Chapter 2
System Response Methods

2.1 Impulse Response

2.1.1 Impulse Response Model Representation

In order to motivate the general applicability of the convolution model to LTI systems, first the *unit impulse function* has to be introduced. The unit impulse function or Dirac (δ) function at time zero is defined heuristically as

$$\delta(t) := 0 \quad \text{for all } t \neq 0, \qquad \int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \tag{2.1}$$

and can be viewed as a rectangular, unit-area pulse with infinitesimally small width. Let the unit impulse function $\delta(t)$ be input to an LTI system and denote the impulse response by $g(t)$. Then, due to the time-invariant behavior of the system, a time-shifted impulse $\delta(t-\tau)$ will result in an output signal $g(t-\tau)$. Moreover, because of the linearity, the impulse $\delta(t-\tau)u(\tau)$ will result in the output $g(t-\tau)u(\tau)$, and after integrating both the input and output impulses over the time interval $[-\infty, \infty]$, that is,

$$\int_{-\infty}^{\infty} \delta(t-\tau)u(\tau)\,\mathrm{d}\tau = u(t)$$

due to the properties of the impulse function, and

$$\int_{-\infty}^{\infty} g(t-\tau)u(\tau)\,\mathrm{d}\tau = y(t)$$

we obtain a relationship between the input $u(t)$ and output $y(t)$. Since only causal systems (see Sect. 1.2.2) are treated, the upper bound of the last convolution integral is set equal to t. In the case where $u(t) = 0$ for $t < 0$ and zero initial condition response, as a result of zero initial conditions or a stable system for which the initial condition response has died to zero by $t = 0$, the lower bound can be set to zero.

K.J. Keesman, *System Identification*,
Advanced Textbooks in Control and Signal Processing,
DOI 10.1007/978-0-85729-522-4_2,

Hence, in the derivation of the practically applicable convolution model

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,\mathrm{d}\tau \tag{2.2}$$

only assumptions have been made with respect to the linearity and time-invariance of the system. Thus the convolution model, fully characterized by the impulse response function $g(t)$, is able to describe the input–output relationship of the large class of LTI systems. Consequently, if $g(t)$ is known, then for a given input signal $u(t)$, the corresponding output signal can be easily computed. This feature explains the interest in impulse response model representations, especially if there is limited prior knowledge about the system behavior.

2.1.2 Transfer Function Model Representation

In the analysis of linear systems the Laplace transformation (see Appendix C for details) forms one of the basic tools. Recall that the Laplace transform is defined as

$$\mathcal{L}\big[f(t)\big] \equiv F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{2.3}$$

Laplace transformation of the convolution model (2.2) gives

$$Y(s) = G(s)U(s) \tag{2.4}$$

which defines an algebraic relationship between transformed output signal $Y(s)$ and transformed input signal $U(s)$. The function $G(s)$ is the Laplace transformed impulse response function, that is, $G(s) \equiv \mathcal{L}[g(t)]$, and is called the transfer function. Consequently, representation (2.4) is called the *transfer function* model representation, which will be treated in more detail in the chapter on frequency response methods. The various model representations with their connections, in terms of transformations and back-transformations, are shown in Fig. 2.1, where the impulse response model has a central place.

Let us further illustrate the application of the transfer function model representation to Example 1.4 and indicate the different connections with the other representations.

Example 2.1 Storage tank: Recall that the input–output relationship of the storage tank, after solving a first-order linear differential equation, was given by

$$y(t) = \int_0^t K\mathrm{e}^{-K(t-\tau)}u(\tau)\,\mathrm{d}\tau$$

Consequently, comparison with the convolution model (2.2) reveals that the impulse response function $g(t)$ is equal to $K\mathrm{e}^{-Kt}$, and thus the transfer function is

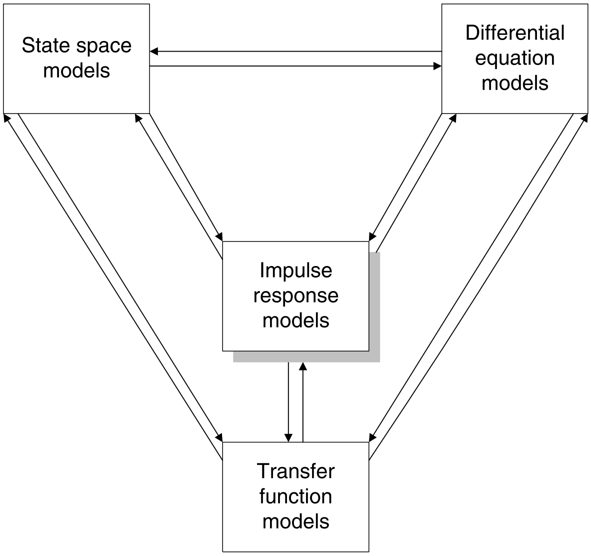

Fig. 2.1 Various model representations for LTI systems

given by $G(s) = \mathcal{L}[K\mathrm{e}^{-Kt}] = \frac{K}{s+K}$, so that

$$Y(s) = G(s)U(s) = \frac{K}{s+K}U(s)$$

An alternative way to find the transfer function and the impulse response function of the storage tank in Example 1.4 is via Laplace transformation of the differential equation

$$\frac{1}{K}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = u(t)$$

as given in Example 1.5. For zero initial conditions, $y(0) = 0$, and after applying the rules of Laplace transformation (see Appendix C for details on the Laplace transform), we find that

$$\frac{1}{K}sY(s) + Y(s) = U(s)$$

Hence, the transfer function, $G(s)$, of this SISO system is found from

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s+K}$$

which, as we have seen before, is the Laplace transform of $g(t)$. Thus $g(t)$ can be directly found by inverse Laplace transformation of $G(s)$. In the same way, $g(t)$ and $G(s)$ can be found from the state-space model.[1] Assuming the zero initial conditions

[1]The transfer function of a general LTI state-space model (1.3) with $x(0) = 0$, possibly obtained after a state correction when $x(0) = x_0 \neq 0$, is given by $G(s) = C[sI - A]^{-1}B + D$ (see, for

on $y(t)$ and $u(t)$ and on all their derivatives and after introducing the differential operator $\pi := \frac{\mathrm{d}}{\mathrm{d}t}$, we can also write the input–output relationship as

$$y(t) = G(\pi)u(t)$$

with $G(\pi) = \frac{K}{\pi+K}$, which shows a clear resemblance with the transfer function $G(s)$.

So far, no real data have been involved; the impulse response function and transfer function have been evaluated on the basis of prior knowledge only. However, in a system identification procedure, this could be the first step in the selection of a proper sampling scheme if there is also some knowledge about the parameter values.

2.1.3 Direct Impulse Response Identification

In what follows, it is indicated how to obtain an estimate of the impulse response function from real data. Since data acquisition is typically performed in discrete time, in the remainder of this chapter and the next chapters, the focus will be on discrete-time representations. In particular, for $u(t) = 0$, $t < 0$, and zero initial condition response, the convolution sum is given by

$$y(t) = \sum_{k=0}^{t} g(t-k)u(k) = \sum_{k=0}^{t} g(k)u(t-k), \quad t \in \mathbb{Z}^+ \tag{2.5}$$

where $g(0)$ is usually equal to zero, because no real system responds instantly to an input. Hence, if we are able to generate a unit pulse, the coefficients of $g(t)$ can be directly found from the measured output. Let, for instance, the pulse input be specified as

$$u(t) = \begin{cases} \alpha, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{2.6}$$

where α is chosen in accordance with the physical limitations on the input signal. The corresponding output will be

$$y(t) = \alpha g(t) + v(t) \tag{2.7}$$

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instance, [GGS01] and, for infinite-dimensional systems, [Zwa04]). For the state correction, introduce $\Delta x(t) := x(t) - \tilde{x}(t)$, where $\tilde{x}(t)$ obeys $\frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = A\tilde{x}(t)$, $\tilde{x}(0) = x_0$, and thus $\Delta x(0) = 0$, while x and $\tilde{x}$ share the same dynamics. Hence, for this specific example with $x(0) = 0$, $A = -K$, $B = 1$, $C = K$, and $D = 0$, we obtain $G(s) = \frac{K}{s+K}$.

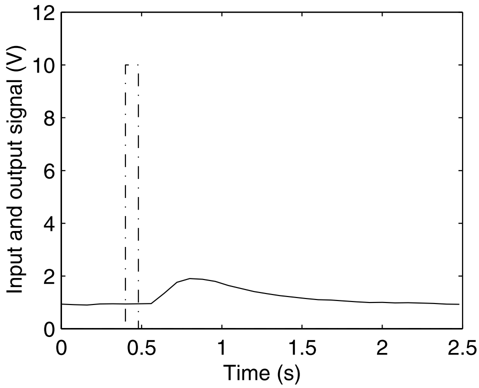

Fig. 2.2 Heating system: pulse input (*dash-dotted line*) at $t = 0.4$ s and measured output (*solid line*)

where $v(t)$ represents the measurement noise of the output signal. Consequently, an estimate of the impulse function, or better the unit-pulse response, is

$$\widehat{g}(t) = \frac{y(t)}{\alpha} \tag{2.8}$$

and the estimation errors are $v(t)/\alpha$. The main advantage of the method is its simplicity, but there are some severe restrictions. Commonly, the estimated unit-pulse response describes the sampled behavior of the continuous-time system. Thus the unit-pulse response may miss significant fast dynamics when the sampling interval is chosen too large, or it may miss the slow dynamics when the duration of the experiment is too small. If dead time (pure delay) is present, it can only be determined within one sampling period. However, its main weakness is that α is limited in practice, which usually prevents a significant reduction of the measurement noise in the estimates, since the estimation errors are inversely proportional with the value of α.

Example 2.2 Heating system: The following pulse response has been measured at a simple lab-scale heating system (see Fig. 2.2). The input of the system is the voltage applied to the heating element. The output is measured with a thermistor. Hence, the output is also in volts. The maximum allowable magnitude of the input is 10 V, and the sampling interval is 0.08 s. To avoid unwanted effects of the initial condition of the system, the pulse input has been applied at $t = 0.4$ s.

The smooth initial curvature in the impulse response indicates that the system is approximately second-order with dead time. Notice from Fig. 2.2 that the dead time is approximately 0.2 s, that is, two to three sampling intervals. After removing the steady-state value, the impulse response coefficients can be directly computed from (2.8).

Consequently, for the identification of LTI systems described by convolution models, the following algorithm can be used.

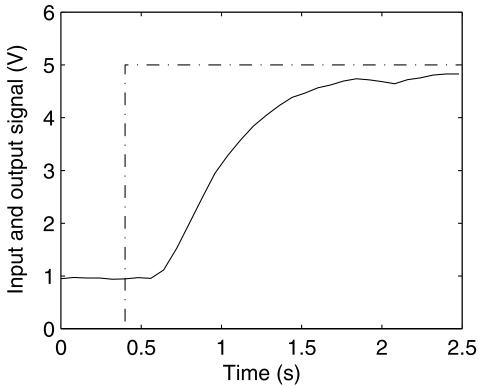

Fig. 2.3 Heating system: step input (*dash-dotted line*) starting at $t = 0.4$ s and measured output (*solid line*)

Algorithm 2.1 Identification of $g(t)$ from a pulse input

1. Generate a pulse with maximum allowable magnitude, α
2. Apply this pulse to the system
3. Use (2.8) to determine estimates of the components of the impulse response $g(t)$

2.2 Step Response

2.2.1 Direct Step Response Identification

A step can be considered as an indefinite succession of contiguous, equal, short, rectangular pulses. Hence, in a similar way as the pulse input, the step input is specified as

$$u(t) = \begin{cases} 0, & t < 0 \\ \alpha, & t \geq 0 \end{cases} \tag{2.9}$$

Example 2.3 Heating system: The effect of applying a step input to the lab-scale heating system can be seen in Fig. 2.3.

Analysis of the step response reveals again that the system is approximately second-order with a dead time of about 0.2 s. For a further analysis of the system, which can be easier obtained from the step response (Fig. 2.2) than from the pulse response (Fig. 2.3), we neglect the second-order dynamics in the graph of the step response. Hence, the dominant time constant, thus neglecting the smooth initial curvature in the step response, can be found by extrapolating the initial slope to the steady-state value. The time intercept is the time constant and is approximately equal to 0.4 s. The static gain is found by dividing the difference between the steady-state values of the output by the difference between the steady-state values of the input, i.e., $(4.8 - 1.0)/(5 - 0) = 0.76$ V/V. Recall that this information about dead time, dominant time constant, and static gain is sufficient for the tuning of PID controllers using the famous Ziegler–Nichols tuning rules (see, for instance, [GGS01]). However, in the design of some predictive controllers for linear systems, as the Dynamic Matrix Controller (DMC), all the step response coefficients are used.

2.2.2 Impulse Response Identification Using Step Responses

Applying the step input of (2.9) to an LTI system described by (2.5) gives

$$y(t) = \alpha \sum_{k=0}^{t} g(k) + v(t) \tag{2.10}$$

Since $y(t-1) = \alpha \sum_{k=0}^{t-1} g(k) + v(t-1)$, estimates of $g(t)$ can be found by taking differences in the step response

$$\widehat{g}(t) = \frac{y(t) - y(t-1)}{\alpha} \tag{2.11}$$

with corresponding error equal to $[v(t) - v(t-1)]/\alpha$. Since differentiation amounts to filtering with a gain proportional to the frequency, differentiation of a noisy step response will generally lead to unacceptable estimates of the impulse response coefficients. Hence, the suggestion is to make α as large as possible.

Summarizing, if for the identification of an LTI system, an impulse input cannot be applied, a step input can be chosen using the following algorithm.

Algorithm 2.2 Identification of $g(t)$ from a step input

1. Generate a step with maximum allowable magnitude, α
2. Apply this step to the system
3. From the step response the dead time, dominant time constant, and static gain can be graphically determined
4. Use (2.11) to determine estimates of the components of the impulse response $g(t)$

However, as stated before, if the goal is to obtain some basic response characteristics, such as dead time, dominant time constant, and static gain, analysis of step responses suffices.

Example 2.4 Heating system: The reconstruction of the impulse response from the previously presented step response (Fig. 2.3), using Algorithm 2.2, shows the following result (see Fig. 2.4). For comparison, the measured impulse response (—) is plotted in Fig. 2.4 as well.

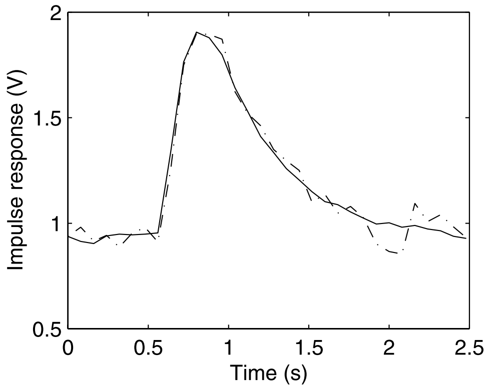

Fig. 2.4 Measured (*solid line*) and reconstructed (*dash-dotted line*) impulse response

2.3 Sine-wave Response

2.3.1 Frequency Transfer Function

Another elementary signal that can identify LTI systems is the sine-wave, which is specified as

$$u(t) = \alpha \sin \omega t \tag{2.12}$$

Before analyzing the output, we must first introduce the frequency transfer function or frequency function, $G(j\omega)$ with j the complex number. This frequency function is the Fourier transform (see Appendix C) of $g(t)$, which can be found by simply substituting $j\omega$ for s in the transfer function $G(s)$. For *sampled systems*, instead of the Laplace or Fourier transform, the discrete Fourier transform (DFT) of $g(t)$ has to be used, that is,

$$G\left(e^{j\omega}\right) = \sum_{t=-\infty}^{\infty} g(t)e^{-j\omega t} \tag{2.13}$$

The DFT can be interpreted as a discrete version of the Fourier transform.

2.3.2 Sine-wave Response Identification

Recall that $\sin \omega t = \mathrm{Im}(e^{j\omega t})$. Since $G(e^{j\omega})$ is a complex number, it can be written as $|G(e^{j\omega})|e^{j\phi}$, where $|G(\cdot)|$ indicates the magnitude and $\phi = \arg(G(\cdot))$. Hence, using (2.5) with $k = -\infty, \ldots, \infty$, the sine-wave input gives an output

$$y(t) = \alpha \sum_{k=-\infty}^{\infty} g(k)\,\mathrm{Im}\left(e^{j\omega(t-k)}\right) = \alpha\,\mathrm{Im} \sum_{k=-\infty}^{\infty} g(k)e^{-j\omega(t-k)}$$

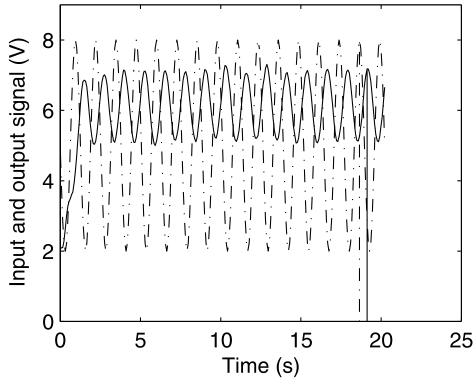

Fig. 2.5 Heating system: sine-wave input (*dash-dotted line*) and output (*solid line*)

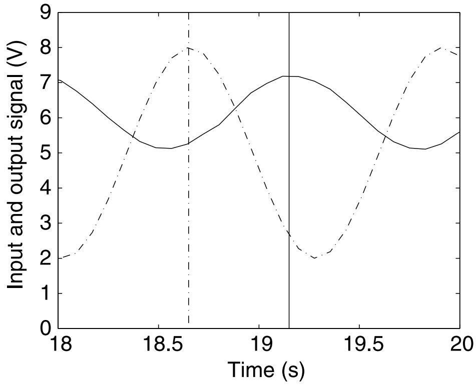

Fig. 2.6 Heating system: snapshot of sine-wave input (*dash-dotted line*) and output (*solid line*)

$$= \alpha \,\mathrm{Im}\left\{ \mathrm{e}^{j\omega t} \sum_{k=-\infty}^{\infty} g(k)\mathrm{e}^{-j\omega k} \right\} = \alpha \,\mathrm{Im}\left\{ \mathrm{e}^{j\omega t} G\left(\mathrm{e}^{j\omega}\right)\right\}$$

$$= \alpha \left| G\left(\mathrm{e}^{j\omega}\right)\right| \sin(\omega t + \phi) \tag{2.14}$$

Consequently, the output is a sine-wave of the same frequency of $u(t)$, but multiplied in magnitude by $|G(\mathrm{e}^{j\omega})|$ and shifted in phase by ϕ. Notice that the result implies that the input is an everlasting sine-wave, which can never be true in practice. Therefore, if it is assumed that $u(t) = 0, \ t < 0$, an initial transient must be accepted in the response. In general, a convenient way to deal with this is neglecting the first part of the response, which is also demonstrated by the following example.

Example 2.5 Heating system: The effect of a sine-wave input signal with a frequency of 5 rad/s on the system output is presented in the following figure (see Fig. 2.5).

The magnitude and phase shift of the frequency function at 5 rad/s is determined at the end of the signal (see Fig. 2.6 for the details). The gain $|G(\mathrm{e}^{j\omega})|$ for $\omega =$ 5 rad/s, is 0.256 V/V, and the phase shift $\phi = -\omega \Delta t = -5 \times 0.50 = -2.50$ rad.

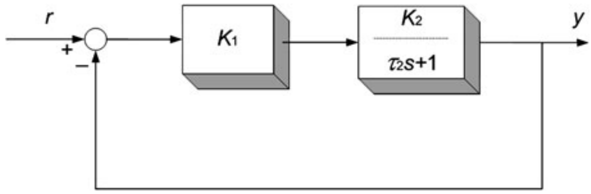

Fig. 2.7 Schematic presentation of closed-loop system under P-control

Since from the signals individual points were taken, this result is very sensitive to noise in both signals, especially at extreme values.

2.4 Historical Notes and References

The methods in this chapter have already a long history with applications on especially electrical and mechanical systems. A general overview of the class of non-parametric identification methods have been given by [Rak80, Wel81]. In particular, impulse response identification has attained a lot of attention in the past and also in recent years [FBT96, SC97, YST97, GCH98, TOS98, SL03, DDk05]. The step response is important in many industrial applications and especially in relation with PID controller tuning. Step response identification methods have been covered by [MR97, WC97]. Sine-wave response identification in the time domain has not received too much attention. Its relevance is much higher in the frequency domain, as we will see in the next chapter.

The more experienced readers, with a background in systems and control theory, may miss the behavioral model representation of Willems [Wil86a, Wil86b, Wil87] in Fig. 2.1. This model representation (see also [PW98]) is out of the scope of this book, as it is too advanced for this introductory text. Nevertheless, the behavioral approach is of interest for further research and application in the system identification field, see, for instance, [JVCR98, JR04].

2.5 Problems

Problem 2.1 In practice we often have to deal with feedback control systems. For instance, in process industry it frequently occurs that a process is controlled by feedback. A schematic example of a first-order system under simple proportional feedback is presented in Fig. 2.7.

On the basis of a priori knowledge of the systems' behavior (see Fig. 2.7), different types of representation will be investigated.

(a) Give the transfer function from (reference) input r to output y.
(b) Give the (set of) differential equation(s) of this system on the basis of the transfer functions presented in the figure.

(c) Derive from the overall transfer function the impulse response of this system, analytically using the inverse Laplace transform (MATLAB: *ilaplace*). Explain/interpret your result.
(d) Represent the system in terms of its convolution or impulse response model.
(e) Plot the unit step response for $K_1 = 1$, $K_2 = 2$, and $\tau_2 = 0.5$ hours. Explain your result.
(f) Represent this system in state-space form.

Problem 2.2 Consider the storage tank example (Example 1.4) with $K = 0.8$.

(a) Define the system (*sys1*) in state-space form using the MATLAB command *ss*.
(b) Define the system (*sys2*) in transfer function form using the MATLAB command *tf*.
(c) Check both representations with the commands *ss2tf* and *tf2ss*.
(d) For this system, determine the impulse response $g(t)$ using the MATLAB command *impulse*.
(e) Determine the step response (y) as well, using the MATLAB command *step*.
(f) Differentiate the step response using the command *diff*. Note: perform a scaling of the differentiated response (yd) by multiplying it with $g(1)/yd(1)$ and add a zero (why?). Plot both impulse responses.
(g) Generate a step input using *zeros* and *ones*. Use the command *lsim* to calculate the corresponding output. Plot the result and explain the result.

Problem 2.3 Let us evaluate the sine-wave response in some more detail. Consider, for this purpose, the system with transfer function

$$G(s) = \frac{2}{10s+1}$$

(a) Define the system in MATLAB
(b) Generate and plot a sine-wave signal with a user-defined frequency.
(c) Determine the sine-wave response using *lsim* and plot it together with the input in one figure. Interpret the result.

Problem 2.4 Investigate the effects of a nonideal input in an impulse response test by plotting the response of the system with impulse response

$$g(t) = \exp(-t) - \exp(-5t)$$

to a rectangular pulse input of unit area and duration (i) 0.1, (ii) 0.2, and (iii) 0.5. Compare each response with $g(t)$ (after [Nor86]).

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