II

Homology and cohomology of manifolds

1 Chains on a manifold (following de Rham). Stokes' formula

1.1

A p -dimensional "*chain element*" $[\sigma]$ on a differentiable manifold X, is defined by a *convex polyhedron* Δ^p of dimension p, an *orientation* ε of Δ^p, and a *differentiable map* $\sigma : \Delta^p \to X$.

More precisely, Δ^p is a compact convex polyhedron in a p-dimensional affine space E^p, and σ is the restriction of a differentiable map defined in a neighbourhood of this compact set.

If φ is a differential form of degree p on X, the *integral of* φ *along the chain element* $[\sigma]$ is defined by

$$\int_{[\sigma]} \varphi = \int_{\Delta^p} \sigma^* \varphi.$$

The integral over the oriented polyhedron Δ^p obviously makes sense, because $\sigma^*\varphi$ is the restriction to the compact set Δ^p of a differential form which is $\mathscr{C}^\infty$ in a neighbourhood of this compact set.

1.2

A *p-dimensional chain* γ in X is defined by a finite formal linear combination of p-dimensional chain elements with integer coefficients. Moreover, we say that two such linear combinations

$$\sum_i n_i[\sigma_i] \quad \text{and} \quad \sum_j n'_j[\sigma'_j]$$

define *the same chain* γ if, for any form φ, they give rise to the same integral:

$$\int_\gamma \varphi = \sum_i n_i \int_{[\sigma_i]} \varphi = \sum_j n'_j \int_{[\sigma'_j]} \varphi.$$

F. Pham, *Singularities of integrals*,
Universitext, DOI 10.1007/978-0-85729-603-0_2,

1.3 The boundary

Let $[\sigma] = (\varDelta^p, \varepsilon, \sigma)$ be a p-dimensional chain element. Every face $\varDelta^p$ of the polyhedron $\varDelta^p$ is contained in a $(p-1)$-plane E_i^p, which we can equip with an orientation ε_i, defined as follows:

$$\varepsilon_i\left(V_1, V_2, \ldots, V_{p-1}\right) = \varepsilon\left(V_0, V_1, V_2, \ldots, V_{p-1}\right),$$

where V_0 is the exterior normal vector to the face E_i^p.

If we set $\sigma_i = \sigma | \varDelta_i^p$, we obtain in this way a chain element

$$[\sigma_i] = (\varDelta_i^p, \varepsilon_i, \sigma_i)\,.$$

The sum of all the chain elements which correspond to all the faces of $\varDelta^p$ is called the *boundary* of $[\sigma]$, and is denoted $\partial[\sigma]$. For any *chain* $\gamma = \sum_i n_i[\sigma_i]$, we set $\partial\gamma = \sum_i n_i \partial[\sigma_i]$. It will follow immediately from Stokes' formula (§1.4) that this chain $\partial\gamma$ does not depend on the particular representation $\sum_i n_i[\sigma_i]$ of γ, and furthermore, that $\partial\partial\gamma = 0$ (which follows easily from $dd = 0$).

1.4 Stokes' formula.

$$\boxed{\int_{\partial\gamma} \varphi = \int_\gamma d\varphi.}$$

It suffices to prove this formula when γ is a chain element $[\sigma]$. The general case follows by linearity. Setting $\psi = \sigma^*\varphi$, we are reduced to the formula

$$\int_{\varDelta^p} d\psi = \sum_i \int_{\varDelta_i^p} \psi,$$

which can be verified in the same way as the formula in §I.7.7 (it is in fact the same formula, except that $\varDelta^p$ is a manifold whose boundary is *piecewise* differentiable, whereas in §I.7.7 we assumed that the boundary was differentiable).

1.5 Chain transformations

Let $f : X \to Y$ be a differentiable map of manifolds. To every chain element $[\sigma] = (\varDelta^p, \varepsilon, \sigma)$ in X, we will associate the chain element $f_*[\sigma] = (\varDelta^p, \varepsilon, f \circ \sigma)$ on Y, and likewise, to every chain $\gamma = \sum_i n_i[\sigma_i]$ in X, we associate the chain $f_*\gamma = \sum_i n_i f_*[\sigma_i]$ in Y.

Obviously, $\int_{f_*\gamma} \varphi = \int_\gamma f^*\varphi$ (which is an immediate consequence of §I.5.3, which says that $(f \circ \sigma)^*\varphi = \sigma^* f^* \varphi$).

This identity proves that the chain $f_*\gamma$ is indeed independent of the choice of representative for γ. Moreover, combined with Stokes' formula and the property $df^* = f^*d$ of §I.5.3, it proves that

$$\boxed{\partial f_* = f_* \partial.}$$

This follows from:

$$\int_{\partial f_*\gamma} = \int_{f_*\gamma} d\varphi = \int_\gamma f^* d\varphi = \int_\gamma df^*\varphi = \int_{\partial\gamma} f^*\varphi = \int_{f^*\partial\gamma} \varphi.$$

1.6 Example. Let $X = S^2$ denote the unit sphere in Euclidean space $\mathbb{R}^3$. The parametric representation of this sphere in polar coordinates defines a two-dimensional chain element $[\sigma] = (\varDelta, \varepsilon, \sigma)$ ($\varDelta$ is a rectangle of length 2π and width π, and $\sigma : \varDelta \to S^2$ is the map defined by "longitudinal–latitudinal" coordinates). The integral of any form φ on this chain element is therefore equal to the integral of φ on the sphere S^2, equipped with the orientation ε_{S^2} "corresponding" to ε ($\varepsilon_{S^2} \circ \sigma = \varepsilon$). In particular, let φ be the differential form corresponding to the "solid angle", in other words, the fundamental form (cf. §I.7.5) associated to the orientation ε_{S^2} and to the metric on S^2 which is induced by the Euclidean metric on $\mathbb{R}^3$. We have

$$\int_{[\sigma]} \varphi = 4\pi.$$

Observe that $d\varphi = 0$ (since φ is of maximal degree) and that $\partial[\sigma] = 0$. In this case we say that φ is a *closed form* and that $[\sigma]$ is a *cycle*. However, φ is not an "exact differential form", nor is $[\sigma]$ a "boundary". This means that there does not exist a form ψ such that $\varphi = d\psi$, nor does there exist a chain γ such that $[\sigma] = \partial\gamma$. If either were to exist, then Stokes' formula would say that $\int_{[\sigma]} \varphi = 0$.

2 Homology

2.1

The main point of the concept of a "chain" is to act as a stepping stone towards the construction of "homology groups", which we are about to define. These days it is preferable to use a slightly different definition of chains from the one given in the previous section, which has the added advantage of being valid for any topological space. On a manifold, all these definitions give rise to the same homology groups, but the modern definition enables us to extend what was previously only possible for differentiable maps, to the case of *continuous maps.*

Let us therefore state some general properties for chains which were obtained in the previous section for differentiable maps between manifolds.

To every topological space X is associated an *abelian group* $C_*(X)$, called the *group of chains* of X. This group is graded by the "dimension" p of the chains:

$$C_*(X) = \bigoplus_p C_p(X)$$

and is equipped with a "*boundary homomorphism*":

$$\partial : C_*(X) \longrightarrow C_*(X)$$

which decreases the dimension by one, and satisfies

$$\partial\partial = 0.$$

To every continuous map $f : X \to Y$ one associates a homomorphism $f_* : C_*(X) \to C_*(Y)$ which preserves the dimension of chains, and which commutes with ∂. The map $f \rightsquigarrow f_*$, is a *covariant functor* (cf. §I.4.2).

2.2 Definition (of homology). Let $Z_*(X) = \operatorname{Ker} \partial$ denote the "*kernel*" of the homomorphism ∂, i.e., the set of chains $\gamma \in C_*(X)$ such that $\partial\gamma = 0$. Such a chain is called a *cycle*, and $Z_*(X)$ is the *group of cycles*.

Let $B_*(X) = \operatorname{Im} \partial$ denote the *image* of the homomorphism ∂, i.e., the set of chains of the form $\partial\gamma$. Such a chain is called a *boundary*, and $B_*(X)$ is the *group of boundaries*.

Since $\partial\partial = 0$, $B_*(X)$ is subgroup of $Z_*(X)$. We can therefore consider the quotient

$$H_*(X) = Z_*(X)/B_*(X)$$

which is called the *homology group* of the space X. Its elements are called "homology classes". Two cycles belong to the same homology class if the difference between them is a boundary, and in this case we say that the cycles are "homologous".

All the above groups are clearly graded by the dimension of the chains which define them. Thus,

$$H_*(X) = \bigoplus_{p=0,1,2,\ldots} H_p(X).$$

In dimension zero, we set $Z_0(X) = C_0(X)$, so that $H_0(X) = C_0(X)/B_0(X)$. But $C_0(X)$ is the *free group generated by the set* X (the group of formal linear combinations of points of X), and $B_0(X)$ is spanned by linear combinations of the form $[x] - [x']$ in $C_0(X)$, where x and x' are the endpoints of a *path* on X. Therefore $H_0(X)$ is the free group generated by the set of *path-connected components of the space* X.

2.3 Examples.

(i) Let $X = S^n$ be the unit sphere of dimension n in $\mathbb{R}^{n+1}$. It is connected, so $H_0(S^n) \approx \mathbb{Z}$ (the group of integers $\gtrless 0$). Otherwise, one shows that $H_*(S^n)$ is zero in all other dimensions, *except in dimension* n, where $H_n(S^n) \approx \mathbb{Z}$. This group is spanned by a cycle which can be constructed in a similar manner to the cycle $[\sigma]$ in example 1.6.

(ii) Let $X = P^n$ be real n-dimensional projective space. Obviously, $H_0(P^n) \approx \mathbb{Z}$. Otherwise, one can prove that $H_{2p}(P^n) = 0$ and that $H_{2p+1}(P^n) \approx \mathbb{Z}_2$ (the cyclic group of order 2), except in the case $2p+1 = n$, where $H_{2p+1}(P^{2p+1}) \approx \mathbb{Z}$. Intuitively, the generator in dimension $2p+1$ "corresponds" to a $(2p+1)$-plane $P^{2p+1} \subset P^n$.[1]
One can also construct chains corresponding to $2p$-planes $P^{2p} \subset P^n$, but, because of the non-orientability of P^{2p} (§7.2), one notices that these chains are not cycles but have a boundary corresponding to a copy of P^{2p-1} which is covered twice. It is this fact which explains why the homology groups in dimension $2p-1$ are cyclic groups of order 2 (P^{2p-1} is not a boundary, but $2\ P^{2p-1}$ is).

2.4 Torsion

In §2.3 (ii) above we gave an example of a homology group with "*torsion*". We say that an abelian group G has torsion if it contains cyclic elements of finite order.

In most cases which arise in practice, homology groups have a *finite* number of generators, and are therefore[2] isomorphic to a direct sum of cyclic groups of finite order ($\mathbb{Z}_r$), or infinite order ($\mathbb{Z}$). If we discard the groups of finite order, we obtain a *free group* (with a finite number of generators), which is called the "homology group *modulo torsion*". The case where there are an infinite number of generators is a little delicate,[3] and we will (implicitly) put this question aside whenever we speak about torsion.

2.5

Let $f : X \to Y$ be a continuous map. Since the homomorphism $f_* : C_*(X) \to C_*(Y)$ of §2.1 commutes with ∂, it maps cycles to cycles, and boundaries to boundaries. It therefore induces a homomorphism $f_* : H_*(X) \to H_*(Y)$ between homology groups which preserves dimensions. The map $f \rightsquigarrow f_*$ is a *covariant functor*. From this it follows in particular that if $f : X \approx Y$ is a homeomorphism, $f_* : H_*(X) \approx H_*(Y)$ is an isomorphism of abelian groups.

2.6 Retractions

A continuous map $r : X \to A$ is called a *retraction* if:

1° A is a subspace of X,
2° $r|A = \mathbb{1}_A$.

[1] In other words, it can be represented by a cycle constructed from a parametric representation of a $(2p+1)$-plane.
[2] This is an exercise in algebra, which is not so straightforward.
[3] For example, a torsion-free group is not necessarily free.

Let $i : A \to X$ be the "inclusion" from A into X. Condition 2° can be written $r \circ i = \mathbb{1}_A$, from which it follows by functoriality that $r_* i_* = \mathbb{1}_{H_*(A)}$

$$\begin{tikzcd} & H_*(X) \arrow[dr, "r_*"] & \\ H_*(A) \arrow[ur, "i_*"] \arrow[rr, "\mathbb{1}"] & & H_*(A) \end{tikzcd}$$

This diagram shows that $H_*(A)$ is a *direct factor* of the group $H_*(X)$ [in other words $\exists\, G \;:\; H_*(X) \approx H_*(A) \times G$].

2.7 Example (Brouwer's fixed point theorem). Let E^n denote the closed unit ball in $\mathbb{R}^n$. Its boundary is a sphere S^{n-1}. The space E^n is "homologically trivial" (§2.9), which is not the case for S^{n-1} [§2.3 (i)]. Therefore, $H_*(S^{n-1})$ cannot be a subgroup of $H_*(E^n)$, so *there cannot exist a retraction from* E^n *onto* S^{n-1}.

Brouwer's "fixed point theorem" states that *every continuous map* $f : E^n \to E^n$ *has at least one fixed point*, which is a corollary of the previous statement. To see this, suppose that the contrary is true, i.e., $\forall\, x \in E^n$, $f(x) \neq x$, and let $r(x)$ denote the point where the sphere S^{n-1} meets the half-line $\overline{f(x), x}$ (Fig. II.1). It is easy to check that the map $r : E^n \to S^{n-1}$ which is defined in this way is continuous, and clearly $r|S^{n-1} = \mathbb{1}_{S^{n-1}}$. This would mean that r is a retraction, contradicting the statement above.

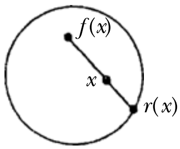

Fig. II.1.

2.8 Homotopy

Two continuous maps f_0 and $f_1 : X \to Y$ are called *homotopic* (in short, $f_0 \simeq f_1$) if they can be "interpolated" by a continuous map $f_\tau : X \to Y$.[4]

One can prove the following fundamental property:

Two homotopic maps give rise to the same homomorphism between homology groups:

$$f_0 \simeq f_1 \Longrightarrow \boxed{f_{0_*} = f_{1_*}} : H_*(X) \longrightarrow H_*(Y).$$

[4] We say that the family of continuous maps $f_\tau : X \to Y$ ($\tau \in [0,1]$) "interpolates" f_0 and f_1, if $f(x,\tau) = f_\tau(x)$ defines a continuous map $f : X \times [0,1] \to Y$.

2.9 Deformation-retractions

A retraction $r : X \to A$ (cf. §2.6) is called a *deformation-retraction* if $i \circ r \simeq \mathbb{1}_X$. It follows from functoriality, together with homotopy invariance 2.8, that $i_* r_* = \mathbb{1}_{H_*(X)}$. By 2.6, we deduce that $r_* : H_*(X) \to H_*(A)$ is an *isomorphism* (and i_* is the inverse isomorphism).

Important special case. The space X is said to be "*contractible*" if there is a deformation retraction onto one of its points. It is therefore "*homologically trivial*", i.e., it has the homology of a point:

$$H_p(X) = 0 \quad \text{if } p > 0,$$
$$H_0(X) = \mathbb{Z}.$$

2.10 Examples. *Euclidean space $\mathbb{R}^n$ is contractible.* If $r : \mathbb{R}^n \to P$ is the retraction from $\mathbb{R}^n$ onto the origin P, the relation

$$f_\tau(x_1, x_2, \ldots, x_n) = (\tau x_1, \tau x_2, \ldots, \tau x_n)$$

defines a *homotopy* between

$$i \circ r = f_0 \quad \text{and} \quad \mathbb{1}_{\mathbb{R}^n} = f_1.$$

The same argument shows that *every star domain in $\mathbb{R}^n$ is contractible.* In the same way, it is easy to see that a cylindrical surface deformation retracts onto its base, and so on.

2.11 Relative homology

Let (X, A) be a *pair*, in other words, a topological space X and a subspace A. Let $i_* : C_*(A) \to C_*(X)$ be the homomorphism induced on chains by the inclusion map $i : A \to X$.[5] The quotient group

$$C_*(X, A) = C_*(X)/i_* C_*(A)$$

is called the group of *relative chains* of the pair (X, A). The homomorphism ∂ clearly induces a homomorphism

$$\partial : C_*(X, A) \longrightarrow C_*(X, A)$$

which satisfies the same properties as in §2.1, and as a result, one can define the *relative homology* group to be

$$H_*(X, A) = Z_*(X, A)/B_*(X, A),$$

[5] This homomorphism is *injective*, and so one can identify $C_*(A)$ with a subgroup of $C_*(X)$.

where
$$Z_*(X,\,A) = \operatorname{Ker}\partial, \qquad B_*(X,\,A) = \operatorname{Im}\partial.$$

Observe that a "relative cycle" [an element of $Z_*(X,\,A)$] is represented by a chain of X whose boundary is a chain in A. This boundary is a cycle in A, and one can easily check that its homology class in A only depends on the relative homology class [in $(X,\,A)$] of the original relative cycle. We have thus defined a homomorphism
$$\partial_* : H_p(X,\,A) \longrightarrow H_{p-1}(A).$$

The *functorial* properties and the *homotopy* property also exist for relative homology: it suffices to replace the word "map" by "map of pairs" in the statements of §§1.5 and 2.8 [a "map of pairs" f: $(X,\,A) \to (Y,\,B)$ consists of two pairs $(X,\,A)$, $(Y,\,B)$ and a map $f : X \to Y$ such that $f(A) \subset B$].

Moreover, the homomorphism ∂_* transforms "naturally": for every map of pairs
$$f : (X,\,A) \longrightarrow (X,\,B),$$
the diagram
$$\begin{tikzcd} H_p(X,\,A) \arrow[r, "f_*"] \arrow[d, "\partial_*"] & H_p(Y,\,B) \arrow[d, "\partial_*"] \\ H_{p-1}(A) \arrow[r, "(f|A)_*"] & H_{p-1}(B) \end{tikzcd}$$
commutes (5).

2.12 Remark (on coefficients). Instead of defining chains as being linear combinations of chain elements *with integer coefficients*, we could obviously have taken linear combinations with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (for example). In this way, by taking coefficients in a *field*, we replace all the *groups* (groups of chains, homology groups, etc.) with *vector spaces*. This has the effect of "killing" the torsion. If the "ordinary" homology group (i.e., with coefficients in $\mathbb{Z}$) has a finite number of generators, it defines, *modulo torsion* (§2.4), a *free group with a finite number of generators*, and the homology with coefficients in a field is simply the *vector space spanned by the same generators*.

3 Cohomology

3.1 Cochains

On every topological space X, there is the notion of a "*cochain*" which is *dual to the notion of a chain*, and this will be used to construct the "cohomology" of X. If X is a differentiable manifold, and if we are interested in its cohomology with coefficients in $\mathbb{R}$ or $\mathbb{C}$, we can replace cochains by *differential forms* which are defined globally on the manifold. The duality is given by integration (this is *de Rham's theorem*).

We will state some general properties of cochains, indicating the corresponding terminology for differential forms in brackets.

To every topological space X is associated an abelian group $C^*(X)$ [or vector space $\Omega(X)$] called the *group of cochains* of X [the space of differential forms on X]. This group is graded by the "*degree*" p of the cochains:

$$C^*(X) = \bigoplus_p C^p(X) \qquad [\Omega(X) = \bigoplus_p \Omega^p(X)],$$

and it is equipped with a "*coboundary homomorphism*" δ (differential d) which increases the degree by one, and satisfies

$$\delta\delta = 0 \qquad [dd = 0].$$

For every continuous map $f : X \to Y$, there corresponds a homomorphism $f : C^*(Y) \to C^*(X)$ which preserves the degree of cochains and commutes with δ. The correspondence $f \rightsquigarrow f^*$ is a *contravariant functor*.

3.2 Definition (of cohomology). In complete analogy with §2.2 we have:

- Cocycles: $Z^*(X) = \operatorname{Ker}\delta$ [closed differential forms: $\Phi(X) = \operatorname{Ker} d$];
- Coboundaries: $B^*(X) = \operatorname{Im}\delta$ [exact differential forms: $d\Omega(X)$];
- Cohomology: $H^*(X) = Z^*(X)/B^*(X)$ [$= \Phi(X)/d\Omega(X)$].

All these groups are graded by the *degree* of cochains:

$$H^*(X) = \bigoplus_{p=0,1,2,\ldots} H^p(X)$$

[if X is a manifold of dimension n, the direct sum stops at n].

In dimension zero, we set $B^0(X) = 0$, so that

$$H^0(X) = Z^0(X) \qquad [= \Phi^0(X)].$$

But $\Phi^0(X)$ is the vector space of differential functions on X with zero differential. Any such function is constant on every connected component of the manifold X. In this way, a basis for the vector space $H^0(X)$ is given by the *set of connected components* of X.

3.3

Every continuous map $f : X \to Y$ induces a homomorphism $f^* : H^*(Y) \to H^*(X)$ between cohomology groups which preserves the degree.

The correspondence $f \rightsquigarrow f^*$ is a *contravariant functor*.

Corollaries.

- *If $f : X \approx Y$ is a homeomorphism, $f^* : H^*(X) \approx H^*(Y)$ is an isomorphism;*
- *if X retracts onto A, $H^*(A)$ is a direct summand of $H^*(X)$.*

3.4 Homotopy

Two homotopic maps induce the same homomorphism between cohomology groups:

$$f_0 \simeq f_1 \Longrightarrow f_0^* = f_1^*.$$

Corollary. *A contractible space is cohomologically trivial, i.e., it has the same cohomology as a point:*

$H^p(X) = 0$ *if* $p > 0$;
$H^0(X) = \mathbb{C}$ *or* $\mathbb{R}$ *for cohomology with coefficients in* $\mathbb{C}$ *or* $\mathbb{R}$.

In particular, the fact that Euclidean space is cohomologically trivial goes by the name of the "*converse of Poincaré's theorem*": every closed differential form on Euclidean space is an exact differential.

3.5 Relative cohomology

Let (X, A) be a pair, and let us consider a *restriction homomorphism*[6] of cochains:

$$i^* : C^*(X) \longrightarrow C^*(A).$$

The *kernel* of this homomorphism

$$C^*(X, A) = \operatorname{Ker} i^*,$$

is called the group of *relative cochains* of the pair (X, A) [in the case of manifolds, this kernel will be denoted $\Omega(X, A)$: it is the vector space of *differential forms of* X *whose restriction to the submanifold* A *is zero*].

This group is obviously equipped with a "coboundary" homomorphism which satisfies the same properties as in §3.1, so that we can define its cohomology:

$$H^*(X, A) = Z^*(X, A)/B^*(X, A) \quad [= \Phi(X, A)/d\Omega(X, A)],$$
$$Z^*(X, A) = \operatorname{Ker} \delta, \qquad B^*(X, A) = \operatorname{Im} \delta.$$

This cohomology is itself equipped with a "*coboundary homomorphism*"

$$\delta^* : H^p(A) \longrightarrow H^{p+1}(X, A)$$

defined as follows: let φ be a cocycle of A, which belongs to the cohomology class $h^p \in H^p(A)$, and let ψ be a cochain of X such that $i^*\psi = \varphi$ (if A is a closed differentiable submanifold of X, such a differential form ψ always exists by §I.6.4; in the general case, one shows that one can choose φ in h^p in such

[6] Recall (§I.6.4) that if A is a *closed* differentiable submanifold of X, the *restriction* homomorphism of differential forms is *surjective*.

a way that ψ exists). Then $\delta\psi$ defines a cocycle on X, whose restriction to A is zero:

$$\delta\psi|A = i^*\delta\psi = \delta i^*\psi = \delta\varphi = 0,$$

and is therefore a relative cocycle on (X, A). One checks that its relative cohomology class, denoted $\delta^* h^p \in H^{p+1}(X, A)$, only depends on the original class h^p.

All the statements of §§3.3, 3.4 can be repeated in the case of relative cohomology by replacing the word "map" by "map of pairs" (6).

3.6 Remark. We can say something more precise about the properties of the differential form $d\psi$ of §3.5, when A is a closed submanifold of X. In §I.6.4 we defined ψ to be equal to $\pi \cdot \mu^*\varphi$ in a tubular neighbourhood of A, where μ was the retraction of this tubular neighbourhood onto A, and π was a $\mathscr{C}^\infty$ function which was equal to 1 *in a neighbourhood of* A. Therefore, in this latter neighbourhood,

$$d\psi = d\mu^*\varphi = \mu^* d\varphi = 0,$$

so that *the support of* $d\psi$ *does not meet* A, which is a stronger property than the property $d\psi|A = 0$ which was established in §3.5.

4 De Rham duality

4.1

Let us make the definition that $C^*(X)$ is the $\mathbb{Z}$-dual of $C_*(X)$ (mentioned in §3.1) more precise. A *cochain* (with coefficients in $\mathbb{Z}$) is a *linear* map $\varphi : C_*(X) \to \mathbb{Z}$. To every chain γ, φ associates an integer denoted $\langle \varphi \mid \gamma \rangle$. We will say that the cochain φ is of *degree* p if $\langle \varphi \mid \gamma \rangle = 0$ for every chain of dimension different from p. The coboundary homomorphism δ is *defined* by

$$\langle \delta\varphi \mid \gamma \rangle = \langle \varphi \mid \partial\gamma \rangle\cdot$$

This relation implies that $\langle \varphi \mid \gamma \rangle$ is zero whenever γ is a cycle and φ is a coboundary, or γ is a boundary and φ a cocycle. As a result, $\langle \varphi \mid \gamma \rangle$ only depends on the homology class of γ and on the cohomology class of φ, and defines a *bilinear form*

$$\langle \mid \rangle : H^p(X) \otimes H_p(X) \longrightarrow \mathbb{Z}.$$

We can then pose the following algebraic problem: since $C^*(X)$ is the $\mathbb{Z}$-dual of $C_*(X)$ by $\langle \mid \rangle$, can we deduce that $H^*(X)$ is the $\mathbb{Z}$-dual of $H_*(X)$ by $\langle \mid \rangle$?

The answer is *yes, modulo torsion.* If, instead of taking coefficients in $\mathbb{Z}$, we took them in a *field*, then we avoid any torsion problems and $H^*(X)$ *is the vector space which is dual to* $H_*(X)$.

4.2

The duality of §4.1 becomes particularly interesting in the light of *de Rham's theorem*. This states that we can replace $C^*(X)$ by $\Omega(X)$, δ by d, and $\langle \mid \rangle$ by integration in the definition of cohomology $H^*(X)$ with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (the formula $\langle \delta\varphi \mid \gamma \rangle = \langle \varphi \mid \partial\gamma \rangle$ then becomes *Stokes' formula*).

We can therefore state the

Theorem (De Rham duality). *The bilinear form given by "integration" identifies the cohomology of differential forms $H^p(X)$ with the space of all linear functions on $H_p(X)$.*

Corollary. *If φ is a closed differential form whose integral vanishes along any cycle, then φ is an exact differential form.* [*As a linear function on $H_p(X)$, φ is zero, and therefore de Rham's duality theorem asserts that its cohomology class is zero.*]

4.3

The considerations of §4.1 can obviously be extended to *relative* homology and cohomology [the relative *cochains* of (X, A) are those whose restriction to A is zero, i.e., the cochains φ such that $\langle \varphi \mid \gamma \rangle = 0$, $\forall \gamma \in i_* C_*(A)$, which shows that $C^*(X, A)$ is indeed the dual of $C_*(X, A) = C_*(X)/i_*C_*(A)$].

De Rham's theorem can also be extended so that *the relative cohomology $H^p(X, A)$ of differential forms is identified, via the bilinear form given by "integration", with the space of all linear functions on $H_p(X, A)$.*

4.4 Transposition of ∂_* and δ^*

We immediately deduce from the transposition formula $\langle \delta\varphi \mid \gamma \rangle = \langle \varphi \mid \partial\gamma \rangle$ (Stokes' formula), the following transposition formula

$$\langle \delta^* k \mid h \rangle = \langle k \mid \partial_* h \rangle,$$

where ∂_* and δ^* are the homomorphisms defined in §§2.11 and 3.5:

$$h \in H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A);$$
$$H^p(X, A) \xleftarrow{\delta^*} H^{p-1}(A) \ni k.$$

5 Families of supports. Poincaré's isomorphism and duality

5.1

We call a *family of supports* in a topological space[7] X, a set Φ of closed subsets of X satisfying the following three properties:

$$(\Phi_1) \qquad A,\, B \in \Phi \Longrightarrow A \cup B \in \Phi;$$

$$(\Phi_2) \qquad \left.\begin{array}{c} A \in \Phi \\ B \text{ closed} \subset A \end{array}\right\} \Longrightarrow B \in \Phi;$$

(Φ_3) [8] every $A \in \Phi$ has a neighbourhood belonging to the family Φ.

Examples. The family F of all closed sets, and the family c of all compact sets, are both families of supports.

If X is a *subspace* of the space Y, and Ψ is a family of supports of Y, one can check that the families

$$\Psi|X = \{A \subset X :\ A \in \Psi\};\text{[9]}$$
$$\Psi \cap X = \{A = B \cap X :\ B \in \Psi\}$$

are families of supports on X.

In particular, $c_Y|X = c_X$, $F_Y \cap X = F_X$.

5.2 Cohomology with supports in Φ

We will not give the definition of the *support of a cochain*, which is rather delicate. Let us only recall, in the case of manifolds, that the *support of a differential form* φ (abbreviated to $\operatorname{supp}\varphi$) is the smallest closed set where $\varphi(x) \neq 0$.

The supports of cochains, as for differential forms, are closed sets which satisfy the following properties:

$$(\text{supp}^1) \qquad \operatorname{supp}(\varphi + \psi) \subset \operatorname{supp}\varphi \cup \operatorname{supp}\psi,$$
$$(\text{supp}^2) \qquad \operatorname{supp}\delta\varphi \subset \operatorname{supp}\varphi;$$
$$(\text{supp}^3) \qquad \forall f : X \longrightarrow Y \text{ and } \forall \psi \in C^*(Y), \quad \operatorname{supp} f^*\psi \subset f^{-1}(\operatorname{supp}\psi).$$

[7] Throughout this section, "topological spaces" will be assumed to be *locally compact and paracompact* (3).

[8] Condition (Φ_3) does not play an essential role, and in any case, will only be used in §§5.6 and 6.5.

[9] In order for this family to satisfy condition (Φ_3), the subspace X must be assumed to be "*locally closed*" (i.e., the intersection of an open and a closed subset of Y).

Let Φ be a family of supports on X. Let $C^*({}^\Phi X)$ denote the *group of cochains with supports in the family* Φ:

$$C^*({}^\Phi X) = \{\varphi \in C^*(X) : \operatorname{supp}\varphi \in \Phi\}\cdot$$

This is a *group*, because of (Φ_1) (supp^1), and it is *stable under the coboundary map* δ, because of (Φ_2) (supp^2). We can therefore define its *cohomology*,

$$H^*({}^\Phi X) = \text{cohomology of } C^*({}^\Phi X).$$

A continuous map

$$f : {}^\Phi X \longrightarrow {}^\Psi Y$$

of topological spaces equipped with families of supports will be called "*cohomologically admissible*" if $f^{-1}(\Psi) \subset \Phi$. It follows from (supp^3) that any such map induces a homomorphism

$$f^* : C^*({}^\Psi Y) \longrightarrow C^*({}^\Phi X),$$

and therefore a homomorphism

$$f^* : H^*({}^\Psi Y) \longrightarrow H^*({}^\Phi X),$$

and the correspondence $f \rightsquigarrow f^*$ is a *contravariant functor* (defined on the category of admissible maps). In particular, if we take the family of all *closed* sets to be the family of supports, all continuous maps are cohomologically admissible (since the inverse image of a closed set under a continuous map is closed). That is why *cohomology with closed supports* $H^*({}^F X)$, which is simply "ordinary cohomology" $H^*(X)$, has a privileged status.

5.3 Homology with supports in Φ

We call the *support of a chain element* $[\sigma]$ (abbreviated to $\mathrm{supp}[\sigma]$) the image of the map σ in X. It is a *compact* subset of X (the image of the compact set Δ under the continuous map σ).

Let us modify the definition of chains in §1.2 by declaring that the formal linear combination

$$\gamma = \sum_i n_i[\sigma_i]$$

will no longer necessarily be finite, but *locally finite*: in other words, every point x has a neighbourhood U_x which only meets a finite number of the supports of the elements $[\sigma_i]$. It is easy to deduce that the set

$$\operatorname{supp}\gamma = \bigcup_i \operatorname{supp}[\sigma_i]$$

is *closed*, which is called the "support of the chain γ".[10]

These closed sets satisfy the following properties:

$$(\mathrm{supp}_1) \qquad \operatorname{supp}(\gamma+\gamma') = \operatorname{supp}\gamma \cup \operatorname{supp}\gamma',$$

$$(\mathrm{supp}_2) \qquad \operatorname{supp}\partial\gamma \subset \operatorname{supp}\gamma.$$

Let us denote by $C_*({}_\Phi X)$ the *group of chains with supports in the family Φ*:

$$C_*({}_\Phi X) = \{\gamma \in C_*(X) : \ \operatorname{supp}\gamma \in \Phi\} \cdot$$

This is indeed a *group*, because of (Φ_1) (supp_1), and it is stable under the *boundary map* ∂, because of (Φ_2) (supp_2). We can therefore define its *homology*,

$$H_*({}_\Phi X) = \text{homology of } C_*({}_\Phi X).$$

A continuous map

$$f : {}_\Phi X \longrightarrow {}_\Psi Y$$

of topological spaces equipped with families of supports will be called "*homologically admissible*" if:

(f_1) f is "Φ-proper", i.e.:
$\forall$ compact sets $K \subset Y$ and $\forall\, A \in \Phi$, $\ f^{-1}(K) \cap A$ is compact.

(f_2) $\quad f(\Phi) \subset \Psi$

It is easy to see that condition (f_1) is equivalent to the following:

(f_1') For every locally finite family $\{A_i \subset X\}$ such that $\bigcup_i A_i \in \Phi$, the family $\{f(A_i)\}$ is locally finite in Y.

This condition therefore allows us to define the image $f_*\gamma$ of a chain γ with support in Φ, and this image will have its support in Ψ because of condition (f_2) along with the following obvious property:

$$(\mathrm{supp}_3) \qquad \forall f : X \longrightarrow Y \quad \text{and} \quad \forall \gamma \in C_*(X), \qquad \operatorname{supp} f_*\gamma = f(\operatorname{supp}\gamma).$$

Any such admissible map therefore induces a homomorphism

$$f_* : H_*({}_\Psi X) \longrightarrow H_*({}_\Psi Y)$$

and the correspondence $f \rightsquigarrow f_*$ is a *covariant functor* on the category of admissible maps.

In particular, if we take the family of *compact sets* to be the family of supports, every continuous map is homologically admissible, since:

1° every closed subset of a compact set is compact,
2° the image of a compact set under a continuous map is compact.

This homology with compact supports $H_*({}_c X)$ coincides incidentally with "ordinary" homology $H_*(X)$ (because every locally finite family of subsets of a compact set is *finite*).

[10] With the definition of chains given in §1.2 we run into the difficulty that the set $\bigcup \operatorname{supp}[\sigma_i]$ will in general *depend* on the choice of representative for γ. This difficulty does not occur for the more modern definitions of chains.

5.4 Poincaré's isomorphism[11]

In an *oriented* manifold X of dimension n, there exists, *for every family of supports Φ*, a *canonical isomorphism*

$$\boxed{H_p({}_\Phi X) \approx H^{n-p}({}^\Phi X)}$$ [12]

between homology and cohomology with the same coefficients.

We will not give the construction of this isomorphism here, since its properties will appear naturally due to the notion of a "*current*", which will be introduced in section 6. In much the same way as "distributions" are a generalization of functions, "currents" are a generalization of differential forms, and happen at the same time to be a generalization of chains.

5.5 Intersection index. Poincaré duality

Let us consider the bilinear form

$$\langle \mid \rangle : H^p(X) \otimes H_p(X) \longrightarrow \mathbb{Z}$$

defined in §1. Poincaré's isomorphism maps the cohomology group $H^p(X)$ (with closed supports) to the homology group $H_{n-p}({}_F X)$, and maps the bilinear form above into the bilinear form

$$\langle \mid \rangle : H_{n-p}({}_F X) \otimes H_p(X) \longrightarrow \mathbb{Z}$$

which can be interpreted as the "*intersection index*" of cycles. In section 7 we will see the precise geometric meaning of this intersection index.

De Rham's duality theorem becomes "*Poincaré duality*":

The homology group with closed supports $H_{n-p}({}_F X)$ is, modulo torsion, the $\mathbb{Z}$-dual to $H_p(X)$ for the "intersection" bilinear form.

Special case. If the manifold X is compact, the family of closed sets coincides with the family of compact sets, and Poincaré duality becomes a duality between the "ordinary" homology groups $H_p(X)$ and $H_{n-p}(X)$.

Corollary. *For every compact connected orientable manifold X of dimension n, $H_n(X) \approx \mathbb{Z}$.*

[11] Many authors call this isomorphism "Poincaré duality".

[12] **Corollary.** *All homology and cohomology groups vanish in dimension $> n$.*

5.6 Leray coboundary

Let X be a differentiable manifold, and let S be a *closed* submanifold of codimension r. We have already noted (remark 3.6) that in the construction of §3.5 which was used to define the homomorphism δ^*, we associated to every closed differential form φ of S, a form $d\psi$ of X *whose support does not meet* S. On the other hand, the support of ψ can be taken to be an arbitrarily small neighbourhood of ψ (by taking the tubular neighbourhood of §I.6.4 to be sufficiently small). Therefore, if $\operatorname{supp}\varphi \in \Phi|S$, where Φ is a family of supports of X, we can use axiom (Φ_3) for families of supports to ensure that $\operatorname{supp}\psi \in \Phi$, and in this way, $\operatorname{supp} d\psi \in \Phi|X - S$.

From this, we deduce a homomorphism (7)

$$\delta^* : H^p({}^{\Phi}|S) \longrightarrow H^{p+1}({}^{\Phi}|X - S).$$

If the manifolds X and S are oriented, Poincaré's isomorphism applied to δ^* gives a homomorphism between homology groups

$$\delta^* : H_{q-r}({}_{\Phi}|S) \longrightarrow H_{q-1}({}_{\Phi}|X - S) \qquad (q = n - p)$$

which is called the "*Leray coboundary*". It can be interpreted geometrically in the following way:

the retraction $\mu : V \to S$ makes the tubular neighbourhood V into a "fibre bundle" (cf. chap. IV, §2) with base S whose fibres are spheres of dimension r. The boundary $\dot{V}$ of V is therefore fibred by $(r-1)$-spheres. The Leray coboundary map "inflates" $(q-r)$-cycles of S inside $X - S$ by fibring them by these $(r-1)$-spheres.

Special case. $\Phi = c$ (the family of compact sets of X).

$\Phi|S$ and $\Phi|X - S$ are therefore simply the families of compact sets of S and $X - S$, and the Leray coboundary can be written

$$\delta^* : H_{q-r}(S) \longrightarrow H_{q-1}(X - S).$$

6 Currents

Throughout this section, X is an *oriented* differentiable manifold of dimension n.

6.1

A *current* j on X is a continuous linear form[13] on $\Omega({}^{c}X)$. In particular:

[13] The word "continuous" should be understood in the following way: if $\varphi_i(i = 1, 2, \dots)$ are $\mathscr{C}^\infty$ differential forms whose supports are all contained in the same compact set (equipped with a chart) and whose coefficients, calculated in this chart, tend uniformly to zero, along with all their derivatives, as $i \to \infty$, then $j[\varphi_i] \to \infty$ with i.

- *Every chain γ defines a current*

$$\gamma[\varphi] = \langle \varphi \mid \gamma \rangle = \int_\gamma \varphi;$$

- *every differential form ω defines a current*

$$\omega[\varphi] = \int_X \omega \wedge \varphi.$$

If $j[\varphi]$ only differs from zero on the forms φ of degree p, we will say that the *dimension* of the current j is p, or that its *degree* is $n-p$. We will denote by $J_p(X) = J^{n-p}(X)$ the vector space of currents of dimension p (of degree $n-p$).

6.2 The support of a current

We will say that the current j is *zero on an open set D if* $j[\varphi] = 0$ for every differential form φ whose (compact) support is contained in D.

The following theorem shows that the notion of a current is *local.*

Theorem. *If $j = 0$ in a neighbourhood U_x of every point $x \in D$, then $j = 0$ on D.*

To see this, let $\{\pi_x\}_{x\in D}$ be a locally finite partition of unity which is subordinate to the covering $\{U_x\}_{x\in D}$. For every form φ supported on D,

$$j[\varphi] = j\Big[\sum_x \pi_x\varphi\Big] = \sum_x j\,[\pi_x\varphi] = 0$$

(since the support of φ is compact, the sum only has a *finite* number of non-zero terms, which then enables us to use the linearity of j).

Definition. The complement of the largest open set on which $j = 0$ is called the *support* of j (the previous theorem gives a meaning to the notion of the “largest open set where $j = 0$”).

If j is a form (resp. chain), this definition coincides with the notion of the support of a form (resp. chain).

Remark. Clearly, the symbol $j[\varphi]$ can even be defined for forms φ with non-compact support, provided that $\operatorname{supp} j \cap \operatorname{supp} \varphi$ is compact. More generally, we can even define the “convergence” of the symbol $j[\varphi]$, in an analogous way to the convergence of an integral (§I.7.6).

Currents with supports in a family Φ. We will let

$$J_p({}_\Phi X) = J^{n-p}({}_\Phi X)$$

denote the vector space of currents of dimension p with supports in a family Φ. Let $f : {}_\Phi X \to {}_\Psi Y$ be a differentiable map, which is *homologically admissible* (§5.3). We will show that it induces a linear map $f_* \,:\, J_p({}_\Phi X) \to J_p({}_\Psi Y)$, which is the transpose of the linear map f^* on differential forms (the correspondence $f \rightsquigarrow f_*$ will therefore be a covariant functor).

Since f is Φ-proper (property (f_1) of §5.3), the closed set

$$\operatorname{supp} j \cap \operatorname{supp} f^*\psi \subset \operatorname{supp} j \cap f^{-1}(\operatorname{supp} \psi)$$

is *compact* for every current j with supports in Φ and every form ψ with compact support. We can therefore define the current f_*j by the formula $(f_*j)[\psi] = j[f^*\psi]$. On the other hand, one can easily check that

$$\operatorname{supp} f_*j \subset f(\operatorname{supp} j),$$

so that, by property (f_2) of §5.3, $\operatorname{supp} f_*j \in \Psi$.

6.3 The boundary and differential of a current

The current ∂j is defined by

$$\boxed{\partial j[\varphi] = j[d\varphi].}$$

By Stokes' formula, this definition coincides with the usual definition of the boundary when the current j is equal to a chain.

The current dj is defined by

$$\boxed{dj = w\,\partial j,}$$

where w is the linear operator which multiplies a current of degree p by $(-)^p$.

Let us check that this definition coincides with the usual definition for the differential when the current j is equal to a form ω:

$$\begin{aligned} d\omega[\varphi] &= \int_X d\omega \wedge \varphi = \int_X d(\omega \wedge \varphi) - w\omega \wedge d\varphi \\ &= -\int_X w\omega \wedge d\varphi = -w\omega[\varphi] = -\partial w\omega[\varphi] = w\partial\omega[\varphi]. \end{aligned}$$

6.4 Homology of currents

This is defined in the obvious way from the operator ∂ or d (which amounts to the same thing), and we will write it

$$H_pJ({}_\Phi X) = H^{n-p}J({}_\Phi X)$$

for currents of dimension p (of degree $n-p$) with supports in the family Φ.

6.5 Homologies between currents and differential forms

We saw in §6.1 that every differential form defines a current: we therefore have a canonical map

$$\Omega({}^{\Phi}X) \longrightarrow J({}_{\Phi}X).$$

Theorem. *This canonical map induces an isomorphism of cohomology groups:*

$$\boxed{H^p({}^{\Phi}X) \approx H^p J({}_{\Phi}X).}$$

The idea of the proof is to construct a "*regularization operator*" R_ε, which commutes with ∂, and maps every current to a $\mathscr{C}^\infty$ form (by "smoothing" it by a $\mathscr{C}^\infty$ function, in the same way that one regularizes a distribution to produce a function). We can take the parameter ε to be small enough that the operator R_ε modifies the currents by as small an amount as we desire. In particular, the support of $\mathrm{R}_\varepsilon j$ can be taken in an arbitrarily small neighbourhood of the support of j, and can therefore be made to belong to the same family of supports [by axiom (Φ_3) for families of supports].

6.6 Homologies between currents and chains

Since every chain defines a current (§6.1), we have a canonical map

$$C_*({}_{\Phi}X) \longrightarrow J({}_{\Phi}X).$$

Theorem. *This canonical map induces an isomorphism of homology groups:*

$$H_p({}_{\Phi}X) \approx H_p J({}_{\Phi}X).$$

By composing this isomorphism with the isomorphism of §6.5, we obtain Poincaré's isomorphism (§5.4).

6.7 A useful example: the current defined by a closed oriented submanifold

A *closed oriented submanifold* S of dimension p (with or without boundary) obviously defines a current of dimension p:

$$S[\varphi] = \int_S \varphi|S$$

[we required that S be closed to ensure that $\operatorname{supp}(\varphi|S)$ is compact if $\operatorname{supp}\varphi$ is].

The *support* of such a current obviously coincides with the set S.

By Stokes' formula as given in §I.7.7, its *boundary* is the current defined by the oriented submanifold ∂S. In particular, if the submanifold S has no boundary, it defines a "*closed*" current, and therefore a *homology class* on the space X. However, the converse, that every homology class of X can be represented by a submanifold, is in general *false* (it depends on the dimension). This difficult question forms part of the theory of "cobordism" due to Thom [35].

7 Intersection indices

7.1

In §5.5 we defined the *intersection index*[14] $\langle \gamma' \mid \gamma \rangle$ of two cycles (or rather their homology classes). More generally, we will try to define, at least in certain cases, the intersection index $\langle k \mid j \rangle$ of two *currents* j and k.

If k is equal to a differential form, we will set $\langle k \mid j \rangle = j[k]$: this is nothing other than the integral $\int_j k$ if j is equal to a chain, whereas if j is also equal to a differential form,

$$\langle k \mid j \rangle = j[k] = \int_X j \wedge k.$$

Now let j and k be two arbitrary currents. We will say that the symbol $\langle k \mid j \rangle$ has a meaning if, whatever the choice of regularizations R_ε and R'_ε (cf. §6.5), $\langle \mathrm{R}'_\varepsilon k \mid \mathrm{R}_\varepsilon j \rangle$ tends to a limit when $\varepsilon \to 0$, and we define $\langle k \mid j \rangle$ to be equal to this limit.

If $\operatorname{supp} j \cap \operatorname{supp} k$ is compact, and if one of the symbols $\langle k \mid \partial j \rangle$, $\langle dk \mid j \rangle$ has a meaning, then the other also has a meaning and they are equal:

$$\langle \mathrm{R}'k \mid \mathrm{R}\,\partial j \rangle = \langle \mathrm{R}'k \mid \partial \mathrm{R} j \rangle = \langle d\mathrm{R}'k \mid \mathrm{R} j \rangle = \langle \mathrm{R}' dk \mid \mathrm{R} j \rangle.$$

In particular, $\langle k \mid j \rangle = 0$ every time that one of the currents is closed and the other is homologous to 0. Therefore if j and k are two closed currents ($\partial j = \partial k = 0$), $\langle k \mid j \rangle$ *only depends on the homology classes of j and k.*

7.2

We say that the current j is $\mathscr{C}^\infty$ *at a point x* if it is equal, in a neighbourhood U_x of x, to a $\mathscr{C}^\infty$ differential form ω (i.e., if $j - \omega = 0$ in U_x, in the sense of §6.2). The set of points where j is $\mathscr{C}^\infty$ is clearly an open set. Its complement is called the "*singular support*" of the current j. If $\operatorname{supp} j \cap \operatorname{supp} k$ is compact, and if the singular supports of j and k do not meet each other, the symbol $\langle k \mid j \rangle$ clearly has a meaning, since this locally reduces to the case where one of the two currents is a $\mathscr{C}^\infty$ form. In fact, it is possible to prove a stronger theorem (due to de Rham), using certain properties of regularizations:

Theorem. *The symbol $\langle k \mid j \rangle$ has a meaning if* $\operatorname{supp} j \cap \operatorname{supp} k$ *is compact, and if the singular support of each current does not meet the singular support of the boundary of the other current.*

[14] Also called the "*Kronecker index*".

7.3

Suppose that the currents j and k are defined by *closed oriented submanifolds with or without boundaries*, S_1 and S_2 (cf. §6.7). By observing that the support of a $\mathscr{C}^\infty$ form cannot be of measure zero, one sees that the singular supports of these two currents are simply the sets S_1 and S_2 (unless the dimension of one of these submanifolds is n).

The preceding theorem can be stated as follows: *in order for the symbol* $\langle S_2 \mid S_1 \rangle$ *to have a meaning, it is enough for* $S_1 \cap S_2$ *to be compact and*

$$\partial S_1 \cap S_2 = S_1 \cap \partial S_2 = \varnothing .$$

We shall give a simple method for calculating this symbol in the case where the submanifolds S_1 and S_2 (of dimension p and $n-p$) *intersect transversally* (10).

Their intersection is therefore a set of isolated points $x^{(1)}, x^{(2)}, \dots, x^{(N)}$, which is finite because $S_1 \cap S_2$ is compact. Let us say that the orientations of S_1 and S_2 *match* at the point $x^{(i)}$ if the system of tangent vectors at the point $x^{(i)}$:

$$(V_1,\, V_2, \dots, V_p,\, W_1,\, W_2, \dots, W_{n-p})\,,$$

obtained by concatenating a system of indicators $(V_1,\, V_2, \dots, V_p)$ for S_1, and a system of indicators $(W_1,\, W_2, \dots, W_{n-p})$ for S_2, is a system of indicators for the orientation on X.

Theorem. $\langle S_2 \mid S_1 \rangle = N^+ - N^-$, *where* N^+ *(resp.* N^-*) is the number of points* $x^{(i)}$ *where the orientations of* S_1 *and* S_2 *match (resp. do not match).*

It clearly suffices to prove it in *a neighbourhood* of a point $x^{(i)}$. In other words, it suffices to check that if

$$\begin{array}{llll} X = \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}, & \text{oriented by the vectors} & & \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dots, \dfrac{\partial}{\partial x_n}\right); \\ S_1 = \mathbb{R}^p, & \text{''} \quad \text{''} \quad \text{''} & & \left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_p}\right); \\ S_2 = \mathbb{R}^{n-p}, & \text{''} \quad \text{''} \quad \text{''} & & \left(\dfrac{\partial}{\partial x_{p+1}}, \dots, \dfrac{\partial}{\partial x_n}\right), \end{array}$$

then $\langle S_2 \mid S_1 \rangle = 1$. This can be checked directly, by using a regularization construction (which we have not given here).

7.4 Example. Let X be the n-dimensional complex sphere (of real dimension $2n$), and let S be the compact submanifold of dimension n defined by the real part of this sphere. We will give X the canonical orientation coming from its complex analytic structure, and give S an arbitrary orientation. Then the intersection index of S with itself is given by *É. Cartan's formula* [2]:

$$\langle S \mid S \rangle = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 2(-)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. S defines a *closed* current since it has no boundary. By §7.1, the intersection index is not changed if we replace the submanifold S by a submanifold S' which defines a homologous current:

$$\langle S' \mid S \rangle = \langle S \mid S \rangle.$$

We can then ensure that S' intersects S transversally, in order to be able to use §7.3.

It is convenient to observe that the complex sphere X is homeomorphic to the "*tangent bundle*" of the real sphere S, i.e., the space of all tangent vectors at every point of S. If

$$z_k = x_k + iy_k \qquad (k = 0, 1, 2, \ldots, n)$$

are the coordinates of $\mathbb{C}^{n+1}$, the equation of X in $\mathbb{C}^{n+1}$ can be written:

$$z \cdot z = \sum_{k=0}^{n} z_k^2 = 1, \qquad \text{i.e.,} \quad \begin{cases} x \cdot x - y \cdot y = 1, \\ \qquad\quad x \cdot y = 0. \end{cases}$$

Now map any point $z \in X$ to the point with coordinates $x_k/\sqrt{1 + y \cdot y}$ in the real Euclidean space $\mathbb{R}^{n+1}$, and to the vector with components y_k which is based at this point. This clearly defines a homeomorphism from X onto the space of tangent vectors of the unit sphere $S \subset \mathbb{R}^{n+1}$. In this way, every vector field which is tangent to the unit sphere represents a submanifold $S' \subset X$, which is clearly homologous[15] to the original submanifold S (represented by the zero vector field). Thus, we are reduced, by §7.3, to study the points where such a vector field vanishes. Let us take, for example, the vector field $y(x)$ defined by the coordinates (in the ambient Euclidean space)

$$\begin{aligned} y_k(x) &= \lambda x_k x_0 \qquad (k = 1, 2, \ldots, n), \\ y_0(x) &= \lambda \left(x_0^2 - 1 \right), \end{aligned}$$

where λ is an arbitrary positive parameter (Fig. II.2).

This vector field is indeed tangent to the sphere, since

$$x \cdot y(x) = \lambda x_0 (x \cdot x - 1) = 0.$$

Furthermore, it only vanishes at the two poles $P^{\pm}$ of the sphere

$$P^{\pm} : x_1 = x_2 = \cdots = x_n = 0, \qquad x_0 = \pm 1.$$

Near these two poles, the coordinates $x_1, x_2, \ldots, x_n$ provide a local chart of the sphere, and the field $y(x)$ is approximately

$$y_k(x) \simeq \pm \lambda x_k \quad \text{near the point } P^{\pm}.$$

[15] In fact, every vector field on a manifold is obviously homotopic to the zero vector field: it suffices to multiply it by a scalar λ which tends to zero.

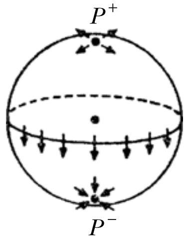

Fig. II.2.

As a result, if the manifold S is described near P^+ by the system of indicators

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n},$$

the manifold S' will be described by the system of indicators

$$\frac{\partial}{\partial x_1} + \lambda \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} + \lambda \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial x_n} + \lambda \frac{\partial}{\partial y_n}$$

(which tends to the first system when $\lambda \to 0$, which shows that we have oriented S' in the correct way) (cf. Fig. II.3).

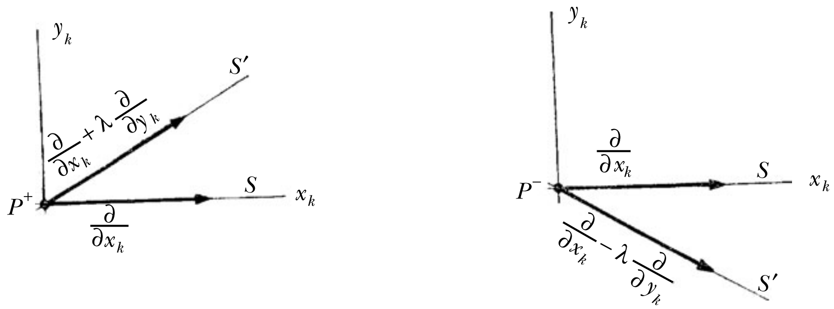

Fig. II.3.

We then see, by recalling that the canonical orientation of a complex manifold is given by the system of indicators

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n},$$

that the orientations of S and S' at the point P^+ match, up to the sign $(-)^{n(n-1)/2}$. A similar argument at the point P^- gives the same result multiplied by $(-)^n$ [because of the minus sign in the equation $y_k(x) = -\lambda x_k$].

We therefore obtain the result stated earlier:

$$\langle S' \mid S \rangle = (-)^{n(n-1)/2} \left(1 + (-)^n\right).$$

Remarks.

(i) The result for odd n could be obtained directly by using the skew symmetry of the intersection index

$$\langle S' \mid S \rangle = (-)^{\dim S \ \dim S'} \langle S \mid S' \rangle,$$

which in this case gives:

$$\langle S \mid S \rangle = (-)^{n^2} \langle S \mid S \rangle \cdot$$

(ii) A corollary of the result for even n is the well-known theorem which states that a continuous vector field on an even-dimensional sphere necessarily vanishes somewhere.

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