)$ are real, the output signal $y(t)$ turns out to be real. If this is the case, considering the decomposition into sinusoidal components, we find

$$\begin{aligned} 2|Y(f)| df \cos[2\pi ft + \varphi_Y(f)] \\ = |G(f)| 2|X(f)| df \cos[2\pi ft + \varphi_X(f) + \varphi_G(f)], \quad f > 0. \end{aligned}$$

Hence, the filter modifies both the amplitude and the phase of the components.

Examples As a first example, we consider the RC filter of Fig. 2.22. To identify the frequency response from its definition (we recall that $G(f)$ is the Fourier transform of the impulse response), the following two steps are needed:

1. Applying a voltage impulse at the input, $e(t) = \delta(t)$, and evaluating the corresponding output voltage $v(t)$ (we need to solve the circuit in a transient regime). Then, the output voltage $v(t)$ gives the impulse response $g(t)$.
2. Evaluating the Fourier transform $G(f)$ of $g(t)$.

As known and as we shall see better in Chap. 9, it is more convenient to carry out the evaluation in a *symbolic form* which yields directly

$$G(f) = 1/(1 + i2\pi f RC).$$

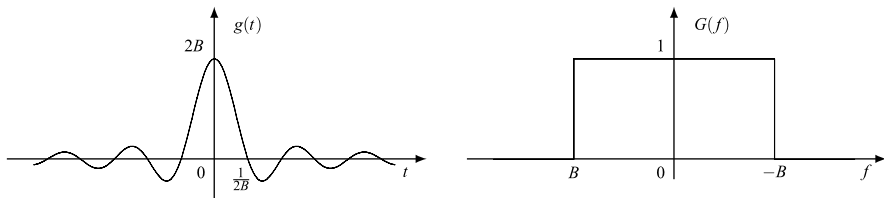


Fig. 2.23 Impulse response and frequency response of an *ideal low-pass* filter

Then, the inverse FT provides the impulse response which is given by

$$g(t) = \alpha 1(t) e^{-\alpha t}, \quad \alpha = 1/(RC).$$

This filter is *causal*, as expected, since it can be physically implemented.

As a second example, we consider the *ideal low-pass filter* which has the following frequency and impulse responses (Fig. 2.23):

$$G(f) = \text{rect}\left(\frac{f}{2B}\right) \xrightarrow{\mathcal{F}^{-1}} g(t) = 2B \text{sinc}(2Bt).$$

This filter is *anticipatory* and cannot be physically implemented. Nevertheless, it is a fundamental tool in Signal Theory (see Sampling Theorem).

2.9 Discrete Time Signals

In this second part of the chapter, we develop the topic of discrete signals.

Definition 2.3 A discrete-time signal is a complex function of a discrete variable

$$s : \mathbb{Z}(T) \rightarrow \mathbb{C}, \quad (2.71)$$

where the domain $\mathbb{Z}(T)$ is the set of the multiples of T

$$\mathbb{Z}(T) = \{\dots, -T, 0, T, 2T, \dots\}, \quad T > 0.$$

The signal (2.71) will usually be denoted in the forms

$$s(nT), \quad nT \in \mathbb{Z}(T) \quad \text{or} \quad s(t), \quad t \in \mathbb{Z}(T). \quad (2.72)$$

For *discrete-time* signals (more briefly, *discrete* signals), we will apply the same development seen for continuous signals. Most of the definitions are substantially the same; the main difference lies on the definitions expressed by integrals for continuous signals, which become sums for discrete signals.

In the final part of the chapter, discrete signals will be related to continuous signals by the Sampling Theorem. Discrete signals will be reconsidered in great detail, after the development of the Unified Theory in Chaps. 11, 12 and 13.

Notations In notations (2.72), the first one has the advantage of evidencing the discrete nature of the signal, whereas the second requires the specification of the domain $\mathbb{Z}(T)$, but is more in agreement with the notation for continuous signals, $s(t)$, $t \in \mathbb{R}$. The quantity $T > 0$ is the *spacing* (or *time-spacing*) between the instants where the signal is defined, and the reciprocal

$$F_p = 1/T \quad (2.73)$$

gives the signal *rate*, that is, the number of signal values per unitary time (values per second or v/s).

In textbooks and in other literature, it is customary to assume a unit spacing ($T = 1$) to simplify the notation in the form $s(n)$ or s_n with $n \in \mathbb{Z}$. We will not follow this consolidate convention for several reasons. First of all, by setting $T = 1$ we loose the application contest and the physical dimensions. Another motivation is that in the applications we often need to compare signals with different time-spacings (see multirate systems of Chap. 7), which is no more possible after the normalization $T = 1$. Finally, normalization represents a serious obstacle to a unified development.

2.9.1 Definitions

Most of the definitions introduced for continuous signals can directly be transferred to discrete signals, but sometimes with unexpected novelties.

Symmetries A discrete signal $s(nT)$ is *even*, if for any n , $s(nT) = s(-nT)$, $n \in \mathbb{Z}$ and it is *odd* if $s(nT) = -s(-nT)$, $n \in \mathbb{Z}$. An arbitrary discrete signal can always be decomposed into an even and an odd component

$$s(nT) = s_p(nT) + s_d(nT) \quad (2.73c)$$

exactly as for continuous signals.

A discrete signal $s(nT)$ is causal (Fig. 2.24) if it is zero for negative n ,

$$s(nT) = 0, \quad n < 0. \quad (2.74)$$

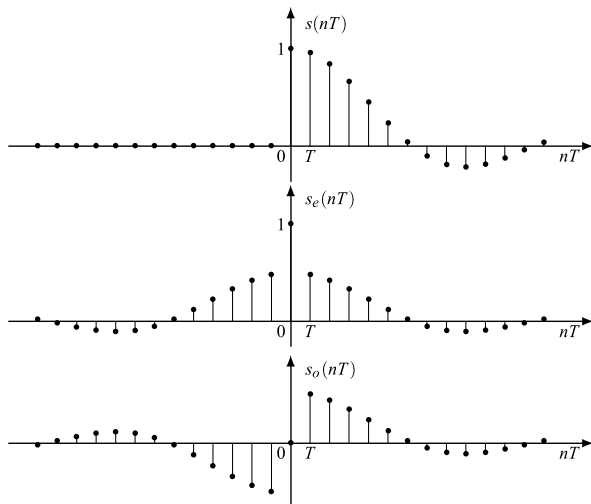
Relationships (2.6) between the even and odd components of a causal signal must be adjusted for discrete signal since $\text{sgn}(0) = 0$. The correct relationships are

$$\begin{aligned} s_d(nT) &= \text{sgn}(nT)s_p(nT), \\ s_p(nT) &= \text{sgn}(nT)s_d(nT) + s(0)\delta_{n0} \end{aligned} \quad (2.74a)$$

whereas in the continuous domain \mathbb{R} a single point has zero measure, and therefore the term related to $s(0)$ is irrelevant.

This is a general difference between the two classes, in so far two continuous signals, which coincide *almost everywhere*, must be considered as the same signal, whereas two discrete signals that differ even in a single point are really different.

Fig. 2.24 Decomposition of a causal discrete signal $s(nT)$ into even and odd parts



Time Shift Given a discrete signal $s(nT)$ and an integer n_0 , the signal $s(nT - n_0T)$ represents a shifted version of $s(nT)$ by the amount n_0T . The difference with respect to the continuous case, where the shift t_0 may be an arbitrary real number, is that now the shift $t_0 = n_0T$ must be a multiple of the spacing T .

Area and Mean Value The application of definition (2.9) would give zero for every discrete signal. To get a useful parameter, the right definition is

$$\text{area}(s) \triangleq \sum_{n=-\infty}^{+\infty} T s(nT). \quad (2.75)$$

In this way, each value $s(nT)$ gives a contribution, $T s(nT)$, to the area.

In the interpretation of this definition (and similar others), it is convenient to refer to a continuous signal $\tilde{s}(t)$, $t \in \mathbb{R}$, which is obtained from the given discrete signal $s(nT)$ by a *hold* operation, namely (Fig. 2.25)

$$\tilde{s}(t) = s(nT), \quad nT \leq t < (n+1)T. \quad (2.76)$$

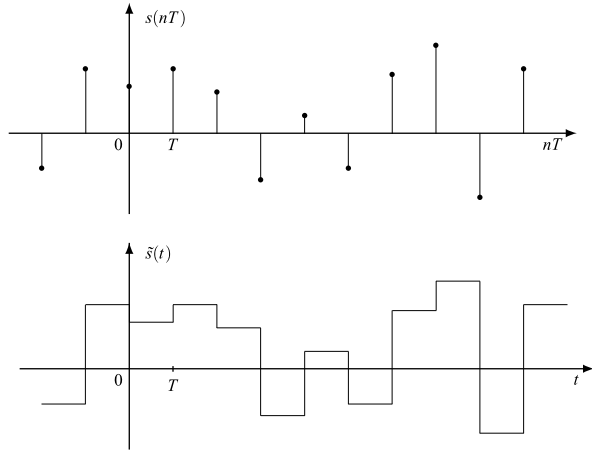
This continuous signal has the same area as $s(nT)$, but the area of $\tilde{s}(t)$ is evaluated according to (2.9) and the area of $s(nT)$ according to (2.75).

The *mean value* of a discrete signal $s(nT)$ is defined by the limit

$$m_s = \lim_{N \rightarrow +\infty} \frac{1}{(2N+1)T} \sum_{n=-N}^{+N} T s(nT). \quad (2.77)$$

Remark The *hold signal* $\tilde{s}(t)$ is not completely useful to study discrete signals using continuous signal definitions. For instance, the FT applied to $\tilde{s}(t)$, $t \in \mathbb{R}$ does not give the FT of $s(t)$, $t \in \mathbb{Z}(T)$.

Fig. 2.25 Discrete signal and correspondent continuous signal obtained by a hold interpolation



Energy and Power A discrete signal has zero energy and zero power, if these parameters are interpreted in the sense of continuous signals. The appropriate definitions for discrete signals are

$$E_s = \lim_{N \rightarrow \infty} \sum_{n=-N}^N T |s(nT)|^2 = \sum_{n=-\infty}^{+\infty} T |s(nT)|^2, \quad (2.78a)$$

$$P_s = \lim_{N \rightarrow \infty} \frac{1}{(2N+1)T} \sum_{n=-N}^N T |s(nT)|^2 \quad (2.78b)$$

which are in agreement with definitions (2.75) and (2.77). Moreover, E_s and P_s defined by (2.78a, 2.78b) equal respectively the energy and the power of the *hold signal* of Fig. 2.25.

Extension and Duration The *extension* $e(s)$ of a discrete signal may be defined as a set of consecutive points nT such that (Fig. 2.26)

$$s(nT) = 0, \quad nT \notin e(s).$$

The difference with respect to the extension of a continuous signal is that $e(s)$ is a subset of the domains $\mathbb{Z}(T)$ and therefore consists of *isolated* points.

The *duration* of a discrete signal is defined by

$$D(s) = \text{meas } e(s) = T \times \text{number of points of } e(s).$$

Here the *measure* is not the Lebesgue measure, which assigns zero to every set of isolated points, but the *Haar measure*, which assigns the finite value T to each point of the extension. Figure 2.26 shows an example of discrete signal with extension, $e(s) = \{-5T, \dots, 11T\}$, whose duration is $D(s) = 17T$.

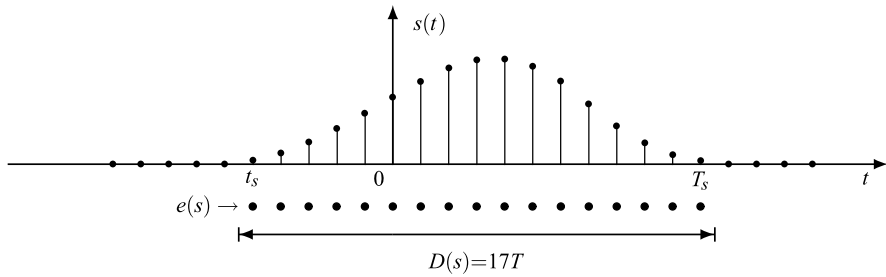


Fig. 2.26 Discrete signal with a limited extension: $e(s) = \{t_s, \dots, T_s\}$ with $t_s = -5T$ and $T_s = 11T$. The duration is $D(s) = 17T$

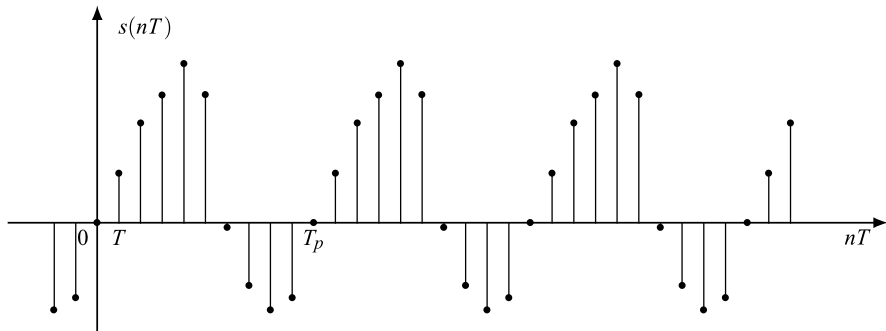


Fig. 2.27 Periodic discrete signal with period $T_p = 10T$

2.9.2 Periodic Discrete Signals

A discrete signal $s(nT)$ is *periodic* if

$$s(nT + NT) = s(nT), \quad \forall n \in \mathbb{Z}$$

where N is a natural number. Clearly, the period $T_p = NT$ must be a multiple of the spacing T . Figure 2.27 shows an example of a periodic discrete signal with period $T_p = 10T$.

As seen for continuous signals, some definitions must be modified for periodic signals. The rule is that the summations extended to the whole domain $\mathbb{Z}(T)$ *must be limited to a period*. For instance, the definition of energy given by (2.78a) for a periodic discrete signal is modified as *energy in a period*, namely

$$E_s = \sum_{n=n_0}^{n_0+N-1} T |s(nT)|^2,$$

where n_0 is an arbitrary integer (usually set to $n_0 = 0$).

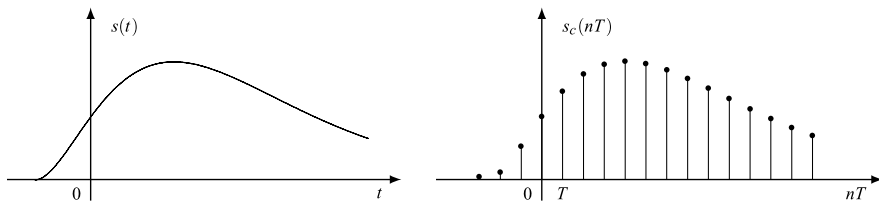


Fig. 2.28 Example of *sampling* of a continuous signal

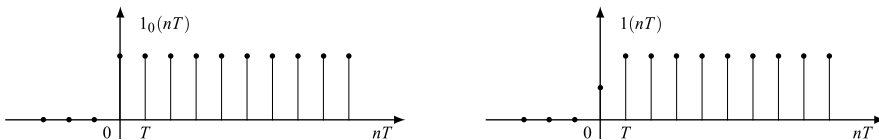


Fig. 2.29 Discrete step signal compared with sampled continuous step signal

As noted in the introduction (see Sect. 1.3), the class of periodic discrete signals is very important in applications, since they are the only signals that can be handled directly on a digital computer. The reason is that a periodic discrete signal $s(nT)$ with the period $T_p = NT$ is *completely specified by its finitely many values in a period*, say $s(0), s(T), \dots, s((N-1)T)$. For all the other classes, the signal specification involves infinitely many values.

2.10 Examples of Discrete Signals

Examples of discrete signals can autonomously be introduced, but frequently they are obtained from continuous signals with a domain *restriction* from \mathbb{R} into $\mathbb{Z}(T)$. This operation, called *sampling*, is stated by the simple relationship (Fig. 2.28)

$$s_c(nT) = s(nT), \quad nT \in \mathbb{Z}(T) \quad (2.79)$$

where $s(t), t \in \mathbb{R}$, is the reference continuous signal and $s_c(nT), nT \in \mathbb{Z}(T)$, is the discrete signal obtained by the sampling operation.

2.10.1 Discrete Step Signal

The *discrete unit step* signal (Fig. 2.29) is defined by

$$1_0(nT) = \begin{cases} 0 & \text{for } n < 0; \\ 1 & \text{for } n \geq 0. \end{cases} \quad (2.80)$$

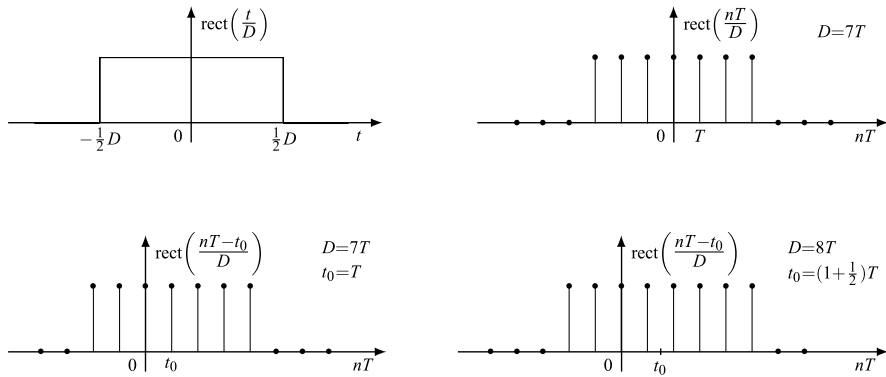


Fig. 2.30 Example of discrete rectangular pulses, compared with its continuous-time version. Above centered at the origin and below centered out of the origin

Note in particular that at the time origin $1_0(nT)$ takes a unit value. Instead, the signal obtained by sampling a unit step continuous signal is given by

$$1(nT) = \begin{cases} 0 & \text{for } n < 0; \\ \frac{1}{2} & \text{for } n = 0; \\ 1 & \text{for } n > 0, \end{cases}$$

as follows from the convention on discontinuities of continuous signals (see Sect. 2.1).

2.10.2 Discrete Rectangular Pulses

The *discrete rectangular pulse* with extension

$$e(r) = \{n_1 T, (n_1 + 1)T, \dots, n_2 T\}, \quad n_1 \leq n_2,$$

can be written in the form

$$r(nT) = \text{rect}\left(\frac{nT - t_0}{D}\right) \quad (2.81)$$

where

$$t_0 = \frac{n_1 + n_2}{2}T, \quad D = (n_2 - n_1 + 1)T \quad (2.81a)$$

are respectively the central instant and the duration. Note that expression (2.81) is not ambiguous since discontinuities of the function $\text{rect}(x)$ are not involved therein. Figure 2.30 shows a few examples of discrete rectangular pulses.

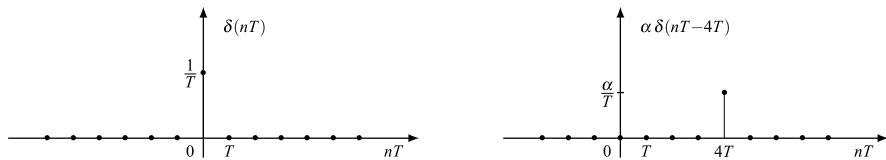


Fig. 2.31 The discrete impulse with unit area at the origin and with area α applied at $n_0T = 4T$

2.10.3 Discrete Impulses

We want a discrete signal with the same properties of the *impulse*, introduced for continuous signal by means of the delta function. However, in the discrete case the formalism of delta function (which is a distribution) is not necessary. In fact, the discrete signal defined by (Fig. 2.31)

$$\delta(nT) = \begin{cases} \frac{1}{T} & \text{for } n = 0; \\ 0 & \text{for } n \neq 0 \end{cases} \quad (2.82)$$

has exactly the same properties as the continuous impulse $\delta(t)$, namely the extension of $\delta(nT)$ is limited to the origin, i.e., $e(\delta) = \{0\}$, $\delta(nT)$ has unit area, $\delta(nT)$ has the *sifting property*

$$\sum_{n=-\infty}^{+\infty} T s(nT) \delta(nT - n_0T) = s(n_0T), \quad (2.83a)$$

the *convolution* (see the next section) of an arbitrary signal $s(nT)$ with the impulse $\delta(nT)$ yields the signal itself

$$s(nT) = \sum_{k=-\infty}^{+\infty} T s(kT) \delta(nT - kT). \quad (2.83b)$$

In general, the impulse with area α and applied at n_0T must be written in the form $\alpha\delta(nT - n_0T)$. Note that a discrete impulse is strictly related to the Kronecker delta, namely

$$T\delta(nT - n_0T) = \delta_{nn_0} = \begin{cases} 1 & \text{for } n = n_0; \\ 0 & \text{for } n \neq n_0. \end{cases} \quad (2.84)$$

2.10.4 Discrete Exponentials and Discrete Sinusoidal Signals

A discrete *exponential* signal has the general form

$$s(nT) = K a^n \quad (2.85a)$$

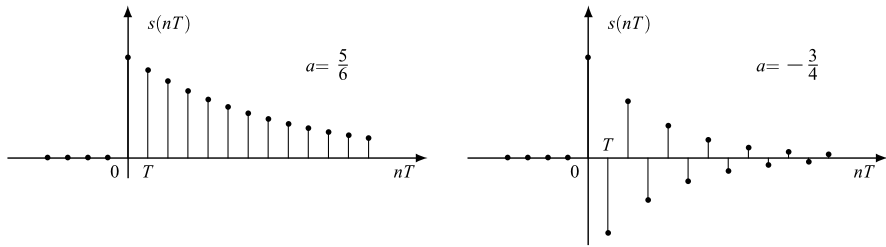


Fig. 2.32 Examples of discrete causal exponential

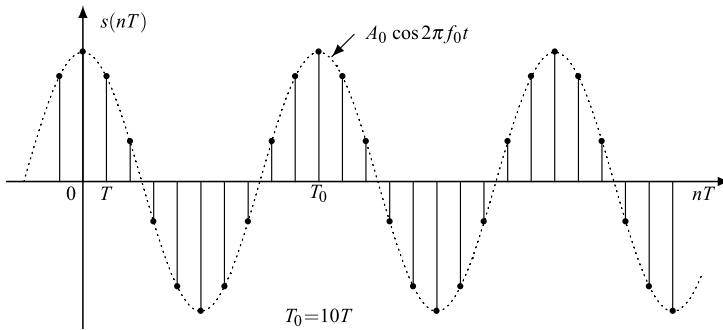


Fig. 2.33 Discrete sinusoidal signal with $f_0 T = 1/10$

where K and a are complex constants. In particular, when $|a| = 1$, it can be written as

$$Ae^{i2\pi f_0 nT} \quad (2.85b)$$

where A is a complex amplitude and f_0 is a real frequency (positive or negative).

A discrete *causal exponential* signal has the general form

$$K 1_0(nT)a^n, \quad (2.86)$$

where K and a are complex constants. Figure 2.32 illustrates this signal for $K = 1$ and two values of a .

A discrete *sinusoidal* signal has the form (Fig. 2.33)

$$A_0 \cos(2\pi f_0 nT + \varphi_0) \quad (2.87)$$

where both A_0 and f_0 are real and positive, and can be expressed as the sum of two exponentials of the form (2.85b) (see (2.21)).

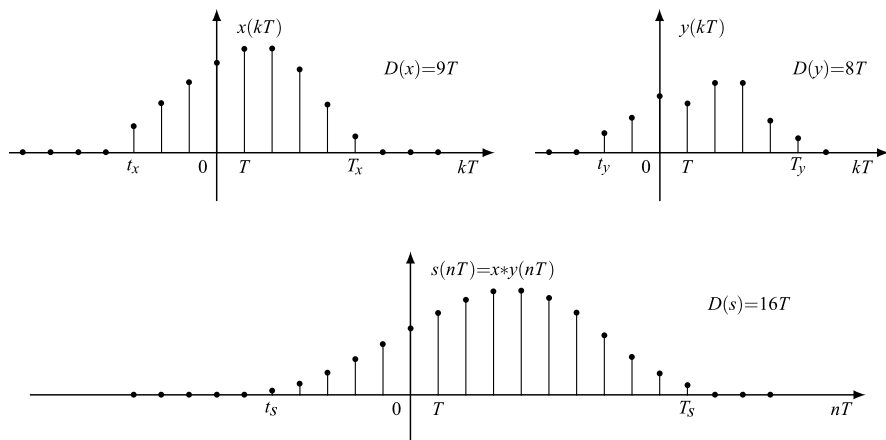


Fig. 2.34 Convolution of two time-limited discrete signals. Note that the convolution duration is $D(s) = D(x) + D(y) - T$

2.11 Convolution of Discrete Signals

As seen for continuous signals, we have different definitions for discrete aperiodic signals and for discrete periodic signals.

2.11.1 Aperiodic Discrete Signals

Given two discrete aperiodic signals $x(nT)$ and $y(nT)$, the convolution defines a new discrete signal $s(nT)$ according to

$$s(nT) = \sum_{k=-\infty}^{+\infty} T x(kT) y(nT - kT). \quad (2.88)$$

This is concisely denoted by $s = x * y$ or, more explicitly, by $s(nT) = x * y(nT)$.

Discrete convolution has the same properties as continuous convolution seen in Sect. 2.4 (rules on commutativity, area, etc.). Here, we outline only the extension rule. If $x(nT)$ and $y(nT)$ have the limited extensions

$$e(x) = \{n_x T, \dots, N_x T\}, \quad e(y) = \{n_y T, \dots, N_y T\}$$

then also their convolution $s(nT) = x * y(nT)$ has a limited extension given by

$$e(s) = \{n_s T, \dots, N_s T\} \quad \text{with } n_s = n_x + n_y, N_s = N_x + N_y. \quad (2.89)$$

Figure 2.34 shows an example, where $e(x) = \{-3T, \dots, 5T\}$ and $e(y) = \{-2T, \dots, 5T\}$. Then $e(s) = \{-5T, \dots, 10T\}$.

2.11.2 Periodic Discrete Signals

At this point, the right definition of the convolution for this class of signals should be evident. Given two periodic discrete signals $x(nT)$ and $y(nT)$ with the same period $T_p = NT$, their convolution is

$$s(nT) = \sum_{k=k_0}^{k_0+N-1} T x(kT) y(nT - kT), \quad (2.90)$$

where the summation is limited to a period. The result is a signal $s(nT)$ with the same period $T_p = NT$.

The “periodic discrete” convolution, often called the *cyclic convolution*, has the same properties as the other kind of convolutions.

2.12 The Fourier Transform of Discrete Signals

Discrete signals can be represented in the frequency domain by means of the FT, as seen for continuous signals. In the discrete case, the physical interpretation of the FT may be less evident, but nevertheless it is a very useful tool.

2.12.1 Definition

A discrete signal $s(nT)$, $nT \in \mathbb{Z}(T)$ can be represented in the form

$$s(nT) = \int_{f_0}^{f_0+F_p} S(f) e^{i2\pi f nT} df, \quad (2.91a)$$

where $S(f)$ is the FT of $s(nT)$, which is given by

$$S(f) = \sum_{n=-\infty}^{+\infty} T s(nT) e^{-i2\pi f nT}. \quad (2.91b)$$

In (2.91a), the integral is extended over an arbitrary period $(f_0, f_0 + F_p)$ of the FT. The FT $S(f)$ is a periodic function of the real variable f (Fig. 2.35) with period

$$F_p = 1/T.$$

This is a consequence of the periodicity of the exponential function $e^{-i2\pi f nT}$ with respect to f . Remarkable is the fact that the *period* of $S(f)$, expressed in cycles per second (or hertz), equals the signal *rate*, expressed in values per second.

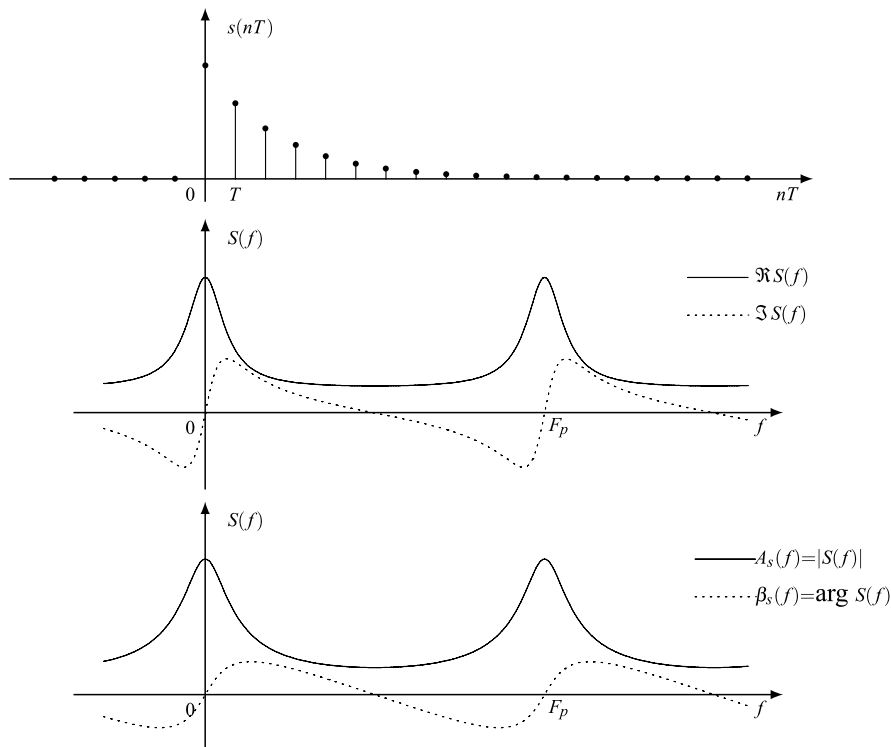


Fig. 2.35 Fourier transform of a real discrete time signal represented by real and imaginary parts, and by modulus and argument

As for continuous signals, we use the notations $S(f) = \mathcal{F}[s \mid f]$ and $s(nT) = \mathcal{F}^{-1}[S \mid nT]$ and also

$$s(nT) \xrightarrow{\mathcal{F}} S(f), \quad S(f) \xrightarrow{\mathcal{F}^{-1}} s(nT).$$

The operator $\xrightarrow{\mathcal{F}}$ represents a complex function of a discrete variable, $s(nT)$, by a periodic function of continuous variable, $S(f)$.

2.12.2 Interpretation

According to (2.91a), a discrete signal $s(nT)$ is represented as the sum of infinitely many exponentials of the form

$$[S(f) df] e^{i2\pi f nT}, \quad f \in [f_0, f_0 + F_p),$$

with infinitesimal amplitude $S(f) df$ and frequency f belonging to a period of the FT. The reason of this frequency limitation is due to the periodicity of discrete

exponentials. In fact, the components with frequency f and $f + kF_p$ are equal

$$[S(f) df] e^{i2\pi f nT} = [S(f + kF_p) df] e^{i2\pi(f + kF_p)nT}, \quad \forall k \in \mathbb{Z}.$$

We can therefore restrict the frequency range to a period, which may be $[0, F_p)$, that is,

$$s(nT) = \int_0^{F_p} S(f) e^{i2\pi f nT} df, \quad F_p = \frac{1}{T}. \quad (2.92)$$

The conclusion is that *the maximum frequency contained in a discrete signal $s(nT)$ cannot exceed the signal rate $F_p = 1/T$.*

For a *real* signal, $s^*(nT) = s(nT)$, the FT $S(f)$ has the *Hermitian symmetry*

$$S(f) = S^*(-f).$$

This symmetry, combined with the periodicity $S(f + F_p) = S(f)$, allows restricting the range from $[0, F_p)$ into $[0, \frac{1}{2}F_p)$. Moreover, from (2.92) we can obtain the form

$$s(nT) = \int_0^{\frac{1}{2}F_p} 2A_S(f) \cos(2\pi f nT + \beta_S(f)) df \quad (2.93)$$

where

$$A_S(f) = |S(f)| \quad \beta_S(f) = \arg S(f).$$

In the sinusoidal form (2.93), the maximum frequency is $\frac{1}{2}F_p$, which is called the *Nyquist frequency*.

2.12.3 Properties of the Fourier Transform

Here we consider only a few of the several properties (or *rules*).

- The shifted version of a discrete signal, $y(nT) = s((n - n_0)T)$, has FT

$$Y(f) = S(f) e^{-i2\pi f n_0 T}. \quad (2.94)$$

- The FT of *convolution*, $s(nT) = x * y(nT)$, is given by the product of the FTs

$$S(f) = X(f)Y(f). \quad (2.95)$$

Note the consistency of this rule: since $X(f)$ and $Y(f)$ are both periodic of period F_p , also their product is periodic with the same period, F_p .

- The FT of the *product* of two signals, $s(nT) = x(nT)y(nT)$, is given by the (cyclic) *convolution* of their FT (see (2.44))

$$S(f) = X * Y(f) = \int_{f_0}^{f_0 + F_p} X(\lambda) Y(f - \lambda) d\lambda. \quad (2.96)$$

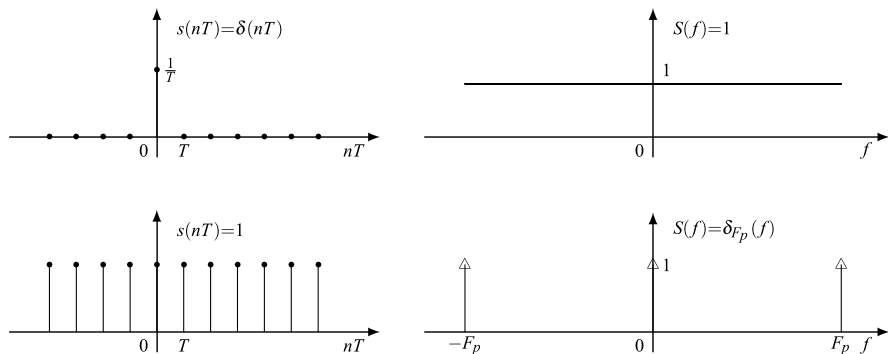


Fig. 2.36 Fourier transforms of the discrete impulse and of the discrete unit signal

- Parseval theorem allows evaluating the signal energy from the Fourier transform according to

$$E_s = \sum_{n=-\infty}^{+\infty} T |s(nT)|^2 = \int_{f_0}^{f_0+F_p} |S(f)|^2 df \quad (2.97)$$

where the integral is over an arbitrary period of $S(f)$.

2.12.4 Examples of Fourier Transforms

The explicit evaluation of the Fourier transform, according to (2.91b), requires the summation of a bilateral series; in the general case, this is not easy. The explicit evaluation of the inverse Fourier transform, according to (2.91a), requires the integration over a period.

Impulses and Constant Signals The FT evaluation of the impulse applied at n_0T is immediate

$$\delta(nT - n_0T) \xrightarrow{\mathcal{F}} e^{-i2\pi f n_0T}.$$

Note that with the notation $\delta(t - t_0)$ instead of $\delta(nT - n_0T)$ the above expression takes the same form as seen for the continuous case (see (2.65))

$$\delta(t - t_0) \xrightarrow{\mathcal{F}} e^{-i2\pi f t_0},$$

where now $t, t_0 \in \mathbb{Z}(T)$. In particular, for $t_0 = n_0T = 0$ we find (Fig. 2.36)

$$\delta(nT) \xrightarrow{\mathcal{F}} 1.$$

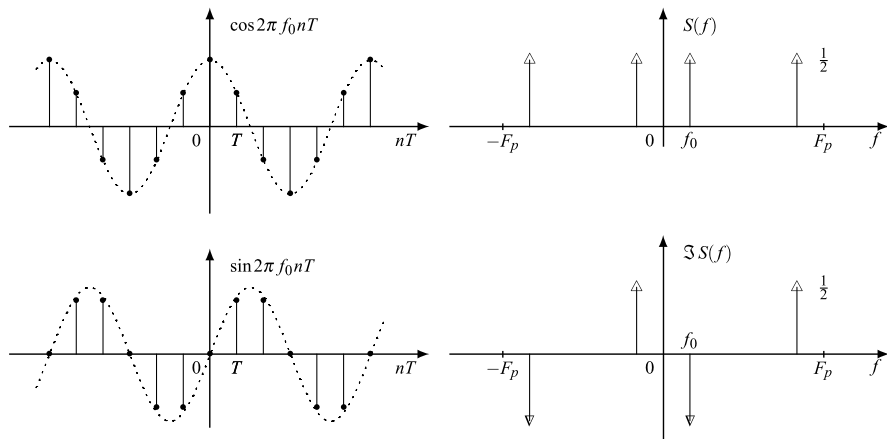


Fig. 2.37 Fourier transform of a cosinusoidal and of sinusoidal discrete signal

Less trivial is the FT evaluation of the unit signal, $s(nT) = 1$, since the definition (2.91b) gives

$$S(f) = T \sum_{n=-\infty}^{+\infty} e^{-i2\pi f nT},$$

where the series is not summable. To overcome the difficulty, we can use the identity (2.53) established in the contest of Fourier series and now rewritten in the form

$$\sum_{n=-\infty}^{+\infty} e^{-i2\pi f nT} = F_p \operatorname{rep}_{F_p} \delta(f), \quad F_p = 1/T. \quad (2.98)$$

Then, we find (Fig. 2.36)

$$1 \xrightarrow{\mathcal{F}} \operatorname{rep}_{F_p} \delta(f) \triangleq \delta_{F_p}(f). \quad (2.99)$$

Hence, the FT of the unit discrete signal, $s(nT) = 1$, consists of the periodic repetition of the frequency impulse $\delta(f)$. Remarkable is the fact that the delta function formalism allows the evaluation of the sum of a divergent series!

Exponential and Sinusoidal Signals If we replace f with $f - f_0$ in identity (2.98), we find the Fourier pair

$$e^{i2\pi f_0 nT} \xrightarrow{\mathcal{F}} \operatorname{rep}_{F_p} \delta(f - f_0) = \delta_{F_p}(f - f_0),$$

which gives the FT of the discrete exponential. Next, using Euler's formulas, we obtain the FT of sinusoidal discrete signals (Fig. 2.37), namely

$$\cos 2\pi f_0 nT \xrightarrow{\mathcal{F}} \frac{1}{2} [\delta_{F_p}(f - f_0) + \delta_{F_p}(f + f_0)],$$

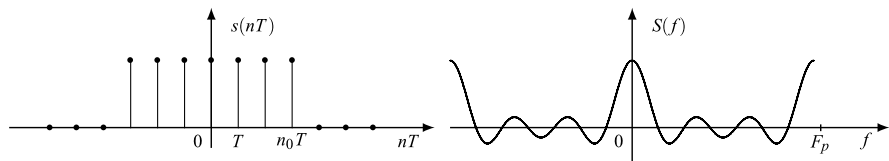


Fig. 2.38 Fourier transform of a discrete rectangular pulse

$$\sin 2\pi f_0 nT \xrightarrow{\mathcal{F}} \frac{1}{2i} [\delta_{F_p}(f - f_0) - \delta_{F_p}(f + f_0)].$$

Rectangular Pulses The discrete rectangular pulse of duration $(2n_0 + 1)T$

$$s(nT) = \begin{cases} A_0 & \text{for } |n| \leq n_0; \\ 0 & \text{for } |n| > n_0 \end{cases}$$

has as FT

$$S(f) = A_0 T \sum_{n=-n_0}^{n_0} e^{-i2\pi f nT}.$$

This finite sum can be expressed by means of the *periodic sinc* function, as seen in Example 2.6 of Sect. 2.5. The result is

$$S(f) = A_0 N T \operatorname{sinc}_N(f N T), \quad N = 2n_0 + 1.$$

Figure 2.38 illustrates $S(f)$ for $n_0 = 3$ ($N = 7$).

Causal Exponentials The FT of the signal $s(nT) = 1_0(n)a^n$ is

$$S(f) = T \sum_{n=0}^{+\infty} a^n e^{-i2\pi f nT} = T \sum_{n=0}^{+\infty} (a e^{-i2\pi f T})^n. \quad (2.100)$$

If $|a| < 1$ the geometrical series is convergent, since

$$|a e^{-i2\pi f T}| = |a| < 1$$

and the FT is given by

$$S(f) = \frac{T}{1 - a \exp(-i2\pi f T)}.$$

If $|a| > 1$ the geometrical series is divergent and the FT does not exist.

2.13 The Discrete Fourier Transform (DFT)

The DFT is commonly introduced to represent a finite sequence of values

$$s_0, s_1, \dots, s_{N-1} \quad (2.101a)$$

by another finite sequence of values

$$S_0, S_1, \dots, S_{N-1}. \quad (2.101b)$$

The two sequences are related by the relationships

$$s_n = \frac{1}{N} \sum_{k=0}^{N-1} S_k W_N^{nk}, \quad S_k = \sum_{n=0}^{N-1} s_n W_N^{-nk}, \quad (2.102)$$

where W_N is the N th root of the unity

$$W_N = \exp(i2\pi/N). \quad (2.103)$$

The first of (2.102) represents the inverse DFT (IDFT) and the second represents the DFT. They are a consequence of the orthogonality condition

$$\frac{1}{N} \sum_{m=0}^{N-1} W_N^{mk} W_N^{-nm} = \delta_{nk}.$$

Comments The DFT works with a finite number of values, and therefore it can be implemented on a digital computer. Its implementation is usually done by a very efficient algorithm, called the FFT (fast Fourier transform) (see Chap. 13).

In Signal Theory, the DFT represents the FT for periodic discrete signals and the finite sequence (2.101a) gives the *signal values in a period* and, analogously, the finite sequence (2.101b) gives the *Fourier transform values in a period*. However, the classical form (2.102) does not show clearly this assertion and the connection (or similarity) with the other FTs.

This will be seen after the development of the Unified Theory, in Chap. 11 and Chap. 13, where the DFT will be obtained as a special case of the unified Fourier transform.

2.14 Filtering of Discrete Signals

A discrete filter (linear, invariant) can be formulated as a system with the following input–output relationship (Fig. 2.39):

$$y(nT) = \sum_{k=-\infty}^{+\infty} T g(nT - kT) x(kT), \quad nT \in \mathbb{Z}(T) \quad (2.104)$$

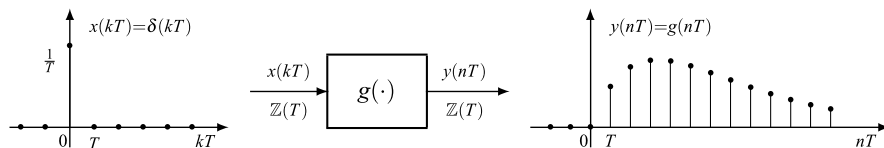


Fig. 2.39 Interpretation of impulse response of a discrete-time filter

where $x(kT)$ is the input signal, $y(nT)$ is the output signal and $g(nT)$ is the *impulse response* which specifies the filter.

We may recognize that (2.104) is a convolution, namely $y(nT) = g * x(nT)$, according to the definition given for (aperiodic) discrete signals in Sect. 2.11. As seen for continuous filters, the meaning of $g(nT)$ is the response of the filter to the discrete impulse $\delta(nT)$ defined by (2.82).

The input–output relationship (2.104) in the frequency domain becomes

$$Y(f) = G(f)X(f) \quad (2.105)$$

where $G(f)$ is the Fourier transform of the impulse response $g(nT)$, called the *frequency response* of the filter.

Thus, we recognize that the frequency-domain analysis of a discrete filter is exactly the same seen for a continuous filter in Sect. 2.8.

2.15 Sampling Theorem

The Sampling Theorem provides a connection between the classes of continuous and discrete signals.

2.15.1 The Operation of Sampling

In Sect. 2.3, we have seen that *sampling* gives a discrete signal $s_c(nT)$ starting from a continuous signal $s(t)$, $t \in \mathbb{R}$ according to the relationship

$$s_c(nT) = s(nT), \quad nT \in \mathbb{Z}(T).$$

The values $s(nT)$ are called the *samples* of $s(t)$, the spacing T is called the *sampling period* and $F_c = 1/T$ is the *sampling frequency* (it gives the number of samples per second).

Since sampling drops a portion of the original signal $s(t)$, it is evident that the recovery of $s(t)$ from its samples $s(nT)$ is not possible, in general. However, for a *band-limited* signal a perfect recovery becomes possible. This is stated by the *Sampling Theorem* which will now be formulated in the classical form. A very different formulation will be seen with the Unified Theory, in Chap. 8.

2.15.2 Formulation and Proof of Sampling Theorem

Theorem 2.1 Let $s(t)$, $t \in \mathbb{R}$ be a band-limited signal according to

$$S(f) = 0 \quad \text{for } |f| > B. \quad (2.106)$$

If the sampling frequency F_c is at least twice the band, $F_c \geq 2B$, then $s(t)$ can be recovered by its samples $s(nT)$, $n \in \mathbb{Z}$ according to the reconstruction formula

$$s(t) = \sum_{n=-\infty}^{+\infty} s(nT) \operatorname{sinc}[F_c(t - nT)]. \quad (2.107)$$

Proof Band-limitation stated by (2.106) allows writing the inverse FT in the form

$$s(t) = \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} S(f) e^{i2\pi f t} df. \quad (2.108a)$$

This, evaluated at $t = nT$, gives

$$s(nT) = \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} S(f) e^{i2\pi f nT} df. \quad (2.108b)$$

Next, consider the periodic repetition of the FT $S(f)$, with period F_c ,

$$S_p(f) = \sum_{k=-\infty}^{+\infty} S(f - kF_c). \quad (2.109)$$

Since $S_p(f)$ is periodic, it can be expanded into a Fourier series (this expansion has been considered for time functions, but it also holds for frequency functions). Considering that the period of $S_p(f)$ is F_c , we have

$$S_p(f) = \sum_{n=-\infty}^{+\infty} S_n e^{i2\pi f nT}, \quad T = 1/F_c, \quad (2.110a)$$

where

$$S_n = \frac{1}{F_c} \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} S_p(f) e^{-i2\pi f nT} df. \quad (2.110b)$$

Now, by the band-limitation, we find that the terms of the periodic repetition do not overlap (Fig. 2.40) and $S_p(f)$ equals $S(f)$ in the interval $(-\frac{1}{2}F_c, \frac{1}{2}F_c)$, that is, $S_p(f) = S(f)$, $-\frac{1}{2}F_c < f < \frac{1}{2}F_c$.

Then, replacing $S_p(f)$ with $S(f)$ in (2.110b) and comparing with (2.108b), we obtain

$$F_c S_n = s(-nT). \quad (2.110c)$$

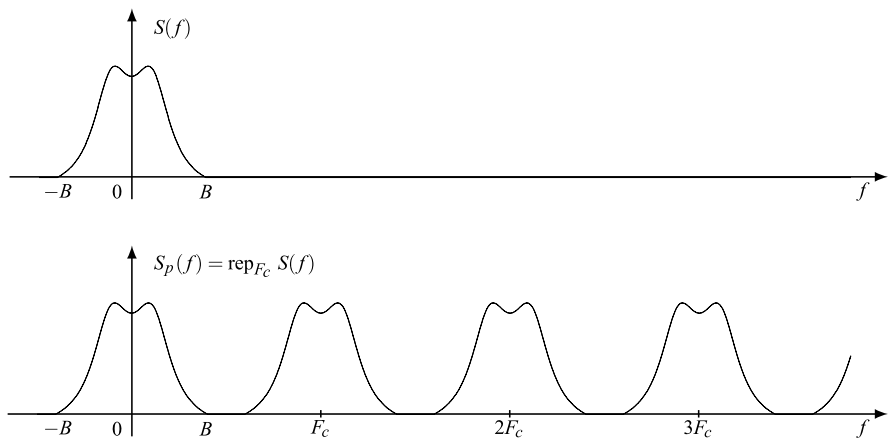


Fig. 2.40 Example of band-limited Fourier transform $S(f)$ and its periodic repetition with $F_c > 2B$

Finally, using series expansion (2.110a) in (2.108a), we find

$$s(t) = \sum_{n=-\infty}^{+\infty} \int_{-\frac{1}{2}F_c}^{\frac{1}{2}F_c} S_n e^{j2\pi f(t+nT)} df = \sum_{n=-\infty}^{+\infty} S_n F_c \operatorname{sinc}[F_c(t+nT)].$$

To complete the proof, it is sufficient to take into account (2.110c). □

2.16 Final Comments on Classical Theory

In this chapter, we have introduced and developed the two signal classes:

1. Continuous-time signals with domain \mathbb{R} , and
2. Discrete-time signals with domain $\mathbb{Z}(T)$.

A systematic comparison of definitions introduced in the time domain for the two classes brings to evidence the strong similarity, with the main difference that in the passage from class 1 to class 2 *integrals are replaced by summations*, specifically

$$\int_{-\infty}^{+\infty} s(t) dt \longrightarrow \sum_{n=-\infty}^{+\infty} T s(nT).$$

In the frequency domain, the two classes give respectively: class 1 of continuous-frequency Fourier transforms, with domain \mathbb{R} , and class 2 of continuous-frequency Fourier transforms with domain \mathbb{R} and period $F_p = 1/T$. In this comparison, the rule of passing from time to frequency domain is not clear. To get this rule, we have to consider not only the domain, but also the periodicity.

On the other hand, we have realized that periodicity plays a fundamental role in definitions. In fact, from class 1 we have extracted the subclass 1(a) of periodic signals and used for them the integral limited to a period instead of the integral over the whole real axis, that is,

$$\int_{-\infty}^{+\infty} s(t) dt \longrightarrow \int_{t_0}^{t_0+T_p} s(t) dt.$$

Analogously, from class 2 we have extracted the subclass 2(a) of periodic signals with the substitution

$$\sum_{n=-\infty}^{+\infty} Ts(nT) \longrightarrow \sum_{n=n_0}^{n_0+N-1} Ts(nT).$$

In the frequency domain, the two subclasses of periodic signals give respectively: class 1(a) of discrete-frequency Fourier transforms, with domain $\mathbb{Z}(F)$, $F = 1/T_p$, and class 2(a) of discrete-frequency Fourier transforms, with domain $\mathbb{Z}(F)$ and period $F_p = 1/T$.

In conclusion, in order to find a link between time and frequency domains it is necessary to consider periodicity or aperiodicity. Only in this way, we find that the global class of signals, consisting of subclasses 1, 2, 1(a) and 2(a), has a full counterpart in the frequency domain consisting of subclasses of exactly the same type. This link will automatically be provided by the Unified Theory.

2.17 Problems

2.1 ★ [Sect. 2.1] Assuming that a continuous-time signal $s(t)$ is the mathematical model of an electrical voltage, find the physical dimensions of the following quantities: area, mean value, (specific) energy, and (specific) power.

2.2 ★ [Sect. 2.2] Show that the *area over a period* of a periodic signal defined by (2.17a) is independent of t_0 .

2.3 ★★ [Sect. 2.2] Show that the *mean value over a period* for a periodic signal, defined by (2.17b), is equal to the *mean value* defined in general by (2.10).

2.4 ★ [Sect. 2.3] Using the functions $1(x)$ and $\text{rect}(x)$ write a concise expression for the signal

$$s(t) = 3 \quad \text{for } t \in (-5, 1), \quad s(t) = t \quad \text{for } t \in (2, 4), \quad s(t) = 0 \quad \text{otherwise.}$$

2.5 ★ [Sect. 2.3] Find the extension, duration, area and energy of the signal of Problem 2.4.

2.6 ★ [Sect. 2.3] Find the energy of the causal exponential with $p_0 = 2 + i2\pi 5$.

2.7 ★ [Sect. 2.3] Write a mathematical expression of a triangular pulse $u(t)$ determined by the following conditions: $u(t)$ is even, has duration 2 and energy 10.

2.8 ★ [Sect. 2.3] An even-symmetric triangular pulse $u(t)$ of duration 4 and amplitude 2 is periodically repeated according to (2.16). Draw the periodic repetition in the following cases: $T_p = 8$, $T_p = 4$ and $T_p = 2$.

2.9 ★★ [Sect. 2.3] Write the derivative $r'(t)$ of the rectangular pulse $r(t)$ defined by (2.26). Verify that the integral of $r'(t)$ from $-\infty$ to t recovers $r(t)$.

2.10 ★★ [Sect. 2.3] Write the first and second derivatives of the rectified sinusoidal signal

$$s(t) = A_0 |\cos \omega_0 t|.$$

2.11 ★★ [Sect. 2.3] Find the (minimum) period of the signal

$$s(t) = 2 \cos \frac{2}{3} \omega_0 t + 3 \sin \frac{4}{5} \omega_0 t.$$

2.12 ★★ [Sect. 2.4] Show that the (acyclic) convolution of an arbitrary signal $x(t)$ with a sinusoidal signal $y(t) = A_0 \cos(\omega_0 t + \phi_0)$ is a sinusoidal signal with the same period as $y(t)$.

2.13 ★ [Sect. 2.4] Show that the derivative of the convolution $s(t)$ of two derivable signals $x(t)$ and $y(t)$ is given by $s' = x' * y = x * y'$.

2.14 ★★★ [Sect. 2.4] Evaluate the convolution of the following pulses:

$$x(t) = A_1 \text{rect}(t/2D), \quad y(t) = A_2 \exp(-t^2/D^2).$$

Hint. Express the result in terms of the normalized Gaussian distribution

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy.$$

2.15 ★ [Sect. 2.4] Evaluate the convolution of the signals

$$x(t) = A_1 \text{sinc}(t/D), \quad y(t) = A_2 \delta(t) + A_3 \delta(t - 2D).$$

2.16 ★★★ [Sect. 2.4] Evaluate the (cyclic) convolution of the signal

$$x(t) = \text{rep}_{T_p} \text{rect}(t/T),$$

with $x(t)$ itself (auto-convolution). Assume $T_p = 4T$.

2.17 ★ [Sect. 2.5] Show that the Fourier coefficients have the same physical *dimensions* as the signal. In particular, if $s(t)$ is a voltage in volts, also S_n must be expressed in volts.

2.18 ★ [Sect. 2.5] Starting from the exponential form of the Fourier series and assuming a *real* signal, prove (2.49a) and (2.49b). Note that in this case S_0 is real.

2.19 ★ [Sect. 2.5] Show that if $s(t)$ is real and even, then its sine–cosine expansion (2.49a) becomes an *only cosine expansion*.

2.20 ★★★ [Sect. 2.5] Assume that a periodic signal has the following symmetry:

$$s(t) = -s(t - T_p/2).$$

Then, show that the Fourier coefficients S_n are zero for n even, i.e., the *even harmonics* disappear. *Hint*: use (2.50).

2.21 ★★ [Sect. 2.5] Evaluate the mean value, the root mean square value and the Fourier coefficients of the periodic signal

$$s(t) = \text{rep}_{T_p} \left[\text{rect} \left(\frac{t}{T_0} \right) A_0 \left(1 - \frac{|t|}{T_0} \right) \right]$$

in the cases $T_p = 2T_0$ and $T_p = T_0$.

2.22 ★ [Sect. 2.5] Check Parseval's Theorem (2.51a) for a sinusoidal signal (see Example 2.4).

2.23 ★ [Sect. 2.5] Evaluate the Fourier coefficients of the signal

$$s(t) = \text{rep}_{T_p} \left[\delta \left(t - \frac{1}{4}T_p \right) - \delta \left(t - \frac{3}{4}T_p \right) \right]$$

and find symmetries (if any).

2.24 ★ [Sect. 2.6] Find the physical dimension of the Fourier transform $S(f)$ when the signal is an electric voltage.

2.25 ★★ [Sect. 2.6] Show that if $s(t)$ is real, $S(f)$ has the Hermitian symmetry. *Hint*: use (2.55a, 2.55b).

2.26 ★★ [Sect. 2.6] Prove rule (2.60b) on the product of two signals.

2.27 ★★ [Sect. 2.6] Prove that the product $s(t) = x(t)y(t)$ of two strictly band-limited signal is strictly band-limited with

$$B(s) = B(x) + B(y).$$

Hence, in particular, the band of $x^2(t)$ is $2B(x)$.

2.28 ★ [Sect. 2.7] Evaluate the Fourier transform of the causal signal

$$s(t) = 1(t)e^{-t/T}, \quad T > 0$$

and then check that it verifies the Hermitian symmetry.

2.29 ★ [Sect. 2.7] Prove the relationship

$$s(t) \cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{2}S(f - f_0) + \frac{1}{2}S(f + f_0) \quad (2.111)$$

called *modulation rule*.

2.30 ★ [Sect. 2.7] Using (2.111) evaluate the Fourier transform of the signal

$$s(t) = \text{rect}(t/T) \cos 2\pi f_0 t.$$

Then, draw graphically $S(f)$ for $f_0 T = 4$, checking that it is an even real function.

2.31 ★★ [Sect. 2.7] Using the rule on the product, prove the relationship

$$1(t) \cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{4} \left[\delta(f - f_0) + \delta(f + f_0) + \frac{1}{i\pi(f - f_0)} + \frac{1}{i\pi(f + f_0)} \right].$$

2.32 ★★∇ [Sect. 2.7] The *scale change* (see Sect. 6.5) has the following rule

$$s(at) \xrightarrow{\mathcal{F}} (1/|a|)S(f/a) \quad a \neq 0. \quad (2.112)$$

Then, giving as known the pair $e^{-\pi t^2} \xrightarrow{\mathcal{F}} e^{-\pi f^2}$, evaluate the Fourier transform of the *Gaussian pulse*

$$u(t) = \frac{A_0}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2\right].$$

2.33 ★★ [Sect. 2.7] Evaluate the Fourier transform of the periodic signal

$$s(t) = \text{rep}_{T_p} \text{rect}\left(\frac{t}{D}\right).$$

2.34 ★★ [Sect. 2.7] Prove the relationship

$$\text{triang}\left(\frac{t}{D}\right) = \text{rect}\left(\frac{t}{2D}\right) \left(1 - \frac{|t|}{D}\right) \xrightarrow{\mathcal{F}} D \text{sinc}^2(fD)$$

where the signal is the 2D-duration *triangular pulse*.

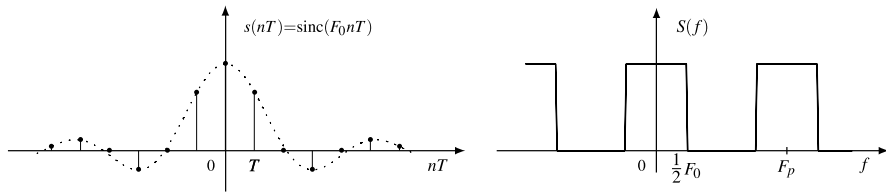


Fig. 2.41 Fourier transform of discrete sinc pulse

2.35 *** [Sect. 2.7] Consider the decomposition of a *real* signal in an even and an odd components

$$s(t) = s_e(t) + s_o(t).$$

Then, prove the relationship

$$s_e(t) \xrightarrow{\mathcal{F}} \Re S(f), \quad s_o(t) \xrightarrow{\mathcal{F}} j\Im S(f).$$

2.36 * [Sect. 2.12] Evaluate the Fourier transforms of the signals

$$s_1(nT) = \begin{cases} A_0, & \text{for } n = \pm 1; \\ 0, & \text{otherwise,} \end{cases} \quad s_2(nT) = \begin{cases} A_0, & \text{for } n = -1, 0, 1; \\ 0, & \text{otherwise,} \end{cases}$$

and check that $S_1(f)$ and $S_2(f)$ are (a) periodic with period $F_p = 1/T$, (b) real and (c) even.

2.37 * [Sect. 2.12] With the signals of the previous problem check the Parseval theorem (2.97).

2.38 * [Sect. 2.12] Show the relationship

$$\text{sinc}(nF_0T) \xrightarrow{\mathcal{F}} (1/F_0) \text{rep}_{F_p} \text{rect}(f/F_0),$$

illustrated in Fig. 2.41 for $F_0T = \frac{1}{2}$. *Hint: show that the inverse Fourier transform of $S(f)$ is $s(nT)$.*

2.39 ** [Sect. 2.15] Apply the Sampling Theorem to the signal

$$s(t) = \text{sinc}^3(Ft), \quad t \in \mathbb{R}$$

with $F = 4$ kHz.

Appendix: Fourier Transform of the Signum Signal $\text{sgn}(t)$

The Fourier transform definition (2.55b) yields

$$\int_{-\infty}^{+\infty} \text{sgn}(t) e^{-i2\pi f t} dt = \frac{2}{i} \int_0^{\infty} \sin 2\pi f t dt.$$

These integrals do not exist. However, $\text{sgn}(t)$ can be expressed as the inverse Fourier transform of the function $1/(i\pi f)$, namely

$$\text{sgn}(t) = \int_{-\infty}^{+\infty} \frac{1}{i\pi f} e^{i2\pi f t} df \triangleq x(t) \quad (2.113a)$$

provided that the integral is interpreted as a *Cauchy principal value*, i.e.,

$$x(t) = \int_{-\infty}^{+\infty} \frac{1}{i\pi f} e^{i2\pi f t} df = \lim_{F \rightarrow \infty} \int_{-F}^F \frac{1}{i\pi f} e^{i2\pi f t} df. \quad (2.113b)$$

Using Euler's formula, we get

$$x(t) = \int_{-\infty}^{+\infty} \frac{1}{i\pi f} \cos(2\pi f t) df + \int_{-\infty}^{+\infty} \frac{1}{\pi f} \sin(2\pi f t) df$$

where the integrand $(1/i2\pi f) \cos(2\pi f)$ is an odd function of f , and therefore the integral is zero. Then

$$x(t) = \int_{-\infty}^{+\infty} \frac{\sin(2\pi f t)}{\pi f} df.$$

Now, for $t = 0$ we find $x(0) = 0$. For $t \neq 0$, letting

$$2ft \rightarrow u, \quad df \rightarrow \frac{du}{2t},$$

we obtain

$$x(t) = \begin{cases} \int_{-\infty}^{+\infty} \frac{\sin(\pi u)}{\pi u} du & \text{for } t > 0; \\ \int_{+\infty}^{-\infty} \frac{\sin(\pi u)}{\pi u} du = - \int_{-\infty}^{+\infty} \frac{\sin(\pi u)}{\pi u} du & \text{for } t < 0. \end{cases}$$

It remains to evaluate the integral

$$I = \int_{-\infty}^{+\infty} \frac{\sin(\pi u)}{\pi u} du = \int_{-\infty}^{+\infty} \text{sinc}(u) du.$$

To this end, we use the rule (2.61) giving for a Fourier pair $s(t), S(f)$

$$\text{area}(S) = \int_{-\infty}^{+\infty} S(f) df = s(0)$$

with $s(t) = \text{rect}(t)$, $S(f) = \text{sinc}(f)$ (see (2.64a, 2.64b)). Hence, we obtain

$$\int_{-\infty}^{+\infty} \text{sinc}(f) \, df = s(0) = \text{rect}(0) = 1.$$

Combination of the above results gives $x(t) = \text{sgn}(t)$.

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