

Chapter 2

Internal Stability: Notions and Characterizations

2.1 Introduction

This chapter is devoted to the analysis of internal global stability notions used in mathematical control and systems theory. The stability notions presented are developed in the system-theoretic framework described in the previous chapter so that one can obtain a wide perspective of the role of stability in various classes of deterministic systems.

The stability notions developed in this chapter are referred to as “internal” because these notions are “uniform” with respect to the effect of external inputs (e.g., disturbances). Therefore, the notions are applicable to both certain and uncertain systems, and can be regarded as direct extensions of the conventional stability notions for systems with no external inputs, i.e., for the disturbance-free case.

A large part of this chapter is also devoted to the presentation of methods of proving stability. For testing global stability properties, there exist at least seven different methods in the literature, as shown below:

- (1) The first method is based upon *analytical solutions*. It attempts to obtain basic estimates for the solutions of the system by actually solving the differential (or difference) equations (or inequalities).
- (2) The second method is based upon *transformation techniques*. With this method basic estimates for the solutions of the system are derived by transforming the system into a different system with special properties.
- (3) The third method is based on *Lyapunov functions and functionals*. With this method basic estimates for the solutions are derived by means of one (or many) Lyapunov functional(s) and comparison lemmas.
- (4) The fourth method is based on *small-gain techniques*. With this method basic estimates for the solutions of the system are derived by means of small-gain arguments.
- (5) The fifth method is based on *Matrosov functions*.
- (6) The sixth method is based on *fixed point theorems*.
- (7) Finally, one has the method based upon *dissipative systems theory*.

Due to the intimate connection of small-gain results to external stability notions (i.e., stability notions which are *not* uniform with respect to the effect of external inputs), the small-gain methods of proving stability will be described separately in the following chapters. However, the first three methods are explained in detail in the present chapter. Finally, it should be emphasized that the different methods of proving stability can be, and often are, combined. For example, one can use Lyapunov functionals or analytical solutions to obtain basic estimates for the solutions and use small-gain arguments for the stability proof.

In what follows, $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ will be a control system with the BIC property, $U = \{0\}$, and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Moreover, $u_0 \in M_U$ will be the identically zero input, i.e., $u_0(t) = 0 \in U$ for all $t \geq 0$. For the output map $H : \mathfrak{N}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$, we assume that either $H : \mathfrak{N}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$ is continuous or that there exists a partition $\pi = \{\tau_i\}_{i=0}^\infty$ of $\mathfrak{N}^+$ with diameter $r > 0$ such that $H : \mathfrak{N}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7 of the previous chapter.

2.2 Definitions of Robust Global Asymptotic Output Stability (RGAOS)

Every stability notion must be defined for systems with solutions that are defined for all times t greater than the initial time t_0 . Consequently, the following definition plays an important role to the stability notions of the present chapter.

Definition 2.1 We say that a system Σ is *Robustly Forward Complete (RFC)* if it has the BIC property and for all $R \geq 0$ and $T \geq 0$, it holds that

$$\sup\{\|\phi(t_0 + s, t_0, x_0, u_0, d)\|_{\mathcal{X}}; s \in [0, T], \|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T], d \in M_D\} < +\infty$$

We next provide the definitions of Robust Global Asymptotic Output Stability (RGAOS) and Uniform Robust Global Asymptotic Output Stability (URGAOS).

Definition 2.2 We say that Σ is *Robustly Globally Asymptotically Output Stable (RGAOS)* if Σ is RFC and the following properties hold:

P1. Σ is *Robustly Lagrange Output Stable*, i.e., for all $\varepsilon > 0$ and $T \geq 0$, it holds that

$$\sup\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}; t \geq t_0, \|x_0\|_{\mathcal{X}} \leq \varepsilon, t_0 \in [0, T], d \in M_D\} < +\infty$$

(Robust Lagrange Output Stability).

P2. Σ is *Robustly Lyapunov Output Stable*, i.e., for all $\varepsilon > 0$ and $T \geq 0$, there exists $\delta := \delta(\varepsilon, T) > 0$ such that

$$\begin{aligned}
& \|x_0\|_{\mathcal{X}} \leq \delta \quad t_0 \in [0, T] \\
& \Rightarrow \quad \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon \quad \forall t \geq t_0, \forall d \in M_D. \\
& \text{(Robust Lyapunov Output Stability).}
\end{aligned}$$

P3. Σ satisfies the *Robust Output Attractivity Property*, i.e., for all $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists $\tau := \tau(\varepsilon, T, R) \geq 0$ such that

$$\begin{aligned}
& \|x_0\|_{\mathcal{X}} \leq R \quad t_0 \in [0, T] \\
& \Rightarrow \quad \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon \quad \forall t \geq t_0 + \tau, \forall d \in M_D.
\end{aligned}$$

Moreover, if $H(t, x, u) \equiv x$, then we say that Σ is *Robustly Globally Asymptotically Stable (RGAS)*.

Definition 2.3 We say that Σ is *Uniformly Robustly Globally Asymptotically Output Stable (URGAOS)* if Σ is RFC and the following properties hold:

P1. Σ is *Uniformly Robustly Lagrange Output Stable*, i.e., for every $\varepsilon > 0$, it holds that

$$\begin{aligned}
& \sup\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}; t \geq t_0, \|x_0\|_{\mathcal{X}} \leq \varepsilon, t_0 \geq 0, d \in M_D\} \\
& < +\infty \\
& \text{(Uniform Robust Lagrange Output Stability).}
\end{aligned}$$

P2. Σ is *Uniformly Robustly Lyapunov Output Stable*, i.e., for every $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ such that

$$\begin{aligned}
& \|x_0\|_{\mathcal{X}} \leq \delta \quad t_0 \geq 0 \\
& \Rightarrow \quad \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon \quad \text{for all } t \geq t_0 \text{ and } d \in M_D \\
& \text{(Uniform Robust Lyapunov Output Stability).}
\end{aligned}$$

P3. Σ satisfies the *Uniform Robust Output Attractivity Property*, i.e., for all $\varepsilon > 0$ and $R \geq 0$, there exists $\tau := \tau(\varepsilon, R) \geq 0$ such that

$$\begin{aligned}
& \|x_0\|_{\mathcal{X}} \leq R \quad t_0 \geq 0 \\
& \Rightarrow \quad \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon \quad \forall t \geq t_0 + \tau, \forall d \in M_D.
\end{aligned}$$

Moreover, if $H(t, x, u) \equiv x$, then we say that Σ is *Uniformly Robustly Globally Asymptotically Stable (URGAS)*.

The following lemma shows that RFC and property P3 in Definition 2.2 are sufficient for RGAOS. In other words, Robust Lagrange and Lyapunov Output Stability are consequences of the Robust Output Attractivity property.

Lemma 2.1 *Suppose that Σ is Robustly Forward Complete (RFC) and satisfies the Robust Output Attractivity Property (property P3 of Definition 2.2). Then Σ is RGAOS.*

The proof of Lemma 2.1 is based on the following fact, which exploits the properties of the output map.

Fact I *For all $T \geq 0$ and $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon, T) > 0$ such that*

$$\sup\{\|H(t, x, 0)\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \delta, t \in [0, T]\} \leq \varepsilon.$$

Proof of Fact I Let $T \geq 0$ and $\varepsilon > 0$. We consider the following cases:

Case 1: $H : \mathbb{R}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$ is continuous.

In this case, the continuity of the map $H : \mathbb{R}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$ and the fact that $H(t, 0, 0) = 0$ for all $t \geq 0$ imply that for every $t \in [0, T]$, there exists $\tilde{\delta} := \tilde{\delta}(t, \varepsilon) > 0$ such that

$$\sup\{\|H(\tau, x, 0)\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \tilde{\delta}, \tau \geq 0, |\tau - t| \leq \tilde{\delta}\} \leq \varepsilon \quad (2.1)$$

Due to the fact that the interval $[0, T]$ is compact, there exists a finite number of times $t_i \in [0, T]$, $i = 1, \dots, N$, with the property $[0, T] \subseteq \bigcup_{i=1, \dots, N} [\max(0, t_i - \tilde{\delta}(t_i, \varepsilon)), t_i + \tilde{\delta}(t_i, \varepsilon)]$. By virtue of (2.1), the selection $\delta(\varepsilon, T) = \min_{i=1, \dots, N} \tilde{\delta}(t_i, \varepsilon) > 0$ guarantees that the statement of Fact I holds.

Case 2: There exists a partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\mathbb{R}^+$ with diameter $r > 0$ such that $H : \mathbb{R}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$ satisfies Hypothesis (L2) in Sect. 1.7 of the previous chapter.

In this case, notice that there exists a finite set $\{\tau_0, \dots, \tau_N\} = \pi \cap [0, T]$ with $\tau_{N+1} > T$. Using the fact that for each fixed $\tau_i \in \pi$, the mapping $\mathcal{X} \times U \ni (x, u) \rightarrow H(\tau_i, x, u)$ is continuous with $H(\tau_i, 0, 0) = 0$, it follows that for every $i = 0, \dots, N$, there exists $\tilde{\delta}_i := \tilde{\delta}_i(\varepsilon) > 0$ such that

$$\sup\{\|H(\tau_i, x, 0)\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \tilde{\delta}_i\} \leq \varepsilon \quad (2.2)$$

By virtue of (2.2) in conjunction with the fact that, for every $(\tau_i, x, u) \in \pi \times \mathcal{X} \times U$ and $t \in [\tau_i, \tau_{i+1})$, it holds that $H(t, x, u) = H(\tau_i, x, u)$, the selection $\delta(\varepsilon, T) = \min_{i=0, \dots, N} \tilde{\delta}_i(\varepsilon) > 0$ guarantees that Fact I holds. The proof is complete. $\square$

We are now in a position to provide the proof of Lemma 2.1.

Proof of Lemma 2.1 It suffices to show that Σ is Robustly Lagrange and Lyapunov Output Stable, i.e., it satisfies properties P1 and P2 of Definition 2.2. First, we show that Σ is Robustly Lagrange Output Stable, by showing that $a(T, s) < +\infty$ for all $T \geq 0, s \geq 0$, where

$$a(T, s) := \sup \left\{ \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : \right. \\ \left. d \in M_D, t \in [t_0, +\infty), \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T] \right\}$$

By virtue of Robust Output Attractivity Property we have, for every $\varepsilon > 0$,

$$a(T, s) \leq \varepsilon + \sup \left\{ \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : \right. \\ \left. d \in M_D, t \in [t_0, t_0 + \tau(\varepsilon, T, s)], \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T] \right\}$$

where $\tau := \tau(\varepsilon, T, s) \geq 0$ is the time involved in the Robust Output Attractivity Property of Definition 2.2. Also, by virtue of Robust Forward Completeness (which implies that the set $\{\phi(t, t_0, x_0, u_0, d); t \in [t_0, t_0 + \tau], \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T], d \in M_D\}$ is bounded) and since $H : \mathbb{R}^+ \times \mathcal{X} \times U \rightarrow \mathcal{Y}$ maps bounded sets of $\mathbb{R}^+ \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, we obtain

$$\sup \left\{ \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : \right. \\ \left. d \in M_D, t \in [t_0, t_0 + \tau(\varepsilon, T, s)], \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T] \right\} < +\infty$$

Combining the above inequalities, we obtain that $a(T, s) < +\infty$ for all $T \geq 0$, $s \geq 0$, or equivalently that Σ is Robustly Lagrange Output Stable.

Next, we show that Σ is Robustly Lyapunov Output Stable. Consider arbitrary $\varepsilon > 0$ and $T \geq 0$. By virtue of the Robust Output Attractivity Property, there exists $\tau := \tau(\varepsilon, T) \geq 0$ such that, whenever $\|x_0\|_{\mathcal{X}} \leq \varepsilon$ and $t_0 \in [0, T]$, it holds that

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon \quad \forall t \in [t_0 + \tau, +\infty), \forall d \in M_D. \quad (2.3)$$

Moreover, by Fact I, there exists $\tilde{\delta} := \tilde{\delta}(\varepsilon, T) > 0$ such that

$$\sup \left\{ \|H(t, x, 0)\|_{\mathcal{Y}} : \|x\|_{\mathcal{X}} \leq \tilde{\delta}, t \in [0, T + \tau(\varepsilon, T)] \right\} \leq \varepsilon \quad (2.4)$$

where $\tau := \tau(\varepsilon, T) \geq 0$ is the time involved in (2.3). Finally, since $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$ for Σ , it follows that, for all $\varepsilon > 0$ and $T \geq 0$, there exists $\delta' > 0$ such that

$$\sup \left\{ \|\phi(\tau, t_0, x, u_0, d)\|_{\mathcal{X}}; d \in M_D, \tau \in [t_0, t_0 + \tau(\varepsilon, T)], t_0 \in [0, T] \right\} \leq \tilde{\delta}(\varepsilon, T)$$

provided that $\|x\|_{\mathcal{X}} \leq \delta'$, where $\tau := \tau(\varepsilon, T) \geq 0$ is the time involved in (2.3), and $\tilde{\delta} := \tilde{\delta}(\varepsilon, T) > 0$ is the positive number involved in (2.4). The above inequality in conjunction with (2.4) gives

$$\sup \left\{ \|H(\tau, \phi(\tau, t_0, x, u_0, d), 0)\|_{\mathcal{Y}}; d \in M_D, \tau \in [t_0, t_0 + \tau(\varepsilon, T)], t_0 \in [0, T] \right\} \leq \varepsilon$$

provided that $\|x\|_{\mathcal{X}} \leq \delta'$.

It is clear from (2.3) and the above inequality that the Robust Lyapunov Output Stability property is satisfied for $\delta(\varepsilon, T) = \min\{\varepsilon, \delta'\}$. The proof is complete. $\square$

The following lemma must be compared to Lemma 1.1, p. 131 in [17], and Proposition 3.2 in [19]. It shows that, for periodic systems, RGAOS is equivalent to URGAS.

Lemma 2.2 *Suppose that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T -periodic. If Σ is RGAOS, then Σ is URGAS.*

Proof The proof is a direct consequence of Definitions 2.2, 2.3 and from the following observation: if $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is T -periodic, then for every $(t_0, x_0, u, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_U \times M_D$, it holds that $H(t, \phi(t, t_0, x_0, u, d), u(t)) = H(t - kT, \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d), (P_{kT}u)(t - kT))$ and $\phi(t, t_0, x_0, u, d) = \phi(t - kT, t_0 - kT, x_0, P_{kT}u, P_{kT}d)$, where $k := [t_0/T]$ denotes the integer part of t_0/T , and the inputs $P_{kT}u \in M_U$, $P_{kT}d \in M_D$ are defined in Definition 1.2. $\square$

Remark 2.1 Notice that Property P3 of Definitions 2.2, 2.3 is not equivalent to $\lim_{t \rightarrow +\infty} \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} = 0$ for all $(t_0, x_0, d) \in (\mathbb{R}^+, \mathcal{X}, M_D)$ (the so-called “weak attractivity property”). In [16] (pp. 191–194) an example of an autonomous finite-dimensional system described by ODEs is reported, where a robust equilibrium point satisfies the weak attractivity property, but it is not URGAS, namely the planar system

$$\begin{cases} \dot{x} = \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)} \\ \dot{y} = \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)} \end{cases}$$

where the right-hand sides are defined to be zero at $x = y = 0$ (and consequently the right-hand sides are locally Lipschitz functions). The above system satisfies Hypotheses (H1–4) in Sect. 1.2 of the previous chapter and is RFC. Hence, by virtue of Lemma 2.1, it follows that the above system cannot satisfy Property P3.

Remark 2.2 Obviously, if a system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, I)$ is RGAS, where I denotes the identity mapping (i.e., $I(t, x) := x$ and $\mathcal{X} = \mathcal{Y}$) and the output function is time-invariant, i.e., $H(t, x) \equiv H(x)$, then the system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is RGAOS. However, this conclusion does not hold if the output function actually depends on time. For example, consider the system with identity output map,

$$\dot{x} = -x \quad x \in \mathbb{R}$$

It is clear that the above system is URGAS. However, if we define the output

$$Y = H(t, x) = \exp(4t)x$$

then it is obvious that the above system is not RGAOS.

Remark 2.3 The need to impose the Robust Lagrange Stability Property and the importance of this property are discussed in [57]. Particularly, in [57] the authors present an example of a scalar time-varying system described by ODEs with identity output map, where properties P2, P3 of Definition 2.3 hold, and the system is not URGAS. Therefore, nonuniform notions of stability are needed for the study of time-varying systems.

We stop at this point in order to emphasize the last phrase of the above comment. Since nonuniform stability notions are seldom discussed in details in most books, we present an example which illustrates the need for such notions in nonperiodic time-varying systems.

Example 2.2.1 Consider the planar, linear, and time-varying system described by the ODEs

$$\begin{aligned}\dot{x}_1(t) &= \exp(t)x_2(t) \\ \dot{x}_2(t) &= -\exp(-t)x_1(t) - 3x_2(t) \\ x(t) &= (x_1(t), x_2(t)) \in \mathfrak{R}^2\end{aligned}\tag{2.5}$$

The solution of system (2.5) with initial condition $x(t_0) = (x_{10}, x_{20})^T \in \mathfrak{R}^2$ is given by the following equations for all $t \geq t_0$:

$$\begin{aligned}x_1(t) &= x_{10}[1 + (t - t_0)]\exp(-(t - t_0)) + x_{20}(t - t_0)\exp(t_0)\exp(-(t - t_0)) \\ x_2(t) &= -x_{10}(t - t_0)\exp(-t_0)\exp(-2(t - t_0)) \\ &\quad + x_{20}[1 - (t - t_0)]\exp(-2(t - t_0))\end{aligned}\tag{2.6}$$

If the output Y of system (2.5) is the state component x_2 , then one can show the URGAS property. Indeed, in this case there exists a constant $M > 0$ such that the solution of (2.5) satisfies $|x_2(t)| \leq M|x_0|\exp(-(t - t_0))$ for all $t \geq t_0$.

However, if the output Y of system (2.5) is the state component x_1 , then one cannot show URGAS. To prove this, we notice that the *Uniform Robust Output Attractivity Property* does not hold. If the Uniform Robust Output Attractivity Property were valid, then for all $\varepsilon > 0$ and $R \geq 0$, we would obtain the existence of $\tau := \tau(\varepsilon, R) \geq 0$ such that for every $t_0 \geq 0$, $x_{20} \in \mathfrak{R}$ with $|x_{20}| \leq R$ and $t - t_0 \geq \tau$, the following inequality would hold:

$$|x_1(t)| = |x_{20}|(t - t_0)\exp(t_0)\exp(-(t - t_0)) \leq \varepsilon\tag{2.7}$$

Notice that we have considered the solution of (2.5) with $x_{10} = 0$. Clearly, (2.7) holds for all $t \geq t_0$ if $R\exp(t_0 - 1) \leq \varepsilon$ and for all $t \geq t_0 + A(\frac{\varepsilon}{R}\exp(-t_0))$ if $R\exp(t_0 - 1) > \varepsilon$, where $A := A(\frac{\varepsilon}{R}\exp(-t_0)) > 1$ is the unique solution of the equation $A\exp(-A) = \frac{\varepsilon}{R}\exp(-t_0)$. Notice that $A(\frac{\varepsilon}{R}\exp(-t_0)) > \ln(\frac{R}{\varepsilon}\exp(t_0))$ and therefore the assumption of Uniform Robust Output Attractivity Property leads to contradiction, because we must have $\tau(\varepsilon, R) \geq A(\frac{\varepsilon}{R}\exp(-t_0))$.

Finally, it should be noted that the Robust Output Attractivity Property holds if the output Y of system (2.5) is the state component x_1 . In this case, we can prove RGAOS using (2.6) (but not URGAOS). Thus, nonuniform stability notions arise naturally in the study of the simple system (2.5).

2.3 KL Characterizations

This section is devoted to the proof of two basic results, which provide characterizations of the RGAOS and URGAOS in terms of KL functions.

Theorem 2.1 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Then the following statements are equivalent:*

- (i) Σ is RGAOS.
- (ii) *There exist functions $\mu, \beta \in K^+$ and $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$, we have*

$$\begin{aligned} & \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} + \mu(t) \|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \\ & \leq \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0) \quad \text{for all } t \geq t_0. \end{aligned} \quad (2.8)$$

- (iii) *There exist functions $\gamma, \beta \in K^+$, $a \in K_\infty$, $\sigma \in KL$ and a constant $R \geq 0$ such that for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$, we have:*

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \sigma(\beta(t_0)(\|x_0\|_{\mathcal{X}} + R), t - t_0) \quad \forall t \geq t_0 \quad (2.9)$$

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \gamma(t)a(\beta(t_0)(\|x_0\|_{\mathcal{X}} + R)) \quad \forall t \geq t_0. \quad (2.10)$$

Theorem 2.2 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Then the following statements are equivalent:*

- (i) Σ is URGAOS.
- (ii) Σ is RFC, and there exists a function $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$, we have

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \sigma(\|x_0\|_{\mathcal{X}}, t - t_0) \quad \text{for all } t \geq t_0. \quad (2.11)$$

For the proof of Theorems 2.1 and 2.2, we need some technical lemmas that illustrate useful properties of the functions of classes K_∞ and K^+ .

Lemma 2.3 *Let $a : \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ be a function with $a(\cdot, 0) = 0$ that satisfies the following properties:*

1. *for each fixed $t \geq 0$, the mapping $a(t, \cdot)$ is nondecreasing,*
2. *for each fixed $s \geq 0$, the mapping $a(\cdot, s)$ is nondecreasing,*
3. *$\lim_{s \rightarrow 0^+} a(t, s) = 0$ for all $t \geq 0$.*

Then there exists a pair of functions $\zeta \in K_\infty$ and $\gamma \in K^+$ such that

$$a(t, s) \leq \zeta(\gamma(t)s) \quad \text{for all } (t, s) \in (\mathbb{R}^+)^2 \quad (2.12)$$

Proof of Lemma 2.3 Without loss of generality we may assume that $a \in C^0(\mathbb{R}^+ \times \mathbb{R}^+)$. Indeed, otherwise we may consider the function

$$\hat{a}(t, s) := \begin{cases} \frac{1}{s} \int_s^{2s} \int_t^{t+1} a(\tau, \xi) d\tau d\xi & \text{for } s > 0 \\ 0 & \text{for } s = 0 \end{cases}$$

which by virtue of the inequality $a(t, s) \leq \hat{a}(t, s) \leq a(t+1, 2s)$ is in $C^0(\mathbb{R}^+ \times \mathbb{R}^+)$ and satisfies $\hat{a}(\cdot, 0) = 0$. Notice that $\hat{a}$ has the same Properties 1, 2, 3 of our statement with a .

By invoking Property 3, for every $t \geq 0$, there exists $\delta(t)$ in $(0, 1)$ such that

$$s \leq \delta(t) \quad \Rightarrow \quad a(t, s) \leq \frac{1}{t+1} \quad (2.13)$$

Define the following function:

$$\tilde{\eta}(t) := ([t] + 1 - t)\delta([t] + 1) + (t - [t])\delta([t] + 2) \quad t \geq 0 \quad (2.14)$$

where $[t]$ denotes the integer part of $t \geq 0$. Notice that by definition (2.14) it follows that $\tilde{\eta}(k) = \delta(k+1)$ and $\lim_{t \rightarrow (k+1)^-} \tilde{\eta}(t) = \delta(k+2)$ for all $k \in \mathbb{Z}^+$, which implies that $\tilde{\eta}$ is continuous. Moreover, definition (2.14) gives $0 < \tilde{\eta}(t) \leq \max\{\delta([t] + 1); \delta([t] + 2)\}$ for all $t \geq 0$, which in conjunction with (2.13) implies:

$$s \leq \tilde{\eta}(t) \quad \Rightarrow \quad \left. \begin{array}{l} s \leq \delta([t] + 1) \text{ or} \\ s \leq \delta([t] + 2) \end{array} \right\} \quad \Rightarrow \quad \left. \begin{array}{l} a([t] + 1, s) \leq \frac{1}{[t]+2} \text{ or} \\ a([t] + 2, s) \leq \frac{1}{[t]+3} \end{array} \right\}$$

The above inequality, combined with the fact that $t \leq [t] + 1 \leq [t] + 2$ and Property 2 for a , gives:

$$s \leq \tilde{\eta}(t) \quad \Rightarrow \quad a(t, s) \leq \frac{1}{t+1} \quad (2.15)$$

Next, define the following function:

$$\eta(t) := \exp(-t) \min_{0 \leq \tau \leq t} \tilde{\eta}(\tau) \quad (2.16)$$

which is a C^0 strictly decreasing function $\eta : \mathbb{R}^+ \rightarrow (0, +\infty)$ with $\lim_{t \rightarrow +\infty} \eta(t) = 0$ and such that

$$s \leq \eta(t) \quad \Rightarrow \quad a(t, s) \leq \frac{1}{t+1} \quad (2.17)$$

Let μ be the inverse function of η defined on $(0, \eta(0)]$ being nonnegative, continuous, and strictly decreasing with $\lim_{t \rightarrow 0^+} \mu(t) = +\infty$. Define

$$\tilde{\mu}(s) := \begin{cases} \mu(s) & \text{if } s \in (0, \eta(0)] \\ 0 & \text{if } s > \eta(0) \end{cases} \quad (2.18)$$

It turns out that $\tilde{\mu} : (0, +\infty) \rightarrow \mathfrak{R}^+$ is nonincreasing, continuous, nonnegative, and satisfies $\lim_{t \rightarrow 0^+} \tilde{\mu}(t) = +\infty$. Additionally, define

$$\beta(s) := s + \begin{cases} 0 & \text{if } s = 0 \\ \sup_{0 < \tau \leq s} a(\tilde{\mu}(\tau), \tau) & \text{if } s > 0 \end{cases} \quad (2.19)$$

We next show that $\beta \in K_\infty$. Indeed, by definition (2.19) it follows that $\beta(0) = 0$ and β is strictly increasing with $\lim_{s \rightarrow +\infty} \beta(s) = +\infty$. The continuity of β on $(0, +\infty)$ follows from the fact that both a and $\tilde{\mu}$ are C^0 on $(0, +\infty)$. Furthermore, notice that (2.17) and (2.18) imply

$$a(\tilde{\mu}(\tau), \tau) \leq \frac{1}{\tilde{\mu}(\tau) + 1} \quad \text{for all } \tau \in (0, \eta(0)]. \quad (2.20)$$

Since $\lim_{s \rightarrow 0^+} \tilde{\mu}(s) = +\infty$, it follows from (2.20) that $\lim_{s \rightarrow 0^+} \beta(s) = 0$, and this establishes the continuity of β at zero. Let $\zeta(s) := a(s, s) + \beta(s)$. Obviously, $\zeta(\cdot)$ is of class K_∞ . Moreover, when $s \geq t$, by virtue of Property 2 for a , it holds that $a(t, s) \leq a(s, s) \leq \zeta(s)$, which implies

$$\sup_{s \geq t > 0} \frac{\zeta^{-1}(a(t, s))}{s} \leq 1 \quad (2.21)$$

Also, when $0 < s \leq \eta(t)$, it follows from (2.18) that $\tilde{\mu}(s) \geq t$, and hence, by Property 2 for a and (2.19), $a(t, s) \leq a(\tilde{\mu}(s), s) \leq \zeta(s)$. The latter implies that

$$\sup_{0 < s \leq \eta(t)} \frac{\zeta^{-1}(a(t, s))}{s} \leq 1 \quad (2.22)$$

Using Property 1, (2.21), and (2.22), we get

$$\sup_{s > 0} \frac{\zeta^{-1}(a(t, s))}{s} \leq 1 + \sup_{\eta(t) \leq s \leq t} \frac{\zeta^{-1}(a(t, s))}{s} \leq 1 + \frac{\zeta^{-1}(a(t, t))}{\eta(t)} \quad (2.23)$$

Let, finally, γ be any function of class K^+ which satisfies

$$\gamma(t) \geq \frac{\zeta^{-1}(a(t, t))}{\eta(t)} + 1 \quad \text{for all } t \geq 0 \quad (2.24)$$

The desired (2.12) is a consequence of (2.23) and (2.24). $\square$

Lemma 2.4 *Let $a : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ be a locally bounded function with $\lim_{s \rightarrow 0^+} a(s) = a(0) = 0$. Then there exists a function $\zeta \in K_\infty$ such that*

$$a(s) \leq \zeta(s) \quad \text{for all } s \geq 0 \quad (2.25)$$

Proof of Lemma 2.4 Define $\tilde{a}(s) := \sup_{0 \leq \tau \leq s} a(\tau)$ and notice that $\tilde{a}$ is nondecreasing with $\lim_{s \rightarrow 0^+} \tilde{a}(s) = \tilde{a}(0) = 0$. Define $\zeta(s) := s + \frac{1}{s} \int_s^{2s} \tilde{a}(\xi) d\xi$ for $s > 0$ and $\zeta(0) := 0$. The desired (2.25) is a consequence of previous definitions. $\square$

The following three technical lemmas provide essential estimates of the output and state behavior for RGAOS systems.

Lemma 2.5 *Suppose that the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is RGAOS. Then there exist functions $\sigma \in KL$ and $\beta \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$:*

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0) \quad (2.26)$$

Proof of Lemma 2.5 Let $\xi, T \geq 0, s \geq 0$, and define:

$$a(T, s) := \sup\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : d \in M_D, t \geq t_0, \\ \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\} \quad (2.27)$$

$$M(\xi, T, s) := \sup\{\|H(t_0 + \xi, \phi(t_0 + \xi, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : d \in M_D, \\ \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\} \quad (2.28)$$

First notice that by virtue of Robust Lagrange Output Stability a is well defined, i.e., $a(T, s) < +\infty$ for all $T \geq 0, s \geq 0$. Furthermore, notice that M is well defined, since by definitions (2.27), (2.28) the following inequality is satisfied for all $\xi, T \geq 0$ and $s \geq 0$:

$$M(\xi, T, s) \leq a(T, s) \quad (2.29)$$

Notice that, a satisfies all hypotheses of the Lemma 2.3, namely, for each fixed $s \geq 0$, $a(\cdot, s)$ is nondecreasing, and for each fixed $T \geq 0$, $a(T, \cdot)$ is nondecreasing and satisfies $a(\cdot, 0) = 0$. Furthermore, Robust Lyapunov Output Stability asserts that for every $T \geq 0$, $\lim_{s \rightarrow 0^+} a(T, s) = 0$. It turns out from Lemma 2.3 that there exist functions $\zeta_1 \in K_\infty$ and $\gamma \in K^+$ such that

$$a(T, s) \leq \zeta_1(\gamma(T)s) \quad \text{for all } (T, s) \in (\mathfrak{R}^+)^2 \quad (2.30)$$

Without loss of generality, we may assume that $\gamma \in K^+$ is nondecreasing. Inequality (2.30) in conjunction with (2.29) implies

$$M(\xi, T, s) \leq \zeta_1(\gamma(T)s) \quad \text{for all } (\xi, T, s) \in (\mathfrak{R}^+)^3 \quad (2.31)$$

Moreover, the Robust Output Attractivity property guarantees that for all $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists $\tau = \tau(\varepsilon, T, R) \geq 0$ such that

$$M(\xi, T, s) \leq \varepsilon \quad \text{for all } \xi \geq \tau(\varepsilon, T, R) \text{ and } 0 \leq s \leq R \quad (2.32)$$

Let

$$g(s) := \sqrt{s} + s^2 \quad (2.33)$$

and let p be a nondecreasing function of class K^+ with $p(0) = 1$ and

$$\lim_{t \rightarrow +\infty} p(t) = +\infty \quad (2.34)$$

Define

$$\mu(\xi) := \sup \left\{ \frac{M(\xi, T, s)}{p(T)g(\zeta_1(\gamma(T)s))} : T \geq 0, s > 0 \right\} \quad (2.35)$$

Obviously, by (2.31) and (2.35) the function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is well defined and satisfies $\mu(\cdot) \leq 1$. We show that $\lim_{\xi \rightarrow +\infty} \mu(\xi) = 0$, or, equivalently, we establish that for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\mu(\xi) \leq \varepsilon \quad \text{for } \xi \geq \delta(\varepsilon) \quad (2.36)$$

Notice first, that for any given $\varepsilon > 0$, there exist constants $a := a(\varepsilon)$ and $b := b(\varepsilon)$ with $0 < a < b$ such that

$$x \notin (a, b) \quad \Rightarrow \quad \frac{x}{\sqrt{x} + x^2} \leq \varepsilon \quad (2.37)$$

We next recall (2.34), which asserts that, for the above ε for which (2.37) holds, there exists $c := c(\varepsilon) \geq 0$ such that $p(T) \geq \frac{1}{\varepsilon}$ for all $T \geq c$. This by virtue of (2.37) and (2.31) yields

$$\frac{M(\xi, T, s)}{p(T)g(\zeta_1(\gamma(T)s))} \leq \varepsilon \quad \text{for all } \xi \geq 0, \text{ when } T \geq c \text{ or } \zeta_1(\gamma(T)s) \notin (a, b) \quad (2.38)$$

Hence, in order to establish (2.36), it remains to consider the case

$$a \leq \zeta_1(\gamma(T)s) \leq b \quad \text{and} \quad 0 \leq T \leq c \quad (2.39)$$

Since, for each fixed $(\xi, s) \in (\mathbb{R}^+)^2$, the mappings $M(\xi, \cdot, s)$, $M(\xi, T, \cdot)$, $\gamma(\cdot)$, and $p(\cdot)$ are nondecreasing, we have

$$\frac{M(\xi, T, s)}{p(T)g(\zeta_1(\gamma(T)s))} \leq \frac{M(\xi, c, \frac{\zeta_1^{-1}(b)}{\gamma(0)})}{g(a)} \quad (2.40)$$

provided that (2.39) holds. By using (2.32) with

$$\varepsilon := \varepsilon g(a) \quad T := c \quad R := \frac{\zeta_1^{-1}(b)}{\gamma(0)}$$

it follows that

$$M\left(\xi, c, \frac{\xi_1^{-1}(b)}{\gamma(0)}\right) \leq \varepsilon g(a) \quad \text{for } \xi \geq \delta(\varepsilon) := \tau\left(\varepsilon g(a), c, \frac{\xi_1^{-1}(b)}{\gamma(0)}\right) \quad (2.41)$$

By taking into account (2.38), (2.40), (2.41), and definition (2.35) of $\mu(\cdot)$ it follows that (2.36) holds with $\delta = \delta(\varepsilon)$ as selected in (2.41). Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{\xi \rightarrow +\infty} \mu(\xi) = 0$. Define

$$\tilde{\mu}(t) := \sup_{\tau \geq t} \mu(\tau) \quad \text{for } t \geq 0; \quad \tilde{\mu}(t) := \sup_{\tau \geq 0} \mu(\tau) \quad \text{for } t < 0 \quad (2.42)$$

Clearly, $\tilde{\mu}(\cdot)$ is a nonincreasing function with $\lim_{\xi \rightarrow +\infty} \tilde{\mu}(\xi) = 0$ and $\tilde{\mu}(t) \geq \mu(t)$ for all $t \geq 0$. Next define, for $t \geq 0$,

$$\bar{\mu}(t) := \exp(-t) + \int_{t-1}^t \tilde{\mu}(w) dw \quad (2.43)$$

Notice that $\bar{\mu} : \mathbb{R}^+ \rightarrow (0, +\infty)$ is a continuous strictly decreasing function such that $\bar{\mu}(\xi) \geq \mu(\xi)$ for all $\xi \geq 0$ and $\lim_{\xi \rightarrow +\infty} \bar{\mu}(\xi) = 0$. Thus, by recalling definition (2.35) we obtain

$$M(\xi, T, s) \leq \bar{\mu}(\xi) \theta(T, s) \quad \text{for all } (T, s) \in (\mathbb{R}^+)^2 \text{ and } \xi \geq 0 \quad (2.44)$$

where $\theta(T, s) := p(T)g(\xi_1(\gamma(T)s))$. Clearly, θ satisfies all hypotheses of Lemma 2.3, and therefore there exist $\zeta \in K_\infty$ and $\beta \in K^+$ such that

$$\theta(T, s) \leq \zeta(\beta(T)s) \quad \text{for all } (T, s) \in (\mathbb{R}^+)^2 \quad (2.45)$$

The desired (2.26) is a consequence of (2.28), (2.44), and (2.45) with $\sigma(s, t) := \bar{\mu}(t)\zeta(s)$ (which is a KL function). $\square$

Lemma 2.6 *Suppose that the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is URGAOS. Then there exists a function $\sigma \in KL$ such that the following estimate holds for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$:*

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \sigma(\|x_0\|_{\mathcal{X}}, t - t_0) \quad (2.46)$$

Proof of Lemma 2.6 Let $\xi \geq 0, s \geq 0$, and define

$$a(s) := \sup\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : d \in M_D, t \geq t_0, \|x_0\|_{\mathcal{X}} \leq s, t_0 \geq 0\} \quad (2.47)$$

$$M(\xi, s) := \sup\{\|H(t_0 + \xi, \phi(t_0 + \xi, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} : d \in M_D, \|x_0\|_{\mathcal{X}} \leq s, t_0 \geq 0\} \quad (2.48)$$

First notice that by virtue of Uniform Robust Lagrange Output Stability, a is well defined, i.e., $a(s) < +\infty$ for every $s \geq 0$. Furthermore, notice that M is well defined,

since by definitions (2.47), (2.48) the following inequality is satisfied for all $\xi \geq 0$ and $s \geq 0$:

$$M(\xi, s) \leq a(s) \quad (2.49)$$

Notice that a is nondecreasing (and consequently locally bounded). Moreover, Uniform Robust Lyapunov Output Stability asserts that $\lim_{s \rightarrow 0^+} a(s) = a(0) = 0$. It turns out from Lemma 2.4 that there exists a function $\zeta_1 \in K_\infty$ such that

$$a(s) \leq \zeta_1(s) \quad \text{for all } s \geq 0 \quad (2.50)$$

Inequality (2.50), in conjunction with (2.49), implies

$$M(\xi, s) \leq \zeta_1(s) \quad \text{for all } (\xi, s) \in (\mathbb{R}^+)^2 \quad (2.51)$$

Moreover, the Uniform Robust Output Attractivity property guarantees that for all $\varepsilon > 0$ and $R \geq 0$, there exists $\tau = \tau(\varepsilon, R) \geq 0$ such that

$$M(\xi, s) \leq \varepsilon \quad \text{for all } \xi \geq \tau(\varepsilon, R) \text{ and } 0 \leq s \leq R \quad (2.52)$$

Let

$$g(s) := \sqrt{s} + s^2 \quad (2.53)$$

and define

$$\mu(\xi) := \sup \left\{ \frac{M(\xi, s)}{g(\zeta_1(s))} : s > 0 \right\} \quad (2.54)$$

Working exactly as in the proof of Lemma 2.5 and using (2.51), (2.52), (2.53), and definition (2.54), we may establish the existence of a continuous strictly decreasing function $\bar{\mu} : \mathbb{R}^+ \rightarrow (0, +\infty)$ with $\bar{\mu}(\xi) \geq \mu(\xi)$ for all $\xi \geq 0$ and $\lim_{\xi \rightarrow +\infty} \bar{\mu}(\xi) = 0$. Thus, by recalling definition (2.54) we obtain

$$M(\xi, s) \leq \bar{\mu}(\xi)\theta(s) \quad \text{for all } \xi, s \geq 0 \quad (2.55)$$

where $\theta(s) := g(\zeta_1(s))$. The desired (2.46) is a consequence of (2.48) and (2.55) with $\sigma(s, t) := \bar{\mu}(t)\theta(s)$ (which is a KL function). $\square$

Finally, the following lemma provides an estimate for the transition map, which turns out to be a necessary and sufficient condition for Robust Forward Completeness for systems which are not necessarily RGAOS.

Lemma 2.7 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with the BIC property, $U = \{0\}$, and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Σ is RFC from the input u if and only if there exist functions $\mu, \beta \in K^+$ and $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, we have*

$$\mu(t) \|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \sigma(\beta(t_0) \|x_0\|_{\mathcal{X}}, t - t_0) \quad \text{for all } t \geq t_0 \quad (2.56)$$

Proof of Lemma 2.7 Suppose first that Σ is RFC from the input u . Define

$$\omega(T, s) := \sup \left\{ \|\phi(t_0 + h, t_0, x_0, u_0, d)\|_{\mathcal{X}}; \|x_0\|_{\mathcal{X}} \leq s, \right. \\ \left. t_0 \in [0, T], h \in [0, T], d \in M_D \right\} \quad (2.57)$$

Notice that by virtue of Robust Forward Completeness from the input u , the function $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is finite valued and for each fixed $t \geq 0$, the mappings $\omega(t, \cdot)$ and $\omega(\cdot, t)$ are nondecreasing. Moreover, definition (2.57) implies that for every $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, the transition map satisfies

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \omega(t, \|x_0\|_{\mathcal{X}}) \quad \text{for all } t \geq t_0 \quad (2.58)$$

Since for each fixed $t \geq 0$, the mappings $\omega(t, \cdot)$ and $\omega(\cdot, t)$ are nondecreasing, inequality (2.58) implies

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \omega(t, t) + \omega(\|x_0\|_{\mathcal{X}}, \|x_0\|_{\mathcal{X}}) \quad \text{for all } t \geq t_0 \quad (2.59)$$

Let $\gamma \in K^+$ be a nondecreasing continuous function that satisfies $1 + \omega(s, s) \leq \gamma(s)$ for all $s \geq 0$ (e.g., $\gamma(s) := 1 + \int_s^{s+1} \omega(\xi, \xi) d\xi$) and define

$$a(s) := s + \begin{cases} s(1 + \gamma(0)) & \text{for } 0 \leq s < 1 \\ 1 + \gamma(s - 1) & \text{for } s \geq 1 \end{cases} \quad \text{and} \quad \mu(t) := \frac{\exp(-t)}{\gamma(t)}$$

with $a \in K_\infty$ and $\mu \in K^+$. It follows that the following inequalities hold for all $s, t \geq 0$:

$$\omega(t, t) + \omega(s, s) \leq \gamma(t) + \gamma(s) \leq \gamma(t)(1 + \gamma(s)) \leq \gamma(t)a(1 + s) \quad (2.60)$$

Inequalities (2.59) and (2.60), in conjunction with (2.58) and definition of $\mu \in K^+$ above, imply that for every $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, the transition map satisfies

$$\mu(t) \|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \exp(-t)a(\|x_0\|_{\mathcal{X}} + 1) \quad \text{for all } t \geq t_0 \quad (2.61)$$

Consider the control system $\Sigma' := (\mathcal{X}, \mathcal{X}, M_U, M_D, \phi, \pi, \tilde{H})$ with $U = \{0\}$, where

$$\tilde{H}(t, x, 0) := \mu(t)x \quad (2.62)$$

It can be immediately verified that $\Sigma' := (\mathcal{X}, \mathcal{X}, M_U, M_D, \phi, \pi, \tilde{H})$ is a control system with outputs and the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. Moreover, by virtue of (2.61) and Lemma 2.1, it follows that Σ' is RGAOS. Consequently, Lemma 2.5 guarantees the existence of functions $\sigma \in KL$ and $\beta \in K^+$ such that (2.56) holds for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$.

Conversely, suppose that (2.56) holds for the transition map of Σ . Clearly, by virtue of the BIC property, for each $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, there exists a maximal existence time, say, $t_{\max} \in \mathbb{R}^+ \cup \{+\infty\}$, such that $[t_0, t_{\max}) \times \{(t_0, x_0, u_0, d)\} \subseteq A_\phi$ and for all $t \geq t_{\max}$, it holds that $(t, t_0, x_0, u_0, d) \notin A_\phi$. In addition, if $t_{\max} <$

$+\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} > M$. Suppose that $t_{\max} < +\infty$ for some $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$. Then there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} > \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, 0) \max_{t \in [0, t_{\max}]} \frac{1}{\mu(t)}$. On the other hand, since (2.56) holds, we obtain

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, 0) \max_{t \in [0, t_{\max}]} \frac{1}{\mu(t)}$$

which is clearly a contradiction. Thus, since (2.56) holds, we must have $t_{\max} = +\infty$ for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$. Moreover, using (2.56) for all $\varepsilon \geq 0$ and $T \geq 0$, we obtain

$$\begin{aligned} & \sup \left\{ \|\phi(t_0 + s, t_0, x_0, u_0, d)\|_{\mathcal{X}}; s \in [0, T], \|x_0\|_{\mathcal{X}} \leq \varepsilon, t_0 \in [0, T], d \in M_D \right\} \\ & \leq \sigma \left(\varepsilon \max_{t \in [0, T]} \beta(t), 0 \right) \max_{t \in [0, 2T]} \frac{1}{\mu(t)} < +\infty \end{aligned}$$

i.e., the property of Robust Forward Completeness from the input u is satisfied. $\square$

We are now in a position to prove the main results of the section.

Proof of Theorem 2.1 The implication (ii) $\Rightarrow$ (iii) is obvious. The implication (iii) $\Rightarrow$ (i) follows immediately by applying the results of Lemma 2.1 and the definition of RFC. To be more precise, notice that inequality (2.10) implies that Σ is RFC and (by virtue of the properties of the KL functions) estimate (2.9) implies that Σ satisfies the Robust Output Attractivity Property. Consequently, since $0 \in \mathcal{X}$ is a robust equilibrium point, it follows by virtue of Lemma 2.1 that Σ is RGAOS.

Next we prove implication (i) $\Rightarrow$ (ii). Suppose that Σ is RGAOS. Then Lemmas 2.5 and 2.7 guarantee that there exist functions $\bar{\sigma} \in KL$, $\tilde{\sigma} \in KL$, and $\bar{\beta}, \tilde{\beta}, \mu \in K^+$ such that the following estimates hold for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$:

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \bar{\sigma}(\bar{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t - t_0) \quad (2.63)$$

$$\mu(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \tilde{\sigma}(\tilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t - t_0) \quad (2.64)$$

Estimates (2.63) and (2.64) imply (2.8) for $\sigma(s, t) := \bar{\sigma}(s, t) + \tilde{\sigma}(s, t)$ and $\beta(t) := \max\{\bar{\beta}(t), \tilde{\beta}(t)\}$. The proof is complete. $\square$

Proof of Theorem 2.2 The implication (i) $\Rightarrow$ (ii) is a direct consequence of Lemma 2.6. The implication (ii) $\Rightarrow$ (i) follows directly from the properties of KL functions. $\square$

2.4 Transformations Preserving RGAOS

The method of verifying RGAOS using transformations is used frequently in Mathematical Control Theory and Stability Theory. Roughly speaking, we want to verify

RGAOS for a system Σ which is the transformation of another system Σ' (in the sense described in Sect. 1.6 of Chap. 1). Suppose that we can establish that Σ' is RGAOS (using another method, e.g., by solving the differential (or difference) equations (or inequalities)). Then RGAOS for Σ is guaranteed under some additional technical assumptions that are presented below.

We start with the following elementary observation, which is a direct consequence of Definitions 1.4, 1.7, and 1.8.

Lemma 2.8 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ which is RFC from the input u . Let $\Phi : \mathbb{R}^+ \times \mathcal{X} \rightarrow \mathcal{X}$ be a change of coordinates, and $q : \mathbb{R}^+ \times V \rightarrow U$ a transformation of V onto U . Then system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ defined by (1.106), (1.107), (1.108) is a control system which is RFC from the input $v \in M_V$.*

Making use of the results of Lemma 1.2, Theorems 2.1, and 2.2, we are in a position to show the following propositions.

Proposition 2.1 *Consider a control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ with $U = \{0\}$, which satisfies the classical semigroup property and the BIC property, and for which $0 \in \mathcal{X}$ is a robust equilibrium point from the input $u \in M_U$. Suppose that Σ is RGAOS. Let $\Phi : \mathbb{R}^+ \times \mathcal{X} \rightarrow \mathcal{X}$ be a change of coordinates. Then, $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ defined by (1.106), (1.107), (1.108) with $V = U = \{0\}$ is RGAOS. Moreover, if Σ is URGAOS and there exists $a \in K_\infty$ such that*

$$\|\Phi^{-1}(t, x)\|_{\mathcal{X}} \leq a(\|x\|_{\mathcal{X}}) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathcal{X} \quad (2.65)$$

then system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ defined by (1.106), (1.107), (1.108) with $V = U = \{0\}$ is URGAOS.

Proof By virtue of Lemma 1.2, $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_V, M_D, \phi', \pi', H')$ is a deterministic control system which satisfies the classical semigroup property and the BIC property. Moreover, $0 \in \mathcal{X}$ is a robust equilibrium point from the input u for Σ' . By virtue of Lemma 2.8, system $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is RFC from the input u .

Finally, notice that by virtue of Lemma 2.5, there exist functions $\sigma \in KL$ and $\beta \in K^+$ such that estimate (2.26) holds for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$. Definitions (1.106) and (1.108), in conjunction with (2.20), imply that the following estimate holds for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$ and $t \geq t_0$:

$$\begin{aligned} \|H'(t, \phi'(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} &= \|H(t, \phi(t, t_0, \Phi^{-1}(t_0, x_0), u_0, d), 0)\|_{\mathcal{Y}} \\ &\leq \sigma(\beta(t_0) \|\Phi^{-1}(t_0, x_0)\|_{\mathcal{X}}, t - t_0) \end{aligned} \quad (2.66)$$

Since $\Phi : \mathbb{R}^+ \times \mathcal{X} \rightarrow \mathcal{X}$ is a change of coordinates, there exists $a \in K_\infty$ and $\gamma \in K^+$ such that $\|\Phi^{-1}(t, x)\|_{\mathcal{X}} \leq a(\gamma(t)\|x\|_{\mathcal{X}})$ for all $(t, x) \in \mathbb{R}^+ \times \mathcal{X}$. Consequently, we

have

$$\|H'(t, \phi'(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \sigma(\beta(t_0)a(\gamma(t_0)\|x_0\|_{\mathcal{X}}), t - t_0) \quad (2.67)$$

The above inequality shows that properties P1, P2, and P3 of Definition 2.2 hold, and consequently $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is RGAOS. Moreover, if Σ is URGAOS and inequality (2.65) holds, then we obtain

$$\|H'(t, \phi'(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \tilde{\sigma}(\|x_0\|_{\mathcal{X}}, t - t_0) \quad (2.68)$$

where $\tilde{\sigma}(s, t) := \sigma(a(s), t)$ is a *KL* function. In this case, Theorem 2.2 guarantees that $\Sigma' := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi', \pi', H')$ is URGAOS. The proof is complete. $\square$

Example 2.4.1 Transformation methods were used in the early stages of Stability Theory to derive conditions for asymptotic stability of linear systems described by ODEs, i.e.,

$$\begin{aligned} \dot{x} &= Ax \quad x \in \mathfrak{R}^n \\ Y &= x \end{aligned} \quad (2.69)$$

where the matrix $A \in \mathfrak{R}^{n \times n}$ is constant. The Jordan theorem on canonical representation of matrices guarantees the existence of a nonsingular matrix $P \in \mathfrak{R}^{n \times n}$ such that

$$PAP^{-1} = \tilde{A} = \begin{bmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & J_{r-1} & 0 \\ 0 & 0 & \dots & 0 & J_r \end{bmatrix} \quad (2.70)$$

where $J_1, \dots, J_r$ are the so-called Jordan blocks corresponding to the eigenvalues $\lambda_k = a_k + i\beta_k$ ($k = 1, \dots, n$) of the matrix $A \in \mathfrak{R}^{n \times n}$ (here i denotes the imaginary unit). Particularly, blocks J_k which correspond to real eigenvalues $\lambda_k = a_k \in \mathfrak{R}$ have the form

$$J_k = [a_k] \quad \text{or} \quad J_k = \begin{bmatrix} a_k & \gamma_k & \dots & 0 & 0 \\ 0 & a_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_k & \gamma_k \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix} \quad \text{with } \gamma_k \neq 0 \quad (2.71)$$

and blocks J_k which correspond to complex eigenvalues $\lambda_k = a_k + i\beta_k$ have the form

$$J_k = \begin{bmatrix} a_k & \beta_k \\ -\beta_k & a_k \end{bmatrix} \quad \text{or} \quad J_k = \begin{bmatrix} K_k & L_k & \dots & 0 & 0 \\ 0 & K_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & K_k & L_k \\ 0 & 0 & \dots & 0 & L_k \end{bmatrix} \quad (2.72)$$

with

$$K_k = \begin{bmatrix} a_k & \beta_k \\ -\beta_k & a_k \end{bmatrix} \quad L_k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \gamma_k \end{bmatrix}$$

Consider the linear system

$$\begin{aligned} \dot{z}_i &= J_i z_i \quad i = 1, \dots, r \\ Y &= P^{-1} z \\ z &= (z'_1, \dots, z'_r)' \in \mathfrak{R}^n \end{aligned} \quad (2.73)$$

where $J_1, \dots, J_r$ are the Jordan blocks of $A \in \mathfrak{R}^{n \times n}$ that appear in (2.70), and $P \in \mathfrak{R}^{n \times n}$ is the nonsingular matrix involved in (2.70). It should be clear from (2.70) that system (2.69) is the transformation of system (2.73) corresponding to the change of coordinates $\Phi(t, z) := P^{-1}z$. On the other hand, system (2.73) is the transformation of system (2.69) corresponding to the change of coordinates $\Phi(t, x) := Px$. Consequently, by virtue of Proposition 2.1, system (2.73) is URGAS if and only if system (2.69) is URGAS.

The reader should notice that the solution of system (2.73) can be found explicitly (due to the simple structure of the Jordan blocks given in (2.71) and (2.72)). The formulae for the solution of the solution of system (2.73) show that if J_k is one of the blocks described in (2.71) and (2.72) and $a_k < 0$, then the solution of $\dot{z}_k = J_k z_k$ satisfies $|z_k(t)| \leq M_k \exp(-\sigma_k(t - t_0))|z_k(t_0)|$ for all $t \geq t_0$, where $M_k \geq 1$ and $\sigma_k \in (0, -a_k]$ are appropriate constants. Consequently, if all eigenvalues of the matrix $A \in \mathfrak{R}^{n \times n}$ have negative real parts, then we have, for all $t \geq t_0$,

$$|Y(t)| \leq \exp(-\sigma(t - t_0))|z(t_0)| \left\| P^{-1} \right\| \sum_{k=1}^r M_k \quad (2.74)$$

where $\sigma := \min\{\sigma_k, k = 1, \dots, r\}$. Estimate (2.74) implies that system (2.73) is URGAS. Moreover, if J_k is one of the blocks described in (2.71) and (2.72) and $a_k \geq 0$, then the solution of $\dot{z}_k = J_k z_k$ does not converge to zero for all initial conditions. Thus if one of the real parts of the eigenvalues of $A \in \mathfrak{R}^{n \times n}$ is nonnegative, then system (2.73) is not URGAS.

Thus, we have shown that system (2.69) is RGAS if and only if all eigenvalues of the matrix $A \in \mathfrak{R}^{n \times n}$ have negative real parts.

2.5 Differential Inequalities and Comparison Lemmas

The purpose of this section is to present basic tools that enable us to handle differential inequalities. Differential inequalities can be used to obtain estimates of the state of a control system. The derived estimates can be further utilized for the proof of RGAOS (this is essentially the first method of proving stability, the method of analytical solutions) or can be combined with the method of Lyapunov functionals (as will be shown in the following section).

We start with two technical lemmas which show how one can obtain inequalities for continuous functions using differential inequalities.

Lemma 2.9 *Let $y : [t_0, t_1) \rightarrow \mathbb{R}$ be an absolutely continuous function, where $t_0 < t_1 \leq +\infty$, and suppose that the following implication holds for certain constants $a > b$:*

“If $b < y(t) < a$ for some $t \in [t_0, t_1)$ and $\dot{y}(t)$ exists, then $\dot{y}(t) \geq 0$.”

Then, the following implication holds:

“If $y(t_0) \geq a$, then $y(t) \geq a$ for all $t \in [t_0, t_1)$.”

Proof The proof is made by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $y(t) < a$. Without loss of generality we may assume that $y(\tau) > b$ for all $\tau \in [t_0, t)$, because if the set $B := \{\tau \in [t_0, t] : y(\tau) \leq b\}$ is nonempty, then we may define $\bar{t} := \inf B$, which satisfies $\bar{t} > t_0$ and $y(\tau) > b$ for all $\tau \in [t_0, \bar{t})$, and replace $t \in (t_0, t_1)$ by $\bar{t} \in (t_0, t_1)$.

Since $y(t_0) \geq a$, by continuity of $y : [t_0, t_1) \rightarrow \mathbb{R}$ it follows that the set $A := \{\tau \in [t_0, t) : y(\tau) = a\}$ is nonempty and closed. Let $T := \sup A$ and notice that by virtue of continuity of $y : [t_0, t_1) \rightarrow \mathbb{R}$ and the fact $y(t) < a$, we get $T < t$. Consequently, we have $y(\tau) \neq a$ for all $\tau \in (T, t)$ and $y(T) = a$. More specifically, we have $y(\tau) < a$ for all $\tau \in (T, t)$ (since the existence of $\tau \in (T, t]$ with $y(\tau) > a$ would imply the existence of certain $\xi \in (\tau, t)$ with $y(\xi) = a$ and thus contradicting the definition of $T := \sup A$). Consequently, the assumed implication gives $\dot{y}(\tau) \geq 0$ for almost all $\tau \in (T, t)$.

Since $y(t) = y(T) + \int_T^t \dot{y}(s) ds$, we obtain $y(t) \geq y(T) = a$, which contradicts the hypothesis $y(t) < a$. The proof is complete. $\square$

Lemma 2.10 *Let $y : [t_0, t_1) \rightarrow \mathbb{R}$ be a continuous function, where $t_0 < t_1 \leq +\infty$, and suppose that the following implication holds for certain constant $a \in \mathbb{R}$:*

“If $y(t) = a$ for some $t \in [t_0, t_1)$, then $D^+ y(t) := \limsup_{h \rightarrow 0^+} \frac{y(t+h) - y(t)}{h} > 0$.”

Then, the following implication holds:

“If $y(t_0) \geq a$, then $y(t) \geq a$ for all $t \in [t_0, t_1)$.”

Proof The proof is made by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $y(t) < a$. Since $y(t_0) \geq a$, by continuity of $y : [t_0, t_1) \rightarrow \mathbb{R}$ it follows that the set $A := \{\tau \in [t_0, t) : y(\tau) = a\}$ is nonempty and closed. Let $T := \sup A$ and notice that by virtue of continuity of $y : [t_0, t_1) \rightarrow \mathbb{R}$ and the fact $y(t) < a$, we get $T < t$. Consequently, we have $y(\tau) \neq a$ for all $\tau \in (T, t]$ and $y(T) = a$. More specifically, we have $y(\tau) < a$ for all $\tau \in (T, t]$ (since the existence of $\tau \in (T, t]$ with $y(\tau) > a$ would imply the existence of certain $\xi \in (\tau, t)$ with $y(\xi) = a$ and thus contradicting the definition of $T := \sup A$).

Thus, for sufficiently small $h > 0$, we obtain $\frac{y(T+h)-y(T)}{h} < 0$. Taking limits, we get $D^+y(T) := \limsup_{h \rightarrow 0^+} \frac{y(T+h)-y(T)}{h} \leq 0$, contradicting the hypothesis that $D^+y(T) := \limsup_{h \rightarrow 0^+} \frac{y(T+h)-y(T)}{h} > 0$. The proof is complete. $\square$

We continue with a technical lemma which provides estimates involving *KL* functions. The reader should recall that a function μ is of class $\mathcal{E}$ if $\mu \in C^0(\mathbb{R}^+; \mathbb{R}^+)$ with $\lim_{t \rightarrow +\infty} \mu(t) = 0$ and $\int_0^{+\infty} \mu(t) dt < +\infty$.

Lemma 2.11 *For every pair of functions $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ being C^0 positive definite and μ being of class $\mathcal{E}$ with $\mu(t) > 0$ for all $t \geq 0$, there exists a *KL* function $\sigma : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ which satisfies the following property:*

(P) *If $c \in [0, 1]$ is a constant, $t_0 \geq 0$, and $y : [t_0, +\infty) \rightarrow \mathbb{R}^+$ is an absolutely continuous function that satisfies the following differential inequality for almost all $t \geq t_0$:*

$$\dot{y}(t) \leq -\rho(y(t)) + c\mu(t) \quad (2.75)$$

then the following estimate holds:

$$y(t) \leq \sigma\left(y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds, t - t_0\right) \quad \text{for all } t \geq t_0. \quad (2.76)$$

Proof Let $G(\rho, \mu, c, t_0)$ be the set of all absolutely continuous functions $y : [t_0, +\infty) \rightarrow \mathbb{R}^+$ which satisfy differential inequality (2.75) a.e. for $t \geq t_0 \geq 0$ and $c \in [0, 1]$.

Define

$$g(s) := \sqrt{s} + s^2 \quad (2.77)$$

$$v(\xi) := \sup \left\{ \frac{y(t_0 + \xi)}{g(y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds)}; y \in G(\rho, \mu, c, t_0), c \in (0, 1], t_0 \geq 0 \right\} \quad (2.78)$$

We will next show that $v(t) \leq 1$ for all $t \geq 0$ and $\lim_{t \rightarrow +\infty} v(t) = 0$. Let $\bar{v} : \mathbb{R}^+ \rightarrow [0, +\infty)$ be a nonincreasing C^0 function with $v(t) \leq \bar{v}(t)$ for all $t \geq 0$ and such that $\lim_{t \rightarrow +\infty} \bar{v}(t) = 0$ (for example, take $\bar{v}(t) := \sup_{s \geq t} v(s)$ for $t \geq 0$ and $\bar{v}(t) := \bar{v}(0)$ for $t < 0$, which is a nonincreasing function with $v(t) \leq \bar{v}(t)$ for all $t \geq 0$

and $\lim_{t \rightarrow +\infty} \tilde{v}(t) = 0$, and define $\bar{v}(t) := \int_{t-1}^t \tilde{v}(s) ds$ for $t \geq 0$. Define the *KL* function

$$\sigma(s, t) := g(s)\bar{v}(t) \quad (2.79)$$

Then, definitions (2.78) and (2.79) imply that (2.76) is fulfilled for all $y \in G(\rho, \mu, c, t_0)$ with $c \in (0, 1]$ and $t_0 \geq 0$. Finally, the case $c = 0$ is implied by the continuity of $\sigma(s, t) := g(s)\bar{v}(t)$.

First, notice that (2.75) yields

$$y(t) \leq y(t_0) + c \int_{t_0}^t \mu(s) ds \quad \text{for all } t \geq t_0 \quad (2.80)$$

for all $y \in G(\rho, \mu, c, t_0)$. Since $\lim_{t \rightarrow +\infty} \mu(t) = 0$ for any constants $r \geq \varepsilon > 0$, there exists a time $\tau := \tau(\varepsilon, r) \geq 0$ such that

$$t \geq \tau \Rightarrow \mu(t) \leq \min \left\{ \frac{1}{2} \rho(s); \frac{\varepsilon}{2} \leq s \leq r \right\} \quad (2.81)$$

Let $0 < \varepsilon < r$. We now show the following implication for all $y \in G(\rho, \mu, c, t_0)$:

$$\text{If } t_1 \geq \tau(\varepsilon, r) \text{ and } y(t_1) \leq \varepsilon, \quad \text{then } y(t) \leq \varepsilon, \quad \forall t \geq t_1 \quad (2.82)$$

To see this, notice that, when $r \geq y(t) \geq \frac{\varepsilon}{2}$ and $t \geq \tau(\varepsilon, r)$, then by (2.75) and (2.81) we have

$$\dot{y}(t) \leq -\rho(y(t)) + \mu(t) \leq -\frac{1}{2} \rho(y(t)) < 0 \quad (2.83)$$

Lemma 2.9 and the above inequality show that implication (2.82) holds. We next establish that, if we define

$$T(\varepsilon, r) := \tau(\varepsilon, r) + \frac{2r}{\min_{\varepsilon \leq s \leq r} \rho(s)} \quad (2.84)$$

then the following is fulfilled for all $R \geq 0$ and $\varepsilon > 0$ with $\varepsilon < R + M$:

$$\text{For all } t \geq t_0 + T(\varepsilon, R + M) \text{ and } y \in G(\rho, \mu, c, t_0) \quad \text{with } y(t_0) \leq R \Rightarrow y(t) \leq \varepsilon \quad (2.85)$$

where $M := \int_0^{+\infty} \mu(t) dt > 0$. Indeed, otherwise, by implication (2.82), there would exist $R \geq 0$ and $\varepsilon > 0$ with $\varepsilon < R + M$, $y \in G(\rho, \mu, c, t_0)$ with $y(t_0) \leq R$, and $t \geq t_0 + T(\varepsilon, R + M)$ such that $y(t) > \varepsilon$. Since $t \geq \tau(\varepsilon, R + M)$, we would have, by virtue of (2.80) and (2.82) with $r = R + M$,

$$y(t) > \varepsilon \quad \text{for all } t \in [t_0 + \tau(\varepsilon, R + M), t_0 + T(\varepsilon, R + M)] \quad (2.86)$$

On the other hand, by (2.75), (2.80), (2.81), and (2.86) it follows that

$$\dot{y}(t) \leq -\frac{1}{2} \min_{\varepsilon \leq s \leq R+M} \rho(s), \quad \text{for all } t \in [t_0 + \tau(\varepsilon, R + M), t_0 + T(\varepsilon, R + M)] \quad (2.87)$$

It turns out from (2.86) and (2.87) that

$$\varepsilon < y(t) \leq R + M - \frac{1}{2}(t - t_0 - \tau) \min_{\varepsilon \leq s \leq R+M} \rho(s) \quad (2.88)$$

for all $t \in [t_0 + \tau(\varepsilon, R + M), t_0 + T(\varepsilon, R + M)]$.

Using (2.88) and taking into account definition (2.84) of $T(\cdot)$, we get $\varepsilon < y(t_0 + T) \leq 0$, which is a contradiction. This establishes (2.85).

Implication (2.82), inequality (2.80), and property (2.85) guarantee that the following attractivity property holds:

$$\begin{aligned} &\text{For all } (\varepsilon, R, t_0, c) \in (0, +\infty) \times \mathbb{R}^+ \times \mathbb{R}^+ \times [0, 1], y \in G(\rho, \mu, c, t_0) \\ &\text{with } y(t_0) \leq R \text{ and } t \geq t_0 + T(\varepsilon, R + M) \Rightarrow 0 \leq y(t) \leq \varepsilon \end{aligned} \quad (2.89)$$

We are now ready to establish that $v(t) \leq 1$ for all $t \geq 0$ and $\lim_{t \rightarrow +\infty} v(t) = 0$ for the function v defined by (2.78). Since $c > 0$ and $\mu(t) > 0$ for all $t \geq 0$, it follows that the denominator in (2.78) is strictly positive and (2.77), (2.80) imply that $v(t) \leq 1$ for all $t \geq 0$. Let $\varepsilon > 0$, and let $a := a(\varepsilon)$, $b := b(\varepsilon)$ be a pair of constants with $0 < a < b$ and being defined in such a way that $x \notin [a, b] \Rightarrow \frac{x}{\sqrt{x+x^2}} < \varepsilon$. Then, by (2.77) and (2.80), it follows that

$$\frac{y(t)}{g(y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds)} < \varepsilon \quad \text{for all } t \geq t_0, y \in G(\rho, \mu, c, t_0)$$

provided that either $y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds < a$ or $y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds > b$.

It remains to consider the case $a \leq y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds \leq b$. By (2.89) we get

$$\frac{y(t)}{g(y(t_0) + c \int_{t_0}^{+\infty} \mu(s) ds)} \leq \frac{y(t)}{g(a)} \leq \varepsilon \quad \text{for all } t \geq t_0 + T(\varepsilon g(a), b + M) \quad (2.90)$$

It turns out from (2.78) and (2.90) that

$$v(t) \leq \varepsilon \quad \text{for all } t \geq T(\varepsilon g(a), b + M) \quad (2.91)$$

Since $\varepsilon > 0$ is arbitrary, (2.91) asserts that $\lim_{t \rightarrow +\infty} v(t) = 0$. The proof is thus complete. $\square$

The following comparison principle can be applied to lower semi-continuous functions and will be useful for the derivation of estimates of values of Lyapunov functionals.

Lemma 2.12 (Comparison principle) *Consider the scalar differential equation*

$$\begin{aligned} \dot{w} &= f(t, w) \\ w(t_0) &= w_0 \end{aligned} \quad (2.92)$$

where $f(t, w)$ is continuous with

$$\sup \left\{ \frac{(x - w)(f(t, x) - f(t, w))}{|x - w|^2} : (t, x) \in S, (t, w) \in S, x \neq w \right\} < +\infty$$

for every compact $S \subset \mathbb{R}^+ \times J$, where $J \subseteq \mathbb{R}$ is an open interval. Let $T > t_0$ be such that the unique solution $w(t)$ of the initial-value problem (2.92) exists and satisfies $w(t) \in J$ for all $t \in [t_0, T)$. Let $v : [t_0, T) \rightarrow \mathbb{R}$ be a lower semi-continuous function that satisfies the differential inequality

$$D^+v(t) := \limsup_{h \rightarrow 0^+} \frac{v(t+h) - v(t)}{h} \leq f(t, v(t)) \quad \text{for all } t \in [t_0, T) \quad (2.93)$$

Suppose furthermore that

$$v(t_0) \leq w_0 \quad (2.94)$$

$$v(t) \in J \quad \text{for all } t \in [t_0, T) \quad (2.95)$$

and that one of the following holds:

- (i) the mapping $f(t, \cdot)$ is nondecreasing on $J \subseteq \mathbb{R}$ for each fixed $t \in [t_0, T)$;
- (ii) there exists $\phi \in C^0(\mathbb{R}^+)$ such that $f(t, w) \leq \phi(t)$ for all $(t, w) \in [t_0, T) \times J$;
- (iii) the function $v : [t_0, T) \rightarrow \mathbb{R}$ is right-continuous.

Then $v(t) \leq w(t)$ for all $t \in [t_0, T)$.

Proof It suffices to show that $v(t) \leq w(t)$ on any interval $[t_0, t_1] \subset [t_0, T)$. Consider the scalar differential equation

$$\begin{aligned} \dot{z} &= f(t, z) + \lambda \\ z(t_0) &= w_0 \end{aligned} \quad (2.96)$$

where λ is a positive constant.

Claim On any compact interval $[t_0, t_1] \subset [t_0, T)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \lambda < \delta$, then (2.96) has a unique solution $z(t, \lambda)$ defined on $[t_0, t_1]$ and satisfies

$$z(t, \lambda) \in J \quad |z(t, \lambda) - w(t)| < \varepsilon \quad \text{for all } t \in [t_0, t_1] \quad (2.97)$$

Proof of Claim Since $J \subseteq \mathbb{R}$ is an open interval and $[t_0, t_1]$ is compact, we can find $\rho > 0$ such that if $x \in \mathbb{R}$ with $|x - w(t)| \leq \rho$ for some $t \in [t_0, t_1]$, then $x \in J$. Let $S := [t_0, t_1] \times [a, b]$, where $a := \min_{t \in [t_0, t_1]} w(t) - \rho$ and $b := \max_{t \in [t_0, t_1]} w(t) + \rho$ (notice that $S \subset [t_0, t_1] \times J$), $L := \sup \left\{ \frac{(x-y)(f(t, x) - f(t, y))}{|x-y|^2} : (t, x) \in S, (t, y) \in S, x \neq y \right\} < +\infty$, and $\lambda < \delta := \min\{\frac{\varepsilon}{4}, \frac{\rho}{2}\} \exp(-\frac{L+1}{2}(t_1 - t_0))$. Clearly, there exists $\tau \in (t_0, t_1]$ such that the solution of (2.96) exists for $t \in [t_0, \tau]$ and satisfies

$(t, z(t, \lambda)) \in S$ for all $t \in [t_0, \tau]$. Define $v(t) := |z(t, \lambda) - w(t)|^2$. Using the above definitions, we get $\dot{v}(t) \leq 2(L+1)v(t) + \lambda^2$ for all $t \in [t_0, \tau]$, which directly implies that $|z(t, \lambda) - w(t)| \leq \lambda \exp(\frac{L+1}{2}(t_1 - t_0))$ for all $t \in [t_0, \tau]$. Using the fact that $\lambda < \delta := \min\{\frac{\varepsilon}{4}, \frac{\rho}{2}\} \exp(-\frac{L+1}{2}(t_1 - t_0))$ and a standard contradiction argument, we can show that $|z(t, \lambda) - w(t)| \leq \min\{\frac{\varepsilon}{4}, \frac{\rho}{2}\}$ for all $t \in [t_0, t_1]$. $\square$

Fact I $v(t) \leq z(t, \lambda)$, for all $t \in [t_0, t_1]$.

Proof of Fact I This fact is shown by contradiction. Suppose that there exists $t \in (t_0, t_1)$ such that $v(t) - z(t, \lambda) > 0$. For the lower semi-continuous function $m(t) := v(t) - z(t, \lambda)$, define the set

$$A^+ := \{\tau \in (t_0, t_1) : m(\tau) > 0\} \quad (2.98)$$

which by assumption is nonempty. The lower semi-continuity of $m(t) := v(t) - z(t, \lambda)$ implies that A^+ is open. Let

$$\tilde{t} := \inf\{t \in A^+\} \quad (2.99)$$

Since A^+ is open, we conclude that $\tilde{t} \notin A^+$ or equivalently that $v(\tilde{t}) \leq z(\tilde{t}, \lambda)$. On the other hand, by definition (2.99) there exists a sequence $\{\tau_i \in A^+\}_{i=1}^\infty$ with $\tau_i \rightarrow \tilde{t}$. Consequently, we obtain

$$v(\tau_i) - v(\tilde{t}) \geq z(\tau_i, \lambda) - z(\tilde{t}, \lambda) \quad (2.100)$$

This implies

$$D^+v(\tilde{t}) \geq \dot{z}(\tilde{t}, \lambda) = f(\tilde{t}, z(\tilde{t}, \lambda)) + \lambda \quad (2.101)$$

We distinguish the following cases:

- (i) If the mapping $f(t, \cdot)$ is nondecreasing on $J \subseteq \mathfrak{R}$, then the inequality $v(\tilde{t}) \leq z(\tilde{t}, \lambda)$ implies $f(\tilde{t}, v(\tilde{t})) \leq f(\tilde{t}, z(\tilde{t}, \lambda))$. The latter inequality, combined with (2.101), implies $D^+v(\tilde{t}) > f(\tilde{t}, v(\tilde{t}))$, which contradicts (2.93).
- (ii) If there exists a continuous function $\phi : [t_0, T] \rightarrow \mathfrak{R}$ such that $f(t, w) \leq \phi(t)$ for all $(t, w) \in [t_0, T] \times J$, then we may define the lower semi-continuous function $\tilde{v}(t) = v(t) - \int_{t_0}^t \phi(s) ds$. This function satisfies the following differential inequality:

$$D^+\tilde{v}(t) \leq D^+v(t) - \phi(t) \leq f(t, v(t)) - \phi(t) \leq 0$$

Consequently, by virtue of Lemma 6.3 in [3], $\tilde{v}(t)$ is nonincreasing. This implies that $\tilde{v}(t+h) \leq \tilde{v}(t)$ for all $h \geq 0$. Moreover, the lower semi-continuity of $\tilde{v}(t)$ implies that for every $\varepsilon > 0$, the inequality $\tilde{v}(t+h) \geq \tilde{v}(t) - \varepsilon$ for sufficiently small $h \geq 0$. It follows that $\tilde{v}(t)$ is a right-continuous function on $[t_0, t_1]$. By virtue of the right-continuity and definition (2.99), we must also have $v(\tilde{t}) \geq z(\tilde{t}, \lambda)$. Thus we must have $v(\tilde{t}) = z(\tilde{t}, \lambda)$, and in this case by virtue of (2.101) we obtain $D^+v(\tilde{t}) > f(\tilde{t}, v(\tilde{t}))$, which contradicts (2.93).

- (iii) By virtue of the right-continuity of the function $v : [t_0, T) \rightarrow \Re$ and definition (2.99), we must also have $v(\tilde{t}) \geq z(\tilde{t}, \lambda)$. Thus we must have $v(\tilde{t}) = z(\tilde{t}, \lambda)$, and in this case by virtue of (2.101) we obtain $D^+v(\tilde{t}) > f(\tilde{t}, v(\tilde{t}))$, which contradicts (2.93). $\square$

Fact II $v(t) \leq w(t)$ for all $t \in [t_0, t_1]$.

Proof of Fact II Again, this claim may be shown by contradiction. Suppose that there exists $a \in (t_0, t_1)$ with $v(a) > w(a)$. Let $\varepsilon = \frac{1}{2}(v(a) - w(a)) > 0$. Furthermore, let $\lambda > 0$ be selected in such a way that (2.97) is satisfied with this particular selection of $\varepsilon > 0$. Then we obtain

$$v(a) = v(a) - w(a) + w(a) = 2\varepsilon + w(a) - z(a, \lambda) + z(a, \lambda) > \varepsilon + z(a, \lambda)$$

which contradicts Fact I. $\square$

Finally, notice that $v(t_1) = \liminf_{t \rightarrow t_1} v(t) \leq \liminf_{t \rightarrow t_1} w(t) = w(t_1)$ and consequently the inequality $v(t) \leq w(t)$ holds for all $t \in [t_0, t_1]$. The proof is complete. $\square$

The following lemma exploits the results of the two previous lemmas and provides a sharper estimate to a subclass of the differential inequalities considered in Lemma 2.11.

Lemma 2.13 *For every continuous positive definite function $\rho : \Re^+ \rightarrow \Re^+$ being locally Lipschitz on $\Re^+ \setminus \{0\}$, there exists a KL function $\sigma : (\Re^+)^2 \rightarrow \Re^+$ with $\sigma(s, 0) = s$ and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ for all $t, s \geq 0$ which satisfies the following property:*

- (P) *If $t_1 > t_0 \geq 0$ and $y : [t_0, t_1] \rightarrow \Re^+$ is an absolutely continuous function that satisfies the following differential inequality for almost all $t \in [t_0, t_1]$:*

$$\dot{y}(t) \leq -\rho(y(t)) \tag{2.102}$$

then the following estimate holds:

$$y(t) \leq \sigma(y(t_0), t - t_0) \quad \text{for all } t \in [t_0, t_1]. \tag{2.103}$$

Proof Consider the autonomous scalar differential equation

$$\begin{aligned} \dot{w}(t) &= -\rho(w(t)) \\ w(t_0) &= w_0 \end{aligned} \tag{2.104}$$

where $w_0 > 0$. We will next show that the above initial-value problem admits a solution denoted by $\zeta(t - t_0, w_0)$ defined for all $t \geq t_0$ which has the following properties:

1. either $\zeta(t - t_0, w_0) > 0$ for all $t \geq t_0$, or there exists $t_{\max} > t_0$ such that $\zeta(t - t_0, w_0) > 0$ for all $t \in [t_0, t_{\max})$ and $\zeta(t - t_0, w_0) = 0$ for all $t \geq t_{\max}$,
2. the function $\sigma : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ defined by $\sigma(s, t) := \zeta(t, s)$ for all $t \geq 0, s > 0$ and $\sigma(0, t) := 0$ for all $t \geq 0$ is a *KL* function with $\sigma(s, 0) = s$ and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ for all $t, s \geq 0$.

Then property (P) is an immediate consequence of Lemma 2.12. Indeed, if $y : [t_0, t_1] \rightarrow \mathbb{R}^+$ is an absolutely continuous function that satisfies differential inequality (2.102) for almost all $t \in [t_0, t_1]$, we have $y(t+h) = y(t) + \int_t^{t+h} \dot{y}(\tau) d\tau \leq y(t) - \int_t^{t+h} \rho(y(\tau)) d\tau$ for all $t \in [t_0, t_1]$ and $h > 0$ sufficiently small. Therefore, we get $D^+y(t) := \limsup_{h \rightarrow 0^+} \frac{y(t+h) - y(t)}{h} \leq -\rho(y(t))$ for all $t \in [t_0, t_1]$. Applying Lemma 2.12 with $J := (0, +\infty)$ and $T := \sup\{t \in [t_0, t_1] : y(t) > 0\}$, we get (2.103) for all $t \in [t_0, T)$ (notice that if $t_{\max} < T$, where $t_{\max} > t_0$ is the time with $\zeta(t - t_0, y(t_0)) > 0$ for all $t \in [t_0, t_{\max})$ and $\zeta(t - t_0, y(t_0)) = 0$ for all $t \geq t_{\max}$, we obtain a contradiction, and consequently we must have $t_{\max} \geq T$). If $T < t_1$, by continuity and monotonicity of $y : [t_0, t_1] \rightarrow \mathbb{R}^+$ we obtain $y(t) = 0$ for all $t \in [T, t_1]$, which shows that (2.103) holds for all $t \in [t_0, t_1]$. Finally, by the continuity and monotonicity of $y : [t_0, t_1] \rightarrow \mathbb{R}^+$ we obtain (2.103) for all $t \in [t_0, t_1]$ and for the case $y(t_0) = 0$.

The theory of ordinary differential equations and the fact that $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is locally Lipschitz on $\mathbb{R}^+ \setminus \{0\}$ guarantees that for all $w_0 > 0$, there exists $t_{\max} > t_0$ such that the unique solution of (2.104) $\zeta(t - t_0, w_0)$ is defined for $t \in [t_0, t_{\max})$ and satisfies $\zeta(t - t_0, w_0) > 0$ for all $t \in [t_0, t_{\max})$. By virtue of (2.104) it follows that $0 < \zeta(t - t_0, w_0) \leq w_0$ for all $t \in [t_0, t_{\max})$. Consequently, if $t_{\max} < +\infty$, we have $\lim_{t \rightarrow t_{\max}^-} \zeta(t - t_0, w_0) = 0$. If $t_{\max} = +\infty$, then Lemma 2.11 implies that $\lim_{t \rightarrow +\infty} \zeta(t - t_0, w_0) = 0$. Notice that the theory of ordinary differential equations guarantees that for each fixed $t \in [t_0, t_{\max})$, the mapping $w_0 \rightarrow \zeta(t - t_0, w_0)$ is continuous. Moreover, for each fixed $w_0 > 0$, the mapping $[t_0, t_{\max}) \ni t \rightarrow \zeta(t - t_0, w_0)$ is nonincreasing. The solution is continuously extended by $\zeta(t - t_0, w_0) = 0$ for all $t \geq t_{\max}$. Using the above properties, it can be shown that the function $\sigma : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ defined by $\sigma(s, t) := \zeta(t, s)$ for all $t \geq 0, s > 0$ and $\sigma(0, t) := 0$ for all $t \geq 0$ is continuous and nonincreasing in $t \geq 0$ with $\lim_{t \rightarrow +\infty} \sigma(s, t) = 0$ for all $s \geq 0$. The reader should also notice that the equalities $\sigma(s, 0) = s$ and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ hold for all $t, s \geq 0$.

The only thing that remains to be shown is that the mapping $\mathbb{R}^+ \ni s \rightarrow \sigma(s, t)$ is nondecreasing. Let $s_1 > s_2 > 0$ arbitrary. Since $\zeta(0, s_1) = s_1 > s_2$ and $\lim_{t \rightarrow +\infty} \zeta(t, s_1) = 0 < s_2$, it follows that there exists unique $\tau > 0$ such that $\zeta(\tau, s_1) = s_2$. For every $t \geq 0$, the semigroup property implies $\zeta(t, s_2) = \zeta(t, \zeta(\tau, s_1)) = \zeta(t + \tau, s_1)$. The fact that the mapping $\mathbb{R}^+ \ni t \rightarrow \zeta(t, s_1)$ is nonincreasing implies that $\zeta(t, s_2) = \zeta(t + \tau, s_1) \leq \zeta(t, s_1)$. Therefore, the mapping $\mathbb{R}^+ \ni s \rightarrow \zeta(t, s)$ is nondecreasing; hence the mapping $\mathbb{R}^+ \ni s \rightarrow \sigma(s, t)$ is nondecreasing. The proof is complete. $\square$

The following comparison lemma provides a “fading memory” estimate for an absolutely continuous mapping.

Lemma 2.14 *For each C^0 positive definite $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, there exists a function σ of class KL , with the following property: if $y : [t_0, t_1] \rightarrow \mathfrak{R}^+$ is an absolutely continuous function, $u : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is a locally bounded mapping, and $I \subset [t_0, t_1]$ a set of Lebesgue measure zero such that $\dot{y}(t)$ is defined on $[t_0, t_1] \setminus I$ and such that the following implication holds for all $t \in [t_0, t_1] \setminus I$:*

$$y(t) \geq u(t) \quad \Rightarrow \quad \dot{y}(t) \leq -\rho(y(t)) \quad (2.105)$$

then the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \leq \max \left\{ \sigma(y(t_0), t - t_0), \sup_{t_0 \leq s \leq t} \sigma(u(s), t - s) \right\} \quad (2.106)$$

Moreover, if $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is locally Lipschitz on $(0, +\infty)$, then $\sigma(s, 0) = s$, and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ for all $t, s \geq 0$.

Proof Notice that by virtue of Lemma 2.11, for each positive definite continuous function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, there exists a continuous function σ of class KL with the following property: if $y : [t_0, t_1] \rightarrow \mathfrak{R}^+$ is an absolutely continuous function and $I \subset [t_0, t_1]$ a set of Lebesgue measure zero such that $\dot{y}(t)$ is defined on $[t_0, t_1] \setminus I$ and such that the following differential inequality holds for all $t \in [t_0, t_1] \setminus I$:

$$\dot{y}(t) \leq -\rho(y(t)) \quad (2.107)$$

then, the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \leq \sigma(y(t_0), t - t_0) \quad (2.108)$$

Indeed, we may consider the absolutely continuous function $\tilde{y} : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ defined as follows:

- for the case $y(t_1) > 0$, $\tilde{y}(t) = y(t)$ for $t \in [t_0, t_1]$, $\tilde{y}(t) = y(t_1) - (t - t_1) \max_{0 \leq w \leq y(t_1)} \rho(w)$ for $\xi \in [t_1, t_1 + \frac{y(t_1)}{\max_{0 \leq w \leq y(t_1)} \rho(w)}]$, and $\tilde{y}(t) = 0$ for $t > t_1 + \frac{y(t_1)}{\max_{0 \leq w \leq y(t_1)} \rho(w)}$,
- for the case $y(t_1) = 0$, $\tilde{y}(t) = y(t)$ for $t \in [t_0, t_1]$, and $\tilde{y}(t) = 0$ for $t > t_1$.

The reader may verify that $\tilde{y} : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is an absolutely continuous function which satisfies $\frac{d}{dt} \tilde{y}(t) \leq -\rho(\tilde{y}(t))$ for almost all $t \geq t_0$. Lemma 2.11 applied for $\tilde{y} : [t_0, +\infty) \rightarrow \mathfrak{R}^+$ implies (2.108) for all $t \in [t_0, t_1]$.

Notice that we may continuously extend σ by defining $\sigma(s, t) := \sigma(s, 0) \exp(-t)$ for $t < 0$. Moreover, notice that Lemma 2.13 guarantees that $\sigma(s, 0) = s$ and $\frac{\partial \sigma}{\partial t}(s, t) = -\rho(\sigma(s, t))$ for all $t, s \geq 0$, provided that $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is locally Lipschitz on $(0, +\infty)$.

Clearly, (2.106) holds for $t = t_0$ and σ the function involved in (2.108). We next show that (2.106) holds for arbitrary $t \in (t_0, t_1]$.

Let arbitrary $t \in (t_0, t_1]$ and define the functions

$$\tilde{u}(\tau) = \begin{cases} u(\tau) & \forall \tau \in [t_0, t] \\ 0 & \text{otherwise} \end{cases} \quad \bar{u}(\tau) := \limsup_{\xi \rightarrow \tau} \tilde{u}(\xi)$$

Notice that $\bar{u}$ is upper semi-continuous on $\tau \in [t_0, t]$, and consequently the function $p(\tau) := y(\tau) - \bar{u}(\tau)$ is lower semi-continuous on $[t_0, t]$. Next define the set

$$\Lambda := \{\tau \in [t_0, t] : y(\tau) \leq \bar{u}(\tau)\} \quad (2.109)$$

We distinguish the following cases:

- (1) $\Lambda = \emptyset$. In this case we have $y(\tau) > \bar{u}(\tau)$ for all $\tau \in [t_0, t]$. Since $\bar{u}(\tau) \geq u(\tau)$ for all $\tau \in [t_0, t]$, the previous inequality, in conjunction with (2.105), implies that $\dot{y}(\tau) \leq -\rho(y(\tau))$ for all $\tau \in [t_0, t] \setminus I$. Thus in this case Lemma 2.11 (or Lemma 2.13) guarantees that estimate (2.108) holds.
- (2) $\Lambda \neq \emptyset$ and $\xi := \sup \Lambda < t$. In this case there exists a sequence $\tau_i \leq \xi$ with $\tau_i \rightarrow \xi$ and $y(\tau_i) - \bar{u}(\tau_i) \leq 0$. Since the function $p(t) = y(t) - \bar{u}(t)$ is lower semi-continuous, we obtain $p(\xi) = \liminf_{\tau \rightarrow \xi} p(\tau) \leq 0$, and consequently $y(\xi) \leq \bar{u}(\xi)$. Moreover, notice that by definition (2.109), implication (2.105), and since $\bar{u}(\tau) \geq u(\tau)$ for all $\tau \in [t_0, t]$, the differential inequality $\dot{y}(\tau) \leq -\rho(y(\tau))$ holds for all $\tau \in (\xi, t] \setminus I$. Consequently, Lemma 2.11 (or Lemma 2.13) implies $y(t) \leq \sigma(y(\tau), t - \tau)$ for all $\tau \in (\xi, t]$. By virtue of the continuity of σ and y , we get

$$y(t) \leq \sigma(y(\xi), t - \xi)$$

which, combined with $y(\xi) \leq \bar{u}(\xi)$, directly implies

$$y(t) \leq \sigma(\bar{u}(\xi), t - \xi) \leq \sup_{t_0 \leq s \leq t} \sigma(\bar{u}(s), t - s) \quad (2.110)$$

- (3) $\Lambda \neq \emptyset$ and $\xi := \sup \Lambda = t$. In this case there exists a sequence $\tau_i \leq t$ with $\tau_i \rightarrow t$ and $y(\tau_i) - \bar{u}(\tau_i) \leq 0$. Since the function $p(t) = y(t) - \bar{u}(t)$ is lower semi-continuous, we obtain $p(t) = \liminf_{\tau \rightarrow t} p(\tau) \leq 0$, and consequently $y(t) \leq \bar{u}(t)$. Moreover, since $\sigma(s, 0) \geq s$ for all $s \geq 0$, it holds that

$$y(t) \leq \bar{u}(t) \leq \sigma(\bar{u}(t), 0) \leq \sup_{t_0 \leq s \leq t} \sigma(\bar{u}(s), t - s)$$

Combining all the above cases, we may conclude that

$$y(t) \leq \max \left\{ \sigma(y(t_0), t - t_0), \sup_{t_0 \leq s \leq t} \sigma(\bar{u}(s), t - s) \right\} \quad (2.111)$$

Let $M := \sup_{t_0 \leq s \leq t} u(s)$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma(s, \tau - \delta) - \sigma(s, \tau) < \varepsilon$ for all $(s, \tau) \in [0, M] \times [0, t]$. Notice that since

$$\bar{u}(\tau) := \limsup_{\xi \rightarrow \tau} \tilde{u}(\xi) \quad \text{and} \quad \tilde{u}(\tau) = \begin{cases} u(\tau) & \forall \tau \in [t_0, t] \\ 0 & \text{otherwise} \end{cases}$$

it follows that $\bar{u}(s) \leq \sup\{u(r) : \max(s - \delta, t_0) \leq r \leq \min(s + \delta, t)\}$ for all $s \in [t_0, t]$. The previous inequalities imply that

$$\begin{aligned} \sigma(\bar{u}(s), t - s) &\leq \sup\{\sigma(u(r), t - s) : \max(s - \delta, t_0) \leq r \leq \min(s + \delta, t)\} \\ &\leq \sup\{\sigma(u(r), t - r - \delta) : \max(s - \delta, t_0) \leq r \leq \min(s + \delta, t)\} \\ &\leq \sup\{\sigma(u(r), t - r) : \max(s - \delta, t_0) \leq r \leq \min(s + \delta, t)\} + \varepsilon \\ &\leq \sup_{t_0 \leq r \leq t} \sigma(u(r), t - r) + \varepsilon \end{aligned}$$

The above inequality, in conjunction with (2.111), implies that for each $\varepsilon > 0$, it holds that

$$y(t) \leq \max\left\{\sigma(y(t_0), t - t_0), \sup_{t_0 \leq s \leq t} \sigma(u(s), t - s)\right\} + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the above estimate directly implies (2.106). The proof is complete. $\square$

Lemma 2.15 *For every pair of functions $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ being locally Lipschitz and positive definite and μ being of class $\mathcal{E}$ with $\mu(t) > 0$ for all $t \geq 0$, there exists a KL function $\sigma : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ which satisfies the following property:*

(P) *If $c \in [0, 1]$ is a constant, $t_0 \geq 0$, $\gamma \in K^+$ with $\int_0^{+\infty} \gamma(t) dt = +\infty$, and $y : [t_0, t_1] \rightarrow \mathbb{R}^+$ is an absolutely continuous function that satisfies the following differential inequality for almost all $t \in [t_0, t_1]$:*

$$\dot{y}(t) \leq -\gamma(t)\rho(y(t)) + c\gamma(t)\mu\left(\int_0^t \gamma(s) ds\right) \quad (2.112)$$

then the following estimate holds for all $t \in [t_0, t_1]$:

$$y(t) \leq \sigma\left(y(t_0) + c \int_{t_0}^{+\infty} \mu\left(\int_0^s \gamma(w) dw\right) \gamma(s) ds, \int_{t_0}^t \gamma(s) ds\right). \quad (2.113)$$

Proof Let $\phi(t, t_0, y_0)$ denote the solution of the scalar differential equation

$$\begin{aligned} \dot{y}(t) &= -\gamma(t)\tilde{\rho}(y(t)) + c\gamma(t)\mu\left(\int_0^t \gamma(\tau) d\tau\right) \\ y(t_0) &= y_0 \in \mathbb{R} \end{aligned} \quad (2.114)$$

where $\tilde{\rho}(y) := \rho(y)$ for $y \geq 0$ and $\tilde{\rho}(y) := 0$ for $y < 0$. Notice that $\tilde{\rho} : \mathbb{R} \rightarrow \mathbb{R}^+$ is a locally Lipschitz function which coincides with $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ on $\mathbb{R}^+$. By performing the time-scaling $\tau = \int_0^t \gamma(s) ds$ it is clear that

$$\phi(t, t_0, y_0) = \zeta\left(\int_0^t \gamma(s) ds, \int_0^{t_0} \gamma(s) ds, y_0\right) \quad (2.115)$$

where $\zeta(\tau, \tau_0, y_0)$ denotes the solution of the scalar differential equation

$$\begin{aligned} \frac{d}{d\tau} y(\tau) &= -\tilde{\rho}(y(\tau)) + c\mu(\tau) \\ y(\tau_0) &= y_0 \in \mathfrak{R} \end{aligned} \quad (2.116)$$

Using a contradiction argument, it can be shown that $\zeta(\tau, \tau_0, y_0)$ exists for all $\tau \geq \tau_0$ and $\zeta(\tau, \tau_0, y_0) \geq 0$ for all $\tau \geq \tau_0$ if $y_0 \geq 0$. Moreover, Lemma 2.11 (applied to the absolutely continuous function $y(\tau) := \zeta(\tau, \tau_0, y_0)$, $y_0 \geq 0$) implies that there exists a *KL* function $\sigma : (\mathfrak{R}^+)^2 \rightarrow \mathfrak{R}^+$ such that the following estimate holds for all $y_0 \geq 0$:

$$0 \leq \zeta(\tau, \tau_0, y_0) \leq \sigma\left(y_0 + c \int_{\tau_0}^{+\infty} \mu(s) ds, \tau - \tau_0\right) \quad \text{for all } \tau \geq \tau_0 \quad (2.117)$$

It follows from (2.115) and (2.117) that the solution $\phi(t, t_0, y_0)$ with $y_0 \geq 0$ satisfies the following estimate: for all $t \in [t_0, t_1]$,

$$0 \leq \phi(t, t_0, y_0) \leq \sigma\left(y_0 + c \int_{t_0}^{+\infty} \mu\left(\int_0^s \gamma(w) dw\right) \gamma(s) ds, \int_{t_0}^t \gamma(s) ds\right) \quad (2.118)$$

By applying Lemma 2.12 with $f(t, y) = -\gamma(t)\tilde{\rho}(y) + c\gamma(t)\mu(\int_0^t \gamma(\tau) d\tau)$ and $J := (-a, +\infty)$, where $a > 0$ and $T := t_1$, we obtain, for all $t \in [t_0, t_1]$,

$$0 \leq y(t) \leq \phi(t, t_0, y_0) \quad (2.119)$$

The conclusion of the lemma follows by combining inequalities (2.118) and (2.119). The proof is complete. $\square$

Remark 2.4 It should be noted that if $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ is a continuous positive definite function, then there exists $\tilde{\rho} : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ which is globally Lipschitz on $\mathfrak{R}^+$ with unit Lipschitz constant and positive definite, and satisfies $0 < \tilde{\rho}(s) \leq \rho(s)$ for all $s > 0$. Indeed, we may define $\tilde{\rho}(s) := \inf\{\rho(y) + |y - s|; y \geq 0\}$, which is globally Lipschitz on $\mathfrak{R}^+$ with unit Lipschitz constant. To see this, first notice that $\tilde{\rho}(s) := \min\{\rho(y) + |y - s|; 0 \leq y \leq s + \rho(s)\}$ for all $s \geq 0$. Let $s_1, s_2 \geq 0$ with $\tilde{\rho}(s_1) := \rho(y_1) + |y_1 - s_1|$ and notice that

$$\tilde{\rho}(s_2) \leq \rho(y_1) + |y_1 - s_2| \leq \rho(y_1) + |y_1 - s_1| + |s_1 - s_2| = \tilde{\rho}(s_1) + |s_1 - s_2|$$

Consequently, Lemma 2.13 guarantees that for every continuous positive definite function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, there exists a *KL* function $\sigma : (\mathfrak{R}^+)^2 \rightarrow \mathfrak{R}^+$ with $\sigma(s, 0) = s$ for all $s \geq 0$ which satisfies Property (P) of Lemma 2.13. Moreover, Lemma 2.14 guarantees that for each C^0 positive definite function $\rho : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, there exists a function σ of class *KL* with $\sigma(s, 0) = s$ for all $s \geq 0$, satisfying the property described in the statement of Lemma 2.14.

2.6 Lyapunov Functionals

One of the most important tools for establishing RGAOS for a control system is the method of Lyapunov functionals. Lyapunov functionals are, roughly speaking, nonnegative functionals of the state of the control system which tend to 0 as $t \rightarrow +\infty$.

The following theorem shows that the existence of a Lyapunov functional is a sufficient condition for RGAOS for systems that satisfy the following additional hypothesis:

(HYP) There exists a constant $r > 0$, a bounded positive continuous function $h : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow (0, r]$, and a partition $\pi = \{T_i\}_{i=0}^\infty$ of $\mathbb{R}^+$ such that for every $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, it holds that $\pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r] \neq \emptyset$, where $b(t, \rho) := \min\{q_\pi(t), t + h(t, \rho)\}$.

Remark 2.5 It is worth noting the following.

1. Hypothesis (HYP) holds for many classes of systems including systems described by ordinary differential equations, systems with variable sampling partition, and systems described by retarded functional differential equations.
2. Hypothesis (HYP) guarantees the existence of $\tau \in (t_0, t_0 + r]$ such that $\tau \in \pi(t_0, x_0, u_0, d)$ for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$. Thus, for all $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, it follows that the set $\pi(t_0, x_0, u_0, d) \cap (t_0, +\infty)$ cannot be empty.

Theorem 2.3 (Lyapunov functionals) *Let $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ be a control system with outputs satisfying Hypothesis (HYP) and the BIC property. Assume that $0 \in \mathcal{X}$ is a robust equilibrium point for Σ . Suppose that there exist mappings $V : \mathbb{R}^+ \times \mathcal{X} \rightarrow \mathbb{R}^+$, $\beta, \gamma, \mu \in K^+$ with $\int_0^{+\infty} \gamma(t) dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$, and a locally Lipschitz positive definite function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, there exists $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r]$, where $b(t, \rho) := \min\{q_\pi(t), t + h(t, \rho)\}$ is the function in (HYP), with $(\tau, t_0, x_0, u_0, d) \in A_\phi$ and the following properties:*

$$\begin{aligned} a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} + \mu(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}) \\ \leq V(t_0, x_0) \leq a_2(\beta(t_0)\|x_0\|_{\mathcal{X}}) \quad \text{for all } t \in [t_0, \tau] \end{aligned} \quad (2.120)$$

$$V(\tau, \phi(\tau, t_0, x_0, u_0, d)) \leq \eta(\tau, t_0, V(t_0, x_0)) \quad (2.121)$$

where $\eta(t, t_0, \eta_0)$ denotes the unique solution of the initial-value problem

$$\dot{\eta} = -\gamma(t)\rho(\eta) + \gamma(t)\varphi\left(\int_0^t \gamma(s) ds\right) \quad \eta(t_0) = \eta_0 \geq 0 \quad (2.122)$$

Then, Σ is RGAOS. Particularly, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$ and if for every $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$, we have, in addition, $a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leq V(t_0, x_0) \leq a_2(\|x_0\|_{\mathcal{X}})$ for all $t \in [t_0, \tau]$, with $\tau \in \pi(t_0, x_0, u_0, d) \cap [b(t_0, \|x_0\|_{\mathcal{X}}), t_0 + r]$ the time for which (2.120) and (2.121) hold, then Σ is URGAOS.

Proof By virtue of Lemma 2.1, it suffices to show that $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$ is RFC and satisfies the Robust Output Attractivity Property (Property P3 of Definition 2.2). Notice that Lemma 2.15 implies that there exist a function $\sigma(\cdot) \in KL$ and a constant $M > 0$ such that the following inequalities are satisfied for all $t_0 \geq 0$:

$$0 \leq \eta(t, t_0, \eta_0) \leq \sigma(\eta_0 + M, q(t, t_0)), \quad \text{for all } t \geq t_0, \text{ for all } \eta_0 \geq 0 \quad (2.123)$$

where $q(t, t_0) := \int_{t_0}^t \gamma(s) ds$. Furthermore, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, it follows from Lemma 2.11 that there exists $\sigma(\cdot) \in KL$ such that the following inequalities are satisfied for all $t_0 \geq 0$:

$$0 \leq \eta(t, t_0, \eta_0) \leq \sigma(\eta_0, t - t_0) \quad \text{for all } t \geq t_0, \text{ for all } \eta_0 \geq 0 \quad (2.124)$$

Pick arbitrary $(t_0, x_0, d) \in \mathbb{R}^+ \times \mathcal{X} \times M_D$ and let $\tau_0 = t_0$. We consider the sequence $x_i = \phi(\tau_i, \tau_{i-1}, x_{i-1}, u_0, d)$, $i \geq 1$, and $\tau_i \in \pi(\tau_{i-1}, x_{i-1}, u_0, d) \cap [b(\tau_{i-1}, \|x_{i-1}\|_{\mathcal{X}}, \tau_{i-1} + r), \tau_{i-1} + r]$, $i \geq 1$, for which (2.121) holds with (τ_{i-1}, x_{i-1}) in place of (t_0, x_0) . By virtue of the weak semigroup property (Property 4 of Definition 1.1), we have $\tau_i \in \pi(t_0, x_0, u_0, d)$, $\tau_i \leq t_0 + ir$ and $x_i = \phi(\tau_i, t_0, x_0, u_0, d)$ for all $i \geq 1$. The semigroup property for $\eta(t, t_0, \eta_0)$, in conjunction with inequality (2.120) and trivial induction arguments, implies that

$$\begin{aligned} a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} + \mu(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}) \\ \leq V(\tau_{i-1}, x_{i-1}) \leq \eta(\tau_{i-1}, t_0, V(t_0, x_0)) \quad \forall i \geq 1, t \in [\tau_{i-1}, \tau_i] \end{aligned} \quad (2.125)$$

Combining (2.123) with (2.125) and using the fact that $\max\{t_0, t - r\} \leq \tau_{i-1}$ for $t \in [\tau_{i-1}, \tau_i]$, $i \geq 1$, we obtain, for all $i \geq 1$ and $t \in [t_0, \tau_i]$,

$$\begin{aligned} a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} + \mu(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}) \\ \leq \sigma(a_2(\beta(t_0)\|x_0\|_{\mathcal{X}}) + M, q(\max\{t_0, t - r\}, t_0)) \end{aligned} \quad (2.126)$$

which directly implies

$$\begin{aligned} a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} + \mu(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}) \\ \leq \sigma(a_2(\beta(t_0)\|x_0\|_{\mathcal{X}}) + M, 0) \quad \text{for all } i \geq 1 \text{ and } t \in [t_0, \tau_i] \end{aligned} \quad (2.127)$$

We next show that $(t, t_0, x_0, u_0, d) \in A_\phi$ for all $t \geq t_0$. By virtue of estimate (2.127) and the BIC property, it suffices to show that $\tau_i \rightarrow +\infty$. Let arbitrary $T > 0$. Since the set $[0, T + r] \times [0, \varepsilon]$ is compact, where $\varepsilon := \frac{a_1^{-1}(\sigma(a_2(\beta(t_0)\|x_0\|_{\mathcal{X}}) + M, 0))}{\min\{\mu(t); t \in [0, T + r]\}}$ and h is continuous, we have $s := \min\{h(t, \rho); (t, \rho) \in [0, T + r] \times [0, \varepsilon]\} > 0$. Consider the infinite sequence $\{y_i\}_{i=0}^\infty$ which satisfies $y_{i+1} = \min\{q_\pi(y_i), y_i + s\}$, $i = 1, 2, \dots$, with $y_0 = 0$. By virtue of a standard contradiction argument we can show that $y_i \rightarrow +\infty$ (as in the proof of the Claim in Sect. 1.2.5), and consequently for every $T > 0$, there exists an integer $N > 0$ such that $y_N > T + r$. Clearly, the sequence $\{\tau_i\}_{i=0}^\infty$

satisfies $\tau_{i+1} \geq \min\{q_\pi(\tau_i), \tau_i + s\}$ for all integers i for which $\tau_i \leq T + r$. By virtue of Theorem 1.6.1 in [34] (Comparison principle) we have $\tau_i \geq y_i$ for all integers i for which $\tau_i \leq T + r$, which implies $\tau_N > T$.

It follows that estimates (2.126) and (2.127) hold for all $t \geq t_0$. Robust Forward Completeness is an immediate consequence of (2.127), and the Robust Output Attractivity Property (Property P3 of Definition 2.2) is an immediate consequence of estimate (2.126).

Notice that if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and if for every $(t_0, x_0, d) \in \mathfrak{N}^+ \times \mathcal{X} \times M_D$, we have, in addition, that $a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leq V(t, x_0) \leq a_2(\|x_0\|_{\mathcal{X}})$ for all $t \in [t_0, \tau]$, then using (2.124) instead of (2.123), we obtain, in addition, the following estimate for all $t \geq t_0$:

$$a_1(\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}) \leq \sigma(a_2(\|x_0\|_{\mathcal{X}}), \max\{0, t - t_0 - r\})$$

The above estimate directly implies that Σ is URGAOS. The proof is complete. $\square$

Theorem 2.3 is a general theorem that can be applied to all deterministic systems of Definition 1.1 which satisfy property (HYP). However, the reader should note that inequalities (2.120), (2.121) are prerequisites for the application of Theorem 2.3 and they must be shown by using another result. Notice that inequalities (2.120) and (2.121) require a substantial amount of knowledge of the transition map of the control system, which is usually not available. This is a serious disadvantage of the method.

There are two ways to overcome the above disadvantage for systems which satisfy the classical semigroup property:

- (1) Lyapunov's approach
- (2) The "discretization approach"

Next, we describe the two approaches in an informal way.

- (1) Lyapunov's approach

For systems which satisfy the classical semigroup property, the above disadvantage can be overcome in an elegant way: we let the constant $r > 0$ involved in Hypothesis (HYP) to become sufficiently small. Then inequality (2.121) becomes a differential inequality,

$$\begin{aligned} V(\tau, \phi(\tau, t_0, x_0, u_0, d)) - V(t_0, x_0) &\leq \eta(\tau, t_0, V(t_0, x_0)) - V(t_0, x_0) \\ \limsup_{h \rightarrow 0^+} \frac{V(t_0 + h, \phi(t_0 + h, t_0, x_0, u_0, d)) - V(t_0, x_0)}{h} \\ &\leq \limsup_{h \rightarrow 0^+} \frac{\eta(t_0 + h, t_0, V(t_0, x_0)) - V(t_0, x_0)}{h} \end{aligned}$$

which implies

$$\begin{aligned} & \limsup_{h \rightarrow 0^+} \frac{V(t_0 + h, \phi(t_0 + h, t_0, x_0, u_0, d)) - V(t_0, x_0)}{h} \\ & \leq -\gamma(t_0)\rho(V(t_0, x_0)) + \gamma(t_0)\varphi\left(\int_0^{t_0} \gamma(s) ds\right) \end{aligned}$$

Notice that an upper bound for the quantity

$$\limsup_{h \rightarrow 0^+} \frac{V(t_0 + h, \phi(t_0 + h, t_0, x_0, u_0, d)) - V(t_0, x_0)}{h}$$

can (usually) be computed without knowledge of the transition map $\phi(\tau, t_0, x_0, u_0, d)$ of the control system $\Sigma := (\mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, \pi, H)$. Moreover, letting $r > 0$ to become arbitrarily small and assuming continuity with respect to time of the transition map $\phi(\tau, t_0, x_0, u_0, d)$, inequality (2.120) requires no knowledge of the transition map:

$$a_1(\|H(t_0, x_0, 0)\|_{\mathcal{Y}} + \mu(t_0)\|x_0\|_{\mathcal{X}}) \leq V(t_0, x_0) \leq a_2(\beta(t_0)\|x_0\|_{\mathcal{X}})$$

The above thoughts lead us to the production of specialized results for two kinds of systems of Definition 1.1, systems described by ODEs and systems described by RFDEs.

(2) The “discretization approach”

The idea in this approach is that we examine the value of the Lyapunov functional only at discrete time instances. For the sequence of the chosen time instances, we can show that the Lyapunov functional decreases in value. However, we do not demand an explicit knowledge of the time instances, and we do not require that the difference between two consecutive time instances must be bounded (by r).

The discretization approach has been developed mostly for systems described by ODEs and has been applied successfully for the solution of difficult control problems. We will present specialized results for systems described by ODEs. However, the method can be applied to various systems which satisfy the classical semigroup property.

Due to the close relation of the discretization approach to the stability analysis of discrete-time systems, we will describe the method in the section devoted to discrete-time systems below. Here we describe Lyapunov’s approach.

2.6.1 Control Systems Described by Ordinary Differential Equations

We start by presenting a general Lyapunov theorem for systems described by ODEs of the form (1.3) under Hypotheses (H1–4).

Let $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a locally bounded function. We define the following generalized Dini derivative for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, $v \in \mathbb{R}^n$:

$$V^0(t, x; v) := \limsup_{h \rightarrow 0^+} \frac{V(t+h, x+hv) - V(t, x)}{h} \quad (2.128)$$

Notice that if $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ is locally Lipschitz, then the above derivative is locally bounded.

Theorem 2.4 Consider system (1.3) under Hypotheses (H1–4) and suppose that it is RFC. Furthermore, suppose that there exist functions $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ being locally Lipschitz, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t) dt = +\infty$, $\varphi \in \mathcal{E}$, $a_1, a_2 \in K_\infty$, and a locally Lipschitz positive definite function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that the following inequalities hold:

$$a_1(|H(t, x, 0)|) \leq V(t, x) \leq a_2(\beta(t)|x|) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \quad (2.129)$$

$$V^0(t, x; f(t, x, 0, d)) \leq -\gamma(t)\rho(V(t, x)) + \gamma(t)\varphi\left(\int_0^t \gamma(s) ds\right) \quad (2.130)$$

for all $(t, x, d) \in \mathbb{R}^+ \times \mathbb{R}^n \times D$

Then system (1.3) is RGAOS. Particularly, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and $\beta(t) \equiv 1$, then (1.3) is URGAOS. Moreover, if there exist functions $a \in K_\infty$, $\mu \in K^+$, and a constant $R \geq 0$ such that $a(\mu(t)|x|) \leq V(t, x) + R$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, then the requirement that (1.3) is RFC is not needed.

Proof By Lemma 2.1, it suffices to show that (1.3) is RFC and satisfies the Robust Output Attractivity Property (Property P3 of Definition 2.2).

Consider a solution $x(t)$ of (1.3) under Hypotheses (H1–4) corresponding to arbitrary $d \in M_D$ with initial condition $x(t_0) = x_0 \in \mathbb{R}^n$. Clearly, the solution exists for $t \in [t_0, t_{\max})$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Notice that the mapping $t \rightarrow V(t, x(t))$ is absolutely continuous on $[t_0, t_{\max})$, and we have

$$\frac{d}{dt} V(t, x(t)) = V^0(t, x(t); f(t, x(t), 0, d(t))) \quad \text{a.e. on } [t_0, t_{\max}) \quad (2.131)$$

where $V^0(t, x; v) := \limsup_{h \rightarrow 0^+} \frac{V(t+h, x+hv) - V(t, x)}{h}$. Indeed, notice that for all $t \in [t_0, t_{\max}) \setminus I$, where $I \subset [t_0, t_{\max})$ is the set of zero Lebesgue measure such that either $\frac{d}{dt} V(t, x(t))$ is not defined on I , or $D^+x(t) := \lim_{h \rightarrow 0^+} h^{-1}(x(t+h) - x(t)) \neq f(t, x(t), 0, d(t))$, we get, for sufficiently small $h > 0$ and any $t \in [t_0, t_{\max}) \setminus I$,

$$x(t+h) - x(t) - hD^+x(t) = hy_h \quad (2.132)$$

where

$$y_h = \frac{x(t+h) - x(t)}{h} - D^+x(t)$$

Since $\lim_{h \rightarrow 0^+} \frac{x(t+h)-x(t)}{h} = D^+x(t)$, we obtain that $y_h \rightarrow 0$ as $h \rightarrow 0^+$. Since $V : \mathfrak{R}^+ \times \mathfrak{R}^n \rightarrow \mathfrak{R}^+$ is locally Lipschitz, we have

$$\begin{aligned} V^0(t, x(t); D^+x(t)) &= \limsup_{h \rightarrow 0^+} \frac{V(t+h, x(t) + hD^+x(t)) - V(t, x(t))}{h} \\ &= \limsup_{h \rightarrow 0^+} \frac{V(t+h, x(t) + hD^+x(t) + hy_h) - V(t, x(t))}{h} \end{aligned}$$

The above equality, in conjunction with (2.132) and the facts that $\frac{d}{dt}V(t, x(t)) = \limsup_{h \rightarrow 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h}$ and $D^+x(t) = f(t, x(t), 0, d(t))$ for all $t \in [t_0, t_{\max}) \setminus I$, implies (2.131).

It follows from Lemma 2.15 and inequalities (2.130) and (2.131) that there exist $\sigma \in KL$ and a constant $M > 0$ such that

$$V(t, x(t)) \leq \sigma \left(V(t_0, x_0) + M, \int_{t_0}^t \gamma(s) ds \right) \quad \text{for all } t \in [t_0, t_{\max}) \quad (2.133)$$

Next, we distinguish the following cases:

1. If (1.3) is RFC, then (2.133) holds for all $t \geq t_0$, then the Robust Output Attractivity Property (Property P3 of Definition 2.2) is a direct consequence of inequalities (2.129) and (2.133) and from the fact that $\int_0^{+\infty} \gamma(t) dt = +\infty$.
2. If there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x|) \leq V(t, x) + R$ for all $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$, then (2.133), in conjunction with (2.129), implies the following estimate:

$$|x(t)| \leq \frac{1}{\mu(t)} a^{-1} \left(R + \sigma \left(a_2(\beta(t_0)|x_0|) + M, 0 \right) \right) \quad \text{for all } t \in [t_0, t_{\max}). \quad (2.134)$$

Estimate (2.134), in conjunction with the BIC property, implies that $t_{\max} = +\infty$. Moreover, estimate (2.134), in conjunction with Definition 2.1, shows that system (1.3) is RFC, since we have, for all $r, T \geq 0$,

$$\begin{aligned} &\sup \{ |x(t_0 + s)|; s \in [0, T], |x_0| \leq r, t_0 \in [0, T], d \in M_D \} \\ &\leq \frac{1}{\min_{0 \leq t \leq 2T} \mu(t)} a^{-1} \left(R + \sigma \left(a_2 \left(r \max_{0 \leq t \leq T} \beta(t) \right) + M, 0 \right) \right) < +\infty \end{aligned}$$

The proof is complete. $\square$

Remark 2.6 If for all $(x, d) \in \mathfrak{R}^n \times D$, the mapping $t \rightarrow f(t, x, 0, d)$ is continuous, then inequality (2.130) is equivalent to the following inequality:

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, 0, d) \leq -\gamma(t) \rho(V(t, x)) + \gamma(t) \varphi \left(\int_0^t \gamma(s) ds \right) \quad (2.135)$$

for almost all $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$ and all $d \in D$.

Indeed, using Corollary 8.2 in [11] (p. 95), we have, for all $v \in \mathfrak{N}^n$ and $(t, x) \in \mathfrak{N}^+ \times \mathfrak{N}^n$,

$$\begin{aligned} & \limsup_{h \rightarrow 0^+, (\tau, y) \rightarrow (t, x)} \frac{V(\tau + h, y + hv) - V(\tau, y)}{h} \\ &= \limsup_{(\tau, y) \rightarrow (t, x)} \left\{ \frac{\partial V}{\partial t}(\tau, y) + \frac{\partial V}{\partial x}(\tau, y)v : (\tau, y) \notin \Omega \cup \Omega_V \right\} \end{aligned}$$

where $\Omega_V \subset \mathfrak{N}^+ \times \mathfrak{N}^n$ is the zero Lebesgue measure set such that $V : \mathfrak{N}^+ \times \mathfrak{N}^n \rightarrow \mathfrak{N}^+$ is not differentiable on Ω_V , and $\Omega \subset \mathfrak{N}^+ \times \mathfrak{N}^n$ is an arbitrary set of zero Lebesgue measure. Taking $v = f(t, x, 0, d)$ and using the facts that $\frac{\partial V}{\partial x}(\tau, y)$ is bounded in a neighborhood of $(t, x) \in \mathfrak{N}^+ \times \mathfrak{N}^n$ and that $f(\tau, y, 0, d) \rightarrow f(t, x, 0, d)$ for $(\tau, y) \rightarrow (t, x)$, we get

$$\begin{aligned} & \limsup_{h \rightarrow 0^+, (\tau, y) \rightarrow (t, x)} \frac{V(\tau + h, y + hf(t, x, 0, d)) - V(\tau, y)}{h} \\ &= \limsup_{(\tau, y) \rightarrow (t, x)} \left\{ \frac{\partial V}{\partial t}(\tau, y) + \frac{\partial V}{\partial x}(\tau, y)f(\tau, y, 0, d) : (\tau, y) \notin \Omega \cup \Omega_V \right\} \end{aligned}$$

The above equality, together with (2.135), implies that, for all $(t, x) \in \mathfrak{N}^+ \times \mathfrak{N}^n$, $d \in D$,

$$\begin{aligned} & \limsup_{h \rightarrow 0^+, (\tau, y) \rightarrow (t, x)} \frac{V(\tau + h, y + hf(t, x, 0, d)) - V(\tau, y)}{h} \\ & \leq -\gamma(t)\rho(V(t, x)) + \gamma(t)\varphi\left(\int_0^t \gamma(s) ds\right) \end{aligned}$$

By means of the fact that

$$V^0(t, x, ; f(t, x, 0, d)) \leq \limsup_{h \rightarrow 0^+, (\tau, y) \rightarrow (t, x)} \frac{V(\tau + h, y + hf(t, x, 0, d)) - V(\tau, y)}{h}$$

(2.130) follows readily.

2.6.2 Control Systems Described by Retarded Functional Differential Equations, RFDEs

We continue by presenting a general Lyapunov theorem for systems described by RFDEs of the form (1.10) under Hypotheses (S1–4). The reader may think that all arguments presented for the case of ODEs can be repeated in this case as well. This is not true. There are many technical complications arising for systems described by RFDEs. The following list presents some complications:

1. the mapping $t \rightarrow V(t, T_r(t)x)$ is not absolutely continuous, even if the functional $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$ is completely locally Lipschitz (or even if it is Fréchet differentiable),
2. it may not be required that the Lyapunov functional bounds a certain function of the norm of the output.

We next show how all the above complications can be overcome. An important class of functionals is presented next.

Definition 2.4 We say that a continuous functional $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}^+$ is “almost Lipschitz on bounded sets” if there exist nondecreasing functions $M : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, $P : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$, and $G : \mathfrak{R}^+ \rightarrow [1, +\infty)$ such that for all $R \geq 0$, the following properties hold:

(P1) For every $x, y \in \{x \in C^0([-r, 0]; \mathfrak{R}^n); \|x\|_r \leq R\}$, it holds that

$$|V(t, y) - V(t, x)| \leq M(R)\|y - x\|_r \quad \text{for all } t \in [0, R].$$

(P2) For every absolutely continuous function $x : [-r, 0] \rightarrow \mathfrak{R}^n$ with $\|x\|_r \leq R$ and essentially bounded derivative, it holds that

$$|V(t+h, x) - V(t, x)| \leq hP(R) \left(1 + \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)|\right)$$

$$\text{for all } t \in [0, R] \text{ and } 0 \leq h \leq \frac{1}{G(R + \sup_{-r \leq \tau \leq 0} |\dot{x}(\tau)|)}.$$

Let $x \in C^0([-r, 0]; \mathfrak{R}^n)$. By $E_h(x; v)$, where $0 \leq h < r$ and $v \in \mathfrak{R}^n$, we denote the following operator:

$$E_h(x; v) := \begin{cases} x(0) + (\theta + h)v & \text{for } -h < \theta \leq 0 \\ x(\theta + h) & \text{for } -r \leq \theta \leq -h \end{cases} \quad (2.136)$$

For mappings $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ that are almost Lipschitz on bounded sets, we will use the following Dini derivative defined for all $(t, x, v) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times \mathfrak{R}^n$:

$$V^0(t, x; v) := \limsup_{h \rightarrow 0^+} \frac{V(t+h, E_h(x; v)) - V(t, x)}{h}$$

The reason for introducing the above derivative becomes apparent by the following lemma.

Lemma 2.16 Let $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ be an almost Lipschitz on bounded sets functional, and let $x \in C^0([t_0 - r, t_{\max}); \mathfrak{R}^n)$ be a solution of (1.10) under Hypotheses (S1–4) corresponding to certain $(d, u) \in M_D \times M_U$, where

$t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then it holds that

$$\limsup_{h \rightarrow 0^+} h^{-1} (V(t+h, T_r(t+h)x) - V(t, T_r(t)x)) \leq V^0(t, T_r(t)x; D^+x(t)) \quad (2.137)$$

for almost all $t \in [t_0, t_{\max})$, where $D^+x(t) = \lim_{h \rightarrow 0^+} h^{-1} (x(t+h) - x(t))$.

Proof It suffices to show that (2.137) holds for all $t \in [t_0, t_{\max}) \setminus I$, where $I \subset [t_0, t_{\max})$ is the set of zero Lebesgue measure such that $D^+x(t) = \lim_{h \rightarrow 0^+} h^{-1} (x(t+h) - x(t))$ is not defined on I . Let $h > 0$ and $t \in [t_0, t_{\max}) \setminus I$. We define

$$T_r(t+h)x - E_h(T_r(t)x; D^+x(t)) = hy_h \quad (2.138)$$

where

$$y_h = h^{-1} \begin{cases} x(t+h+\theta) - x(t) - (\theta+h)D^+x(t) & \text{for } -h < \theta \leq 0 \\ 0 & \text{for } -r \leq \theta \leq -h \end{cases}$$

and notice that $y_h \in C^0([-r, 0]; \mathfrak{R}^n)$ (as a difference of continuous functions, see (2.138) above). Equivalently, y_h satisfies

$$y_h := \begin{cases} \frac{\theta+h}{h} \left(\frac{x(t+\theta+h) - x(t)}{\theta+h} - D^+x(t) \right) & \text{for } -h < \theta \leq 0 \\ 0 & \text{for } -r \leq \theta \leq -h \end{cases}$$

with

$$\|y_h\|_r \leq \sup \left\{ \left| \frac{x(t+s) - x(t)}{s} - D^+x(t) \right|; 0 < s \leq h \right\}.$$

Since $\lim_{h \rightarrow 0^+} \frac{x(t+h) - x(t)}{h} = D^+x(t)$, we obtain that $y_h \rightarrow 0$ as $h \rightarrow 0^+$. Since $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ is almost Lipschitz on bounded sets and since $y_h \rightarrow 0$ as $h \rightarrow 0^+$, it follows that

$$\begin{aligned} & \limsup_{h \rightarrow 0^+} h^{-1} (V(t+h, E_h(T_r(t)x; D^+x(t)) + hy_h) - V(t, T_r(t)x)) \\ &= V^0(t, T_r(t)x; D^+x(t)) \end{aligned}$$

Moreover, definition (2.138) and the above equality imply

$$\begin{aligned} & \limsup_{h \rightarrow 0^+} h^{-1} (V(t+h, T_r(t+h)x) - V(t, T_r(t)x)) \\ &= \limsup_{h \rightarrow 0^+} h^{-1} (V(t+h, E_h(T_r(t)x; D^+x(t)) + hy_h) - V(t, T_r(t)x)) \\ &= V^0(t, T_r(t)x; D^+x(t)) \end{aligned}$$

The proof is complete. □

The reason for introducing the class of almost Lipschitz on bounded sets functionals is the following result, which shows that the mapping $t \rightarrow V(t, T_r(t)x)$ is absolutely continuous for a special class of initial conditions.

Lemma 2.17 *Let $V : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ be a functional which is almost Lipschitz on bounded sets, and let $x \in C^0([t_0 - r, t_{\max}); \mathfrak{R}^n)$ be a solution of (1.10) under Hypotheses (S1–4) corresponding to certain $(d, u) \in M_D \times M_U$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$, where $t_{\max} \in (t_0, +\infty]$ is the maximal existence time of the solution. Then for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \rightarrow V(t, T_r(t)x)$ is absolutely continuous.*

Proof It suffices to show that for all $T \in (t_0, t_{\max})$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{k=1}^N |V(b_k, T_r(b_k)x) - V(a_k, T_r(a_k)x)| < \varepsilon$ for every finite collection of pairwise disjoint intervals $[a_k, b_k] \subset [t_0, T]$ ($k = 1, \dots, N$) with $\sum_{k=1}^N (b_k - a_k) < \delta$. Let $T \in (t_0, t_{\max})$ and $\varepsilon > 0$ (arbitrary). Since the solution $x \in C^0([t_0 - r, T]; \mathfrak{R}^n)$ of (1.10) under Hypotheses (S1–4) corresponding to certain $(d, u) \in M_D \times M_U$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$ is bounded on $[t_0 - r, T]$, there exists $R_1 > 0$ such that $\sup_{t_0 \leq \tau \leq T} \|T_r(\tau)x\|_r \leq R_1$. Moreover, by Hypothesis (S2) and since $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathfrak{R}^n)$, there exists $R_2 > 0$ such that $\sup_{t_0 - r \leq \tau \leq T} |\dot{x}(\tau)| \leq R_2$. The previous observations, in conjunction with properties (P1)–(P2) of Definition 2.4 imply, for every interval $[a, b] \subset [t_0, T]$ with $b - a \leq \frac{1}{G(R_1 + R_2)}$,

$$\begin{aligned} & |V(b, T_r(b)x) - V(a, T_r(a)x)| \\ & \leq (b - a)P(R_1)(1 + R_2) + M(R_1)\|T_r(b)x - T_r(a)x\|_r \end{aligned}$$

In addition, the estimate $\sup_{t_0 - r \leq \tau \leq T} |\dot{x}(\tau)| \leq R_2$ implies $\|T_r(b)x - T_r(a)x\|_r \leq (b - a)R_2$ for every interval $[a, b] \subset [t_0, T]$. Consequently, we obtain, for every interval $[a, b] \subset [t_0, T]$ with $b - a \leq \frac{1}{G(R_1 + R_2)}$,

$$|V(b, T_r(b)x) - V(a, T_r(a)x)| \leq (b - a)[P(R_1)(1 + R_2) + M(R_1)R_2]$$

The previous inequality implies that for every finite collection of pairwise disjoint intervals $[a_k, b_k] \subset [t_0, T]$ ($k = 1, \dots, N$) with $\sum_{k=1}^N (b_k - a_k) < \delta$, where $\delta = \frac{1}{2} \min\{\frac{1}{G(R_1 + R_2)}; \frac{\varepsilon}{P(R_1)(1 + R_2) + M(R_1)R_2}\} > 0$, it holds that

$$\sum_{k=1}^N |V(b_k, T_r(b_k)x) - V(a_k, T_r(a_k)x)| < \varepsilon.$$

The proof is complete. $\square$

The following technical lemma shows that, without loss of generality, we can restrict our attention to continuously differentiable initial conditions.

Lemma 2.18 Consider system (1.3) under Hypotheses (S1–4) and suppose that there exist mappings $\beta_1 : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathfrak{R}$ and $\beta_2 : \mathfrak{R}^+ \times \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A \rightarrow \mathfrak{R}$, where $A \subseteq M_D \times M_U$, with the following properties:

- (i) for each $(t, t_0, d, u) \in \mathfrak{R}^+ \times \mathfrak{R}^+ \times A$, the mappings $x \rightarrow \beta_1(t, x)$, $x \rightarrow \beta_2(t, t_0, x, d, u)$ are continuous,
- (ii) there exists a continuous function $M : \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ such that

$$\sup \left\{ \beta_2(t_0 + \xi, t_0, x_0, d, u); \sup_{t \geq 0} |u(\tau)| \leq s, \xi \in [0, T], x_0 \in C^0([-r, 0]; \mathfrak{R}^n), \right. \\ \left. \|x_0\|_r \leq s, t_0 \in [0, T], (d, u) \in A \right\} \leq M(T, s),$$

- (iii) for every $(t_0, x_0, d, u) \in \mathfrak{R}^+ \times C^1([-r, 0]; \mathfrak{R}^n) \times A$, the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ satisfies

$$\beta_1(t, T_r(t)x) \leq \beta_2(t, t_0, x_0, d, u) \quad \text{for all } t \geq t_0. \quad (2.139)$$

Moreover, suppose that one of the following properties holds:

- (iv)

$$c(T, s) := \sup \left\{ \|T_r(t_0 + \xi)x\|_r; \sup_{t \geq 0} |u(\tau)| \leq s, \xi \in [0, T], x_0 \in C^0([-r, 0]; \mathfrak{R}^n), \right. \\ \left. \|x_0\|_r \leq s, t_0 \in [0, T], (d, u) \in A \right\} < +\infty,$$

- (v) there exist functions $a \in K_\infty$, $\mu \in K^+$ and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq \beta_1(t, x) + R$ for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n)$.

Then for all $(t_0, x_0, d, u) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A$, the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ exists for all $t \geq t_0$ and satisfies (2.139).

Proof We distinguish the following cases:

- (a) Property (iv) holds.

The proof will be made by contradiction. Suppose on the contrary that there exist $(t_0, x_0, d, u) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \times A$ and $t_1 > t_0$ such that the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ satisfies

$$\beta_1(t_1, T_r(t_1)x) > \beta_2(t_1, t_0, x_0, d, u)$$

(1.9) and property (iv) implies that, for all $\tilde{x}_0 \in C^0([-r, 0]; \mathfrak{R}^n)$ with $\|x_0 - \tilde{x}_0\|_r \leq 1$,

$$\|T_r(t_1)x - T_r(t_1)\tilde{x}\|_r \leq \sqrt{\frac{K_2}{K_1}} \|x_0 - \tilde{x}_0\|_r \exp(K_1^{-1}L(t_1, \bar{c})(t_1 - t_0)) \quad (2.140)$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d, u) \in A$, and $\bar{c} = 2c(t_1, \|x_0\|_r + 1 + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)|) + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)|$.

Let $\varepsilon := \beta_1(t_1, T_r(t_1)x) - \beta_2(t_1, t_0, x_0, u) > 0$. Using property (iv), (2.140), the fact that $C^1([-r, 0]; \mathfrak{H}^n)$ is dense in $C^0([-r, 0]; \mathfrak{H}^n)$, and the continuity of the mappings $x \rightarrow \beta_1(t_1, x)$ and $x \rightarrow \beta_2(t_1, t_0, x, d, u)$, we conclude that there exists $\tilde{x}_0 \in C^1([-r, 0]; \mathfrak{H}^n)$ such that

$$\begin{aligned} \|x_0 - \tilde{x}_0\|_r &\leq 1; \quad |\beta_2(t_1, t_0, x_0, d, u) - \beta_2(t_1, t_0, \tilde{x}_0, d, u)| \leq \frac{\varepsilon}{2} \\ |\beta_1(t_1, T_r(t_1)x) - \beta_1(t_1, T_r(t_1)\tilde{x})| &\leq \frac{\varepsilon}{2} \end{aligned}$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d, u) \in A$. Combining property (iii) for $\tilde{x}(t)$ with the above inequalities and the definition of ε , we obtain $\beta_1(t_1, T_r(t_1)x) > \beta_1(t_1, T_r(t_1)\tilde{x})$, a contradiction.

(b) Property (v) holds.

It suffices to show that property (iv) holds. Since there exist functions $a \in K_\infty$, $\mu \in K^+$, and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq \beta_1(t, x) + R$ for all $(t, x) \in \mathfrak{H}^+ \times C^0([-r, 0]; \mathfrak{H}^n)$, it follows that from property (iii) that for every $(t_0, x_0, d, u) \in \mathfrak{H}^+ \times C^1([-r, 0]; \mathfrak{H}^n) \times A$, the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ satisfies

$$a(\mu(t)|x(t)|) \leq R + \beta_2(t, t_0, x_0, d, u) \quad \text{for all } t \geq t_0$$

Moreover, making use of property (ii) and the above inequality, we obtain that for every $(t_0, x_0, d, u) \in \mathfrak{H}^+ \times C^1([-r, 0]; \mathfrak{H}^n) \times A$, the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ satisfies, for all $t \geq t_0$,

$$\|T_r(t)x\|_r \leq \|x_0\|_r + 1 + \frac{1}{\mu(t)} a^{-1} \left(R + M \left(t, \|x_0\|_r + \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right) \right) \quad (2.141)$$

Notice that in order to obtain inequality (2.141), we have also used the causality argument that the solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ depends only on the values of the input u on the interval $[t_0, t]$.

We claim that estimate (2.141) holds for all $(t_0, x_0, d, u) \in \mathfrak{H}^+ \times C^0([-r, 0]; \mathfrak{H}^n) \times A$. Notice that this claim implies directly that property (iv) holds with

$$c(T, s) := s + 1 + \frac{1}{\min_{0 \leq \tau \leq 2T} \mu(\tau)} a^{-1} \left(R + \max_{0 \leq x \leq 2s, 0 \leq \tau \leq 2T} M(\tau, x) \right)$$

The proof of the claim will be made by contradiction. Suppose on the contrary that there exists $(t_0, x_0, d, u) \in \mathfrak{H}^+ \times C^0([-r, 0]; \mathfrak{H}^n) \times A$ and $t_1 > t_0$ such that the

solution $x(t)$ of (1.10) with initial condition $T_r(t_0)x = x_0$ corresponding to input $(d, u) \in A$ satisfies, for all $t \geq t_0$,

$$\|T_r(t_1)x\|_r > \|x_0\|_r + 1 + \frac{1}{\mu(t_1)} a^{-1} \left(R + M(t_1, \|x_0\|_r + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)|) \right) \quad (2.142)$$

Let $B := \sup_{t_0 \leq \tau \leq t_1} \|T_r(\tau)x\| < +\infty$. Using (1.9) and (2.141), it follows that (2.140) holds for all

$$\tilde{x}_0 \in C^1([-r, 0]; \mathfrak{R}^n) \quad \text{with } \|x_0 - \tilde{x}_0\|_r \leq 1$$

and

$$\begin{aligned} \bar{c} = B + \|x_0\|_r + 2 + \max & \left\{ \frac{a^{-1}(R + M(t, s))}{\mu(t)}; 0 \leq s \leq \|x_0\|_r + 1 \right. \\ & \left. + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)|, t_0 \leq t \leq t_1 \right\} + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)| \end{aligned}$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d, u) \in A$. Let

$$\varepsilon := \|T_r(t_1)x\|_r - \|x_0\|_r - 1 - \frac{1}{\mu(t_1)} a^{-1} \left(R + M(t_1, \|x_0\|_r + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)|) \right) > 0$$

Using (2.140), (2.141), the denseness of $C^1([-r, 0]; \mathfrak{R}^n)$ in $C^0([-r, 0]; \mathfrak{R}^n)$, and the continuity of the mapping

$$x \rightarrow g(x) := \|x\|_r + 1 + \frac{1}{\mu(t_1)} a^{-1} \left(R + M(t_1, \|x\|_r + \sup_{t_0 \leq \tau \leq t_1} |u(\tau)|) \right)$$

we conclude that there exists $\tilde{x}_0 \in C^1([-r, 0]; \mathfrak{R}^n)$ such that

$$\|x_0 - \tilde{x}_0\|_r \leq 1 \quad |g(x_0) - g(\tilde{x}_0)| \leq \frac{\varepsilon}{2} \quad \left| \|T_r(t_1)x\|_r - \|T_r(t_1)\tilde{x}\|_r \right| \leq \frac{\varepsilon}{2}$$

where $\tilde{x}(t)$ denotes the solution of (1.10) with initial condition $T_r(t_0)x = \tilde{x}_0$ corresponding to input $(d, u) \in A$. Combining (2.141) for $\tilde{x}(t)$ with the above inequalities and the definition of ε , we obtain $\|T_r(t_1)x\|_r > \|T_r(t_1)\tilde{x}\|_r$, a contradiction. The proof is complete. $\square$

Finally, we show how we can overcome the fact that it may not be required that the Lyapunov functional bounds a certain function of the norm of the output. The following definition introduces an important relation between output mappings. The equivalence relation defined next will be used extensively in the following sections of the present work.

Definition 2.5 Suppose that there exists a continuous mapping $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ with $h(t, 0) = 0$ for all $t \geq -r$ and functions $a_1, a_2 \in K_\infty$ such that

$a_1(|h(t, x(0))|) \leq \|H(t, x)\|_{\mathcal{Y}} \leq a_2(\sup_{\theta \in [-r, 0]} |h(t + \theta, x(\theta))|)$ for all $(t, x) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n)$. Then we say that $H : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \rightarrow \mathcal{Y}$ is equivalent to the finite-dimensional mapping h .

For example, the identity output mapping $H(t, x) = x \in C^0([-r, 0]; \mathbb{R}^n)$ is equivalent to finite-dimensional mapping $h(t, x) = x \in \mathbb{R}^n$.

We are now in a position to state the main Lyapunov sufficient condition for RGAOS.

Theorem 2.5 *Consider system (1.10) under Hypotheses (S1–4) and suppose that system (1.10) is RFC. Moreover, suppose that there exist functions $a_1, a_2 \in K_\infty$, $\beta, \gamma \in K^+$ with $\int_0^{+\infty} \gamma(t) dt = +\infty$, $\varphi \in \mathcal{E}$, a positive definite locally Lipschitz function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and a mapping $V : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^+$, which is almost Lipschitz on bounded sets, such that*

$$a_1(\|H(t, x)\|_{\mathcal{Y}}) \leq V(t, x) \leq a_2(\beta(t)\|x\|_r) \quad \forall (t, x) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \quad (2.143)$$

$$V^0(t, x; f(t, x, 0, d)) \leq -\gamma(t)\rho(V(t, x)) + \gamma(t)\varphi\left(\int_0^t \gamma(s) ds\right) \quad \text{for all } (t, x, d) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \times D \quad (2.144)$$

Then system (1.10) is RGAOS. Furthermore,

- (i) if $H : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \rightarrow \mathcal{Y}$ is equivalent to the finite-dimensional continuous mapping $h : [-r, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^p$, then inequality (2.143) can be replaced by the following inequality:

$$a_1(|h(t, x(0))|) \leq V(t, x) \leq a_2(\beta(t)\|x\|_r) \quad \forall (t, x) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \quad (2.145)$$

- (ii) if there exist functions $a \in K_\infty$, $\mu \in K^+$, and a constant $R \geq 0$ such that $a(\mu(t)|x(0)|) \leq V(t, x) + R$ for all $(t, x) \in \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n)$, then the requirement that (1.10) is RFC is not needed.

Finally, if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and $\beta(t) \equiv 1$, then system (1.10) is URGAOS. In this case, if $H : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \rightarrow \mathcal{Y}$ is equivalent to the time-periodic finite-dimensional continuous mapping $h : [-r, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^p$, then inequality (2.143) can be replaced by inequality (2.145).

Proof Case 1: (1.10) is RFC.

Consider a solution $x(t)$ of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathbb{R}^n)$. By Lemma 2.17, for every $T \in (t_0, +\infty)$, the mapping $[t_0, T] \ni t \rightarrow V(t, T_r(t)x)$ is absolutely continuous. It follows from (2.144) and Lemma 2.16 that $\frac{d}{dt}(V(t, T_r(t)x)) \leq$

$-\gamma(t)\rho(V(t, T_r(t)x)) + \gamma(t)\varphi(\int_0^t \gamma(s) ds)$ a.e. on $[t_0, +\infty)$. The previous differential inequality, in conjunction with the comparison Lemma 2.15, shows that there exist $\sigma \in KL$ and a constant $M > 0$ such that

$$V(t, T_r(t)x) \leq \sigma \left(V(t_0, x_0) + M, \int_{t_0}^t \gamma(s) ds \right) \quad \text{for all } t \geq t_0 \quad (2.146)$$

It follows from Lemma 2.18 that the solution $x(t)$ of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \mathfrak{R}^n)$ satisfies (2.146) for all $t \geq t_0$. Next, we distinguish the following cases:

1. If (2.143) holds, then the Robust Output Attractivity Property (Property P3 of Definition 2.2) is a direct consequence of (2.143) and the fact that $\int_0^{+\infty} \gamma(t) dt = +\infty$. It follows from Lemma 2.1 that system (1.10) is RGAOS.
2. If (2.145) holds, then (2.146) implies the following estimate:

$$|h(t, x(t))| \leq a_1^{-1} \left(\sigma \left(a_2(\beta(t_0)\|x_0\|_r) + M, \int_{t_0}^t \gamma(s) ds \right) \right) \quad \text{for all } t \geq t_0.$$

Since $h : [-r, +\infty) \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ is continuous with $h(t, 0) = 0$ for all $t \geq -r$, it follows from Lemma 2.3 with $a(\tau, s) := \max\{|h(t - r, x)| : t \in [0, \tau], |x| \leq s\}$ that there exist functions $\zeta \in K_\infty$ and $\delta \in K^+$ such that

$$|h(t - r, x)| \leq \zeta(\delta(t)|x|) \quad \text{for all } (t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$$

Combining the two previous inequalities yields

$$\begin{aligned} & \sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| \\ & \leq \max \left\{ a_1^{-1} \left(\sigma \left(a_2(\phi(t_0)\|x_0\|_r) + M, 0 \right) \right), \zeta(\phi(t_0)\|x_0\|_r) \right\} \quad \text{for all } t \in [t_0, t_0 + r] \\ & \sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| \\ & \leq a_1^{-1} \left(\sigma \left(a_2(\beta(t_0)\|x_0\|_r) + M, \int_{t_0}^{t-r} \gamma(s) ds \right) \right) \quad \text{for all } t \geq t_0 + r \end{aligned}$$

where $\phi(t) := \beta(t) + \max_{0 \leq \tau \leq t+r} \delta(\tau)$. The above estimates, in conjunction with the facts that $\int_0^{+\infty} \gamma(t) dt = +\infty$ and $H : \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \rightarrow \mathcal{Y}$ is equivalent to the finite-dimensional mapping h (Definition 2.5), show that the Robust Output Attractivity Property (Property P3 of Definition 2.2) holds. Hence, system (1.10) is nonuniformly RGAOS.

Case 2: There exist functions $a \in K_\infty$, $\mu \in K^+$, and a constant $R \geq 0$ such that

$$a(\mu(t)|x(0)|) \leq V(t, x) + R \quad \text{for all } (t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^n) \quad (2.147)$$

Consider a solution of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with initial condition $T_r(t_0)x = x_0 \in C^1([-r, 0]; \mathbb{R}^n)$. By Lemma 2.17, for every $T \in (t_0, t_{\max})$, the mapping $[t_0, T] \ni t \rightarrow V(t, T_r(t)x)$ is absolutely continuous. It follows from (2.144) and Lemma 2.16 that for every $T \in (t_0, t_{\max})$, it holds that $\frac{d}{dt}(V(t, T_r(t)x)) \leq -\gamma(t)\rho(V(t, T_r(t)x)) + \gamma(t)\varphi(\int_0^t \gamma(s)ds)$ a.e. on $[t_0, T]$. The previous differential inequality, in conjunction with the comparison Lemma 2.15, shows that there exist $\sigma \in KL$ and a constant $M > 0$ such that (2.146) holds for all $t \in [t_0, T]$.

Combining (2.143), (2.146), and (2.147), we obtain

$$|x(t)| \leq \frac{1}{\mu(t)} a^{-1} \left(\sigma \left(a_2(\beta(t_0)\|x_0\|_r) + M, \int_{t_0}^t \gamma(s)ds \right) + R \right) \quad \text{for all } t \in [t_0, T] \quad (2.148)$$

Estimate (2.148) shows that $t_{\max} = +\infty$, and consequently estimates (2.146) and (2.148) hold for all $t \geq t_0$.

It follows from Lemma 2.18 that the solution $x(t)$ of (1.10) under Hypotheses (S1–4) corresponding to arbitrary $d \in M_D$ and $u \equiv 0$ with arbitrary initial condition $T_r(t_0)x = x_0 \in C^0([-r, 0]; \mathbb{R}^n)$ satisfies (2.146) and (2.148) for all $t \geq t_0$. Therefore, system (1.10) is RFC, and the Robust Output Attractivity Property (Property P3 of Definition 2.2) is a direct consequence of (2.146) and (2.143) (or (2.145)), as in the previous case.

Finally, notice that if $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, then Lemma 2.13 shows that there exists $\sigma \in KL$ such that

$$V(t, T_r(t)x) \leq \sigma(V(t_0, x_0), t - t_0) \quad \text{for all } t \geq t_0 \quad (2.149)$$

instead of (2.146). Utilizing (2.143) with $\beta(t) \equiv 1$ and (2.149), we obtain (2.11). Therefore system (1.10) is URGAOS. If (2.145) holds, then (2.149) implies the following estimate:

$$|h(t, x(t))| \leq a_1^{-1}(\sigma(a_2(\beta(t_0)\|x_0\|_r), t - t_0)) \quad \text{for all } t \geq t_0$$

Since $h : [-r, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^p$ is continuous and time-periodic (say, with period $T > 0$) with $h(t, 0) = 0$ for all $t \geq -r$, it follows from Lemma 2.4 (applied to the continuous function $a(s) := \max\{|h(t - r, x)| : t \in [0, T], |x| \leq s\}$) that there exists a function $\zeta \in K_\infty$ such that

$$|h(t - r, x)| \leq \zeta(|x|) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

Combining the previous two inequalities, we obtain

$$\begin{aligned} \sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| &\leq \max\{\zeta(\|x_0\|_r), a_1^{-1}(\sigma(a_2(\|x_0\|_r), 0))\} \\ &\quad \text{for all } t \in [t_0, t_0 + r] \\ \sup_{\theta \in [-r, 0]} |h(t + \theta, x(t + \theta))| &\leq a_1^{-1}(\sigma(a_2(\|x_0\|_r), t - t_0)) \\ &\quad \text{for all } t \geq t_0 + r \end{aligned}$$

The above estimates, in conjunction with the fact $H : \mathbb{R}^+ \times C^0([-r, 0]; \mathbb{R}^n) \rightarrow \mathcal{Y}$ is equivalent to the finite-dimensional mapping h (Definition 2.5), show that Properties P1, P2, and P3 of Definition 2.3 hold. Hence, system (1.10) is URGAOS. The proof is complete. $\square$

2.7 Examples of the Method of Lyapunov Functionals

In this section we present some examples of the use of Lyapunov functionals for the proof of RGAOS. In all cases we will show RGAOS without assuming any knowledge of the transition map; this is the advantage of the method of Lyapunov functionals.

Example 2.7.1 Consider the system

$$\begin{aligned} \dot{x}_1(t) &= -2g_1(t)x_1(t) + d_1(t)g_2(t)x_1^2(t) - g_3(t)x_1^3(t) + d_2(t)b(t)|x_2(t - \tau(t))|^p \\ \dot{x}_2(t) &= c(t)x_2(t) \\ Y(t) &= x_1(t) \in \mathbb{R} \\ x(t) &= (x_1(t), x_2(t))' \in \mathbb{R}^2 \quad d = (d_1, d_2) \in D := [-1, 1] \times [-1, 1] \end{aligned} \tag{2.150}$$

where $g_1 \in K^+$, $g_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$, $g_3 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $b : \mathbb{R}^+ \rightarrow \mathbb{R}$, and $c : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and $p > \frac{1}{2}$ is a constant. We consider system (2.150) under the following hypotheses:

- (A1) There exist a continuous function $g \in K^+$ with $\int_0^{+\infty} g(s) ds = +\infty$ and a constant $M > 0$ such that $Mg(t) \leq g_1(t)$ for all $t \geq 0$. Moreover, $g_2^2(t) \leq 4g_1(t)g_3(t)$ for all $t \geq 0$.
- (A2) The function $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuously differentiable, nonnegative, and bounded from above by a constant $r > 0$.
- (A3) There exist constants $K, m > 0$ such that, for all $t \geq 0$,

$$\frac{|b(t)|^2}{g_1(t)} \exp\left(2p \int_0^{t-\tau(t)} c(s) ds\right) \leq K(1 - \dot{\tau}(t)) \exp\left(-mt - \int_0^t g(s) ds\right). \tag{2.151}$$

For example, Hypotheses (A1–3) are satisfied for $p = 2$, $\tau(t) := 2 + \frac{1}{2} \sin(t)$, $c(t) \equiv 1$, $b(t) \equiv 1$, $g_1(t) = \exp(6t)$, $g_2(t) = \exp(t)$, $g_3(t) \equiv 1$, with $g(t) \equiv 1$, $M := 1$, $K := 2$, $m := 1$. We next show that system (2.150) under Hypotheses (A1–3) is nonuniformly in time RGAOS. We consider the functional

$$\begin{aligned} V(t, x) &:= \frac{1}{2}x_1^2(0) + \exp\left(-p^{-1}mt - \int_0^t (p^{-1}g(s) + 2c(s)) ds\right)x_2^2(0) \\ &\quad + \frac{K}{2m} \exp\left(-mt - \int_0^t (g(s) + 2pc(s)) ds\right)|x_2(0)|^{2p} \end{aligned}$$

$$\begin{aligned}
& + \frac{K}{2} \exp\left(-mt - \int_0^t g(s) ds\right) \\
& \times \int_{-\tau(t)}^0 \exp\left(-ms - 2p \int_0^{t+s} c(\xi) d\xi\right) |x_2(s)|^{2p} ds \quad (2.152)
\end{aligned}$$

Since $p > \frac{1}{2}$, it follows that the functional defined by (2.152) is almost Lipschitz on bounded sets. Moreover, inequalities (2.143) hold with $H(t, x) = x_1(0) \in \mathcal{Y} := \mathfrak{R}$, $a_1(s) := 2^{-1}s^2$, $a_2(s) := s^2 + 2^{-1}K(m^{-1} + r \exp(mr))s^{2p}$, and $\beta(t) := 1 + \max_{0 \leq \xi \leq t} \exp(-\frac{m}{2p}\xi - \frac{1}{2p} \int_0^\xi g(s) ds) \max_{-\tau(\xi) \leq s \leq 0} \exp(-\int_0^{\xi+s} c(w) dw)$. Furthermore, we obtain, for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2) \times D$,

$$\begin{aligned}
& V^0(t, x; -2g_1(t)x_1(0) + d_1g_2(t)x_1^2(0) - g_3(t)x_1^3(0) + d_2b(t)|x_2(-\tau(t))|^p, \\
& \quad c(t)x_2(0)) \\
& \leq -2g_1(t)x_1^2(0) + |g_2(t)||x_1(0)|^3 - g_3(t)x_1^4(0) + |b(t)||x_1(0)||x_2(-\tau(t))|^p \\
& \quad - p^{-1}(m + g(t)) \exp\left(-p^{-1}mt - \int_0^t (p^{-1}g(s) + 2c(s)) ds\right) x_2^2(0) \\
& \quad - g(t) \frac{K}{2m} \exp\left(-mt - \int_0^t (g(s) + 2pc(s)) ds\right) |x_2(0)|^{2p} \\
& \quad - g(t) \frac{K}{2} \exp\left(-mt - \int_0^t g(s) ds\right) \int_{-\tau(t)}^0 \\
& \quad \times \exp\left(-ms - 2p \int_0^{t+s} c(\xi) d\xi\right) |x_2(s)|^{2p} ds \\
& \quad - \frac{K}{2}(1 - \dot{\tau}(t)) \exp\left(-m(t - \tau(t)) - \int_0^t g(s) ds - 2p \int_0^{t-\tau(t)} c(s) ds\right) \\
& \quad \times |x_2(-\tau(t))|^{2p}
\end{aligned}$$

Hypothesis (A1) implies that $-g_1(t)x_1^2 + |g_2(t)||x_1|^3 - g_3(t)x_1^4 \leq 0$ for all $(t, x_1) \in \mathfrak{R}^+ \times \mathfrak{R}$. Using the inequality $|b(t)||x_1(0)||x_2(-\tau(t))|^p \leq 2^{-1}g_1(t)x_1^2(0) + \frac{|b(t)|^2}{2g_1(t)}|x_2(-\tau(t))|^{2p}$ in conjunction with (2.151), we obtain:

$$\begin{aligned}
& V^0(t, x; -2g_1(t)x_1(0) + d_1g_2(t)x_1^2(0) - g_3(t)x_1^3(0) + d_2b(t)|x_2(-\tau(t))|^p, \\
& \quad c(t)x_2(0)) \\
& \leq -\frac{1}{2}g_1(t)x_1^2(0) - p^{-1}g(t) \exp\left(-p^{-1}mt - \int_0^t (p^{-1}g(s) + 2c(s)) ds\right) x_2^2(0) \\
& \quad - g(t) \frac{K}{2m} \exp\left(-mt - \int_0^t (g(s) + 2pc(s)) ds\right) |x_2(0)|^{2p}
\end{aligned}$$

$$\begin{aligned}
& -g(t) \frac{K}{2} \exp\left(-mt - \int_0^t g(s) ds\right) \\
& \times \int_{-\tau(t)}^0 \exp\left(-ms - 2p \int_0^{t+s} c(\xi) d\xi\right) |x_2(s)|^{2p} ds
\end{aligned}$$

for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2) \times D$.

Since $Mg(t) \leq g_1(t)$ for all $t \geq 0$, the above inequality, in conjunction with definition (2.152), gives, for all $(t, x, d) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2) \times D$,

$$\begin{aligned}
& V^0(t, x; -2g_1(t)x_1(0) + d_1g_2(t)x_1^2(0) - g_3(t)x_1^3(0) + d_2b(t)|x_2(-\tau(t))|^p, \\
& \quad c(t)x_2(0)) \\
& \leq -\min\{M, p^{-1}, 1\}g(t)V(t, x)
\end{aligned}$$

Consequently, inequality (2.144) holds with

$$\rho(s) := s \quad \varphi(t) \equiv 0 \quad \text{and} \quad \gamma(t) := \min\{M, p^{-1}, 1\}g(t)$$

If system (2.150) were RFC, we would have showed all hypotheses of Theorem 2.5. However, since the inequality $a(\mu(t)|x(0)|) \leq V(t, x) + R$ holds for all $(t, x) \in \mathfrak{R}^+ \times C^0([-r, 0]; \mathfrak{R}^2)$ with $R := 0$, $a(s) := s^2$ and

$$\mu(t) := \left(\min\left(2^{-1}; \exp\left(-\frac{m}{p}t - \int_0^t (p^{-1}g(s) + 2c(s))ds\right)\right) \right)^{\frac{1}{2}}$$

the requirement that system (2.150) were RFC is not needed. Thus we can conclude that system (2.152) is RGAOS.

Moreover, if, in addition to Hypotheses (A1–3), the following hypothesis holds as well:

(A4) There exist constants $\Gamma, A > 0$ such that $g(t) \geq \Gamma$ and

$$\frac{m}{2p}t + \frac{1}{2p} \int_0^t g(s) ds + \min_{-\tau(t) \leq w \leq 0} \int_0^{t+w} c(s) ds \geq -A \quad \text{for all } t \geq 0$$

then we can conclude that system (2.150) is URGAOS. Notice that if (A4) holds, inequalities (2.143) and (2.144) hold with $\rho(s) := \min\{M, p^{-1}, 1\}gs$, $H(t, x) = x_1(0) \in \mathcal{Y} := \mathfrak{R}$, $a_1(s) := 2^{-1}s^2$, $a_2(s) := \exp(2A)s^2 + 2^{-1}K(m^{-1} + r \exp(mr)) \exp(2pA)s^{2p}$, $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and $\beta(t) \equiv 1$. Consequently, all hypotheses of Theorem 2.5 hold (without the requirement that system (2.150) is RFC).

Theorems 2.4 and 2.5 can be used for the proof of RFC (simply consider the system with zero output map $H(t, x) \equiv 0$). The following example illustrates this point.

Example 2.7.2 Consider the following system described by ODEs:

$$\begin{aligned}\dot{x} &= c_1(t, d)x + c_2(t, d)x^2 + c_3(t, d)x^3 \\ x &\in \mathfrak{R}, t \geq 0, d \in D\end{aligned}\tag{2.153}$$

where $D \subset \mathfrak{R}^m$ is a compact set, $c_i(\cdot) \in C^0(\mathfrak{R}^+ \times D)$, $i = 1, 2, 3$, are continuous mappings for which there exists a function $\phi \in K^+$ such that

$$2\phi(t)c_3(t, d) \leq -|c_2(t, d)| \quad \text{for all } (t, d) \in \mathfrak{R}^+ \times D\tag{2.154}$$

We next prove that this system is RFC. Consider the Lyapunov function candidate

$$W(t, x) := \exp\left(-\int_0^t \left(1 + 2\max_{d \in D} |c_1(\tau, d)| + \phi(\tau) \max_{d \in D} |c_2(\tau, d)|\right) d\tau\right) |x|^2\tag{2.155}$$

Clearly, we have, for all $(t, x, d) \in \mathfrak{R}^+ \times \mathfrak{R} \times D$,

$$\begin{aligned}\frac{\partial W}{\partial t}(t, x) + \frac{\partial W}{\partial x}(t, x)(c_1(t, d)x + c_2(t, d)x^2 + c_3(t, d)x^3) \\ = -\left(1 + 2\max_{d \in D} |c_1(t, d)| + \phi(t) \max_{d \in D} |c_2(t, d)|\right) W(t, x) \\ + (2c_1(t, d)x^2 + 2c_2(t, d)x^3 + 2c_3(t, d)x^4) \\ \times \exp\left(-\int_0^t \left(1 + 2\max_{d \in D} |c_1(\tau, d)| + \phi(\tau) \max_{d \in D} |c_2(\tau, d)|\right) d\tau\right)\end{aligned}\tag{2.156}$$

It follows from inequality (2.154), in conjunction with the Young inequality $2c_2(t, d)x^3 \leq \phi(t)|c_2(t, d)|x^2 + \frac{|c_2(t, d)|}{\phi(t)}x^4$, that the following inequality holds:

$$\begin{aligned}\frac{\partial W}{\partial t}(t, x) + \frac{\partial W}{\partial x}(t, x)(c_1(t, d)x + c_2(t, d)x^2 + c_3(t, d)x^3) \leq -W(t, x) \\ \text{for all } (t, x, d) \in \mathfrak{R}^+ \times \mathfrak{R} \times D\end{aligned}$$

The reader should notice that all hypotheses of Theorem 2.4 hold with

$$\begin{aligned}H(t, x) &\equiv 0 \quad a(s) = a_2(s) := s^2 \\ \mu(t) &:= \exp\left(-\frac{1}{2} \int_0^t \left(1 + 2\max_{d \in D} |c_1(\tau, d)| + \phi(\tau) \max_{d \in D} |c_2(\tau, d)|\right) d\tau\right) \\ \beta(t) = \gamma(t) &\equiv 1 \quad \rho(s) := s \quad \varphi(t) \equiv 0 \quad \text{and} \quad R := 0\end{aligned}$$

Thus system (2.153) with zero output map is URGAS. Particularly, this implies that system (2.153) is RFC.

Example 2.7.3 Consider system (1.7) which arises in the study of the chemostat model (1.4) of Example 1.2.1 in Chap. 1. Here we assume that $s_{\text{in}}(t) \equiv s_{\text{in}}^*$ (i.e.,

$u_2(t) \equiv 0$ from (1.6)), $D(t) \equiv D^*$ (i.e., $u_1(t) \equiv 0$ from (1.6)), $b = 0$ (i.e., $R := 1$), and $K(s) \equiv K$ (i.e., $p(x_2) \equiv 1$). Therefore the transformed chemostat model (1.7) takes the simple form

$$\begin{aligned}\dot{x}_1 &= D^* g(x_2) \\ \dot{x}_2 &= D^* \exp(-x_2) [M(1 - \exp(x_1)) - M g(x_2) \exp(x_1) + 1 - \exp(x_2)] \\ x &= (x_1, x_2) \in \mathfrak{N}^2\end{aligned}\tag{2.157}$$

Moreover, we will assume that the specific growth rate $\mu(s)$ of the microbial species is a nondecreasing, locally Lipschitz, bounded function $\mu : [0, +\infty) \rightarrow [0, +\infty)$ with $\mu(0) = 0$, $\mu(s) \neq D^*$ for all $s \neq s^*$, and $\mu(s) > 0$ for all $s > 0$. Notice that the previous requirements are automatically satisfied if $\mu : [0, +\infty) \rightarrow [0, +\infty)$ is strictly increasing, but here we will avoid to make such an assumption. Consequently, the function $g : \mathfrak{N} \rightarrow (-1, +\infty)$ is a nondecreasing, locally Lipschitz, bounded function with $g(0) = 0$, $\lim_{x_2 \rightarrow -\infty} g(x_2) = -1$, and $g(x_2) \neq 0$ for all $x_2 \neq 0$.

We will show that system (2.157) is URGAS (although in this case the adjective “robust” does not apply since no disturbances appear in model (2.157)). To this end, we will use the Lyapunov function

$$V(x) = \exp(x_1) - x_1 - 1 + B(x_2) + Q(1 - \exp(x_2) + M - M \exp(x_1))^2 \tag{2.158}$$

where $B(x) := M^{-1} \int_0^x \frac{g(w)}{1+g(w)} \exp(w) dw = \frac{1}{Ms^*} \int_{s^*}^{s^* \exp(x)} \frac{\mu(s) - D^*}{\mu(s)} ds$, and $Q > 0$ is a constant. We first notice that the function $V : \mathfrak{N}^2 \rightarrow [0, +\infty)$ is a continuously differentiable function with $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. For every $a \geq 0$, the set $\{x \in \mathfrak{N}^2 : V(x) \leq a\}$ is compact. Indeed, this follows from the equalities $\lim_{x_1 \rightarrow \pm\infty} (\exp(x_1) - x_1 - 1) = +\infty$ and $\lim_{x_2 \rightarrow \pm\infty} B(x_2) = +\infty$. Notice that for all $x_2 \geq 0$, it holds that $B(x_2) \geq \frac{1}{Ms^*} \int_{s^*}^{s^* \exp(x_2)} \frac{\mu(s) - D^*}{\mu_{\max}} ds$, where $\mu_{\max} = \sup_{s>0} \mu(s)$, which shows that $\lim_{x_2 \rightarrow +\infty} B(x_2) = +\infty$. The equality $\lim_{x_2 \rightarrow -\infty} B(x_2) = +\infty$ follows from the fact that the function $\mu : [0, +\infty) \rightarrow [0, +\infty)$ is locally Lipschitz and thus there exist $p, L > 0$ with $\mu(s) \leq Ls$ for all $0 \leq s \leq p$. Consequently, for all $x_2 \leq \ln(\frac{p}{s^*})$, we obtain the following inequalities which show that $\lim_{x_2 \rightarrow -\infty} B(x_2) = +\infty$:

$$\begin{aligned}B(x_2) &= \frac{1}{Ms^*} \int_{s^* \exp(x_2)}^{s^*} \frac{D^* - \mu(s)}{\mu(s)} ds \geq \frac{1}{Ms^*} \int_{s^* \exp(x_2)}^p \frac{D^* - \mu(s)}{\mu(s)} ds \\ &\geq \frac{D^*}{Ms^*} \int_{s^* \exp(x_2)}^p \frac{ds}{\mu(s)} - \frac{p}{Ms^*} \geq \frac{D^*}{Ms^* L} \int_{s^* \exp(x_2)}^p \frac{ds}{s} - \frac{p}{Ms^*} \\ &= \frac{D^*(\ln(p) - \ln(s^*) - x_2)}{Ms^* L} - \frac{p}{Ms^*}\end{aligned}$$

The following proposition helps us at this point.

Proposition 2.2 *Let $V : \mathbb{R}^n \rightarrow [0, +\infty)$ be a continuous function with $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. Moreover, assume that for every $a \geq 0$, the set $\{x \in \mathbb{R}^n : V(x) \leq a\}$ is compact. Then there exist functions $a_1, a_2 \in K_\infty$ such that $a_1(|x|) \leq V(x) \leq a_2(|x|)$ for all $x \in \mathbb{R}^n$. Furthermore, if $P : \mathbb{R}^n \rightarrow [0, +\infty)$ is a continuous function with $P(0) = 0$ and $P(x) > 0$ for all $x \neq 0$, then there exists a locally Lipschitz positive definite function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\rho(V(x)) \leq P(x)$ for all $x \in \mathbb{R}^n$.*

Proof of Proposition 2.2 The function $a(s) := \max\{V(x) : x \in \mathbb{R}^n, |x| \leq s\}$ is a nondecreasing (and thus locally bounded) function with $\lim_{s \rightarrow 0^+} a(s) = a(0) = 0$. Therefore, Lemma 2.4 implies the existence of $a_2 \in K_\infty$ such that $a(s) \leq a_2(s)$ for all $s \geq 0$ and consequently, $V(x) \leq a_2(|x|)$ for all $x \in \mathbb{R}^n$. On the other hand, since for every $a \geq 0$, the set $\{x \in \mathbb{R}^n : V(x) \leq a\}$ is compact, it follows that the function $\beta(s) := \min\{V(x) : x \in \mathbb{R}^n, |x| \geq s\}$ is a well-defined, nondecreasing function with $\lim_{s \rightarrow +\infty} \beta(s) = +\infty$, $\beta(0) = 0$ and $\beta(s) > 0$ for all $s > 0$. The reader should verify all previous statements (which require knowledge of real analysis!). The function $a_1(s) := \frac{1}{s+1} \int_0^s \beta(w) dw = \frac{s}{s+1} \int_0^1 \beta(\lambda s) d\lambda$ defined for $s \geq 0$ is a K_∞ function which satisfies all requirements.

Finally, consider $\tilde{\rho}(s) := \min\{P(x) : x \in \mathbb{R}^n, 1 \leq V(x) \leq s\}$ for all $s \geq 1$ and $\tilde{\rho}(s) := \min\{P(x) : x \in \mathbb{R}^n, s \leq V(x) \leq 1\}$ for all $0 \leq s \leq 1$, which is positive definite, nondecreasing on $[0, 1]$ and nonincreasing on $[1, +\infty)$. Define the constant

$$a := \min \left\{ \frac{1}{2} \int_0^1 \tilde{\rho}(w) dw, \int_1^2 \tilde{\rho}(w) dw \right\}$$

and the function

$$\bar{\rho}(s) := \frac{2a}{s+1} \frac{\int_0^s \tilde{\rho}(w) dw}{\int_0^1 \tilde{\rho}(w) dw} \quad \text{for } 0 \leq s \leq 1$$

and

$$\bar{\rho}(s) := a \frac{\int_s^{s+1} \tilde{\rho}(w) dw}{\int_1^2 \tilde{\rho}(w) dw} \quad \text{for } s > 1$$

The function $\bar{\rho} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, positive definite, and satisfies $\bar{\rho}(s) \leq \tilde{\rho}(s)$ for all $s \geq 0$. By virtue of Remark 2.4, the function $\rho(s) := \inf\{\bar{\rho}(y) + |y - s|; y \geq 0\}$ satisfies all requirements. $\square$

Therefore, by Proposition 2.2 it follows that there exist $a_1, a_2 \in K_\infty$ such that (2.129) holds with $\beta(t) \equiv 1$ and $H(t, x, 0) := x$.

It is a matter of easy manipulation to show that

$$V^0(x; f(x)) = -P(x)$$

where

$$\begin{aligned}
 P(x) &:= M^{-1} \frac{g(x_2)}{1 + g(x_2)} D^* [Mg(x_2) + \exp(x_2) - 1] \\
 &\quad + 2D^* Q(1 - \exp(x_2) + M - M \exp(x_1))^2, \\
 f(x) &:= D^* [g(x_2), \exp(-x_2) (M(1 - \exp(x_1)) - Mg(x_2) \exp(x_1) + 1 - \exp(x_2))]'.
 \end{aligned}$$

Notice that $P : \mathbb{R}^2 \rightarrow [0, +\infty)$ is a continuous function with $P(0) = 0$ and $P(x) > 0$ for all $x \neq 0$. It follows from Proposition 2.2 that (2.130) holds with $\varphi(t) \equiv 0$, $\gamma(t) \equiv 1$, and appropriate locally Lipschitz positive definite function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. The proof that system (2.157) is URGAS is finished with the help of Theorem 2.4.

2.8 RGAOS for Discrete-Time Systems

In this section, the method of Lyapunov functionals is developed for discrete-time systems of the form (1.110) under Hypotheses (L1–3). Particularly, we have the following technical result.

Proposition 2.3 *Consider system (1.110) under Hypotheses (L1–3). Suppose that there exist functions $V : \pi \times \mathcal{X} \rightarrow \mathbb{R}^+$, $a_1, a_2 \in K_\infty$, $a_3 \in C^0(\mathbb{R}^+; \mathbb{R}^+)$ being positive definite with $a_3(s) \leq s$ for all $s \geq 0$, $\beta \in K^+$, and $q : \pi \rightarrow \mathbb{R}^+$ with $\lim_{i \rightarrow +\infty} q(\tau_i) = 0$ such that, for all $(i, x, d) \in Z^+ \times \mathcal{X} \times D$,*

$$a_1(\|H(\tau_i, x, 0)\|_{\mathcal{Y}}) \leq V(\tau_i, x) \leq a_2(\beta(\tau_i)\|x\|_{\mathcal{X}}) \quad (2.159)$$

$$V(\tau_{i+1}, f(\tau_i, d, x, 0)) \leq V(\tau_i, x) - a_3(V(\tau_i, x)) + q(\tau_i) \quad (2.160)$$

Then system (1.110) is RGAOS if one of the following holds:

- (i) $a_3 \in K$, or
- (ii) $\sum_{i=0}^{\infty} q(\tau_i) < +\infty$.

Moreover, if, in addition, $q \equiv 0$ and $\beta(t) \equiv 1$, then system (1.110) is URGAS.

For the proof of Proposition 2.3, we need the following technical result.

Lemma 2.19 *Let $a \in K$ with $a(s) \leq s$ for all $s \geq 0$, $M \in (0, \frac{1}{2} \lim_{s \rightarrow +\infty} a(s))$, and consider a sequence $\{V_i \in \mathbb{R}^+\}_{i=0}^{\infty}$ which satisfies the following inequality:*

$$V_{i+1} \leq V_i - a(V_i) + M \quad \text{for all } i \geq k_0 \in Z^+ \quad (2.161)$$

Then the following inequalities hold:

$$V_i \leq V_{k_0} + a^{-1}(M) + M \quad \text{for all } i \geq k_0 \in Z^+ \quad (2.162)$$

$$V_i < a^{-1}(2M) + M \quad \text{for all } i \geq k_0 + \frac{V_{k_0}}{M} \quad (2.163)$$

Proof of Lemma 2.19 We first prove (2.162) by induction. Notice that (2.162) holds for $i = k_0$. Suppose that (2.162) holds for some $i \in \mathbb{Z}^+$ with $i \geq k_0$. Consider the cases:

- if $a(V_i) \geq M$, then (2.161) implies $V_{i+1} \leq V_i$, and consequently (2.162) holds for $i + 1$,
- if $a(V_i) < M$ or equivalently if $V_i < a^{-1}(M)$, then (2.161) implies $V_{i+1} \leq V_i + M < a^{-1}(M) + M$, and consequently (2.162) holds for $i + 1$.

Next, we prove the following claim: if (2.163) holds for some $i = k \in \mathbb{Z}^+$ with $k \geq k_0$, then (2.163) holds for all $i \geq k$. Consider the cases:

- if $a(V_i) \geq M$, then (2.161) implies $V_{i+1} \leq V_i$, and consequently (2.163) holds for $i + 1$,
- if $a(V_i) < M$ or equivalently if $V_i < a^{-1}(M)$, then (2.161) implies $V_{i+1} \leq V_i + M < a^{-1}(M) + M \leq a^{-1}(2M) + M$, and consequently (2.163) holds for $i + 1$.

The proof of inequality (2.163) is made by contradiction. Suppose that there exists $k \in \mathbb{Z}^+$ with $k \geq k_0 + \frac{V_{k_0}}{M}$ such that $V_k \geq a^{-1}(2M) + M$. By virtue of the previous claim, this implies that $V_i \geq a^{-1}(2M) + M$ for all $i = k_0, \dots, k$. Consequently, we have $-a(V_i) + M \leq -M$ for all $i = k_0, \dots, k$. Thus, we obtain from (2.161):

$$V_{i+1} \leq V_i - M \quad \text{for all } i = k_0, \dots, k \quad (2.164)$$

Clearly, (2.164) implies that $V_k \leq V_{k_0} - M(k - k_0)$, and this estimate along with $k \geq k_0 + \frac{V_{k_0}}{M}$ gives $V_k \leq 0$. Clearly, this implication is in contradiction with the assumption $V_k \geq a^{-1}(2M) + M > 0$. The proof is complete. $\square$

We are now in a position to provide the proof of Proposition 2.3.

Proof of Proposition 2.3 Notice that Lemma 1.3 implies that system (1.110) is RFC. Thus, by virtue of Lemma 2.1, it suffices to show the Robust Output Attractivity Property (property P3 of Definition 2.2). Let $\varepsilon > 0$, $R, T \geq 0$ arbitrary and consider the solution $x(t)$ of (1.110) with initial condition $x(t_0) = x_0$ with $|x_0| \leq R$, $t_0 \in [0, T]$ corresponding to $d \in M_D$ and $u \equiv 0 \in M_U$. Let $J(T) \in \mathbb{Z}^+$ with $q_\pi(T) := \tau_{J(T)}$.

We distinguish the following cases:

Case of (i): Let $M = M(\varepsilon) \in (0, \frac{1}{2} \lim_{s \rightarrow +\infty} a_3(s))$ with $a_3^{-1}(2M) + M = a_1^{-1}(\varepsilon)$. There exists $N(\varepsilon) \in \mathbb{Z}^+$ such that $q(\tau_i) \leq M(\varepsilon)$ for all $i \geq N(\varepsilon)$. Notice that (2.160) implies $V(\tau_{i+1}, x(\tau_{i+1})) \leq V(\tau_i, x(\tau_i)) - a_3(V(\tau_i, x(\tau_i))) + M$ for all $i \geq N(\varepsilon)$ with $\tau_i \geq \tau_s = p_\pi(t_0)$. Lemma 2.19 and the fact $a_3^{-1}(2M) + M = a_1^{-1}(\varepsilon)$ imply that $V(\tau_i, x(\tau_i)) \leq a_1^{-1}(\varepsilon)$ for all $i \geq j + \frac{V(\tau_j, x(\tau_j))}{M(\varepsilon)}$ with $j = j(\varepsilon, T) := \max\{N(\varepsilon), J(T)\}$. Moreover, (2.160) implies that $V(\tau_j, x(\tau_j)) \leq V(\tau_s, x_0) + \sum_{k=0}^{j(\varepsilon, T)} q(\tau_k)$. Using (2.159), in conjunction with previous inequalities, implies that

$$V(\tau_i, x(\tau_i)) \leq a_1^{-1}(\varepsilon)$$

$$\text{for all } i \geq \tilde{N}(\varepsilon, T, R) := j(\varepsilon, T) + \frac{a_2(R \max_{k=0, \dots, J(T)} \beta(\tau_k)) + \sum_{k=0}^{j(\varepsilon, T)} q(\tau_k)}{M(\varepsilon)}$$

with $j = j(\varepsilon, T) := \max\{N(\varepsilon), J(T)\}$. Therefore, inequality (2.159) gives $\|H(t, x(t))\|_Y \leq \varepsilon$ for all $t \geq t_0 + \tau(\varepsilon, T, R)$ with $\tau(\varepsilon, T, R) := \tau_{\tilde{N}(\varepsilon, T, R)}$.

Case of (ii): Define $B := \sum_{i=0}^{\infty} q(\tau_i) < +\infty$. Notice that (2.160) implies

$$V(\tau_{i+1}, x(\tau_{i+1})) \leq V(\tau_i, x(\tau_i)) + q(\tau_i) \quad \text{for all } i \in \mathbb{Z}^+ \text{ with } \tau_i \geq p_\pi(t_0)$$

Consequently, we obtain (inductively) $V(\tau_i, x(\tau_i)) \leq V(\tau_s, x_0) + B$ for all $i \in \mathbb{Z}^+$ with $\tau_i \geq \tau_s = p_\pi(t_0)$. This implies (using (2.159)) $V(\tau_i, x(\tau_i)) \leq G = G(T, R) := a_2(R \max_{k=0, \dots, J(T)} \beta(\tau_k)) + B + 1$ for all $i \in \mathbb{Z}^+$ with $\tau_i \geq \tau_s = p_\pi(t_0)$. Also, define

$$\rho(v) := \exp(v - G) \min_{v \leq y \leq G} a_3(y) \quad (2.165)$$

which obviously is a strictly increasing, continuous, and positive definite function on $[0, G]$. Clearly, there exists $\tilde{\rho} \in K$ with $\tilde{\rho}(v) = \rho(v)$ for all $v \in [0, G]$ (e.g., $\tilde{\rho}(v) := \rho(G) + v - G$ for $v > G$). Notice that (2.160) and definition (2.165) implies $V(\tau_{i+1}, x(\tau_{i+1})) \leq V(\tau_i, x(\tau_i)) - \tilde{\rho}(V(\tau_i, x(\tau_i))) + q(\tau_i)$ for all $i \in \mathbb{Z}^+$ with $\tau_i \geq p_\pi(t_0)$. From this point the proof continues with the same procedure as in the previous case.

Case of $q \equiv 0$ and $\beta(t) \equiv 1$: Notice that (2.160) implies $V(\tau_{i+1}, x(\tau_{i+1})) \leq V(\tau_i, x(\tau_i))$ for all $i \in \mathbb{Z}^+$ with $\tau_i \geq p_\pi(t_0)$. Consequently, we obtain (inductively) $V(\tau_i, x(\tau_i)) \leq V(\tau_s, x_0)$ for all $i \in \mathbb{Z}^+$ with $\tau_i \geq \tau_s = p_\pi(t_0)$. The previous inequality, in conjunction with (2.159) with $\beta(t) \equiv 1$, implies Uniform Robust Lagrange and Lyapunov Output Stability. Thus we are left with the proof of the Uniform Robust Output Attractivity Property.

Notice that if $V(\tau_k, x(\tau_k)) \leq a_1^{-1}(\varepsilon)$ for some $k \in \mathbb{Z}^+$ with $\tau_k \geq \tau_s = p_\pi(t_0)$, then we get $V(\tau_i, x(\tau_i)) \leq a_1^{-1}(\varepsilon)$ for all $i \geq k$.

Let $S(\varepsilon, R) := \{a_3(y) : y \in [a_1^{-1}(\varepsilon), a_1^{-1}(\varepsilon) + a_2(R)]\} > 0$. We claim that

$$V(\tau_i, x(\tau_i)) \leq a_1^{-1}(\varepsilon) \quad \text{for all } i \geq s + \frac{a_2(R)}{S(\varepsilon, R)}$$

The proof is made by contradiction. Suppose that there exists $k \in \mathbb{Z}^+$ with $k \geq s + \frac{a_2(R)}{S(\varepsilon, R)}$ such that $V_k > a_1^{-1}(\varepsilon)$. By virtue of the previous observation, this implies that $V_i > a_1^{-1}(\varepsilon)$ for all $i = s, \dots, k$. Consequently, we have $V_{i+1} \leq V_i - S(\varepsilon, R)$ for all $i = s, \dots, k$. Thus we obtain $V_k \leq V_s - S(\varepsilon, R)(k - s)$, and this estimate, in conjunction with our assumption $k \geq s + \frac{a_2(R)}{S(\varepsilon, R)} \geq s + \frac{V_s}{S(\varepsilon, R)}$, gives $V_k \leq 0$. Clearly, this implication is in contradiction with the assumption $V_k > a_1^{-1}(\varepsilon)$. The reader

should verify that the inequality $t \geq t_0 + r(\lceil \frac{a_2(R)}{S(\varepsilon, R)} \rceil + 2)$ implies that $t \geq \tau_i$ with $i \geq s + \frac{a_2(R)}{S(\varepsilon, R)}$. The proof is complete. $\square$

Example 2.8.1 Consider the nonlinear finite-dimensional discrete-time time-varying system

$$\begin{aligned} x_1(t+1) &= d(t)x_1(t) \\ x_2(t+1) &= 2^{-t}d(t)|x_1(t)|^{\frac{1}{2}} \\ Y(t) &= H(t, x(t)) := x_2(t) \\ x(t) &:= (x_1(t), x_2(t)) \in \mathfrak{R}^2, t \in Z^+, d(t) \in [-2, 2] \end{aligned} \quad (2.166)$$

with $\pi = Z^+$. Consider the continuous function $V(t, x) := \exp(-t)|x_1| + |x_2|$, which clearly satisfies the following inequality:

$$|Y| = |x_2| \leq V(t, x) \leq 2|x| \quad \text{for all } (t, x) \in Z^+ \times \mathfrak{R}^2 \quad (2.167)$$

Moreover, notice that for all $(t, x, d) \in Z^+ \times \mathfrak{R}^2 \times [-2, 2]$, we obtain

$$\begin{aligned} V(t+1, dx_1, 2^{-t}d|x_1|^{\frac{1}{2}}) &= \exp(-t-1)|d||x_1| + 2^{-t}|d||x_1|^{\frac{1}{2}} \\ &\leq 2e^{-1}\exp(-t)|x_1| + 2^{-t+1}|x_1|^{\frac{1}{2}} \\ &\leq \frac{2+e}{2e}\exp(-t)|x_1| + \frac{2e}{e-2}\left(\frac{e}{4}\right)^t \\ &\leq \lambda V(t, x) + q(t) \end{aligned} \quad (2.168)$$

where $\lambda := \frac{2+e}{2e} \in (0, 1)$ and $q(t) := \frac{2e}{e-2}(\frac{e}{4})^t$ with $\lim_{t \rightarrow +\infty} q(t) = 0$. By virtue of (2.167) and (2.168), it follows that statement (i) of Proposition 2.3 is satisfied with $\beta(t) \equiv 1$, $a_1(s) := s$, $a_2(s) := 2s$, and $a_3(s) := (1 - \lambda)s$. We conclude that system (2.166) is RGAOS.

Example 2.8.2 Consider again the cobweb model (1.123) with buffer stocks. For the case with no government intervention, the corresponding dynamical system (1.123) with $u(t) \equiv 0$ is actually one-dimensional (the state variable $x_2(t)$ is constant and irrelevant) and is given by the difference equation

$$x_1(t+1) = h(x_1(t) + x_{\text{eq}}) - x_{\text{eq}} \quad (2.169)$$

where

$$h(y) = \begin{cases} 1 & \text{if } y \leq c_1 \\ 1 + c_1 r - r y & \text{if } c_1 < y < c_1 + c_2 \\ 1 - c_2 r & \text{if } y \geq c_1 + c_2 \end{cases} \quad (2.170)$$

We consider the following cases:

Case 1: If $c_1 + c_2 > 1 - c_2 r$ and $r < 1$, then we can show that system (1.123) with $u(t) \equiv 0$ is URGAOS.

Indeed, we can prove this fact by using Proposition 2.3 and by considering the Lyapunov function $V(x_1) := |x_1|$. It is a matter of simple but tedious calculations to show that the following inequality holds:

$$V(h(x_1 + x_{\text{eq}}) - x_{\text{eq}}) < V(x_1) \quad \text{for all } x \in \mathfrak{R}, x_1 \neq 0 \quad (2.171)$$

Define $P(x_1) := V(x_1) - V(h(x_1 + x_{\text{eq}}) - x_{\text{eq}})$. Clearly, inequality (2.171) implies that $P : \mathfrak{R} \rightarrow [0, +\infty)$ is a continuous function with $P(0) = 0$ and $P(x) > 0$ for all $x \neq 0$. By Proposition 2.2 there exists a locally Lipschitz positive definite function $a_3 : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ such that $a_3(V(x)) \leq P(x)$ for all $x \in \mathfrak{R}$. Consequently, inequalities (2.159) and (2.160) hold with $a_1(s) = a_2(s) := s$, $q \equiv 0$, and $\beta(t) \equiv 1$.

Case 2: If $c_1 + c_2 \leq 1 - c_2 r$ (or $S_{\max} \leq \frac{ad-bc}{b+d}$), then we notice that $h(y) \geq c_1 + c_2$ for all $y \in \mathfrak{R}$. Thus, for every initial condition $x_1(0) \in \mathfrak{R}$, we obtain

$$x_1(t) = 0 \quad (\text{or } P(t) = b^{-1}(a - S_{\max})) \quad \text{for all } t \geq 2. \quad (2.172)$$

In this case, we can show that system (1.123) with $u(t) \equiv 0$ is URGAOS, by combining Lemmas 1.3, 2.1, and 2.2 (notice that system (1.123) is autonomous). In fact, in this case the phenomenon of finite-time stability appears as (2.172) shows (i.e., the equilibrium point is approached in finite time).

Thus, in any of the above two cases, system (1.123) with $u(t) \equiv 0$ is URGAOS. The characterization of the set of the parameters for which URGAOS occurs is sharp. Indeed, if $c_1 + c_2 > 1 - c_2 r$ and $r \geq 1$ (or $S_{\max} > \frac{ad-bc}{b+d}$ and $d \geq b$), the dynamics of system (1.123) present the well-known phenomenon of the “hog-cycles” or nontrivial period-2 solutions, as described in the economic literature. For the mildly nonlinear case (1.123), the “hog-cycles” can be given explicitly:

- If $c_1 + c_2 > 1$, a nontrivial period-2 solution is

$$\begin{aligned} x_{1,1} &= 1 - x_{\text{eq}} \\ x_{1,2} &= 1 - r(1 - c_1) - x_{\text{eq}} \leq c_1 - x_{\text{eq}}. \end{aligned}$$

- If $1 \geq c_1 + c_2$ and $c_1 + rc_2 \geq 1$, a nontrivial period-2 solution is

$$\begin{aligned} x_{1,1} &= 1 - x_{\text{eq}} \\ x_{1,2} &= 1 - rc_2 - x_{\text{eq}} \leq c_1 - x_{\text{eq}}. \end{aligned}$$

- If $c_1 + rc_2 < 1$, a nontrivial period-2 solution is

$$\begin{aligned} x_{1,1} &= 1 - r(1 - c_1) + r^2 c_2 - x_{\text{eq}} \geq c_1 + c_2 - x_{\text{eq}} \\ x_{1,2} &= 1 - rc_2 - x_{\text{eq}} < c_1 + c_2 - x_{\text{eq}}. \end{aligned}$$

- If $r = 1$, there are infinite nontrivial period-2 solutions

$$\begin{aligned} x_{1,1} &\in (\max\{c_1; 1 - c_2\} - x_{\text{eq}}, \min\{1; c_1 + c_2\} - x_{\text{eq}}) \quad \text{arbitrary} \\ x_{1,2} &= 1 + c_1 - x_{1,1} - 2x_{\text{eq}}. \end{aligned}$$

Thus, we have performed a complete parametric stability analysis for system (1.123).

As suggested previously, the stability analysis for discrete-time systems is strongly related to the discretization approach for Lyapunov stability analysis of systems described by ODEs.

The following result is the main result for the discretization approach for systems described by ODEs.

Proposition 2.4 *Consider system (1.3) under Hypotheses (H1–4) and suppose that there exist a function $V \in C^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and functions $a_1, a_2 \in K_\infty$, and $\beta, \mu \in K^+$ satisfying (2.129) and the following inequality:*

$$a_1(\mu(t)|x|) \leq V(t, x) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \quad (2.173)$$

Moreover, suppose that there exist a locally bounded function $T : (0, +\infty) \rightarrow (0, +\infty)$, a positive definite function $q \in C^0(\mathbb{R}^+; \mathbb{R}^+)$, and a function $a \in K_\infty$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n \setminus \{0\}$, and $d \in M_D$, the solution $x(t, t_0, x_0; d)$ of (1.3) with initial condition $x(t_0, t_0, x_0; d) = x_0$ corresponding to $d \in M_D$ exists on $[t_0, t_0 + T(V(t_0, x_0))]$ and satisfies the following inequalities:

$$V(t, x(t, t_0, x_0; d)) \leq a(V(t_0, x_0)) \quad \text{for all } t \in [t_0, t_0 + T(V(t_0, x_0))] \quad (2.174)$$

$$\min_{t \in [t_0, t_0 + T(V(t_0, x_0))]} V(t, x(t, t_0, x_0; d)) \leq V(t_0, x_0) - q(V(t_0, x_0)) \quad (2.175)$$

Then system (1.3) is RGAOS. Moreover, if $\beta(t) \equiv 1$, then system (1.3) is URGAOS.

Proof Let $\sigma \in KL$ be the function with the following property.

(P) If $\{V_i \geq 0\}_{i=0}^\infty$ is a sequence with $V_{i+1} \leq V_i - q(V_i)$, then $V_i \leq \sigma(V_0, i)$ for all $i \geq 0$.

The existence of $\sigma \in KL$ which satisfies property (P) is guaranteed by Lemma 4.3 in [19]. In order to present a complete proof, we notice that the function $\sigma \in KL$ can be constructed by following the procedure in the proof of Lemma 2.11 and defining $v(i) := \sup\{\frac{V_i}{g(V_0)} : \{V_i\}_0^\infty \in G(V_0), V_0 > 0\}$, where $g(s) = s^2 + \sqrt{s}$, and $G(V_0)$ denotes the set of all sequences $\{V_i \geq 0\}_{i=0}^\infty$ with $V_{i+1} \leq V_i - q(V_i)$ for all $i \geq 0$. Then we show that $v(i) \rightarrow 0$ and the function $\sigma \in KL$ can be defined as $\sigma(s, i) := g(s)v(i)$ for all $s > 0, i \in \mathbb{Z}^+$ and $\sigma(0, i) := 0$ for all $i \in \mathbb{Z}^+$.

Define $T(0) = 1$. Let $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $d \in M_D$ arbitrary and define the following sequences:

$$\begin{aligned} x_{i+1} &= x(\tau_i + s_i, \tau_i, x_i; d) & T_i &= T(V(\tau_i, x_i)) \\ \tau_{i+1} &= \tau_i + s_i & V_i &= V(\tau_i, x_i) \quad i \geq 0 \end{aligned} \quad (2.176)$$

with $\tau_0 = t_0$, where $s_i \in [0, T_i]$ satisfies

$$V(x(\tau_i + s_i, \tau_i, x_i; d)) = \min_{s \in [0, T_i]} V(x(\tau_i + s, \tau_i, x_i; d)) \quad (2.177)$$

for the case $x_i \neq 0$ and $s_i = T_i = 1$ for the case $x_i = 0$. Notice that by virtue of the semigroup property we obtain that $x_i = x(\tau_i, t_0, x_0; d)$.

Inequality (2.175) and definitions (2.176), (2.177) imply that

$$V_{i+1} \leq V_i - q(V_i) \quad (2.178)$$

for the case $x_i \neq 0$. For the case $x_i = 0$, by the uniqueness of solution of (1.3) we have $x_{i+1} = 0$, and consequently inequality (2.178) holds as well in this case. Therefore, property (P) guarantees that

$$V_i \leq \sigma(V_0, i) \quad \text{for all } i \geq 0 \quad (2.179)$$

where $\sigma \in KL$ is the function involved in property (P).

Inequality (2.174), definitions (2.176) and the semigroup property guarantee that $V(x(t, t_0, x_0; d)) = V(x(t, \tau_i, x_i; d)) \leq a(V_i)$ for all $t \in [\tau_i, \tau_{i+1}]$ for the case $x_i \neq 0$. By the uniqueness of solution of (1.3), it follows that $V(x(t, t_0, x_0; d)) = V(x(t, \tau_i, x_i; d)) \leq a(V_i)$ for all $t \in [\tau_i, \tau_{i+1}]$ for the case $x_i = 0$ as well. Since $\{V_i \geq 0\}_{i=0}^\infty$ is nonincreasing (a consequence of (2.178)), we obtain

$$V(x(t, t_0, x_0; d)) \leq a(V(t_0, x_0)) \quad \text{for all } t \in [t_0, \sup \tau_i] \quad (2.180)$$

Next, we show that

$$V(x(t, t_0, x_0; d)) \leq a(V(t_0, x_0)) \quad \text{for all } t \geq 0 \quad (2.181)$$

It should be noticed that Robust Forward Completeness, Robust Lyapunov and Lagrange stability follow directly from inequality (2.181) in conjunction with (2.129) and (2.173). Moreover, if $\beta(t) \equiv 1$, then Uniform Robust Lyapunov and Uniform Lagrange stability follows directly from inequality (2.181).

For the proof of inequality (2.181), we distinguish two cases:

Case 1: $\sup \tau_i < +\infty$.

By virtue of inequality (2.179) we obtain that $\lim V_i = 0$ and consequently $\lim_{t \rightarrow (\sup \tau_i)^-} V(x(t, t_0, x_0; d)) = 0$. This implies that $\lim_{t \rightarrow (\sup \tau_i)^-} x(t, t_0, x_0; d) = 0$, which implies $x(t, t_0, x_0; d) = 0$ for all $t \geq \sup \tau_i$. Therefore inequality (2.181) is a consequence of (2.180) and the fact that $V(x(t, t_0, x_0; d)) = 0$ for all $t \geq \sup \tau_i$.

Case 2: $\sup \tau_i = +\infty$.

In this case inequality (2.181) is a direct consequence of inequality (2.180).

We next show Robust Attractivity. Let $\varepsilon > 0$, $S \geq 0$, $R \geq 0$, $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ with $t_0 \in [0, S]$, $|x_0| \leq R$, and $d \in M_D$ be arbitrary. By virtue of (2.129), (2.178), and the semigroup property it follows that if $V_i \leq a^{-1}(a_1(\varepsilon))$ for some $i \geq 0$, then we have $|H(t, x(t, t_0, x_0; d), 0)| \leq \varepsilon$ for all $t \geq \tau_i$. Define $J := \min\{i \geq 0 : V_i \leq a^{-1}(a_1(\varepsilon))\}$. Let $N_\varepsilon(S, R) \in \mathbb{Z}^+$ such that $\sigma(a_2(R \max_{0 \leq t \leq S} \beta(t)), N_\varepsilon(S, R)) \leq a^{-1}(a_1(\varepsilon))$ and notice that inequalities (2.129) and (2.179) imply that $J \leq N_\varepsilon(S, R)$.

Next suppose that $J \geq 1$. Since $V_i \geq a^{-1}(a_1(\varepsilon))$ for all $i \leq J-1$ (a consequence of definition $J := \min\{i \geq 0 : V_i \leq a^{-1}(a_1(\varepsilon))\}$), we get from (2.176) and the facts that $\{V_i \geq 0\}_{i=0}^\infty$ is nonincreasing, $V(t_0, x_0) \leq a_2(R \max_{0 \leq t \leq S} \beta(t))$, and that

$$\tau_{i+1} = \tau_i + s_i \leq \tau_i + T_i \leq \tau_i + \tilde{T}_\varepsilon(S, R) \quad \text{for all } i \leq J-1$$

where

$$\tilde{T}_\varepsilon(S, R) := \sup \left\{ T(V) : a^{-1}(a_1(\varepsilon)) \leq V \leq a^{-1}(a_1(\varepsilon)) + a_2 \left(R \max_{0 \leq t \leq S} \beta(t) \right) \right\}$$

Therefore $\tau_{i+1} \leq i \tilde{T}_\varepsilon(S, R)$ for all $i \leq J-1$, and therefore inequality $J \leq N_\varepsilon(S, R)$ implies $\tau_J \leq N_\varepsilon(S, R) \tilde{T}_\varepsilon(S, R)$. It follows that $V(x(t, t_0, x_0; d)) \leq a^{-1}(a_1(\varepsilon))$ for all $t \geq N_\varepsilon(S, R) \tilde{T}_\varepsilon(S, R)$.

The above conclusion holds as well in the case $J = 0$; namely we have

$$V(x(t, t_0, x_0; d)) \leq a^{-1}(a_1(\varepsilon)) \quad \text{for all } t \geq N_\varepsilon(S, R) \tilde{T}_\varepsilon(S, R)$$

Notice that, if $\beta(t) \equiv 1$, then $N_\varepsilon(S, R) \in \mathbb{Z}^+$ and

$$\tilde{T}_\varepsilon(S, R) := \sup \left\{ T(V) : a^{-1}(a_1(\varepsilon)) \leq V \leq a^{-1}(a_1(\varepsilon)) + a_2 \left(R \max_{0 \leq t \leq S} \beta(t) \right) \right\}$$

are independent of $S \geq 0$. The proof is complete. $\square$

Remark 2.7 The reader should notice that the converse of Proposition 2.4 holds for the case of Uniform Robust Global Asymptotic Stability (URGAS), i.e., if system (1.3) is URGAS, then for every function $V \in C^0(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ which satisfies (2.129) for certain $a_1, a_2 \in K_\infty$ with $\beta(t) \equiv 1$, there exist a function $a \in K_\infty$ and a locally bounded function $T : (0, +\infty) \rightarrow (0, +\infty)$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n \setminus \{0\}$, and $d \in M_D$, the solution $x(t, t_0, x_0; d)$ of (1.3) with initial condition $x(t_0, t_0, x_0; d) = x_0$ corresponding to $d \in M_D$ exists on $[0, T(V(t_0, x_0))]$ and satisfies inequalities (2.173), (2.174), and (2.175). Notice that (2.173) with $\mu(t) \equiv 1$ is a direct consequence of (2.129) and the fact that $H(t, x, 0) := x$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$. Moreover, since system (1.3) is URGAS, Theorem 2.2 implies the existence of $\sigma \in KL$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $d \in M_D$, the solution $x(t, t_0, x_0; d)$ of (1.3) with initial condition $x(t_0, t_0, x_0; d) = x_0$ corresponding to $d \in M_D$ satisfies (2.11) with $H(t, x, 0) := x$. Without loss of generality, we may

assume that for each $s > 0$, the mapping $t \rightarrow \sigma(s, t)$ is strictly decreasing (if not, replace $\sigma(s, t)$ by $\sigma(s, t) + s \exp(-t)$).

Combining (2.129) and (2.11) with $H(t, x, 0) := x$, we obtain

$$V(x(t, t_0, x_0; d)) \leq a_2(\sigma(a_1^{-1}(V(x_0)), t)) \quad \text{for all } t \geq 0 \quad (2.182)$$

Let $q \in (0, 1)$, and let $T(s) > 0$ be the unique solution of $a_2(\sigma(a_1^{-1}(s)T(s))) = (1 - q)s$ for each $s > 0$. It can be shown that the mapping $(0, +\infty) \ni s \rightarrow T(s)$ is bounded on every compact set $S \subset (0, +\infty)$. Therefore, by virtue of (2.182), we conclude that inequalities (2.174) and (2.175) hold with $a(s) := a_2(\sigma(a_1^{-1}(s), 0))$ and $q(s) := qs$.

For the case of nonuniform RGAOS, we can present a result which is useful for the solution of important control problems (see Chap. 6).

Proposition 2.5 *Consider system (1.3) under Hypotheses (H1–4). Suppose that there exists a function $\gamma \in K^+$ satisfying*

$$\sum_{j=0}^{+\infty} \gamma(2j) < +\infty \quad (2.183)$$

$$\lim_{t \rightarrow +\infty} \gamma(t) = 0 \quad (2.184)$$

It is further assumed that there exist functions $a, a_1, a_2 \in K_\infty$, $\beta, \mu \in K^+$, $\rho \in C^0(\mathfrak{R}^+; \mathfrak{R}^+)$ being positive definite such that (2.129) and (2.173) hold and the following properties are fulfilled for all $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times M_D$ and $j \in \mathbb{Z}^+$:

$$\sup_{t \in [t_0, t_0] + 2j} V(t, x(t, t_0, x_0; d)) \leq a(V(t_0, x_0)) + \gamma(t_0) \quad (2.185)$$

$$V(2j + 2, x(2j + 2, 2j, x_0; d)) \leq V(2j, x_0) - \rho(V(2j, x_0)) + \gamma(2j) \quad (2.186)$$

where $x(t, t_0, x_0; d)$ denotes the unique solution of (1.3) with initial condition $x(t_0) = x_0$ corresponding to $d \in M_D$. Then system (1.3) is RGAOS.

Proof Let $(t_0, x_0, d) \in \mathfrak{R}^+ \times \mathfrak{R}^n \times M_D$, and let $j \in \mathbb{Z}^+$ be the smallest integer which satisfies $t_0 \leq 2j$. Inequality (2.186) implies that, for any pair of integers $i \geq j$,

$$V(2i, x(2i, t_0, x_0, d)) \leq V(2j, x(2j, t_0, x_0, d)) + \sum_{k=0}^i \gamma(2k) \quad (2.187)$$

Let $M := \sum_{k=0}^{+\infty} \gamma(2k)$ and $B := \sup_{t \geq 0} \gamma(t)$. Then, by (2.129), (2.185), and (2.186), we get

$$V(t, x(t, t_0, x_0, d)) \leq a(a_2(\beta(t_0)|x_0|)) + B + M + B \quad \text{for all } t \geq t_0 \quad (2.188)$$

Inequality (2.188), in conjunction with (2.129), implies RFC and Robust Lagrange Output Stability. Therefore, according to Lemma 2.1, in order to establish RGAOS, it suffices to show that system (1.3) satisfies the property of Uniform Output Attractivity on compact sets of initial data. To establish this property, consider arbitrary constants $\varepsilon > 0$, $R \geq 0$, $T \geq 0$ and let $(t_0, x_0, d) \in \mathfrak{N}^+ \times \mathfrak{N}^n \times M_D$ with $t_0 \in [0, T]$ and $|x_0| \leq R$. Define $K := a(a_2(R \max_{t \in [0, T]} \beta(t)) + B + M) + B$. Then by (2.188) it holds that

$$V(t, x(t, t_0, x_0, d)) \leq K \quad \text{for all } t \geq t_0 \quad (2.189)$$

Also, define

$$\tilde{\rho}(s) := \min_{s \leq y \leq K} \rho(y) \quad (2.190)$$

which obviously is a nondecreasing and continuous function, and let $J \geq 0$ be an integer with $\frac{1}{2}\tilde{\rho}(K) \geq \gamma(2i)$ for all integers $i \geq J$, whose existence is guaranteed from (2.184). Next, consider the sequence

$$q_i := \inf \left\{ s \in [0, K] : \frac{1}{2}\tilde{\rho}(s) \geq \gamma(2i) \right\} \quad \text{for } i \geq J \quad (2.191)$$

By virtue of (2.184) and (2.191), $q_i \rightarrow 0$. Consequently, there exists an integer $N := N(\varepsilon, K) \geq J$ such that

$$q_i + \gamma(2i) \leq S(\varepsilon) \quad \text{and} \quad \gamma(2i) \leq \frac{1}{2}a_1(\varepsilon) \quad \text{for all } i \geq N \quad (2.192)$$

where $a_1 \in K_\infty$ is the function involved in (2.129), and

$$S(\varepsilon) := a^{-1} \left(\frac{1}{2}a_1(\varepsilon) \right) \quad (2.193)$$

Notice next that (2.186) asserts that for all integers $i \geq \max(N, j)$, the following holds:

$$\begin{aligned} & V(2(i+1), x(2(i+1), t_0, x_0; d)) \\ & \leq \max \left(S(\varepsilon), V(2i, x(2i, t_0, x_0; d)) - \frac{1}{2}\rho(V(2i, x(2i, t_0, x_0; d))) \right) \end{aligned} \quad (2.194)$$

Indeed, in order to establish (2.194), we consider two cases. For the first case, assume that $V(2i, x(2i, t_0, x_0, d)) \geq q_i$. Then it follows from (2.190), (2.191), and (2.192) that $\frac{1}{2}\rho(V(2i, x(2i, t_0, x_0, d))) \geq \gamma(2i)$, and this, in conjunction with (2.186), implies (2.194). The other case is $V(2i, x(2i, t_0, x_0, d)) \leq q_i$. Then the latter, in conjunction with (2.186) and (2.192), implies again (2.194).

The following property is a consequence of (2.194):

$$\begin{aligned} & V(2i, x(2i, t_0, x_0, d)) \leq S(\varepsilon) \\ & \text{for all integers } i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1 \end{aligned} \quad (2.195)$$

To show (2.195), suppose on the contrary that there exists integer $i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1$ with $V(2i, x(2i, t_0, x_0, d)) > S(\varepsilon)$. Then, (2.194) would imply

$$\begin{aligned} V(2k, x(2k, t_0, x_0, d)) &> S(\varepsilon) \\ \text{for all } k &= \max(N, j), \max(N, j) + 1, \dots, \max(N, j) + i \end{aligned} \quad (2.196)$$

By (2.189), (2.190), (2.194), and (2.196) we would have

$$\begin{aligned} V(2(k+1), x(2(k+1), t_0, x_0, d)) &\leq V(2k, x(2k, t_0, x_0, d)) - \frac{1}{2}\tilde{\rho}(S(\varepsilon)) \\ \text{for all } k &= \max(N, j), \max(N, j) + 1, \dots, \max(N, j) + i - 1 \end{aligned}$$

and therefore

$$\begin{aligned} V(2k, x(2k, t_0, x_0, d)) &\leq K - (k - \max(N, j))\frac{1}{2}\tilde{\rho}(S(\varepsilon)) \\ \text{for all } k &= \max(N, j), \max(N, j) + 1, \dots, \max(N, j) + i - 1 \end{aligned}$$

The previous inequality for $k = i$ gives $V(2i, x(2i, t_0, x_0, d)) \leq S(\varepsilon)$, which is a contradiction, and this establishes (2.195).

Finally, by (2.185) and (2.195) we obtain $\sup_{t \in [2i, 2i+2]} V(t, \phi(t, t_0, x_0; d)) \leq a(S(\varepsilon)) + \gamma(2i)$ for all integers $i \geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1$. This, in conjunction with (2.192) and (2.193), gives

$$\begin{aligned} \sup_{t \geq 2i} V(t, \phi(t, t_0, x_0, d)) &\leq a_1(\varepsilon) \\ \text{for all integers } i &\geq \max(N, j) + \frac{2K}{\tilde{\rho}(S(\varepsilon))} + 1 \end{aligned} \quad (2.197)$$

Using the inequality above and (2.129), we may conclude that the property of Uniform Output Attractivity on compact sets of initial data holds for system (1.3). This completes the proof of Proposition 2.5. $\square$

Propositions 2.4 and 2.5 show that the discretization approach extends the classical Lyapunov analysis, since it does not assume that a certain differential inequality holds for all times. However, it is important to be able to estimate accurately the value of the Lyapunov function along the trajectory of the solution. This disadvantage can be overcome in certain cases.

2.9 Bibliographical and Historical Notes

1. For proving stability by means of Fixed Point Theorems, see [7] and references therein. In our opinion this method of proving stability is flexible and in most of the cases is very similar to small-gain arguments (that are presented later in Chap. 5).

2. The notion of RFC for systems described by ODEs was introduced in [22] and was extended to general systems in [21, 26]. It should be noted that the notion of RFC implies the standard notion of forward completeness, which requires that for every initial condition the solution of the system exists for all times greater than the initial time, or equivalently, the solutions of the system do not present finite escape time. Conversely, an extension of Proposition 5.1 in [39] to the time-varying case shows that every forward complete system described by ODEs, whose dynamics are locally Lipschitz with respect to (t, x) , uniformly in $d \in D$, is RFC. A Lyapunov characterization for usual forward completeness for systems described by ODEs is given in [2].
3. The notion of URGAS for systems described by ODEs was used in [54, 55] (see also [20, 36, 56, 59]). For recent results on uniform global asymptotic stability of nonlinear and time-varying switched systems, see [38] and references therein. Particularly, for systems with identity output mapping, the notion of URGAS was used in [32, 39]. Nonuniform notions for RGAOS (and RGAS) were developed in [21–26, 29–31] in complete analogy with the uniform notions. Many results of the present chapter have appeared in [4–8, 13, 18] For systems described by RFDEs, similar notions have appeared in [4-07, [18]] (see also [33]).
4. The notion of inf-convolutions (used in Remark 2.4) is studied in [11].
5. Differential inequalities are exploited in the book [34] (where vector differential inequalities are used as well). A useful comparison lemma appears in [32]. The reader should notice that the use of differential inequalities is a basic tool for the derivation of useful results for systems (see, for example, the frequent use of various differential inequalities in [15]).
6. The problem of the absolute continuity of the Lyapunov functional for systems described by RFDEs was studied in [47, 48] (see also [24, 30, 31]). The problem of using a Dini derivative for the Lyapunov functional which can be estimated without knowledge of the solution is crucial for systems described by RFDEs: many times the derivatives used depend on the knowledge of the solution (see [17, 33, 46, 59]). We have decided to work with Dini derivatives which can be estimated without knowledge of the solution (because this is the big advantage of the Lyapunov method; see [24, 30, 31, 47, 48] as well as [13]).
7. Excellent introductions to the method of Lyapunov functions and functionals for systems described by ODEs are given in [32, 53] and for systems described by RFDEs in [17, 33]. The books [16, 33] are classical in stability studies.
8. Chemostat models, like the chemostat model (2.157) studied in Example 2.7.3, are very frequently analyzed by means of monotone system theory (see [52]). Usually, it is assumed that the specific growth rate for the microbial species is a strictly increasing, continuously differentiable, bounded function $\mu : [0, +\infty) \rightarrow [0, +\infty)$ with $\mu(0) = 0$.
9. The question of whether (U)RGAS for a system implies the existence of a Lyapunov functional is crucial: see [3, 22, 24, 29, 39, 55, 56]. The following chapter is devoted to this important question.
10. The Lyapunov method for discrete-time systems is developed in [19, 23, 25, 35]; also see [37].

11. The phenomenon of “hog cycles” or nontrivial period-2 solutions (which means that the system is not RGAOS) for economic cobweb models like the model studied in Example 2.8.4, are known for a long time; see, for instance, [14]. The “hog cycles” have interesting economic interpretations.
12. The “discretization approach” for Lyapunov functions was described in [1] (see also [32, 44, 45, 49, 50], the Appendix in [12], and the proof of the main result in [28]). Clearly, the discretization approach does not require a Lyapunov function with a negative definite derivative. Instead, the discretization approach requires that the difference of the values of the Lyapunov function at two consecutive time instances is negative. Therefore, in this approach the Lyapunov function can even have a positive derivative in certain regions of the state space. However, the main difficulty in the application of this approach is the estimation of the difference of the values of the Lyapunov function at two consecutive time instances. The application of this approach to feedback stabilization problems gave very important results in [12] (see also recent extensions in [28]). The recent work [27] attempts the application of the discretization approach in a way that the difference can be estimated without knowledge of the solution map. The analysis in [27] shows that the discretization approach can be utilized with ideas that exploit higher derivatives of Lyapunov functions (see [9, 10, 60]) and ideas from the original theorem of Matrosov (see [51]).
13. In the present book we will not study the Matrosov methodology of proving stability (see [51]). However, it should be noted that the original result by Matrosov has been generalized recently in various directions (see [40–43, 51, 58]).

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