

Generating Functions Count

2.1 Counting – from Polynomials to Power Series

Consider the outcomes when a pair of dice are thrown and our interest is the sum of the numbers showing. One way to model the situation is by means of a grid in which each point (whose coordinates are non-negative integers $1 \leq x, y \leq 6$) represents one outcome. In the grid below, the corresponding sum is shown alongside some of the resulting points:

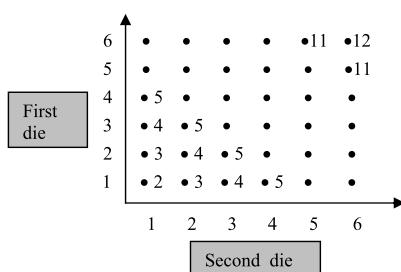


Figure 2.1 Sum of two dice.

The problem with this model is that it does not help us to portray 3 dice, 4 dice or even more dice – the geometric insights become first difficult and then impossible.

We need to change our model. We may represent the first die by a polynomial

$$z + z^2 + z^3 + z^4 + z^5 + z^6.$$

The symbol z here is called an **indeterminate** which means that it is not a variable that takes on values – its only role is to keep track of aspects of an enumeration. It may be replaced by X , by y or by any convenient symbol. In this instance it holds two items of information:

- the **powers** of z keep track of the different faces of the dice;
- the **coefficients** of the powers of z show the number of occurrences of each face.

The second die is represented by the same polynomial and the outcome of throwing two dice is represented – quite naturally – by

$$(z + z^2 + z^3 + z^4 + z^5 + z^6)(z + z^2 + z^3 + z^4 + z^5 + z^6).$$

By expanding this

$$z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 + 6z^7 + 5z^8 + 4z^9 + 3z^{10} + 2z^{11} + z^{12}$$

we find that there is one way of obtaining a score of 2; there are two ways of getting a score of 3; three ways of getting a score of 4 and so on. But – and this is the crucial advantage of this algebraic model – the same method may be employed to find the number of ways that a particular score may be obtained with any number of dice. It is important to ask why this works. We can obtain a score of 9 by getting the combinations (3, 6), (6, 3), (4, 5) and (5, 4). This is precisely the same as how many ways we can get z^9 when multiplying out the product above. So the coefficients enumerate. This is an important idea that we use time and time again. We formulate this in Subsection 5.4.2 as Theorem 5.17.

This ingenious idea of a polynomial counting things is the fundamental idea behind generating functions. Very often we will write down the generating function of an enumerative example in a form that needs to be “expanded”. There are two powerful tools that enable us to do this.

Theorem 2.1 (Binomial Theorem)

For a given integer r

$$(1+z)^r = \begin{cases} \sum_{k \geq 0} \binom{r}{k} z^k & \text{if } r > 0; \\ 1 & \text{if } r = 0; \\ \sum_{k \geq 0} (-1)^k \binom{-r+k-1}{k} z^k & \text{if } r < 0. \end{cases}.$$

Moreover, the Binomial coefficients involved are given by the explicit form

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} = \frac{r!}{k!(r-k)!}.$$

Convention: the sum $\sum_{k=0}^r \binom{r}{k} z^k$ may be written $\sum_{k \geq 0} \binom{r}{k} z^k$ in which the summation index k takes the values $k = 0$ to $k = \infty$. As soon as $k > r$ the Binomial coefficients are zero. This convention leads to concise ways of writing and dealing with such sums.

Convergence: the first form of the sum is a finite sum because $\binom{r}{k} = 0$ if $k > r$. It is therefore valid for all values of z . This second form is valid for all values of z except -1 , since 0^0 is undefined. The third form is a non-terminating expression (so it is an infinite expansion); it is only valid when $|z| < 1$.

Example 2.2

The Binomial Theorem makes it easy to expand powers of expressions. For example:

(i)

$$\begin{aligned} (2+3z)^7 &= 2^7 \left(1 + \frac{3z}{2}\right)^7 = 2^7 \sum_{r \geq 0} \binom{7}{r} \left(\frac{3z}{2}\right)^r \\ &= 2^7 + 2^7 \binom{7}{1} \frac{3z}{2} + 2^7 \binom{7}{2} \frac{3^2 z^2}{2^2} + \cdots + 2^7 \binom{7}{7} \frac{3^7 z^7}{2^7} \\ &= 128 + 1344z + 6048z^2 + \cdots + 2187z^7; \end{aligned}$$

(ii)

$$\begin{aligned} (2+3z)^{-7} &= 2^{-7} \left(1 + \frac{3z}{2}\right)^{-7} = 2^{-7} \sum_{r \geq 0} \binom{7+r-1}{r} \left(\frac{3z}{2}\right)^r \\ &= 2^{-7} + 2^{-7} \binom{7}{1} \frac{3z}{2} + 2^{-7} \binom{8}{2} \frac{3^2 z^2}{2^2} + \cdots \\ &= \frac{1}{128} + \frac{21}{256}z + \frac{63}{128}z^2 + \cdots. \end{aligned}$$

This expansion is valid only when $\left|\frac{3z}{2}\right| < 1$ that is, when $|z| < \frac{2}{3}$.

Theorem 2.3 (The sum of a geometric progression (GP))

In the finite case, we have:

$$1 + z + z^2 + \cdots + z^n = \sum_{r=0}^n z^r = \begin{cases} \frac{1-z^{n+1}}{1-z} & \text{if } z \neq 1; \\ n+1 & \text{otherwise.} \end{cases} \quad (2.1)$$

If $|z| < 1$ we can evaluate the infinite sum:

$$1 + z + z^2 + \cdots = \sum_{r \geq 0} z^r = \frac{1}{1-z}. \quad (2.2)$$

We return to generating functions and the way in which these results may be exploited.

Example 2.4

We explore the number of ways there are to obtain a score of 12 with the throw of five dice. Simple: it is the coefficient of z^{12} in the product of five polynomials, each of which enumerates the outcomes of a single die:

$$(z + z^2 + z^3 + z^4 + z^5 + z^6)^5.$$

Now we proceed to expand and simplify this using Theorem 2.1 and Equation (2.1). We have,

$$\begin{aligned} (z + z^2 + z^3 + z^4 + z^5 + z^6)^5 &= z^5 (1 + z + z^2 + z^3 + z^4 + z^5)^5 \\ &= \frac{z^5 (1 - z^6)^5}{(1 - z)^5} \\ &= z^5 (1 - z^6)^5 (1 - z)^{-5} \\ &= (z^5 - 5z^{11} + \cdots - z^{35}) \sum_{r \geq 0} \binom{r+4}{4} z^r. \end{aligned}$$

So the coefficient of z^{12} is just

$$\binom{7+4}{4} - 5 \binom{1+4}{4} = 330 - 25 = 305.$$

There are 305 ways to obtain a score of 12 when five dice are thrown.

Example 2.5

A generous father wishes to divide £20 between his daughters Emma and Pippa, and son Leon, so that they each receive at least £5, nobody receives more than £10, and Emma gets an even amount. How many ways are there of doing this? What we seek are the non-negative, integer solutions to the equation $e + p + l = 20$ subject to the

conditions that $5 \leq e, p, l \leq 10$ and e is even. Viewed in this way, we may associate a polynomial with each recipient:

$$z^5 + z^6 + z^7 + z^8 + z^9 + z^{10}$$

for Pippa and Leon, together with

$$z^6 + z^8 + z^{10}$$

for Emma. Each of these enumerates the amounts they might receive. The answer to our problem is the coefficient of z^{20} in the product

$$(z^6 + z^8 + z^{10})(z^5 + z^6 + z^7 + z^8 + z^9 + z^{10})^2$$

Once again, thanks to Theorem 2.1 and Equation (2.1) it is easier to find than it looks. The expression may be re-written, and then manipulated

$$\begin{aligned} (z^6 + z^8 + z^{10})(z^5 + z^6 + z^7 + z^8 + z^9 + z^{10})^2 &= z^{16}(1 + z^2 + z^4) \left(\frac{1 - z^6}{1 - z} \right)^2 \\ &= z^{16} \frac{(1 + z^2 + z^4)(1 - 2z^6 + z^{12})}{(1 - z)^2} \\ &= (z^{16} + z^{18} + \dots + z^{32}) \sum_{r \geq 0} \binom{r+1}{1} z^r. \end{aligned}$$

The required coefficient of z^{20} is now easy to pick out. It is

$$\binom{5}{1} + \binom{3}{1} + \binom{1}{1} = 5 + 3 + 1 = 9.$$

The expressions we have used to help us count in Examples 2.4 and 2.5 were made up from polynomials. This is because the terms in which we were interested only had a finite number of possibilities. In Example 2.4 the only possible scores are 5, 6, 7, ..., 28, 29, 30. They make up a finite sequence $\{1, 2, \dots, 30\}$.

Many enumerations are not finite and result in a sequence that does not terminate: for example the number of subsets of a set with r elements. In dealing with an infinite sequence, we need infinite expressions. This leads us to the idea of a **generating function**.

Definition 2.6 (Generating function)

Given any sequence $\{u_r\} = \{u_0, u_1, u_2, \dots\}$, a generating function $U(z)$ for the sequence is an expression (called a **power series** in the infinite case and a polynomial in the finite case) in which,

$$U(z) = u_0 + u_1 z + u_2 z^2 + \dots = \sum_{r \geq 0} u_r z^r.$$

This definition has two parts:

- (i) the **power** series (or polynomial) on the right, each of whose coefficients is a term of the sequence placed against a power of the indeterminate z that matches its position in the sequence;
- (ii) the **function** $U(z)$ that explicitly represents the power series, or polynomial following some re-arrangement.

If we can find the function $U(z)$, that is manipulate the power series into a new, simpler form using tools such as the Binomial Theorem and sums of a GP, then we may employ other results from analysis on it: in so doing, we unearth information about the sequence itself. Think of a generating function as a reel of magnetic tape, divided up into a number of segments – possibly infinite. Each segment is coded with a term of the sequence, which is also the coefficient of the corresponding power of z ; each power of z is simply the address of an individual segment.

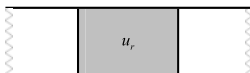


Figure 2.2 Generating function as a reel of magnetic tape.

The name for the reel of magnetic tape is the new form of the function. The importance of a generating function is that it is also “hard wired” with many other important aspects of the sequence. In ways we will explore and discover, the generating function $U(z)$ encapsulates crucial information about the sequence $\{u_r\}$. Problems of counting – enumeration – are the inspiration for generating functions, and they provide a powerful, versatile and robust tool to solve and explore these problems.

$$P(z) = (1 + z + z^{1+1} + z^{1+1+1} + \dots)(1 + z^2 + z^{2+2} + z^{2+2+2} + \dots)(1 + z^4 + z^{4+4} + z^{4+4+4} + \dots)$$

The number of
1¢ used

The number of
2¢ used

The number of
4¢ used

Figure 2.3 Giving change for 100¢.

Example 2.7

Imagine a country in which there are only three coins: a 1¢ coin, a 2¢ coin and a 4¢ coin – this is a very mathematical country. We determine the generating function for the enumeration of the number of ways that change may be given for 100¢. We need to solve the equation $r + 2s + 4t = 100$ for non-negative integers r , s and t . We do this by finding the coefficient of z^{100} in a generating function (Figure 2.3).

We could, of course, multiply out initial terms of each bracket, and with some perseverance we might eventually find the coefficient of z^{100} . But now we exploit the generating function by concentrating on the right-hand side – seeking an easier form for it.

We start by writing the generating function in the form

$$P(z) = (1 + z + z^2 + z^3 + \cdots)(1 + z^2 + z^4 + z^6 + \cdots)(1 + z^4 + z^8 + z^{12} + \cdots).$$

And now for the decisive step: each of the bracketed expressions is a GP, so when $|z| < 1$, we have

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^4)}.$$

Now we use partial fractions and find that

$$P(z) = \frac{1}{8(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{9}{32(1-z)} + \frac{1}{16(1+z)^2} + \frac{5}{32(1+z)} + \frac{1+z}{8(1+z^2)}.$$

We may expand each term on the right using the Binomial Theorem; the coefficient of z^{100} is simply:

$$\frac{1}{8} \binom{102}{2} + \frac{1}{4} \binom{102}{1} + \frac{9}{32} + \frac{(-1)^{100}}{16} \binom{101}{1} + \frac{5(-1)^{100}}{32} + \frac{(-1)^{50}}{8} = 676.$$

That's the power of generating functions – it even provides the answer for any amount of money, but we'll pick that up again later. This example made use of another powerful tool: **partial fractions**. This is a technique that comes into its own with generating functions.

2.1.1 Exercises

Exercise 2.1

What is the generating function for the score obtained when ten dice are thrown? Use this to find the number of ways that a score of 25 may be obtained. (Leave your answer in binomial form.)

Exercise 2.2

The equation $x + 2y + 4z = 100$ determines a plane in three-dimensional Euclidean space. How many non-negative lattice points (a lattice point is a point each of whose coordinates is an integer) lie on this plane?

Exercise 2.3

How many ways are there to give change for £2 if the coinage is 1p and 3p.

Exercise 2.4

Show that the distinct divisors of $p_1^2 p_2^3$ (where p_1 and p_2 are primes) are generated by the expression

$$(1 + p_1 + p_1^2) (1 + p_2 + p_2^2 + p_2^3).$$

Deduce that an arbitrary positive integer r whose prime factorization is

$$r = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

has

$$\prod_{m=1}^k (1 + a_m)$$

distinct divisors.

2.2 Recurrence Relations and Enumeration

When we examine a particular enumeration, we frequently resort to breaking down one of its configurations into smaller parts, so that we can understand how it is made up. Mathematically this may be expressed as a relation between configurations of different “sizes”. Such an expression is called a **recurrence relation** and they come in all sorts of varieties.

Definition 2.8 (Recurrence relation (informal))

The recurrence relation satisfied by a sequence is a recipe that uses initial terms as the ingredients for subsequent terms. It is usually “seeded” by some initial values.

Although recurrences are often the starting point in the analysis of an enumeration, deriving a recurrence relation is by no means easy or obvious. We have a number of tools in the enumerative toolbox that help, such as the principle of exhaustion, that were set out in Chapter 1. This section concentrates on this crucial first step – constructing a recurrence by analyzing how the objects of different sizes fit together.

Example 2.9

We start by finding a recurrence satisfied by the number of denary strings of length r , see Definition 1.3, in which between them the digits $\{3, 6, 9\}$ occur an even number of times. The strategy is to construct an object of “size” $r + 1$ from those of “size” r .

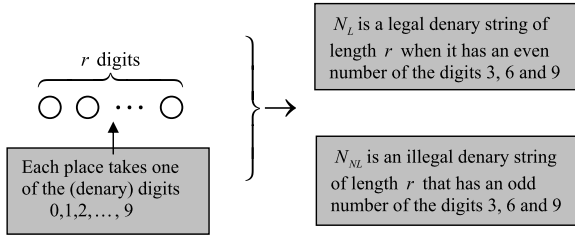


Figure 2.4 Denary strings.

Call the number of legal strings of length r , n_r . A legal string of length $r + 1$ has one of the forms $N_L a$ or $N_{NL} b$ where:

- N_L is a legal string of length r so $a \in \{0, 1, 2, 4, 5, 7, 8\}$;
- N_{NL} is not a legal string of length r so $b \in \{3, 6, 9\}$.

It follows that

$$\begin{aligned} n_{r+1} &= \#(a) \cdot \#(N_L) + \#(b) \cdot \#(N_{NL}) = 7\#(N_L) + 3\#(N_{NL}) \\ &= 3[\#(N_L) + \#(N_{NL})] + 4\#(N_L). \end{aligned}$$

The total number of strings of length r is 10^r and so

$$n_{r+1} = 3 \cdot 10^r + 4\#(N_L) = 3 \cdot 10^r + 4n_r.$$

This is the required recurrence. Once we know one initial value we can use the recurrence to creep forwards, finding successive terms of the sequence, one at a time.

Example 2.10

We have $n_1 = 7$ so that $n_2 = 3 \cdot 10^1 + 4n_1 = 58$ and $n_3 = 3 \cdot 10^2 + 4n_2 = 532$; we may continue this as far as we please.

Here is another example, this time with a geometric flavour – and with a sequence as answer that turns out to be very important.

Example 2.11

A car park consists of a row of r spaces; a motorbike (m) takes one space, a car (c) two.

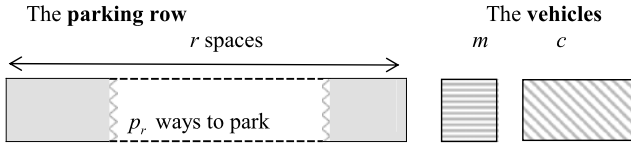


Figure 2.5 Parking cars and motorbikes.

We seek the number of ways there are of filling the spaces. Call the number of ways of filling r spaces, p_r . The strategy is to use the principle of exhaustion on the final parking space. *Either* there is a car *or* a motorbike in the final space and so the form of any parking arrangement has one of the two possibilities $P_{r-1}m$ or $P_{r-2}c$ where

- P_{r-1} is a parking arrangement of $r-1$ places, and m is a motorbike;
- P_{r-2} is a parking arrangement of $r-2$ places and c is a car.

Hence,

$$\#(P_r) = \#(m) \cdot \#(P_{r-1}) + \#(c) \cdot \#(P_{r-2})$$

which means that the required recurrence is

$$p_r = p_{r-1} + p_{r-2}.$$

We easily find that $p_1 = 1$ and $p_2 = 2$, so then $p_3 = 3$, $p_4 = 5$, $p_5 = 8$, and the sequence proceeds, each term being the sum of the two preceding terms.

The terms of this sequence are called **Fibonacci numbers** and we meet them on many occasions. We define $p_0 = 1$ as this is the only value consistent with the recurrence relation. The sequence $\{p_r\}$ is *not* the Fibonacci sequence, which we denote by $\{F_r\}$. In fact we have

$$p_r = F_{r+1}$$

and so the sequence $\{p_r\}$ *does* consist of Fibonacci numbers, but they are displaced by one step.

Example 2.12

Next we explore another situation that also leads to Fibonacci numbers, again displaced. We seek to count the number of ways there are to select subsets (including the empty set) from the set $\{1, 2, 3, \dots, r\}$ in which no consecutive numbers occur. Suppose the number of ways is s_r . Then in selecting such a subset it *either* includes the element r *or not* (Figure 2.6).

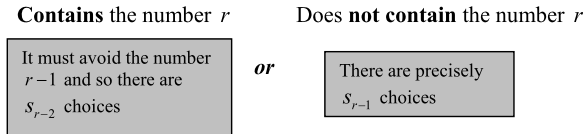


Figure 2.6 Subsets without consecutive terms.

Overall then, the number of subsets s_r is

$$s_r = s_{r-1} + s_{r-2}.$$

However, for the set $\{1\}$ there are two subsets, $\emptyset$ and $\{1\}$, neither of which contains consecutive numbers. So $s_1 = 2$. Similarly, $s_2 = 3$. If we want the recurrence to be true for $r = 0$ then we must set $s_0 = 1$. The sequence continues (each term being the sum of the two previous terms) $\{s_r\} = \{1, 2, 3, 5, 8, \dots\}$ and each term is also a Fibonacci number, this time displaced two places:

$$s_r = F_{r+2}.$$

Time to define the Fibonacci sequence.

Definition 2.13 (Fibonacci sequence)

The Fibonacci sequence $\{F_r\}$ satisfies the recurrence $F_r = F_{r-1} + F_{r-2}$ with the initial terms $F_0 = 0$ and $F_1 = 1$.

The Fibonacci sequence has the initial terms $\{F_r\} = \{0, 1, 1, 2, 3, 5, 8, \dots\}$.

Example 2.14

This time we count the number of ways there are to toss a coin until it ends (for the first time) on double heads. Suppose that there are t_r ways to do this. So we require the number of alphabetic strings of length r in which the final two places are H followed

by another H, preceded by a string of Hs and Ts in which the string “HH” does not occur. Such a string **starts** *either* with a tail *or* a head (which is then followed by a tail): Figure 2.7.

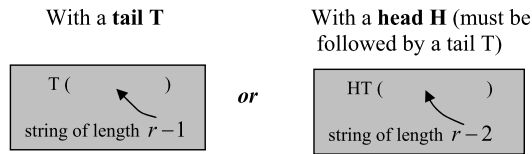


Figure 2.7 Alphabetic strings that end on HH.

Overall then, the number of strings t_r is

$$t_r = t_{r-1} + t_{r-2}.$$

For one toss of a coin the outcomes are either T or H. So $t_1 = 0$ since neither of these ends in a double head. Similarly, $t_2 = 1$, $t_3 = 1$ and hence

$$t_r = F_{r-1}.$$

If we define $F_{-1} = 1$ (a value consistent with the Fibonacci recurrence relation) then this is true when $r \geq 0$.

The next example exploits Fibonacci numbers but results in another sequence of importance.

Example 2.15

We count the number of ways we can select a subset from the **circular** set $\{1, 2, 3, \dots, r\}$ that does not contain consecutive numbers. (Circular here, means that we regard 1 and r as consecutive, and again, we include the empty set as an allowable subset choice.) Suppose the number of such selections is L_r . A selection *either* contains the element r , *or not* (Figure 2.8).

From Example 2.12 we know the number of ways that subsets (without consecutive elements) may be chosen from a set of r elements is F_{r+2} . Overall then, we have

$$L_r = F_{r-3+2} + F_{r-1+2} = F_{r-1} + F_{r+1}.$$

The sequence that appears here is a sibling sequence to the Fibonacci sequence.

Contains the number r

Both the 'neighbours' 1 and $r-1$ must be avoided. So we can choose from those remaining $\{2, 3, \dots, r-2\}$ in $F_{r-3+2} = F_{r-1}$ ways.

or

Does **not contain** the number r

We can choose from those remaining $\{1, 2, \dots, r-1\}$ in $F_{r-1+2} = F_{r+1}$ ways.

Figure 2.8 Subsets of a circular set.

Definition 2.16 (Lucas sequence)

The sequence $\{L_r\}$, defined as the sum of pairs of Fibonacci numbers:

$$L_r = F_{r-1} + F_{r+1}$$

is called the Lucas sequence, and its terms are Lucas numbers. It has the initial terms $\{L_r\} = \{2, 1, 3, 4, 7, 11, 18, \dots\}$ and satisfies the recurrence

$$L_r = L_{r-1} + L_{r-2}.$$

Note: the Lucas and Fibonacci sequences share the same recurrence relation (you will be asked to prove this in the exercises). They are distinguished by the two terms with which they begin: $\{0, 1\}$ for the Fibonacci sequence and $\{2, 1\}$ for the Lucas sequence. The same is true for the sequences of Example 2.11, $\{p_r\}$, and of Example 2.12, $\{s_r\}$: they too obey the same recurrence – they are distinguished by their two initial terms.

Example 2.17

Suppose there is an inexhaustible number of coloured counters – red, yellow, green and blue. We seek a recurrence for the number of ways that r of them can be stacked so that there are no adjacent red counters (Figure 2.9).

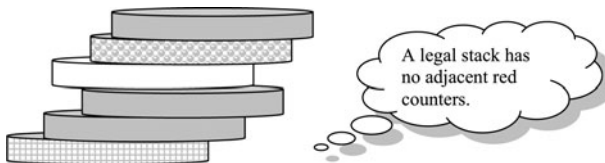


Figure 2.9 No adjacent red counters.

The enumerative strategy is to construct a stack of “size” $r+1$ from one of “size” r . Suppose that we denote the number of legal stacks by s_r . In constructing s_{r+1} for $r \geq 2$ we use the principle of exhaustion on the topmost counter; every legal stack *either* has the form $S_R c_1$ or $S_{NR} c_2$, where:

- (i) S_R is a legal stack (of size r) that ends with a red counter, so that $c_1 \in \{Y, G, B\}$;
- (ii) S_{NR} is a legal stack (of size r) that does not end with a red counter, so that $c_2 \in \{R, Y, G, B\}$.

It follows that,

$$\begin{aligned} s_{r+1} &= 3\#(S_R) + 4\#(S_{NR}) \\ &= 3[\#(S_R) + \#(S_{NR})] + \#(S_{NR}) = 3s_r + \#(S_{NR}). \end{aligned}$$

However $\#(S_{NR}) = 3s_{r-1}$ since it can be made from a legal stack of size $r-1$ followed by a final counter that may be chosen from three pieces: $\{Y, G, B\}$. So

$$s_{r+1} = 3s_r + 3s_{r-1}.$$

We note that $s_1 = 4, s_2 = 15 = (4 \cdot 4 - 1)$ so that $s_3 = 3 \cdot s_2 + 3 \cdot s_1 = 57$ and $s_4 = 3 \cdot 57 + 3 \cdot 15 = 216$. Again, we can continue calculating as many terms of the sequence as desired – though it does become very tedious.

Example 2.18

A geometric example: finding the number of regions created when r lines are drawn in the plane – no two of which are parallel, and no three of which are coincident (Figure 2.10).

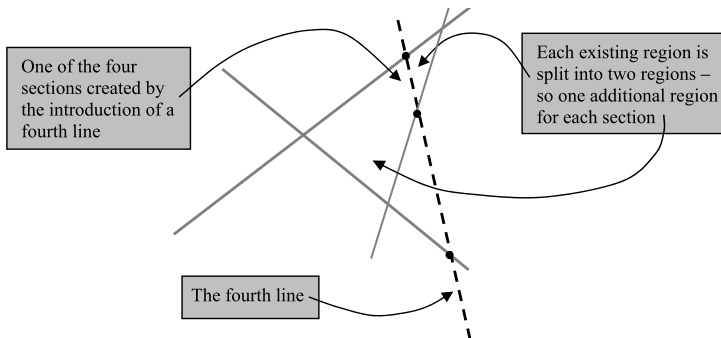


Figure 2.10 Regions created by four lines.

Suppose we denote the number of regions by R_r . The r th line meets each of the other $r - 1$ lines in a distinct point (because of the conditions) and these $r - 1$ points of intersection divide the r th line into r sections. Each of these sections divides an existing region into two regions, adding one new region.

So, when the r th line is drawn, the number of regions increases by r . We have the recurrence:

$$R_r = R_{r-1} + r \text{ and } R_0 = 1.$$

We met the derangement sequence $\{d_r\}$ in Equation (1.2); there we showed that its terms, called derangement numbers, satisfy a rather complicated recurrence relation:

$$r! = \sum_{k=0}^r \binom{r}{k} d_{r-k}.$$

In fact, it also satisfies a much simpler recurrence relation. This next example shows how this may be constructed enumeratively.

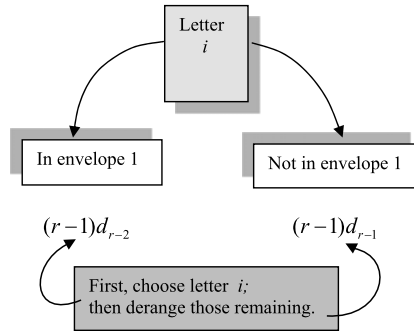


Figure 2.11 Where does letter i go?

Example 2.19

The recurrence relation is

$$d_r = (r-1)d_{r-2} + (r-1)d_{r-1}.$$

Suppose that we have a set of r letters $l_1, l_2, \dots, l_r$ and r envelopes correctly (and distinctly) addressed $e_1, e_2, \dots, e_r$. We seek the number of derangements in which no letter l_i goes into the correct envelope e_i . We call this number d_r . In any of these

derangements letter 1 must be placed in envelope i where $2 \leq i \leq r$. Now we exhaust the possibilities for letter i (see Figure 2.11):

- *either* letter i is in envelope 1
- *or* letter i is not in envelope 1.

Overall then, we have

$$d_r = (r-1)d_{r-2} + (r-1)d_{r-1}.$$

The importance of this new recurrence is its simplicity. It may also be derived directly from the explicit formula (Equation 1.1, page 12) – see Exercise 2.8.

2.2.1 Exercises

Exercise 2.5

Let a_r be the number of legal arithmetic “expressions” that are made up of r items chosen from the operations $\{+, \times, /\}$ and the digits $\{0, 1, 2, \dots, 9\}$. A legal arithmetic expression is one that can be evaluated: $2 + 3 \times 5$ is legal, as is 57; while $8 + \times 9$ is not. Find a recurrence relation for a_r .

Exercise 2.6

Find a recurrence for the number of binary strings, made up of r digits drawn from 0, 1 which do not have consecutive 0s.

Exercise 2.7

Let u_r be the number of ways that the natural number r may be written as a sum of 1s and 2s, in which the order of the summands is counted. For example: $u_2 = 2$ since $2 \equiv 2 = 1 + 1$, while $u_3 = 3$ since $3 \equiv 2 + 1 = 1 + 2 = 1 + 1 + 1$. Find a recurrence relation satisfied by the sequence u_r .

Exercise 2.8

Given the explicit formula (Equation 1.1)

$$d_r = \sum_{k=0}^r (-1)^k \binom{r}{k} (r-k)!$$

prove that the derangement sequence satisfies the recurrence $d_r = rd_{r-1} + (-1)^r$. Then show that this recurrence leads to the recurrence of Example 2.19.

Exercise 2.9

Prove that the Lucas sequence satisfies the same recurrence as that for the Fibonacci sequence:

$$L_r = L_{r-1} + L_{r-2}.$$

2.3 Sequence to Generating Function

There are two routes to a generating function. The first starts with the sequence itself, and assumes that we know all of its terms.

2.3.1 Sequence to Generating Function

We start with a simple example of a sequence with known terms and seek to find its generating function – that is, we want this as a function in an explicit form rather than as a power series.

Example 2.20

We can easily find the generating function of the sequence $\{u_r\} = \{1, 1, 1, \dots\}$. The generating function is the power series

$$1 + 1z + 1z^2 + \dots = 1 + z + z^2 + \dots$$

However, we note that if $|z| < 1$ this is a convergent geometric progression, so by Equation (2.2)

$$U(z) = \frac{1}{1-z}.$$

The function $U(z)$ is the generating function for the sequence $\{u_r\} = \{1, 1, 1, \dots\}$. We may use this generating function to find generating functions for other sequences.

Example 2.21

The generating function of the last example may be written like this

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

If we differentiate both sides then we obtain:

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \dots$$

(Term-by-term differentiation of a power series is valid when it is absolutely convergent; in this case when $|z| < 1$.) This new expression is simply another generating function for another sequence: the function

$$\frac{1}{(1-z)^2}$$

is the generating function for the sequence

$$\{1, 2, 3, \dots\} = \{r + 1\}.$$

We can also integrate absolutely convergent power series to create another “new sequence for old”. Using operations like this we can build a library of sequences and their generating functions. There are other operations we can use as well.

Example 2.22

If we multiply the generating function of the last example by z we find that

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \dots$$

We may conclude that the sequence

$$\{0, 1, 2, 3, \dots\} = \{r\}$$

has the generating function

$$\frac{z}{(1-z)^2}.$$

So our generating function library now consists of

Sequence	Generating Function
$\{1, 1, 1, \dots\}$	$\frac{1}{1-z}$
$\{r+1\} = \{1, 2, 3, \dots\}$	$\frac{1}{(1-z)^2}$
$\{r\} = \{0, 1, 2, 3, \dots\}$	$\frac{z}{(1-z)^2}$

Table 2.1 Generating function library.

We will build the library as the book progresses and it is summarized in Appendix A.

2.3.2 Recurrence to Generating Function

Most enumerations start with a recurrence relation that has been derived, rather than the enumerative sequence itself. Once we have such a recurrence relation for a sequence of interest, we have seen that it can be used to find successive terms of the

sequence. But there are much more powerful ways of exploiting the recurrence to find this and other properties of the sequence – the route to these methods goes through a generating function. So we turn to the problem of converting a recurrence into a generating function. There is a standard way that we go about this task – and it follows a recipe with three steps.

Algorithm 2.23 (Recurrence to generating function – three-step recipe)

The three parts of the recipe to convert a recurrence into a generating function are:

- (i) write out recurrence;
- (ii) multiply through by z^r and sum for valid r ;
- (iii) invoke generating and other library functions.

Example 2.24

We illustrate the recipe by converting the recurrence $u_r = 5u_{r-1} - 6u_{r-2}$ with the initial values $u_0 = 5$ and $u_1 = 12$ for the sequence $\{u_r\}$ into a generating function.

- (i) The first step is $u_r = 5u_{r-1} - 6u_{r-2}$.
- (ii) The second step then becomes,

$$\sum_{r \geq 2} u_r z^r = \sum_{r \geq 2} 5u_{r-1} z^r - \sum_{r \geq 2} 6u_{r-2} z^r. \quad (2.3)$$

(Note that as there is a term $r-2$ in the recurrence, we take $r \geq 2$.)

- (iii) The final step is where all the action takes place. The generating function we seek is, $U(z) = \sum_{r \geq 0} u_r z^r$ which is almost the term on the left of Equation (2.3); so we write,

$$\begin{aligned} U(z) &= u_0 + u_1 z + \sum_{r \geq 2} u_r z^r = 5 + 12z + \sum_{r \geq 2} u_r z^r \\ \Rightarrow \sum_{r \geq 2} u_r z^r &= U(z) - 5 - 12z. \end{aligned}$$

Now for the first term on the right of Equation (2.3). We have,

$$\begin{aligned} \sum_{r \geq 2} 5u_{r-1} z^r &= 5z \sum_{r \geq 2} u_{r-1} z^{r-1} = 5z \left(\sum_{r \geq 1} u_{r-1} z^{r-1} - u_0 \right) \\ &= 5zU(z) - 25z. \end{aligned}$$

The final term is now easy

$$\sum_{r \geq 2} 6u_{r-2}z^r = 6z^2 \sum_{r \geq 2} u_{r-2}z^{r-2} = 6z^2 U(z).$$

We put each of these results into Equation (2.3):

$$U(z) - 5 - 12z = 5zU(z) - 25z - 6z^2U(z)$$

and then solve this for $U(z)$. We find that

$$U(z) = \frac{5 - 13z}{1 - 5z + 6z^2}.$$

We have quickly passed over a very important idea which we now make explicit. This important result, whose proof is immediate, should become second nature.

Theorem 2.25 (Re-indexing a sum)

We have

$$U(z) = \sum_{r \geq 0} u_r z^r = \sum_{r \geq 1} u_{r-1} z^{r-1} = \sum_{r \geq 2} u_{r-2} z^{r-2} \dots$$

Most of the recurrences that we encountered in the last section can be converted into generating functions by the three-step recipe. A notable exception is the derangement sequence – we must wait for Chapter 7 for that.

Example 2.26

The recurrence relation satisfied by the Fibonacci sequence $\{F_r\}$ is $F_r = F_{r-1} + F_{r-2}$ with the initial values $F_0 = 0$ and $F_1 = 1$. Applying the three-step recipe gives:

(i) The recurrence is $F_r = F_{r-1} + F_{r-2}$.

(ii) Then we sum on the index r with corresponding powers of z :

$$\sum_{r \geq 2} F_r z^r = \sum_{r \geq 2} F_{r-1} z^r + \sum_{r \geq 2} F_{r-2} z^r.$$

(iii) Finally

$$\sum_{r \geq 0} F_r z^r - F_0 - F_1 z = z \left(\sum_{r \geq 0} F_{r-1} z^{r-1} - F_0 \right) + z^2 \sum_{r \geq 0} F_r z^r.$$

Using the initial values and denoting the generating function by $F(z)$, we have

$$F(z) - 0 - z = z(F(z) - 0) + z^2 F(z)$$

and we can solve this for the required generating function $F(z)$:

$$F(z) = \sum_{r \geq 0} F_r z^r = \frac{z}{1 - z - z^2}.$$

Working in the same way with the recurrence $L_r = L_{r-1} + L_{r-2}$ with the initial values $L_0 = 2$ and $L_1 = 1$ of the Lucas sequence and denoting the generating function by $L(z)$, we find that

$$L(z) = \sum_{r \geq 0} L_r z^r = \frac{2 - z}{1 - z - z^2}.$$

Note: the generating functions of these two sequences have the same denominator; also note that they obey the same recurrence. We will explore this later.

Example 2.27

We will convert the recurrence of Example 2.18 into a generating function.

(i) The recurrence is:

$$R_r = R_{r-1} + r \text{ and } R_0 = 1.$$

(ii) The next step is

$$\sum_{r \geq 1} R_r z^r = \sum_{r \geq 1} R_{r-1} z^r + \sum_{r \geq 1} r z^r.$$

(iii) In this final step we draw on our library of generating functions for the last term on the right. We can also re-write the other terms. We then have:

$$\sum_{r \geq 0} R_r z^r - R_0 = z \sum_{r \geq 0} R_r z^r + \frac{z}{(1 - z)^2}.$$

If we let the generating function be $R(z)$ then we have

$$R(z) - 1 = zR(z) + \frac{z}{(1 - z)^2}$$

and hence

$$R(z) = \frac{1 - z + z^2}{(1 - z)^3}.$$

2.3.3 Exercises

Exercise 2.10

Find generating functions for the sequences

- (i) $\{v_r\} = \{1, -1, 1, -1, \dots\}$;
- (ii) $\{u_r\} = \{1, -2, 3, -4, \dots\}$;
- (iii) $\{o_r\} = \{1, 0, 3, 0, 5, 0, \dots\}$.

Exercise 2.11

Integrate the expression $\frac{1}{1-z} = 1 + z + z^2 + \dots$ and determine the constant of integration by assigning the value $z = 0$. What is the generating function for the Reciprocal sequence, $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$?

Exercise 2.12

A sequence $\{u_r\}$ obeys the recurrence relation, $u_r = u_{r-1} + 2u_{r-2}$ with $u_0 = 4$ and $u_1 = 5$. Find the generating function for the sequence.

Exercise 2.13

The sequence $\{a_r\}$ satisfies the recurrence relation $a_r = 2a_{r-1} + 15a_{r-2}$ with $a_0 = 4$ and $a_1 = 4$. Find the generating function of the sequence $\{a_r\}$.

Exercise 2.14

Use the recurrence and the initial terms of the Lucas sequence $\{L_r\}$ to confirm that its generating function is as given in Example 2.26.

2.4 Miscellaneous Exercises

Exercise 2.15

Find a generating function for the number of strings of length r made up from the digits $\{0, 1, 2, 3\}$ in which there is never a 3 anywhere to the right of 0.

Exercise 2.16

We define the matrix M as the 2×2 array:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The powers of this matrix have a number of surprising connections with Fibonacci and Lucas numbers.

(i) Show that

$$M^r = \begin{pmatrix} F_{r-1} & F_r \\ F_r & F_{r+1} \end{pmatrix}.$$

(ii) show that the trace of M^r is given by $\text{tr}(M^r) = L_r$;

(iii) by considering the determinant of M^r prove **Cassini's identity**

$$F_{r-1}F_{r+1} - F_r^2 = (-1)^r.$$

Exercise 2.17

Find a generating function for the number of non-negative integer solutions to the equation $a + 2b = r$ for each positive integer r .

Exercise 2.18

A mathematics examination consists of six modules each with m marks. Show that the number of ways a candidate may score precisely $3m$ marks (the “pass mark”) overall is

$$\binom{3m+5}{5} - 6\binom{2m+4}{5} + 15\binom{m+3}{5}.$$

Exercise 2.19

In this exercise we investigate the number of ways p_r in which a positive integer may be written as the **ordered** sums of the summands $\{1, 2, 3\}$. For example: one can only be written as 1 so $p_1 = 1$; however two may be written as $2 = 2 = 1 + 1$ and hence $p_2 = 2$. Three has four ways of being written, $3 = 3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$ and hence $p_3 = 4$. Find a recurrence relation for the terms of the sequence $\{p_r\}$.

Exercise 2.20

You are given two tiles – one a unit square, and the other a rectangle made up from two unit squares. The rectangle can be laid vertically or horizontally. We denote the number of ways of tiling a $2 \times r$ rectangle by f_r . Show that $f_r = 2f_{r-1} + 3f_{r-2}$ and hence find a generating function for the number of tilings.

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